If $a,b,c>0$ and $(a+b)(b+c)(c+a)=8$, then prove that :

$$\frac{\sqrt{a^2+ab+b^2}}{\sqrt{ab}+2}+\frac{\sqrt{b^2+bc+c^2}}{\sqrt{bc}+2}+\frac{\sqrt{c^2+ca+a^2}}{\sqrt{ca}}\ge\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z$ and then : $\sum_{cyc}\frac{\sqrt{b^2+bc+c^2}}{\sqrt{bc}+2}\ge\sum_{cyc}\frac{\frac{\sqrt{3}}{2}(y^2+z^2)}{yz+2}$

$$=\sqrt{3}.\sum_{cyc}\frac{(y^2+z^2)^2}{2yz(y^2+z^2)+4(y^2+z^2)}\overset{\text{Bergstrom}}{\ge}\sqrt{3}.\frac{\left(\sum_{cyc}(y^2+z^2)\right)^2}{\sum_{cyc}\left(2yz\left(\sum_{cyc}x^2-x^2\right)\right)+8\sum_{cyc}x^2}$$

$$\overset{\prod_{cyc}(x^2+y^2)=8}{=}\sqrt{3}.\frac{2\left(\sum_{cyc}x^2\right)^2}{\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)-xyz\sum_{cyc}x+2\left(\sum_{cyc}x^2\right).\sqrt[3]{\left(\prod_{cyc}(x^2+y^2)\right)}}\overset{?}{\ge}\sqrt{3}$$

$$\Leftrightarrow 2\left(\sum_{cyc}x^2\right)^2-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)+xyz\sum_{cyc}x\overset{?}{\ge}2\left(\sum_{cyc}x^2\right).\sqrt[3]{\left(\prod_{cyc}(x^2+y^2)\right)}$$

$$\Leftrightarrow\left(2\left(\sum_{cyc}x^2\right)^2-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)+xyz\sum_{cyc}x\right)^3$$

$$\overset{?}{\ge}8\left(\prod_{cyc}(x^2+y^2)\right)\left(\sum_{cyc}x^2\right)^3$$

$$\Leftrightarrow\left(2\left(\sum_{cyc}x^2\right)^2-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)+xyz\sum_{cyc}x\right)^3$$

$$\overset{?}{\ge}_{(*)}8\left(\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2y^2\right)-x^2y^2z^2\right)\left(\sum_{cyc}x^2\right)^3$$

Assigning $y+z=X,z+x=Y,x+y=Z\Rightarrow X+Y-Z=2z>0,Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - X, y = s - Y, z = s - Z$$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and such substitutions} \Rightarrow \sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y)$$

$$\Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and } \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and also,}$$

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R + r)^2 - 2s^2) \therefore (\bullet), (\bullet\bullet), (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet) \Rightarrow (*) \Leftrightarrow$$

$$(2(s^2 - 8Rr - 2r^2)^2 - (s^2 - 8Rr - 2r^2)(4Rr + r^2) + r^2s^2)^3 \overset{?}{\geq}$$
$$8(r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2) - r^4s^2)(s^2 - 8Rr - 2r^2)^3$$
$$\Leftrightarrow s^{12} - (54Rr + 10r^2)s^{10} + r^2(1196R^2 + 480Rr + 47r^2)s^8$$
$$-r^3(13960R^3 + 8976R^2r + 1908Rr^2 + 134r^3)s^6$$
$$+r^4(90816R^4 + 82400R^3r + 27996R^2r^2 + 4224Rr^3 + 239r^4)s^4$$
$$-r^5(312832R^5 + 373248R^4r + 178240R^3r^2 + 42592R^2r^3 + 5094Rr^4 + 244r^5)s^2$$
$$+r^6\begin{pmatrix} 446464R^6 + 669696R^5r + 418560R^4r^2 + 139520R^3r^3 + 26160R^2r^4 \\ +2616Rr^5 + 109r^6 \end{pmatrix} \boxed{\underset{(**)}{\overset{?}{\geq}}} 0$$

$$\text{Now, } 3132t^3 - 9088t^2 + 9000t - 3037 \left(t = \frac{R}{r}\right)$$
$$= (t - 2)(3132t^2 - 2824t + 3352) + 3667 \overset{\text{Euler}}{\geq} 3667 > 0,$$
$$14496t^4 - 57364t^3 + 86922t^2 - 59604t + 15587$$
$$= (t - 2)(14496t^3 - 28372t^2 + 30178t + 752) + 17091 \overset{\text{Euler}}{\geq} 17091 > 0 \text{ and}$$
$$8000t^5 - 41552t^4 + 87477t^3 - 92947t^2 + 49798t - 10773$$
$$= (t - 2)(t^2(8000t^2 - 25552t + 26272) + 10101t(t - 2) + t + 9396) + 8019$$
$$\overset{\text{Euler}}{\geq} 8019 \begin{pmatrix} \because \text{discriminant of } (8000t^2 - 25552t + 26272) = \\ -187799296 < 0 \Rightarrow t^2(8000t^2 - 25552t + 26272) > 0 \end{pmatrix} > 0$$
$$\therefore P = (s^2 - 16Rr + 5r^2)^6 + 16r(21R - 20r)(s^2 - 16Rr + 5r^2)^5$$
$$+16r^2(358R^2 - 685Rr + 336r^2)(s^2 - 16Rr + 5r^2)^4$$
$$+16r^3(3132R^3 - 9088R^2r + 9000Rr^2 - 3037r^3)(s^2 - 16Rr + 5r^2)^3$$
$$+16r^4\begin{pmatrix} 14496R^4 - 57364R^3r + 86922R^2r^2 \\ -59604Rr^3 + 15587r^4 \end{pmatrix}(s^2 - 16Rr + 5r^2)^2$$
$$+64r^5\begin{pmatrix} 8000R^5 - 41552R^4r + 87477R^3r^2 - 92947R^2r^3 \\ +49798Rr^4 - 10773r^5 \end{pmatrix}(s^2 - 16Rr + 5r^2)$$

$\overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove : $\boxed{\text{LHS of } (**) \overset{?}{\geq} P}$

$$\Leftrightarrow 5632t^6 - 40256t^5 + 117000t^4 - 177919t^3$$

$$+150265t^2 - 67251t + 12538 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)\Big((t-2)\big((t-2)(5632t^3 - 6464t^2 + 10632t + 8497) + 21951\big) + 3645\Big)$$

$$\overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\sqrt{a^2+ab+b^2}}{\sqrt{ab}+2} + \frac{\sqrt{b^2+bc+c^2}}{\sqrt{bc}+2} + \frac{\sqrt{c^2+ca+a^2}}{\sqrt{ca}} \geq \sqrt{3}$$

$$\forall\, a,b,c > 0 \,\Big|\, (a+b)(b+c)(c+a) = 8, \text{''} = \text{''} \text{ iff } a = b = c = 1 \text{ (QED)}$$