ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b,c\in\left[0;\frac{1}{2}\right] \text{ and } a+b+c=1, \text{ then prove that :}$$

$$a^3+b^3+c^3+4abc\le\frac{9}{32}$$

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$$a^3+b^3+c^3+4abc-\frac{9}{32}=$$

$$=3abc+\left(\sum_{cyc}a\right)\left(\left(\sum_{cyc}a\right)^2-3\sum_{cyc}ab\right)+4abc-\frac{9}{32}$$

$$\overset{a+b+c=1}{=}7abc+1-3\big(bc+a(1-a)\big)-\frac{9}{32}=3a^2-3a+1+bc(7a-3)-\frac{9}{32}$$

$$\therefore a^3+b^3+c^3+4abc-\frac{9}{32}=\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}\to(m)$$

$$\boxed{\text{Case 1}}\ 7a-3\ge0 \text{ and then : } a^3+b^3+c^3+4abc-\frac{9}{32}\overset{via\,(m)}{=}$$

$$\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}\overset{A-G}{\le}\frac{32(3a^2-3a)+8(b+c)^2(7a-3)+23}{32}$$

$$\overset{a+b+c=1}{=}\frac{32(3a^2-3a)+8(1-a)^2(7a-3)+23}{32}=\frac{56a^3-40a^2+8a-1}{32}$$

$$=\frac{(2a-1)(19a^2+(3a-1)^2)}{32}\le0\ \left(\because a\in\left[0;\frac{1}{2}\right]\right)\therefore a^3+b^3+c^3+4abc\le\frac{9}{32}$$

$$\boxed{\text{Case 2}}\ 7a-3<0 \text{ and } \because\left(b-\frac{1}{2}\right),\left(c-\frac{1}{2}\right)\le0\ \therefore\left(b-\frac{1}{2}\right)\left(c-\frac{1}{2}\right)\ge0$$

$$\Rightarrow bc-\frac{b+c}{2}+\frac{1}{4}\ge0\overset{a+b+c=1}{\Rightarrow}bc-\frac{1-a}{2}+\frac{1}{4}\ge0\Rightarrow bc-\frac{1-2a}{4}\ge0$$

$$\Rightarrow(7a-3)\left(bc-\frac{1-2a}{4}\right)\le0\Rightarrow bc(7a-3)\le\frac{(7a-3)(1-2a)}{4} \text{ and then :}$$

$$a^3+b^3+c^3+4abc-\frac{9}{32}\overset{via\,(m)}{=}\frac{32(3a^2-3a)+32bc(7a-3)+23}{32}$$

$$\le\frac{32(3a^2-3a)+8(7a-3)(1-2a)+23}{32}=\frac{-(4a-1)^2}{32}\le0$$

$$\therefore a^3+b^3+c^3+4abc\le\frac{9}{32}\ \therefore \text{combining both cases,}$$

$$a^3+b^3+c^3+4abc\le\frac{9}{32}\ \forall a,b,c\in\left[0;\frac{1}{2}\right]\wedge a+b+c=1,$$

$$''='' \text{ iff }\left(a=\frac{1}{2},b=c=\frac{1}{4}\right)\text{ or }\left(b=\frac{1}{2},c=a=\frac{1}{4}\right)\left(c=\frac{1}{2},a=b=\frac{1}{4}\right)\text{ (QED)}$$