ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC, if $a^2+b^2+c^2=3$, then prove that :

$$\frac{1}{3a+bc}+\frac{1}{3b+ca}+\frac{1}{3c+ab}+\frac{ab+bc+ca}{2}\le\frac{9}{4}$$

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$$\frac{1}{3a+bc}\overset{a^2+b^2+c^2=3}{=}\frac{1}{\sqrt{3\sum_{cyc}a^2}\,.\,a+bc}\le\frac{1}{a(a+b+c)+bc}$$

$$=\frac{1}{a(a+b)+c(a+b)}=\frac{1}{(a+b)(c+a)}$$

$$\Rightarrow\frac{1}{3a+bc}\le\frac{b+c}{2s(s^2+2Rr+r^2)}\text{ and analogs}$$

$$\Rightarrow\frac{1}{3a+bc}+\frac{1}{3b+ca}+\frac{1}{3c+ab}+\frac{ab+bc+ca}{2}\overset{a^2+b^2+c^2=3}{\le}$$

$$\left(\sum_{cyc}a^2\right).\sum_{cyc}\frac{b+c}{6s(s^2+2Rr+r^2)}+\frac{3\sum_{cyc}ab}{2\sum_{cyc}a^2}$$

$$=\frac{4s(s^2-4Rr-r^2)}{3s(s^2+2Rr+r^2)}+\frac{3(s^2+4Rr+r^2)}{4(s^2-4Rr-r^2)}$$

$$=\frac{16(s^2-4Rr-r^2)^2+9(s^2+4Rr+r^2)(s^2+2Rr+r^2)}{12(s^2+2Rr+r^2)(s^2-4Rr-r^2)}\overset{?}{\le}\frac{9}{4}$$

$$\Leftrightarrow s^4+(10Rr+7r^2)s^2-r^2(272R^2+172Rr+26r^2)\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}(26Rr+2r^2)s^2-r^2(272R^2+172Rr+26r^2)$$

$$\overset{\text{Gerretsen}}{\ge}(26Rr+2r^2)(16Rr-5r^2)-r^2(272R^2+172Rr+26r^2)$$

$$=18r^2(8R^2-15Rr-2r^2)=18r^2(8R+5r)(R-2r)\overset{\text{Euler}}{\ge}0\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{1}{3a+bc}+\frac{1}{3b+ca}+\frac{1}{3c+ab}+\frac{ab+bc+ca}{2}\le\frac{9}{4}$$

$$\forall\,\Delta\,ABC\,\big|\,a^2+b^2+c^2=3,''=''\text{ iff }a=b=c=1\text{ (QED)}$$