ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2=12$, then prove that:

$$(a^3+4a+8)(b^3+4b+8)(c^3+4c+8)\le 24^3$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $a^3+4a+8=\dfrac{(a^2+8)^2-(a-2)^2(a^2-2a+4)}{6}\le\dfrac{(a^2+8)^2}{6},\quad \forall a>0$, then

$$(a^3+4a+8)(b^3+4b+8)(c^3+4c+8)\le\frac{1}{6^3}\left[(a^2+8)(b^2+8)(c^2+8)\right]^2$$

$$\overset{AM-GM}{\le}\frac{1}{6^3}\left[\left(\frac{a^2+8+b^2+8+c^2+8}{3}\right)^3\right]^2=24^3$$

Equality holds iff $a=b=c=2$.