ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca + 2abc = 1$, then prove that:

$$a + b + c \geq 2(ab + bc + ca)$$

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$$ab + bc + ca + 2abc = 1 \Rightarrow a(b + c + 2bc) = 1 - bc \Rightarrow b + c = \frac{1 - bc}{a} - 2bc \text{ and so, } a + b + c \geq 2(ab + bc + ca) \text{ becomes:}$$

$$a + \frac{1 - bc}{a} - 2bc \geq 2(1 - 2abc) \left(\because \sum_{cyc} ab = 1 - 2abc \right)$$

$$\Leftrightarrow a^2 + 1 - bc - 2abc \geq 2a - 4a^2bc \Leftrightarrow (1 + 4bc)a^2 - 2(1 + bc)a + 1 - bc \overset{(*)}{\geq} 0$$

Now, LHS of $(*)$ is a quadratic polynomial in a with discriminant, $\delta = 4(1 + bc)^2 - 4(1 + 4bc)(1 - bc) = 4(5b^2c^2 - bc)$ and if $bc \leq \frac{1}{5}$, $(*)$ is trivially true and so, we now focus on the scenario when: $bc > \frac{1}{5}$ and in order to prove $(*)$, it suffices to prove: $a \overset{(**)}{\leq} \frac{1 + bc - \sqrt{5b^2c^2 - bc}}{1 + 4bc}$

$$\text{Now, } 1 - bc = a(b + c) + 2abc \overset{A-G}{\geq} 2a.\sqrt{bc} + 2abc = 2a.\sqrt{bc}\left(1 + \sqrt{bc}\right)$$

$$\Rightarrow \left(1 + \sqrt{bc}\right)\left(1 - \sqrt{bc}\right) \geq 2a.\sqrt{bc}\left(1 + \sqrt{bc}\right) \Rightarrow 1 - \sqrt{bc} \geq 2a.\sqrt{bc}$$

$$\Rightarrow a \leq \frac{1 - \sqrt{bc}}{2\sqrt{bc}} \overset{?}{\leq} \frac{1 + bc - \sqrt{5b^2c^2 - bc}}{1 + 4bc} \Leftrightarrow \frac{1 - m}{2m} \overset{?}{\leq} \frac{1 + m^2 - \sqrt{5m^4 - m^2}}{1 + 4m^2}$$

$$\left(m = \sqrt{bc}\right) \Leftrightarrow 6m^3 - 4m^2 + 3m - 1 \underset{(***)}{\overset{?}{\geq}} 2m.\sqrt{5m^4 - m^2}$$

$$\text{We have: } 6m^3 - 4m^2 + 3m - 1 = \frac{1}{81}\left(\frac{2(9m - 2)^3}{3} + 171\left(m - \frac{227}{513}\right)\right) > 0$$

$$\because bc > \frac{1}{5} \Rightarrow m > \frac{1}{\sqrt{5}} > \frac{227}{513} > \frac{2}{9} \therefore (***) \Leftrightarrow$$

$$(6m^3 - 4m^2 + 3m - 1)^2 \overset{?}{\geq} 4m^2(5m^4 - m^2)$$

$$\Leftrightarrow 16m^6 - 48m^5 + 56m^4 - 36m^3 + 17m^2 - 6m + 1 \overset{?}{\geq} 0$$

$$\Leftrightarrow (m - 1)^2(2m - 1)^2(4m^2 + 1) \overset{?}{\geq} 0 \rightarrow \text{true} \therefore (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore a + b + c \geq 2(ab + bc + ca)\ \forall\, a, b, c > 0 \,\middle|\, ab + bc + ca + 2abc = 1,$$

$$'' ='' \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$