ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, *then prove that* :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{\sqrt{a^2+b^2+c^2}} \geq 3+\sqrt{3}$$

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$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2} \therefore \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{\sqrt{a^2+b^2+c^2}}$$

$$\geq \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}} \overset{?}{\geq} 3+\sqrt{3} \Leftrightarrow \frac{9\sum_{cyc} a^2}{\left(\sum_{cyc} a\right)^2}-3 \overset{?}{\geq} \sqrt{3}-\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}$$

$$\Leftrightarrow \frac{9\sum_{cyc} a^2-3\sum_{cyc} a^2-6\sum_{cyc} ab}{\left(\sum_{cyc} a\right)^2} \overset{?}{\geq} \frac{3-\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2}}{\sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}}$$

$$\Leftrightarrow \frac{6\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)}{\left(\sum_{cyc} a\right)^2} \boxed{\overset{?}{\geq} (*)} \frac{2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)}{\left(\sum_{cyc} a^2\right)\left(\sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}\right)}$$

$$\text{Now, } \sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}} > 1 \text{ and}$$

$$\because \sum_{cyc} a^2-\sum_{cyc} ab \geq 0$$

$$\therefore \frac{2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)}{\left(\sum_{cyc} a^2\right)\left(\sqrt{3}+\frac{\sum_{cyc} a}{\sqrt{\sum_{cyc} a^2}}\right)} \leq \frac{2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)}{\sum_{cyc} a^2} \leq \frac{6\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)}{\left(\sum_{cyc} a\right)^2}$$

$$\Rightarrow (*) \text{ is true} \therefore \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a+b+c}{\sqrt{a^2+b^2+c^2}} \geq 3+\sqrt{3}\ \forall\, a, b, c > 0,$$

$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$