ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c \geq 0, a^2 + b^2 + c^2 = 3$. Prove that:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + abc \geq 4$$

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WLOG, we assume that b is between a and c. By AM − GM inequality, we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + abc = \left(\frac{a^2}{b} + a^2b\right) + \left(\frac{b^2}{c} + b^2c\right) + \left(\frac{c^2}{a} + c^2a\right) + abc - \left(a^2b + b^2c + c^2a\right) \geq$$

$$\geq 2a^2 + 2b^2 + 2c^2 + abc - \left(a^2b + b^2c + c^2a\right) = 6 + c(a-b)(b-c) - b\left(a^2 + c^2\right) \geq$$

$$\geq 6 - b\left(3 - b^2\right) = 4 + (b+2)(b-1)^2 \geq 4,$$

as desired. Equality holds iff $a = b = c = 1$.