ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 + 2abc = 1$, then prove that :

$$\left(\frac{a}{bc}\right)^3 + \left(\frac{b}{ca}\right)^3 + \left(\frac{c}{ab}\right)^3 \ge 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

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$$a^2 + b^2 + c^2 + 2abc = 1 \Rightarrow a^2 + 2bc.a + b^2 + c^2 - 1 = 0$$

$$\Rightarrow a = \frac{-2bc \pm 2\sqrt{b^2c^2 - b^2 - c^2 + 1}}{2} = -bc \pm \sqrt{(1 - b^2)(1 - c^2)}$$

$$(\because 1 - b^2 = c^2 + a^2 + 2abc > 0 \text{ and analogously}, 1 - c^2 > 0)$$

$$= -bc + \sqrt{(1 - b^2)(1 - c^2)}\ (\because a > 0)$$

$$= -bc + \frac{1}{3}.\sqrt{9(1 - b)(1 - c) * (1 + b)(1 + c)}$$

$$\overset{A-G}{\le} -bc + \frac{9(1 - b)(1 - c) + (1 + b)(1 + c)}{6} = -bc + \frac{10 - 8(b + c) + 10bc}{6}$$

$$= \frac{5 - 4(b + c) + 2bc}{3} \Rightarrow 3a \le 5 - 4(b + c) + 2bc \text{ and analogs}$$

$$\therefore 3\sum_{cyc} a \le 15 - 8\sum_{cyc} a + 2\sum_{cyc} ab \Rightarrow 11\sum_{cyc} a \le 15 + 2\sum_{cyc} ab \le 15 + \frac{2}{3}\left(\sum_{cyc} a\right)^2$$

$$\Rightarrow 2t^2 - 33t + 45 \ge 0\ \left(t = \sum_{cyc} a\right) \Rightarrow (2t - 3)(t - 15) \ge 0 \Rightarrow t = \sum_{cyc} a \le \frac{3}{2}$$

$$\left(\because a, b, c < 1 \Rightarrow \sum_{cyc} a < 3 \Rightarrow t \ngeq 15\right) \therefore 1 \ge \frac{2}{3}\sum_{cyc} a \rightarrow (\blacksquare)$$

$$\text{Now, } \left(\frac{a}{bc}\right)^3 + \left(\frac{b}{ca}\right)^3 + \left(\frac{c}{ab}\right)^3 \overset{\text{via } (\blacksquare)}{\ge} \frac{2}{3a^3b^3c^3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^6\right) \overset{?}{\ge} 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$= \frac{2}{a^2b^2c^2}\left(\sum_{cyc} a^2b^2\right) \Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^6\right) \overset{?}{\underset{(*)}{\ge}} 3abc\left(\sum_{cyc} a^2b^2\right)$$

$$\text{But, } \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^6\right) \overset{\text{Chebyshev}}{\ge} \frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) \ge$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right) \overset{A-G}{\ge} \frac{1}{3}.9abc.\left(\sum_{cyc} a^2b^2\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\frac{a}{bc}\right)^3 + \left(\frac{b}{ca}\right)^3 + \left(\frac{c}{ab}\right)^3 \ge 2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

$$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 + 2abc = 1, \text{''}=\text{''} \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$