ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2+2abc=1$, then prove that :

$$\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+1\right)^2\ge 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+1$$

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$$a^2+b^2+c^2+2abc=1\Rightarrow a^2+2bc.a+b^2+c^2-1=0$$
$$\Rightarrow a=\frac{-2bc\pm 2\sqrt{b^2c^2-b^2-c^2+1}}{2}=-bc\pm\sqrt{(1-b^2)(1-c^2)}$$
$$\left(\because 1-b^2=c^2+a^2+2abc>0 \text{ and analogously}, 1-c^2>0\right)$$
$$=-bc+\sqrt{(1-b^2)(1-c^2)}\ (\because a>0)$$
$$=-bc+\frac{1}{3}.\sqrt{9(1-b)(1-c)*(1+b)(1+c)}$$
$$\overset{A-G}{\le}-bc+\frac{9(1-b)(1-c)+(1+b)(1+c)}{6}=-bc+\frac{10-8(b+c)+10bc}{6}$$
$$=\frac{5-4(b+c)+2bc}{3}\Rightarrow 3a\le 5-4(b+c)+2bc \text{ and analogs}$$
$$\therefore 3\sum_{cyc}a\le 15-8\sum_{cyc}a+2\sum_{cyc}ab\Rightarrow 11\sum_{cyc}a\le 15+2\sum_{cyc}ab\le 15+\frac{2}{3}\left(\sum_{cyc}a\right)^2$$
$$\Rightarrow 2t^2-33t+45\ge 0\ \left(t=\sum_{cyc}a\right)\Rightarrow (2t-3)(t-15)\ge 0\Rightarrow t\le\frac{3}{2}$$
$$\left(\because a,b,c<1\Rightarrow\sum_{cyc}a<3\Rightarrow t\ngeq 15\right)\therefore\frac{3}{4}\ge\frac{1}{2}\sum_{cyc}a\to (1)$$
$$\text{Again}, 1=a^2+b^2+c^2+2abc\overset{A-G}{\ge}3p^2+2p^3\ \left(p=\sqrt[3]{abc}\right)$$
$$\Rightarrow(2p-1)(p+1)^2\le 0\Rightarrow p=\sqrt[3]{abc}\le\frac{1}{2}\Rightarrow 1-2abc\ge\frac{3}{4}$$
$$\Rightarrow\sum_{cyc}a^2\ge\frac{3}{4}\overset{\text{via }(1)}{\ge}\frac{1}{2}\sum_{cyc}a\to (2)$$
$$\text{We have}:\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+1\right)^2\ge 4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+1$$
$$\Leftrightarrow\left(\sum_{cyc}a^2+abc\right)^2\ge 4\sum_{cyc}a^2b^2+a^2b^2c^2\overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow}$$
$$(1-abc)^2\ge 4\sum_{cyc}a^2b^2+a^2b^2c^2\Leftrightarrow 1-2abc\ge 4\sum_{cyc}a^2b^2\overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow}$$
$$\sum_{cyc}a^2\ge 4\sum_{cyc}a^2b^2\overset{a^2+b^2+c^2+2abc=1}{\Leftrightarrow}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2+2abc\right)\ge 4\sum_{cyc}a^2b^2$$

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$$\Leftrightarrow \sum_{cyc} a^4 + 2\sum_{cyc} a^2b^2 + 2abc\left(\sum_{cyc} a^2\right) \geq 4\sum_{cyc} a^2b^2$$

$$\Leftrightarrow \sum_{cyc} a^4 + 2abc\left(\sum_{cyc} a^2\right) \overset{(*)}{\geq} 2\sum_{cyc} a^2b^2$$

$$\text{Now, via ②, } \sum_{cyc} a^4 + 2abc\left(\sum_{cyc} a^2\right) \geq \sum_{cyc} a^4 + abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{Schur}}{\geq} \sum_{cyc} a^3b + \sum_{cyc} ab^3 \overset{A-G}{\geq} 2\sum_{cyc} a^2b^2 \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + 1\right)^2 \geq 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

$$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 + 2abc = 1, '' ='' \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$