ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2+abc=4$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$$

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$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\Leftrightarrow\sum_{\text{cyc}}a^2b^2\ge abc\sum_{\text{cyc}}a^2$$

$$\overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow}a^2(b^2+c^2)+b^2c^2\ge abc(4-abc)$$

$$\overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow}a^2(4-a^2-abc)+b^2c^2\ge 4abc-a^2b^2c^2$$

$$\Leftrightarrow(1+a^2)b^2c^2-(a^3+4a)bc+4a^2-a^4\overset{(*)}{\ge}0$$

Now, LHS of $(*)$ is a quadratic polynomial in bc with discriminant, $\delta=(a^3+4a)^2-4(1+a^2)(4a^2-a^4)=a^4(5a^2-4)$ and if $a^2\le\frac{4}{5}$, $(*)$ is trivially true and so, we now focus on the scenario when : $a^2>\frac{4}{5}$ and in order to prove $(*)$, it suffices to prove : $bc\overset{(**)}{\le}\frac{a^3+4a-a^2.\sqrt{5a^2-4}}{2(1+a^2)}$

Now, $4-a^2=b^2+c^2+abc\overset{A-G}{\ge}bc(2+a)\Rightarrow bc\le 2-a\overset{?}{\le}\frac{a^3+4a-a^2.\sqrt{5a^2-4}}{2(1+a^2)}\Leftrightarrow 3a^3-4a^2+6a-4\underset{(***)}{\overset{?}{\ge}}a^2.\sqrt{5a^2-4}$

We have : $3a^3-4a^2+6a-4=\frac{1}{81}\left(\frac{(9a-4)^3}{3}+342\left(a-\frac{454}{513}\right)\right)>0$

$\because a>\frac{2}{\sqrt{5}}>\frac{454}{513}>\frac{4}{9}\therefore(***)\Leftrightarrow(3a^3-4a^2+6a-4)^2\overset{?}{\ge}a^4(5a^2-4)$

$$\Leftrightarrow a^6-6a^5+14a^4-18a^3+17a^2-12a+4\overset{?}{\ge}0$$

$\Leftrightarrow(a-1)^2(a-2)^2(a^2+1)\overset{?}{\ge}0\rightarrow$ true $\therefore(***)\Rightarrow(**)\Rightarrow(*)$ is true

$\therefore\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\ \forall\, a,b,c>0\,\middle|\,a^2+b^2+c^2+abc=4$, " = " iff $a=b=c=1$ (QED)