ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c, d \geq 0$, prove that :

$$3(a+b+c+d)^3 + 18(abc+bcd+cda+dab) \geq 11(a+b+c+d)(ab+bc+cd+da+ac+bd)$$

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Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

WLOG, we assume that $d = min(a, b, c, d)$.

Let $p := a+b+c, q := ab+bc+ca, r := abc$. The desired inequality becomes :

$$3(p+d)^3 + 18(r+qd) \geq 11(p+d)(q+pd).$$

By Schur's inequality, we have $9r \geq 4pq - p^3$, so it suffices to prove that :

$$p^3 - 2p^2d - 2pd^2 + 3d^3 \geq q(3p - 7d).$$

Since $q \leq \frac{p^2}{3}$ and $p \geq 3d$, so it suffices to prove that :

$$p^3 - 2p^2d - 2pd^2 + 3d^3 \geq \frac{p^2}{3}(3p - 7d) \text{ or } \frac{d}{3}(p - 3d)^2 \geq 0,$$

which is true and the proof is complete.

Equality holds iff $(a = b = c = d)$ and $(a = b = c, d = 0)$ and its permutations.