ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0,(\sqrt{a}+1)(\sqrt{b}+1)=4$ then:

$$\frac{a^3}{(a+1)^2}+\frac{b^3}{(b+1)^2}\ge\frac{1}{2}$$

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$$(\sqrt{a}+1)(\sqrt{b}+1)=4\ or\ \sqrt{ab}+(\sqrt{a}+\sqrt{b})+1\ge 4\ or$$

$$\frac{a+b}{2}+\sqrt{2(a+b)}\ge 3\ (\ Am-Gm\ \&\ CBS)$$

$$u^2+2\sqrt{2}u-6\overset{u=\sqrt{a+b}>0}{\ge}0\ \ or\ (u+3\sqrt{2})(u-\sqrt{2})\ge 0\ or$$

$$u-\sqrt{2}\ge 0\ (as\ \ u>0)\ or\ \ u\ge\sqrt{2}\ or\ \ u^2\ge 2\ or,a+b\ge 2\ (1)$$

$$\frac{a^3}{(a+1)^2}+\frac{b^3}{(b+1)^2}\overset{CBS}{\ge}\left(\frac{a^3}{2(a^2+1)}+\frac{b^3}{2(b^2+1)}\right)=$$

$$=\frac{1}{2}\left(a-\frac{a}{a^2+1}+b-\frac{b}{b^2+1}\right)\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2}\left((a+b)-\left(\frac{a}{2a}+\frac{b}{2b}\right)\right)\overset{(1)}{\ge}\frac{1}{2}\left(\ 2-\left(\frac{1}{2}+\frac{1}{2}\right)\right)=\frac{1}{2}$$

$$Equality\ holds\ \ for\ a=b=1$$