ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $xy + yz + zx = 3xyz$ then:

$$\frac{1}{x^3+y+z}+\frac{1}{y^3+z+x}+\frac{1}{z^3+x+y}\le 1$$

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$$xy + yz + zx = 3xyz$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$$

$$3=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\overset{Bergstrom}{\ge}\frac{(1+1+1)^2}{x+y+z}$$

$$3(x+y+z)\ge 9$$

$$x+y+z\ge 3\ (1)$$

$$x+y+z\ge 3$$

$$y+z\ge 3-x$$

$$x^3+y+z\overset{(1)}{\ge}x^3+3-x=(x^3+1+1)+1-x\overset{AM-GM}{\ge}3x+1-x=2x+1\ \ (2)$$

$$\frac{1}{x^3+y+z}+\frac{1}{y^3+z+x}+\frac{1}{z^3+x+y}=\sum\frac{1}{x^3+y+z}\overset{(2)}{\le}$$

$$\le\sum\frac{1}{2x+1}=\sum\frac{1}{x+x+1}\overset{AM-HM}{\le}\frac{1}{9}\sum\left(\frac{1}{x}+\frac{1}{x}+1\right)=$$

$$=\frac{1}{9}\left(\sum\frac{1}{x}+\sum\frac{1}{x}+\sum 1\right)\overset{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3}{=}\frac{1}{9}(3+3+3)=1$$

$$Equality\ holds\ for\ x=y=z=1$$