ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2+2abc=1$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 2\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right)$$

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Let $x:=\frac{bc}{a}, y:=\frac{ca}{b}, z:=\frac{ab}{c}$. The given condition becomes $xy+yz+zx+2xyz=1$ or

$\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=2$, and we need to prove that $x+y+z\geq 2(xy+yz+zx)$.

By CBS inequality, we have :

$$1=\frac{x}{1+x}+\frac{y}{1+y}+\frac{z}{1+z}\geq\frac{(x+y+z)^2}{x(1+x)+y(1+y)+z(1+z)}$$

then $x+y+z\geq 2(xy+yz+zx)$

Equality holds iff $a=b=c=\frac{1}{2}$.