ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ then prove that:

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \le 1$$

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$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \le 1 \Leftrightarrow \sum_{cyc}\left((b^2+2)(c^2+2)\right)$$

$$\le (a^2+2)(b^2+2)(c^2+2) \Leftrightarrow a^2b^2c^2 + \sum_{cyc} a^2b^2 \ge 4$$

$$\Leftrightarrow \frac{\sum_{cyc} ab}{abc\sum_{cyc} a}.a^2b^2c^2 + \sum_{cyc} a^2b^2 \ge 4\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^2 \left(\because 1 = \frac{\sum_{cyc} ab}{abc\sum_{cyc} a}\right)$$

$$\Leftrightarrow \left(abc\sum_{cyc} ab + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right)\right)\left(\sum_{cyc} ab\right)^2 \overset{(*)}{\ge} 4a^2b^2c^2\left(\sum_{cyc} a\right)^3$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow$ x, y, z form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3) \text{ and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \rightarrow (4)$$

and via (1), (2), (3) and (4), $(*) \Leftrightarrow$

$$\left(r^2 s(4Rr + r^2) + sr^2((4R+r)^2 - 2s^2)\right)\left((4Rr + r^2)\right)^2 \ge 4r^4 s^5$$

$$\Leftrightarrow ((4R+r)^2 - 2s^2)(4R+r)^2 + r(4R+r)^3 \overset{(**)}{\ge} 4s^4$$

Now, via Doucet (or Trucht) and via Gerretsen, LHS of $(**)$ − RHS of $(**) \ge$ $s^2(16R^2 + 8Rr + r^2) + r(4R+r)^3 - 4s^2(4R^2 + 4Rr + 3r^2)$

$$= r(4R+r)^3 - s^2(8Rr + 11r^2) \overset{\text{Gerretsen}}{\ge}$$

$$\mathbf{r(4R+r)^3-(4R^2+4Rr+3r^2)(8Rr+11r^2)=4r(R-2r)(8R^2+9Rr+4r^2)}$$

$$\overset{\text{Euler}}{\geq} \mathbf{0} \Rightarrow (**) \Rightarrow (*) \textbf{ is true} \therefore \frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2} \leq 1$$

$$\forall\, a,b,c>0 \,\Big|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}, ''='' \textbf{ iff } a=b=c=1 \text{ (QED)}$$