ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 1$ and $a + b + c = 6$, then prove that :

$$(a^2+2)(b^2+2)(c^2+2) \leq 216$$

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$\boxed{\text{Case 1}}$ Exactly 2 variables equal to 1 and WLOG we may assume $b = c = 1$ $(a = 4)$ and then : $(a^2+2)(b^2+2)(c^2+2) = 162 < 216$

$\boxed{\text{Case 2}}$ Exactly 1 variable equals to 1 and WLOG we may assume $a = 1$ $\left(b, c > 1 \,\middle|\, b + c = 5\right)$ and then : $(a^2+2)(b^2+2)(c^2+2)$

$$= 3(b^2+2)((5-b)^2+2) \overset{?}{<} 216 \Leftrightarrow b^4 - 10b^3 + 29b^2 - 20b - 18 \overset{?}{<} 0$$

$$\Leftrightarrow \frac{b(b-5)(2b-5)^2}{4} - \frac{9(2b-5)^2}{16} - \frac{63}{16} \overset{?}{<} 0 \to \text{true} \because b = 5 - c \overset{c>1}{<} 4 < 5$$

$$\therefore (a^2+2)(b^2+2)(c^2+2) < 216$$

$\boxed{\text{Case 3}}$ $a, b, c > 1$ and then : $a = x + 1, b = y + 1, c = z + 1$ where $x, y, z > 0$ such that : $\sum_{\text{cyc}} x = 3$ and assigning $y + z = M, z + x = N, x + y = P$

$$\Rightarrow M + N - P = 2z > 0, N + P - M = 2x > 0 \text{ and } P + M - N = 2y > 0$$

$\Rightarrow M + N > P, N + P > M, P + M > N \Rightarrow M, N, P$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding

$$2\sum_{\text{cyc}} x = \sum_{\text{cyc}} M = 2s \Rightarrow \sum_{\text{cyc}} x = s \to (1) \Rightarrow x = s - M, y = s - N, z = s - P$$

$$\Rightarrow xyz = r^2 s \to (2) \text{ and via such substitutions, } \sum_{\text{cyc}} xy = \sum_{\text{cyc}} (s-M)(s-N)$$

$$\Rightarrow \sum_{\text{cyc}} xy = 4Rr + r^2 \to (3) \text{ and } \sum_{\text{cyc}} x^2 = \left(\sum_{\text{cyc}} x\right)^2 - 2\sum_{\text{cyc}} xy$$

$$= \left(\sum_{\text{cyc}} x\right)^2 - 2\sum_{\text{cyc}} xy \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\therefore \sum_{\text{cyc}} x^2 = s^2 - 8Rr - 2r^2 \to (4) \text{ and finally, } \sum_{\text{cyc}} x^2y^2 = \left(\sum_{\text{cyc}} xy\right)^2 - 2xyz\sum_{\text{cyc}} x$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s^2 \Rightarrow \sum_{\text{cyc}} x^2y^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\text{Now, } (a^2+2)(b^2+2)(c^2+2) - 216 = \prod_{\text{cyc}} (x^2 + 2x + 3) - 216$$

ROMANIAN MATHEMATICAL MAGAZINE

$$= x^2y^2z^2 + 2xyz\sum_{cyc} xy + 3\sum_{cyc} x^2y^2 + 4xyz\sum_{cyc} x + 6\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 18xyz$$

$$+8xyz + 9\sum_{cyc} x^2 + 12\sum_{cyc} xy + 18\sum_{cyc} x + 27 - 216 \overset{\sum_{cyc} x = 3}{=}$$

$$x^2y^2z^2 + \frac{2}{3}xyz\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + \frac{3}{9}\left(\sum_{cyc} x\right)^2\left(\sum_{cyc} x^2y^2\right) + \frac{4}{9}xyz\left(\sum_{cyc} x\right)^3$$

$$+\frac{6}{27}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} xy\right) - \frac{10}{27}xyz\left(\sum_{cyc} x\right)^3 + \frac{9}{81}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} x^2\right)$$

$$+\frac{12}{81}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} xy\right) + \frac{18}{243}\left(\sum_{cyc} x\right)^6 - \frac{189}{729}\left(\sum_{cyc} x\right)^6 \overset{?}{\le} 0$$

$$\overset{\text{via } (1),(2),(3),(4) \text{ and } (5)}{\Leftrightarrow} 5s^4 - 10(4Rr + r^2)s^2 - 3s^2(s^2 - 8Rr - 2r^2) - 2r^2s^2$$

$$-9r^2((4R + r)^2 - 2s^2) - 18r^2(4Rr + r^2) - 27r^4 \overset{?}{\ge} 0$$

$$\Leftrightarrow s^4 - (8Rr - 6r^2)s^2 - r^2(72R^2 + 72Rr + 27r^2) \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\ge} (8Rr + r^2)s^2 - r^2(72R^2 + 72Rr + 27r^2) \overset{\text{Gerretsen}}{\ge}$

$(8Rr + r^2)(16Rr - 5r^2) - r^2(72R^2 + 72Rr + 27r^2) = 8(R - 2r)(7R + 2r) \overset{\text{Euler}}{\ge} 0$

$\Rightarrow (*)$ is true $\therefore (a^2 + 2)(b^2 + 2)(c^2 + 2) \le 216$ and so, combining *all* 3 *cases*,

$(a^2 + 2)(b^2 + 2)(c^2 + 2) \le 216 \ \forall\, a, b, c \ge 1 \,|\, a + b + c = 6,$

$'' ='' $ iff $a = b = c = 2$ (QED)