ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, ab+bc+ca+2abc=1$ then:

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

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$$Let\ a=\frac{x}{y+z}, b=\frac{y}{z+x}, c=\frac{z}{x+y}\ then\ ab+bc+ca+2abc=1$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\left(\frac{1}{a}\right)^3+\left(\frac{1}{b}\right)^3+\left(\frac{1}{c}\right)^3\overset{Chebycv}{\geq}\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2\right)\geq$$

$$\geq\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$We\ need\ to\ show:$$

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$or\ \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$or\ \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 2\ or\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 6\ or\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\geq 6\ true$$

$$since,\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}=\left(\frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z}\right)\overset{AM-GM}{\geq}$$

$$\geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}}=6$$

$$Equality\ holds\ for\ x=y=z\ or\ a=b=c=\frac{1}{2}.$$