ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 + abc = 4$, then prove that :

$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

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$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1 \overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow}$$

$$\frac{1}{a^2b^2c^2}\left(2\sum_{cyc} a^2 + 4 - \sum_{cyc} a^2\right)^2 \geq \frac{1}{a^2b^2c^2}\left(16\sum_{cyc} a^2b^2 + \left(4 - \sum_{cyc} a^2\right)^2\right)$$

$$\Leftrightarrow 16 + \left(\sum_{cyc} a^2\right)^2 + 8\sum_{cyc} a^2 \geq 16\sum_{cyc} a^2b^2 + 16 + \left(\sum_{cyc} a^2\right)^2 - 8\sum_{cyc} a^2$$

$$\Leftrightarrow a^2 + b^2 + c^2 \geq a^2(b^2 + c^2) + b^2c^2 \overset{a^2+b^2+c^2+abc=4}{\Leftrightarrow}$$

$$a^2 - b^2c^2 + (1 - a^2)(4 - a^2 - abc) \geq 0$$

$$\Leftrightarrow a^2 + (1 - a^2)(4 - a^2) \boxed{\overset{(*)}{\geq}} b^2c^2 + a(1 - a^2)bc$$

Now, $4 - a^2 = b^2 + c^2 + abc \overset{A-G}{\geq} bc(2 + a) \Rightarrow bc \leq 2 - a \Rightarrow$ LHS of $(*) \leq$ $(2 - a)^2 + a(1 - a^2)(2 - a) \overset{?}{\leq} a^2 + (1 - a^2)(4 - a^2) \Leftrightarrow 2a(a^2 - 2a + 1) \overset{?}{\geq} 0$

$$\Leftrightarrow 2a(a - 1)^2 \overset{?}{\geq} 0 \to \text{true} \therefore (*) \text{ is true} \Rightarrow$$

$$\left(\frac{2a}{bc} + \frac{2b}{ca} + \frac{2c}{ab} + 1\right)^2 \geq 16\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + 1$$

$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 + abc = 4$, " = " iff $a = b = c = 1$ (QED)