ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ then prove that :

$$a + b + c \ge \frac{3}{a + b + c} + \frac{2}{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a + b + c \ge \frac{3}{a + b + c} + \frac{2}{abc} \Leftrightarrow \frac{(\sum_{cyc} a)(\sum_{cyc} ab)}{abc \sum_{cyc} a} \ge \frac{3}{\sum_{cyc} a} + \frac{2}{abc} \cdot \frac{abc \sum_{cyc} a}{\sum_{cyc} ab} \left(\because 1 = \frac{\sum_{cyc} ab}{abc \sum_{cyc} a}\right)$$

$$\Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)^2 \overset{(*)}{\ge} 3abc \sum_{cyc} ab + 2abc\left(\sum_{cyc} a\right)^2$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - x)(s - y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \text{ and via } (1), (2) \text{ and } (3), (*) \Leftrightarrow$$

$$s(4Rr + r^2)^2 \ge 3r^2 s(4Rr + r^2) + 2r^2 s^3 \Leftrightarrow s^2 \le 8R^2 - 2Rr - r^2 \to \text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\le} 4R^2 + 4Rr + 3r^2 = 8R^2 - 2Rr - r^2 - 2(R - 2r)(2R + r) \overset{\text{Euler}}{\le} 8R^2 - 2Rr - r^2 \Rightarrow (*) \text{ is true} \therefore a + b + c \ge \frac{3}{a + b + c} + \frac{2}{abc}$$

$$\forall\, a, b, c > 0 \,\Big|\, a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$