ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$ and $(\sqrt{a}+2)(\sqrt{b}+2) \geq 9$ then:

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1} \geq 1$$

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$$(\sqrt{a}+2)(\sqrt{b}+2) \geq 9$$

$$\sqrt{ab}+2(\sqrt{a}+\sqrt{b})+4 \geq 9 \text{ or } \sqrt{ab}+2(\sqrt{a}+\sqrt{b}) \geq 5$$

$$\frac{a+b}{2}+2\sqrt{2(a+b)} \overset{AM-GM\&CBS}{\overbrace{\geq}} \sqrt{ab}+2(\sqrt{a}+\sqrt{b}) \geq 5$$

$$a+b+4\sqrt{2(a+b)}-10 \geq 0$$

$$u^2+4\sqrt{2}u-10 \overset{u=\sqrt{a+b}>0}{\geq} 0 \text{ or } (u+5\sqrt{2})(u-\sqrt{2}) \geq 0 \text{ or}$$

$$u-\sqrt{2} \geq 0 \ (as\ u>0) \text{ or } u \geq \sqrt{2} \text{ or } u^2 \geq 2 \text{ or}$$

$$a+b \geq 2 \ (1)$$

$$\frac{a^3}{a^2+1}+\frac{b^3}{b^2+1} = (a+b)-\left(\frac{a}{a^2+1}+\frac{b}{b^2+1}\right) \overset{AM-GM}{\geq}$$

$$\geq (a+b)-\left(\frac{a}{2a}+\frac{b}{2b}\right) \overset{(1)}{\geq} 2-\left(\frac{1}{2}+\frac{1}{2}\right) = 1$$

$$Equality\ holds\ for\ a=b=1$$