ROMANIAN MATHEMATICAL MAGAZINE

**For $a, b, c \in \mathbb{R}$ such that $ab + bc + ca \geq 0, a + b + c = 3$. Prove that:**

$$(2a + b - 3c)(2b + c - 3a)(2c + a - 3b) \leq 42\sqrt{21}$$

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Since $ab + bc + ca \geq 0$, then we have $18 = 2(a + b + c)^2 \geq (a - b)^2 + (b - c)^2 + (c - a)^2$ so it suffices to prove that:

$$(2a + b - 3c)(2b + c - 3a)(2c + a - 3b) \leq 2\sqrt{7 . \frac{(a - b)^2 + (b - c)^2 + (c - a)^2}{6}}^{\,3} \quad (1)$$

WLOG, we assume that $c = min\{a, b, c\}$. Let $x := a - c \geq 0, y := b - c \geq 0$.

The inequality (1) becomes:

$$(2x + y)(2y - 3x)(x - 3y) \leq 2\sqrt{7 \cdot \frac{x^2 - xy + y^2}{3}}^{\,3}$$

The inequality is true for $y = 0$. Now, we assume that $y > 0$,

let $t := \frac{x}{y}$, the inequality becomes

$$(2t + 1)(2 - 3t)(t - 3) \leq 2\sqrt{7 \cdot \frac{t^2 - t + 1}{3}}^{\,3} \quad (2)$$

This inequality is true if $LHS_{(2)} < 0$. We assume now that $LHS_{(2)} \geq 0$.

After squaring, the inequality is equivalent to

$$(t + 4)^2(5t - 1)^2(4t - 5)^2 \geq 0,$$

which is true and the proof is complete. Equality holds iff

$$(a, b, c) \in \left\{\left(1 + \frac{2\sqrt{21}}{7}, 1 + \frac{\sqrt{21}}{7}, 1 - \frac{3\sqrt{21}}{7}\right), \left(1 - \frac{3\sqrt{21}}{7}, 1 + \frac{2\sqrt{21}}{7}, 1 + \frac{\sqrt{21}}{7}\right), \left(1 + \frac{\sqrt{21}}{7}, 1 - \frac{3\sqrt{21}}{7}, 1 + \frac{2\sqrt{21}}{7}\right)\right\}.$$