ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = ab + bc + ca$, then prove that :

$$2(a+b+c) - abc + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq -9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0$,
$y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r\ (say);\ \text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1)$$

$\Rightarrow a = s - x, b = s - y, c = s - z \therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow$

$$\sum_{cyc} ab = \sum_{cyc} (s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3) \text{ and}$$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4)$$

$$\text{Now, } 2(a+b+c) - abc + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} + 9 \overset{abc = ab+bc+ca}{=}$$

$$\frac{2abc(\sum_{cyc} a)}{\sum_{cyc} ab} - \sum_{cyc} ab + \frac{1}{2}\left(2\sum_{cyc} a^2 - 2\sum_{cyc} ab\right) + \frac{9a^2b^2c^2}{(\sum_{cyc} ab)^2}$$

$$\overset{\text{via (1),(2),(3),(4)}}{=} \frac{2r^2s^2}{4Rr + r^2} - (4Rr + r^2) + s^2 - 12Rr - 3r^2 + \frac{9r^4s^2}{(4Rr + r^2)^2}$$

$$= \frac{2r(4R+r)s^2 - r(4R+r)^3 + (s^2 - 12Rr - 3r^2)(4R+r)^2 + 9r^2s^2}{(4R+r)^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)s^2 \ \boxed{\overset{?}{\geq} \text{①}}\ r(4R+r)^3$$

$$\text{Now, } (4R^2 + 4Rr + 3r^2)s^2 \overset{\text{Rouche}}{\geq}$$

$$(4R^2 + 4Rr + 3r^2)\left(2R^2 + 10Rr - r^2 - 2(R-2r).\sqrt{R^2 - 2Rr}\right) \overset{?}{\geq} r(4R+r)^3$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)(2R^2 + 10Rr - r^2) - r(4R+r)^3$$

$$\overset{?}{\geq} 2(R-2r).\sqrt{R^2 - 2Rr}.(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 2(R-2r)(4R^3 - 3R^2r + r^3)\ \boxed{\overset{?}{\geq} \text{②}}\ 2(R-2r).\sqrt{R^2 - 2Rr}.(4R^2 + 4Rr + 3r^2)$$

$$\boldsymbol{a}\mathbf{nd} \because \mathbf{R}-\mathbf{2r} \overset{\mathbf{Euler}}{\geq} \mathbf{0} \therefore \text{in order to prove } ②, \text{it suffices to prove :}$$

$$(\mathbf{4R^3 - 3R^2r + r^3})^2 \overset{?}{>} (\mathbf{R^2 - 2Rr})(\mathbf{4R^2 + 4Rr + 3r^2})^2 \Leftrightarrow \mathbf{r^3(4R + r)^3} \overset{?}{>} \mathbf{0} \rightarrow \text{true}$$

$$\Rightarrow ② \Rightarrow ① \text{ is true} \therefore 2(a+b+c) - abc + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq -9$$

$$\forall\, a, b, c > 0 \,\Big|\, abc = ab + bc + ca, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$