ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a^3(a+b)}{a^2+b^2}+\frac{b^3(b+c)}{b^2+c^2}+\frac{c^3(c+a)}{c^2+a^2}\ge a^2+b^2+c^2$$

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$$\frac{a^3(a+b)}{a^2+b^2}+\frac{b^3(b+c)}{b^2+c^2}+\frac{c^3(c+a)}{c^2+a^2}-(a^2+b^2+c^2)=$$

$$=\sum_{cyc}\left(\frac{b^3(b+c)}{b^2+c^2}-\frac{b^2+c^2}{2}\right)=\sum_{cyc}\frac{2b^4+2b^3c-b^4-c^4-2b^2c^2}{2(b^2+c^2)}$$

$$=\sum_{cyc}\frac{(b^2+c^2)(b^2-c^2)}{2(b^2+c^2)}+\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\sum_{cyc}\left(b^2c(b-c)\left(\sum_{cyc}a^2b^2+a^4\right)\right)$$

$$=\frac{1}{2}\sum_{cyc}(b^2-c^2)+\frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}\cdot\left(\begin{array}{c}\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\right)\\+abc\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2\right)\end{array}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\left(\sum_{cyc}a^2b^2\right)\left(\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\right)+abc\left(\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2\right)\boxed{\overset{?}{\underset{(*)}{\ge}}}0$$

$$\left(\because\sum_{cyc}(b^2-c^2)=0\right)$$

Let $s-a=x, s-b=y, s-c=z$ **& then :** $a=y+z, b=z+x, c=x+y\to(m)$

$$\text{Now, }(m)\Rightarrow\sum_{cyc}b^3c-\sum_{cyc}b^2c^2=\sum_{cyc}(z+x)^3(x+y)-\sum_{cyc}(z+x)^2(x+y)^2$$

$$=\sum_{cyc}xy^3-xyz\sum_{cyc}x=xyz\sum_{cyc}\frac{y^2}{z}-xyz\sum_{cyc}x$$

$$\overset{\text{Bergstrom}}{\ge}\;xyz\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x}-xyz\sum_{cyc}x=0\;\therefore\sum_{cyc}b^3c-\sum_{cyc}b^2c^2\ge0\to①$$

$$\text{Again, }(m)\Rightarrow\sum_{cyc}a^3b^2-abc\sum_{cyc}a^2=$$

$$\sum_{cyc}(y+z)^3(z+x)^2-(x+y)(y+z)(z+x)\sum_{cyc}(y+z)^2=\sum_{cyc}(x^5+xy^4-2x^3y^2)$$

$$= \sum_{cyc} x(x^2 - y^2)^2 \geq 0 \therefore \sum_{cyc} a^3b^2 - abc\sum_{cyc} a^2 \geq 0 \to ② \therefore ① \text{ and } ② \Rightarrow$$

$$(*) \text{ is true } \therefore \frac{a^3(a+b)}{a^2+b^2} + \frac{b^3(b+c)}{b^2+c^2} + \frac{c^3(c+a)}{c^2+a^2} \geq a^2 + b^2 + c^2$$

$\forall$ ABC, with equality iff Δ ABC is equilateral (QED)