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2401. Prove that:

$$\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y^2+y+x^4y^2+x^4y}dxdy=-\frac{\pi^2\zeta(3)}{36}(3+2\sqrt{3})$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$I=\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y^2+y+x^4y^2+x^4y}dxdy$$

$$=\int_0^\infty\int_0^\infty \frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{(y+y^2)(x^8+x^4+1)}dxdy=$$

$$=2\int_0^\infty \frac{(1+x^4)\ln(x)}{x^8+x^4+1}dx\int_0^\infty \frac{\ln(y)\ln(1+y)}{y(1+y)}dy=2JK$$

$$J=\int_0^\infty \frac{(1+x^4)\ln(x)}{x^8+x^4+1}dx=\int_0^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=$$

$$=\int_0^1 \frac{(1-x^8)\ln(x)}{1-x^{12}}dx+\int_1^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=J_1+J_2$$

$$J_1=\int_0^1 \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=\sum_{n=0}^\infty\int_0^1 (x^{12n}-x^{8+12n})\ln(x)\,dx=$$

$$=\sum_{n=0}^\infty \frac{1}{(12n+9)^2}-\sum_{n=0}^\infty \frac{1}{(12n+1)^2}=\frac{1}{144}\left(\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{1}{12}\right)\right)$$

$$J_2=\int_1^\infty \frac{(1-x^8)\ln(x)}{1-x^{12}}dx=\int_0^1 \frac{\left(1-\frac{1}{x^8}\right)\ln(x)}{1-\frac{1}{x^{12}}}\frac{dx}{x^2}=\int_0^1\frac{(x^{10}-x^2)\ln(x)}{1-x^{12}}dx=$$

$$=\sum_{n=0}^\infty\int_0^1 (x^{12n+10}-x^{12n+2})\ln(x)dx=\sum_{n=0}^\infty \frac{1}{(12n+3)^2}-\sum_{n=0}^\infty\frac{1}{(12n+11)^2}=$$

$$=\frac{1}{144}\left(\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)$$

$$J=J_1+J_2=\frac{1}{144}\left(\psi^{(1)}\left(\frac{3}{4}\right)-\psi^{(1)}\left(\frac{1}{12}\right)+\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)$$

Notes:

polygamma reflection formula:

$$\left\{(-1)^m\psi^{(m)}(1-x)-\psi^{(m)}(x)=\pi\frac{d}{dx^m}\cot(\pi x)\right\}$$

$$J=J_1+J_2=\frac{1}{144}\left(\psi^{(1)}\left(1-\frac{1}{4}\right)+\psi^{(1)}\left(\frac{1}{4}\right)-\psi^{(1)}\left(1-\frac{11}{12}\right)-\psi^{(1)}\left(\frac{11}{12}\right)\right)=$$

$$=\frac{1}{144}\left(2\pi^2-4\pi^2(2+\sqrt{3})\right)=-\frac{\pi^2}{72}(3+2\sqrt{3})$$

$$K=\int_0^\infty\frac{\ln(y)\ln(1+y)}{y(1+y)}dy=\int_0^1\frac{\ln(y)\ln(1+y)}{y(1+y)}dy+\int_1^\infty\frac{\ln(y)\ln(1+y)}{y(1+y)}dy=K_1+K_2$$

$$K_1=\int_0^1\frac{\ln(y)\ln(1+y)}{y(1+y)}dday=\int_0^1\frac{\ln(y)\ln(1+y)}{y}dy-\int_0^1\frac{\ln(y)\ln(1+y)}{1+y}dy=$$

$$=-\sum_{n=1}^\infty\frac{(-1)^n}{n}\int_0^1 y^{n-1}\ln(y)dy-[\frac{\ln(y)\ln^2(1+y)}{2}]_0^1+\frac{1}{2}\int_0^1\frac{\ln^2(1+y)}{y}dy=$$

$$=\sum_{n=1}^\infty\frac{(-1)^n}{n^3}+\frac{1}{2}\int_0^1\frac{\ln^2(1+y)}{y}dy,\left\{\frac{1}{1+y}=t,dt=-t^2dy,t\left[\frac{1}{2};1\right]\right\}$$

$$=-\eta(3)+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln^2(t)}{\frac{1}{t}-1}\frac{dt}{t^2}=-\frac{3}{4}\zeta(3)+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln^2(t)}{t(1-t)}dt=$$

$$=-\frac{3}{4}\zeta(3)+\frac{1}{2}\sum_{n=0}^\infty\int_{\frac{1}{2}}^1 t^{n-1}\ln^2(t)dt=-\frac{3}{4}\zeta(3)+\frac{\zeta(3)}{8}=-\frac{5\zeta(3)}{8}$$

$$K_2=\int_1^\infty\frac{\ln(y)\ln(1+y)}{y(1+y)}dy=\int_0^1\frac{\ln\left(\frac{1}{y}\right)\ln\left(1+\frac{1}{y}\right)}{y\left(1+\frac{1}{y}\right)}dy=\int_0^1\frac{\ln^2(y)-\ln(y)\ln(1+y)}{1+y}dy=$$

$$=\sum_{n=0}^\infty(-1)^n\int_0^1 y^n\ln^2(y)dy-[\frac{\ln(y)\ln^2(1+y)}{2}]_0^1+\frac{1}{2}\int_0^1\frac{\ln^2(1+y)}{y}dy=2\sum_{n=0}^\infty\frac{(-1)^n}{(n+1)^3}$$

$$+\frac{\zeta(3)}{8}=2\eta(3)+\frac{\zeta(3)}{8}=\frac{13\zeta(3)}{8}$$

$$K=K_1+K_2=\frac{13\zeta(3)}{8}-\frac{5\zeta(3)}{8}=\zeta(3)$$

$$\int_0^\infty\int_0^\infty\frac{(1+x^4)\ln(x^2)\ln(y)\ln(1+y)}{x^8y^2+x^8y+y+y^2+x^4y^2+x^4y}dxdy=2JK=-\frac{\pi^2\zeta(3)(3+2\sqrt{3})}{36}$$

2402. Prove that:

$$\int_0^1\int_0^1\frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}dxdy=\frac{3G}{4}$$

Proposed by Amin Hajiyev-Azerbaijan

Solution 1 by proposer

$$I=\int_0^1\int_0^1\frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}dxdy=\int_0^1\int_0^1 f(x;y)dxdy$$

$$f(x;y)=\frac{x^2y^2+4xy+1}{x^4y^4-x^4-y^4+8x^3y^3+16x^2y^2+8xy+1}=$$

$$=\frac{x^2y^2+4xy+1}{(x^2y^2)^2+(4xy)^2+1+2(4xy)(x^2y^2)+2(4xy)+2x^2y^2-x^4-y^4-2x^2y^2}=$$

$$=\frac{x^2y^2+4xy+1}{(x^2y^2+4xy+1)^2-(x^2+y^2)^2}$$

$$=\frac{x^2y^2+4xy+1+x^2y^2+4xy+1+y^2+x^2-x^2-y^2}{2(x^2y^2+4xy+1+x^2+y^2)(x^2y^2+4xy+1-x^2-y^2)}=$$

$$=\frac{1}{2}\left(\frac{1}{x^2y^2+4xy+1+x^2+y^2}+\frac{1}{x^2y^2+4xy+1-x^2-y^2}\right)$$

$$I=\frac{1}{2}(I_1+I_2)\quad \begin{Bmatrix} I_1=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy \\ I_2=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy \end{Bmatrix}$$

$$I_1=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy=\int_0^1\int_0^1\frac{1}{(1+xy)^2+(x+y)^2}dxdy=$$

$$=\int_0^1\int_0^1\frac{1}{(1+xy)^2\left(1+\left(\frac{x+y}{1+xy}\right)^2\right)}dxdy$$

$$let:\left\{\frac{x+y}{1+xy}=t,x=\frac{t-y}{1-yt},\frac{dx}{dt}=\frac{1-y^2}{(1-yt)^2},1+xy=\frac{1-y^2}{1-yt};\ t[1;\ y]\right\}$$

$$I_1=\int_0^1\int_y^1\frac{1-y^2}{\left(\frac{1-y^2}{1-yt}\right)^2(1+t^2)(1-yt)^2}dtdy=\int_0^1\int_y^1\frac{1}{(1+t^2)(1-y^2)}dtdy$$

$$=\int_0^1\frac{1}{1-y^2}[\tan^{-1}(t)]_y^1dy=$$

$$=\frac{\pi}{4}\int_0^1\frac{1}{1-y^2}dy-\int_0^1\frac{\tan^{-1}(y)}{1-y^2}dy.\ \rightarrow\ \ IBP\ \begin{Bmatrix} v=\int\frac{1}{1-y^2}dy=\ln\left(\frac{1+y}{1-y}\right). \\ u=\tan^{-1}(y),\frac{du}{dy}=\frac{1}{1+y^2} \end{Bmatrix}$$

$$I_1=\frac{\pi}{4}\int_0^1\frac{1}{1-y^2}dy-\frac{\pi}{4}\int_0^1\frac{1}{1-y^2}dy+\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1+y}{1-y}\right)}{1+y^2}dy,\left\{\frac{1-y}{1+y}=\theta;\quad \theta[0;1]\right\}$$

$$I_1=-\frac{1}{2}\int_0^1\frac{\ln(\theta)}{1+\theta^2}d\theta=-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1\theta^{2n}\ln(\theta)d\theta=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{G}{2}$$

$$I_2=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy=\int_0^1\int_0^1\frac{1}{(1+xy)^2-(x-y)^2}dxdy$$

$$=\int_0^1\int_0^1\frac{1}{(1+xy)^2\left(1-\left(\frac{x-y}{1+xy}\right)^2\right)}dxdy$$

$$let:\left\{\frac{x-y}{1+xy}=m,x=\frac{m+y}{1-my},\frac{dx}{dm}=\frac{1+y^2}{(1-my)^2},1+xy=\frac{1+y^2}{1-my},m\left[\frac{1-y}{1+y};-y\right]\right\}$$

$$I_2=\int_0^1\int_{-y}^{\frac{1-y}{1+y}}\frac{1+y^2}{\left(\frac{1+y^2}{1-my}\right)^2(1-m^2)(1-my)^2}dmdy=\int_0^1\int_{-y}^{\frac{1-y}{1+y}}\frac{1}{(1+y^2)(1-m^2)}dmdy$$

$$=\int_0^1\frac{1}{1+y^2}[\text{arctanh}(m)]_{-y}^{\frac{1-y}{1+y}}dy=$$

$$=\int_0^1\frac{\text{arctanh}\left(\frac{1-y}{1+y}\right)}{1+y^2}dy+\int_0^1\frac{\text{arctanh}(y)}{1+y^2}dy=2\int_0^1\frac{\text{arctanh}(y)}{1+y^2}dy=-\int_0^1\frac{\ln\left(\frac{1-y}{1+y}\right)}{1+y^2}dy=$$

$$=-\int_0^1\frac{\ln(y)}{1+y^2}dy=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=G$$

$$I=\frac{1}{2}(I_1+I_2)=\frac{1}{2}\left(G+\frac{G}{2}\right)=\frac{3G}{4}$$

Solution 2 by Pham Duc Nam-Vietnam

$$*\ x^4y^4+8x^3y^3+16x^2y^2+8xy-x^4-y^4=$$

$$=(x^2y^2)^2+16x^2y^2+1++2x^2y^2\bullet 4xy+2\bullet 4xy\bullet 1+2\bullet x^2y^2\bullet 1-(x^4+y^4+2x^2y^2)=$$

$$=(x^2y^2+4xy+1)^2-(x^2+y^2)^2$$

$$=\left(x^2y^2+4xy+1+(x^2+y^2)\right)\left(x^2y^2+4xy+1-(x^2+y^2)\right)$$

$$I=\frac{1}{2}\left(\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+x^2+y^2}dxdy+\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-x^2-y^2}dxdy\right)=\frac{1}{2}(J+K)$$

$$J=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1+(x^2+y^2)}dxdy=\int_0^1[\frac{\arctan\left(\frac{x+y}{1+xy}\right)}{1-y^2}]_0^1dy$$

$$=\int_0^1\frac{1}{1-y^2}\left(\frac{\pi}{4}-\arctan(y)\right)dy=$$

$$K=\int_0^1\int_0^1\frac{1}{x^2y^2+4xy+1-(x^2+y^2)}dxdy=\frac{1}{2}\int_0^1[\frac{\ln\left(\frac{xy+x-y+1}{xy-x+y+1}\right)}{1+y^2}]_0^1dy=$$

$$=\frac{1}{2}\int_0^1\frac{1}{1+y^2}\left(\ln\left(\frac{1+y}{1-y}\right)-\ln(y)\right)dy$$

$$J=\int_0^1\frac{1}{1-y^2}\left(\frac{\pi}{4}-\arctan(y)\right)dy,\ let:\left\{t=\frac{1-y}{1+y},y=\frac{1-t}{1+t},dy=-\frac{2}{(1+t)^2}dt\right\}$$

$$J=2\int_0^1\frac{1}{\frac{4t(1+t)^2}{(1+t)^2}}\left(\frac{\pi}{4}-\arctan\left(\frac{1-t}{1+t}\right)\right)dt=\frac{1}{2}\int_0^1\frac{1}{t}\left(\frac{\pi}{4}-\frac{\pi}{4}+\arctan(t)\right)dt=$$

$$=\frac{1}{2}\int_0^1\frac{\arctan(t)}{t}dt=\frac{G}{2}$$

$$K=\frac{1}{2}\int_0^1\frac{1}{1+y^2}\left(\ln\left(\frac{1+y}{1-y}\right)-\ln(y)\right)dy=-\frac{1}{2}\int_0^1\frac{\ln(y)}{1+y^2}dy+\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1+y}{1-y}\right)}{1+y^2}dy$$

$$=\frac{G}{2}-\frac{1}{2}\int_0^1\frac{\ln\left(\frac{1-y}{1+y}\right)}{1+y^2}dy,\quad let:\left\{t=\frac{1-y}{1+y},y=\frac{1-t}{1+t},dy=-\frac{2}{(1+t)^2}dt\right\}$$

$$k=\frac{G}{2}-\frac{1}{2}\int_0^1\frac{\ln(t)}{1+t^2}dt=\frac{G}{2}+\frac{G}{2}=G$$

$$I=\frac{1}{2}(J+K)=\frac{1}{2}\left(G+\frac{G}{2}\right)=\frac{3G}{4}$$

Notes:

$$\int_0^1\frac{\arctan(t)}{t}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 t^{2n}dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=\beta(2)=G$$

$$\int_0^1\frac{\ln(t)}{1+t^2}dt=\sum_{n=0}^{\infty}(-1)^n\int_0^1 t^{2n}\ln(t)dt=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G;\ arctan\left(\frac{1-t}{1+t}\right)=\frac{\pi}{4}-arctan(t)$$

2403. ***Prove that:***

$$\int_0^1\frac{\sinh^{-1}(x)}{x}dx=-\frac{Li_2(3-2\sqrt{2})}{2}+\frac{\pi^2}{12}-\frac{ln^2(1+\sqrt{2})}{2}+\ln(1+\sqrt{2})\ln(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{\sinh^{-1}(x)}{x}dx=\int_0^1\frac{\ln\left(\sqrt{1+x^2}+x\right)}{x}dx=-\int_0^1\frac{\ln\left(\sqrt{1+x^2}-x\right)}{x}dx$$

$$substitution\ \left\{\sqrt{1+x^2}-x=t,x=\frac{1-t^2}{2t},dx=-\frac{1+t^2}{2t^2}dt,t[\sqrt{2}-1;1]\right\}$$

$$\Omega=-\int_{\sqrt{2}-1}^1\frac{\ln(t)}{\left(\frac{1-t^2}{2t}\right)}\frac{(1+t^2)dt}{2t^2}=-\int_{\sqrt{2}-1}^1\frac{(1+t^2)\ln(t)}{t(1-t^2)}dt=$$

$$=-\int_{\sqrt{2}-1}^1\frac{\ln(t)}{t(1-t^2)}dt-\int_{\sqrt{2}-1}^1\frac{t\ln(t)}{1-t^2}dt=-(\Omega_1+\Omega_2)$$

$$\Omega_1=\int_{\sqrt{2}-1}^1\frac{\ln(t)}{t(1-t^2)}dt=\int_{\sqrt{2}-1}^1\frac{\ln(t)}{t}dt+\int_{\sqrt{2}-1}^1\frac{t\ln(t)}{1-t^2}dt=\int_{\sqrt{2}-1}^1\frac{\ln(t)}{t}dt+\Omega_2$$

$$\Omega_1=\left[\frac{ln^2(t)}{2}\right]_{\sqrt{2}-1}^1+\sum_{n=0}^{\infty}\int_{\sqrt{2}-1}^1 t^{2n+1}\ln(t)\,dt=$$

$$=-\frac{ln^2(1+\sqrt{2})}{2}+\sum_{n=0}^{\infty}\left(\frac{t^{2n+2}}{2n+2}\ln(t)\Big|_{\sqrt{2}-1}^1-\frac{1}{2n+2}\int_{\sqrt{2}-1}^1 t^{2n+1}\,dt\right)=$$

$$=-\frac{ln^2(1+\sqrt{2})}{2}-\frac{\ln(\sqrt{2}-1)}{2}\sum_{n=1}^{\infty}\frac{(\sqrt{2}-1)^{2n}}{n}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(\sqrt{2}-1)^{2n}}{n^2}=$$

$$=-\frac{ln^2(1+\sqrt{2})}{2}+\frac{\ln(\sqrt{2}-1)\ln(2\sqrt{2}-2)}{2}-\frac{\pi^2}{24}+\frac{1}{4}\mathrm{Li}_2(3-2\sqrt{2})=$$

$$=-\frac{ln^2(1+\sqrt{2})}{2}+\frac{ln^2(1+\sqrt{2})}{2}-\frac{\ln(1+\sqrt{2})\ln(2)}{2}-\frac{\pi^2}{24}+\frac{1}{4}Li_2(3-2\sqrt{2})=$$

$$=\frac{1}{4}Li_2(3-2\sqrt{2})-\frac{\pi^2}{24}-\frac{\ln(1+\sqrt{2})\ln(2)}{2}$$

$$\Omega_2=\Omega_1+\frac{ln^2(1+\sqrt{2})}{2}=\frac{1}{4}Li_2(3-2\sqrt{2})-\frac{\pi^2}{24}-\frac{\ln(1+\sqrt{2})\ln(2)}{2}+\frac{ln^2(1+\sqrt{2})}{2}$$

$$\Omega=-(\Omega_1+\Omega_2)=-\frac{1}{2}Li_2(3-2\sqrt{2})+\frac{\pi^2}{12}+\ln(1+\sqrt{2})\ln(2)-\frac{ln^2(1+\sqrt{2})}{2}$$

$$\int_0^1\frac{\sinh^{-1}(x)}{x}dx=-\frac{Li_2(3-2\sqrt{2})}{2}+\frac{\pi^2}{12}-\frac{ln^2(1+\sqrt{2})}{2}+\ln(1+\sqrt{2})\ln(2)$$

2404. Prove that:

$$\int_0^1\sinh^{-1}(x)\cosh^{-1}(x)dx=\frac{\pi}{2}\left(1-\frac{4\sqrt{2\pi}}{\Gamma^2(1/4)}\right)i$$

Proposed by Ankush Kumar Parcha-India

Solution by Cosghun Memmedov-Azerbaijan

$$\int_0^1\sinh^{-1}(x)\cosh^{-1}(x)dx=\int_0^1 ln(x+\sqrt{x^2+1})ln(x+\sqrt{x^2-1})\mathrm{d}x\overset{IBP}{=}$$

$$=-\left(\int_0^1\frac{xln(x+\sqrt{x^2-1})}{\sqrt{x^2+1}}\mathrm{d}x+\int_0^1\frac{xln(x+\sqrt{x^2+1})}{\sqrt{x^2-1}}\mathrm{d}x\right)\overset{IBP}{=}\ln(i)+\int_0^1\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}\mathrm{d}x+\int_0^1\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\mathrm{d}x=$$

$$=\frac{\pi i}{2}-2i\int_0^1\frac{x}{\sqrt{1-x^4}}dx\overset{(x^4\to x)}{=}\frac{\pi i}{2}-\int_0^1 x^{-\frac{1}{4}}(1-x)^{-\frac{1}{2}}\mathrm{d}x=\frac{\pi i}{2}-\frac{i}{2}\mathrm{B}\left(\frac{3}{4},\frac{1}{2}\right)=$$

$$=\frac{\pi i}{2}-\frac{i}{2}\times\frac{\Gamma(\frac{3}{4})\times\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})}=\frac{\pi i}{2}-\frac{\pi i}{2}\times\frac{4\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})}=\frac{\pi}{2}\left(1-\frac{4\sqrt{2\pi}}{\Gamma^2(1/4)}\right)i$$

Notes: $\Gamma(\frac{5}{4})=\Gamma(1+\frac{1}{4})=\frac{1}{4}\Gamma(\frac{1}{4});\ \Gamma(\frac{3}{4})=\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})};\ \Gamma(\frac{1}{2})=\sqrt{\pi}$

2405. Prove that:

$$\int_0^1\int_0^1\frac{e^{x+y}-1}{e^{x+y}+1}dxdy=4\chi_2(e^2)-4Li_2(e)+\frac{\pi^2}{6}-1$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\sigma=\int_0^1\int_0^1\frac{e^{x+y}-1}{e^{x+y}+1}dxdy$$

$$\text{subtitation } \{e^{x+y}=t, dt=tdx, t[e^{1+y};e^{y}]\}$$

$$\sigma=\int_0^1\int_{e^y}^{e^y+1}\frac{t-1}{t(t+1)}dtdy=\int_0^1\int_{e^y}^{e^{1+y}}\left(\frac{2}{1+t}-\frac{1}{t}\right)dtdy=$$

$$=\int_0^1[2\ln(1+t)-\ln(t)]_{e^y}^{e^{1+y}}dy=$$

$$=2\int_0^1\ln(1+e^{1+y})dy-2\int_0^1\ln(1+e^y)dy-\int_0^1 dy=$$

$$=-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 e^{n(1+y)}dy+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 e^{ny}dy-1=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{e^{ny}}{n}\right]_0^1-2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n}\left[\frac{e^{ny}}{n}\right]_0^1-1=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-2\sum_{n=1}^{\infty}\frac{(-1)^n e^{2n}}{n^2}+2\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n^2}-1=$$

$$=4Li_2(-e)+2\eta(2)-2Li_2(-e^2)-1$$

Notes:

$$\text{Polylogarithm function}: Li_a(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^a}, |z|\le 1$$

$$Li_a(z^2)=\frac{1}{2^{1-a}}\left(Li_a(z)+Li_a(-z)\right)$$

$$\text{Legendre chi function}: \chi_a(z)=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)^a}=\frac{1}{2}\left(Li_a(z)-Li_a(-z)\right)$$

$$\sigma=4Li_2(-e)-4Li_2(e)-4Li_2(-e)+4\chi_2(e^2)+\frac{\pi^2}{6}-1=$$

$$=4\chi_2(e^2)-4Li_2(e)+\frac{\pi^2}{6}-1$$

2406. ***Prove that***

$$\int_0^{\infty}\int_0^{\infty}e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)\,dxdy=\pi(2-\gamma)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^{\infty}\int_0^{\infty}e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)\,dxdy, \left\{\frac{y}{x}=t, dt=dy, t[\infty;0]\right\}$$

$$\Omega=\int_0^{\infty}\int_0^{\infty}\frac{e^{-x}x\ln(1+t^2)\ln(xt)}{t^2}dxdt=$$

$$=\int_0^\infty e^{-x}x\ln(x)\int_0^\infty \frac{\ln(1+t^2)}{t^2}dxdt+\int_0^\infty e^{-x}x\int_0^\infty \frac{\ln(t)\ln(1+t^2)}{t^2}dxdt=$$

$$=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^\infty e^{-x}x\ln(x)\int_0^\infty \frac{\ln(1+t^2)}{t^2}dxdt=I\bullet J$$

$$I=\int_0^\infty e^{-x}x\ln(x)\,dx=\lim_{a\to 1}\frac{d}{da}\int_0^\infty x^a e^{-x}dx=\lim_{a\to 1}\frac{d}{da}\Gamma(a+1)=$$

$$=\lim_{a\to 1}\psi^{(0)}(a+1)\Gamma(a+1)=\psi^{(0)}(2)=1-\gamma$$

$$note:\left\{Gamma\ function:\int_0^\infty e^{-x}x^{a-1}dx=\Gamma(a)\right\}$$

$$J=\int_0^\infty \frac{\ln(1+t^2)}{t^2}dt=\int_0^1 \frac{\ln(1+t^2)}{t^2}dt+\int_1^\infty \frac{\ln(1+t^2)}{t^2}dt=J_1+J_2$$

$$J_1=\int_0^1 \frac{\ln(1+t^2)}{t^2}dt=-\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_0^1 t^{2n-2}\,dt=-\sum_{n=1}^\infty \frac{(-1)^n}{n(2n-1)}=$$

$$=\sum_{n=1}^\infty \frac{(-1)^n}{n}-2\sum_{n=1}^\infty \frac{(-1)^n}{2n-1}=\frac{\pi}{2}-\ln(2)$$

$$J_2=\int_1^\infty \frac{\ln(1+t^2)}{t^2}dt,\qquad \left\{\frac{1}{t}=u\,,du=-u^2dt,u[0;1]\right\}$$

$$J_2=\int_0^1 \ln(1+u^2)du-2\int_0^1 \ln(u)\,du=-\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_0^1 u^{2n}du-2[u\ln(u)-u]_0^1=$$

$$=-\sum_{n=1}^\infty \frac{(-1)^n}{n(2n+1)}+2=2\sum_{n=1}^\infty \frac{(-1)^n}{2n+1}-\sum_{n=1}^\infty \frac{(-1)^n}{n}+2=\frac{\pi}{2}-2+\ln(2)+2=\frac{\pi}{2}+\ln(2)$$

$$J=J_1+J_2=\frac{\pi}{2}-\ln(2)+\frac{\pi}{2}+\ln(2)=\pi,\qquad \Omega_1=J\bullet I=\pi(1-\gamma)$$

$$\Omega_2=\int_0^\infty e^{-x}x\int_0^\infty \frac{\ln(t)\ln(1+t^2)}{t^2}dtdx=M\bullet K$$

$$M=\int_0^\infty e^{-x}xdx=\Gamma(2)=1$$

$$K=\int_0^\infty \frac{\ln(t)\ln(1+t^2)}{t^2}dt=\int_0^1 \frac{\ln(t)\ln(1+t^2)}{t^2}dt+\int_1^\infty \frac{\ln(t)\ln(1+t^2)}{t^2}dt=K_1+K_2$$

$$K_1=\int_0^1 \frac{\ln(t)\ln(1+t^2)}{t^2}dt=-\sum_{n=1}^\infty \frac{(-1)^n}{n}\int_0^1 t^{2n-2}\ln(t)\,dt=\sum_{n=1}^\infty \frac{(-1)^n}{n(2n-1)^2}=$$

$$=\sum_{n=1}^\infty \frac{(-1)^n}{n}-2\sum_{n=1}^\infty \frac{(-1)^n}{2n-1}+2\sum_{n=1}^\infty \frac{(-1)^n}{(2n-1)^2}=\frac{\pi}{2}-2G-\ln(2)$$

$$K_2=\int_1^\infty \frac{\ln(t)\ln(1+t^2)}{t^2}dt,\left\{\frac{1}{t}=u,du=-u^2dt,u[0;1]\right\}$$

$$K_2=-\int_0^1 \ln\left(1+\frac{1}{u^2}\right)\ln(u)\,du=-\int_0^1 \ln(u)\ln(1+u^2)du+2\int_0^1 \ln^2(u)du=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 u^{2n}\ln(u)\,du+2\left[2u+u\ln^2(u)-2u\ln(u)\right]_0^1=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(2n+1)^2}+4=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+4=2G-4+\frac{\pi}{2}+\ln(2)+4=$$

$$=2G+\frac{\pi}{2}+\ln(2)$$

$$K=K_1+K_2=\frac{\pi}{2}-2G-\ln(2)+2G+\frac{\pi}{2}+\ln(2)=\pi$$

$$\Omega_2=K\bullet M=\pi$$

$$\Omega=\Omega_1+\Omega_2=\pi+\pi(1-\gamma)=\pi(2-\gamma)$$

$$\int_0^{\infty}\int_0^{\infty}e^{-x}\left(\frac{x}{y}\right)^2\ln\left(1+\frac{y^2}{x^2}\right)\ln(y)\,dxdy=\pi(2-\gamma)$$

2407. Prove that:

$$\int\int_{[0;1]^2}\ln\left(\frac{1}{e^x+e^y}\right)dxdy=2Li_3(-e)+\frac{3\zeta(3)}{2}+\frac{\pi^2}{6}-\frac{1}{3}$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int\int_{[0;1]^2}\ln\left(\frac{1}{e^x+e^y}\right)dxdy=\int\int_{[0;1]^2}\ln\left(\frac{e^{-x}}{1+e^{y-x}}\right)dxdy=$$

$$=\int\int_{[0;1]^2}\ln(e^{-x})dxdy-\int\int_{[0;1]^2}\ln(1+e^{y-x})\,dxdy=$$

$$=-\int\int_{[0;1]^2}x\,dxdy+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int\int_{[0;1]^2}\frac{e^{ny}}{e^{nx}}dxdy=$$

$$=-\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_{[0;1]}e^{ny}\left[\frac{e^{-nx}}{n}\right]_0^1dy=-\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\int_{[0;1]}e^{ny}dy=$$

$$=-\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\left[\frac{e^{ny}}{n}\right]_0^1=-\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{e^{-n}}{n}-\frac{1}{n}\right)\left(\frac{e^{n}}{n}-\frac{1}{n}\right)=$$

$$=-\frac{1}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{2}{n^2}-\frac{e^{-n}}{n^2}-\frac{e^{n}}{n^2}\right)=$$

$$=-\frac{1}{2}-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)^n e^{-n}}{n^3}+\sum_{n=1}^{\infty}\frac{(-1)e^{n}}{n^3}=$$

$$=-\frac{1}{2}+2\eta(3)+Li_3\left(-\frac{1}{e}\right)+Li_3(-e)=2Li_3(-e)+\frac{3\zeta(3)}{2}+\frac{\pi^2}{6}-\frac{1}{3}$$

$$notes:\left\{\begin{array}{c} Li_s(z)+(-1)^s Li_s\left(\frac{1}{z}\right)=\frac{(2\pi i)^s}{\Gamma(s)}\zeta\left(1-s;\frac{1}{2}+\frac{\ln(-z)}{2\pi i}\right) \\ Li_3(-z)-Li_3\left(-\frac{1}{z}\right)=-\frac{1}{6}\ln^3(z)-\zeta(2)\ln(z);\ Li_3(-e)-Li_3\left(-\frac{1}{e}\right)=-\frac{1}{6}-\zeta(2) \\ \eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}=(1-2^{1-z})\zeta(z) \end{array}\right\}$$

2408. **If:**

$$\Omega=\int_0^\infty\frac{\ln(1+x^2)\ln(x)}{x^2}dx+\frac{1}{4}\int_0^\infty\frac{\ln\left(1+\frac{y^2}{4}\right)\ln(y)}{y^2}dy+\frac{1}{9}\int_0^\infty\frac{\ln\left(1+\frac{z^2}{9}\right)\ln(z)}{z^2}dz+\cdots$$

$$Then, show\ that:\ \Omega=\pi\big(\zeta(3)-\zeta'(3)\big)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^\infty\frac{\ln\left(1+\frac{x^2}{n^2}\right)\ln(x)}{x^2}dx=\sum_{n=1}^{\infty}\frac{1}{n^2}f\left(\frac{1}{n}\right)$$

$$f\left(\frac{1}{n}\right)=\int_0^\infty\frac{\ln\left(1+\frac{x^2}{n^2}\right)\ln(x)}{x^2}dx,\left\{\frac{1}{n}=a\right\}\to\ f(a)=\int_0^\infty\frac{\ln(1+a^2x^2)\ln(x)}{x^2}dx$$

$$\frac{d}{da}f(a)=2a\int_0^\infty\frac{\ln(x)}{(1+a^2x^2)}dx;\left\{\tan^{-1}(ax)=t;\frac{dt}{dx}=\frac{a}{1+a^2x^2};\ t\left[\frac{\pi}{2};0\right]\right\}$$

$$\frac{d}{da}f(a)=2\int_0^{\frac{\pi}{2}}\ln\left(\frac{\tan(t)}{a}\right)dt=2\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt-2\ln(a)\int_0^{\frac{\pi}{2}}dt=2I-\pi\ln(a)$$

$$I=\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt,\left\{\frac{\pi}{2}-t\to t\right\}\ I=\int_0^{\frac{\pi}{2}}\ln\left(\tan\left(\frac{\pi}{2}-t\right)\right)dt=\int_0^{\frac{\pi}{2}}\ln\big(\cot(t)\big)dt$$

$$I=\frac{1}{2}\left(\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\big)dt+\int_0^{\frac{\pi}{2}}\ln\big(\cot(t)\big)dt\right)=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln\big(\tan(t)\cot(t)\big)dt=0$$

$$\frac{d}{da}f(a)=-\pi\ln(a);f(a)=-\pi\int\ln(a)da=\pi a-\pi a\ln(a),\left\{a=\frac{1}{n}\right\}$$

$$f\left(\frac{1}{n}\right)=\frac{\pi}{n}-\frac{\pi}{n}\ln\left(\frac{1}{n}\right)=\pi\left(\frac{\ln(n)}{n}+\frac{1}{n}\right)$$

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{n^2}f\left(\frac{1}{n}\right)=\pi\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}+\pi\sum_{n=1}^{\infty}\frac{1}{n^3}=\pi\lim_{a\to0}\frac{d}{da}\sum_{n=1}^{\infty}\frac{1}{n^{3-a}}+\pi\zeta(3)=$$

$$=\pi\lim_{a\to0}\frac{d}{da}\zeta(3-n)+\pi\zeta(3)=-\pi\zeta'(3)+\pi\zeta(3)=\pi\big(\zeta(3)-\zeta'(3)\big)$$

2409. Prove that:

$$\int_0^1 \int_0^\infty \frac{x/y}{\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy = \frac{\pi}{2\sqrt{2}}\varpi$$

where ϖ, is the lemniscate constant .

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1 \int_0^\infty \frac{x}{y\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy;\ subtitation\ \left\{\frac{x}{y} = t, dt = \frac{dx}{y}, t[\infty; 0]\right\}$$

$$\Omega = \int_0^1 y\int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dtdy = K \bullet M, \quad K = \int_0^1 ydy = \frac{1}{2}, \quad M = \int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dt$$

$$M = \int_0^\infty \frac{t}{\sqrt{\sinh(t)}}dt = \sqrt{2}\int_0^\infty \frac{t}{\sqrt{e^t - e^{-t}}}dt$$

$$subtitation: \{e^{-t} = u, du = -udt;\ u[0;1]\}$$

$$M = -\sqrt{2}\int_0^1 \frac{\sqrt{u}\ln(u)}{u\sqrt{1-u^2}}du, \qquad \{u^2 \to u\}$$

$$M = -\frac{\sqrt{2}}{4}\int_0^1 \frac{\ln(u)}{u^{-\frac{3}{4}}\sqrt{1-u}}du = -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\int_0^1 \frac{u^a}{(1-u)^{\frac{1}{2}}}du =$$

$$= -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\beta\left(a+1;\frac{1}{2}\right) = -\frac{\sqrt{2}}{4}\lim_{a\to-\frac{3}{4}}\frac{d}{da}\left(\frac{\Gamma(a+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{3}{2}\right)}\right) =$$

$$= -\frac{\sqrt{2\pi}}{4}\lim_{a\to-\frac{3}{4}}\left(\frac{\Gamma(a+1)\left(\psi^{(0)}(a+1) - \psi^{(0)}\left(a+\frac{3}{2}\right)\right)}{\Gamma\left(a+\frac{3}{2}\right)}\right) =$$

$$= -\frac{\sqrt{2\pi}}{4}\left(\frac{\Gamma\left(\frac{1}{4}\right)\left(\psi^{(0)}\left(\frac{1}{4}\right) - \psi^{(0)}\left(1-\frac{1}{4}\right)\right)}{\Gamma\left(\frac{3}{4}\right)}\right) = \frac{\pi\sqrt{2\pi}}{4}\frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\Gamma^2\left(\frac{1}{4}\right) =$$

$$= \frac{\pi\sqrt{2}}{2}\varpi = \frac{\pi}{\sqrt{2}}\varpi, \qquad \int_0^1 \int_0^\infty \frac{\frac{x}{y}}{\sqrt{\sinh\left(\frac{x}{y}\right)}}dxdy = K \bullet M = \frac{\pi}{2\sqrt{2}}\varpi$$

$$notes:\begin{cases} polygamma: (-1)^m\psi^{(m)}(1-z)-\psi^{(m)}(z)=\pi\dfrac{d^m}{dz^m}\cot(\pi z) \\ gamma: \Gamma(1-z)\Gamma(z)=\pi\csc(\pi z);\ \Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi} \\ lemniscate\ constant: \varpi=\dfrac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \end{cases}$$

2410. A Reciprocal integral relation: If we define the function $\psi(q)$

$$\psi(q)=\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)-q}{\cosh^2(2\pi x)-q}dx$$

then prove the relation

$$\frac{\psi(q)}{q}=2q(q-1)\frac{\partial^2\psi(q)}{\partial q^2}+(2q+1)\frac{\partial\psi(q)}{\partial q}$$

Proposed by Srinivasa Raghava – AIRMC – India

Solution by Rana Ranino – Algeria

$$\psi(q)=\underbrace{\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)}{\cosh^2(2\pi x)-q}dx}_{odd\ function}-q\int_{-\infty}^{\infty}\frac{dx}{\cosh^2(2\pi x)-q}$$

$$\psi(q)=2q\int_0^{\infty}\frac{dx}{\cosh^2(2\pi x)-q}\overset{t=\tanh(2\pi x)}{\cong}$$

$$=-\frac{1}{\pi}\int_0^1\frac{dt}{\frac{1-q}{q}+t^2}=-\frac{1}{\pi}\sqrt{\frac{q}{1-q}}\tan^{-1}\left(\sqrt{\frac{q}{1-q}}\right)$$

$$\frac{d\psi}{dq}=-\frac{1}{2\pi(1-q)\sqrt{q(1-q)}}\tan^{-1}\sqrt{\frac{q}{1-q}}-\frac{1}{2\pi(1-q)}$$

$$\frac{d\psi}{dq}=\frac{1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)}$$

$$(2q+1)\frac{d\psi}{dq}=\frac{2q+1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)^2}$$

$$\frac{d^2\psi}{dq^2}=\frac{1}{2q(1-q)}\frac{d^2\psi}{dq^2}+\frac{2q-1}{2q^2(1-q)^2}\psi(q)-\frac{1}{2\pi(1-q)^2}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}=-\frac{d\psi}{dq}-\frac{2q-1}{q(1-q)}\psi(q)+\frac{q}{\pi(1-q)}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}=-\frac{4q-1}{2q(1-q)}\psi(q)+\frac{2q+1}{2\pi(1-q)}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=-\frac{4q-1}{2q(1-q)}\psi(q)+\frac{2q+1}{2\pi(1-q)}+\frac{2q+1}{2q(1-q)}\psi(q)-\frac{2q+1}{2\pi(1-q)}$$

$$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=\frac{\psi(q)}{q}\ \text{Proved}$$

$$\psi(q)=\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)-q}{\cosh^2(2\pi x)-q}dx=-\frac{1}{\pi}\sqrt{\frac{q}{1-q}}\tan^{-1}\left(\sqrt{\frac{q}{1-q}}\right)$$

$$2q(q-1)\frac{d^2\psi}{dq^2}+(2q+1)\frac{d\psi}{dq}=\frac{\psi(q)}{q}$$

2411. ***Prove that:***

$$\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\frac{\pi^2}{12}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\sigma=\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k};$$

Notes:

$$\left\{\text{Skew harmonic series}: H_n^-=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k},\ H_{2n}^-=\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\int_0^1\frac{1-x^{2n}}{1+x}dx\right.$$

$$\left.=H_{2n}-H_n\right\}$$

$$\left\{\text{Generating Binomial series}: \sum_{n=1}^{\infty}\frac{x^{2n-1}}{4^n}\binom{2n}{n}=\frac{1}{x\sqrt{1-x^2}}-\frac{1}{x};\int\sum_{n=1}^{\infty}\frac{x^{2n-1}}{4^n}\binom{2n}{n}dx\right.$$

$$\left.=\int\frac{1-\sqrt{1-x^2}}{x\sqrt{1-x^2}}dx\right\}$$

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{n4^n}\binom{2n}{n}=-2\ln(x)-2\tanh^{-1}\sqrt{1-x^2}$$

$$\sigma=\sum_{n=1}^{\infty}\frac{H_{2n}^{-}}{n4^n}\binom{2n}{n}=\sum_{n=1}^{\infty}\frac{H_{2n}-H_n}{n4^n}\binom{2n}{n}=\sum_{n=1}^{\infty}\frac{1}{n4^n}\binom{2n}{n}\int_0^1\frac{1-x^{2n}}{1+x}dx=$$

$$=\int_0^1\frac{1}{1+x}\left(\sum_{n=1}^{\infty}\frac{1}{n4^n}\binom{2n}{n}-\sum_{n=1}^{\infty}\frac{x^{2n}}{n4^n}\binom{2n}{n}\right)dx$$

$$=\int_0^1\frac{-2\ln(1)-2\tanh^{-1}(0)+2\tanh^{-1}\sqrt{1-x^2}+2\ln(x)}{1+x}dx=$$

$$=2\int_0^1\frac{\tanh^{-1}\sqrt{1-x^2}}{1+x}dx-2\int_0^1\frac{\ln(x)}{1+x}dx\quad Using\ \dot{I}BP\ method:\{v=\ln(1+x),u$$

$$=\tanh^{-1}\sqrt{1-x^2},\frac{du}{dx}=-\frac{1}{x\sqrt{1-x^2}}\}$$

$$\sigma=2[\ln(1+x)\tanh^{-1}\sqrt{1-x^2}]_0^1+2\int_0^1\frac{\ln(1+x)}{x\sqrt{1-x^2}}dx-\frac{\pi^2}{6}=2\int_0^1\frac{\ln(1+x)}{x\sqrt{1-x^2}}dx-\frac{\pi^2}{6}$$

$$Using\ Feynman's\ integration\ method$$

$$I(a)=2\int_0^1\frac{\ln(1+ax)}{x\sqrt{1-x^2}}dx,\frac{d}{da}I(a)=2\int_0^1\frac{1}{(1+ax)\sqrt{1-x^2}}dx$$

$$Subtitation\ \{x=\sin(t),dx=\cos(t)\,dt\}$$

$$I'(a)=2\int_0^{\frac{\pi}{2}}\frac{1}{a\sin(t)+1}dt,\left\{\tan\left(\frac{t}{2}\right)=u,dt=\frac{2du}{1+u^2},\sin(t)=\frac{2u}{1+u^2},u[1;0]\right\}$$

$$I'(a)=4\int_0^1\frac{1}{\frac{2au}{1+u^2}+1}\frac{du}{1+u^2}=4\int_0^1\frac{1}{u^2+2ua+1}du=4\left[\frac{\tan^{-1}\left(\frac{u+a}{\sqrt{1-a^2}}\right)}{\sqrt{1-a^2}}\right]_0^1=$$

$$=\frac{4}{\sqrt{1-a^2}}\left(\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right);\ \{I(1)=\sigma\quad I(0)=0\}$$

$$I=4\int_0^1\frac{1}{\sqrt{1-a^2}}\left(\tan^{-1}\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right)\right)da$$

$$Substitution:\left\{\tan^{-1}\left(\frac{a+1}{\sqrt{1-a^2}}\right)-\tan^{-1}\left(\frac{a}{\sqrt{1-a^2}}\right);\ d\theta=-\frac{1}{2\sqrt{1-a^2}};\theta\left[0;\frac{\pi}{4}\right]\right\}$$

$$I=8\int_0^{\frac{\pi}{4}}\theta d\theta=8\left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{4}}=\frac{8\pi^2}{32}=\frac{\pi^2}{4}$$

$$\sigma=\sum_{n=1}^{\infty}\frac{1}{n2^{2n}}\binom{2n}{n}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=\frac{\pi^2}{4}-\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

2412. If:

$$\sum_{m=0}^{\infty}\left((-1)^{\left[\frac{m+1}{5}\right]}+(-1)^{\left[\frac{m-1}{5}\right]}\right)x^m=2$$

$$then\ find\ the\ value\ of\ the\ expression\ \ x^5-x^3-x^2-x$$

$$[*]\ is\ the\ floor\ function$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\sum_{m=0}^{\infty}(-1)^{\left[\frac{m+1}{5}\right]}x^m+\sum_{m=0}^{\infty}(-1)^{\left[\frac{m-1}{5}\right]}x^m=\Omega_1+\Omega_2=2$$

$$0<x<1;\ k=\left[\frac{m+1}{5}\right], n=\left[\frac{m-1}{5}\right], m\epsilon N_0;\ \begin{matrix} m+1<5, k=0 \\ m+1>5, k\epsilon N \\ m<1, \quad k\epsilon Z^- \end{matrix}$$

$$\Omega_1=\sum_{m=0}^{\infty}(-1)^{\left[\frac{m+1}{5}\right]}x^m=1+x+x^2+x^3-x^4-x^5-x^6-x^7-x^8+x^9\dots$$

$$=\sum_{m=0}^{3}x^m-\sum_{m=4}^{8}x^m+\sum_{m=9}^{13}x^m-\cdots$$

$$\Omega_2=\sum_{m=0}^{\infty}(-1)^{\left[\frac{m-1}{5}\right]}x^m=-1+x+x^2+x^3+x^4+x^5-x^6-x^7-\cdots-x^{10}+x^{11}+\cdots$$

$$\Omega_1+\Omega_2=2,\quad 2x+2x^2+2x^3-2x^6-2x^7-2x^8+\cdots=2$$

$$(x+x^2+x^3)-x^5(x+x^2+x^3)+x^{10}(x+x^2+x^3)\dots=1$$

$$(x+x^2+x^3)\left(1-x^5+x^{10}-x^{15}+x^{20}-\cdots\right)=1$$

$$(x+x^2+x^3)\sum_{m=0}^{\infty}(-1)^m x^{5m}=1;\frac{x+x^2+x^3}{1+x^5}=1$$

$$-x^5+x+x^2+x^3=1\ ;\ x^5-x^3-x^2-x=-1$$

2413. Prove that:

$$\int_0^1\int_0^1\frac{e^{x+y}-1}{e^{x+y}+1}dxdy=4\chi_2(e^2)-4\text{Li}_2(e)+\frac{\pi^2}{6}-1$$

where χ_v is the Legendre's chi function, $\text{Li}_2(z)$ is the dilogarithm or Spence's function.

Proposed by Ankush Kumar Parcha-India

Solution by Cosghun Memmedov-Azerbaijan

$$\int_0^1\int_0^1\frac{e^{x+y}-1}{e^{x+y}+1}dxdy\overset{\{x+y=t\}}{\triangleq}\int_0^1\int_y^{y+1}\frac{e^t-1}{e^t+1}dtdy=\int_0^1\int_y^{y+1}\frac{1-e^{-t}}{1+e^{-t}}dtdy=$$

$$=\int_0^1\int_y^{y+1}dtdy-2\int_0^1\int_y^{y+1}\frac{e^{-t}}{1+e^{-t}}dtdy=\int_0^1 dy+2\sum_{n=1}^{\infty}(-1)^n\int_0^1\int_y^{y+1}e^{-tn}dtdy=$$

$$=1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1\left(e^{-yn}-e^{-(y+1)n}\right)dtdy=1+2\left(-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}-\right.$$

$$\left.2\sum_{n=1}^{\infty}\frac{\left(-e^{-1}\right)^n}{n^2}+\sum_{n=1}^{\infty}\frac{\left(-e^{-2}\right)^n}{n^2}\right)=1+2\left(-\frac{\pi^2}{12}+\mathrm{Li}_2\left(-\frac{1}{e^2}\right)-2\mathrm{Li}_2\left(-\frac{1}{e}\right)\right)=$$

$$=1+2\left(-\frac{\pi^2}{12}-2-\frac{\pi^2}{6}-\mathrm{Li}_2\left(-e^2\right)+1+\frac{\pi^2}{3}+\mathrm{Li}_2\left(e^2\right)-2\mathrm{Li}_2(\mathrm{e})\right)=1+\frac{\pi^2}{6}+4\chi_2\left(e^2\right)-4\mathrm{Li}_2(\mathrm{e})-2=$$

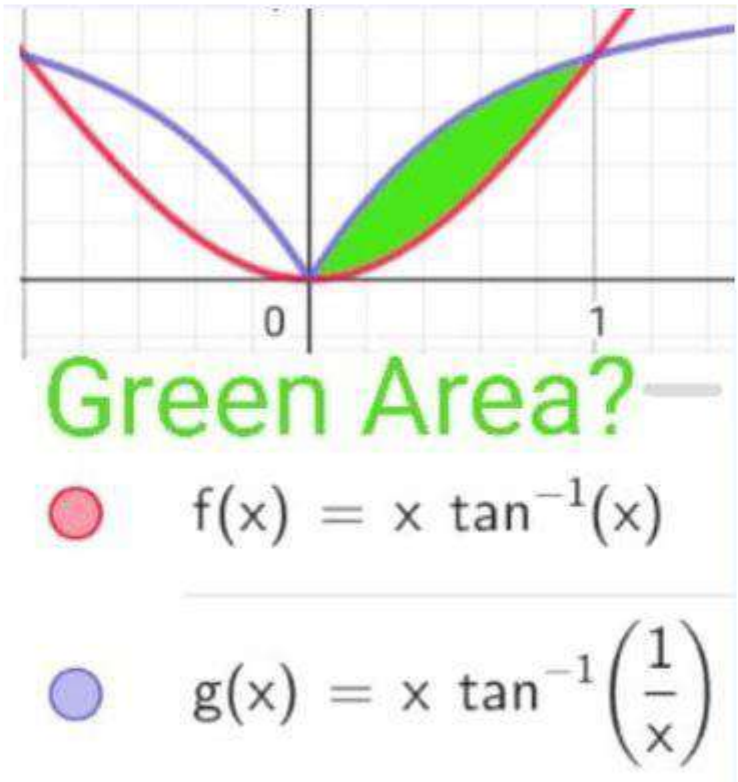
$$=4\chi_2\left(e^2\right)-4\mathrm{Li}_2(\mathrm{e})+\frac{\pi^2}{6}-1$$

Notes: Dilogarithm formula

$$\mathrm{Li}_2(-\mathrm{z})+\mathrm{Li}_2\left(-\frac{1}{z}\right)=-\frac{ln^2(z)}{2}+2\mathrm{Li}_2(-1),\quad \mathrm{Li}_2(\mathrm{z})+\mathrm{Li}_2(-\mathrm{z})=\frac{1}{2}\mathrm{Li}_2\left(z^2\right)$$

Legendres chi function $\chi_v(z)=\frac{1}{2}\left[\mathrm{Li}_{\mathrm{v}}(\mathrm{z})-\mathrm{Li}_{\mathrm{v}}(-\mathrm{z})\right]$

2414.



Proposed by Sonu Aarnav-India

Solution by Daniel Sitaru-Romania

$$Green\ area=\int_0^1(g(x)-f(x))\,dx=\int_0^1\left(xtan^{-1}\left(\frac{1}{x}\right)-xtan^{-1}x\right)dx=$$

$$=\int_0^1\left(x\left(\frac{\pi}{2}-\tan^{-1}x\right)-x\tan^{-1}x\right)dx=\int_0^1\left(\frac{\pi x}{2}-2x\tan^{-1}x\right)dx=$$

$$=\frac{\pi}{2}\left(\frac{1}{2}-\frac{0}{2}\right)-2\int_0^1\left(\frac{x^2}{2}\right)'\tan^{-1}xdx=\frac{\pi}{4}-2\left(\frac{1}{2}\right)\tan^{-1}1+2\int_0^1\frac{x^2}{2}\cdot\frac{1}{x^2+1}dx=$$

$$=\frac{\pi}{4}-\frac{\pi}{4}+\int_0^1\frac{x^2+1-1}{x^2+1}dx=\int_0^1\frac{x^2+1}{x^2+1}dx-\int_0^1\frac{1}{x^2+1}dx=$$

$$=1-\tan^{-1}1+\tan^{-1}0=1-\frac{\pi}{4}$$

2415. Find a closed form:

$$\int_0^\infty\int_0^\infty\frac{\arctan(x^2)\arctan(y^4)}{x^2y^3}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\int_0^\infty\int_0^\infty\frac{\arctan(x^2)\arctan(y^4)}{x^2y^3}dxdy=$$

$$=\int_0^\infty\frac{\arctan(x^2)}{x^2}dx\,.\int_0^\infty\frac{\arctan(y^4)}{y^3}dy=X\,.Y$$

$$X=\int_0^\infty\frac{\arctan(x^2)}{x^2}dx\overset{IBP}{\cong}\left(-\frac{\arctan(x^2)}{x^2}\right)_0^\infty+2\int_0^\infty\frac{dx}{1+x^4}=\frac{1}{2}\int_0^\infty\frac{x^{-\frac{3}{4}}}{1+x}dx=$$

$$=\frac{1}{2}\beta\left(\frac{1}{4};\frac{3}{4}\right)=\frac{\pi}{\sqrt{2}}$$

$$Y=\int_0^\infty\frac{\arctan(y^4)}{y^3}dy\overset{IBP}{\cong}\left(-\frac{\arctan(y^4)}{2y^2}\right)_0^\infty+2\int_0^\infty\frac{y}{1+y^8}dy=\int_0^\infty\frac{dy}{1+y^4}=$$

$$=\frac{1}{4}\int_0^\infty\frac{y^{-\frac{3}{4}}}{1+y}dy=\frac{1}{4}\beta\left(\frac{1}{4};\frac{3}{4}\right)=\frac{\pi}{2\sqrt{2}}$$

$$\int_0^\infty\frac{\arctan(x^2)}{x^2}dx\,.\int_0^\infty\frac{\arctan(y^4)}{y^3}dy=X\,.\,Y=\frac{\pi}{\sqrt{2}}\cdot\frac{\pi}{2\sqrt{2}}=\frac{\pi^2}{4}$$

2416. Find a closed form:

$$\int_{-\infty}^{\infty}\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)(1+y+y^2)}dxdy$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\int_{-\infty}^{\infty}\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)(1+y+y^2)}dxdy=\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}\cdot\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx=Y\,.\,X$$

$$Y=\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}=\int_{-\infty}^{\infty}\frac{dy}{(\frac{1}{4}+y+y^2+\frac{3}{4})^2}=\int_{-\infty}^{\infty}\frac{dy}{(\left(\frac{1}{2}+y\right)^2+\frac{3}{4})^2}$$

$$\underbrace{\overbrace{=}^{y+1=\frac{\sqrt{3}}{2}\tan(z)}}_{dy=\frac{\sqrt{3}}{2}sec^2(z)}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\frac{\sqrt{3}}{2}sec^2(z)}{((\frac{\sqrt{3}}{2}\sec(z))^2)^2}dz=\frac{\sqrt{3}}{2}\cdot\frac{16}{9}\cdot2\int_{0}^{\frac{\pi}{2}}cos^2(z)dz=\frac{4\pi}{3\sqrt{3}}$$

$$X=\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx\overset{x\to\frac{1}{x}}{\cong}-\int_{0}^{1}\frac{xln^3(x)}{(1-x)^2}dx=-\sum_{n=0}^{\infty}n\int_{0}^{1}x^nln^3(x)dx$$

$$6\sum_{n=0}^{\infty}\frac{n}{(n+1)^4}=6\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^3}-\frac{1}{(n+1)^4}\right)=6\sum_{n=0}^{\infty}\frac{1}{(n+1)^3}-6\sum_{n=0}^{\infty}\frac{1}{(n+1)^4}$$

$$=6\zeta(3)-6\zeta(4)=6(\zeta(3)-\frac{\pi^4}{90})$$

$$\int_{-\infty}^{\infty}\frac{dy}{(1+y+y^2)^2}\cdot\int_{1}^{\infty}\frac{ln^3(x)}{(x^3-2x^2+x)}dx=Y.X=\frac{8\pi}{\sqrt{3}}(\zeta(3)-\frac{\pi^4}{90})$$

2417. If $0<a\le b$, then :

$$4\int_a^b \tanh x\,dx \ge \cos 2a - \cos 2b$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$4\int_a^b \tanh x\,dx = 4\int_a^b \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = 4\int_a^b \frac{(e^x + e^{-x})'}{e^x + e^{-x}}dx$$

$$= 4\ln(e^b + e^{-b}) - 4\ln(e^a + e^{-a}) \overset{?}{\ge} \cos 2a - \cos 2b$$

$$\Leftrightarrow 4\ln(e^b + e^{-b}) + \cos 2b \overset{?}{\ge} 4\ln(e^a + e^{-a}) + \cos 2a$$

$$\Leftrightarrow f(b) - f(a) \overset{?}{\ge} 0 \ (f(x) = 4\ln(e^x + e^{-x}) + \cos 2x\ \forall\, x > 0) \overset{\text{via MVT}}{\Leftrightarrow}$$

$$(b-a)f'(c) \overset{?}{\ge} 0\ (a \le c \le b) \Leftrightarrow f'(c) \overset{?}{\ge} 0\ (\because b \ge a) \Leftrightarrow 4.\frac{e^c - e^{-c}}{e^c + e^{-c}} - 2\sin 2c \overset{?}{\ge} 0$$

$$\Leftrightarrow 4.\frac{e^{2c} + 1 - 2}{e^{2c} + 1} - 2\sin 2c \overset{?}{\ge} 0 \Leftrightarrow \boxed{2 - \sin 2c - \frac{4}{e^{2c} + 1} \overset{?}{\ge} 0} \to (1)$$

$$\text{Let } F(y) = \sin y - y + \frac{y^3}{6} - \frac{y^5}{120}\ \forall\, y \in [0,\infty)$$

$$\therefore F'(y) = \cos y - \frac{y^4}{24} + \frac{y^2}{2} - 1 \text{ and } F''(y) = -\left(\sin y - \left(y - \frac{y^3}{6}\right)\right)$$

$$\text{Let } P(y) = \sin y - y + \frac{y^3}{6}\ \forall\, y \in [0,\infty) \therefore P'(y) = \cos y + \frac{y^2}{2} - 1 \text{ and}$$

$$P''(y) = y - \sin y \ge 0 \Rightarrow P'(y) \text{ is} \uparrow \text{on } [0,\infty) \Rightarrow P'(y) \ge P'(0) = 0 \Rightarrow P(y) \text{ is} \uparrow$$

$$\text{on } [0,\infty) \Rightarrow P(y) \ge P(0) = 0 \Rightarrow \sin y \ge y - \frac{y^3}{6}\ \forall\, y \in [0,\infty)$$

$$\Rightarrow F''(y) \le 0 \Rightarrow F'(y) \text{ is} \downarrow \text{on } [0,\infty) \Rightarrow F'(y) \le F'(0) = 0 \Rightarrow F(y) \text{ is} \downarrow \text{on } [0,\infty)$$

$$\Rightarrow F(y) \le F(0) = 0 \Rightarrow \sin y \le y - \frac{y^3}{6} + \frac{y^5}{120}\ \forall\, y \in [0,\infty)$$

$$\therefore \forall\, y \in (0,\infty), \sin y < y - \frac{y^3}{6} + \frac{y^5}{120} \to (i)$$

$$\text{Let } h(y) = e^y - 1 - y - \frac{y^2}{2}\ \forall\, y \in [0,\infty) \therefore h'(y) = e^y - 1 - y \ge 0 \Rightarrow h(y) \text{ is} \uparrow$$

$$\text{on } [0,\infty) \Rightarrow h(y) \ge h(0) = 0 \Rightarrow \forall\, y \in (0,\infty), e^y > 1 + y + \frac{y^2}{2} \to (ii)$$

$$\boxed{Case\ 1}\ c \ge 1 \text{ and we have} : 2 - \sin 2c - \frac{4}{e^{2c} + 1} > 1 - \sin 2c + 1 - \frac{4}{2c + 1 + 1}$$

$$= 1 - \sin 2c + 1 - \frac{2}{c+1} = (1 - \sin 2c) + \frac{c-1}{c+1} \ge 0 \Rightarrow (1) \text{ is true}$$

$$\boxed{Case\ 2}\ 0 < c < 1 \text{ and via } (i),(ii), 2 - \sin 2c - \frac{4}{e^{2c} + 1} >$$

$$2-\left(2c-\frac{8c^3}{6}+\frac{32c^5}{120}\right)-\frac{4}{\left(1+c+\frac{c^2}{2}\right)^2+1}$$

$$=2\left(\frac{15-15c+20c^3-4c^5}{15}-\frac{8}{4(1+c)^2+c^4+4c^2(1+c)+4}\right)$$

$$=\frac{-2c^3\left(4c^6+16c^5+12c^4-48c^3-113c^2-115c-100\right)}{15(4(1+c)^2+c^4+4c^2(1+c)+4)}$$

$$=\frac{-2c^3\left(4(c^6-1)+16(c^5-1)+12(c^4-1)-48c^3-113c^2-115c-68\right)}{15(4(1+c)^2+c^4+4c^2(1+c)+4)}>0$$

$\because 0<c<1\Rightarrow 4(c^6-1)+16(c^5-1)+12(c^4-1)-48c^3-113c^2-115c-68<0\Rightarrow$ (1) is true $\therefore$ combining both cases, (1) is true $\forall\, c\in(0,\infty)$

$$\therefore 4\int_a^b \tanh x\,dx\ge\cos 2a-\cos 2b \text{ whenever } 0<a\le b \text{ (QED)}$$

2418. Find a closed form:

$$\Omega=\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2 dxdy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2 dxdy,\qquad \left\{\int_0^1\int_0^1 f(xy)dxdy=-\int_0^1 \ln(x)f(x)\,dx\right\}$$

$$\Omega=-\int_0^1\frac{\ln(x)\ln^2(1+x)}{(1+x)^2}dx, \text{using İBP method}$$

$$\left\{\begin{array}{l} u=\ln(x)\ln^2(1+x),\ \ du=\left(\frac{2\ln(x)\ln(1+x)}{1+x}+\frac{\ln^2(1+x)}{x}\right)dx \\ v=\int\frac{1}{(1+x)^2}dx=-\frac{1}{1+x}\end{array}\right\}$$

$$\Omega=\left[\frac{\ln(x)\ln^2(1+x)}{1+x}\right]_0^1-2\int_0^1\frac{\ln(x)\ln(1+x)}{(1+x)^2}dx-\int_0^1\frac{\ln^2(1+x)}{x(1+x)}dx$$

$$\Omega=2\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\frac{\ln(x)\ln(1+x)}{(1+x)^2}dx, \text{using IBP method}$$

$$\left\{\begin{array}{l} u=\ln(x)\ln(1+x),\ \ du=\left(\frac{\ln(x)}{1+x}+\frac{\ln(1+x)}{x}\right)dx \\ v=\int\frac{1}{(1+x)^2}dx=-\frac{1}{1+x}\end{array}\right\}$$

$$\Omega_1=[-\frac{\ln(x)\ln(1+x)}{1+x}]_0^1+\int_0^1\frac{\ln(x)}{(1+x)^2}dx+\int_0^1\frac{\ln(1+x)}{x}dx-\int_0^1\frac{\ln(1+x)}{1+x}dx=$$

$$=-\sum_{n=1}^{\infty}n(-1)^n\int_0^1 x^{n-1}\ln(x)\,dx-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{n-1}dx-[\frac{\ln^2(1+x)}{2}]_0^1=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}-\frac{\ln^2(2)}{2}=-\ln(2)+\frac{\zeta(2)}{2}-\frac{\ln^2(2)}{2}$$

$$\Omega_2=\int_0^1\frac{\ln^2(1+x)}{x(1+x)}dx,\quad \textit{using IBP method}$$

$$\left\{\begin{matrix} u=\ln^2(1+x), & du=\frac{2\ln(1+x)}{1+x}dx \\ v=\int\frac{1}{x(1+x)}dx=\int\frac{1}{x}dx-\int\frac{1}{1+x}dx=\ln(x)-\ln(1+x) \end{matrix}\right\}$$

$$\Omega_2=[\ln^2(1+x)(\ln(x)-\ln(1+x))]_0^1-2\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx+2\int_0^1\frac{\ln^2(1+x)}{1+x}dx$$

$$=$$

$$=-\ln^3(2)-2I+2J$$

$$I=\int_0^1\frac{\ln(x)\ln(1+x)}{1+x}dx=-\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1 x^n\ln(x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{(n+1)^2}=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_{n+1}-\frac{1}{n+1}\right)}{(n+1)^2}=\sum_{n=1}^{\infty}\frac{(-1)^nH_{n+1}}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^3}=$$

$$=1-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^2}-\frac{3\zeta(3)}{4}-1=\frac{5\zeta(3)}{8}-\frac{3\zeta(3)}{4}=-\frac{\zeta(3)}{8}$$

$$J=\int_0^1\frac{\ln^2(1+x)}{1+x}dx=\int_1^2\frac{\ln^2(x)}{x}dx=[\ln^3(x)]_1^2-2\int_1^2\frac{\ln^2(x)}{x}dx=\ln^3(2)-2J$$

$$3J=\ln^3(2)\quad J=\frac{\ln^3(2)}{3}$$

$$\Omega_2=-\ln^3(2)+\frac{\zeta(3)}{4}+\frac{2\ln^3(2)}{3}=\frac{\zeta(3)}{4}-\frac{\ln^3(2)}{3}$$

$$\int_0^1\int_0^1\left(\frac{\ln(1+xy)}{1+xy}\right)^2dxdy=-2\Omega_1-\Omega_2=2\ln(2)-\frac{\pi^2}{6}+\ln^2(2)-\frac{\zeta(3)}{4}+\frac{\ln^3(2)}{3}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\mathrm{I}=\int_0^1\int_0^1\frac{\log^2(1+xy)}{(1+xy)^2}dx\,\mathrm{dy}\Bigg]_{xy=m}=\int_0^1\frac{1}{x}\int_0^1\frac{\log^2(1+m)}{(1+m)^2}dm\,\mathrm{dx}\overset{IBP}{=}$$

$$\overset{IBP}{=} \log(x)\int_0^x \frac{\log^2(1+m)}{(1+m)^2}dm\Bigg]_{x=0}^{x=1} - \int_0^1 \frac{\log(x)\log^2(1+x)}{(1+x)^2}dx$$

$$= -\int_0^1 \frac{\log(x)\log^2(1+x)}{(1+x)^2}dx \overset{IBP}{=}$$

$$\overset{IBP}{=} \frac{\log(x)\log^2(1+x)}{(1+x)^2}\Bigg]_{x=0}^{x=1} - \int_0^1 \frac{\log^2(1+x)}{x(1+x)}dx - 2\int_0^1 \frac{\log(x)\log(1+x)}{(1+x)^2}dx =$$

$$= -\int_0^1 \frac{\log^2(1+x)}{x(1+x)}dx - 2\int_0^1 \frac{\log(x)\log(1+x)}{(1+x)^2}dx = -I_1 - 2I_2$$

$$I_1 = \int_0^1 \frac{\log^2(1+x)}{x(1+x)}dx = \int_0^1 \frac{\log^2(1+x)}{x}dx - \int_0^1 \frac{\log^2(1+x)}{1+x}dx = \frac{1}{4}\zeta(3) - \frac{1}{3}\log^3(2)$$

$$I_2 = \int_0^1 \frac{\log(x)\log(1+x)}{(1+x)^2}dx \overset{IBP}{=} -\frac{\log(x)\log(1+x)}{1+x}\Bigg]_{x=0}^{x=1} + \int_0^1 \frac{\log(1+x)}{x(1+x)}dx + \int_0^1 \frac{\log(x)}{(1+x)^2}dx$$

$$= \int_0^1 \frac{\log(1+x)}{x}dx - \int_0^1 \frac{\log(1+x)}{1+x}dx + \int_0^1 \frac{\log(x)}{(1+x)^2}dx = \frac{1}{2}\zeta(2) - \frac{1}{2}\log^2(2) - \log(2)$$

$$I = -I_1 - 2I_2 = -\frac{1}{4}\zeta(3) - \zeta(2) + \frac{1}{3}\log^3(2) + \log^2(2) + 2\log(2)$$

2419. Find a closed form:

$$\Omega = \int_0^1\int_0^1 \frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^1\int_0^1 \frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy = -\int_0^1 \frac{\ln(x)\tanh^{-1}(x)}{1-x^2}dx =$$

$$= \frac{1}{2}\int_0^1 \frac{\ln(x)\ln\left(\frac{1-x}{1+x}\right)}{1-x^2}dx = -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2k+1}\int_0^1 x^{2k+2n+1}\ln(x)\,dx =$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2k+2n+2)^2}=\frac{1}{4}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(2k+1)(n+k+1)^2}$$

$$\left\{polygamma\ function: \psi^{(a)}(n+1)=\sum_{k=1}^{\infty}\frac{(-1)^{n+1}a!}{(k+n)^{a+1}};\ a\epsilon\{Z\geq 1\}\right\}$$

$$\Omega=\frac{1}{4}\sum_{k=0}^{\infty}\frac{\psi^{(1)}(k+1)}{2k+1}=\frac{1}{4}\sum_{k=0}^{\infty}\frac{\zeta(2)-H_k^{(2)}}{2k+1}=\frac{1}{4}\left(\zeta(2)\sum_{k=0}^{\infty}\frac{1}{2k+1}-\sum_{k=0}^{\infty}\frac{H_k^{(2)}}{2k+1}\right)=$$

$$=\frac{1}{4}\left(\zeta(2)\int_0^1\frac{1}{1-x^2}dx-\int_0^1\sum_{k=0}^{\infty}H_k^{(2)}x^{2k}\,dx\right)=\frac{1}{4}\left(\frac{1}{2}\zeta(2)[\ln\left(\frac{1-x}{1+x}\right)]_0^1-I\right)$$

$$\left\{we\ know\ \leq>\ \sum_{n=0}^{\infty}x^nH_n^{(a)}=\frac{Li_a(x)}{1-x},\ \ Li_a(1)=\zeta(a)\right\}$$

$$I=\int_0^1\sum_{k=0}^{\infty}x^{2k}H_k^{(2)}dx=\int_0^1\frac{Li_2(x^2)}{1-x^2}dx, using\ IBP\ method$$

$$I=[\frac{1}{2}Li_2(x^2)]_0^1\int_0^1\frac{1}{1-x^2}dx-\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx=$$

$$=\frac{\zeta(2)}{2}\int_0^1\frac{1}{1-x^2}dx-\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx$$

$$\Omega=\frac{1}{4}\left(\frac{1}{2}\zeta(2)\int_0^1\frac{1}{1-x^2}dx-I\right)=\frac{1}{4}\int_0^1\frac{\ln(1-x^2)\ln\left(\frac{1-x}{1+x}\right)}{x}dx=$$

$$=\frac{1}{4}\int_0^1\frac{\ln^2(1-x)-\ln^2(1+x)}{x}dx=\frac{1}{4}\int_0^1\frac{\ln^2(x)}{1-x}dx-\frac{1}{4}\int_0^1\frac{\ln^2(1+x)}{x}dx=$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}\int_0^1x^n\ln^2(x)dx-\frac{1}{4}J=\frac{1}{2}\zeta(3)-\frac{1}{4}J$$

$$J=\int_0^1\frac{\ln^2(1+x)}{x}dx, substitution\left\{\frac{1}{1+x}=y,\qquad y\left[\frac{1}{2};\ 1\right]\right\}$$

$$J=\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{y(1-y)}dy=\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{y}dy+\int_{\frac{1}{2}}^1\frac{\ln^2(y)}{1-y}dy=$$

$$=\frac{\ln^3(2)}{3}+\int_0^1\frac{\ln^2(y)}{1-y}dy-\int_0^{\frac{1}{2}}\frac{\ln^2(y)}{1-y}dy=\frac{\ln^3(2)}{3}+2\zeta(3)-\frac{7}{4}\zeta(3)-\frac{\ln^3(2)}{3}=$$

$$=\frac{\zeta(3)}{4};\quad \Omega=\frac{1}{2}\zeta(3)-\frac{1}{4}J=\frac{1}{2}\zeta(3)-\frac{1}{16}\zeta(3)=\frac{7}{16}\zeta(3)$$

$$\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dxdy=\frac{7}{16}\zeta(3)$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\frac{\tanh^{-1}(xy)}{1-x^2y^2}dx\,dy\Bigg]_{xy=m}=\int_0^1\frac{1}{x}\int_0^x\frac{\tanh^{-1}(m)}{1-m^2}dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\log(x)\int_0^x\frac{\tanh^{-1}(m)}{1-m^2}dm\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log(x)\tanh^{-1}(x)}{1-x^2}dx=-\int_0^1\frac{\log(x)\tanh^{-1}(x)}{1-x^2}dx$$

$$\tanh^{-1}(x)=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$I=-\frac{1}{2}\int_0^1\frac{\log(x)\log\left(\frac{1+x}{1-x}\right)}{1-x^2}dx=\frac{1}{2}\int_0^1\frac{\log(x)\log\left(\frac{1-x}{1+x}\right)}{1-x^2}dx\Bigg]_{\frac{1-x}{1+x}=z}=$$

$$=\frac{1}{4}\int_0^1\frac{\log(z)\log\left(\frac{1-z}{1+z}\right)}{z}dz=$$

$$=\frac{1}{4}\int_0^1\frac{\log(z)\log(1-z)}{z}dz-\frac{1}{4}\int_0^1\frac{\log(z)\log(1+z)}{z}dz=$$

$$=\frac{1}{4}\varsigma(3)-\frac{1}{4}\left\{-\frac{3}{4}\varsigma(3)\right\}=\frac{7}{16}\varsigma(3)$$

2420. Find a closed form:

$$\Omega=\int_0^1\int_0^1\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy\,;IBP\to\quad u=\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right),\ du=\frac{y(y^2-x^2)dx}{x^4+x^2y^2+y^4}$$

$$v=\int dx=x$$

$$\Omega=\int_0^1\left([\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)x]_0^1-\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dx\right)dy=$$

$$\Omega=\int_0^1\tanh^{-1}\left(\frac{1}{y}+y\right)dy-\int_0^1\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dxdy=\Omega_1-\Omega_2$$

$$\Omega_1=\int_0^1 \tanh^{-1}\left(y+\frac{1}{y}\right)dy\,,IBP\to \quad u=\tanh^{-1}\left(y+\frac{1}{y}\right),\ du=\frac{(1-y^2)}{1+y^2+y^4}dy \quad v=\int dy=y$$

$$\Omega_1=[\tanh^{-1}\left(y+\frac{1}{y}\right)y]_0^1-\int_0^1\frac{y(1-y^2)}{1+y^2+y^4}dy=\tanh^{-1}(2)-\int_0^1\frac{y-2y^3+y^5}{1-y^6}dy=$$

$$=\tanh^{-1}(2)-\sum_{n=0}^{\infty}\int_0^1 y^{6n+1}-2y^{6n+3}+y^{6n+5}dy=\frac{1}{2}\ln\left(\frac{1+2}{1-2}\right)-$$

$$-\sum_{n=0}^{\infty}\left(\frac{1}{6n+2}-\frac{2}{6n+4}+\frac{1}{6n+6}\right)=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{(3n+1)(3n+2)}+$$

$$+\frac{1}{6}\sum_{n=0}^{\infty}\frac{1}{(n+1)(3n+2)}=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{1}{6}\left(\psi^{(0)}\left(1-\frac{1}{3}\right)-\psi^{(0)}\left(\frac{1}{3}\right)\right)+$$

$$+\frac{1}{6}\left(\psi^{(0)}(1)-\psi^{(0)}\left(\frac{2}{3}\right)\right)=\frac{1}{2}\ln(3)-\frac{i\pi}{2}-\frac{\pi}{6\sqrt{3}}+\frac{\ln(3)}{4}-\frac{\pi}{12\sqrt{3}}=$$

$$=\frac{3\ln(3)}{4}-\frac{\pi}{4\sqrt{3}}-\frac{i\pi}{2}=-\frac{\pi\sqrt{3}}{12}+\frac{3\ln(3)}{4}-\frac{i\pi}{2}$$

$$\Omega_2=\int_0^1\int_0^1\frac{xy(y^2-x^2)}{x^4+x^2y^2+y^4}dxdy=$$

$$=\int_0^1\int_0^1\frac{xy^3}{x^4+x^2y^2+y^4}dxdy-\int_0^1\int_0^1\frac{x^3y}{x^4+x^2y^2+y^4}dxdy=0\,\{symmetry\}$$

$$\int_0^1\int_0^1\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dxdy=\Omega_1-\Omega_2=-\frac{\pi\sqrt{3}}{12}+\frac{3\ln(3)}{4}-\frac{i\pi}{2}$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dx\,dy\Bigg]_{\frac{y}{x}=m}=\int_0^1 x\int_0^{\frac{1}{x}}\tanh^{-1}\left(m+\frac{1}{m}\right)dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\frac{x^2}{2}\int_0^{\frac{1}{x}}\tanh^{-1}\left(m+\frac{1}{m}\right)dm\Bigg|_{x=0}^{x=1}+\frac{1}{2}\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx=\frac{1}{2}\int_0^1\tanh^{-1}\left(m+\frac{1}{m}\right)dm+$$

$$+\frac{1}{2}\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx=\int_0^1\tanh^{-1}\left(x+\frac{1}{x}\right)dx$$

$$\tanh^{-1}\left(x+\frac{1}{x}\right)=z\Rightarrow\tanh\left(\tanh^{-1}\left(x+\frac{1}{x}\right)\right)=\tanh(z)\Rightarrow x+\frac{1}{x}=\tanh(z)\Rightarrow x+\frac{1}{x}$$
$$=\frac{e^{2z}-1}{e^{2z}+1}\Rightarrow e^{2z}=\frac{x^2+x+1}{x-1-x^2}\Rightarrow\log(e^{2z})=\log\left(\frac{x^2+x+1}{x-1-x^2}\right)\Rightarrow z$$
$$=\frac{1}{2}\log\left(\frac{x^2+x+1}{x-1-x^2}\right)$$
$$\tanh^{-1}\left(x+\frac{1}{x}\right)=\frac{1}{2}\log\left(\frac{x^2+x+1}{x-1-x^2}\right)$$
$$I=\frac{1}{2}\int_0^1\log\left(\frac{x^2+x+1}{x-1-x^2}\right)dx=\frac{1}{2}\left\{\int_0^1\log(x^2+x+1)\,dx-\int_0^1\log(x-1-x^2)\,dx\right\}$$
$$=\frac{1}{2}\left\{\left(\frac{3}{2}\log(3)-2+\frac{\pi}{2\sqrt{3}}\right)-\left(i\pi-2+\frac{\pi}{\sqrt{3}}\right)\right\}=-\frac{\pi\sqrt{3}}{12}+\frac{3}{4}\log(3)-\frac{i\pi}{2}$$

2421. Find a closed form:

$$\Omega=\int_0^1\frac{x(ln\sqrt{1+x^2}+\ arctan^2(x))}{1+x^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{x(ln\sqrt{1+x^2}+\ arctan^2(x))}{1+x^2}dx=\ \Omega_1+\Omega_2\ \begin{cases}\Omega_1=\int_0^1\frac{xln\sqrt{1+x^2}}{1+x^2}dx\\ \Omega_2=\int_0^1\frac{xarctan^2(x)}{1+x^2}dx\end{cases}$$

$$\Omega_1=\int_0^1\frac{xln\sqrt{1+x^2}}{1+x^2}dx\overset{IBP}{=}\frac{1}{2}\left[\frac{1}{2}ln^2(1+x^2)\right]_0^1-\frac{1}{2}\int_0^1\frac{xln(1+x^2)}{1+x^2}dx=\frac{1}{4}ln^2(2)-\Omega_1$$
$$2\Omega_1=\frac{1}{4}ln^2(2)\ \ll=\gg\ \Omega_1=\frac{1}{8}ln^2(2)$$
$$\Omega_2=\int_0^1\frac{xarctan^2(x)}{1+x^2}dx\overset{IBP}{=}\left[\frac{1}{2}arctan^2(x)\ln(1+x^2)\right]_0^1$$
$$-\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx=$$
$$\frac{\pi^2}{32}\ln(2)-\int_0^1\frac{\arctan(x)\ln(1+x^2)}{1+x^2}dx\ll=\gg\left\{\arctan(x)=t;\ \frac{dt}{dx}=\frac{1}{1+x^2};x\right.$$
$$\left.=\tan(t)\,t\left[\frac{\pi}{4};0\right]\right\}$$

$$\Omega_2 = \frac{\pi^2}{32}\ln(2) - \int_0^{\frac{\pi}{4}} t\ln\left(1+\tan^2(t)\right)dt = \frac{\pi^2}{32}\ln(2) + 2\int_0^{\frac{\pi}{4}} t\ln(\cos(t))dt$$

$$= \frac{\pi^2}{32}\ln(2) + 2I$$

Note :

$$\left\{We\ know \to \ln(\cos(t)) = -\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nt)}{n}\ \ Fourier\ series\right\}$$

$$I = \int_0^{\frac{\pi}{4}} t\ln(\cos(t))dt = -\ln(2)\int_0^{\frac{\pi}{4}} tdt - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)dt = -\frac{\pi^2}{32}\ln(2) -$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\cos(2nt)}{4n^2} + \frac{t\sin(2nt)}{2n}\right]_0^{\frac{\pi}{4}}$$

$$= -\frac{\pi^2}{32}\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\pi\sin\left(\frac{\pi n}{2}\right)}{8n} + \frac{\cos\left(\frac{\pi n}{2}\right)}{4n^2} - \frac{1}{4n^2}\right] =$$

$$-\frac{\pi^2}{32}\ln(2) - \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{\pi n}{2}\right) - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\cos\left(\frac{\pi n}{2}\right) + \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} =$$

$$-\frac{\pi^2}{32}\ln(2) + \frac{\pi}{8}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2} + \frac{1}{4}\left(\frac{1}{2^3}\left(1 - \frac{1}{2^3} + \frac{1}{3^3} - \cdots\right)\right) - \frac{1}{4}(1-2^{1-3})\zeta(3) =$$

$$-\frac{\pi^2}{32}\ln(2) + \frac{\pi}{8}G + \frac{3\zeta(3)}{16} + \frac{1}{32}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3} = \frac{\pi}{8}G - \frac{\pi^2}{32}\ln(2) - \frac{3\zeta(3)}{16} + \frac{3\zeta(3)}{128} =$$

$$\frac{\pi}{8}G - \frac{\pi^2}{32}\ln(2) - \frac{21\zeta(3)}{128}$$

$$\Omega_2 = \frac{\pi^2}{32}\ln(2) + 2I = \frac{\pi^2}{32}\ln(2) + \frac{\pi}{4}G - \frac{\pi^2}{16}\ln(2) - \frac{21\zeta(3)}{64} = \frac{\pi}{4}G - \frac{\pi^2}{32}\ln(2) - \frac{21\zeta(3)}{64}$$

$$\int_0^1 \frac{x(\ln\sqrt{1+x^2} + \arctan^2(x))}{1+x^2}dx = \Omega_1 + \Omega_2 = \frac{\pi}{4}G - \frac{\pi^2}{32}\ln(2) - \frac{21\zeta(3)}{64} + \frac{1}{8}\ln^2(2)$$

Solution 2 by Ankush Kumar Parcha-India

$$We\ have,\ \underbrace{\frac{1}{2}\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx}_{X} + \underbrace{\int_0^1\frac{x\arctan^2(x)}{1+x^2}dx}_{Y} \qquad (1)$$

$$X = \underbrace{\frac{1}{2}\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx}\ \overset{x^2\to x}{\triangleq}\ \frac{1}{4}\int_0^1\frac{\ln(1+x)}{1+x}dx \Rightarrow \left(\frac{\ln^2(1+x)}{8}\right)\Big|_0^1$$

$$X = \underbrace{\frac{1}{2}\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx} = \frac{\ln^2(2)}{8}$$

$$Y=\underbrace{\int_0^1 \frac{x\arctan^2(x)}{1+x^2}dx}\overset{\tan^{-1}(x)\to x}{\triangleq}\int_0^{\frac{\pi}{4}} x^2\tan(x)\,dx\overset{IBP}{\triangleq}-\left(x^2\int\frac{d}{dx}ln\cos(x)dx\right)\Big|_0^{\frac{\pi}{4}}+$$

$$2\int_0^{\frac{\pi}{4}} xln\cos(x)dx\underbrace{\overset{Note\ section}{=}}_{(1)}\frac{\pi^2}{32}ln(2)-2\ln(2)\int_0^{\frac{\pi}{4}}xdx-2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\int_0^{\frac{\pi}{4}}x\cos(2nx)dx=$$

$$\overset{IBP}{\triangleq}\frac{\pi^2}{32}ln(2)-\frac{\pi^2\ln(2)}{16}$$

$$-2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\left[\left(\frac{x\sin(2nx)}{2n}\right)\Big|_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{\sin(2nx)}{2n}dx\right]=-\frac{\pi^2}{32}ln(2)-$$

$$2\sum_{n\in N}\frac{(-1\text{-})^n}{n}\left[\frac{\pi}{8n}\sin\left(\frac{\pi n}{2}\right)+\left(\frac{\cos(2nx)}{4n^2}\right)\Big|_0^{\frac{\pi}{4}}\right]=-\frac{\pi^2}{32}ln(2)-\frac{\pi}{4}\underbrace{\sum_{n\in N}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2}}_{n\to 2n+1}-$$

$$\underbrace{\frac{1}{2}\sum_{n\in N}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3}}_{n\to 2n}+\frac{1}{2}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}\underbrace{\overset{Note\ section}{=}}_{(2)}-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^2}+$$

$$\frac{1}{16}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}-\frac{1}{2}\sum_{n\in N}\frac{(-1)^{n+1}}{n^3}=-\frac{\pi^2}{32}ln(2)+\frac{\pi}{4}G-\frac{7}{16}\eta(3)$$

$$X+Y=\int_0^1\frac{x(ln\sqrt{1+x^2}+\arctan^2(x))}{1+x^2}dx=\frac{\pi}{4}G-\frac{21}{64}\zeta(3)+\frac{ln^2(2)}{8}-\frac{\pi^2\ln(2)}{32}$$

Note section :

1. $\ln(2\cos(x))=\sum_{n\in N}\frac{(-1)^{n+1}}{n}\cos(2nx),\quad |x|<\frac{\pi}{2}$
2. $\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^2}=G\ \ (Catalan's\ constant)$
3. $\eta(s)=(1-2^{1-s}).\ \zeta(3)$

2422. Find a closed form:

$$\Omega=\int_0^1\frac{\ln(1+x^2)(\arctan(x)+x)}{1+x^2}dx$$

Proposed by Shirvan Tahirov, Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\frac{\ln(1+x^2)\arctan(x)}{1+x^2}dx+\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx=\Omega_1+\Omega_2$$

$$\Omega_1 = \int_0^1 \frac{\ln(1+x^2)\tan^{-1}(x)}{1+x^2}dx, \left\{\tan^{-1}(x) = t, dt = \frac{dx}{1+x^2}, t\left[\frac{\pi}{4};0\right]\right\}$$

$$\Omega_1 = \int_0^{\frac{\pi}{4}} t\ln(1+\tan^2(t))\,dt = -2\int_0^{\frac{\pi}{4}} t\ln(\cos(t))\,dt$$

$$\left\{\textit{Fourier series of } \ln(\cos(z)) = -\ln(2) - \sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nz)}{n}\right\}$$

$$\Omega_1 = 2\ln^2(2)\int_0^{\frac{\pi}{4}} tdt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}} t\cos(2nt)\,dt = \frac{\pi^2\ln(2)}{16} +$$

$$+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{t\sin(2nt)}{2n}\Big]_0^{\frac{\pi}{4}} - \frac{1}{2n}\int_0^{\frac{\pi}{4}}\sin(2nt)\,dt\right] = \pi^2\frac{\ln(2)}{16} + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n\sin\left(\frac{\pi n}{2}\right)}{n^2} +$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[\frac{\cos(2nt)}{2n}\right]_0^{\frac{\pi}{4}} =$$

$$= \pi^2\frac{\ln(2)}{16} + \frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\left(\frac{\pi n}{2}\right)}{n^3} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} =$$

$$= \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G - \frac{3}{64}\zeta(3) + \frac{3}{8}\zeta(3) = \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G + \frac{21}{64}\zeta(3)$$

$$\Omega_2 = \int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx, \qquad \{x^2 = t, dt = 2xdx, \qquad t[1;0]\}$$

$$\Omega_2 = \int_0^1\frac{\ln(1+t)}{1+t}dt = \frac{1}{2}\left[\frac{\ln^2(1+t)}{2}\right]_0^1 = \frac{\ln^2(2)}{4}$$

$$\int_0^1\frac{\ln(1+x^2)(\tan^{-1}(x)+x)}{1+x^2}dx = \Omega_1+\Omega_2 = \frac{\pi^2\ln(2)}{16} - \frac{\pi}{4}G + \frac{21}{64}\zeta(3) + \frac{\ln^2(2)}{4}$$

2423. Find a closed form:

$$\Omega = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\Omega_{a>0} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{a^{\alpha n+\beta m}(m+1)^k} = -\sum_{n=1}^{\infty}\left(\frac{-1}{a^\alpha}\right)^n\sum_{m=1}^{\infty}\frac{1}{(m+1)^k}\left(\frac{1}{a^\beta}\right)^{m+1} =$$

$$\left(1-\sum_{n=0}^{\infty}\left(-\frac{1}{a^\alpha}\right)^n\right)a^\beta\left(-\frac{1}{a^\beta}+\sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\left(\frac{1}{a^\beta}\right)^{m+1}\right)$$

$$= \left(1-\frac{1}{1+\frac{1}{a^\alpha}}\right)\left(-1+a^\beta Li_k\left(\frac{1}{a^\beta}\right)\right) = \frac{1}{a^\alpha+1}\left(a^\beta Li_k\left(\frac{1}{a^\beta}\right)-1\right)$$

$$As: a = 2\,, \qquad \alpha = \beta = 1\,, \qquad k = 3.$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3} = \frac{1}{2+1}\left(2Li_3\left(\frac{1}{2}\right)-1\right) = \frac{1}{3}\left\{2\left(\frac{7}{8}\zeta(3)+\frac{ln^3(2)}{6}-\frac{\pi^2\ln(2)}{12}\right)-1\right\}$$

$$= \frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3} = -\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\sum_{n=1}^{\infty}\frac{1}{2^m(m+1)^3} = \frac{\frac{1}{2}}{1+\frac{1}{2}}\sum_{m=1}^{\infty}\frac{2}{2^{m+1}(m+1)^3} =$$

$$\frac{2}{3}\left(\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{2}\right) = \frac{2}{3}Li_3\left(\frac{1}{2}\right)-\frac{1}{3} = \frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Solution 3 by Pham Duc Nam-Vietnam

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}\Big(2\sum_{m=1}^{\infty}\frac{1}{2^{m+1}(m+1)^3}\Big) = \frac{1}{3}\left(2\sum_{m=2}^{\infty}\frac{1}{2^m m^3}\right) =$$

$$\frac{1}{3}\left(2\left(\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{2}\right)\right) = \frac{2}{3}\sum_{m=1}^{\infty}\frac{1}{2^m m^3}-\frac{1}{3}$$

$$*\sum_{m=1}^{\infty}\frac{1}{2^m m^3} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{2^m}\int_0^1 x^{m-1}ln^2(x)dx = \frac{1}{2}\int_0^1 ln^2(x)dx\left(\sum_{m=1}^{\infty}\frac{x^{m-1}}{2^m}\right)$$

$$= \frac{1}{2}\int_0^1\frac{ln^2(x)}{2-x}dx =$$

$$\frac{1}{2}\int_0^1\frac{ln^2(1-x)}{1+x}dx = \frac{1}{2}I$$

$$*\,I = \int_0^1\frac{ln^2(1-x)}{1+x}dx, \qquad J = \int_0^1\frac{ln^2(1+x)}{1+x}dx = \frac{1}{3}ln^2(2)$$

$$I-J = \int_0^1\frac{ln^2(1-x)-ln^2(1+x)}{1+x}dx = \int_0^1\frac{\ln\left(\frac{1-x}{1+x}\right)\ln(1-x^2)}{1+x}dx\,, x\to\frac{1-x}{1+x} =$$

$$\int_0^1\frac{\ln(x)\ln\left(\frac{4x}{(1+x)^2}\right)}{1+x}dx = \int_0^1\frac{\ln(x)(2\ln(2)+\ln(x)-2\ln(1+x))}{1+x}dx$$

$$= 2\ln(2)\underbrace{\int_0^1\frac{\ln(x)}{1+x}dx}_{-\frac{\pi^2}{12}}+$$

$$\underbrace{\int_0^1 \frac{ln^2(x)}{1+x}dx}_{\frac{3}{2}\zeta(3)} - 2\int_0^1 \frac{\ln(x)\ln(1+x)}{1+x}dx$$

$$= -\frac{\pi^2}{6}\ln(2) + \frac{3}{2}\zeta(3) + 2\sum_{n=0}^{\infty}(-1)^n H_n \int_0^1 x^n \ln(x)\,dx =$$

$$= -\frac{\pi^2}{6}\ln(2) + \frac{3}{2}\zeta(3) - 2\sum_{n=0}^{\infty}\frac{(-1)^n H_n}{(n+1)^2} = -\frac{\pi^2}{6}\ln(2) + \frac{3}{2}\zeta(3) + 2\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_n - \frac{1}{n}\right)}{n^2}$$

$$=$$

$$-\frac{\pi^2}{6}\ln(2) + \frac{3}{2}\zeta(3) + 2\left(-\frac{5}{8}\zeta(3) + \frac{3}{4}\zeta(3)\right) = -\frac{\pi^2}{6}\ln(2) + \frac{7}{4}\zeta(3) \Rightarrow$$

$$I = \frac{7}{4}\zeta(3) - \frac{\pi^2}{6}\ln(2) + \frac{ln^2(2)}{3} \Rightarrow$$

$$S = \frac{1}{3}\left(\frac{7}{4}\zeta(3) - \frac{\pi^2}{6}\ln(2) + \frac{ln^2(2)}{3}\right) - \frac{1}{3} = \frac{7}{12}\zeta(3) + \frac{ln^3(2)}{9} - \frac{\pi^2\ln(2)}{18} - \frac{1}{3}$$

Solution 4 by Ankush Kumar Parcha-India

$$\sum_{n\in N} x^n = \frac{x}{1-x}\ ,\quad |x|<1 \Rightarrow \sum_{n,m\in N}\frac{(-1)^{n+1}}{2^{n+m}(m+1)^3} \underset{m\to m-1}{=} \frac{2}{3}\left(-\frac{1}{2} + \sum_{n,m\in N}\frac{1}{2^m m^3}\right) =$$

$$-\frac{1}{3} + Li_3\left(\frac{1}{2}\right) \quad (1)$$

$$\left(Li_2(1-z) + Li_2\left(\frac{z-1}{z}\right) = -\frac{ln^2(z)}{2}\ ,\quad z\in C\backslash\{0\}\right)$$

Divide both sides by $-z(1-z)$ and integrate if with respect to z. We get.

$$-\int\frac{Li_2(1-z)}{z(1-z)}dz - \int Li_2\left(\frac{z-1}{z}\right)\frac{dz}{z(1-z)} = \frac{1}{2}\int\frac{ln^2(z)}{z(1-z)}dz \Rightarrow$$

$$\int d\,Li_3\left(\frac{z-1}{z}\right) - \underbrace{\int\frac{Li_2(1-z)}{z}dz}_{IBP} - \int\frac{Li_2(1-z)}{1-z}dz$$

$$= \frac{1}{2}\int\frac{ln^2(z)}{z}dz + \frac{1}{2}\int\frac{ln^2(z)}{1-z}dz\,Li_3\left(\frac{z-1}{z}\right) + \int d\,Li_3(1-z)$$

$$-Li_2(1-z)\ln(z) + \frac{1}{2}\underbrace{\int\frac{ln^2(z)}{1-z}}_{IBP}dz = \int\frac{dln^3(z)}{6} \Rightarrow$$

$$Li_3\left(\frac{z-1}{z}\right) + Li_3(1-z) - Li_2(1-z)\ln(z) - \frac{ln^2(z)\ln(1-z)}{2} + \underbrace{\int\frac{ln(z)\ln(1-z)}{z}dz}_{IBP} = \frac{ln^3(z)}{6}$$

$$\Rightarrow Li_3\left(\frac{z-1}{z}\right) + Li_3(1-z) - Li_2(1-z)\ln(z) - Li_2(z)\ln(z) + Li_3(z) \Rightarrow$$

$$\frac{ln^3(z)}{6} + \frac{ln^2(z)\ln(1-z)}{2} + C$$

$$\left(*\,Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - \ln(z)\ln(1-z)\,, z\in C\right) \Rightarrow$$

$$Li_3(z)+Li_3(1-z)+Li_3\left(\frac{z-1}{z}\right)=\frac{\pi^2}{6}\ln(2)+\frac{ln^3(z)}{6}-\frac{ln^2(z)\ln(1-z)}{2}+C\overset{set\ z=1}{\Longrightarrow}$$

$$Li_3(1)+0=-\underbrace{\lim_{x\to1}\frac{ln^2(z)\ln(1-z)}{2}}_{=0}+C\Rightarrow C=\zeta(3)$$

$$Li_3(z)+Li_3(1-z)+Li_3\left(1-\frac{1}{z}\right)=\zeta(3)+\frac{\pi^2}{6}\ln(2)+\frac{ln^3(z)}{6}-\frac{ln^2(z)\ln(1-z)}{2}\overset{at\ z=\frac{1}{2}}{\Longrightarrow}$$

$$Li_3\left(\frac{1}{2}\right)=\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln(2)+\frac{ln^3(2)}{6}$$

Put the value of $Li_3\left(\frac{1}{2}\right)$ *in equation* $-$ (1). *We get*:

$$\Rightarrow -\frac{1}{3}+\frac{2}{3}\left(\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}\ln(2)+\frac{ln^3(2)}{6}\right)\Rightarrow\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}}{2^{m+n}(m+1)^3}$$

$$=\frac{7}{12}\zeta(3)+\frac{ln^3(2)}{9}-\frac{\pi^2\ln(2)}{18}-\frac{1}{3}$$

Note $:\ \zeta(3)\to$ ***Apery's constant***

2424. Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy$$

Proposed by Lamiye Quliyeva, Abbaszade Yusif-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\frac{\ln\big((1+x)(1+y)\big)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\int_0^1\frac{\ln(1+x)\tan^{-1}(1+x)}{(1+x)(1+y)}dxdy=\ln(1+y)\,|_0^1\int_0^1\frac{\ln(1+x)tan^{-1}(1+x)}{1+x}dx$$

$$=$$

$$=\ln(2)\bullet I$$

$$I=\int_0^1\frac{\ln(1+x)\tan^{-1}(1+x)}{1+x}dx=\int_1^2\frac{\ln(x)\tan^{-1}(x)}{x}dx=$$

$$=\int_0^2\frac{\ln(x)\tan^{-1}(x)}{x}dx-\int_0^1\frac{\ln(x)\tan^{-1}(x)}{x}dx=I_1-I_2$$

$$I_1=\int_0^2\frac{\ln(x)\tan^{-1}(x)}{x}dx,\qquad using\ \dot{I}BP\ method$$

$$\left\{\begin{array}{c} u=\ln(x), \qquad du=\frac{dx}{x} \\ v=\int\frac{\tan^{-1}(x)}{x}dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int x^{2n}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2}\text{İBP}\end{array}\right\}$$

$$note:\left\{\sum_{n=0}^{\infty}a_{2n+1}=\frac{1}{2}\left(\sum_{n=1}^{\infty}a_n-\sum_{n=0}^{\infty}(-1)^n a_n\right)\right\}$$

$$v=\frac{1}{2i}\sum_{n=1}^{\infty}\frac{i^n x^n}{n^2}-\frac{1}{2i}\sum_{n=1}^{\infty}\frac{(-1)^n i^n x^n}{n^2}=\frac{i}{2}\left(Li_2(-ix)-Li_2(ix)\right)$$

$$I_1=[\frac{i\ln(x)}{2}(Li_2(-ix)-Li_2(ix)]_0^2-\frac{i}{2}\int_0^2\frac{Li_2(-ix)-Li_2(ix)}{x}dx=$$

$$=\frac{i\ln(2)}{2}\left(Li_2(-2i)-Li_2(2i)\right)-\frac{i}{2}[Li_3(-ix)-Li_3(ix)]_0^2=$$

$$=\frac{i}{2}\ln(2)\left(Li_2(-2i)-Li_2(2i)\right)-\frac{i}{2}\left(Li_3(-2i)-Li_3(2i)\right)$$

$$I_2=\int_0^1\frac{\ln(x)\tan^{-1}(x)}{x}dz=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}\ln(x)\,dx=$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=-\beta(3)=-\frac{\pi^3}{32}$$

$$\Omega_1=\ln(2)\,(I_1-I_2)=$$

$$=\frac{i\ln^2(2)}{2}\left(Li_2(-2i)-Li_2(2i)\right)-\frac{i\ln(2)}{2}\left(Li_3(-2i)-Li_3(2i)\right)+\frac{\pi^3}{32}\ln(2)$$

$$\Omega_2=\int_0^1\int_0^1\frac{\ln(1+y)\tan^{-1}(1+x)}{(1+y)(1+x)}dx=$$

$$=\left[\frac{\ln^2(1+y)}{2}\right][0;1]\int_0^1\frac{\tan^{-1}(1+x)}{1+x}dx=\frac{\ln^2(2)}{2}\int_1^2\frac{\tan^{-1}(x)}{x}dx=$$

$$=\frac{\ln^2(2)}{2}\left(\int_0^2\frac{\arctan(x)}{x}dx-\int_0^1\frac{\arctan(x)}{x}dx\right)=$$

$$=\frac{i\ln^2(2)}{4}\left(Li_2(-2i)-Li_2(2i)\right)-\frac{\ln^2(2)}{2}\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^2}=$$

$$=i\frac{\ln^2(2)}{4}\left(\mathrm{Li}_2(-2i)-\mathrm{Li}_2(2i)\right)-\frac{\ln^2(2)}{2}\mathrm{G}$$

$$\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\arctan(1+\mathrm{x})}{(1+\mathrm{x})(1+y)}dxdy=\Omega_1+\Omega_2=$$

$$=3i\frac{\ln^2(2)}{2}\left(Li_2(-2i)-Li_2(2i)\right)-\frac{i\ln(2)}{2}\left(Li_3(-2i)-Li_3(2i)\right)+\frac{\pi^3}{32}\ln(2)-\frac{\ln^2(2)}{2}\mathrm{G}$$

2425. Find a closed form:

$$\Omega=\int_0^1\int_1^\infty\frac{\sqrt{x}\ln(x)+y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dxdy+\int_0^1\int_1^\infty\frac{y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dxdy, substitution\left\{\frac{1}{x}=t,dt=-t^2dx,t[0;1]\right\}$$

$$\Omega_1=-\int_0^1\frac{\ln(x)}{\sqrt{x}(1+x)^2}dx=\sum_{n=1}^\infty n(-1)^n\int_0^1 x^{n-\frac{3}{2}}\ln(x)\,dx=-4\sum_{n=1}^\infty\frac{(-1)^n n}{(2n-1)^2}=$$

$$=-2\left(\sum_{n=1}^\infty\frac{(-1)^n}{(2n-1)}+\sum_{n=1}^\infty\frac{(-1)^n}{(2n-1)^2}\right)=-2\left(-G-\frac{\pi}{4}\right)=2G+\frac{\pi}{2}$$

$$\Omega_2=\int_0^1\int_1^\infty\frac{y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=-[\frac{1}{1+x}]_1^\infty\int_0^1 y\ln\left(\arccos^2(1-y)\right)dy=$$

$$=\int_0^1\ln(\arccos(y))\,dy-\int_0^1 y\ln\left(\arccos^2(y)\right)dy=I-J$$

$$I=\int_0^1\ln(\arccos(y))\,dy,\left\{\cos^{-1}(y)=t,dt=-\frac{dy}{\sqrt{1-y^2}}=-\frac{dy}{\sin(t)},t\left[0;\frac{\pi}{2}\right]\right\}$$

$$I=\int_0^{\frac{\pi}{2}}\sin(t)\ln(t)\,dt=[-\ln(\mathrm{t})\cos(\mathrm{t})]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\frac{\cos(t)-1+1}{t}dt=Ci\left(\frac{\pi}{2}\right)-\gamma$$

$$J=\int_0^1 y\ln(\arccos(y))\,dy=\int_0^{\frac{\pi}{2}}\sin(t)\cos(t)\ln(t)\,dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(2t)\ln(t)\,dt=$$

$$=\frac{1}{2}[-\frac{\ln(t)\cos(2t)}{2}]_0^{\frac{\pi}{2}}+\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{\cos(2t)-1}{t}dt+\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1}{t}dt=$$
$$\{2t\to t\}$$

$$=\frac{1}{4}\left(\ln\left(\frac{\pi}{2}\right)+\ln(0)\right)+\frac{1}{4}(Ci(\pi)-\gamma-\ln(\pi))+\frac{1}{4}\ln\left(\frac{\pi}{2}\right)-\frac{1}{4}\ln(0)=$$

$$=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)+\frac{1}{4}Ci(\pi)-\frac{\gamma}{4}-\frac{\ln(\pi)}{4}=\frac{1}{4}Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}\ln\left(\frac{4}{\pi}\right)-\frac{\gamma}{4}$$

$$\Omega_2=I-J=Ci\left(\frac{\pi}{2}\right)-\gamma-\frac{1}{4}Ci(\pi)+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)+\frac{\gamma}{4}=Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}Ci(\pi)-\frac{3\gamma}{4}+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)$$

$$\int_0^1\int_1^\infty \frac{\sqrt{x}\ln(x)+y\ln\left(\arccos^2(1-y)\right)}{(1+x)^2}dxdy=\Omega_1+\Omega_2=$$

$$=2G+\frac{\pi}{2}+Ci\left(\frac{\pi}{2}\right)-\frac{1}{4}Ci(\pi)-\frac{3\gamma}{4}+\frac{1}{4}\ln\left(\frac{4}{\pi}\right)$$

$$note\ \left\{cosine\ integral: Ci(x)=\gamma+\ln(x)+\int_0^x\frac{\cos(z)-1}{z}dz\right\}$$

Solution 2 by Exodo Halcalias-Angola

$$H=\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dx+\int_0^1 yln(\arccos(1-y))^2dy$$

$$H_1=\int_1^\infty\frac{\sqrt{x}\ln(x)}{(1+x)^2}dx\overset{x\to 1/x}{\triangleq}\int_0^1\frac{\ln(\frac{1}{x})}{\sqrt{x}(1+x)^2}dx\overset{IBP}{\triangleq}\int_0^1\frac{\arctan(\sqrt{x})}{x}dx+\int_0^1\frac{\sqrt{x}}{(1+x)x}dx=$$

$$2\int_0^1\frac{\arctan(x)}{x}dx+1\int_0^1\frac{1}{1+x^2}dx=2G+\frac{\pi}{2}$$

$$H_2=\int_0^1 yln(\arccos(1-y))^2dy=2\int_0^1 yln(\arccos(1-y))dy\overset{1-y\to y}{\triangleq}$$

$$2\left(\int_0^1\ln(\arccos(y))dy-\int_0^1 yln(\arccos(y))dy\right)$$

$$E_1=\int_0^1\ln(\arccos(y))dy\overset{\arccos(y)\to y}{\triangleq}\int_0^{\frac{\pi}{2}}\sin(y)\ln(y)\,dy\overset{IBP}{\triangleq}[\ln(x)(1-\cos(y))]_0^{\frac{\pi}{2}}+$$

$$\int_0^{\frac{\pi}{2}}\frac{\cos(y)-1}{y}dy=\ln\left(\frac{\pi}{2}\right)+\int_0^{\frac{\pi}{2}}\frac{\cos(y)-1}{y}dy,$$

$$Ci(z)-\gamma=\ln(z)+\int_0^z\frac{\cos(y)-1}{y}dy,\qquad E_1=Ci\left(\frac{\pi}{2}\right)-\gamma$$

$$E_2=\int_0^1 yln(\arccos(y))dy\overset{\arccos(y)\to y}{\triangleq}\int_0^{\frac{\pi}{2}}\ln(y)\sin(y)\cos(y)\,dy=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(y)\sin(2y)\,dy$$

$$\overset{2y\to y}{\triangleq}\frac{1}{4}\int_0^\pi\ln(y/2)\sin(y)\,dy=\frac{1}{4}\int_0^\pi\ln(y)\sin(y)\,dy-\frac{1}{4}\ln(2)\int_0^\pi\sin(y)\,dy\overset{IBP}{\triangleq}$$

$$\frac{1}{4}\left\{[\ln(y)(1-\cos(y))]_0^\pi+\int_0^\pi\frac{\cos(y)-1}{y}dy\right\}-\frac{2}{4}\ln(2)=$$

$$Recalling\ \ that:\ Ci(z)-\gamma=\ln(z)+\int_0^z\frac{\cos(y)-1}{y}dy$$

$$E_2=\frac{1}{4}(Ci(\pi)-\gamma+\ln(\pi)-\ln(4))=\frac{1}{4}(Ci(\pi)-\gamma+\ln(\frac{\pi}{4}))$$

$$H_2=2(E_1-E_2)=2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}-\frac{3}{2}\gamma-\frac{1}{2}\ln(\frac{\pi}{4})$$

$$H=H_1+H_2=2G+2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}-\frac{3}{2}\gamma+\frac{1}{2}\ln\left(\frac{4}{\pi}\right)+\frac{\pi}{2}$$

$$\int_0^1\int_1^\infty \frac{\sqrt{x}\ln(x)+y\ln(\arccos(1-y))^2}{(1+x)^2}dxdy$$
$$=2G+2Ci\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}+\frac{1}{2}\left(\pi-3_\gamma+\ln\left(\frac{4}{\pi}\right)\right)$$

2426. Find a closed form:

$$\Omega=\int_0^1 Li_2(-x^2)\arctan(x)\,dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by proposer

$$Li_2(-x^2)=-\int_0^1\frac{\ln(1+x^2y)}{y}dy \Rightarrow \Omega=-\int_0^1\int_0^1\frac{\arctan(x)\ln(1+x^2y)}{y}dxdy$$

$$\begin{cases} u=\ln(1+x^2y) & du=\dfrac{2xy}{1+x^2y}\\ dv=\arctan(x) & v=\arctan(x)-\dfrac{1}{2}\ln(1+x^2)\end{cases}$$

$$\Omega=-\int_0^1\frac{1}{y}\left[\ln(1+y)\left(\frac{\pi}{4}-\frac{1}{2}\ln(2)\right)-\int_0^1\frac{2x^2y\arctan(x)}{1+x^2y}dx+\int_0^1\frac{xy\ln(1+x^2)}{1+x^2y}dx\right]dy$$

$$\Omega=-\frac{\pi}{4}\int_0^1\frac{\ln(1+y)}{y}dy$$
$$+\frac{\ln(2)}{2}\int_0^1\frac{\ln(1+y)}{y}dy+2\int_0^1\int_0^1\frac{x^2\arctan(x)}{1+x^2y}dxdy-\int_0^1\int_0^1\frac{x\ln(1+x^2)}{1+x^2y}dxdy=$$

$$=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}+2\int_0^1\ln(1+x^2)\arctan(x)\,dx-\int_0^1\frac{\ln^2(1+x^2)}{x}dx$$
$$=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}-\frac{\zeta(3)}{8}+\Omega_1,$$

$$\Omega_1=2\int_0^1\ln(1+x^2)\arctan(x)\,dx$$
$$=2[\arctan(x)\,(x\ln(1+x^2)+2\arctan(x)-2x)]_0^1$$
$$-2\int_0^1\frac{x\ln(1+x^2)+2\arctan(x)-2x}{1+x^2}dx$$

$$\Omega_1=\frac{\pi}{2}\left(\ln(2)+\frac{\pi}{2}-2\right)-2\int_0^1\frac{x\ln(1+x^2)}{1+x^2}dx-4\int_0^1\frac{\arctan(x)}{1+x^2}dx+4\int_0^1\frac{x}{1+x^2}dx$$

$$\Omega_1=\frac{\pi\ln(2)}{2}+\frac{\pi^2}{4}-\pi-\frac{\ln^2(2)}{2}-\frac{\pi^2}{8}+2\ln(2)$$

$$\Omega=-\frac{\pi^3}{48}+\frac{\pi^2\ln(2)}{24}-\frac{\zeta(3)}{8}+\frac{\pi\ln(2)}{2}+\frac{\pi^2}{4}-\pi-\frac{\ln^2(2)}{2}-\frac{\pi^2}{8}+2\ln(2)$$

$$\Omega = -\frac{\pi^3}{48} + \frac{\pi^2 \ln(2)}{24} - \frac{\zeta(3)}{8} + \frac{\pi \ln(2)}{2} - \pi - \frac{\ln^2(2)}{2} + \frac{\pi^2}{8} + 2\ln(2)$$

2427. Prove that:

$$I = \int_0^{\frac{\pi}{2}} \log\sqrt{1+\sin(x)+\cos(x)}\,dx = G - \frac{\pi}{8}\log(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}}\log\sqrt{1+\sin(x)+\cos(x)}\,dx$$

$$=\int_0^{\frac{\pi}{2}}\log\sqrt{\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}}\log\sqrt{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^2+\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}}\log\sqrt{2\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)\cos\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}}\log\sqrt{2\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log\left(2\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)dx$$

$$=\frac{1}{2}\left\{\int_0^{\frac{\pi}{2}}\log(2\sqrt{2})\,dx+\int_0^{\frac{\pi}{2}}\log\left(\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+\int_0^{\frac{\pi}{2}}\log\left(\sin\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)dx+\int_0^{\frac{\pi}{2}}\log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+2\int_0^{\frac{\pi}{2}}\log\left(\cos\left(\frac{x}{2}\right)\right)dx\right\}$$

$$=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+4\int_0^{\frac{\pi}{4}}\log(\cos(x))\,dx\right\}=\frac{1}{2}\left\{\frac{3\pi}{4}\log(2)+4\left\{\frac{1}{2}G-\frac{\pi}{4}\log(2)\right\}\right\}$$

$$=G-\frac{\pi}{8}\log(2)\ \text{ Note: }\int_0^{\frac{\pi}{4}}\log(\cos(x))\,dx=\frac{1}{2}G-\frac{\pi}{4}\log(2)$$

2428. Prove that

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(x)}{\cos(y)}+\frac{\cos(x)}{\sin(y)}\right)dx\,dy=\frac{7}{8}\varsigma(3)+\frac{\pi^2}{4}\log(2)$$

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin(x)}{\cos(y)}+\frac{\cos(x)}{\sin(y)}\right)dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log\left(\frac{2\cos(x-y)}{\sin(2y)}\right)dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(2)\,dx\,dy+$$

$$+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(x-y))\,dx\,dy-\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(2y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+I_1-I_2$$

$$I_1=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(x-y))\,dx\,dy\Bigg|_{x-y=m}=\int_0^{\frac{\pi}{2}}\int_{-y}^{\frac{\pi}{2}-y}\log(\cos(m))\,dm\,dy\overset{IBP}{=}$$

$$\overset{IBP}{=}y\int_{-y}^{\frac{\pi}{2}-y}\log(\cos(m))\,dm\Bigg|_{y=0}^{y=\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}y\log(\cos(y))\,dy=$$

$$=\frac{\pi}{2}\int_{-\frac{\pi}{2}}^{0}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}y\log(\cos(y))\,dy=$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-y\right)\log(\sin(y))\,dy=$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\cos(m))\,dm+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy-\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dy+\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy=$$

$$=2\int_0^{\frac{\pi}{2}}y\log(\sin(y))\,dy=2\left\{-\frac{\pi^2}{8}\log(2)+\frac{7}{16}\varsigma(3)\right\}=-\frac{\pi^2}{4}\log(2)+\frac{7}{8}\varsigma(3)$$

$$I_2=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(2y))\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(2)\,dx\,dy+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dx\,dy+$$

$$+\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\cos(y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dx\,dy=\frac{\pi^2}{4}\log(2)+$$

$$+\pi\int_0^{\frac{\pi}{2}}\log(\sin(y))\,dy=\frac{\pi^2}{4}\log(2)+\pi\left(-\frac{\pi}{2}\log(2)\right)=-\frac{\pi^2}{4}\log(2)$$

$$I=\frac{\pi^2}{4}\log(2)+I_1-I_2=\frac{7}{8}\varsigma(3)+\frac{\pi^2}{4}\log(2)$$

$$\text{Note: } \int_0^{\frac{\pi}{2}} y\log(\sin(y))\,dy - \frac{\pi^2}{8}\log(2) + \frac{7}{16}\varsigma(3)$$

2429. Prove that

$$I = \int_0^{\infty} \frac{\log(1+x)}{x(1+x)^4}dx = \varsigma(2) - \frac{49}{36}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$I = \int_0^{\infty} \frac{\log(1+x)}{x(1+x)^4}dx\Bigg]_{1+x=y} = \int_1^{\infty} \frac{\log(y)}{(y-1)y^4}dy\Bigg]_{\frac{1}{y}=z} = -\int_0^1 \frac{z^3\log(z)}{1-z}dz$$

$$= -\sum_{k=0}^{\infty}\int_0^1 z^{k+3}\log(z)\,dz =$$

$$= -\sum_{k=0}^{\infty}\left\{-\frac{1}{(k+4)^2}\right\} = \sum_{k=0}^{\infty}\frac{1}{(k+4)^2} = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} - \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right) = \varsigma(2) - \frac{49}{36}$$

$$\text{Note: } \int_0^1 x^m\log^n(x)\,dx = (-1)^n\frac{n!}{(m+1)^{n+1}}$$

2430. Find a closed form:

$$\Omega = \int_0^1\int_1^{\infty}\frac{yx^2(\ln(x)+1)^2 ln^2(1+y^2)}{(x^2+1)^2(y^2+1)}dxdy$$

Proposed by Shirvan Tahirov, Abbaszade Yusif-Azerbaijan

Solution 1 by Djamel Arrouche-Algeria

$$\int_1^{\infty}\frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx\,.\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy = \int_0^1\frac{(1-\ln(x))^2}{(x^2+1)^2}dx.\int_0^1\frac{ln^2(1+t)}{1+t}.\frac{dt}{2} =$$

$$= \frac{1}{6}ln^3(2).\int_0^1\frac{(ln^2(x)-2\ln(x)+1)}{(x^2+1)^2}dx = \frac{1}{6}ln^3(2)\,.\,\Omega$$

$$\Omega = \int_0^1\frac{ln^2(x)}{(x^2+1)^2}dx - 2\int_0^1\frac{\ln(x)}{(x^2+1)^2}dx + \int_0^1\frac{dx}{(x^2+1)^2}$$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\frac{\pi}{4}} \frac{dy}{1+tan^2(y)} = \int_0^{\frac{\pi}{4}} cos^2(y)dy = \int_0^{\frac{\pi}{4}} \frac{1+\cos(2y)}{2}dy = \frac{\pi}{8}+\frac{1}{4}$$

$$\frac{1}{(x^2+1)^2} = \sum_{n\geq 1} n(-1)^{n-1}x^{2n-2}$$

$$-2\int_0^1 \frac{\ln(x)}{(x^2+1)^2}dx$$

$$= \sum_{n\geq 1} n(-1)^{n-1}\int_0^1 x^{2n-2}\ln(x)\,dx = -2\sum_{n\geq 1} n(-1)^{n-1}\left[-\frac{1}{(2n-1)^2}\right] =$$

$$\sum_{n\geq 1}\frac{n(-1)^{n-1}(2n-1+1)}{(2n-1)^2} = \sum_{n\geq 1}\frac{(-1)^{n-1}}{(2n-1)^2} + \sum_{n\geq 1}\frac{(-1)^{n-1}}{(2n-1)^2} = G + \frac{\pi}{4}$$

$$\int_0^1 \frac{ln^2(x)}{(x^2+1)^2}dx = \int_0^1 ln^2(x)\sum_{n\geq 1} n(-1)^{n-1}x^{2n-2}\,dx = \sum_{n\geq 1} n(-1)^{n-1}.\frac{1}{(2n-1)^3} =$$

$$\sum_{n\geq 1}\frac{(-1)^{n-1}}{(2n-1)^3} + \sum_{n\geq 1}\frac{(-1)^{n-1}}{(2n-1)^2} = G + \frac{\pi^3}{32}$$

$$\Omega = \frac{\pi}{8}+\frac{1}{4}+G+\frac{\pi}{4}+G+\frac{\pi^3}{32} = \frac{1}{32}(64G+\pi^3+12\pi+8)$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I = \underbrace{\int_1^\infty \frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx}_{A}.\underbrace{\int_0^1 \frac{yln^2(1+y^2)}{(y^2+1)}dy}_{B}$$

$$A = \int_1^\infty \frac{x^2(\ln(x)+1)^2}{(x^2+1)^2}dx \underset{dx=-\frac{dx}{x^2}}{\overset{x\to\frac{1}{x}}{=}} = \int_0^1 \frac{[-\ln(x)+1]^2}{x^2\left(\frac{x^2+1}{x^2}\right)^2}\frac{dx}{x^2} = \int_0^1 \frac{[-\ln(x)+1]^2}{(x^2+1)^2}dx =$$

$$[-\ln(x)+1]^2\left[\frac{\arctan(x)}{2}+\frac{x}{2(x^2+1)}\right]_0^1 + \int_0^1 \frac{dx}{x^2+1} + \int_0^1 \frac{\arctan(x)}{x}dx$$

$$-\int_0^1 \frac{\ln(x)}{x^2+1}dx - \int_0^1 \frac{\ln(x)\arctan(x)}{x}dx$$

$$A = \frac{\pi}{8}+\frac{1}{4}+\sum_{n=0}^\infty (-1)^n\int_0^1 x^{2n}dx + \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\int_0^1 x^{2n-1+1}dx$$

$$-\sum_{n=0}^\infty (-1)^n\int_0^1 x^{2n}\ln(x)dx -$$

$$\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\int_0^1 x^{2n+1-1}\ln(x)dx = \frac{\pi}{8}+\frac{1}{4}+\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}+\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}+\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=2G+\frac{\pi^3}{32}+\frac{3\pi}{8}+\frac{1}{4}$$

$$B=\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy\overset{IBP}{\cong}\frac{1}{2}ln^3(y^2+1)\Big|_0^1-2\underbrace{\int_0^1\frac{yln^2(1+y^2)}{(y^2+1)}dy}_{B}$$

$$B+2B=\frac{1}{2}ln^3(y^2+1)\Big|_0^1\ ,\ 3B=\frac{1}{2}ln^3(2)\quad B=\frac{1}{6}ln^3(2)\quad This\quad I=A\ .\ B$$

$$I=\frac{1}{6}ln^3(2)\left(2G+\frac{\pi^3}{32}+\frac{3\pi}{8}+\frac{1}{4}\right)=\frac{1}{6}ln^3(2)\ .\ \frac{1}{32}(64G+\pi^3+12\pi+8)=$$

$$\frac{1}{192}ln^3(2)(64G+\pi^3+12\pi+8)\quad Note: G\to Catalan's\ constant\ldots$$

2431. Find a closed form:

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dx\,dy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution 1 by Ankush Kumar Parcha-India

$$We\ know,\int_0^1\int_0^1 f(xy)dx\,dy=-\int_0^1\ln(t)f(t)\,dt$$

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dx\,dy=-\int_0^1\frac{\ln^2(x)\ln(1+2x)}{x}dx=$$

$$=^{IBP}\left[\ln^2(x)\int d(\mathrm{Li}_2(-2x))\right]_0^1-2\int_0^1\frac{\ln(x)Li_2(-2x)}{x}dx=^{IBP}-2\left[\ln(x)\int d(Li_3(-2x))\right]_0^1$$

$$+2\int_0^1\frac{Li_3(-2x)}{x}dx=2\int_0^1 d(Li_4(-2x)dx=2Li_4(-2)$$

$$Li_4(-z)+Li_4\left(-\frac{1}{z}\right)=-\frac{7\pi^4}{360}-\pi^2\ln^2(z)-\frac{\ln^4(z)}{24}$$

$$I=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dx\,dy$$

$$=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)-\frac{7\pi^4}{360}-\frac{\pi^2\ln^2(2)}{12}-\frac{\ln^4(2)}{24}$$

Solution 2 by Arowolo Isaiah-Nigeria

$$I=-\int_0^1\frac{\ln^2(x)\ln(1+2x)}{x}=^{IBP}-\left(\left[\frac{\ln^3(x)\ln(1+2x)}{3}\right]_0^1-\frac{2}{3}\int_0^1\frac{\ln^3(x)}{1+2x}\mathrm{dx}\right)$$

$$I=\frac{2}{3}\int_0^1\frac{\ln^3(x)}{x}dx=\frac{2}{3}(\sum_{n=0}^{\infty}(-2)^n\int_0^1 x^{3n}\ln^3(x)\,\mathrm{dx})$$

$$=-\frac{1}{3}(\sum_{n=1}^{\infty}(-2)^n\int_0^1 x^{n-1}\ln^3(x)\,dx)$$

$$I=-\frac{1}{3}\sum_{n=1}^{\infty}(-2)^n\left(\frac{d^3}{dn^3}\left(\frac{1}{n}\right)\right)=2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}=2Li_4(-2)=Li_4(-2)+Li_4(-2)$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)+\ln^2(2)\mathrm{Li}_2(-1)+2\mathrm{Li}_4(-1)-\frac{\ln^4(2)}{24}$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)+\ln^2(2)\left(-\frac{1}{2}\zeta(2)\right)+2\left(-\frac{7}{8}\zeta(4)\right)-\frac{\ln^4(2)}{24}$$

$$I=Li_4(-2)-Li_4\left(-\frac{1}{2}\right)-\frac{\pi^2\ln^2(2)}{12}-\frac{7\pi^4}{360}-\frac{\ln^4(2)}{24}$$

2432. Find a closed form:

$$\mathbf{I}=\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)^2(x+3)^3}dx$$

Proposed by Ankush Kumar Parcha-India

Solution by Alireza Askari-Iran

$$\mathrm{I}=\int_0^{\infty}\frac{\ln(x+1)}{(x+1)^2}dx-\frac{5}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+3)^2}dx+\frac{9}{4}\int_0^{\infty}\frac{x\ln(x+1)}{(x+1)(x+2)(x+3)}dx$$
$$+\frac{10}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)(x+3)}dx=A+B+C+D+W$$

$$W+D=\frac{9}{4}\int_0^{\infty}\frac{\ln(x+1)\,x}{(x+1)(x+2)(x+3)}dx$$
$$+\frac{10}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+1)(x+2)(x+3)}dx\underbrace{=}_{x+2\to x}-2\int_2^{\infty}\frac{\ln(x-1)}{x(x^2-1)}dx$$
$$+\frac{9}{4}\int_2^{\infty}\frac{\ln(x-1)}{x^2-1}dx=$$

$$\mathrm{x}\to\frac{1}{x}=-2\int_0^{\frac{1}{2}}\frac{x\ln(1-x)}{1-x^2}dx+\frac{9}{4}\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x^2}dx+\frac{9}{4}\int_0^{\frac{1}{2}}\frac{(x-1)\ln(x)}{1-x^2}dx$$
$$-\frac{1}{4}\int_0^{\frac{1}{2}}\frac{x\ln(x)}{1-x^2}dx=\Phi+\Upsilon+\Psi+\Lambda=W+D$$

$$\Phi=-2\int_0^{\frac{1}{2}}\frac{xln(1-x)}{1-x^2}dx\underbrace{=}_{IBP}-2\ln(2)\ln\left(\frac{3}{4}\right)+\int_0^{\frac{1}{2}}\frac{\ln(1-x^2)}{1-x}dx=-2\ln(2)\ln\left(\frac{3}{4}\right)$$

$$+\int_0^{\frac{1}{2}}\frac{\ln(1-x)}{1-x}dx+\int_0^{\frac{1}{2}}\frac{\ln(1+x)}{1-x}dx=-2\ln(2)\ln\left(\frac{3}{4}\right)-\frac{\ln(2)^2}{2}+P\to$$

$$P=\int_0^{1/2}\frac{ln(1+x)}{1-x}dx\underbrace{=}_{1-x\to x}\int_{1/2}^{1}\frac{\ln(2-x)}{x}dx=\ln(2)\int_{\frac{1}{2}}^{1}\frac{\ln\left(1-\frac{x}{2}\right)}{x}dx$$

$$=(\ln(2))^2-\sum_{n=1}^{\infty}\frac{1}{n2^n}\int_{\frac{1}{2}}^{1}x^{n-1}\,dx=\ln(2))^2-\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^2}+\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\right)^n}{n^2}=$$

$$=(\ln 2)^2-Li_2\left(\frac{1}{2}\right)+Li_2\left(\frac{1}{4}\right)\to\ \Phi=-\ln(2)\ln(3)+3\ln(2)^2+Li_2\left(\frac{1}{4}\right)-\frac{\pi^2}{12}\qquad note: Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{(ln2)^2}{2}$$

$$\Upsilon=\frac{9}{4}\int_0^{1/2}\frac{\ln(1-x)}{1-x^2}dx$$

$$=\frac{9}{8}\int_0^{1/2}\frac{\ln(1-x)}{1-x}dx+\frac{9}{8}\int_0^{1/2}\frac{\ln(1-x)}{1+x}dx=-\frac{9}{16}(ln2)^2+\frac{9\ln(2)}{32}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)^n}{n+1}$$

$$-\frac{9}{16}\sum_{n=0}^{\infty}\frac{(1-\left(\frac{1}{2}\right)^{n+1})}{2^n(n+1)^2}$$

$$\to\ \Upsilon=\frac{9}{8}\ln(2)\ln(3)+\frac{9}{4}(ln2)^2-\frac{9\pi^2}{96}+\frac{9}{8}Li_2(\frac{1}{4})$$

$$\Lambda=\frac{-1}{4}\int_0^{\frac{1}{2}}\frac{xln(x)}{1-x^2}dx$$

$$=\frac{-1}{4}\sum_{n=0}^{\infty}\int_0^{\frac{1}{2}}x^{2n+1}\ln(x)\,dx\underbrace{=}_{IBP}\frac{\ln(2)}{8}\sum_{n=0}^{\infty}\frac{1}{4^{n+1}(n+1)}$$

$$+\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{2n+2}\int_0^{\frac{1}{2}}x^{2n+1}dx=\frac{1}{16}\sum_{n=0}^{\infty}\frac{2ln2(n+1)+1}{4^{n+1}(n+1)^2}=$$

$$\Lambda=\frac{-\ln(2)\ln(3)}{8}+\frac{(\ln 2)^2}{4}+\frac{1}{16}Li_2(\frac{1}{4})$$

$$\Psi=\frac{9}{4}\int_0^{1/2}\frac{(x-1)\ln(x)}{1-x^2}dx$$

$$=\frac{-9}{4}\int_0^{1/2}\frac{\ln(x)}{x+1}dx\underbrace{=}_{IBP}\frac{9\ln(2)}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{n+1}$$

$$+\frac{9}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\int_0^{1/2}x^n dx=\frac{9\ln(2)}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{n+1}+\frac{9}{8}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{(n+1)^2}$$

$$\Psi=\frac{9\ln(2)\ln(3)}{4}-\frac{9(\ln 2)^2}{4}-\frac{9}{4}Li_2\left(-\frac{1}{2}\right)$$

$$=\frac{9\ln(2)\ln(3)}{4}-\frac{27(ln2)^2}{8}-\frac{9}{8}Li_2\left(\frac{1}{4}\right)+\frac{9\pi^2}{48}\qquad note: Li_2(z)$$

$$+Li_2(-z)=\frac{1}{2}Li_2(z^2)$$

$$\to W+D=\Phi+\Upsilon+\Psi+\Lambda=\frac{17}{8}(ln2)^2+\frac{17}{8}Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{96}$$

$$A=-\int_0^{\infty}\frac{\ln(x+1)}{(x+2)^2}dx\underbrace{=}_{IBP}\int_0^{\infty}\frac{dx}{(x+1)(x+2)}=\ln(\frac{x+1}{x+2}]_0^{\infty}=-\ln(2)$$

$$B=\frac{-5}{4}\int_0^{\infty}\frac{\ln(x+1)}{(x+3)^2}dx\underbrace{=}_{IBP}-\frac{5}{4}\int_0^{\infty}\frac{dx}{(x+1)(x+3)}=-\frac{5}{8}\ln(\frac{x+1}{x+3}]_0^{\infty}=\frac{5}{8}\ln(3)$$

$$C=\int_0^{\infty}\frac{\ln(x+1)}{(x+3)^3}dx\underbrace{=}_{IBP}-\frac{1}{4}\int_0^{\infty}\frac{1}{(x+1)(x+3)^2}dx$$

$$=\frac{-1}{16}\int_0^{\infty}(\frac{1}{x+1}-\frac{1}{x+3})dx+\frac{1}{8}\int_0^{\infty}\frac{1}{(x+3)^2}dx$$

$$=\frac{-1}{16}\ln(\frac{x+1}{x+3}]_0^{\infty}-\frac{1}{8(x+3)}]_0^{\infty}=-\frac{\ln(3)}{16}+\frac{1}{24}$$

ANSWER: I=A+B+C+D+W=$\frac{17}{16}Li_2\left(\frac{1}{4}\right)+\frac{\pi^2}{96}+\frac{17}{8}(\ln 2)^2-\frac{11}{16}\ln(3)-\ln(2)+\frac{1}{24}$

2433. Prove that

$$I=\int_0^1\int_0^1\frac{y\log(1+xy^2)}{1-xy^2}dx\,dy=\int_0^1\int_0^1\frac{x\log(2-x^2)}{1-x^2y}dx\,dy=\frac{a}{b}\varsigma(b)\log(\sqrt[b]{b})-\frac{1}{b}\varsigma(a)$$

For: $a,b\in Z^+$ and $GCD(a,b)=1$. Find a,b.

Proposed by Bui Hong Suc-Vietnam

Solution by Togrul Ehmedov-Azerbaijan

$$I=\int_0^1\int_0^1\frac{y\log(1+xy^2)}{1-xy^2}dx\,dy\Bigg]_{xy^2=m}=\frac{1}{2}\int_0^1\frac{1}{x}\int_0^x\frac{\log(1+m)}{1-m}dm\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\frac{1}{2}\left[\log(x)\int_0^x\frac{\log(1+m)}{1-m}dm\Bigg]_{x=0}^{x=1}-\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx\right]=$$

$$=-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx=-\frac{1}{2}\left\{\varsigma(3)-\frac{\pi^2}{4}\log(2)\right\}=-\frac{1}{2}\varsigma(3)+\frac{\pi^2}{8}\log(2)$$

$$=-\frac{1}{2}\varsigma(3)+\frac{3}{4}\varsigma(2)\log(2)=-\frac{1}{2}\varsigma(3)+\frac{3}{2}\varsigma(2)\log(\sqrt{2})\Rightarrow a=3;b=2$$

$$\text{Note: }\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx=\varsigma(3)-\frac{\pi^2}{4}\log(2)$$

$$I=\int_0^1\int_0^1\frac{x\log(2-x^2)}{1-x^2y}dx\,dy=-\int_0^1\frac{\log(1-x^2)\log(2-x^2)}{x}dx\Bigg]_{x^2\to x}$$

$$=-\frac{1}{2}\int_0^1\frac{\log(1-x)\log(2-x)}{x}dx\Bigg]_{x\to 1-x}=-\frac{1}{2}\int_0^1\frac{\log(x)\log(1+x)}{1-x}dx$$

$$=-\frac{1}{2}\left\{\varsigma(3)-\frac{\pi^2}{4}\log(2)\right\}=-\frac{1}{2}\varsigma(3)+\frac{\pi^2}{8}\log(2)$$

$$=-\frac{1}{2}\varsigma(3)+\frac{3}{4}\varsigma(2)\log(2)=-\frac{1}{2}\varsigma(3)+\frac{3}{2}\varsigma(2)\log(\sqrt{2})\Rightarrow a=3;b=2$$

2434. Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-xy)\,Li_4(1-x)}{x(1-x)(1-xy)}dx\,dy$$

Proposed by Bui Hong Suc-Vietnam

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\frac{\ln(1-xy)\,Li_4(1-x)}{x(1-x)(1-xy)}dx\,dy=-\frac{1}{2}\int_0^1\frac{\ln^2(1-x)\,Li_4(1-x)}{x^2(1-x)}dx$$

$$=-\frac{1}{2}\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x(1-x)^2}dx$$

$$=-\frac{1}{2}\left\{\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x}dx+\int_0^1\frac{\ln^2(x)\,Li_4(x)}{1-x}dx+\int_0^1\frac{\ln^2(x)\,Li_4(x)}{(1-x)^2}dx\right\}$$

$$\Omega_1=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^1 x^{k-1}\ln^2(x)\,dx=2\sum_{k=1}^{\infty}\frac{1}{k^7}=2\varsigma(7)$$

$$\Omega_2=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{1-x}dx=2\varsigma(3)\varsigma(4)+20\varsigma(2)\varsigma(5)-36\varsigma(7)$$

$$\Omega_3=\int_0^1\frac{\ln^2(x)\,Li_4(x)}{(1-x)^2}dx\overset{IBP}{=}-\int_0^1\frac{Li_3(x)\ln^2(x)}{x(1-x)}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{x(1-x)}dx=$$

$$=-\int_0^1\frac{Li_3(x)\ln^2(x)}{x}dx-\int_0^1\frac{Li_3(x)\ln^2(x)}{1-x}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{x}dx-2\int_0^1\frac{Li_4(x)\ln(x)}{1-x}dx$$

$$\Omega_{3a}=\int_0^1\frac{Li_3(x)\ln^2(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^3}\int_0^1 x^{k-1}\ln^2(x)\,dx=2\sum_{k=1}^{\infty}\frac{1}{k^6}=2\varsigma(6)$$

$$\Omega_{3b}=\int_0^1\frac{Li_3(x)\ln^2(x)}{1-x}dx=\varsigma^2(3)-\varsigma(6)$$

$$\Omega_{3c}=\int_0^1\frac{Li_4(x)\ln(x)}{x}dx=\sum_{k=1}^{\infty}\frac{1}{k^4}\int_0^1 x^{k-1}\ln(x)\,dx=-\sum_{k=1}^{\infty}\frac{1}{k^6}=-\varsigma(6)$$

$$\Omega_{3d}=\int_0^1\frac{Li_4(x)\ln(x)}{1-x}dx=\varsigma^2(3)-\frac{25}{12}\varsigma(6)$$

$$\Omega_3=\Omega_{3a}-\Omega_{3b}-2\Omega_{3c}-2\Omega_{3d}=\frac{31}{6}\varsigma(6)-3\varsigma^2(3)$$

$$\Omega=-\frac{1}{2}\{\Omega_1+\Omega_2+\Omega_3\}=17\varsigma(7)-\varsigma(3)\varsigma(4)-10\varsigma(2)\varsigma(5)-\frac{31}{12}\varsigma(6)+\frac{3}{2}\varsigma^2(3)$$

2435. Find a closed form:

$$I=\int_e^{e^2}\int_e^{e^2}\frac{\log_x(y^x)+\log_y(x^y)}{\sqrt{xy}}dxdy$$

Proposed by Lamiye Quliyeva-Azerbaijan

Solution by Abbaszade Yusif-Azerbaijan

$$I=\int_e^{e^2}\int_e^{e^2}\frac{\log_x(y^x)+\log_y(x^y)}{\sqrt{xy}}dxdy=\int_e^{e^2}\int_e^{e^2}\frac{\frac{x\ln(y)}{\ln(x)}+\frac{y\ln(x)}{\ln(y)}}{\sqrt{xy}}dxdy$$

$$I=\int_e^{e^2}\int_e^{e^2}\left(\frac{\sqrt{x}\ln(y)}{\sqrt{y}\ln(x)}+\frac{\sqrt{y}\ln(x)}{\sqrt{x}\ln(y)}\right)dxdy=_{symmetry}^{by}2\int_e^{e^2}\int_e^{e^2}\frac{\sqrt{x}\ln(y)}{\sqrt{y}\ln(x)}dxdy$$

$$=2\int_e^{e^2}\frac{\sqrt{x}}{\ln(x)}\int_e^{e^2}\frac{\ln(y)}{\sqrt{y}}dxdy=2\,I_1\times I_2$$

$$I_1=\int_e^{e^2}\frac{\sqrt{x}}{\ln(x)}dx=_{\ln(x)=t}^{t[1;2]}=\int_1^2\frac{e^{\frac{3t}{2}}}{t}dt=Ei\left[\frac{3x}{2}\right]_1^2=Ei(3)-Ei\left(\frac{3}{2}\right)$$

$$I_2=\int_e^{e^2}\frac{\ln(y)}{\sqrt{y}}dy=^{IBP}\left[2\sqrt{y}\ln(y)\right]_e^{e^2}-\int_e^{e^2}\frac{2}{\sqrt{y}}dy=\left[2\sqrt{y}\ln(y)-4\sqrt{y}\right]_e^{e^2}=2\sqrt{e}$$

$$I=2I_1\times I_2=4\sqrt{e}\left(Ei(3)-Ei\left(\frac{3}{2}\right)\right)$$

Note section :

$$\int\left(\frac{e^{ax}}{x}\right)dx=Ei(ax)+c$$

2436. Prove that

$$I=\int_1^{\infty}\frac{x\sqrt{x}\log(x)}{x^3+x\sqrt{x}+1}dx=\frac{4}{81}\left(\varphi_1\left(\frac{1}{9}\right)-\varphi_1\left(\frac{4}{9}\right)\right)$$

where $\varphi_1(s)$ denotes trigamma function

Proposed by Vasile Mircea Popa-Romania

Solution by Togrul Ehmedov-Azerbaijan

$$\text{Let } x^3=m$$

$$I=\frac{4}{9}\int_1^{\infty}\frac{m^{\frac{2}{3}}\log(m)}{m^2+m+1}dm=\frac{4}{9}\int_1^{\infty}\frac{m^{\frac{2}{3}}(1-m)\log(m)}{1-m^3}dm$$

$$\text{Let } m=1/z$$

$$I=-\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}(z-1)\log(z)}{z^3-1}dz=-\frac{4}{9}\int_0^1\frac{z^{\frac{1}{3}}\log(z)}{z^3-1}dz+\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}\log(z)}{z^3-1}dz$$

$$=\frac{4}{9}\int_0^1\frac{z^{\frac{1}{3}}\log(z)}{1-z^3}dz-\frac{4}{9}\int_0^1\frac{z^{-\frac{2}{3}}\log(z)}{1-z^3}dz$$

$$=\frac{4}{9}\sum_{k=0}^{\infty}\int_0^1 z^{3k+\frac{1}{3}}\log(z)\,dz-\frac{4}{9}\sum_{k=0}^{\infty}\int_0^1 z^{3k-\frac{2}{3}}\log(z)\,dz$$

$$=-\frac{4}{9}\sum_{k=0}^{\infty}\frac{1}{\left(3k+\frac{4}{3}\right)^2}+\frac{4}{9}\sum_{k=0}^{\infty}\frac{1}{\left(3k+\frac{1}{3}\right)^2}$$

$$=-\frac{4}{81}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{4}{9}\right)^2}+\frac{4}{81}\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{1}{9}\right)^2}=\frac{4}{81}\left(\varphi_1\left(\frac{1}{9}\right)-\varphi_1\left(\frac{4}{9}\right)\right)$$

2437. Find a closed form:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-(xyz)^2}dxdydz$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution 1 by Amin Hajiyev-Azerbaijan

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-x^2y^2z^2}dxdydz,\left\{\int_0^1\int_0^1 f(xy)dxdy=-\int_0^1\ln(x)f(x)dx\right\}$$

$$\Omega=-\int_0^1\int_0^1\frac{x^2z\ln(xz)ln(x)}{1-x^2z^2}dxdz$$

$$=-\sum_{n=0}^{\infty}\int_0^1\int_0^1 x^{2n+2}z^{2n+1}\left(\ln^2(x)+\ln(x)\ln(z)\right)dxdz=$$

$$=-\sum_{n=0}^{\infty}\int_0^1 z^{2n+1}dz\int_0^1 x^{2n+2}\ln^2(x)\,dx-\sum_{n=0}^{\infty}\int_0^1 z^{2n+1}\ln(z)dz\int_0^1 x^{2n+2}\ln(x)\,dx=$$

$$=-2\sum_{n=0}^{\infty}\frac{1}{(2n+2)(2n+3)^3}-\sum_{n=0}^{\infty}\frac{1}{(2n+2)^2(2n+3)^2}=$$

$$=\sum_{n=0}^{\infty}\frac{1}{(2n+3)^2}+2\sum_{n=0}^{\infty}\frac{1}{(2n+3)^3}-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}$$

$$we\ have.\left\{\sum_{n=1}^{\infty}a_n=\sum_{n=0}^{\infty}a_{2n+1}+\sum_{n=1}^{\infty}a_{2n}\right\}$$

$$\Omega=\sum_{n=1}^{\infty}\frac{1}{(n+2)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n+2)^2}+2\sum_{n=1}^{\infty}\frac{1}{(n+2)^3}-2\sum_{n=1}^{\infty}\frac{1}{(2n+2)^3}-\frac{\zeta(2)}{4}=$$

$$=\frac{\pi^2}{6}-\frac{5}{4}-\frac{\pi^2}{24}+\frac{1}{4}+2\zeta(3)-\frac{9}{4}-\frac{1}{4}\zeta(3)+\frac{1}{4}-\frac{\pi^2}{24}=$$

$$=\frac{\pi^2}{12}+\frac{7\zeta(3)}{4}-3$$

$$\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\ln(xyz)}{1-x^2y^2z^2}dxdydz=\frac{\pi^2}{12}+\frac{7\zeta(3)}{4}-3$$

Solution 2 by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{x^2y^2z\log(xyz)}{1-x^2y^2z^2}dx\,dy\,dz=\sum_{k=0}^{\infty}\int_0^1\int_0^1\int_0^1 xy(xyz)^{2k+1}\log(xyz)\,dx\,dy\,dz$$

$$\text{Let } xyz=m$$

$$\Omega=\sum_{k=0}^{\infty}\int_0^1\int_0^1\int_0^{xy} m^{2k+1}\log(m)\,dm\,dy\,dx\overset{IBP}{=}$$

$$\overset{IBP}{=}\sum_{k=0}^{\infty}\int_0^1\left\{\left[y\int_0^{xy} m^{2k+1}\log(m)\,dm\right]_{y=0}^{y=1}-\int_0^1(xy)^{2k+2}\log(xy)\,dy\right\}dx=$$

$$=\sum_{k=0}^{\infty}\int_0^1\int_0^x m^{2k+1}\log(m)\,dm\,dx-\sum_{k=0}^{\infty}\int_0^1\int_0^1(xy)^{2k+2}\log(xy)\,dy\,dx=$$

$$=\sum_{k=0}^{\infty}\left\{\left[x\int_0^x m^{2k+1}\log(m)\,dm\right]_{x=0}^{x=1}-\int_0^1 x^{2k+2}\log(x)\,dx\right\}$$

$$-\sum_{k=0}^{\infty}\int_0^1\frac{1}{x}\int_0^x p^{2k+2}\log(p)\,dp\,dx=$$

$$=\sum_{k=0}^{\infty}\int_0^1 m^{2k+1}\log(m)\,dm-\sum_{k=0}^{\infty}\int_0^1 x^{2k+2}\log(x)\,dx+\sum_{k=0}^{\infty}\int_0^1 x^{2k+2}\log^2(x)\,dx=$$

$$=-\sum_{k=0}^{\infty}\frac{1}{(2k+2)^2}+\sum_{k=0}^{\infty}\frac{1}{(2k+3)^2}+2\sum_{k=0}^{\infty}\frac{1}{(2k+3)^3}=$$

$$=-\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}+\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}-1\right)+2\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}-1\right)=$$

$$=-\frac{\pi^2}{24}+\left(\frac{\pi^2}{8}-1\right)+2\left(\frac{7}{8}\varsigma(3)-1\right)=\frac{\pi^2}{12}+\frac{7}{4}\varsigma(3)-3=\frac{1}{2}\varsigma(2)+\frac{7}{4}\varsigma(3)-3$$

2438. Find a closed form:

$$\int_0^1 \left(x\ln\left(\cos^{-1}(1-x^2)\right)+\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Djamel Arrouche-Algeria

$$\int_0^1 x\ln\left(\cos^{-1}(1-x^2)\right)dx=\Omega_1\quad 1-x^2=y\,;\ \Omega=\frac{1}{2}\int_0^1\ln\left(\cos^{-1}(y)\right)dy\,;\ y=\cos(s)$$

$$\Omega=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(s)\sin(s)\,ds\overset{IBP}{=}u=\ln(s),v'=\sin(s)\,;u'=\frac{1}{s}\quad v(s)=1-\cos(s)$$

$$\Omega=\frac{1}{2}\left[(1-\cos(s)\ln(s)\right]_0^{\frac{\pi}{2}}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos(s)}{s}ds=\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos(s)}{s}ds$$

$$Ci(z)=\ln(z)+\gamma+\int_0^z\frac{\cos(x)-1}{x}dx$$

$$\Omega_1=\frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right)-\gamma\right]$$

$$\Omega_2=\int_0^1\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2dx\,;\ \frac{x}{1+x}=y\to x=\frac{y}{1-y}\Rightarrow dx=\frac{dy}{(1-y)^2}$$

$$A=\int_0^{\frac{1}{2}}(\ln(y)+1)^2.\frac{dy}{(1-y)^2}=\int_0^{\frac{1}{2}}\frac{\ln^2(y)}{(1-y)^2}+2\int_0^{\frac{1}{2}}\frac{\ln(y)}{(1-y)^2}+\frac{dy}{(1-y)^2}$$

$$\lim_{x\to 0}\left[\frac{\ln^2(y)}{1-y}\right]_x^{\frac{1}{2}}-\int_x^{\frac{1}{2}}\frac{2\ln(y)}{y(1-y)}dy+2\left[\frac{\ln(y)}{1-y}dy\right]_x^{\frac{1}{2}}-\int_x^{\frac{1}{2}}\frac{2}{y(1-y)}dy+1=2\ln^2(2)+$$

$$\lim_{x\to 0}-\frac{\ln^2(x)}{1-x}-\ln^2\left(\frac{1}{2}\right)+\ln^2(x)-4\ln(2)-\frac{2\ln(x)}{1-x}-2\ln\left(\frac{1}{2}\right)+2\ln\left(\frac{1}{2}\right)+2\ln(x)+$$

$$2\ln(1-x)-2\int_0^{\frac{1}{2}}\frac{\ln(y)}{1-y}dy=\ln^2(2)-4\ln(2)+1+2\int_0^{\frac{1}{2}}\frac{\ln(1-(1-y))}{1-y}d(1-y)$$

$$+$$

$$\lim_{x\to}\left[-\frac{\ln^2(x)}{1-x}+\ln^2(x)+2\ln(x)-\frac{2\ln(x)}{1-x}+2kn(1-x)\right]_{x=0}\ ,\ ''\lim_{x\to 0}x\ln^n(x)=0''$$

$$=\ln^2(2)-4\ln(2)+1-2\left[Li_2\left(\frac{1}{2}\right)-Li_2(1)\right]\quad Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\ln^2(2)}{2},$$

$$Li_2(1)=\frac{\pi^2}{6}$$

$$\Omega_2=2\ln^2(2)-4\ln(2)+1+\frac{\pi^2}{6}$$

$$\Omega = \Omega_1 + \Omega_2 = \frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right) - \gamma\right] + 2ln^2(2) - 4\ln(2) + 1 + \frac{\pi^2}{6}$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1 (xln\left(cos^{-1}(1-x^2)\right) + (\ln\left(\frac{x}{x+1}\right) + 1)^2)dx = \underbrace{\int_0^1 xln\left(cos^{-1}(1-x^2)\right)dx}_{A} + \underbrace{+\int_0^1 (\ln\left(\frac{x}{x+1}\right)+1)^2 dx}_{B}$$

$$A = \int_0^1 xln\left(cos^{-1}(1-x^2)\right)dx \underbrace{\overset{x^2=p}{=}}_{dx=\frac{dp}{2x}} \int_0^1 xln\left(cos^{-1}(1-p)\right)\frac{dp}{2x}$$

$$= \frac{1}{2}\int_0^1 ln\left(cos^{-1}(1-p)\right)dp =$$

$$\frac{1}{2}Ci\left(cos^{-1}(1-p)\right) + \frac{1}{2}(p-1)\ln\left(cos^{-1}(1-x^2)\right)\Big|_0^1 =$$

$$= \frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right) - \gamma\right]$$

$$B = \int_0^1 (\ln\left(\frac{x}{x+1}\right)+1)^2 dx = \int_0^1 ln^2\left(\frac{x}{x+1}\right)dx + 2\int_0^1 \ln\left(\frac{x}{x+1}\right)dx + \int_0^1 dx =$$

$$\int_0^1 ln^2(x)dx - 2\int_0^1 \ln(x+1)\ln(x)\,dx + \int_0^1 ln^2(x+1)dx$$

$$+2\int_0^1 \ln(x)\,dx - 2\int_0^1 \ln(x+1)dx$$

$$+\int_0^1 dx = 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^n \ln(x)\,dx + \int_1^2 ln^2(x)dx - 2$$

$$+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^n dx + 1$$

$$B = 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{d}{dn}\left(\frac{x^{n+1}}{n+1}\Big|_0^1\right) + 2(\ln(2)-1)^2 - 2 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{x^{n+1}}{n+1}\Big|_0^1\right) + 1$$

$$B = 1 - 4 + \frac{\pi^2}{6} + 4\ln(2) + 2ln^2(2) - 4\ln(2) + 2 + 2 - 4\ln(2)$$

$$B = 1 + \frac{\pi^2}{6} - 4\ln(2) + 2ln^2(2) \quad This: \quad I = A + B$$

$$\int_0^1 \left(x\ln\left(\cos^{-1}(1-x^2)\right)+\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx$$

$$=\frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right)-\gamma\right]+2\ln^2(2)-4\ln(2)+1+\frac{\pi^2}{6}$$

$Note:\ ''Ci(z)''\ Cosine\ integral\ \dots\ Ci(z)=\ln(z)+\gamma+\int_0^z \frac{\cos(x)-1}{x}dx$

Solution 3 by Exodo Halcalias-Angola

$$\int_0^1 \left(x\ln\left(\cos^{-1}(1-x^2)\right)+\left(\ln\left(\frac{x}{x+1}\right)+1\right)^2\right)dx$$

$$H_1=\int_0^1 x\ln\left(\cos^{-1}(1-x^2)\right)dx \overset{1-x^2\to x}{\cong} \frac{1}{2}\int_0^1 \ln\left(\cos^{-1}(1-x^2)\right)dx \overset{\cos^{-1}(x)\to x}{\cong} \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(x)\sin(x)\,dx$$

$$\overset{IBP}{\cong}\frac{1}{2}\left[(\ln(x)(1-\cos(x)))\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\frac{\cos(x)-1}{x}dx\right]=\frac{1}{2}\left[\ln\left(\frac{\pi}{2}\right)+\int_0^{\frac{\pi}{2}}\frac{\cos(x)-1}{x}dx\right]=$$

$$\frac{1}{2}\left(\mathrm{Ci}\left(\frac{\pi}{2}\right)+\gamma\right)\qquad H_1=\frac{1}{2}\left(\mathrm{Ci}\left(\frac{\pi}{2}\right)+\gamma\right)$$

$$H_2=\int_0^1\left(\ln\left(\frac{x}{1+x}\right)+1\right)^2dx=\int_0^1 \ln^2\left(\frac{x}{1+x}\right)dx+2\int_0^1\ln\left(\frac{x}{x+1}\right)dx+\int_0^1 dx$$

$$A=\int_0^1 \ln^2\left(\frac{x}{1+x}\right)dx=\int_0^1 \ln^2(x)dx+\int_0^1 \ln^2(1+x)dx-2\int_0^1\ln(x)\ln(x+1)\,dx=$$

$$A_1=\int_0^1 \ln^2(x)dx+\int_0^1 \ln^2(1+x)dx=\int_0^1 \ln^2(x)dx+\int_1^2 \ln^2(x)dx$$

$$\int \ln^2(z)dz=z(\ln^2(z)-2\ln(z)-2)\qquad A_1=2\ln^2(2)-4\ln(2)+4$$

$$A_2=\int_0^1\ln(x)\ln(1+x)\,dx\overset{IBP}{\cong}[x\ln(1+x)(\ln(x)-1)]_0^1-\int_0^1\frac{x(\ln(x)-1)}{1+x}dx$$

$$A_2=-\ln(2)+\int_0^1\frac{x}{1+x}dx-\int_0^1\frac{x\ln(x)}{1+x}dx=-\ln(2)+\int_0^1\frac{1+x-1}{1+x}dx-$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\int_0^1 x^k\ln(x)\,dx=-\ln(2)+1-\ln(2)+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+1)^2}=-2\ln(2)+2-\frac{\pi^2}{12}$$

$$A=A_1-2A_2=\frac{\pi^2}{6}+2\ln^2(2)$$

$$B=\int_0^1\ln\left(\frac{x}{1+x}\right)dx=\int_0^1\ln(x)\,dx-\int_0^1\ln(1+x)\,dx=\int_0^1\ln(x)\,dx-\int_1^2\ln(x)\,dx=$$
$$-2\ln(2)$$

$$H_2=A+2B+1=\frac{\pi^2}{6}+2\ln^2(2)-4\ln(2)+1$$

$$H=H_1+H_2=\frac{1}{2}\left[Ci\left(\frac{\pi}{2}\right)-\gamma\right]+2\ln^2(2)-4\ln(2)+1+\frac{\pi^2}{6}$$

2439. Find a closed form:

$$\Omega=\int_0^1\frac{\ln(1+x^2)(\tan^{-1}(x)+x)}{1+x^2}dx$$

Proposed by Abbaszade Yusif, Shirvan Tahirov-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1\frac{\ln(1+x^2)\tan^{-1}x}{1+x^2}dx+\int_0^1\frac{\ln(1+x^2)\,x}{1+x^2}dx=A+B$$

$$A=\int_0^1\frac{\ln(1+x^2)\tan^{-1}x}{1+x^2}dx$$

$$\underset{\tan^{-1}(x)\to x}{\underbrace{=}}-2\int_0^{\frac{\pi}{4}}x\ln\cos x\,dx=2\int_0^{\frac{\pi}{4}}x\ln 2dx+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x\cos(2nx)dx$$

$$=\frac{\pi^2}{16}\ln 2+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\frac{\pi}{4}}x\cos(2nx)dx=\{\text{IBP METHOD}\}$$

$$\frac{\pi^2}{16}\ln 2+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\frac{\pi}{4}}\sin(2nx)\,dx=$$

$$\frac{\pi^2}{16}\ln 2+\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}\cos\frac{n\pi}{2}-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n^3}$$

$$A=\frac{\pi^2}{16}\ln 2+\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+1)^2}-\frac{7}{16}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n)^3}=\frac{\pi^2}{16}\ln 2-\frac{\pi}{4}C+\frac{21}{64}\zeta(3)$$

$$B=\int_0^1\frac{\ln(1+x^2)\,x}{1+x^2}dx=\frac{(\ln(1+x^2))^2}{4}\Big]_0^1=\frac{(\ln 2)^2}{4}$$

$$\text{ANSWER}=\text{A}+\text{B}=\frac{\pi^2}{16}\ln 2-\frac{\pi}{4}\text{C}+\frac{21}{64}\zeta(3)+\frac{(\ln 2)^2}{4}$$

2440. Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dxdy$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1\int_0^1\frac{\ln(xy)\ln(1+2xy)}{xy}dxdy=$$

$$\underbrace{=}_{xy\to t}\int_0^1\int_0^y\frac{\ln(t)\ln(1+2t)}{ty}dtdy=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n}\int_0^1\int_0^y\frac{1}{y}\ln(t)\,t^{n-1}dtdy=\{IBP\ method\}$$

$$=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\ln(y)\,y^{n-1}dy+\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\int_0^y\frac{t^{n-1}}{y}dtdy=A+B$$

$$A=-\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\ln(y)\,y^{n-1}dy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}\quad\left\{note\int_0^1 x^m\left(lnx\right)^n dx=\frac{(-1)^n n!}{(m+1)^{n+1}}\right\}$$

$$B=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}\int_0^1\int_0^y\frac{t^{n-1}}{y}dtdy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^3}\int_0^1 y^{n-1}dy=\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}$$

$$I=A+B=2\sum_{n=1}^{\infty}\frac{(-2)^n}{n^4}=2Li_4(-2)\quad\left\{note\ \ Li_4(-2)=-Li_4\left(\frac{1}{2}\right)-\frac{7\pi^4}{360}-\frac{\pi^2(ln2)^2}{12}-\frac{(ln2)^4}{24}\right\}$$

$$\text{I}=Li_4(-2)-Li_4\left(\frac{-1}{2}\right)-\frac{\pi^2(ln2)^2}{12}-\frac{7\pi^4}{360}-\frac{(ln2)^4}{24}$$

2441. Find a closed form:

$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy$$

Proposed by Abbaszade Yusif, Lamiye Quliyeva-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1\int_0^1\frac{\ln(1+x+y+xy)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy=$$

$$=\int_0^1\int_0^1\frac{\ln(1+y)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy+\int_0^1\int_0^1\frac{\ln(1+x)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy=\text{A}+\text{B}$$

$$\text{B}=\int_0^1\frac{dy}{1+y}\int_0^1\frac{\ln(1+x)\tan^{-1}(x+1)}{x+1}dx\underbrace{=}_{x+1\to x}\ln(2)\int_1^2\frac{\ln(x)\tan^{-1}(x)}{x}dx\quad\{\text{IBP}\}$$

$$=\frac{(ln2)^3\tan^{-1}(2)}{2}-\frac{ln2}{2}\int_1^2\frac{(lnx)^2}{1+x^2}dx=\frac{(ln2)^3\tan^{-1}(2)}{2}-\frac{ln2}{2}C$$

$$C=\int_1^2\frac{(lnx)^2}{1+x^2}dx\underbrace{=}_{x\to 1/x}\frac{1}{2}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{1+ix}dx+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{1-ix}dx$$

$$=\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{2}\int_{\frac{1}{2}}^1(lnx)^2x^n\,dx\underbrace{=}_{IBP}$$

$$-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}(\frac{i^n+(-i)^n}{n+1})(2^{-n})-\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{n+1}\int_{\frac{1}{2}}^1x^n\ln(x)\,dx\quad\{\text{IBP}\}$$

$$=-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}(\frac{i^n+(-i)^n}{n+1})(2^{-n})-\frac{ln2}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}(2^{-n})+\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}\int_{\frac{1}{2}}^1x^ndx$$

$$=$$

$$C=-\frac{(ln2)^2}{4}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{n+1}(2^{-n})-\frac{ln2}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^2}(2^{-n})-\frac{1}{2}\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^3}(2^{-n})+\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{(n+1)^3}$$

$$B=\frac{(ln2)^3\tan^{-1}(2)}{2}+\frac{(ln2)^3}{8}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n}+\frac{(ln2)^2}{4}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n^2}+\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{\left(\frac{i}{2}\right)^{n-1}+\left(-\frac{i}{2}\right)^{n-1}}{n^3}-\frac{ln2}{2}\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{n^3}$$

$$B=\frac{(ln2)^3\tan^{-1}(2)}{2}+\frac{(ln2)^3}{2}\tan^{-1}\left(\frac{1}{2}\right)+\frac{(ln2)^2}{2i}\left(Li_2\left(\frac{i}{2}\right)-Li_2\left(\frac{-i}{2}\right)\right)-\frac{ln2}{2i}(Li_3(i)-Li_3(-i)+Li_3\left(\frac{-i}{2}\right)-Li_3\left(\frac{i}{2}\right))$$

$$\int_0^1\int_0^1\frac{\ln(1+y)\tan^{-1}(x+1)}{(x+1)(y+1)}dxdy\underbrace{=}_{1+y\to y}\int_1^2\frac{\ln(y)}{y}dy\int_1^2\frac{\tan^{-1}(x)}{x}dx$$

$$=\frac{ln2)^2}{2}\int_1^2\frac{\tan^{-1}(x)}{x}dx$$

$$\frac{(ln2)^2}{2}\int_1^2\frac{\tan^{-1}(x)}{x}dx \underbrace{=}_{IBP} \frac{(ln2)^3}{2}\tan^{-1}(2)-\frac{(ln2)^2}{2}\int_1^2\frac{\ln(x)}{x^2+1}dx \underbrace{=}_{x\to\frac{1}{x}}$$

$$\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^2}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{x^2+1}\mathrm{dx}=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^2}{2}W$$

$$W=\int_{\frac{1}{2}}^1\frac{\ln(x)}{x^2+1}\mathrm{dx}=\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{1+ix}dx+\frac{1}{2}\int_{\frac{1}{2}}^1\frac{\ln(x)}{1-ix}dx)=\sum_{n=0}^{\infty}\frac{(i^n+(-i)^n)}{2}\int_{1/2}^1 x^n\ln(x)\,dx$$

$$\{IBP\}\quad=\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\sum_{n=0}^{\infty}\frac{i^n+(-i)^n}{2(n+1)}\int_{1/2}^1 x^n dx$$

$$W=\frac{ln2}{4}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{2n^2}+\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{4n^2}$$

$$A=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^3}{8}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n}-\frac{(ln2)^2}{4}\sum_{n=1}^{\infty}\frac{i^{n-1}+(-i)^{n-1}}{n^2}$$

$$+\frac{(ln2)^2}{8}\sum_{n=1}^{\infty}\frac{(\frac{i}{2})^{n-1}+(\frac{-i}{2})^{n-1}}{n^2}$$

$$A=\frac{(ln2)^3}{2}\tan^{-1}2+\frac{(ln2)^3}{2}\tan^{-1}\left(\frac{1}{2}\right)$$

$$-\frac{(ln2)^2}{4i}\left(Li_2(i)-Li_2(-i)-Li_2\left(\frac{i}{2}\right)+Li_2\left(\frac{-i}{2}\right)\right)$$

$$\text{ANSWER}=A+B=\frac{\pi(ln2)^3}{2}+\frac{3(ln2)^2}{4i}\left(Li_2\left(\frac{i}{2}\right)-Li_2\left(\frac{-i}{2}\right)\right)$$

$$-\frac{ln2}{2i}\left(Li_3(i)-Li_3(-i)+Li_3\left(\frac{-i}{2}\right)-Li_3\left(\frac{i}{2}\right)\right)$$

2442. Solve for real numbers:

$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

$$First\,,\quad let's\ look\ at\ the\ solution\ integral\ldots$$

$$\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx\overset{I.B.P}{\triangleq}\left(-2x\sqrt{\cos(x)}\right)\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\sqrt{\cos(x)}\,dx=2\int_0^{\frac{\pi}{2}}\sqrt{\cos(x)}\,dx$$

$$Substitution\ :\ \cos(x)=t\,,-\sin(x)\,dx=dt\,,\qquad dx=-\frac{1}{\sqrt{1-t^2}}\ t[0;1]$$

$$\Omega=2\int_0^1\frac{\sqrt{t}}{\sqrt{1-t^2}}dt\underset{dt=\frac{dy}{2\sqrt{y}}}{\overset{2tdt=dy}{\triangleq}}\int_0^1\frac{\sqrt[4]{y}}{\sqrt{1-y}}\frac{dy}{\sqrt{y}}=\int_0^1 y^{\frac{1}{4}-\frac{1}{2}}(1-y)^{-\frac{1}{2}}dy$$

$$=\int_0^1 y^{-\frac{1}{4}}(1-y)^{-\frac{1}{2}}dy=\beta\left(\frac{3}{4},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}$$

$$The\ Solution\ of\ our\ integral\ is\ completed\,,now\ let's\ return\ to\ the\ previous\ expression\ldots$$

$$x^2+\sqrt{\frac{\pi}{2}}\Gamma^{-2}\left(\frac{3}{4}\right)\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma^2\left(\frac{3}{4}\right)}\int_0^{\frac{\pi}{2}}\frac{x\sin(x)}{\sqrt{\cos(x)}}dx=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{1}{\Gamma^2\left(\frac{3}{4}\right)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{5}{4}\right)}=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{\sqrt{\pi}}{\frac{1}{4}\cdot\Gamma\left(1-\frac{1}{4}\right)\cdot\Gamma\left(1+\frac{1}{4}\right)}=3x$$

$$x^2+\sqrt{\frac{\pi}{2}}\cdot\frac{4\sqrt{\pi}}{\sqrt{2}\,\pi}-3x=0$$

$$x^2-3x+2=0\ \rightarrow(x-1)(x-2)=0\rightarrow\ Answer\ :\ x=1\ and\ x=2$$

2443. Find a closed form:

$$\Omega=\int_{-1}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)\left(1+\frac{1-x}{1+x}e^x\right)}dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_{-1}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)\left(1+\frac{1-x}{1+x}e^x\right)}dx\underbrace{=}_{x\to -x}\int_{-1}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)\,e^x}{x(1+x^2)\left(e^x+\frac{1+x}{1-x}\right)}dx=I$$

$$2I=\int_{-1}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$=\int_{-1}^{0}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_{0}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx=A+B=$$

$$A\underbrace{=}_{x\to -x}\int_{0}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx\quad\&\quad B=\int_{0}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$2I=A+B=2\int_{0}^{1}\frac{(\ln(x^2))^2\tan^{-1}(x)}{x(1+x^2)}dx\to\frac{I}{4}=\int_{0}^{1}\frac{(\ln(x))^2\tan^{-1}(x)}{x(1+x^2)}dx\Rightarrow \mathrm{I}=?$$

$$\frac{I}{4}=\int_{0}^{1}\frac{(lnx)^2\tan^{-1}(x)}{x}dx-\int_{0}^{1}\frac{x(lnx)^2\tan^{-1}(x)}{1+x^2}dx=\Omega-\Psi$$

$$\Omega=\int_{0}^{1}\frac{(lnx)^2\tan^{-1}(x)}{x}dx=\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\int_{0}^{1}x^{a-1}\tan^{-1}(x)\,dx$$

$$=\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_{0}^{1}x^{a+2n}\,dx$$

$$\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(a+2n+1)}=2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^4}=2\beta(4)$$

$$\Psi=\int_{0}^{1}\frac{x(lnx)^2\tan^{-1}(x)}{1+x^2}dx\underbrace{=}_{x\to\frac{1}{x}}\int_{1}^{\infty}\frac{(lnx)^2\tan^{-1}\left(\frac{1}{x}\right)}{x(1+x^2)}dx$$

$$=\frac{\pi}{2}\int_{1}^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_{1}^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$=\frac{\pi}{2}\int_{1}^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_{0}^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_{0}^{1}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx$$

$$\Psi=\frac{\pi}{2}\int_{1}^{\infty}\frac{(lnx)^2}{x(1+x^2)}dx-\int_{0}^{\infty}\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_{0}^{1}\frac{(lnx)^2\tan^{-1}(x)}{x}dx-\Psi$$

$$2\Psi=\frac{\pi}{2}\int_1^\infty\frac{(lnx)^2}{x(1+x^2)}dx-\int_0^\infty\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx+\int_0^1\frac{(lnx)^2\tan^{-1}(x)}{x}dx=M-N+\Omega$$

$$\Omega=2\beta(4)$$

$$M=\frac{\pi}{2}\int_1^\infty\frac{(lnx)^2}{x(1+x^2)}dx\underbrace{=}_{x\to\frac{1}{x}}\frac{\pi}{2}\frac{\partial^2}{\partial a^2}\Big|_{a=0}\sum_{n=0}^\infty(-1)^n\int_0^1x^{2n+a+1}dx=$$

$$=\frac{\pi}{2}\frac{\partial^2}{\partial a^2}\Big|_{a=0}\sum_{n=0}^\infty\frac{(-1)^n}{a+2n+2}=-\frac{\pi}{8}\sum_{n=1}^\infty\frac{(-1)^n}{(n)^3}=-\frac{\pi}{8}\sum_{n=1}^\infty\frac{-3}{4n^3}=\frac{3\pi\zeta(3)}{32}=M$$

$$N=\int_0^\infty\frac{(lnx)^2\tan^{-1}(x)}{x(1+x^2)}dx\Rightarrow f(a)=\int_0^\infty\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}dx\Rightarrow f(1)=N$$

$$(\text{Feynman trick}\}\to\frac{d}{da}f(a)=\frac{d}{da}\int_0^\infty\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}dx$$

$$=\int_0^\infty\frac{\partial}{\partial a}\left(\frac{(lnx)^2\tan^{-1}(ax)}{x(1+x^2)}\right)dx$$

$$\frac{d}{da}f(a)=\int_0^\infty\frac{(lnx)^2}{(1+(ax)^2)(1+x^2)}dx=\frac{a^2}{a^2-1}\int_0^\infty\frac{(lnx)^2}{1+(ax)^2}dx-\frac{1}{a^2-1}\int_0^\infty\frac{(lnx)^2}{1+x^2}dx$$
$$=K-H$$

$$K=\frac{a^2}{a^2-1}\int_0^\infty\frac{(lnx)^2}{1+(ax)^2}dx\underbrace{=}_{ax\to x}\frac{a}{a^2-1}\int_0^\infty\frac{\left(\ln\left(\frac{x}{a}\right)\right)^2}{1+x^2}dx=$$

$$\frac{a}{a^2-1}\int_0^\infty\frac{(lnx)^2}{1+x^2}dx-\frac{2aln(a)}{a^2-1}\int_0^\infty\frac{\ln(x)}{1+x^2}dx+\frac{a(lna)^2}{a^2-1}\int_0^\infty\frac{1}{x^2+1}dx$$

$$\{note:\int_0^\infty\frac{(lnx)^2}{1+x^2}dx=\frac{\partial^2}{\partial a^2}\Big|_{a=0}\int_0^{\frac{\pi}{2}}(\tan x)^a\,dx=\frac{\partial^2}{\partial a^2}\Big|_{a=0}\frac{1}{2}\beta(\frac{a+1}{2}\cdot\frac{1-a}{2}=\frac{\pi^3}{8}\}$$

$$\left\{note:\int_0^\infty\frac{ln(x)}{1+x^2}dx=\frac{\partial}{\partial a}\Big|_{a=0}\int_0^{\frac{\pi}{2}}(\tan x)^a\,dx=\frac{\partial}{\partial a}\Big|_{a=0}\frac{1}{2}\beta(\frac{a+1}{2}\cdot\frac{1-a}{2}=0\right\}$$

$$H=\frac{1}{a^2-1}\int_0^\infty\frac{(lnx)^2}{1+x^2}dx=\frac{\pi^3}{8(a^2-1)}$$

$$\underbrace{=}_{f(0)=0\,\&\,f(1)=N}f(1)=\int_0^1df(a)=\int_0^1(K-H)da=\frac{\pi^3}{8}\int_0^1\frac{da}{a+1}+\frac{\pi}{2}\int_0^1\frac{a(lna)^2}{a^2-1}da$$

$$N=f(1)=\frac{\pi^3}{8}\ln(2)-\frac{\pi}{2}\frac{\partial^2}{\partial b^2}\Big|_{b=0}\sum_{n=0}^{\infty}\int_0^1 a^{2n+b+1}da$$

$$\frac{\pi^3}{8}\ln(2)-\frac{\pi}{2}\frac{\partial^2}{\partial b^2}\Big|_{b=0}\sum_{n=0}^{\infty}\frac{1}{2n+b+2}=\frac{\pi^3}{8}\ln(2)-\pi\sum_{n=1}^{\infty}\frac{1}{8n^3}=\frac{\pi^3}{8}\ln(2)-\frac{\pi\zeta(3)}{8}=N$$

$$I=4\Omega-4\Psi=4\Omega-4\left(\frac{M}{2}-\frac{N}{2}+\frac{\Omega}{2}\right)=2\Omega-2M+2N$$

$$I=4\beta(4)-\frac{3\pi}{16}\zeta(3)+\frac{\pi^3}{4}ln(2)-\frac{\pi}{4}\zeta(3)$$

$$\text{ANSWER}=I=\frac{\psi^{(3)}(\frac{1}{4})}{192}-\frac{\pi^4}{24}-\frac{7\pi}{16}\zeta(3)+\frac{\pi^3}{4}ln(2)$$

{note section}

$$\beta(4)=\frac{\psi^{(3)}(\frac{1}{4})}{768}-\frac{8\pi^4}{768}$$

2444. Find a closed form:

$$\Omega=\int_0^1 Li_2(-x^2)\tan^{-1}(x)dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=\int_0^1 Li_2(-x^2)\tan^{-1}(x)dx=$$

$$\underbrace{=}_{IBP}Li_2(-1)\left(\frac{\pi}{4}-\frac{ln2}{2}\right)+2\int_0^1\ln(1+x^2)\tan^{-1}(x)dx-\int_0^1\frac{(\ln(1+x^2))^2}{x}dx$$

$$A=\int_0^1\ln(1+x^2)\tan^{-1}(x)dx\quad\&\quad B=\int_0^1\frac{(\ln(1+x^2))^2}{x}dx$$

$$A=\int_0^1\ln(1+x^2)\tan^{-1}(x)dx\underbrace{=}_{IBP}\frac{\pi}{4}ln2-\frac{(ln2)^2}{2}-\int_0^1\frac{2x^2\tan^{-1}(x)-xln(1+x^2)}{1+x^2}dx$$

$$\frac{\pi}{4}ln2-\frac{(ln2)^2}{2}-2\int_0^1\tan^{-1}x\,dx+2\int_0^1\frac{\tan^{-1}x}{1+x^2}dx+\int_0^1\frac{xln(1+x^2)}{1+x^2}dx$$

$$A=\left(\frac{\pi}{4}+1\right)ln2-\frac{(ln2)^2}{4}+\frac{\pi^2}{16}-\frac{\pi}{2}$$

$$B=\int_0^1\frac{(\ln(1+x^2))^2}{x}dx\underbrace{=}_{x^2\to x}\int_0^1\frac{(\ln(1+x))^2}{2x}dx\underbrace{=}_{1+x\to x}\int_1^2\frac{(lnx)^2}{2(x-1)}dx$$

$$\underbrace{=}_{x\to\frac{1}{x}}\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x(1-x)}dx=\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2x}dx+\int_{\frac{1}{2}}^1\frac{(lnx)^2}{2(1-x)}dx=$$

$$\frac{(ln2)^3}{6}+\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\int_{\frac{1}{2}}^1\frac{x^{a+n}}{2}dx=\frac{(ln2)^3}{6}+\frac{\partial^2}{\partial a^2}\bigg|_{a=0}\sum_{n=0}^{\infty}\frac{1-\left(\frac{1}{2}\right)^{a+n+1}}{2(a+n+1)}=$$

$$B=\frac{(ln2)^3}{6}+\sum_{n=1}^{\infty}\frac{1}{(n)^3}-\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{(n)^3}-\frac{(ln2)^2}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n}-ln2\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n^2}$$

$$B=\frac{(ln2)^3}{6}+\zeta(3)-Li_3\left(\frac{1}{2}\right)-\frac{(ln2)^3}{2}-ln2Li_2\left(\frac{1}{2}\right)\Rightarrow B=\frac{\zeta(3)}{8}$$

$$\{\text{note}: \mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{(\ln2)^2}{2}\ \&\ \mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{(\ln2)^3}{6}-\frac{\pi^2}{12}\ln2+\frac{7\zeta(3)}{8}\ \&\ \mathrm{Li}_2(-1)=-\frac{\pi^2}{12}$$

$$ANSWER=-\frac{\pi^3}{48}+\frac{\pi^2ln2}{24}+2A-B$$

$$=-\frac{\pi^3}{48}+\frac{\pi^2ln2}{24}+\left(\frac{\pi}{2}+2\right)ln2-\frac{(ln2)^2}{2}+\frac{\pi^2}{8}-\pi-\frac{\zeta(3)}{8}$$

2445. Find a closed form:

$$I=\int_0^1\frac{Li_2(-x)\ln(1+x)}{x(x+1)}dx$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by Alireza Askari-Iran

$$\Omega=A-B=\int_0^1\frac{Li_2(-x)\ln(1+x)}{x}dx-\int_0^1\frac{Li_2(-x)\ln(1+x)}{(x+1)}dx$$

$$A=\int_0^1\frac{Li_2(-x)\ln(1+x)}{x}dx=-\frac{(Li_2(-1))^2}{2}=\{note: dLi_2(-x)=-\frac{\ln(1+x)}{x}dx$$

$$B=\int_0^1\frac{Li_2(-x)\ln(1+x)}{(x+1)}dx\underbrace{=}_{IBP}\frac{Li_2(-1)(\ln(2))^2}{2}+\int_0^1\frac{(\ln(1+x))^3}{2x}dx$$

$$C=\int_0^1\frac{(\ln(1+x))^3}{2x}dx\underbrace{=}_{1+x\to x}\int_1^2\frac{(\ln(x))^3}{2(x-1)}dx\underbrace{=}_{x\to\frac{1}{x}}\int_{1/2}^1\frac{(\ln(x))^3}{2(x-x^2)}dx$$

$$=\int_{1/2}^{1}\frac{(\ln(x))^3}{2x}dx+\int_{1/2}^{1}\frac{(\ln(x))^3}{2(1-x)}dx=\frac{(\ln(2))^4}{8}+\sum_{n=0}^{\infty}\frac{1}{2}\int_{\frac{1}{2}}^{1}x^n(\ln(x))^3\,dx=$$

$$\to D=\sum_{n=0}^{\infty}\frac{1}{2}\int_{\frac{1}{2}}^{1}x^n(\ln(x))^3\,dx=\sum_{n=0}^{\infty}\frac{1}{2}\left(\int_0^1 x^n(\ln(x))^3dx-\int_0^{\frac{1}{2}}x^n(\ln(x))^3dx\right)=\Psi-\Phi$$

$$\left\{note:\int_0^1 x^m(\ln(x))^n\,dx=\frac{(-1)^n n!}{(m+1)^{n+1}}\ \{n>-1\wedge m\neq-1\}\right\}$$

$$\Psi=\sum_{n=0}^{\infty}\frac{1}{2}\int_0^1 x^n(\ln(x))^3dx=\sum_{n=0}^{\infty}\frac{3}{(n+1)^4}=3\zeta(4)$$

$$\Phi=\sum_{n=0}^{\infty}\frac{1}{2}\int_0^{\frac{1}{2}}x^n(\ln(x))^3dx=$$

$$\underbrace{=}_{IBP}-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)}\int_0^{\frac{1}{2}}x^n\,(\ln(x))^2dx$$

$$\underbrace{=}_{IBP}-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3(ln2)^2}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^2}+\sum_{n=0}^{\infty}\frac{3}{(n+1)^2}\int_0^{\frac{1}{2}}x^n\ln(x)dx$$

$$\underbrace{=}_{IBP}-3ln2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^3}-\sum_{n=0}^{\infty}\frac{3}{(n+1)^3}\int_0^{\frac{1}{2}}x^n\,dx=$$

$$\Phi=-\frac{(\ln(2))^3}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{n+1}-\frac{3(ln2)^2}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^2}-3ln2\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^3}-3\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)^4}$$

$$\Phi=-\frac{(ln2)^3}{2}Li_1\left(\frac{1}{2}\right)-\frac{3(ln2)^2}{2}Li_2\left(\frac{1}{2}\right)-3ln2Li_3\left(\frac{1}{2}\right)-3Li_4\left(\frac{1}{2}\right)$$

$$\Phi=\frac{-(ln2)^4}{4}+\frac{\pi^2(ln2)^2}{8}-\frac{21}{8}ln2\,\zeta(3)-3Li_4\left(\frac{1}{2}\right)$$

$$C=-\frac{(\ln(2))^4}{8}+\Psi-\Phi=3\zeta(4)+\frac{(ln2)^4}{8}+\frac{\pi^2(ln2)^2}{8}+\frac{21}{8}ln2\,\zeta(3)+3Li_4\left(\frac{1}{2}\right)$$

$$\begin{cases}B=\dfrac{Li_2(-1)(\ln(2))^2}{2}+C=-\dfrac{\pi^2(ln2)^2}{24}+C\\ A=-\dfrac{\pi^4}{288}\end{cases}$$

ANSWER: $A-B=\frac{21}{8}ln2\,\zeta(3)+3Li_4\left(\frac{1}{2}\right)+\frac{(ln2)^4}{8}-\frac{\pi^2(ln2)^2}{12}-\frac{53\pi^4}{1440}$

$\{note\ section\}$

$\zeta(4)=\frac{\pi^4}{90}\ .\ Li_3\left(\frac{1}{2}\right)=\frac{(ln2)^3}{6}-\frac{\pi^2}{12}ln2+\frac{7}{8}\zeta(3)\ .\ Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{(ln2)^2}{2}$

2446. Find a closed form:

$$\Omega=\int_0^\infty\int_0^\infty\int_0^\infty\frac{\log_e\left(\frac{1}{(x+y+z)^{x+y+z}}\right)}{1+e^{x+y+z}}dxdydz$$

Proposed by Amin Hajiyev-Azerbaijan

Solution by Pham Duc Nam-Vietnam

$$*\ \Omega=-\int_0^\infty\int_0^\infty\int_0^\infty\frac{(x+y+z)\ln(x+y+z)}{1+e^{x+y+z}}dxdydz$$

$$Let: x\to\ e^{-x}, y\to\ e^{-y}, z\to\ e^{-z}\to\ \Omega$$

$$=-\int_0^1\int_0^1\int_0^1\frac{\ln(xyz)\ln\left(-\ln(xyz)\right)}{1+xyz}dxdydz$$

$$symmetry\to\frac{1}{2}\int_0^1\frac{\ln^3(x)\ln\left(-\ln(x)\right)}{1+x}dx=-\frac{1}{2}\int_0^\infty\frac{x^3\ln(x)}{1+e^{-x}}e^{-x}dx$$

$$=-\frac{1}{2}\int_0^\infty x^3\ln(x)e^{-x}\left(\sum_{n=0}^\infty(-1)^n e^{-nx}\right)dx$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\int_0^\infty x^3\ln(x)e^{-x(n+1)}dx=$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}\int_0^\infty x^{s-1}e^{-(n+1)x}\,dx$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}M\{e^{-(n+1)x}\}(s)=$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\frac{d}{ds}|_{s=4}\frac{\Gamma(s)}{(n+1)^s}$$

$$=-\frac{1}{2}\sum_{n=0}^\infty(-1)^n\left(\frac{\psi(s)\Gamma(s)}{(n+1)^s}-\frac{\ln(n+1)\,\Gamma(s)}{(n+1)^s}\right)|_{s=4}=$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{-6\ln(n+1)-6\gamma+11}{(n+1)^4}\right)=$$

$$=-\frac{11}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}+3\gamma\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^4}+3\sum_{n=0}^{\infty}\frac{(-1)^n\ln(n+1)}{(n+1)^4}=$$

$$=-\frac{11}{2}\frac{7\pi^4}{720}+3\gamma\frac{7\pi^4}{720}+3\left(-\frac{7\zeta'(4)}{8}-\frac{\pi^4}{720}\ln(2)\right)=$$

$$=\frac{7\pi^4}{240}\gamma-\frac{77\pi^4}{1440}-\frac{21\zeta'(4)}{8}-\frac{\pi^4}{240}\ln(2)$$

notes:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s}=\zeta(s)(1-2^{1-s})\to\frac{d}{ds}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s}=\frac{d}{ds}\zeta(s)(1-2^{1-s})$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\ln(n+1)}{(n+1)^s}=-2^{-s}\big((2^s-2)\zeta'(s)+2\ln(2)\zeta(s)\big)$$

2447. Prove that:

$$\Omega=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\sin(x+y+z)\ .\ \cos(x+y+z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz=-\pi^{5/2}\frac{\sqrt[4]{15+11\sqrt{2}-2\sqrt{116+82\sqrt{2}}}}{4\sqrt[4]{8}\ .\ \Gamma\left(\frac{1}{4}\right).\Gamma\left(\frac{1}{8}\right)}$$

Proposed by Shirvan Tahirov-Azerbaijan, Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\sin(x+y+z)\ .\ \cos(x+y+z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz$$

$$=\frac{1}{2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{\sin(2x+2y+2z)}{\sqrt{x\sqrt{y\sqrt{z}}}}dxdydz=$$

$$=\frac{1}{2}Im\left\{\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{e^{2xi}e^{2yi}e^{2zi}}{x^{\frac{1}{2}}.y^{\frac{1}{4}}.z^{\frac{1}{8}}}dxdydz=\right\}$$

$$=\frac{1}{2}Im\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{2ix}x^{-\frac{1}{2}}e^{2iy}y^{-\frac{1}{4}}e^{2iz}z^{-\frac{1}{6}}dxdydz=$$

$$=\frac{1}{2}Im(K\,.\;M\;.N)$$

$$K=\int_0^\infty e^{2xi}\,x^{-\frac{1}{2}}dx=\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{2i^3}}=\frac{\sqrt{\pi}}{i\sqrt{2i}}=i^{-\frac{3}{2}}\sqrt{\frac{\pi}{2}}=e^{-\frac{3}{4}\ln(i^2)}\sqrt{\frac{\pi}{2}}=-e^{-\frac{3}{4}i\pi}\sqrt{\frac{\pi}{2}}$$

$$M=\int_0^\infty e^{2yi}y^{-\frac{1}{4}}dy=\frac{\Gamma\left(\frac{3}{4}\right)}{(-2i)^{\frac{3}{4}}}=\frac{\Gamma\left(\frac{3}{4}\right)i^{\frac{3}{4}}}{\sqrt[4]{8}}=\frac{\Gamma\left(\frac{3}{4}\right)e^{\frac{3}{8}\ln(i^2)}}{\sqrt[4]{8}}=e^{\frac{3}{8}i\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt[4]{8}}$$

$$N=\int_0^\infty e^{2zi}z^{-\frac{1}{8}}dz=\frac{\Gamma\left(\frac{7}{8}\right)}{(-2i)^{\frac{7}{8}}}=\frac{\Gamma\left(\frac{7}{8}\right)i^{\frac{7}{8}}}{2^{\frac{7}{8}}}=e^{\frac{7}{16}i\pi}\frac{\Gamma\left(\frac{7}{8}\right)}{2^{\frac{7}{8}}}$$

$$\Omega=\frac{1}{2}Im(K\,.\;M\;.N)=-\frac{1}{2}Im\left\{\sqrt{\frac{\pi}{2}}\,.e^{\frac{7}{16}i\pi}\frac{\Gamma\left(\frac{7}{8}\right)}{2^{\frac{7}{8}}}\;.e^{-\frac{3}{4}i\pi}.e^{\frac{3}{8}i\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\sqrt[4]{8}}\right\}=$$

$$-\frac{1}{2}Im\left\{\sqrt{\frac{\pi}{2}}.\;\frac{\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{3}{4}\right)}{2^{\frac{7}{8}}.\;\sqrt[4]{8}}\;.e^{\frac{1}{16}i\pi}\right\}=-\frac{1}{2}.\;\sqrt{\frac{\pi}{2}}.\frac{\Gamma\left(\frac{7}{8}\right)\Gamma\left(\frac{3}{4}\right)}{2^{\frac{7}{8}}.\;\sqrt[4]{8}}\;.\sin\left(\frac{\pi}{16}\right)$$

$$\Omega=-\frac{1}{2}.\sqrt{\frac{\pi}{2}}\frac{\pi^2\sin\left(\frac{\pi}{16}\right)}{2^{\frac{7}{8}}.\;\sqrt[4]{8}\,.\sin\left(\frac{\pi}{4}\right).\sin\left(\frac{\pi}{8}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{8}\right)}$$

$$=-\pi^{5/2}\frac{\sqrt[4]{15+11\sqrt{2}-2\sqrt{116+82\sqrt{2}}}}{4\sqrt[4]{8}\,.\;\Gamma\left(\frac{1}{4}\right)\;.\Gamma\left(\frac{1}{8}\right)}$$

2448. Prove that:

$$\int_R\left(\frac{1}{x}-\frac{\sin(x)}{x^2}\right)^2dx=\frac{\pi}{3}$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega=\int_{-\infty}^\infty\left(\frac{x-\sin(x)}{x^2}\right)^2dx=2\int_0^\infty\frac{x^2-2x\sin(x)+sin^2(x)}{x^4}dx=$$

$$=2\int_0^\infty L^{-1}\left\{\frac{1}{x^4}\right\}(s)L\{x^2-2x\sin(x)+sin^2(x)\}(s)ds$$

$$Laplace\;transform: L\{f\}(s)=\int_0^\infty f(t)e^{-st}dt$$

$$Inverse\ Laplace\ transform: L^{-1}\{F(s)\}(t) = f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds$$

$$L\{t^n\}(s) = \frac{n!}{s^{n+1}};\ L^{-1}\left\{\frac{a}{s^m}\right\}(t) = \frac{a\,t^{m-1}}{(m-1)!} \ \rightarrow\ L^{-1}\left\{\frac{1}{x^4}\right\}(s) = \frac{s^3}{3!} = \frac{s^3}{6}$$

$$L\{x^2\}(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

$$L\{t^n\sin(at)\}(s) = (-1)^n\frac{d^n}{ds^n}\left(\frac{a}{s^2+a^2}\right) \rightarrow L\{2x\sin(x)\}(s) = \frac{4s}{(s^2+1)^2}$$

$$L\{\cos(at)\}(s) = \frac{s}{s^2+a^2} \ \rightarrow\ L\{sin^2(x)\}(s) = L\left\{\frac{1}{2} - \frac{\cos(2x)}{2}\right\}(s) =$$

$$= \frac{1}{2s} - \frac{s}{2(s^2+4)} = \frac{2}{s(s^2+4)}$$

$$\Omega = 2\int_0^\infty \frac{s^3}{6}\left(\frac{2}{s^3} - \frac{4s}{(s^2+1)^2} + \frac{2}{s(s^2+4)}\right)ds =$$

$$= \frac{8}{3}\int_0^\infty \frac{1}{s^2+1}ds - \frac{8}{3}\int_0^\infty \frac{1}{s^2+4}ds - \frac{4}{3}\int_0^\infty \frac{1}{(s^2+1)^2}ds = \frac{8}{3}\cdot\frac{\pi}{2} - \frac{8}{3}\cdot\frac{\pi}{4} - \frac{4}{3}\cdot\frac{\pi}{4} = \frac{\pi}{3}$$

2449. Find:

$$\Omega = \int_1^{\sqrt{3}}\left(\frac{\tan^{-1}x}{x-\tan^{-1}x}\right)^2 dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Yen Tung Chung-Taiwan

$$\underbrace{\int_1^{\sqrt{3}}\left(\frac{\tan^{-1}x}{x-\tan^{-1}x}\right)^2 dx}_{let\ y=\tan^{-1}x\Rightarrow x=\tan y, dx=\sec^2 y dy} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(\frac{y}{\tan y - y}\right)^2 \sec^2 y\,dy =$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{y^2}{(\sin y - y\cos y)^2}dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{y^2}{(1+y^2)\left(\frac{1}{\sqrt{1+y^2}}\sin y - \frac{y}{\sqrt{1+y^2}}\cos y\right)}dy =$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1-\frac{1}{1+y^2}}{(\cos(\tan^{-1}y)\sin y-\sin(\tan^{-1}y)\cos y)^2}dy$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{\sin^2(y-\tan^{-1}y)}d(y-\tan^{-1}y)=$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\csc^2(y-\tan^{-1}y)\,d(y-\tan^{-1}y)=-\cot(y-\tan^{-1}y)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$=\frac{1+\tan y\tan(\tan^{-1}y)}{\tan(\tan^{-1}y)-\tan y}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{1+y\tan y}{1-\tan y}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{1+\frac{\pi}{\sqrt{3}}}{\frac{\pi}{3}-\sqrt{3}}-\frac{1+\frac{\pi}{4}}{\frac{\pi}{4}-1}=$$

$$=\frac{\pi\sqrt{3}+3}{\pi-3\sqrt{3}}+\frac{4+\pi}{4-\pi}$$

Solution 2 by Pham Duc Nam-Vietnam

$$\int\frac{\arctan^2(x)}{(x-\arctan^2(x))^2}dx=\int\frac{(\arctan(x)-x)^2-(x^2-2x\arctan(x))}{(\arctan(x)-x)^2}dx$$

$$=x-\int\frac{x^2-2x\arctan(x)}{(\arctan(x)-x)^2}dx=x-\int\frac{x^2\left(1-\frac{2\arctan(x)}{x}\right)(1+x^2)}{(1+x^2)(\arctan(x)-x)^2}dx$$

$$\begin{cases}u=\left(1-\frac{2\arctan(x)}{x}\right)(1+x^2)\\ dv=\frac{x^2}{1+x^2}\frac{1}{(\arctan(x)-x)^2}dx\end{cases}\Rightarrow\begin{cases}du=\frac{2(1-x^2)(\arctan(x)-x)}{x^2}\\ v=\frac{1}{\arctan(x)-x}\end{cases}$$

$$\Rightarrow\int\frac{\arctan^2(x)}{(x-\arctan^2(x))^2}dx=$$

$$=x-\frac{1}{\arctan(x)-x}\left(1-\frac{2\arctan(x)}{x}\right)(1+x^2)+2\int\frac{1-x^2}{x^2}dx$$

$$=\frac{x\arctan(x)+1}{\arctan(x)-x}+C\Rightarrow\int_1^{\sqrt{3}}\frac{\arctan^2(x)}{(x-\arctan^2(x))^2}dx=$$

$$=\frac{x\arctan(x)+1}{\arctan(x)-x}\Bigg|_1^{\sqrt{3}}=\frac{4+\pi}{4-\pi}-\frac{\pi\sqrt{3}+3}{3\sqrt{3}-\pi}$$

Solution 3 by Ravi Prakash-India

Put $\tan^{-1} x = \theta, x = \tan\theta$

$$\Omega = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{\theta}{\tan\theta - \theta}\right)^2 \sec^2\theta\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x^2}{(\sin x - x\cos x)^2} dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x\sin x}{(\sin x - x\cos x)^2} \cdot \frac{x}{\sin x} dx$$

$$= \frac{-1}{\sin x - x\cos x} \cdot \frac{x}{\sin x}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin x - x\cos x} \frac{(\sin x - x\cos x)dx}{\sin^2 x}$$

$$= \frac{-1}{\sin x - x\cos x} \cdot \frac{x}{\sin x}\Big|_{\frac{\pi}{3}}^{\frac{\pi}{3}} - \cot x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{-1}{\frac{\sqrt{3}}{2} - \frac{\pi}{3} \cdot \frac{1}{2}} \cdot \frac{\frac{\pi}{3}}{\frac{\sqrt{3}}{2}} + \frac{1}{\frac{1}{\sqrt{2}} - \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}}} \cdot \frac{\frac{\pi}{4}}{\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{3}} + 1$$

$$= \frac{2\pi}{4-\pi} - \frac{4\pi}{9-\sqrt{3}\pi} + 1 - \frac{1}{\sqrt{3}} = \frac{\pi+4}{4-\pi} - \left[\frac{4\pi\sqrt{3} + 9 - \sqrt{3}\pi}{\sqrt{3}(9-\sqrt{3}\pi)}\right]$$

$$= \frac{\pi+4}{4-\pi} - \frac{9+3\sqrt{3}\pi}{3(3\sqrt{3}-\pi)} = \frac{\pi+4}{4-\pi} - \frac{3+\sqrt{3}\pi}{3\sqrt{3}-\pi}$$

Solution 4 by Hikmat Mammadov-Azerbaijan

$$\Omega = \int_1^{\sqrt{3}} \left(\frac{\tan^{-1} x}{x - \tan^{-1} x}\right)^2 dx$$

$$= \int_1^{\sqrt{3}} \left(1 + \frac{x^2}{(x - \tan^{-1}(x))^2} - \frac{2x}{(x - \tan^{-1}(x))}\right) dx$$

$$= (\sqrt{3} - 1) + \int_1^{\sqrt{3}} \frac{-x^2 + 2x \cdot \tan^{-1}(x)}{(x - \tan^{-1}(x))^2} dx \overset{say}{\longrightarrow} S$$

Note: $\frac{1}{(x - \tan^{-1}(x))} = u \to u' = \frac{-\frac{x^2}{1+x^2}}{(x-\tan^{-1}(x))^2}$

$$S = (\sqrt{3} - 1) + \int_1^{\sqrt{3}} \frac{\frac{x^2}{(1+x^2)}(1+x^2) - (1+x^2) \cdot (-\tan^{-1}(x) + x)}{(x - \tan^{-1}(x))^2} dx$$

$$= (\sqrt{3}-1) + \int_1^{\sqrt{3}} \frac{-(\tan^{-1}(x)-x)'\cdot(1+x^2)+(1+x^2)\cdot(\tan^{-1}(x)-x)}{(x-\tan^{-1}(x))}dx$$

$$= \sqrt{3}-1+\left[\frac{x^2+1}{\tan^{-1}(x)-x}\right]_1^{\sqrt{3}} = \sqrt{3}-1+\frac{12}{\pi-3\sqrt{3}}-\frac{8}{\pi-4}$$

$$\Rightarrow \Omega = \sqrt{3}-1+\frac{12}{\pi-3\sqrt{3}}-\frac{8}{\pi-4}$$

2450. Find:

$$\Omega = \int_2^4 \frac{\sqrt[2024]{ln(9-x)}}{\sqrt[2024]{ln(9-x)}+\sqrt[2024]{ln(x+3)}}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$y = 6-x \Rightarrow x = 6-y \Rightarrow dx = -dy$$
$$x = 2 \Rightarrow y = 4, \qquad x = 4 \Rightarrow y = 2$$

$$\Omega = \int_4^2 \frac{\sqrt[2024]{ln(9-6+y)}}{\sqrt[2024]{ln(9-6+y)}+\sqrt[2024]{ln(6-y+3)}}(-dy)$$

$$\Omega = \int_2^4 \frac{\sqrt[2024]{ln(3+y)}}{\sqrt[2024]{ln(3+y)}+\sqrt[2024]{ln(9-y)}}dy,$$

$$\Omega = \int_2^4 \frac{\sqrt[2024]{ln(3+x)}}{\sqrt[2024]{ln(3+x)}+\sqrt[2024]{ln(9-x)}}dx$$

$$2\Omega = \int_2^4 \frac{\sqrt[2024]{ln(9-x)}+\sqrt[2024]{ln(x+3)}}{\sqrt[2024]{ln(9-x)}+\sqrt[2024]{ln(x+3)}}dx$$

$$2\Omega = 4-2 \Leftrightarrow \Omega = 1$$

2451. Find a closed form:

$$\Omega = \int_0^1\int_0^\infty \frac{\ln(1+x^3)(\ln(1+y)+tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$We\ have:\int_0^1\int_0^\infty\frac{\ln(1+x^3)(\ln(1+y)+tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy\to$$

$$\underbrace{\int_0^\infty\frac{\ln(1+x^3)}{1+x^2}dx}_{\Omega_1}\underbrace{\int_0^1\frac{\ln(1+y)+tan^{-1}(y)}{(1+y)^2}dy}_{\Omega_2}\ (1)$$

$$Consider,\quad \Omega(a)=\int_0^\infty\frac{\ln(a+x)}{1+x^2}dx\ (2)$$

Differentiate above equation both sides with respect to 'a'. We get,

$$\overset{Leibniz\ Integral\ (rule)}{\Longrightarrow}\quad \frac{d}{da}\Omega(a)=\int_0^\infty\frac{\partial}{\partial a}\left(\frac{\ln(a+x)}{1+x^2}\right)dx\to\int_0^\infty\underbrace{\frac{dx}{(a+x)(1+x^2)}}_{Partical\ fraction}$$

$$\to\frac{a}{1+a^2}\int_0^\infty\frac{dx}{1+x^2}+\frac{1}{1+a^2}\int_0^\infty\left(\frac{1}{a+x}-\frac{1}{1+x^2}\right)dx\to\frac{a}{1+a^2}\int_0^\infty darctan(x)$$

$$+\frac{a}{1+a^2}\int_0^\infty dln\left(\frac{a+x}{\sqrt{1+x^2}}\right)\to\frac{d}{da}\Omega(a)=\frac{\pi a}{2(1+a^2)}-\frac{\ln(a)}{1+a^2}$$

Integrate above equation both sides with respect to. We get

$$\to\int\frac{d}{da}\Omega(a)\ da=\frac{\pi}{2}\int\frac{a}{1+a^2}da-\underbrace{\int\frac{\ln(a)}{1+a^2}da}_{I.B.P}\to\Omega(a)+C=$$

$$\frac{\pi}{4}\int dln(1+a^2)-\ln(a)\arctan(a)+\Im\underbrace{\int\frac{\ln(1+ia)}{a}}_{=-Li_2(ia)}=\Omega(a)+C=$$

$$\frac{\pi}{4}\ln(1+a^2)-\ln(a)\arctan(a)-\Im\{Li_2(-ia\}\ \underbrace{\to}_{Set\ a=0}\ \Omega(a=0)+C=-C$$

$$\int_0^\infty\frac{\ln(x)}{1+x^2}dx=C\to\int_0^\infty\frac{\ln(x)}{1+x^2}dx-\int_0^\infty\frac{\ln(x)}{1+x^2}dx=-2C\to C=0$$

$$\Omega(a)=\int_0^\infty\frac{\ln(a+x)}{1+x^2}dx=\frac{\pi}{4}\ln(1+a^2)-\ln(a)\arctan(a)-\Im\{Li_2(-ia\}\ (3)$$

$$Now,\Omega_1=\int_0^\infty\frac{\ln(1+x^3)}{1+x^2}dx\to\int_0^\infty\frac{\ln(1+x)}{1+x^2}dx+\int_0^\infty\ln\left(x-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\frac{dx}{1+x^2}$$

$$+\int_0^\infty\ln\left(x-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\frac{dx}{1+x^2}\ \overset{Utilising\ Equation-(3)}{\Longrightarrow}\ \frac{\pi}{4}\ln(2)-\Im\{Li_2(-i)\}+\frac{\pi}{4}\ln\left(\frac{1+i\sqrt{3}}{2}\right)-$$

$$-\ln\left(\frac{-1-i\sqrt{3}}{2}\right)arctan\left(\frac{-1-i\sqrt{3}}{2}\right)-\Im\left\{Li_2\left(\frac{i-\sqrt{3}}{2}\right)\right\}+\frac{\pi}{4}\ln\left(\frac{1+i\sqrt{3}}{2}\right)-$$

$$\ln\left(\frac{-1+i\sqrt{3}}{2}\right)arctan\left(\frac{-1+i\sqrt{3}}{2}\right)-\Im\left\{Li_2\left(\frac{i+\sqrt{3}}{2}\right)\right\}$$

$$(\ln(x\pm iy)=\frac{\ln(x^2+y^2)}{2}\pm iarctan\left(\frac{y}{x}\right),x>0\ and\ y>0)$$

$$(Li_s(\pm i)=-2^{-s}\eta(s)\pm i\beta(s))$$
$$(Cl_{2m}(\theta)=\Im\{Li_{2m}(e^{i\theta})\}\ m\geq 1\}$$

$$\frac{\pi}{4}\ln(2)+\beta(2)-\frac{2i\pi}{3}\left(\frac{\pi}{4}+i\frac{\ln(2+\sqrt{3})}{2}\right)-\frac{2i\pi}{3}\left(-\frac{\pi}{4}+i\frac{\ln(2+\sqrt{3})}{2}\right)-Cl_2\left(\frac{\pi}{6}\right)-Cl_2\left(\frac{5\pi}{6}\right)$$

$$\overset{\beta(2)=G}{\overbrace{-->}}-2\sum_{n\in N}\frac{\sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n}{3}\right)}{n^2}+G+\frac{\pi}{4}\ln(2)-\frac{i\pi^2}{6}+\frac{\pi}{3}\ln(2+\sqrt{3})+\frac{i\pi^2}{6}+\frac{\pi}{3}\ln(2+\sqrt{3})=$$

$$-2\sum_{n\in Z_0^+}\frac{(-1)^n}{(6n+3)^2}-\sum_{n\in Z_0^+}\frac{(-1)^n}{(2n+1)^2}-\sum_{n\in N}\frac{(-1)^{n-1}}{(6n-3)^2}+G+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)\longrightarrow$$

$$-\frac{2G}{9}-G-\frac{G}{9}+G+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)$$

$$Now,\Omega_2=\int_0^1\frac{\ln(1+y)+\tan^{-1}(y)}{(1+y)^2}dy\overset{I.B.P}{\cong}-\left(\frac{\ln(1+y)+\arctan(y)}{1+y}\right)_0^1+\int_0^1\frac{dy}{(1+y)^2}+$$

$$+\int_0^1\underbrace{\frac{dy}{(1+y)(1+y^2)}}_{Partial\ fraction}=-\frac{1}{2}\left(\frac{\pi}{4}+\ln(2)\right)-\int_0^1 d\left(\frac{1}{1+y}\right)-\frac{1}{2}\int_0^1\frac{y}{1+y^2}dy+\frac{1}{2}\int_0^1\frac{dy}{1+y}+$$

$$+\frac{1}{2}\int_0^1\frac{1}{1+y^2}dy=-\frac{\pi}{8}-\frac{\ln(2)}{2}-\frac{\ln(2)}{4}+\frac{\ln(2)}{2}+\frac{1}{2}+\frac{\pi}{8}$$

$$\Omega_2=\int_0^1\frac{\ln(1+y)+\tan^{-1}(y)}{(1+y)^2}dy=\frac{2-\ln(2)}{4}$$

Put the value of Ω_1 *and* Ω_2 *in equation* $-(1)$. *We get:*

$$\Omega_1\cdot\Omega_2=\int_0^1\int_0^\infty\frac{\ln(1+x^3)(\ln(1+y)+\tan^{-1}(y))}{(1+y)^2(1+x^2)}dxdy=$$
$$\left(\frac{2-\ln(2)}{4}\right)\left(-\frac{G}{3}+\frac{2\pi}{3}\ln(2+\sqrt{3})+\frac{\pi}{4}\ln(2)\right)$$

Note: G → Catalan's constant

2452. Prove that:

$$\int_0^\infty\frac{x}{\sinh(x)}dx-\int_0^\infty\frac{y}{\sinh(3y)}dy+\int_0^\infty\frac{z}{\sinh(5z)}dz-\cdots=\frac{\pi^2}{4}G$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\sum_{n=0}^{\infty}(-1)^n\int_0^\infty\frac{x}{\sinh(x(2n+1))}dx=\sum_{n=0}^{\infty}(-1)^n\int_0^\infty\frac{x}{\frac{e^{x(2n+1)}-e^{-x(2n+1)}}{2}}dx=$$

$$=\sum_{n=0}^{\infty}(-1)^n\int_0^\infty\frac{2x}{e^{x(2n+1)}-e^{-x(2n+1)}}dx=2\sum_{n=0}^{\infty}(-1)^n\int_0^\infty\frac{x}{e^{x(2n+1)}-e^{-x(2n+1)}}dx=$$

$$= -2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^1\frac{\ln(t)}{1-t^2}dt = -2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sum_{m=0}^{\infty}\int_0^1 t^{2m}\ln(t)\,dt =$$

$$2G\left(\zeta(2)-\frac{1}{4}-\frac{1}{16}-\cdots\right) = 2G\left(\zeta(2)-\frac{\zeta(2)}{4}\right) = \frac{\pi^2}{4}G$$

Notes :

1) $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} = \zeta(2)$

2) $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = G$

3) $\sinh(x) = \frac{e^x - e^{-x}}{2}$

4) $\sinh\big(x(2n+1)\big) = \frac{e^{x(2n+1)} - e^{-x(2n+1)}}{2}$

2453. Find a closed form:

$$\Omega = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin(y)}{\sin(x)}\sqrt{\frac{\sin(2x)}{\sin(2y)}}\,dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin(y)}{\sin(x)}\sqrt{\frac{\sin(2x)}{\sin(2y)}}\,dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(2x)}{\sin^2(x)}}\cdot\sqrt{\frac{\sin^2(y)}{\sin(2y)}}\,dxdy =$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(2x)}{\sin^2(x)}}\,dx\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin^2(y)}{\sin(2y)}}\,dy = \int_0^{\frac{\pi}{2}}\sqrt{\frac{2\sin(x)\cos(x)}{\sin(x).\sin(x)}}\,dx\;.\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(y).\sin(y)}{2\sin(y)\cos(y)}}\,dy$$

$$\int_0^{\frac{\pi}{2}}\sqrt{\frac{2\cos(x)}{\sin(x)}}\,dx.\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin(y)}{2\cos(y)}}\,dy = \int_0^{\frac{\pi}{2}}\sqrt{2}\cos(x)^{\frac{1}{2}}\sin(x)^{-\frac{1}{2}}dx.\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2}}\sin(y)^{\frac{1}{2}}\cos(y)^{-\frac{1}{2}}dy =$$

$$\frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2},\frac{-\frac{1}{2}+1}{2}\right).\frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2},\frac{\frac{1}{2}+1}{2}\right) = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{4}\right).\beta\left(\frac{1}{4},\frac{3}{4}\right) =$$

$$\frac{1}{4}\Gamma\left(\frac{1}{4},\frac{3}{4}\right).\Gamma\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}\cdot\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{1}{4}\cdot\frac{4\pi^2}{2}=\frac{\pi^2}{2}$$

Note :

$$\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}\quad,\quad \Gamma(n).\Gamma(1-n)=\frac{\pi}{\sin(n\pi)}$$

2454. Find a closed form:

$$\int_0^1\int_0^1\frac{\ln(1-x^2y^2)-xyln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan, Ankush Kumar Parcha-India
Solution by Quadri Faruk Temitope-Nigeria

$$\int_0^1\int_0^1\frac{\ln(1-x^2y^2)-xyln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy=$$

$$I=\int_0^1\int_0^1\frac{\ln(1-x^2y^2)}{1-x^2y^2}dxdy-\int_0^1\int_0^1\frac{xyln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dxdy=$$

$$\int_0^1\int_0^1\frac{\ln(1-xy)+\ln(1+xy)}{1-x^2y^2}dxdy-\int_0^1\int_0^1\frac{xy(\ln(1-xy)-\ln(1+xy))}{1-x^2y^2}dxdy=$$

$$\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1-xy)}{1+xy}dxdy\,\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}dxdy+\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1+xy)}{1+xy}dxdy+$$

$$+\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1+xy)}{1-xy}dxdy+\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1-xy)}{1+xy}dxdy\,\frac{-1}{2}\int_0^1\int_0^1\frac{\ln(1-xy)}{1-xy}dxdy-$$

$$\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1+xy)}{1+xy}dxdy+\frac{1}{2}\int_0^1\int_0^1\frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1\int_0^1\frac{\ln(1-xy)}{1+xy}dxdy+\int_0^1\int_0^1\frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1\left[\frac{\ln(1-xy)\ln(1+xy)}{x}\Big|_0^1+\int_0^1\frac{\ln(1+xy)}{1-xy}dy\right]dx+\int_0^1\int_0^1\frac{\ln(1+xy)}{1-xy}dxdy$$

$$I=\int_0^1\frac{\ln(1-x)\ln(1+x)}{x}dx\;+\;2\int_0^1\int_0^1\frac{\ln(1+xy)}{1-xy}dxdy$$

Recall that :

$$\int_0^1 \frac{\ln(1-x)\ln(1+x)}{x}dx = -\frac{5}{8}\zeta(3)$$

Hence :

$$\int_0^1\int_0^1 \frac{\ln(1+xy)}{1-xy}dxdy = \frac{\pi^2}{4}\ln(2) - \zeta(3)$$

$$I = -\frac{5}{8}\zeta(3) + 2\left[\frac{\pi^2}{4}\ln(2) - \zeta(3)\right] = \frac{\pi^2}{2}\ln(2) - \frac{21}{8}\zeta(3)$$

2455. Find a closed form:

$$\int_0^1 \left((ln^4(x) + x^4) + \frac{x^3}{3-x^3}\right)dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$I = \int_0^1 \left((ln^4(x) + x^4) + \frac{x^3}{3-x^3}\right)dx = \underbrace{\int_0^1 ln^4(x)dx}_{A} + \underbrace{\int_0^1 x^4dx}_{B} + \underbrace{\int_0^1 \frac{x^3}{3-x^3}dx}_{C}$$

$$A = \int_0^1 ln^4(x)dx \underbrace{\overset{\substack{\ln(x)=-p \\ x=e^{-p}}}{=}}_{\substack{dx=-e^{p}dp \\ [-\infty;0]}} \int_{-\infty}^0 (-p)^4.(-e^{-p})dp = \int_0^{+\infty} p^{5-1}e^{-p} = \Gamma(5) = 24$$

$$B = \int_0^1 x^4dx = \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}$$

$$C = \int_0^1 \frac{x^3}{3-x^3}dx = \frac{1}{3}\int_0^1 \frac{x^3}{1-(\frac{x}{\sqrt{3}})^3}dx = \frac{1}{3}\sum_{n=0}^{\infty}\int_0^1 (\frac{x}{\sqrt{3}})^{3n}.x^3\,dx =$$

$$\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{(\sqrt{3})^{3n}}\int_0^1 x^{3n+3}\,dx = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\left(\frac{x^{3n+4}}{3n+4}\right)\Big|_0^1 = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n}\cdot\frac{1}{(3n+4)} =$$

$$= -\frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$

Hence :

$$I = A + B + C = 24 + \frac{1}{5} - \frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)} = \frac{121}{5} - \frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$

$$\int_0^1 \left((ln^4(x)+x^4)+\frac{x^3}{3-x^3}\right)dx=\frac{121}{5}-\frac{4\sqrt{3\pi}\,.\,\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(-\frac{1}{6}\right)}$$

2456. Prove that:

$$\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-\pi x}dx=\frac{2}{\pi}+\frac{4\sqrt{2}\Gamma\left(\frac{5}{4}\right)^2}{\pi^{\frac{3}{2}}}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-\pi x}dx=\frac{2}{\pi}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right)?$$

$$\Omega=\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)}+1\right)e^{-x}dx\overset{x\to\pi x}{=}$$

$$=\frac{1}{\pi}\int_0^\infty\left(\sqrt{\coth(x)}+\sqrt{\tanh(x)}+2\right)e^{-x}dx$$

$$=\frac{2}{\pi}+\frac{1}{\pi}\int_0^\infty\left(\sqrt{\coth(x)}+\sqrt{\tanh(x)}\right)e^{-x}dx$$

Let: $t=\sqrt{\coth(x)}\Rightarrow x=\text{arccoth}(t^2)\Rightarrow dx=\frac{2t}{1-t^4}dt$, also: $e^{-\text{arccoth}(t^2)}=\sqrt{\frac{t^2-1}{t^2+1}}$

$$\Rightarrow\Omega=\frac{2}{\pi}+\frac{1}{\pi}\int_1^\infty\left(t+\frac{1}{t}\right)\sqrt{\frac{t^2-1}{t^2+1}}\frac{2t}{t^4-1}dt=\frac{2}{\pi}+\frac{2}{\pi}\int_1^\infty\frac{1}{\sqrt{t^4-1}}dt\overset{t\to\frac{1}{t}}{\longrightarrow}$$

$$\frac{2}{\pi}+\frac{2}{\pi}\int_0^1\frac{1}{\sqrt{1-t^4}}dt=$$

$$=\frac{2}{\pi}+\frac{1}{2\pi}\int_0^1(t^4)^{\frac{1}{4}-1}(1-t^4)^{\frac{1}{2}-1}d(t^4)=\frac{2}{\pi}+\frac{1}{2\pi}B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{2}{\pi}+\frac{1}{2\pi}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\because\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\pi\sqrt{2}\Rightarrow\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}\text{ and }\Gamma\left(\frac{5}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Rightarrow\Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{4\Gamma\left(\frac{5}{4}\right)}$$

$$\Rightarrow\Omega=\frac{2}{\pi}+\frac{\sqrt{\pi}}{2\pi}\frac{4\Gamma\left(\frac{5}{4}\right)}{\pi\sqrt{2}}4\Gamma\left(\frac{5}{4}\right)=\frac{2}{\pi}+\frac{4\sqrt{2}}{\pi\sqrt{\pi}}\Gamma^2\left(\frac{5}{4}\right),\text{ hence proved.}$$

2457. If $0<a\le b$ then:

$$e^a+e^{-a}+2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{x^2+1}\right)}dx \le e^b+e^{-b}$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$0<a\le b;\ c^a+c^{-a}+2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{x^2+1}\right)}dx \le c^b+c^{-b}$$

Since sinh convex on $\mathbb{R}_+ \Rightarrow \forall t\in\mathbb{R}_+, \sinh(t)\ge t\ge 0$

$$\forall t\in\mathbb{R}_+, \sinh^2(t)\ge t^2, \qquad \forall t\in\mathbb{R}_+, 1+\sinh^2(t)\ge 1+t^2$$

$$\forall t\in\mathbb{R}_+, \cosh^2(t)\ge 1+t^2$$

$$\forall t\in\mathbb{R}_+, \sqrt{\cosh(t)}\ge \sqrt[4]{1+t^2}$$

So $\Rightarrow \forall t\in\mathbb{R}_+, \frac{\sqrt{\cosh(t)}}{\sqrt[4]{1+t^2}}\ge 1$

The Cauchy – Schwarz inequality gives $\Rightarrow$

$$\forall x\in\mathbb{R}_+, \int_0^x \cosh(t)\,dt \int_0^x \frac{dt}{\sqrt{1+t^2}} \ge \left(\int_0^x \frac{\sqrt{\cosh(t)}}{\sqrt[4]{1+t^2}}\right)^2$$

$$\forall x\in\mathbb{R}_+, \int \cosh(t)\,dt \int_0^x \frac{dt}{\sqrt{1+t^2}} \ge \left(\int_0^x dt\right)^2$$

i.e: $\forall x\in\mathbb{R}_+, \sinh(x)\,\text{arcsinh}(x)\ge x^2$

So $\Rightarrow \forall x\in[a;b], \frac{x^2}{\text{arcsinh}(x)}\le \sinh(x)$

$$\int_a^b \frac{x^2}{\text{arcsinh}(x)}dx \le \int_a^b \sinh(x)\,dx$$

i.e: $\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le \cosh(a)-\cosh(b)$

$$2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le e^b+e^{-b}-e^a-e^{-a}$$

Finally $\Rightarrow c^a+c^{-a}+2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le c^b+c^{-b}$

2458. Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\sqrt{\frac{(k-1)k}{2k-1+2\sqrt{(k-1)k}}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$\Omega=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\sqrt{\frac{(k-1)k}{2k-1+2\sqrt{(k-1)k}}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=2}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{\left(\sqrt{k-1}+\sqrt{k}\right)^2}}\cdot\left(\sum_{k=1}^{n}\sqrt{k}\right)^{-1}$$

$$S_1=\sum_{k=2}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{k-1}+k}=\sum_{k=1}^{n}\frac{\sqrt{k\cdot(k-1)}}{\sqrt{k-1}+\sqrt{k}}=\sum_{k=2}^{n}\left(k\cdot\sqrt{k-1}-(k-1)\cdot\sqrt{k}\right)$$

$$=\sum_{k=2}^{n}\left((k-1)\cdot\sqrt{k-1}+\sqrt{k-1}-k\cdot\sqrt{k}+\sqrt{k}\right)$$

$$=\sum_{k=2}^{n}\left((k-1)\cdot\sqrt{k-1}-k\cdot\sqrt{k}\right)+\sum_{k=2}^{n}\left(\sqrt{k}+\sqrt{k-1}\right)$$

$$=-n\cdot\sqrt{n}+1+\left(2\sum_{k=2}^{n}\sqrt{k}-\sqrt{n-1}+1\right)\Rightarrow\sqrt{k-1}\le\int_{k-1}^{k}\sqrt{x}\le\sqrt{k}$$

$$\Rightarrow\sum_{k=2}^{n}\sqrt{k-1}\le\int_{1}^{n}\sqrt{x}\le\sum_{k=2}^{n}\sqrt{k}=S\Rightarrow S-\sqrt{n}+1\le\frac{2}{3}\cdot n\sqrt{n}-\frac{2}{3}\le S\Rightarrow$$

$$\Rightarrow S\sim\frac{2}{3}\cdot n\sqrt{n}$$

$$\Rightarrow\Omega=\frac{\frac{1}{n}\cdot\left(-n\sqrt{n}+2+2S-2\sqrt{n-1}+1\right)}{S}\Rightarrow$$

$$\Rightarrow\Omega\sim\frac{1}{n}\cdot\left(\frac{1}{3}\cdot n\sqrt{n}\right)\cdot\left(\frac{2}{3}\cdot n\sqrt{n}\right)^{-1}=0\Rightarrow\Omega=0$$

2459.

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right).\ \Gamma\left(\frac{x+1}{2}\right).\ \Gamma\left(\frac{1-x}{2}\right).\Gamma\left(\frac{2-x}{2}\right).\sin(\pi x)$$

Solve for real numbers:

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0$$

Proposed by Daniel Sitaru-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right).\ \Gamma\left(\frac{x+1}{2}\right).\ \Gamma\left(\frac{1-x}{2}\right).\Gamma\left(\frac{2-x}{2}\right).\sin(\pi x)$$

$$Note:\ \Gamma(x)\ .\ \Gamma(1-x)\ =\frac{\pi}{\sin(\pi x)}$$

$$\Gamma\left(\frac{x}{2}\right).\ \Gamma\left(\frac{2-x}{2}\right)=\Gamma\left(\frac{x}{2}\right).\ \Gamma\left(1-\frac{x}{2}\right)=\frac{\pi}{\sin(\frac{\pi}{2}x)}$$

$$\Gamma\left(\frac{x+1}{2}\right).\Gamma\left(\frac{1-x}{2}\right)=\Gamma\left(\frac{x+1}{2}\right).\Gamma\left(1-\frac{x+1}{2}\right)=\frac{\pi}{\sin(\frac{\pi}{2}-\frac{\pi}{2}x)}=\frac{\pi}{\cos(\frac{\pi}{2}x)}$$

$$\Omega(x)=\Gamma\left(\frac{x}{2}\right).\ \Gamma\left(\frac{x+1}{2}\right).\ \Gamma\left(\frac{1-x}{2}\right).\Gamma\left(\frac{2-x}{2}\right).\sin(\pi x)$$

$$=\frac{\pi}{\sin(\frac{\pi}{2}x)}.\frac{\pi}{\cos(\frac{\pi}{2}x)}.\sin(\pi x)=$$

$$\frac{2\pi^2}{\sin(2.\ \frac{\pi}{2}x)}.\sin(\pi x)=2\pi^2$$

$$x^2-\frac{4x}{\Omega(x)}+\frac{1}{\pi^4}=0\rightarrow x^2-\frac{4x}{2\pi^2}+\frac{1}{\pi^4}=0\rightarrow\left(x-\frac{1}{\pi^2}\right)^2=0$$

$$Answer:\ x=\frac{1}{\pi^2}$$

2460. Find a closed form:

$$\xi=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Abbaszade Yusif-Azerbaijan

$$\xi=\int_0^1\int_0^1\int_0^1\sum_{x,y,z}\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz \underset{symmetry}{\overset{by}{=}} 3\int_0^1\int_0^1\int_0^1\frac{\sqrt{xy}}{\sqrt{yz}+\sqrt{x}}dxdydz$$

$$a^2=x,b^2=y,c^2=z \qquad dx=2ada,dy=2bdb,dz=2cdc \qquad a,b,c\in[0,1]$$

$$\xi=3\int_0^1\int_0^1\int_0^1\frac{a^2b^2c}{a+bc}\times 8dadbdc$$

$$=24\int_0^1\int_0^1\int_0^1\frac{a^2b^2c}{a+bc}dadbdc \qquad \{a+bc=t,bdc=dt\}$$

$$\xi=24\int_0^1\int_0^1\int_a^{a+b}\frac{a^2b^2}{t}\times\frac{t-a}{b}\times\frac{dt}{b}dadb=24\int_0^1\int_0^1\int_a^{a+b}\frac{a^2(t-a)}{t}dtdadb$$

$$\xi=24\int_0^1\int_0^1\left(a^2(a+b-a)-a^3\ln\left(1+\frac{b}{a}\right)\right)dadb=$$

$$=24\left(\int_0^1\int_0^1 a^2bdadb-\int_0^1 a^3\ln\left(1+\frac{1}{a}\right)da+\int_0^1\int_0^1\frac{a^3b}{a+b}dadb\right) \ \{a+b=t\}$$

$$\xi=24\left(\frac{1}{6}-\frac{a^4}{4}\ln\left(1+\frac{1}{a}\right)_0^1+\int_0^1\frac{a^4}{4}\left(\frac{1}{1+a}-\frac{1}{a}\right)da+\int_0^1\int_a^{a+1}\frac{a^3}{t}\times(t-a)dtda\right)$$

$$\xi=24\left(\frac{1}{6}-\frac{1}{4}\ln(2)+\frac{1}{4}\ln(2)-\frac{1}{4}\times\frac{7}{12}-\frac{1}{16}+\int_0^1\left(a^3(a+1-a)-a^4\ln\left(1+\frac{1}{a}\right)\right)da\right)$$

$$\xi=24\left(\frac{1}{6}-\frac{7}{48}-\frac{1}{16}+\int_0^1 a^3da-\frac{a^5}{5}\ln\left(1+\frac{1}{a}\right)_0^1+\int_0^1\frac{a^5}{5}\left(\frac{1}{1+a}-\frac{1}{a}\right)da\right)$$

$$\xi=24\left(-\frac{1}{24}+\frac{1}{4}-\frac{1}{5}\ln(2)+\frac{1}{5}\times\frac{47}{60}-\frac{1}{5}\ln(2)-\frac{1}{25}\right)$$

$$\xi=24\left(\frac{39}{120}-\frac{2}{5}\ln(2)\right)$$

$$\xi=\frac{39}{5}-\frac{48}{5}\ln(2)$$

2461. **Find:**

$$\Omega=\sum_{k=1}^{\infty}\frac{H_k2^{-k}}{(k+1)}$$

Proposed by Vincenzo Dima-Italy

Solution by Shirvan Tahirov-Azerbaijan

$$\sum_{k=1}^{\infty}\frac{H_k2^{-k}}{(k+1)}=\sum_{k=1}^{\infty}\frac{H_k}{2^k(k+1)}=\sum_{k=1}^{\infty}\left(\frac{H_k}{2^k}\cdot\frac{1}{(k+1)}\right)=\sum_{k=1}^{\infty}\frac{H_k}{2^k}\int_0^1 x^k dx=$$

$$\int_0^1\left(\sum_{k=1}^{\infty}H_k\left(\frac{x}{2}\right)^k\right)dx=-\int_0^1\frac{\ln\left(1-\frac{x}{2}\right)}{\left(1-\frac{x}{2}\right)}dx=-\left(-\ln\left(1-\frac{x}{2}\right)\right)\Big|_0^1=$$

$$=-\left(-ln^2(2)\right)=ln^2(2)$$

Note :

$$\sum_{k=1}^{\infty}H_kx^k=-\frac{\ln(1-x)}{1-x}$$

Where H_k is the harmonic number.

2462. Prove that:

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{2^{6n-1}(4n-3)}\binom{4n}{2n}=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Pham Duc Nam-Vietnam

$$S=\sum_{n=1}^{\infty}\frac{F_{2n}}{2^{6n-1}(4n-3)}\binom{4n}{2n}=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$$

$$S=2\sum_{n=1}^{\infty}\frac{F_{2n}}{(8)^{2n}(4n-3)}\binom{4n}{2n}=\sum_{n=1}^{\infty}\sum_{k=1}^{2}e^{\pi ink}\frac{F_n}{(8)^n(2n-3)}\binom{2n}{n}$$

Now, using the ordinary generating function of central binomial coefficients:

$$\sum_{n=0}^{\infty}x^n\binom{2n}{n}=\frac{1}{\sqrt{1-4x}}\Rightarrow\sum_{n=1}^{\infty}x^{2n-4}\binom{2n}{n}=\frac{1}{x^4\sqrt{1-4x^2}}-\frac{1}{x^4}\Rightarrow$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{x^{2n-3}}{2n-3}\binom{2n}{n}=-\frac{8x^2\sqrt{1-4x^2}+\sqrt{1-4x^2}-1}{3x^3}+C$$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{x^{2n}}{2n-3}\binom{2n}{n}=-\frac{8x^2\sqrt{1-4x^2}+\sqrt{1-4x^2}-1}{3}\Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{2n-3}\binom{2n}{n} = -\frac{8x\sqrt{1-4x}+\sqrt{1-4x}-1}{3}$$

And the Binet's formula for Fibonacci sequence: $F_n = \frac{1}{\sqrt{5}}(\varphi^n - (1-\varphi)^n)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

$$\Rightarrow S = \frac{1}{\sqrt{5}}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\left(\left(\frac{\varphi e^{\pi i k}}{8}\right)^n - \left(\frac{(1-\varphi)e^{\pi i k}}{8}\right)^n\right)\frac{1}{2n-3}\binom{2n}{n}$$

$$= \frac{1}{\sqrt{5}}\sum_{k=1}^{2}\left(\begin{array}{c}\frac{1}{3}\left(e^{i\pi k}(1-\varphi)\sqrt{1+\frac{1}{2}e^{i\pi k}(\varphi-1)}+\sqrt{1+\frac{1}{2}e^{i\pi k}(\varphi-1)}-1\right)+\\ +\frac{1}{3}\left(-e^{i\pi k}\varphi\sqrt{1-\frac{1}{2}e^{i\pi k}\varphi}-\sqrt{1-\frac{1}{2}e^{i\pi k}\varphi}+1\right)\end{array}\right)$$

$$= \frac{1}{3}\frac{1}{\sqrt{5}}\left(\sqrt{\frac{1}{2}(2\sqrt{5}+5)}-\frac{1}{\sqrt{2}}\right) = \frac{1}{3}\frac{1}{\sqrt{5}}\sqrt{\left(\sqrt{\frac{1}{2}(2\sqrt{5}+5)}-\frac{1}{\sqrt{2}}\right)^2} =$$

$= \frac{1}{3}\frac{1}{\sqrt{5}}\sqrt{3+\sqrt{5}-\sqrt{2\sqrt{5}+5}} = \frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$, hence proved.

2463. Find a closed form:

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{\log(x^2)\log(y^2)\log(z^2)\log(xyz)}{1+xyz}\,dxdydz$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{\ln(x^2)\ln(y^2)\ln(z^2)\,xyz}{1+xyz}\,dxdydz =$$

$$\lim_{a\to 0}\frac{\partial}{\partial a}\int_0^1\int_0^1\int_0^1 \frac{\ln(x^2)\ln(y^2)\ln(z^2)\,(xyz)^a}{1+xyz}\,dxdydz =$$

$$8\lim_{a\to 0}\frac{\partial}{\partial a}\sum_{n\ge 0}(-1)^n\int_0^1\int_0^1\int_0^1 x^a\ln(x)\,y^a\ln(y)\,z^a\ln(z)\,(xyz)^n dxdydz=$$

$$8\lim_{a\to 0}\frac{\partial}{\partial a}\sum_{n\ge 0}(-1)^n\int_0^1\int_0^1\int_0^1 \ln(x)\ln(y)\ln(z)\,x^{a+n}y^{a+n}z^{a+n}dxdydz=$$

$$8\lim_{a\to 0}\frac{\partial}{\partial a}\sum_{n\ge 0}(-1)^n(\int_0^1 x^{a+n}\ln(x)\,dx+\int_0^1 y^{a+n}\ln(y)\,dx+\int_0^1 z^{a+n}\ln(z)\,dz)=$$

$$\Delta:\int_0^1 x^{a+n}\ln(x)\,dx\overset{I.B.P}{\cong}\left(\frac{\ln(x)\,.\,x^{n+a+1}}{n+a+1}\right)\begin{matrix}1\\0\end{matrix}$$

$$-\int_0^1\frac{x^{n+a}}{n+a+1}dx=0-\frac{1}{(n+a+1)}\int_0^1 x^{n+a}dx=-\frac{1}{(n+a+1)^2}$$

The other two integral's are solved in the same form ...

$$8\lim_{a\to 0}\frac{\partial}{\partial a}\sum_{n\ge 0}(-1)^n\left(\left(-\frac{1}{(n+a+1)^2}\right).\left(-\frac{1}{(n+a+1)^2}\right).\left(-\frac{1}{(n+a+1)^2}\right)\right)=$$

$$-8\lim_{a\to 0}\frac{\partial}{\partial a}\sum_{n\ge 0}\frac{(-1)^n}{(n+a+1)^6}=48\lim_{a\to 0}\sum_{n\ge 0}\frac{(-1)^n}{(n+a+1)^7}=48\sum_{n\ge 0}\frac{(-1)^n}{(n+1)^7}=$$

$$=48\sum_{n\ge 1}\frac{(-1)^{n+1}}{(n)^7}=48\eta(7)$$

Where $\eta(v)$ is the Dirichlet eta function.

2464. Prove the below closed form:

$$\int_0^{\frac{\pi}{4}}\frac{\sin(x)+\cos(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))(1+\tan(x))}dx=\ln\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)-2\sqrt{2}+3$$

Proposed by Ankush Kumar Parcha-India

Solution by Pham Duc Nam-Vietnam

$$\Omega=\int_0^{\frac{\pi}{4}}\frac{\sin(x)+\cos(x)+\tan(x)}{(1+\sin(x))(1+\cos(x))(1+\tan(x))}dx=?$$

*** By letting: $t=\tan(x)$, we have: $\Omega=\int_0^1\left(-\frac{t}{\sqrt{1+t^2}}+\frac{1}{1+\sqrt{1+t^2}}+\frac{t}{1+t}\right)dt$**

$$= \left(-\sqrt{1+t^2}+t-\ln(1+t)\right)\Big|_0^1 + \underbrace{\int_0^1 \frac{1}{1+\sqrt{1+t^2}}dt}_{\sqrt{1+t^2}=u+t} =$$

$$= 2-\sqrt{2}-\ln(2)+\int_{\sqrt{2}-1}^{1}\frac{u^2+1}{u(1+u)^2}du$$

$$= 2-\sqrt{2}-\ln(2)+\int_{\sqrt{2}-1}^{1}\left(\frac{1}{u}-\frac{2}{(1+u)^2}\right)du = 2-\sqrt{2}-\ln(2)+\left(\frac{2}{1+u}+\ln(u)\right)\Big|_{\sqrt{2}-1}^{1}$$

$$= 2-\sqrt{2}-\ln(2)-\sqrt{2}+1-\ln(\sqrt{2}-1) = 3-2\sqrt{2}-\ln(2\sqrt{2}-2) =$$

$$= 3-2\sqrt{2}+\ln\left(\frac{1}{2\sqrt{2}-2}\right) = \ln\left(\frac{1}{2}+\frac{1}{\sqrt{2}}\right)-2\sqrt{2}+3$$

2465. Prove that:

$$\int_{R^+}\frac{\log^2(x)\sin^3(x)}{x}dx = \frac{\pi\gamma^2}{4}-\frac{\pi\gamma}{4}\log(3)+\frac{\pi^3}{48}-\frac{\pi}{8}\log^2(3)$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\Omega = \int_0^\infty \frac{\log^2(x)}{x}\sin^3(x)\,dx = \lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty x^{a-1}\sin^3(x)dx =$$

$$= \lim_{a\to 0}\frac{d^2}{da^2}\int_0^\infty L_x^{-1}\{x^{a-1}\}(s)L_x\{\sin^3(x)\}(s)ds =$$

$$= \lim_{a\to 0}\frac{d^2}{da^2}\frac{6}{\Gamma(1-a)}\int_0^\infty \frac{s^{-a}}{(1+s^2)(9+s^2)}ds =$$

$$= \frac{3}{4}\lim_{a\to 0}\frac{d^2}{da^2}\frac{1}{\Gamma(1-a)}\left(\int_0^\infty \frac{s^{-a}}{s^2+1}ds - \int_0^\infty \frac{s^{-a}}{s^2+9}ds\right)$$

$$I(a) = \int_0^\infty \frac{s^{-a}}{1+s^2}ds, \quad \{s^2=t, \quad dt = 2\sqrt{t}ds, \quad t[0;\infty]\}$$

$$I(a) = \frac{1}{2}\int_0^\infty \frac{t^{\frac{1}{2}-\frac{a}{2}-1}}{(1+t)^{\frac{1}{2}-\frac{a}{2}+\frac{a}{2}+\frac{1}{2}}}dt = \frac{1}{2}\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)$$

$$J(a)=\int_0^\infty \frac{s^{-a}}{9+s^2}ds=\frac{1}{9}\int_0^\infty \frac{s^{-a}}{\frac{s^2}{9}+1}ds\ \left\{\frac{s^2}{9}=t,\quad dt=\frac{2}{3}\sqrt{t}ds,\quad t[0;\infty]\right\}$$

$$J(a)=\frac{3^{-a}}{6}\int_0^\infty \frac{t^{\frac{1}{2}-\frac{a}{2}-1}}{(1+t)^{\frac{1}{2}-\frac{a}{2}+\frac{1}{2}+\frac{a}{2}}}dt=\frac{3^{-a-1}}{2}\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)$$

$$\Omega=\frac{3}{8}\lim_{a\to 0}\frac{d^2}{da^2}\frac{(1-3^{-a-1})\beta\left(\frac{1}{2}-\frac{a}{2};\frac{1}{2}+\frac{a}{2}\right)}{\Gamma(1-a)}=\frac{3}{8}\lim_{a\to 0}\frac{d^2}{da^2}(1-3^{-1-a})\frac{\Gamma\left(\frac{1}{2}-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{a}{2}\right)}{\Gamma(1-a)}$$

$$note:\left\{\Gamma(1-\alpha)\Gamma(1+\alpha)=\pi\alpha\csc(\pi\alpha),\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)=\pi\sec\left(\frac{\pi\alpha}{2}\right)\right\}$$

$$\Omega=\frac{3\pi}{8}\lim_{a\to 0}\frac{d^2}{da^2}\frac{(1-3^{-1-a})\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}=\frac{3\pi}{8}(\Omega_1-\Omega_2)$$

$$\Omega_1=\lim_{a\to 0}\left(\frac{\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}\right)^{(2)}$$

$$=\lim_{a\to 0}\frac{\sec\left(\frac{\pi a}{2}\right)}{4\Gamma(1-a)}\left(4\psi^{(0)}(1-a)^2-4\psi^{(1)}(1-a)+\pi^2\left(\tan^2\left(\frac{\pi a}{2}\right)+\sec^2\left(\frac{\pi a}{2}\right)\right)\right.$$
$$\left.+4\pi\tan\left(\frac{\pi a}{2}\right)\psi^{(0)}(1-a)\right)=$$

$$=\psi^{(0)}(1)^2-\psi^{(1)}(1)+\frac{\pi^2}{4}=\gamma^2-\frac{\pi^2}{6}+\frac{\pi^2}{4}=\gamma^2+\frac{\pi^2}{12}$$

$$\Omega_2=\frac{1}{3}\lim_{a\to 0}\left(\frac{3^{-a}\sec\left(\frac{\pi a}{2}\right)}{\Gamma(1-a)}\right)^{(2)}$$

$$=\frac{1}{3}\lim_{a\to 0}\frac{3^{-a}\sec^3\left(\frac{\pi a}{2}\right)}{4\Gamma(1-a)}\left(\pi^2 sin^2\left(\frac{\pi a}{2}\right)+\pi^2+2\pi\sin(\pi a)\left(\psi^{(0)}(1-a)-\log(3)\right)\right.$$
$$\left.+4\cos^2\left(\frac{\pi a}{2}\right)\left(\psi^{(0)}(1-a)^2-\psi^{(1)}(1-a)-2\log(3)\psi^{(0)}(1-a)+\log^2(3)\right)\right)=$$

$$=\frac{1}{12}\left(\pi^2+4\psi^{(0)}(1)^2-4\psi^{(1)}(1)-8\log(3)\psi^{(0)}(1)+4\log^2(3)\right)=$$

$$=\frac{\pi^2}{12}+\frac{\gamma^2}{3}-\frac{\pi^2}{18}+\frac{2\gamma\log(3)}{3}+\frac{\log^2(3)}{3}=\frac{\pi^2}{36}+\frac{\gamma^2}{3}+\frac{2\gamma\log(3)}{3}+\frac{\log^2(3)}{3}$$

$$\Omega = \frac{3\pi}{8}(\Omega_1 - \Omega_2) = \frac{3\pi}{8}\left(\frac{\pi^2}{18} + \frac{2\gamma^2}{3} - \frac{2\gamma\log(3)}{3} - \frac{\log^2(3)}{3}\right) =$$

$$= \frac{\pi^3}{48} + \frac{\pi\gamma^2}{4} - \frac{\gamma\pi\log(3)}{4} - \frac{\pi\log^2(3)}{8}$$

2466.

$$\Omega_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{1}{\cos(x)} + \frac{1}{\sin(x)}\right)dx \ \ and$$

$$\Omega_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\ln\left(\frac{1}{\cos(y)} - \frac{1}{\sin(y)}\right)dy$$

$$Prove\ \ that:\ \Omega_1 + \Omega_2 = \frac{21}{64}\zeta(3) + \frac{9}{32}\pi^2\ln(2)$$

Proposed by Shirvan Tahirov-Azerbaijan, Ankush Kumar Parcha-India

Solution by Quadri Faruk Temitope-Nigeria

$$N = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{1}{\cos(x)} + \frac{1}{\sin(x)}\right)dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\frac{\sin(x)+\cos(x)}{\sin(x)\cos(x)}\right)dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)\cos(x))dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\cos(x))dx \ = A_1 + A_2 + A_3$$

$$This:$$

$$A_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x)+\cos(x))dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln\left(\sqrt{2}\sin\left(\frac{3}{4}\pi - x\right)\right)dx =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sqrt{2})\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin\left(\frac{3}{4}\pi - x\right))dx = \frac{1}{2}\ln(2)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} xdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{3}{4}\pi - x\right)\ln(sin(x))\,dx =$$

$$\frac{x^2}{4}\ln(2) + \frac{3}{4}\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln(\sin(x))\,dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))\,dx = \frac{1}{4}\ln(2)\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) + \frac{3}{4}\pi\left(\frac{G}{2} - \frac{\pi}{4}\ln(2)\right) -$$

$$\left(\frac{\pi}{8}G + \frac{21}{128}\zeta(3) - \frac{3}{32}\pi^2\ln(2)\right),$$

$$A_1 = \frac{\pi}{4}G - \frac{21}{128}\zeta(3) - \frac{3}{64}\pi^2\ln(2)$$

$$A_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\sin(x))dx = \frac{\pi}{8}G + \frac{21}{128}\zeta(3) - \frac{3}{32}\pi^2\ln(2)$$

$$A_3 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x\ln(\cos(x))dx \ = -\frac{\pi}{8}G - \frac{3}{32}\pi^2\ln(2) - \frac{35}{128}\zeta(3)$$

$$I=A_1-A_2-A_3=\left(\frac{\pi}{4}G-\frac{21}{128}\zeta(3)-\frac{3}{64}\pi^2\ln(2)\right)-\left(\frac{\pi}{8}G+\frac{21}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)-$$

$$\left(-\frac{\pi}{8}G-\frac{3}{32}\pi^2\ln(2)-\frac{35}{128}\zeta(3)\right)=\frac{\pi}{4}G-\frac{7}{128}\zeta(3)+\frac{9}{64}\pi^2\ln(2)$$

Hence:

$$M=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\ln\left(\frac{1}{\cos(y)}-\frac{1}{\sin(y)}\right)dy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\ln\left(\frac{\sin(y)-\cos(y)}{\sin(y)\cos(y)}\right)dy=$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\ln(\sin(y)-\cos(y))dy-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\ln(\sin(y)\cos(y))dy=B_1-B_2-B_3$$

Since the M − integral is symmetric with the N − integral, we can directly write the answer of the M − integral, the integral's are similar …

$$B_1=\frac{1}{4}\ln(2)\left(\frac{\pi^2}{4}-\frac{\pi^2}{16}\right)+\frac{3}{4}\pi\left(-\frac{G}{2}-\frac{\pi}{4}\ln(2)\right)-\left(-\frac{\pi}{8}G-\frac{35}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)$$

$$B_1=-\frac{\pi}{4}G+\frac{35}{128}\zeta(3)-\frac{3}{64}\pi^2\ln(2)\ ,\ \ M=B_1-B_2-B_3$$

$$M=\left(-\frac{3}{64}\pi^2\ln(2)-\frac{\pi}{4}G+\frac{35}{128}\zeta(3)\right)-\left(\frac{\pi}{8}G+\frac{21}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)-$$

$$\left(-\frac{\pi}{8}G-\frac{35}{128}\zeta(3)-\frac{3}{32}\pi^2\ln(2)\right)=\frac{49}{128}\zeta(3)-\frac{\pi}{4}G+\frac{9}{64}\pi^2\ln(2)$$

$$\textit{Then}:\Omega_1+\Omega_2=N+M$$

$$\Omega_1+\Omega_2=\frac{\pi}{4}G-\frac{7}{128}\zeta(3)+\frac{9}{64}\pi^2\ln(2)+\frac{49}{128}\zeta(3)-\frac{\pi}{4}G+\frac{9}{64}\pi^2\ln(2)$$

$$\Omega_1+\Omega_2=\frac{21}{64}\zeta(3)+\frac{9}{32}\pi^2\ln(2)$$

2467. Suppose:

$$f(x)=\int_0^{\infty}\ln(1+e^{-2t})\,dt$$

Prove without any software:

$$\frac{3}{8}\le f(x)\le\frac{1}{4}(1+\ln(2))$$

Proposed by Khaled Abd Imouti-Syria

Solution by Pham Duc Nam-Vietnam

$$*\ I=\int_0^{\infty}\ln(1+e^{-2t})\,dt,u=e^{-2t}\Rightarrow -2t=\ln(u)\Rightarrow dt=-\frac{1}{2}\frac{du}{u}\Rightarrow$$

$$\Rightarrow I=\frac{1}{2}\int_0^1\frac{\ln(1+u)}{u}du=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1u^{n-1}\,du=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{24}$$

*** We need to prove:** $\frac{3}{8} \le \frac{\pi^2}{24} \le \frac{1}{4}(1+\ln(2))$

a) $\frac{3}{8} \le \frac{\pi^2}{24} \Leftrightarrow \pi^2 \ge 9$

Since the series expansion of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots \Rightarrow$

$\Rightarrow \sin(x) \le x \ \ \forall x \in \mathbb{R} \Rightarrow \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \le \frac{\pi}{6} \Rightarrow \pi > 3 \Leftrightarrow \pi^2 \ge 9$**, done.**

b) $\frac{\pi^2}{24} \le \frac{1}{4}(1+\ln(2)) \Leftrightarrow \pi^2 \le 6 + 6\ln(2)$. **We will show that:** $\pi^2 \le 10$

Indeed, we have: $0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx = \frac{22}{7} - \pi \Rightarrow \pi < \frac{22}{7} \Rightarrow$

$\Rightarrow \pi^2 < \frac{484}{49} \le 10$**, true since:** $10 = \frac{490}{49}$

Now, we will show that: $10 \le 6 + 6\ln(2) \Leftrightarrow \ln(2) \ge \frac{2}{3}$,

Since the series expansion of: $\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots \Rightarrow$

$\Rightarrow \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \frac{2}{3} + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \cdots \Rightarrow \ln(2) = \frac{2}{3} + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \cdots \Rightarrow \ln(2) \ge \frac{2}{3}$**, true**

$\Rightarrow \pi^2 \le 10 \le 6 + 6\ln(2)$**, done. Combine all results we conclude that:**

$$\frac{3}{8} \le \frac{\pi^2}{24} \le \frac{1}{4}(1+\ln(2))$$

2468. Find:

$$\Omega = \lim_{n\to\infty} \log n \cdot \left(\sum_{k=2}^{n} \frac{\log k}{k}\right)^{\frac{1}{n}} \cdot \left(\int_0^n \frac{|\sin x|}{x}\right)^{-1}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Hikmat Mammadov-Azerbaijan

Euler – Maclaurin summation gives us

$$\sum_{k=2}^{n} \frac{\log k}{k} = \sum_{k=1}^{n} \frac{\log k}{k} \sim \frac{\log^2 n}{2}$$

as $n \to \infty$**, so that**

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n}\frac{\log k}{k}\right)^{\frac{1}{n}}=1.$$

We also have $\int_1^n \frac{|\sin x|}{x}dx \sim \frac{2}{\pi}\log n$ as $n\to\infty$ (proved below), and it follows that

$$\lim_{n\to\infty}\log n\left(\sum_{k=2}^{n}\frac{\log k}{k}\right)^{\frac{1}{n}}\left(\int_1^n\frac{|\sin x|}{x}dx\right)^{-1}=\frac{\pi}{2}$$

Let $m=\left\lfloor\frac{n}{\pi}\right\rfloor$ so that $m\pi\le n<m\pi+\pi$. We have

$$\int_1^n\frac{|\sin x|}{x}dx=\int_1^\pi\frac{|\sin x|}{x}dx+\sum_{k=2}^{m}\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{x}dx+\int_{m\pi}^{n}\frac{|\sin x|}{x}dx \quad (1)$$

Using $\int_{j\pi}^{j\pi+\pi}|\sin x|\,dx=2$, we can find the following upper and lower bounds for the terms on the RHS of (1):

$$0\le\int_1^\pi\frac{|\sin x|}{x}dx\le\log\pi,\ \frac{2}{\pi}\sum_{k=2}^{m}\frac{1}{k}\le\sum_{k=2}^{m}\int_{(k-1)\pi}^{k\pi}\frac{|\sin x|}{x}dx\le\frac{2}{\pi}\sum_{k=2}^{m}\frac{1}{k-1} \quad (2)$$

$$0\le\int_{m\pi}^{n}\frac{|\sin x|}{x}dx\le\int_{m\pi}^{m\pi+\pi}dx\le\frac{2}{m\pi+\pi}$$

Dividing both sides of (1) by $\log n$ and applying the bounds in (2), we get

$$\frac{2}{\pi\log n}\sum_{k=2}^{m}\frac{1}{k}\le\frac{1}{\log n}\int_1^n\frac{|\sin x|}{x}dx\le\frac{\log\pi}{\log n}+\frac{2}{\pi\log n}\sum_{k=1}^{m-1}\frac{1}{k}+\frac{2}{(m\pi+\pi)\log n}$$

Taking the limit as $n\to\infty$, the first and third terms on the RHS of (3) tend to 0. We also have the following asymptotic behavior for the harmonic sums, where γ is the Euler-Mascheroni constant,

$$\sum_{k=2}^{m}\frac{1}{k}\sim\gamma+\log m\le\gamma+\log\frac{n}{\pi}\sim\log n,\ \sum_{k=1}^{m-1}\frac{1}{k}\sim\gamma+\log(m-1)\le\gamma+\log\left(\frac{n}{\pi}-1\right)$$
$$\sim\log n$$

and it follows from the squeeze theorem applied to (3) that

$$\lim_{n\to\infty}\frac{1}{\log n}\int_1^n\frac{|\sin x|}{x}dx=\frac{2}{\pi}$$

2469. Find:

$$\Omega=\lim_{n\to\infty}\left(\frac{1}{\sqrt[n+1]{(n+1)!}}\cdot\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}}\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Adrian Popa – Romania

First, we will calculate:

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}}\overset{C.D.}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^n\cdot(n+1)\cdot n!}{n!\,(n+1)n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}=e$$

$$\Omega=\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(n+1)!}}\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}}=\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}\cdot\frac{1}{n+1}\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}}}$$

$$=\lim_{n\to\infty}e\sqrt[5]{\frac{(n+1)H_n}{nH_{n+1}(n+1)^5}}=e\cdot\lim_{n\to\infty}\sqrt[5]{\frac{H_n}{H_{n+1}\cdot n(n+1)^4}}=0$$

2470. Find:

$$\Omega=\lim_{x\to0}\frac{(1-2x)^{-\frac{3}{x}}-(1-3x)^{-\frac{2}{x}}}{x}$$

Proposed by Vasile Mircea Popa – Romania

Solution by Pham Duc Nam-Vietnam

$$L=\lim_{x\to0}\frac{(1-2x)^{-\frac{3}{x}}-(1-3x)^{-\frac{2}{x}}}{x}=\lim_{x\to0}\frac{e^{-\frac{3}{x}\ln(1-2x)}-e^{-\frac{2}{x}(1-3x)}}{x}=$$

$$=\lim_{x\to0}e^{-\frac{2}{x}\ln(1-3x)}\lim_{x\to0}\frac{e^{-\frac{3}{x}\ln(1-2x)+\frac{2}{x}\ln(1-3x)}-1}{x}$$

$$= e^6 \lim_{x\to 0} \frac{e^{-\frac{3}{x}\ln(1-2x)+\frac{2}{x}\ln(1-3x)}-1}{\left(-\frac{3}{x}\ln(1-2x)+\frac{2}{x}\ln(1-3x)\right)} \frac{\left(-\frac{3}{x}\ln(1-2x)+\frac{2}{x}\ln(1-3x)\right)}{x} =$$

$$= e^6 \lim_{x\to 0} \frac{2\ln(1-3x)-3\ln(1-2x)}{x^2}$$

Let: $K = \lim_{x\to 0} \frac{2\ln(1-3x)-3\ln(1-2x)}{x^2}, x \to -x$

$$\Rightarrow K = \lim_{x\to 0} \frac{2\ln(1+3x)-3\ln(1+2x)}{x^2} \Rightarrow 2K = \lim_{x\to 0} \frac{2\ln(1-9x^2)-3\ln(1-4x^2)}{x^2}$$

$$= 2\cdot 9 \lim_{x\to 0} \frac{\ln(1-9x^2)}{9x^2} - 3\cdot 4 \lim_{x\to 0} \frac{\ln(1-4x^2)}{4x^2} = 18+12 = -6 \Rightarrow K = -3$$

$$\Rightarrow L = -3e^6$$

$\because \lim_{x\to 0} \frac{e^x-1}{x} = 1, \lim_{x\to 0} \frac{\ln(1-x)}{x} = -1$ are basic limits.

2471.

Find the general form of the integral and if $a_1 a_2 a_3 \dots a_k = 1$, $2k > 2n+1$ prove that:

$$\xi = \int_0^\infty \frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)} dx \le$$

$$\le \frac{\pi}{2^k} \sum_{m=1}^{k} \frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\setminus m}(a_t - a_m)}$$

Proposed by Abbaszade Yusif-Azerbaijan

Solution by proposer

$$let\ f(z) = \frac{z^{2n}}{(z^2+a_1^2)(z^2+a_2^2)(z^2+a_3^2)\dots(z^2+a_k^2)} = \frac{z^{2n}}{\Pi_{i=1}^{k}(z^2+a_i^2)}$$

Consider : Γ $U(-R;R)$ is the anti − clockwise and semi − circular contour in the upper half of C plane

$$\oint_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx$$

$$\oint_C f(z)\,dz = 2\pi i \sum Res(f; poles\ of\ f) = 2\pi i \sum_{m=1}^{k} \frac{(a_m i)^{2n}}{2a_m i\ \Pi_{t\in[1;k]\setminus m}(a_t^2 - a_m^2)} =$$

$$= \pi \sum_{m=1}^{k} \frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}(a_t^2 - a_m^2)}$$

$$f(-x)=f(x)=>\int_{-R}^{R}f(x)\,dx=2\int_{0}^{R}f(x)\,dx=2\xi$$

$$z=Re^{i\theta},dz=izd\theta\ ,\theta[0;\pi]=>\int_{\Gamma}f(z)\,dz=\int_{0}^{\pi}\frac{\left(Re^{i\theta}\right)^{2n}}{\Pi_{i=1}^{k}\left(R^2e^{2i\theta}+a_i^2\right)}\times iRe^{i\theta}d\theta$$

$$=\int_{0}^{\pi}\frac{iR^{2n+1}e^{2in\theta+i\theta}}{\Pi_{i=1}^{k}\left(R^2e^{2i\theta}+a_i^2\right)}d\theta$$

$$\left|\int_{\Gamma}f(z)\,dz\right|\le\int_{\Gamma}|f(z)|\,dz=\int_{0}^{\pi}\frac{|i|\left|R^{2n+1}\right|\left|e^{i\theta(2n+1)}\right|}{\Pi_{i=1}^{k}\left|R^2e^{2i\theta}+a_i^2\right|}d\theta\le\int_{0}^{\pi}\frac{R^{2n+1}}{\Pi_{i=1}^{k}(R^2)}d\theta=\frac{R^{2n+1}}{R^{2k}}\pi$$
$$=0$$

$$2\xi=\pi\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}\left(a_t^2-a_m^2\right)}$$

$$\xi=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)(a_t+a_m)}\le\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}\left(2\sqrt{a_ta_m}(a_t-a_m)\right)}$$

$$=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{2^{k-1}a_m^{\frac{k-1}{2}}\times\sqrt{\frac{\Pi_{j=1}^{k}(a_j)}{a_m}}\ \Pi_{t\in[1;k]\setminus m}(a_t-a_m)}=$$

$$=\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{a_m^{\frac{k}{2}-1}\ \Pi_{t\in[1;k]\setminus m}(a_t-a_m)}=\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}$$

$$\int_0^\infty\frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx=\frac{\pi}{2}\sum_{m=1}^{k}\frac{(-1)^n a_m^{2n-1}}{\Pi_{t\in[1;k]\setminus m}\left(a_t^2-a_m^2\right)}$$

$$\int_0^\infty\frac{x^{2n}}{(x^2+a_1^2)(x^2+a_2^2)(x^2+a_3^2)\dots(x^2+a_k^2)}dx\le\frac{\pi}{2^k}\sum_{m=1}^{k}\frac{(-1)^n a_m^{\frac{4n-k}{2}}}{\Pi_{t\in[1;k]\setminus m}(a_t-a_m)}$$

2472. ***Let be*** $f,g:[2;+\infty)\to R^+:f^3(x))=x+3f(x),f(2)=2$
and $g(x+1)+4x+3=g(x)+4x^3+6x^2,g(2)=2$

Find : $\Omega=\frac{\int_2^{2024}g[f(x)]}{\int_2^{2024}f[g(x)]}$

Proposed by Bui Hong Suc-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

The solution of the functional equation:

$$g(x+1)+4x+3=g(x)+4x^3+6x^2,g(2)=2$$

is the polynomial

$$g(x) = ax^4 + bx^3 + cx^2 + dx + e$$

Let's find the coefficients a,b,c,d and e from here.Because of this

$$g(x+1) - g(x) = a((x+1)^4 - x^4) + b((x+1)^3 - x^3) + c((x+1)^2 - x^2) +$$
$$+d((x+1) - x) = 4ax^3 + (6a + 3b)x^2 + (4a + 3b + 2c)x + (a + b + c + d) =$$
$$= 4x^3 + 6x^2 - 4x - 3$$

From here

$$\begin{cases} 4a=4 \\ 6a+3b=6 \\ 4a+3b+2c=-4 \\ a+b+c+d=-3 \end{cases} \Rightarrow a = 1; b = 0; c = -4; d = 0$$

We have

$$g(x) = x^4 - 4x^2 + e\,, g(2) = 2 \Rightarrow g(x) = x^4 - 4x^2 + 2 = (x^2 - 2)^2 - 2$$
$$g(x) = (x^2 - 2)^2 - 2$$

The solution of the functional equation

$$f^3(x)) = x + 3f(x), f(2) = 2$$

is

$$f(x) = 2$$

So ,

$$g(x) = (x^2 - 2)^2 - 2\ \ , f(x) = 2$$

Therefore

$$g(f(x)) = (f^2(x) - 2)^2 - 2 = (4 - 2)^2 - 2 = 2$$
$$f(g(x)) = 2$$

Then

$$\Omega = \frac{\int_2^{2024} g[f(x)]}{\int_2^{2024} f[g(x)]} = 1$$

2473. Find all values $\alpha \in [-2024, 2024]$ such that:

$$\int_\alpha^{2024} (|2025x| - x^2 + 2026x)\, dx \le \alpha^2 + 2025\ (1)$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Khanh Hung Vu-Vietnam

Put $m = 2024$ for easy demonstration. We consider 2 cases:

Case 1. $-m \le \alpha < 0$

We have $\alpha < 0 < m$, so we have:

$$\int_\alpha^m (|(m+1)x| - x^2 + (m+2)x)\, dx = \int_0^m \left((2m+3)x - x^2\right)\, dx + \int_\alpha^0 (x - x^2)\, dx$$

$$\to \int_\alpha^m (|(m+1)x| - x^2 + (m+2)x)\, dx = \frac{m^2}{6}(4m+9) + \frac{1}{6}\alpha^2(2\alpha - 3)$$

So we can rewrite the inequality (1) as:

$$\frac{m^2}{6}(4m+9)+\frac{1}{6}\alpha^2(2\alpha-3)\le\alpha^2+m+1$$
$$\rightarrow\frac{\alpha^3}{3}-\frac{3\alpha^2}{2}+\frac{2m^3}{3}+\frac{3m^2}{2}-m-1\ge0$$

Consider the function $f(\alpha)=\frac{\alpha^3}{3}-\frac{3\alpha^2}{2}+\frac{2m^3}{3}+\frac{3m^2}{2}-m-1$ in range $[-m,0)$

We have $f'(\alpha)=\alpha^2-3\alpha=\alpha(\alpha-3)>0$.So the function $f(\alpha)$ is a increasing function in range $[-m,0)\rightarrow f(\alpha)\ge f(-m)\rightarrow f(\alpha)\ge\frac{m^3}{3}-m-1>0$.So for all $-m\le\alpha<0$, the inequality (1) is satisfied.

Case 2. $0\le\alpha\le m$

We have $0\le\alpha\le m$, so we have:

$$\int_\alpha^m(|(m+1)x|-x^2+(m+2)x)\,dx=\int_\alpha^m(x-x^2)\,dx=\frac{\alpha^3}{3}-\frac{\alpha^2}{2}+\frac{m^2}{2}-\frac{m^3}{3}$$

So we can rewrite the inequality (1) as:

$$\frac{\alpha^3}{3}-\frac{\alpha^2}{2}+\frac{m^2}{2}-\frac{m^3}{3}\le\alpha^2+m+1$$
$$\rightarrow\frac{\alpha^3}{3}-\frac{3\alpha^2}{2}-\frac{m^3}{3}+\frac{m^2}{2}-m-1\le0$$

Consider the function $g(\alpha)=\frac{\alpha^3}{3}-\frac{3\alpha^2}{2}-\frac{m^3}{3}+\frac{m^2}{2}-m-1$ in range $[0,m]$

We have $g'(\alpha)=\alpha^2-3\alpha=\alpha(\alpha-3)$. So $g'(\alpha)=0\rightarrow\alpha=3$

We have:

$$\begin{cases}g(0)=-\frac{m^3}{3}+\frac{m^2}{2}-m-1<0\\ g(3)=-\frac{m^3}{3}+\frac{m^2}{2}-m<0\\ g(m)=-m^2-m-1<0\end{cases}$$

$\rightarrow g(\alpha)\le0\ \forall\ \alpha\in[0,m]$. So for all $0\le\alpha\le m$, the inequality (1) is satisfied.

In conclusion, all real numbers $\alpha\in[-2024,2024]$ satisfy the inequality (1).

2474. Prove that:

$$\Omega=\int_0^\infty\int_0^1\frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy=-\frac{1}{48}(\pi^2-9\zeta(3))(4G+\pi(\ln(2)-1))$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_0^\infty\int_0^1\frac{x\ln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy=\int_0^1\frac{x\ln^2(x)}{(x+1)^2}dx\int_0^\infty\frac{\ln(y+1)}{(1+y^2)^2}dy=$$

$$\left(\int_0^1 \frac{ln^2(x)}{x+1}dx - \int_0^1 \frac{ln^2(x)}{(x+1)^2}dx\right)\int_0^\infty \frac{\ln(y+1)}{(1+y^2)^2}dy =$$

$$\left(\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{k-1}ln^2(x)dx + \sum_{n\in N}(-1)^n n\int_0^1 x^{n-1}ln^2(x)dx\right)$$

$$\cdot\left(\int_1^\infty + \int_0^1\right)\frac{\ln(y+1)}{(1+y^2)^2}dy = 2\left(\sum_{k\in N}\frac{(-1)^{k-1}}{k^3} - \sum_{k\in N}\frac{(-1)^{n-1}}{n^2}\right)\left(\int_1^\infty \frac{\ln(y+1)}{(1+y^2)^2}dy + \right.$$

$$\left.\int_0^1 \frac{\ln(y+1)}{(1+y^2)^2}dy\right) = 2(\eta(3)-\eta(2))\left(\int_0^1 \frac{\ln\left(\frac{1}{y}+1\right)}{\left(\frac{1}{y^2}+1\right)^2 y^2}dy + \int_0^1 \frac{\ln(y+1)}{(1+y^2)^2}dy\right) =$$

$$2\left((1-2^{1-3})\zeta(3)-(1-2^{1-2})\zeta(2)\right)\left(\int_0^1 \frac{y^2\ln(y+1)-\ln(y)}{(y^2+1)}dy + \int_0^1 \frac{\ln(y+1)}{(1+y^2)^2}dy\right)$$

$$=$$

$$\left(\frac{3\zeta(3)}{2}-\zeta(2)\right)\left(\int_0^1 \frac{\ln(y+1)}{1+y^2}dy\right.$$

$$\left.-\int_0^1 \frac{\ln(y+1)}{(1+y^2)^2}dy - \int_0^1 \frac{y^2\ln(y)}{(1+y^2)^2}dy + \int_0^1 \frac{\ln(y+1)}{(1+y^2)^2}dy\right) =$$

$$\left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\int_0^1 \frac{\ln(y+1)}{1+y^2}dy - \int_0^1 \frac{y^2\ln(y)}{(1+y^2)^2}dy\right) = \left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\int_0^1 \frac{\ln\left(\frac{2}{y+1}\right)}{y^2+1}dy\right.$$

$$-$$

$$-\frac{1}{2}\left(-\frac{y\ln(y)}{y^2+1}+\ln(y)\,tan^{-1}(y)\right)\Bigg|_0^1$$

$$\left.+\frac{1}{2}\int_0^1\left(-\frac{1}{1+y^2}+\frac{tan^{-1}(y)}{y}\right)dy\right) = \left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)$$

$$\left(\frac{\ln(2)}{2}\int_0^1 \frac{dy}{y^2+1} - \frac{1}{2}\int_0^1 \frac{dy}{y^2+1} + \frac{1}{2}\int_0^1 \frac{tan^{-1}(y)}{y}dy\right) = \left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\frac{\ln(2)}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot\frac{\pi}{4}\right.$$

$$+$$

$$\left.+\frac{Ti_2(1)}{2}\right) = \left(\frac{3\zeta(3)}{2}-\frac{\pi^2}{6}\right)\left(\pi\frac{\ln(2)}{8} - \frac{\pi}{8} + \frac{G}{2}\right)$$

$$\int_0^\infty\int_0^1 \frac{xln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy = -\frac{1}{48}(\pi^2-9\zeta(3))(4G+\pi(\ln(2)-1)$$

Solution 2 by Quadri Faruk Temitope-Nigeria

$$I = \int_0^\infty\int_0^1 \frac{xln^2(x)\ln(y+1)}{(x+1)^2(1+y^2)^2}dxdy = \int_0^1 \frac{xln^2(x)}{(x+1)^2}dx\int_0^\infty \frac{\ln(y+1)}{(1+y^2)^2}dy = A\ .\ B$$

$working\ \ or\ A:$

$$A=\int_0^1 \frac{xln^2(x)}{(x+1)^2}dx=-\sum_{n=1}^{\infty}(-1)^n n\int_0^1 x^n ln^2(x)dx=-\sum_{n=1}^{\infty}(-1)^n n\frac{d^2}{dn^2}(\int_0^1 x^n dx)=$$

$$-\sum_{n=1}^{\infty}(-1)^n n\left(\frac{2}{(1+n)^3}\right)=-2\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{(1+n)^3}\right)$$

$$=-2\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{(1+n)^2}-\frac{1}{(1+n)^3}\right)=$$

$$-2\sum_{n=1}^{\infty}\frac{(-1)^n}{(1+n)^2}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{(1+n)^3}=-2\left(\frac{\pi^2}{12}-1\right)+2\left(\frac{3}{4}\zeta(3)-1\right)=-\frac{\pi^2}{6}+2+\frac{3}{2}\zeta(3)-2=$$

$$=\frac{3}{2}\zeta(3)-\frac{\pi^2}{6}=\frac{1}{6}(9\zeta(3)-\pi^2)$$

$$A=\int_0^1 \frac{xln^2(x)}{(x+1)^2}dx=\frac{1}{6}(9\zeta(3)-\pi^2)$$

$working\ \ or\ \ B:$

$$B=\int_0^{\infty}\frac{\ln(y+1)}{(1+y^2)^2}dy\quad Recall\ \ that:\ln(y+1)=\int_0^1\frac{y}{1+xy}dy$$

$$\int_0^1\int_0^{\infty}\frac{y}{(1+xy)(1+y^2)^2}dxdy=\int_0^{\infty}\int_0^1\frac{x^2y}{(1+x^2)^2(y^2+1)}dxdy-$$

$$-\int_0^{\infty}\int_0^1\frac{x}{(1+x^2)^2(y^2+1)}dxdy+\int_0^{\infty}\int_0^1\frac{x}{(1+y^2)^2(x^2+1)}dxdy+$$

$$+\int_0^{\infty}\int_0^1\frac{y}{(1+y^2)^2(x^2+1)}dxdy-\int_0^{\infty}\int_0^1\frac{x^3}{(1+x^2)^2(xy+1)}dxdy$$

$$B=\int_0^{\infty}\frac{x^2}{(1+x^2)^2}dx\int_0^1\frac{y}{y^2+1}dy-\int_0^{\infty}\frac{dy}{y^2+1}\int_0^1\frac{x}{(1+x^2)^2}dx+$$

$$+\int_0^{\infty}\frac{dy}{(1+y^2)^2}\int_0^1\frac{x}{1+x^2}dx$$

$$+\int_0^{\infty}\frac{dx}{1+x^2}\int_0^1\frac{y}{(1+y^2)^2}dy-\int_0^{\infty}\int_0^1\frac{x^3}{(1+x^2)^2(xy+1)}dxdy$$

$$B=\left[\frac{1}{2}tan^{-1}(x)-\frac{x}{2(x^2+1)}\right]_0^{\infty}.\frac{1}{2}\ln(y^2+1)\Big|_0^1-tan^{-1}(y)\Big|_0^{\infty}.\frac{(-1)}{2(x^2+1)}\Big|_0^1+$$

$$+\frac{1}{2}\left[\frac{y}{y^2+1}+tan^{-1}(y)\right]_0^{\infty}-\frac{1}{2}\ln(x^2+1)\Big|_0^1+tan^{-1}(x)\Big|_0^{\infty}.\frac{(-1)}{2(y^2+1)}\Big|_0^1-$$

$$-\int_0^{\infty}\int_0^1\frac{x^3}{(1+x^2)^2(xy+1)}dxdy$$

$$B=\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)-\frac{\pi}{2}.\frac{1}{4}+\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)+\frac{\pi}{2}.\frac{1}{4}-\frac{\pi}{2}.\frac{1}{4}+\frac{G}{2}-\frac{\pi}{4}\left(\frac{\ln(2)}{2}\right)$$

$$B=\frac{\pi}{8}\ln(2)-\frac{\pi}{8}+\frac{G}{2}=\frac{1}{8}(4G+\pi(\ln(2)-1)$$

$$B=\int_0^{\infty}\frac{\ln(y+1)}{(1+y^2)^2}dy=\frac{1}{8}(4G+\pi(\ln(2)-1)$$

$$I=A\,.\,B=\int_0^1\frac{x\ln^2(x)}{(x+1)^2}dx\int_0^{\infty}\frac{\ln(y+1)}{(1+y^2)^2}dy=\left(\frac{1}{6}(9\zeta(3)-\pi^2)\right).(\frac{1}{8}(4G+\pi(\ln(2)-1)\,)$$

$$=-\frac{1}{48}(\pi^2-9\zeta(3))(4G+\pi(\ln(2)-1)$$

Note : $G-$ *Catalan's constant* , $\zeta(3)-$ *Apery's constant*

2475. Prove that:

$$\psi=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left(\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)+1\right)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$\therefore Si(z)=\int_0^1\frac{\sin t}{t}dt;\,Ci(z)=-\int_z^{\infty}\frac{\cos t}{t}dt=\gamma+\ln(z)+\int_0^z\frac{\cos t-1}{t}dt$$

$$\psi=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=2\int_0^1 x\ln(arcsin(1-x))dx-$$

$$-\int_0^1 x\ln(x^2+1)dx-2A-B$$

$$\therefore A=\int_0^1 x\ln(arcsin(1-x))dx\overset{1-x\to 1}{\cong}\int_0^1(1-x)\ln(arcsin(x))dx=$$

$$=\int_0^1\ln(arcsin(x))dx-\int_0^1 x\ln(arcsin(x))dx\overset{x=\sin t}{\cong}\underbrace{\int_0^1\ln(t)d(\sin t)}_{IBP}-$$

$$-\int_0^{\frac{\pi}{2}}\cos(t)\sin(t)\,ln(t)dt=sin(t)ln(t)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\sin t}{t}dt-\frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}}\sin(2t)ln(t)dt}_{2t\to t}=$$

$$=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+\frac{1}{4}\int_0^{\pi}\ln\left(\frac{t}{2}\right)d(\cos(t))=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+$$

$$+\frac{1}{4}\left\{\underbrace{\int_0^{\pi}\ln(t)d(\cos(t)-1)}_{IBP}-ln(2)\underbrace{\int_0^{\pi}d(\cos(t))}_{=-2}\right\}=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+$$

$$+\frac{1}{4}\left\{cos(t)-1)ln(t)\Big|_0^{\pi}-\int_0^z\frac{\cos t-1}{t}dt+2\ln(2)\right\}=\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)+\frac{1}{4}(-2ln(\pi)$$

$$-Ci(\pi)+\gamma+\ln(\pi)+2\ln(2))=\frac{1}{4}\left\{\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{4}\right)\right\}$$

$$\therefore B=\int_0^1 x\ln(x^2+1)dx\overset{IBP}{\cong}\frac{(x^2+1)\ln(x^2+1)}{2}\Big|_0^1-\int_0^1\frac{(x^2+1)x}{(x^2+1)}dx$$

$$=\ln(2)-\int_0^1 xdx=\ln(2)-\frac{1}{2}$$

$$Therefore:\psi=2A-B=2\left\{\frac{1}{4}\left\{\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{4}\right)\right\}\right\}-\left(\ln(2)-\frac{1}{2}\right)=$$

$$=\frac{1}{2}\left(1+\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)\right)$$

Solution 2 by Ankush Kumar Parcha-India

$$We\ have,\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx$$

$$=2\underbrace{\int_0^1 xlnsin^{-1}(1-x)dx}_{x\to1-x}-\underbrace{\int_0^1 xln(1+x^2)dx}_{x^2\to x}$$

$$=2\underbrace{\int_0^1(1-x)lnsin^{-1}dx}_{x\to\sin(x)}-\frac{1}{2}\underbrace{\int_0^1 ln(1+x)dx}_{I.B.P}\overset{\sin(2x)=2\sin(x)\cos(x)}{\cong}2\underbrace{\int_0^{\frac{\pi}{2}}ln(x)\cos(x)\,dx}_{I.B.P}-$$

$$\underbrace{\int_0^{\frac{\pi}{2}}ln(x)\sin(2x)\,dx}_{I.B.P}$$

$$-\left(\frac{\ln(1+x)}{2}\int dx\right)\Big|_0^1+\frac{1}{2}\int_0^1\frac{x+1-1}{x+1}dx=(2\ln(x)\int dsin(x))\Big|_0^{\frac{\pi}{2}}-$$

$$-2\int_0^{\frac{\pi}{2}}dSi(x)+\left(\frac{\ln(x)}{2}\int dcos(2x)\right)\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{dCi(x)}{2}-\frac{\ln(2)}{2}+\int_0^1\frac{dx}{2}-\frac{1}{2}\int_0^1\frac{dx}{1+x}$$

$$\left(\int_0^x\frac{1-\cos(t)}{t}dt=\gamma+\ln(x)-Ci(x),\ |Arg(x)|<\pi\right)$$

$$=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}\ln\left(\frac{\pi}{2}\right)-\lim_{x\to0}\frac{\ln(x)}{2}-\frac{Ci(\pi)}{2}+\frac{\gamma}{2}+\lim_{x\to0}\frac{\ln(2x)}{2}-\frac{\ln(2)}{2}+\frac{1}{2}-$$

$$\int_0^1\frac{dln(1+x)}{2}=\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}+\frac{3}{2}\ln\left(\frac{\pi}{2}\right)+\frac{\ln(2)}{2}-\ln(2)+\frac{1}{2}=$$

$$\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}+\ln\left(\frac{\pi\sqrt{\pi}}{2}\right)-\ln\left(\frac{2}{\sqrt{e}}\right)$$

$$,\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}+\ln\left(\frac{\pi}{4}\sqrt{\pi e}\right)$$

$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)=\psi$$

$$\psi=\frac{1}{2}\left(\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)+1\right)$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx\quad Recall\ That:\ \ln\left(\frac{A}{B}\right)=ln(A)-ln(B)$$

$$I=\underbrace{\int_0^1 x\ln\left(arcsin^2(1-x)\right)dx}_{A}-\underbrace{\int_0^1 x\ln(1+x^2)\,dx}_{B}$$

$$Working\ \ or\ \ A:$$

$$A=\int_0^1 x\ln\left(arcsin^2(1-x)\right)dx=2\int_0^1 x\ln(arcsin(1-x))dx\quad let:\ 1-x=x$$

$$A=2\int_0^1(1-x)\ln(arcsin(1-x))dx=2\int_0^1\ln(arcsin(x))dx$$

$$-2\int_0^1 x\ln(arcsin(x))dx$$

$$Very\ \ integration\ \ by\ \ parts:$$

$$A=2[xln(\sin^{-1}x))-Si(\sin^{-1}(x))]\begin{matrix}1\\0\end{matrix}-2[x(x\ln(\sin^{-1}(x))-Si(\sin^{-1}(x))]\begin{matrix}1\\0\end{matrix}-$$

$$-\int_0^1 x^2\ln(\sin^{-1}(x))dx+\int_0^1 x\,Si\,(\sin^{-1}(x))dx$$

$$A=2\left[\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)\right]-2\left[ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)-\frac{\gamma}{4}+\frac{1}{4}ln\left(\frac{\pi}{4}\right)+\frac{Ci(\pi)}{4}\right.$$

$$\left.+\sin\left(\frac{\pi}{2}\right)-\ln\left(\frac{\pi}{2}\right)\right]$$

$$A=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)+\frac{\gamma}{2}-\frac{1}{2}\ln\left(\frac{\pi}{4}\right)-\frac{Ci(\pi)}{2}$$

$$A=2\ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2\,\mathrm{Si}\left(\frac{\pi}{2}\right)-\frac{Ci(\pi)}{2}$$

- $Working\ \ or\ \ B:$

$$B=\int_0^1 x\ln(1+x^2)\,dx=-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1 x^{2n+1}dx$$

$$=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

$$B=-\frac{1}{2}[-\ln(2)]+\frac{1}{2}[\ln(2)-1]=\frac{ln(2)}{2}+\frac{ln(2)}{2}-\frac{1}{2}=ln(2)-\frac{1}{2}$$

$$This\ ,\qquad I=A-B$$

$$I=2ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)-\left[\ln(2)-\frac{1}{2}\right]$$

$$I=2ln\left(\frac{\pi}{2}\right)-\frac{1}{2}ln\left(\frac{\pi}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)-\ln(2)+\frac{1}{2}$$

$$Recall\ that:\ \ln A^B=B\ln A$$

$$\ln(A)+\ln(B)=ln(AB)$$

$$ln(A)-ln(B)=ln\left(\frac{A}{B}\right)$$

$$I=ln\left(\frac{\pi^2}{4}\right)-ln\left(\frac{\sqrt{\pi}}{2}\right)-\ln(2)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)+\frac{1}{2}$$

$$I=ln\left(\frac{\pi^{3/2}}{4}\right)+\frac{\gamma}{2}-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}Ci(\pi)+\frac{1}{2}$$

$$I=\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left[ln\left(\frac{\pi^3}{16}\right)+\gamma-4Si\left(\frac{\pi}{2}\right)-Ci(\pi)+1\right]$$

Solution 4 by Exodo Halcalias-Angola

$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\underbrace{\int_0^1 xlnarcsin^2(1-x)dx}_{\arcsin(1-x)\to x}-\underbrace{\int_0^1 xln(x^2+1)dx}_{x^2+1\to y}=$$

$$\int_0^{\frac{\pi}{2}}(1-\sin(x))dxcos(x)\ln(x^2)dx-\frac{1}{2}\int_1^2\ln(x)\,dx=2\underbrace{\int_0^{\frac{\pi}{2}}\cos(x)\ln(x)\,dx}_{I.B.P}-$$

$$2\int_0^{\frac{\pi}{2}}\sin(x)\cos(x)\ln(x)\,dx-\frac{1}{2}(yln(y)-y)\Big|_1^2$$

$$=2\left((\ln(x)\sin(x))\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\sin(x)}{x}dx\right)-$$

$$\int_0^{\frac{\pi}{2}}\ln(x)\sin(2x)\,dx\ [2x\to t]-\frac{1}{2}(2\ln(2)-2+1)=2(\ln\left(\frac{\pi}{2}\right)-sin\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right))-$$

$$\frac{1}{2}\int_0^{\pi}\sin(t)\ln\left(\frac{t}{2}\right)-\ln(2)+\frac{1}{2}=2(\left(\ln\left(\frac{\pi}{2}\right)-Si\left(\frac{\pi}{2}\right)\right)-\frac{1}{2}\underbrace{\int_0^{\pi}\sin(t)\ln(t)\,dt}_{I.B.P}+$$

$$\frac{1}{2}\int_0^{\pi}\sin(t)\ln(2)\,dt+\ln\left(\frac{1}{2}\right)+\frac{1}{2}$$

$$=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}\left(-\ln(t)\cos(t)\Big|_0^{\pi}+\int_0^{\pi}\frac{cost}{t}dt\right)$$

$$-\frac{\ln(2)}{2}\int_0^\pi dcost+\ln\left(\frac{1}{2}\right)+\frac{1}{2}=2\ln(\frac{\pi}{2})-2Si(\frac{\pi}{2})-\frac{1}{2}(\ln(\pi)$$
$$+\int_0^\pi\frac{dt}{t}+\int_0^\pi\frac{cost-1}{t}dt)-$$
$$\frac{\ln(2)}{2}(-2)+\ln\left(\frac{1}{2}\right)+\frac{1}{2}$$
$$=2\ln\left(\frac{\pi}{2}\right)-2Si\left(\frac{\pi}{2}\right)-\frac{1}{2}(\ln(\pi)+\ln(\pi)+\int_0^\pi\frac{cost-1}{t}dt)+\frac{1}{2}=$$
$$\frac{1}{2}\left(4\ln\left(\frac{\pi}{2}\right)-4Si\left(\frac{\pi}{2}\right)+1\right)-\frac{1}{2}(\ln(\pi)+Ci(\pi)-\gamma)$$
$$\int_0^1 x\ln\left(\frac{arcsin^2(1-x)}{x^2+1}\right)dx=\frac{1}{2}\left(\gamma-Ci(\pi)-4Si\left(\frac{\pi}{2}\right)+\ln\left(\frac{\pi^3}{16}\right)+1\right)$$

2476. Prove that:

$$\int_0^1(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2})dx=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Bui Hong Suc-Vietnam

$$Let:\ S_{n,k}=\int_0^1 x^{k-1}Li_n(x)dx\overset{I.B.P}{=}\frac{x^kLi_n(x)}{k}\Big|_0^1$$
$$-\frac{1}{k}\int_0^1 x^{k-1}Li_{n-1}(x)dx=\frac{Li_n(1)}{k}-\frac{1}{k}S_{n-1,k}=$$
$$\frac{\zeta(n)}{k}-\frac{1}{k}\left(\frac{\zeta(n-1)}{k}-\frac{1}{k}S_{n-2,k}\right)=\frac{\zeta(n)}{k}-\frac{\zeta(n-1)}{k^2}+\frac{1}{k^2}S_{n-2,k}$$
$$=\frac{\zeta(n)}{k}-\frac{\zeta(n-1)}{k^2}+\frac{\zeta(n-2)}{k^3}-$$
$$\frac{1}{k^3}S_{n-3,k}=\cdots=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{n-2}}{k^{n-1}}S_{1,k}=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+$$
$$\frac{(-1)^{n-1}}{k^{n-1}}\int_0^1 x^{k-1}\ln(1-x)\,dx=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{i-1}H_k}{k^n}$$
$$\therefore S_1=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx$$
$$=\sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^1 x^{2n}ln^2(x)dx\overset{I.B.P}{=}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{2n+1}(x^{2n+1}ln^2(x)$$
$$-$$

$$\left.\frac{2x^{2n+1}\ln(x)}{2n+1}+\frac{2x^{2n+1}}{(1+2n)^2})\right|_0^1$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+2)}{(1+2n)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1+1)}{(1+2n)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+2n)^3}=\beta(2)+\beta(3)=G+\frac{\pi^3}{32}$$

$$Then:\ \Omega=\int_0^1(x^{k-1}\sum_{a=1}^{\infty}\frac{x^a}{a^n}+\frac{ln^2(x)}{(1+x^2)^2})dx=\int_0^1 x^{k-1}Li_n(x)dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=$$

$$S_{n,k}+S_1=\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\zeta(n+1-i)}{k^i}+\frac{(-1)^{i-1}H_k}{k^n}+G+\frac{\pi^3}{32}$$

$$As\ n=3,k=2\ :\ \Omega=\int_0^1(x\sum_{a=1}^{\infty}\frac{x^a}{a^3}+\frac{ln^2(x)}{(1+x^2)^2})dx$$

$$=\int_0^1 xLi_3(x)dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=$$

$$\sum_{i=1}^{\infty}(-1)^{i-1}\frac{\zeta(4-i)}{2^i}+\frac{H_2}{8}+G+\frac{\pi^3}{32}=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 2 by Exodo Halcalias-Angola

$$\int_0^1(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2})dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n+1}dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}+$$

$$\left(\frac{ln^2(x)}{2}\left(\frac{x}{x^2+1}+tan^{-1}(x)\right)\right)\Big|_0^1-\int_0^1\frac{\ln(x)}{x^2+1}dx-\int_0^1\frac{\ln(x)\,tan^{-1}(x)}{x}dx=$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n(n+2)}-\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{2n-2}\ln(x)\,dx-$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\int_0^1 x^{2n-2}\ln(x)\,dx=\frac{\zeta(3)}{2}-\frac{\zeta(2)}{4}+\frac{1}{4}\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n}dx+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}=\frac{\zeta(3)}{2}-\frac{1}{4}.\frac{\pi^2}{6}-\frac{1}{4}\int_0^1 xln(1-x)dx\ +\ \beta(2)+\beta(3)=\frac{\zeta(3)}{2}-\frac{\pi^2}{24}-$$

$$\frac{1}{4}.(2(x^2-1)\ln(1-x)-x(x+2))\Big|_0^1+G+\frac{\pi^3}{32}=\frac{\zeta(3)}{2}-\frac{\pi^2}{24}+G+\frac{\pi^3}{32}+\frac{3}{16}$$

$$\therefore\int_0^1(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2})dx=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$I=\int_0^1(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2})dx=\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n^3}dx+\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=A+B$$

Working or A:

$$A=\int_0^1 x\sum_{n=1}^{\infty}\frac{x^n}{n^3}dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n+1}dx=\sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\frac{1}{(n+2)}=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}=$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}+\frac{1}{8}\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+2}\right]=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{2}\left(\frac{1}{8}\right)=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}$$

$$A=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}$$

Working or B :

$$B=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx\ \left\{dv=\frac{1}{(1+x^2)^2},\quad v=\frac{1}{2}\left(\frac{x}{1+x^2}+tan^{-1}(x)\right\}\right.$$

$$\left\{u=ln^2(x),\quad \frac{du}{dx}=\frac{2\ln(x)}{x}\right\}$$

$$B=\int_0^1\frac{ln^2(x)}{(1+x^2)^2}dx=\left(\frac{1}{2}\left(\frac{x}{x^2+1}+tan^{-1}(x)\right)ln^2(x)\right)\Big|_0^1-\int_0^1\frac{\ln(x)}{1+x^2}dx-$$

$$\int_0^1\frac{\ln(x)tan^{-1}(x)}{x}dx=-\int_0^1\frac{\ln(x)}{1+x^2}dx-\int_0^1\frac{\ln(x)tan^{-1}(x)}{x}dx=$$

$$-\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)dx-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1)}\int_0^1 x^{2n+1-1}\ln(x)\,dx=$$

$$-\sum_{n=0}^{\infty}(-1)^n\frac{d}{dn}\left(\int_0^1 x^{2n}dx\right)-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\frac{d}{dn}\left(\int_0^1 x^{2n}dx\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}+$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}=G+\frac{\pi^3}{32}$$

$$I=A+B=\frac{\zeta(3)}{2}-\frac{1}{4}\zeta(2)+\frac{3}{16}+G+\frac{\pi^3}{32}=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

$$\therefore\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\frac{ln^2(x)}{(1+x^2)^2}\right)dx=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{3}{16}$$

Solution 4 by Abbaszade Yusif-Azerbaijan

$$\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\left(\frac{\ln x}{1+x^2}\right)^2\right)=\int_0^1\sum_{n=1}^{\infty}\frac{x^{n+1}}{n^3}dx+\int_0^1\left(\frac{\ln x}{1+x^2}\right)^2dx=\xi_1+\xi_2$$

$$\xi_1=\int_0^1\sum_{n=1}^{\infty}\frac{x^{n+1}}{n^3}dx=\sum_{n=1}^{\infty}\frac{1}{n^3(n+2)}=\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{n^2-2n+4}{n^3}-\frac{1}{n+2}\right)$$

$$\xi_1=\frac{1}{8}\sum_{n=1}^{\infty}\left(\left(\frac{1}{n}-\frac{1}{n+2}\right)-\frac{2}{n^2}+\frac{4}{n^3}\right)=\frac{1}{8}\left(1+\frac{1}{2}-2\times\frac{\pi^2}{6}+4\zeta(3)\right.$$

$$\xi_1=\frac{1}{8}\left(\frac{3}{2}-\frac{\pi^2}{3}+4\zeta(3)\right)=\frac{3}{16}-\frac{\pi^2}{24}+\frac{\zeta(3)}{2}$$

$$\xi_2=\int_0^1\left(\frac{\ln x}{1+x^2}\right)^2dx=\frac{\ln^2 x}{2}\left(\frac{x}{1+x^2}+\arctan x\right)_0^1-\int_0^1\left(\frac{\ln x}{1+x^2}+\frac{\ln x\arctan x}{x}\right)dx$$

$$\xi_2=-\int_0^1\frac{\ln x}{1+x^2}dx-\ln x\,Ti_2(x)_0^1+\int_0^1\frac{Ti_2(x)}{x}dx\qquad x=\tan(\theta)$$

$$\xi_2=-\int_0^{\frac{\pi}{4}}\frac{\ln(\tan\theta)}{\sec^2\theta}\times\sec^2\theta\,d\theta+Ti_3(1)=\frac{\pi^3}{32}-\int_0^{\frac{\pi}{4}}\ln(tan\,\theta)\,d\theta=\frac{\pi^3}{32}+G$$

$$\int_0^1\left(x\sum_{n=1}^{\infty}\frac{x^n}{n^3}+\left(\frac{\ln x}{1+x^2}\right)^2\right)dx=\xi_1+\xi_2=G+\frac{\zeta(3)}{2}+\frac{\pi^3}{32}-\frac{\pi^2}{24}+\frac{16}{3}$$

Note Section :

$$\int_0^x\frac{Ti_{n-1}(t)}{t}dt=Ti_n(x)\ ,\quad \int_0^{\frac{\pi}{4}}\ln(tanx)\,dx=-G$$

2477. Prove that:

$$\int_1^{\infty}\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx=G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_1^{\infty}\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx\overset{x\to\frac{1}{x}}{=}\int_0^1\left(\frac{1}{x^2}-\frac{2}{x}\ln(x)+ln^2(x)\right)\frac{x^2}{(1+x^2)(1+x)^2}dx$$

$$=\int_0^1\frac{1}{(1+x^2)(1+x)^2}dx-2\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx+$$

$$\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx\ \rightarrow\ H=H_1-2H_2+H_3$$

$$H_1=\int_0^1\frac{1}{(1+x^2)(1+x)^2}dx$$

$$=\frac{1}{2}\int_0^1\frac{dx}{x+1}+\frac{1}{2}\int_0^1\frac{dx}{(1+x)^2}-\frac{1}{2}\int_0^1\frac{x}{x^2+1}dx=\frac{1}{2}\int_0^1d\ln(1+x)+\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{4}\int_0^1dln((1+x))$$

$$=$$

$$=\frac{\ln(2)}{2}+\frac{1}{2}\cdot\frac{1}{2}-\frac{\ln(2)}{4}=\frac{\ln(2)}{4}+\frac{1}{4}$$

$$H_2=\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx=$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x^2}dx$$

$$=\frac{1}{2}\int_0^1\frac{\ln(x)}{(1+x)^2}dx$$

$$=\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1x^{2k-2}\ln(x)\,dx-\frac{1}{2}\sum_{k\in N}(-1)^{k-1}k\int_0^1x^{k-1}\ln(x)\,dx=$$

$$-\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}}{(2k-1)^2}+\frac{1}{2}\sum_{k\in N}\frac{(-1)^{k-1}k}{k^2}$$
$$=-\frac{\beta(2)}{2}+\frac{1}{2}\int_0^1\sum_{k\in N}(-1)^{k-1}x^{k-1}dx=-\frac{G}{2}+\frac{1}{2}\int_0^1\frac{1}{1+x}dx=$$
$$-\frac{G}{2}+\frac{1}{2}\int_0^1 dl(1+x)=-\frac{G}{2}+\frac{\ln(2)}{2}$$
$$H_3=\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx=\frac{1}{2}\int_0^1\frac{xln^2(x)}{1+x^2}dx$$
$$-\frac{1}{2}\int_0^1\frac{ln^2(x)}{x+1}dx+\int_0^1\frac{ln^2(x)}{(1+x)^2}dx=\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{2k-1}ln^2(x)dx-$$
$$\frac{1}{2}\sum_{k\in N}(-1)^{k-1}\int_0^1 x^{k-1}ln^2(x)dx+\frac{1}{2}\sum_{k\in N}(-1)^{k-1}k\int_0^1 x^{k-1}ln^2(x)dx$$
$$=\sum_{k\in N}\frac{(-1)^{k-1}}{(2k)^3}-\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}+\sum_{k\in N}\frac{k(-1)^{k-1}}{k^3}=$$
$$-\frac{7}{8}\sum_{k\in N}\frac{(-1)^{k-1}}{k^3}+\sum_{k\in N}\frac{(-1)^{k-1}}{k^2}=-\frac{7}{8}(1-2^{1-3})\zeta(3)+(1-2^{1-2})\zeta(2)=\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)$$
$$H=H_1-2H_2+H_3=\frac{\ln(2)}{4}+\frac{1}{4}-\frac{G}{2}+\frac{\ln(2)}{2}+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)$$
$$=G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

Solution 2 by Ankush Kumar Parcha-India

$$\int_1^\infty\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx\overset{x\to\frac{1}{x}}{\cong}\int_0^1\frac{(1-xln(x))^2}{(1+x^2)(1+x)^2}dx$$
$$=\int_0^1\frac{x^2ln^2(x)}{(1+x^2)(1+x)^2}dx-2\int_0^1\frac{xln(x)}{(1+x^2)(1+x)^2}dx+\int_0^1\frac{1}{(1+x^2)(1+x)^2}dx$$
$$=\frac{1}{2}\int_0^1\frac{xln^2(x)}{1+x^2}-\frac{1}{2}\int_0^1\frac{ln^2(x)}{x+1}dx+\frac{1}{2}\int_0^1\frac{ln^2(x)}{(1+x)^2}dx+\int_0^1\frac{\ln(x)}{(1+x)^2}dx-\int_0^1\frac{\ln(x)}{1+x^2}dx-\frac{1}{2}\int_0^1\frac{x}{1+x^2}dx+$$
$$\frac{1}{2}\int_0^1\frac{dx}{1+x}+\frac{1}{2}\int_0^1\frac{dx}{(1+x)^2}$$
$$\left(\because\sum_{n\in N}n(-x)^n=-\frac{x}{(1+x)^2},\quad |x|<1\right)$$

$$\frac{1}{2}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n-1}ln^2(x)dx-\frac{1}{2}\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{n-1}ln^2(x)dx+\frac{1}{2}\sum_{n\in N}(-1)^{n-1}n\int_0^1 x^{n-1}ln^2(x)dx+$$

$$+\sum_{n\in N}(-1)^{n-1}n\int_0^1 x^{n-1}\ln(x)\,dx-\sum_{n\in N}(-1)^{n-1}\int_0^1 x^{2n}\ln(x)\,dx-\int_0^1\frac{d(1+x^2)}{4}+\int_0^1\frac{d(1+x)}{2}$$

$$-\frac{1}{2}\int_0^1 d\left(\frac{1}{1+x}\right)$$

$$(\because\int_0^1 t^m ln^n(t)dt=\frac{(-1)^n.\ n!}{(m+1)^{n+1}}, n>-1\bigwedge m\neq 1)$$

$$=\frac{1}{8}\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}-\sum_{n\in N}\frac{(-1)^{n-1}}{n^3}+\sum_{n\in N}\frac{(-1)^{n-1}}{n^2}+\underbrace{\sum_{n\in N}\frac{(-1)^n}{(2n+1)^2}}_{G(Catalan's\ constant)}-\frac{\ln(2)}{4}+\frac{\ln(2)}{2}+\frac{1}{4}$$

$$(\because\sum_{n\in N}\frac{x^n}{n}=-\ln(1-x),\quad |x|\leq 1\bigwedge x\neq 1)$$

$$\int_1^\infty\frac{(\ln(x)+x)^2}{(1+x^2)(1+x)^2}dx=G+\frac{\zeta(2)}{2}-\frac{21}{32}\zeta(3)-\frac{3}{4}\ln(2)+\frac{1}{4}$$

$G\to Catalan's\ constant$, $\zeta(3)\to Apery's\ constant$

2478.

$$Let:\ a\in R_{\geq 0};m,n,k,b\in Z^+$$

$$Find:S=\int_0^1\left(x^k Li_n(x)+\frac{ln^m(x)}{(1+x^a)^b}\right)dx$$

Proposed by Bui Hong Suc-Vietnam

Inspired by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$\int_0^1\left(x^k Li_n(x)+\frac{ln^m(x)}{(1+x^a)^b}\right)dx=\int_0^1 x^k Li_n(x)dx+\int_0^1\frac{ln^m(x)}{(1+x^a)^b}dx=I_1+I_2$$

$$Working\ or\ I_1$$

$$I_2=\int_0^1\frac{ln^m(x)}{(1+x^a)^b}dx\ \ y=x^a,\qquad x=y^{1/a},\qquad dx=\frac{dy}{ay^{1-\frac{1}{a}}}$$

$$I_2=\int_0^1\frac{ln^m\left(y^{\frac{1}{a}}\right)}{(1+y)^b}\frac{dy}{ay^{1-\frac{1}{a}}}=\frac{1}{a}\int_0^1\frac{\left(\frac{1}{a}\right)^m}{(1+y)^b y^{1-\frac{1}{a}}}ln^m(y)$$

$$Recall\ that:(1+y)^b=\sum_{t=0}^\infty\binom{-b}{t}y^t$$

$$I_2=\frac{\left(\frac{1}{a}\right)^m}{a}\sum_{t=0}^{\infty}\binom{-b}{t}\int_0^1 y^t\frac{ln^m(y)}{y^{1-\frac{1}{a}}}dy=a^{-m-1}\sum_{t=0}^{\infty}\binom{-b}{t}\int_0^1 y^{t+\frac{1}{a}-1}ln^m(y)dy$$

$$=\left(\frac{1}{a}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}\underbrace{\int_0^1 y^{t+\frac{1}{a}-1}ln^m(y)\,dy}_{A}$$

Working on A

$$A=\int_0^1 y^{t+\frac{1}{a}-1}ln^m(y)dy\quad let:\ln(y)=p\ \ dy=-e^{-p},\qquad y=e^{-p}\ \ [\infty;0]$$

$$\int_\infty^0 (e^{-p})^{t+\frac{1}{a}-1}(-p)^m.-e^{-p}dp$$

$$=\int_0^\infty(-p)^m(e^{-p})^{(t+\frac{1}{a}-1+1)}dp=(-1)^m\int_0^\infty p^m e^{-p(t+\frac{1}{a})}dp$$

$$let\ :\ p\left(t+\frac{1}{a}\right)=x\ \ ;\ p=\frac{x}{t+\frac{1}{a}}\ :dp=\frac{dx}{t+\frac{1}{a}}\quad[0;\infty]$$

$$A=(-1)^m\int_0^\infty\left(\frac{x}{t+\frac{1}{a}}\right)^m e^{-p}\frac{dx}{t+\frac{1}{a}}$$

$$=\frac{(-1)^m}{(t+\frac{1}{a})^{m+1}}\int_0^\infty x^m e^{-x}dx=\frac{(-1)^m}{(t+\frac{1}{a})^{m+1}}\int_0^\infty x^{m+1-1}e^{-x}dx=$$

$$\frac{(-1)^m}{\left(t+\frac{1}{a}\right)^{m+1}}\Gamma(m+1)\ \ \textit{Note that}:$$

$$\int_0^\infty x^{m-1}e^{-x}=\Gamma(m)\quad\Gamma(m+1)=m\Gamma(m)\quad A=\frac{(-1)^m m\Gamma(m)}{\left(t+\frac{1}{a}\right)^{m+1}}$$

Hence :

$$I_2=\left(\frac{1}{a}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}\underbrace{\int_0^1 y^{t+\frac{1}{a}-1}ln^m(y)\,dy}_{A}=\left(\frac{1}{a}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}\frac{(-1)^m m\Gamma(m)}{\left(t+\frac{1}{a}\right)^{m+1}}dy=$$

$$\left(\frac{1}{a}\cdot\frac{1}{t+\frac{1}{a}}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}(-1)^m m\Gamma(m)=\left(\frac{1}{a}\cdot\frac{1}{at+1}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}(-1)^m m\Gamma(m)=$$

$$\left(\frac{1}{at+1}\right)^{m+1}\sum_{t=0}^{\infty}\binom{-b}{t}(-1)^m m\Gamma(m)=(-1)^m m\Gamma(m)\sum_{t=0}^{\infty}\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$

$$I_2=(-1)^m m\Gamma(m)\sum_{t=0}^{\infty}\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$

Working on I_1

$$I_1=\int_0^1 x^k Li_n(x)dx \quad u=Li_n(x),\qquad \frac{du}{dx}=\frac{Li_{n-1}(x)}{x},\qquad dv=x^k,\qquad v=\frac{x^{k+1}}{k+1}$$

$$I_1=\int_0^1 x^k Li_n(x)dx$$

$$=\frac{x^{k+1}}{k+1}Li_n(x)\Big|_0^1$$

$$-\int_0^1\frac{Li_{n-1}(x)x^{k+1}}{(k+1)x}dx=\frac{\zeta(n)}{k+1}-\frac{1}{k+1}\left[\frac{Li_{n-1}(x)x^{k+1}}{k+1}\Big|_0^1-\frac{1}{k+1}\int_0^1 x^k Li_{n-2}(x)dx\right]=$$

$$I_1=\frac{\zeta(n)}{k+1}-\frac{\zeta(n)}{(k+1)^2}+\cdots+(-1)^n\frac{\zeta(2)}{(k+1)^{n-1}}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$=\sum_{p=1}^{n-1}(-1)^{p-1}\frac{\zeta(n-p+1)}{(k+1)^p}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$I=I_1+I_2=\sum_{p=1}^{n-1}(-1)^{p-1}\frac{\zeta(n-p+1)}{(k+1)^p}+(-1)^{n-1}\frac{H_{k+1}}{(k+1)^n}$$

$$+(-1)^m m\Gamma(m)\sum_{t=0}^{\infty}\binom{-b}{t}\frac{1}{(at+1)^{m+1}}$$

2479. Prove the below closed form

$$\Omega=\int_0^1\int_0^1\int_0^1\log\left(\frac{1}{x^y y^z z^x}\right)\frac{dxdydz}{1+xyz}=\frac{9}{4}\zeta(3)-\frac{1}{4}\zeta(2)+3\log\left(\frac{4}{e}\right)$$

Where, $\zeta(3)$ is the Apery's constant.

Proposed by Ankush Kumar Parcha-India

Solution by Togrul Ehmedov-Azerbaijan

$$\Omega=\int_0^1\int_0^1\int_0^1\log\left(\frac{1}{x^y y^z z^x}\right)\frac{dxdydz}{1+xyz}=-\int_0^1\int_0^1\int_0^1\log(x^y y^z z^x)\frac{dxdydz}{1+xyz}=$$

$$=-3\int_0^1\int_0^1\int_0^1\log(x^y)\frac{dxdydz}{1+xyz}=-3\int_0^1\int_0^1\int_0^1 y\log(x)\frac{dxdydz}{1+xyz}=$$

$$=-3\sum_{k=0}^{\infty}(-1)^k\int_0^1\int_0^1\int_0^1 y^{k+1}z^k x^k\log(x)\,dx\,dy\,dz=$$

$$=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\int_0^1\int_0^1 y^{k+1}z^k\,dy\,dz=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2(k+2)}\int_0^1 z^k dz=$$

$$=3\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^3(k+2)}=3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3(k+1)}=$$

$$=3\sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{1}{k^3}-\frac{1}{k^2}+\frac{1}{k}-\frac{1}{k+1}\right]=$$

$$=3\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}-\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k+1}\right\}=$$

$$=3\left\{\frac{3}{4}\zeta(3)-\frac{1}{2}\zeta(2)+2\log(2)-1\right\}=\frac{9}{4}\zeta(3)-\frac{1}{4}\zeta(2)+3\log\left(\frac{4}{e}\right)$$

2480. Find a closed form:

$$\int_1^{\infty}\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Quadri Faruk Temitope-Nigeria

$$I=\int_1^{\infty}\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx\overset{\substack{x\to\frac{1}{x}\\ dx=-\frac{dx}{x^2}}}{\cong}\int_1^0\frac{\ln\left(\frac{1}{x}+1\right)\ln\left(\frac{1}{x^2}+1\right)}{(x+1)^2}\cdot-\frac{dx}{x^2}=\int_0^1\frac{\ln\left(\frac{1+x}{x}\right)\ln\left(\frac{1+x^2}{x^2}\right)}{\frac{(x+1)^2}{x^2}}\frac{dx}{x^2}=$$

$$=\int_0^1\frac{\ln\left(\frac{1+x}{x}\right)\ln\left(\frac{1+x^2}{x^2}\right)}{(x+1)^2}dx$$

$$\textit{Recall that}:\ \ln\left(\frac{A}{B}\right)=\ln(A)-\ln(B)$$

$$I=\int_0^1\frac{[\ln(1+x)-\ln(x)][\ln(x^2+1)-\ln(x^2)]}{(x+1)^2}dx=\int_0^1\frac{[\ln(1+x)-\ln(x)][\ln(x^2+1)-2\ln(x)]}{(x+1)^2}dx=$$

$$\int_0^1\frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx-2\int_0^1\frac{\ln(x+1)\ln(x)}{(x+1)^2}dx-\int_0^1\frac{\ln(x)\ln(x^2+1)}{(x+1)^2}dx+2\int_0^1\frac{\ln^2(x)}{(x+1)^2}dx$$

$$\textit{Integration by parts}:\int udv=uv-\int vdu$$

$$I=\left[-\frac{\ln(x+1)+1}{x+1}.\ln(1+x^2)\Big|_0^1+2\int_0^1\frac{x(1+\ln(1+x))}{(1+x)(1+x^2)}dx\right]$$

$$-2\left[-\frac{\ln(1+x)+1}{1+x}.\ln(x)\Big|_0^1+\int_0^1\frac{\ln(x+1)+1}{x(1+x)}dx\right]-$$

$$\left[\left(\frac{x\ln(x)}{1+x}-\ln(1+x)\right)\ln(1+x^2)\Big|_0^1-2\int_0^1\frac{x^2\ln(x)}{(1+x)(1+x^2)}dx+2\int_0^1\frac{x\ln(1+x)}{1+x^2}dx\right]+$$

$$2\left[\left(\frac{xln(x)}{x+1}-\ln(x+1)\right).\ln(x)-\int_0^1\frac{\ln(x)}{x+1}dx+\int_0^1\frac{\ln(x+1)}{x}dx\right]$$

$$I=-\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}$$

$$+2\int_0^1\frac{x}{(1+x^2)}dx+2\int_0^1\frac{xln(1+x)}{(1+x)(1+x^2)}dx-2\int_0^1\frac{\ln(1+x)}{x(1+x)}dx-2\int_0^1\frac{1}{x(1+x)}dx+ln^2(2)+$$

$$2\int_0^1\frac{x^2\ln(x)}{(1+x)(1+x^2)}dx-2\int_0^1\frac{xln(1+x)}{1+x^2}dx-2\int_0^1\frac{\ln(x)}{x+1}dx+2\int_0^1\frac{\ln(1+x)}{x}dx$$

$$I=\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}-\int_0^1\frac{dx}{1+x}+\int_0^1\frac{dx}{1+x^2}+\int_0^1\frac{xdx}{1+x^2}-\int_0^1\frac{\ln(x+1)}{1+x}dx+\int_0^1\frac{\ln(x+1)}{1+x^2}dx+\int_0^1\frac{xln(x+1)}{1+x^2}dx-$$

$$-2\int_0^1\frac{\ln(1+x)}{x}dx$$

$$+2\int_0^1\frac{\ln(1+x)}{1+x}dx$$

$$+2\int_0^1\frac{dx}{1+x}+\int_0^1\frac{\ln(x)}{1+x}dx-\int_0^1\frac{\ln(x)}{1+x^2}dx+\int_0^1\frac{xln(x)}{1+x^2}dx-2\int_0^1\frac{xln(1+x)}{1+x^2}dx-$$

$$2\int_0^1\frac{\ln(x)}{1+x}dx-2\int_0^1\frac{\ln(1+x)}{x}dx$$

$$This:\ I=\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}-\ln(2)+\arctan(x)\Big|_0^1+\frac{1}{2}\ln(1+x^2)\Big|_0^1-\frac{1}{2}ln^2(1+x)\Big|_0^1$$

$$+\int_0^1\frac{\ln(x+1)}{x^2+1}dx-\int_0^1\frac{xln(1+x)}{1+x^2}dx-$$

$$4\int_0^1\frac{\ln(1+x)}{x}dx+ln^2(1+x)\Big|_0^1+2\ln(1+x)\Big|_0^1-\int_0^1\frac{\ln(1+x)}{1+x}dx-\int_0^1\frac{\ln(x)}{1+x^2}dx+\int_0^1\frac{xln(x)}{1+x^2}dx$$

$$I=\frac{ln^2(2)}{2}-\frac{\ln(2)}{2}-\ln(2)+\frac{\pi}{4}+\frac{1}{2}\ln(2)-\frac{1}{2}ln^2(2)$$

$$+\int_0^1\frac{\ln(1+x)}{1+x^2}dx-\int_0^1\frac{xln(1+x)}{1+x^2}dx-2\int_0^1\frac{\ln(1+x)}{x}dx+ln^2(2)+$$

$$2ln(2)-\frac{1}{2}ln^2(2)-\int_0^1\frac{\ln(x)}{1+x^2}dx+\int_0^1\frac{xln(x)}{1+x^2}dx$$

$$Note\ \ that:\ \int_0^1\frac{\ln(1+x)}{1+x^2}dx=\frac{\pi}{8}\ln(2)$$

$$I=\ln(2)+ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\sum_{n=1}^{\infty}(-1)^n\int_0^1x^{2n-1}\ln(1+x)\,dx+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^1x^{n-1}dx-$$

$$\sum_{n=1}^{\infty}(-1)^n\int_0^1x^{2n}\ln(x)\,dx-\sum_{n=1}^{\infty}(-1)^n\int_0^1x^{2n-1}\ln(x)\,dx$$

$$I=\ln(2)+\frac{3}{2}ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\sum_{n=1}^{\infty}(-1)^n\left[\frac{\ln(2)}{2n}-\frac{1}{2n}\int_0^1\frac{x^{2n}}{1+x}dx\right]+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$$

$$+\frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$I=\ln(2)+ln^2(2)+\frac{\pi}{4}+\frac{\pi}{8}\ln(2)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_{2n}}{n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n}+2\left[-\frac{\pi^2}{12}\right]+G-\frac{\pi^2}{48}$$

$$I=G+\ln(2)+\frac{7}{8}ln^2(2)+\frac{5\pi^2}{96}+\frac{\pi}{8}(2+\ln(2))$$

2481. Find a closed form:

$$\int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Exodo Halcalias-Angola

$$I=\int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx \overset{I.B.P}{\cong} -\frac{\ln(x+1)\ln(x^2+1)}{x+1}\Big|_0^1+2\int_1^\infty \frac{x\ln(x+1)}{(x^2+1)(x+1)}dx$$

$$I=\frac{\ln(2)}{2}+2I_1+I_2$$

$$I_1=\int_1^\infty \frac{x\ln(x+1)}{(x^2+1)(x+1)}dx \overset{x\to\frac{1}{x}}{\cong}\int_0^1 \frac{\ln(x+1)}{(x^2+1)(x+1)}dx-\int_0^1\frac{\ln(x)}{(x^2+1)(x+1)}dx=\frac{1}{2}\int_0^1\frac{(x+1)\ln\left(\frac{2}{x+1}\right)}{x^2+1}dx-$$

$$\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx+\frac{1}{2}\int_0^1\frac{x\ln(x)}{x^2+1}dx$$

$$-\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx=\frac{\ln(2)}{2}\int_0^1\frac{x+1}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{(x+1)\ln(x+1)}{x^2+1}dx+\frac{\beta(2)}{2}-\frac{3}{8}\int_0^1\frac{\ln(x)}{x+1}dx$$

$$=\frac{\ln(2)}{2}\left(\frac{\ln(2)}{2}+\frac{\pi}{4}\right)-\frac{1}{2}\int_0^1\frac{x\ln(x+1)}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x+1)}{x^2+1}dx+\frac{\beta(2)}{2}+\frac{3}{8}\int_0^1\frac{\ln(x+1)}{x}dx=\frac{ln^2(2)}{4}+\frac{\pi\ln(2)}{8}-$$

$$\frac{1}{2}\left(\frac{\ln(x+1)\ln(x^2+1)}{2}\Big|_0^1\right.$$

$$\left.-\frac{1}{2}\int_0^1\frac{\ln(x^2+1)}{x+1}dx\right)-\frac{1}{2}\int_0^1\frac{\ln\left(\frac{2}{x+1}\right)}{x^2+1}dx+G-\frac{3}{8}\int_0^1 dLi_2(-x)=\frac{ln^2(2)}{4}+\frac{\pi\ln(2)}{8}-\frac{ln^2(2)}{4}+$$

$$\frac{1}{4}\int_0^1\frac{\ln(x^2+1)}{x+1}dx-\frac{\ln(2)}{4}\int_0^1\frac{dx}{x^2+1}+\frac{G}{2}-\frac{3Li_2(-1)}{8}$$

$$=\frac{\pi\ln(2)}{8}+\frac{1}{4}\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}}\frac{\ln(2x^2-2\sqrt{2}x+2)}{x}dx-\frac{\pi\ln(2)}{16}+\frac{G}{2}+\frac{\pi^2}{32}$$

$$\left(\because\ Li_2(r,\theta)=-\frac{1}{2}\int_0^r\frac{\ln(x^2-2zcos\theta+1}{z}dz\right)$$

$$=\frac{\pi\ln(2)}{16}+\frac{ln^2(2)}{4}+\frac{1}{4}\left(\frac{1}{2}Li_2\left(cos^2\left(\frac{\pi}{4}\right)\right)-\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2\right)+\frac{G}{2}+\frac{\pi^2}{32}=$$

$$=\frac{\pi\ln(2)}{16}+\frac{ln^2(2)}{4}+\frac{1}{4}\left(\frac{1}{2}\left(\frac{\pi^2}{12}-\frac{ln^2(2)}{2}\right)-\frac{\pi^2}{16}\right)+\frac{G}{2}+\frac{\pi^2}{32}$$

$$I_1=\frac{\pi\ln(2)}{16}+\frac{3ln^2(2)}{16}+\frac{5\pi^2}{192}+\frac{G}{2}$$

$$I_2=\int_1^\infty \frac{\ln(x^2+1)}{(x+1)^2}dx \overset{x\to\frac{1}{x}}{\cong} \int_0^1 \frac{\ln(x^2+1)}{(x+1)^2}dx$$

$$-2\int_0^1 \frac{\ln(x)}{(x+1)^2}dx=-\frac{\ln(x^2+1)}{x+1}\Big|_0^1+\int_0^1 \frac{x+1}{x^2+1}dx-\int_0^1 \frac{dx}{x+1}+2\ln(2)$$

$$I_2=\frac{3\ln(2)}{2}+\int_0^1 \frac{dx}{x^2+1}-\frac{1}{2}\int_0^1 \frac{dx}{x+1}=\frac{\pi}{4}+\ln(2)$$

$$I=\frac{\ln(2)}{2}+2I_1+I_2$$

$$I=\frac{\ln(2)}{2}+2\left(\frac{\pi\ln(2)}{16}+\frac{3ln^2(2)}{16}+\frac{5\pi^2}{192}+\frac{G}{2}\right)+\frac{\pi}{4}+\ln(2)$$

$$\therefore \int_1^\infty \frac{\ln(x+1)\ln(x^2+1)}{(x+1)^2}dx=G+\frac{5\pi^2}{96}+\frac{7ln^2(2)}{8}+\frac{\pi\ln(2)}{8}+\frac{\pi}{4}+\ln(2)$$

$G\to$ is the Catalan's constant ...

2482. Prove that:

$$\int_0^1\int_0^\infty \frac{ln^2(x)ln^2(1+y^2)}{y(1+x^2)}dx=\frac{\pi^3}{64}\zeta(3)$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$We\ \ have\,;\ \underbrace{\int_0^\infty \frac{ln^2(x)}{(1+x^2)}dx}_{K}\underbrace{\int_0^1 \frac{ln^2(1+y^2)}{y}dy}_{N}\ (1)$$

$$K=\int_0^\infty \frac{ln^2(x)}{(1+x^2)}dx=\int_0^1 \frac{ln^2(x)}{1+x^2}dx+\underbrace{\int_1^\infty \frac{ln^2(x)}{1+x^2}dx}_{x\to\frac{1}{x}}=2\int_0^1 \frac{ln^2(x)}{1+x^2}dx \overset{\therefore|x|<1}{\cong}$$

$$2\sum_{n\in N\cup\{0\}}(-1)^n\int_0^1 x^{2n}ln^2(x)dx \overset{Note\ Section\ (1)}{\cong} 4\sum_{n\in N\cup\{0\}}\frac{(-1)^n}{(2n+1)^3}=4\underbrace{\beta(2)}_{\frac{\pi^3}{32}}=4.\frac{\pi^3}{32}$$

$$=\frac{\pi^3}{8}$$

$$N=\int_0^1 \frac{ln^2(1+y^2)}{y}dy \overset{y^2\to x}{\cong} \frac{1}{2}\int_0^1 \frac{ln^2(1+x)}{x}dx=\frac{1}{4}\underbrace{\int_0^1 \frac{1}{x}ln^2\left(\frac{1-x}{1+x}\right)dx}_{\frac{1-x}{1+x}\to x}$$

$$+\frac{1}{4}\underbrace{\int_0^1 \frac{ln^2(1-x^2)}{x}dx}_{x^2\to x}-$$

$$\frac{1}{2}\underbrace{\int_0^1\frac{ln^2(1-x)}{x}dx}_{x\to 1-x}=\frac{1}{2}\int_0^1\frac{ln^2(x)}{1-x^2}dx+\frac{1}{8}\underbrace{\int_0^1\frac{ln^2(1-x)}{x}dx}_{x\to 1-x}-\frac{1}{2}\int_0^1\frac{ln^2(x)}{1-x}dx=$$

$$\frac{1}{2}\int_0^1\frac{ln^2(x)}{1-x^2}dx$$

$$-\frac{3}{8}\int_0^1\frac{ln^2(x)}{1-x}dx$$

$$=\frac{1}{2}\sum_{n\in N\cup\{0\}}\int_0^1 x^{2n}ln^2(x)dx-\frac{3}{8}\sum_{n\in N}x^{n-1}ln^2(x)dx\overset{Note\ Section\ (1)}{\cong}$$

$$\sum_{n\in N\cup\{0\}}\frac{1}{(2n+1)^3}-\frac{3}{4}\sum_{n\in N}\frac{1}{n^3}=\frac{7}{8}\zeta(3)-\frac{3}{4}\zeta(3)=\frac{\zeta(3)}{8}$$

$$\underbrace{\int_0^\infty\frac{ln^2(x)}{(1+x^2)}dx}_{K}\underbrace{\int_0^1\frac{ln^2(1+y^2)}{y}dy}_{N}=\frac{\pi^3}{8}\cdot\frac{\zeta(3)}{8}=\frac{\pi^3}{64}\zeta(3)$$

2483. **Find a closed form:**

$$\int_0^1\frac{xln(x)}{(x+1)(x^2+1)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Arowolo Isaiah-Nigeria

$$M=\int_0^1\frac{xln(x)}{(x+1)(x^2+1)}dx$$

$$M=\frac{1}{2}\int_0^1\frac{xln(x)}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx$$

$$M\overset{x\to x^2}{\cong}\frac{1}{8}\int_0^1\frac{\ln(x)}{x+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx$$

$$M=-\frac{3}{8}\int_0^1\frac{\ln(x)}{x+1}dx+\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln(tan(x))\,dx$$

$$M=-\frac{3}{8}(\int_0^1\frac{\ln(x)}{1-x}dx+\int_0^1\frac{2xln(x)}{1-x^2}dx)+\frac{1}{2}(-\frac{1}{2}CI_2\left(\frac{\pi}{2}\right)-\frac{1}{2}CI_2\left(\frac{\pi}{2}\right))$$

$$M=-\frac{3}{8}\int_0^1\frac{\ln(x)}{1-x}dx+\frac{3}{16}\int_0^1\frac{\ln(x)}{1-x}dx-\frac{1}{2}CI_2\left(\frac{\pi}{2}\right)\to M$$

$$=-\frac{3}{16}\int_0^1\frac{\ln(x)}{1-x}dx-\frac{1}{2}CI_2\left(\frac{\pi}{2}\right)$$

$$M=-\frac{3}{16}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(x)\,dx-\frac{G}{2}\to\frac{3}{16}\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{G}{2}$$

$$M=\frac{3}{16}\zeta(2)-\frac{G}{2}=\frac{3}{16}\cdot\frac{\pi^2}{6}-\frac{G}{2}=\frac{\pi^2}{32}-\frac{G}{2}$$

2484. Find a closed form:

$$\int_0^1\int_0^1\frac{x^2\ln(x^2+1)ln^2(y)}{(x^2+1)(y+1)}dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Izumi Ainsworth-Japan

$$I=\int_0^1\int_0^1\frac{x^2\ln(x^2+1)ln^2(y)}{(x^2+1)(y+1)}dxdy=\underbrace{\int_0^1\frac{x^2\ln(x^2+1)}{(x^2+1)}dx}_{I_1}\underbrace{\int_0^1\frac{ln^2(y)}{(y+1)}dy}_{I_2}\dots(\alpha)$$

$$I_1\overset{x^2\to x}{\cong}\frac{1}{2}\int_0^1\frac{\sqrt{x}\ln(x+1)}{x+1}dx=-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^nH_n\int_0^1x^{n+\frac{1}{2}}dx=-\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+3}=$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left(H_n-\frac{1}{n}\right)}{2n+1}=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{2n+1}-\left(\frac{1}{3}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n(2n+1)}\right)=\underbrace{\beta(2)}_{G}-\frac{\pi\ln(2)}{2}-$$

$$\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{n}}_{-\ln(2)}+2\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}}_{\frac{\pi}{4}-1}=G-\frac{\pi\ln(2)}{2}+\ln(2)+\frac{\pi}{2}-2\dots(\beta)$$

$$I_2=\int_0^1\frac{ln^2(y)}{(y+1)}dy=\sum_{n=0}^{\infty}(-1)^n\int_0^1y^nln^2(y)dy=2\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}}_{\eta(3)}=\frac{3\zeta(3)}{2}\dots(\gamma)$$

$$*(\beta)\wedge(\gamma)\to(\alpha)$$

$$I=\underbrace{\int_0^1\frac{x^2\ln(x^2+1)}{(x^2+1)}dx}_{I_1}\underbrace{\int_0^1\frac{ln^2(y)}{(y+1)}dy}_{I_2}=\frac{3\zeta(3)}{2}\left(G-\frac{\pi\ln(2)}{2}+\ln(2)+\frac{\pi}{2}-2\right)$$

$$I=\frac{3}{4}\zeta(3)(2G+\pi+\ln(4)-\pi\ln(2)-4)$$

2485. Find a closed form:

$$\int_0^1\int_{\frac{\pi}{4}}^{\pi}\left(\frac{\ln(\sqrt{x})}{x}-\frac{ln^3(y+1)}{y^3}\right)dxdy$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Izumi Ainsworth-Japan

$$I=\underbrace{\int_{\frac{\pi}{4}}^{\pi}\frac{\ln(\sqrt{x})}{x}dx}_{I_1}-\frac{3\pi}{4}\underbrace{\int_0^1\frac{ln^3(y+1)}{y^3}dy}_{I_2}$$

$$I_1=\int_{\frac{\pi}{4}}^{\pi}\frac{\ln(\sqrt{x})}{x}dx=\frac{1}{2}\int_{\frac{\pi}{4}}^{\pi}\frac{\ln(x)}{x}dx\overset{\ln(x)\to x}{\cong}\frac{1}{2}\int_{\ln\left(\frac{\pi}{4}\right)}^{\ln(\pi)}xdx=\frac{x^2}{4}\Big|_{\ln\left(\frac{\pi}{4}\right)}^{\ln(\pi)}=$$

$$=\frac{ln^2(\pi)-ln^2\left(\frac{\pi}{4}\right)}{4}=\ln(2)\ln\left(\frac{\pi}{2}\right)$$

$$I_2=\int_0^1\frac{ln^3(y+1)}{y^3}dy\overset{\frac{1}{1+y}=y}{\cong}-\int_{\frac{1}{2}}^1\frac{yln^3(y)}{(1-y)^3}dy=-\underbrace{\int_0^1\frac{yln^3(y)}{(1-y)^3}dy}_{I_3}+\underbrace{\int_0^{\frac{1}{2}}\frac{yln^3(y)}{(1-y)^3}dy}_{I_4}$$

$$I_3=\int_0^1\frac{yln^3(y)}{(1-y)^3}dy=\frac{1}{2}\sum_{n=2}^{\infty}n(n-1)\int_0^1y^{n-1}ln^3(y)dy=-3\sum_{n=2}^{\infty}\frac{n(n-1)}{n^4}=$$

$$=3\sum_{n=1}^{\infty}\left(\frac{1}{n^3}-\frac{1}{n^2}\right)=3\zeta(3)-\frac{\pi^2}{2}$$

$$I_4=\int_0^{\frac{1}{2}}\frac{yln^3(y)}{(1-y)^3}dy\overset{y\to\frac{y}{2}}{\cong}2\int_0^1\frac{yln^3\left(\frac{y}{2}\right)}{(2-y)^3}dy$$

$$=\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{2^n}\int_0^1y^{n-2}(\ln(y)-\ln(2))^3dy=$$

$$\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{2^n}\Big(\int_0^1y^{n-2}ln^3(y)dy$$

$$-3\ln(2)\int_0^1y^{n-2}ln^2(y)dy+3\,ln^2(y)\int_0^1y^{n-2}\ln(y)\,dy-$$

$$-ln^3(2)\int_0^1y^{n-2}dy\Big)$$

$$=\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{2^n}\left(-\frac{3!}{(n-1)^4}-\frac{3\ln(2)\,2!}{(n-1)^3}-\frac{3ln^2(2)}{(n-1)^2}-\frac{ln^3(2)}{n-1}\right)=$$

$$-6\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)^3}-6\ln(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)^2}$$

$$-3ln^2(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n(n-1)}-ln^3(2)\sum_{n=3}^{\infty}\frac{n-2}{2^n}=$$

$$= 3\sum_{n=2}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n^2}-\frac{\left(\frac{1}{2}\right)^n}{n^3}\right)$$
$$-3\ln(2)\sum_{n=2}^{\infty}\left(\frac{\left(\frac{1}{2}\right)^n}{n}-\frac{\left(\frac{1}{2}\right)^n}{n^2}\right)-\frac{3ln^2(2)}{2}\sum_{n=2}^{\infty}\left(\frac{1}{2^n}-\frac{\left(\frac{1}{2}\right)^n}{n}\right)-\frac{ln^3(2)}{4}\sum_{n=1}^{\infty}\frac{n}{2^n}=$$
$$3\left[Li_3\left(\frac{1}{2}\right)-Li_2\left(\frac{1}{2}\right)\right]-3\ln(2)\left[Li_1\left(\frac{1}{2}\right)-Li_2\left(\frac{1}{2}\right)\right]-\frac{3ln^2(2)}{2}\left[1-Li_1\left(\frac{1}{2}\right)\right]$$
$$-\frac{ln^3(2)}{4}Li_{-1}\left(\frac{1}{2}\right)=$$
$$3\left(\frac{ln^2(2)}{6}-\frac{\pi^2\ln(2)}{12}+\frac{7}{8}\zeta(3)-\frac{\pi^2}{12}+\frac{ln^2(2)}{2}\right)-3\ln(2)\left(\ln(2)-\frac{\pi^2}{12}+\frac{ln^2(2)}{2}\right)$$
$$-\frac{3ln^2(2)}{2}+$$
$$+\frac{3ln^3(2)}{2}-\frac{ln^3(2)}{2}=\frac{21}{8}\zeta(3)-\frac{\pi^2}{4}-3ln^2(2)$$
$$I_1=\ln(2)\ln\left(\frac{\pi}{2}\right)$$
$$I_2=\int_0^1\frac{ln^3(y+1)}{y^3}dy=-\underbrace{\int_0^1\frac{yln^3(y)}{(1-y)^3}dy}_{I_3}+\underbrace{\int_0^{\frac{1}{2}}\frac{yln^3(y)}{(1-y)^3}dy}_{I_4}=-\left(3\zeta(3)-\frac{\pi^2}{2}\right)+$$
$$+\frac{21}{8}\zeta(3)-\frac{\pi^2}{4}-3ln^2(2)=-\frac{3}{8}\zeta(3)+\frac{\pi^2}{4}-3ln^2(2)$$
$$I=\underbrace{\int_{\frac{\pi}{4}}^{\pi}\frac{\ln(\sqrt{x})}{x}dx}_{I_1}-\frac{3\pi}{4}\underbrace{\int_0^1\frac{ln^3(y+1)}{y^3}dy}_{I_2}$$
$$=\ln(2)\ln\left(\frac{\pi}{2}\right)-\frac{3\pi}{4}\left(-\frac{3}{8}\zeta(3)+\frac{\pi^2}{4}-3ln^2(2)\right)$$
$$=\frac{9}{32}\pi\left(\zeta(3)+8ln^2(2)\right)+\ln(2)\ln\left(\frac{\pi}{2}\right)-\frac{3\pi^3}{16}$$

2486. Find a closed form:

$$\int_0^1 x^n\sqrt{1-x^2}\,dx$$

Proposed by Marin Chirciu-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$I=\int_0^1 x^n\sqrt{1-x^2}\,dx\overset{x\to x^2}{\Leftarrow}\frac{1}{2}\int_0^1 x^{\frac{n}{2}-\frac{1}{2}}(1-x)^{\frac{1}{2}}dx$$

$$I=\frac{1}{2}\beta\left(\frac{n+1}{2},\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}=\frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$$

$$I=\int_0^1 x^n\sqrt{1-x^2}\,dx=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$$

2487. Find a closed form:

$$\int_0^\infty\frac{dx}{x^4+x^2+1}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{dx}{x^4+x^2+1}=\frac{1}{(x^2+1)^2-x^2}=\frac{1}{(x^2-x+1)(x^2+x+1)}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+x+1}=$$
$$\frac{(A+C)x^3+(A+B-C+D)x^2+(A+B+C-D)x+(B+D)}{(x^2-x+1)(x^2+x+1)}$$

$$\begin{cases}A+C=0\\A+B-C+D=0\\A+B+C-D=0\\B+D=0\end{cases}\to\begin{cases}A+C=0\\A+B=0\\B+D=0\\C-D=0\end{cases}\to\begin{cases}A=-\frac{1}{2}\\B=\frac{1}{2}\\C=\frac{1}{2}\\D=\frac{1}{2}\end{cases}$$

$$\frac{1}{x^4+x^2+1}=\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2-x+1}+\frac{\frac{1}{2}x+\frac{1}{2}}{x^2+x+1}$$

$$\int_0^\infty\frac{dx}{x^4+x^2+1}=-\frac{1}{2}\int_0^\infty\frac{x-1}{x^2-x+1}dx+\frac{1}{2}\int_0^\infty\frac{x+1}{x^2+x+1}dx=-\frac{1}{4}\int_0^\infty\frac{2x-1}{x^2-x+1}dx+$$
$$\frac{1}{4}\int_0^\infty\frac{1}{x^2-x+1}dx+\int_0^\infty\frac{2x+1}{x^2+x+1}dx+\frac{1}{4}\int_0^\infty\frac{1}{x^2+x+1}dx=$$
$$-\frac{1}{4}\int_0^\infty\frac{d(x^2-x+1)}{x^2-x+1}+\frac{1}{4}\int_0^\infty\frac{d(x^2+x+1)}{x^2+x+1}+\frac{1}{4}\int_0^\infty\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{4}\int_0^\infty\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}=$$
$$\frac{1}{4}(-\ln(x^2-x+1)+\ln(x^2+x+1))\Big|_0^\infty+\frac{1}{4}\left(\int_0^\infty\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}+\int_0^\infty\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)=$$

$$\frac{1}{4}\left(ln\frac{x^2+x+1}{x^2-x+1}\right)\Bigg|_0^{\infty}+\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\left(arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^{\infty}=$$

$$=\frac{1}{4}\lim_{x\to\infty}\left(\frac{x^2+x+1}{x^2-x+1}-ln1\right)+\frac{\sqrt{3}}{6}\left(\frac{\pi}{2}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{6}\right)=0+\frac{\sqrt{3}}{6}.\pi=\pi\frac{\sqrt{3}}{6}$$

$$\int_0^{\infty}\frac{dx}{x^4+x^2+1}=\pi\frac{\sqrt{3}}{6}\ (proved)$$

2488. Find a closed form:

$$\int_1^{\sqrt{2}}\frac{dx}{x\sqrt{-x^4+3x^2-2}}$$

Proposed by Kader Tapsoba-Burkina Faso

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_1^{\sqrt{2}}\frac{dx}{x\sqrt{-x^4+3x^2-2}}\overset{x\to\frac{1}{t}}{\cong}\int_1^{\frac{1}{\sqrt{2}}}\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{-\frac{1}{t^4}+\frac{3}{t^2}-2}}=\int_{\frac{1}{\sqrt{2}}}^{1}\frac{tdt}{\sqrt{-2t^4+3t^2-1}}=$$

$$=\frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{1}\frac{dt^2}{\sqrt{-2t^4+3t^2-1}}\overset{u\to t^2}{\cong}\frac{1}{2}\int_{\frac{1}{2}}^{1}\frac{du}{\sqrt{-2u^2+3u-1}}\overset{v=-4u+3}{\cong}$$

$$\left\{v=-4u+3\to u=\frac{3-v}{4}\to du=-\frac{1}{4}dv\right\}$$

$$=-2u^2+3u-1=-2.\frac{(3-v)^2}{16}+3.\frac{3-v}{4}-1=-\frac{(9-6v+v^2)}{8}+\frac{9-3v}{4}-1=$$

$$=\frac{-9-v^2+6v+18-6v-8}{8}=\frac{1-v^2}{8}=\frac{1}{2}\int_1^{-1}\frac{-\frac{1}{4}dv}{\sqrt{\frac{1-v^2}{8}}}=\frac{1}{2}\cdot\left(\frac{1}{4}\right).\sqrt{8}\int_{-1}^{1}\frac{dv}{\sqrt{1-v^2}}=$$

$$=\frac{\sqrt{2}}{4}.2\int_0^1\frac{dv}{\sqrt{1-v^2}}=\frac{\sqrt{2}}{2}(\arcsin(v)|_0^1=\frac{\sqrt{2}}{2}(\arcsin(1)-\arcsin(0))=\frac{\pi\sqrt{2}}{4}(proved)$$

$$\int_1^{\sqrt{2}}\frac{dx}{x\sqrt{-x^4+3x^2-2}}=\frac{\pi\sqrt{2}}{4}$$

2489. Find a closed form:

$$\int_{-1}^{1}\frac{dx}{x^2+x+1+\sqrt{x^4+3x^2+1}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{x^2+x+1+\sqrt{x^4+3x^2+1}}=\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{\left(x^2+x+1+\sqrt{x^4+3x^2+1}\right)\left(x^2+x+1-\sqrt{x^4+3x^2+1}\right)}=$$

$$\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{(x^2+x+1)^2-(x^4+3x^2+1)}=\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{x^4+x^2+1+2x^3+2x^2+2x-x^4-3x^2-1}=$$

$$=\frac{x^2+x+1-\sqrt{x^4+3x^2+1}}{2x(x^2+1)}=\frac{1}{2x}+\frac{1}{2(x^2+1)}-\frac{\sqrt{x^4+3x^2+1}}{2x(x^2+1)}$$

$$\int_{-1}^{1}\left(\frac{1}{2x}+\frac{1}{2(x^2+1)}-\frac{\sqrt{x^4+3x^2+1}}{2x(x^2+1)}\right)dx=\frac{1}{2}\int_{-1}^{1}\frac{1}{x}dx+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-\frac{1}{2}\int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx$$

$$f(x)=\frac{1}{x}\ ,\quad g(x)=\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}\to f(x)\ and\ g(x)\ odd\ function$$

$$\int_{-1}^{1}\frac{1}{x}dx=0\ ,\quad \int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx=0$$

$$\frac{1}{2}\int_{-1}^{1}\frac{1}{x}dx+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-\frac{1}{2}\int_{-1}^{1}\frac{\sqrt{x^4+3x^2+1}}{x(x^2+1)}dx=0+\frac{1}{2}\int_{-1}^{1}\frac{dx}{x^2+1}-0=\frac{\arctan(x)}{2}\Big|_{-1}^{1}=$$

$$=\frac{1}{2}(\arctan(1)-\arctan(-1))=\frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{\pi}{4}$$

$$\int_{-1}^{1}\frac{dx}{x^2+x+1+\sqrt{x^4+3x^2+1}}=\frac{\pi}{4}$$

2490. Find a closed form:

$$\int_1^4\frac{dx}{x^2+x\sqrt{x}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$Let\ \sqrt{x}=t\to dt=\frac{dx}{2\sqrt{x}},\quad \int_1^2\frac{2t}{t^4+t^3}dt=2\int_1^2\frac{dt}{t^3+t^2}$$

$$\frac{1}{t^3+t^2}=\frac{1}{t^2(t+1)}=\frac{At+B}{t^2}+\frac{C}{t+1}$$

$$(At+B)(t+1)+Ct^2=1,\ \ At^2+At+Bt+B+Ct^2=1$$

$$(A+B)t^2+(A+B)t+B=1,\ \begin{cases}A+C=1\\A+B=0\\B=1\end{cases}\to\begin{cases}A=-1\\B=1\\C=1\end{cases}$$

$$Then:\ \frac{1}{t^3+t^2}=\frac{-t+1}{t^2}+\frac{1}{t+1}=-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t+1}$$

$$2\int_1^2 \frac{dt}{t^3+t^2} = 2\int_1^2 \left(-\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t+1}\right)dt = 2\left(\ln\left(\frac{t+1}{t}-\frac{1}{t}\right)\right)\Big|_2^1 =$$

$$= 2\left(\left(\ln\left(\frac{3}{2}\right)-\frac{1}{2}\right)-(\ln(2)-1)\right) = 2\ln\left(\left(\frac{3}{4}\right)+\frac{1}{2}\right) = 2\ln\left(\frac{3}{4}\right)+1$$

$$\int_1^4 \frac{dx}{x^2+x\sqrt{x}} = 2\ln\left(\frac{3}{4}\right)+1$$

2491. Prove that:

$$\int_{-\pi}^{\pi} ln^2\left(\cos\left(\frac{x}{2}\right)\right)dx = 2\pi ln^2(2)+\frac{\pi^3}{6}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Shirvan Tahirov-Azerbaijan

$$\int_{-\pi}^{\pi} ln^2\left(\cos\left(\frac{x}{2}\right)\right)dx$$

$$= 4\int_0^1 \frac{ln^2(2)}{\sqrt{1-x^2}}dx = \int_0^1 \lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\frac{x^a}{\sqrt{1-x^2}}dx = \lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{x^a}{\sqrt{1-x^2}}dx =$$

$$\left\{x^2=t,\quad \frac{dt}{dx}=2x=2\sqrt{t}\right\}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{t^{\frac{a}{2}}}{\sqrt{1-t}}\frac{dt}{2\sqrt{t}} = \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 \frac{t^{\frac{a-1}{2}}}{\sqrt{1-t}}dt =$$

$$\frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\int_0^1 t^{\frac{a+1}{2}-1}(1-t)^{\frac{1}{2}-1}dt = \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\beta\left(\frac{a+1}{2};\frac{1}{2}\right)$$

$$= \frac{1}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\left(\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right) =$$

$$\frac{\sqrt{\pi}}{2}\lim_{a\to 0}\frac{\partial^2}{\partial^2 a}\left(\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}+1\right)}\right) = \frac{\sqrt{\pi}}{2}\lim_{a\to 0}\frac{\Gamma\left(\frac{a+1}{2}\right)}{4\Gamma\left(\frac{a}{2}+1\right)}\left(\left(\psi^{(0)}\left(\frac{a}{2}+1\right)-\psi^{(0)}\left(\frac{a+1}{2}\right)\right)\right)^2$$

$$-\psi^{(1)}(1)+$$

$$+\psi^{(1)}\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}\left(\left(\psi^{(0)}(1)-\psi^{(0)}\left(\frac{1}{2}\right)\right)\right)^2-\psi^{(1)}(1)+\psi^{(1)}\left(\frac{1}{2}\right) =$$

$$\frac{\sqrt{\pi}}{8}\left((\partial-(-\partial-2\ln(2)))^2-\frac{\pi^2}{6}+\frac{\pi^2}{2}\right) = 4\cdot\frac{\pi}{8}\left(4ln^2(2)+\frac{\pi^2}{3}\right) = 2\pi ln^2(2)+\frac{\pi^3}{6}$$

2492. Find a closed form:

$$\int_1^2 \frac{x\sqrt{x-1}}{x-5}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_1^2 \frac{x\sqrt{x-1}}{x-5}dx \overset{\sqrt{x-1}\to t}{=} \int_0^1 \frac{2t^2(t^2+1)}{t^2-4}dt = 2\int_0^1 \frac{t^4+t^2}{t^2-4}dt =$$

$$Let\ :\ \sqrt{x-1}\to t \Rightarrow x = t^2+1 \Rightarrow dx = 2tdt$$

$$= 2\left(\int_0^1 \left(t^2+5-\frac{20}{4-t^2}\right)dt = 2\left(\left(\frac{t^3}{3}+5t\right)\Bigg|_0^1 - 20.\frac{1}{4}ln\left|\frac{2+t}{2-t}\right|\Bigg|_0^1\right) =\right.$$

$$= 2\left(\left(\frac{1}{3}+5\right)-5(\ln(3)-\ln(1))\right) = 2\left(\frac{16}{3}-5\ln(3)\right) = \frac{32}{3}-10\ln(3)$$

$$\int_1^2 \frac{x\sqrt{x-1}}{x-5}dx = \frac{32}{3}-10\ln(3)\ \ (proved)$$

2493. Find a closed form:

$$\int_0^1 \frac{xln(x)}{(1+x)(1+x^2)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Ankush Kumar Parcha-India

$$\int_0^1 \frac{xln(x)}{(1+x)(1+x^2)}dx \overset{Partial\ fraction}{=} \frac{1}{2}\int_0^1 \frac{\ln(x)}{1+x^2}dx + \frac{1}{2}\underbrace{\int_0^1 \frac{xln(x)}{1+x^2}dx}_{x^2\to x} -$$

$$\frac{1}{2}\int_0^1 \frac{\ln(x)}{1+x}dx \overset{Note\ section}{=} -\frac{G}{2}-\frac{3}{8}\int_0^1 \frac{\ln(x)}{1+x}dx = -\frac{G}{2}+\frac{3}{8}\eta(2) =$$

$$= -\frac{G}{2}+\frac{3}{16}\zeta(2) = \frac{\pi^2}{32}-\frac{G}{2}$$

$$\int_0^1 \frac{xln(x)}{(1+x)(1+x^2)}dx = \frac{\pi^2}{32}-\frac{G}{2}$$

Note section :

1) $\int_0^1 \frac{\ln\left(\frac{1}{t}\right)}{1+t^2}dt = G\ \ (Catalan's\ constant)$

2) $\eta(z) = \frac{(-1)^{z-1}}{\Gamma(z)}\int_0^1 \frac{ln^{z-1}(x)}{1+x}dx,\ \ \Re(z) > 0$

3) $\eta(s) = (1-2^{1-s}).\zeta(s)$

2494. Find a closed form:

$$\Omega=\int_0^1\frac{x^2\ln(x)+ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Exodo Halcalias-Angola

$$\int_0^1\frac{x^2\ln(x)+ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx=\int_0^1\frac{x^2\ln(x)}{1+x+x^2+x^3}dx+\frac{1}{4}\int_0^1\frac{ln^2(\sqrt{x})}{1+x+x^2+x^3}dx=$$

$$\int_0^1\frac{(1-x)x^2\ln(x)}{(1-x)(1+x+x^2+x^3)}dx+\frac{1}{4}\int_0^1\frac{(1-x)ln^2(\sqrt{x})}{(1-x)(1+x+x^2+x^3)}dx=$$

$$\int_0^1\frac{(1-x)x^2\ln(x)}{1-x^4}dx$$

$$+\frac{1}{4}\int_0^1\frac{(1-x)ln^2(\sqrt{x})}{1-x^4}dx=\frac{1}{16}\left(\int_0^1\frac{x^{\frac{3}{4}-1}\ln(x)}{1-x}dx-\int_0^1\frac{\ln(x)}{1-x}dx\right)+$$

$$\frac{1}{256}\left(\int_0^1\frac{x^{\frac{1}{4}-1}ln^2(x)}{1-x}dx-\int_0^1\frac{x^{\frac{1}{2}-1}ln^2(x)}{1-x}dx\right)$$

$$\left\{\therefore\int_0^1\frac{z^{n-1}ln^k(z)}{1-z}dz=-\psi^{(k)}(n),\forall\, k\in N\right\}$$

$$=\frac{1}{16}\left(-\psi^{(1)}\left(\frac{3}{4}\right)+\psi^{(1)}(1)\right)+\frac{1}{256}\left(-\psi^{(2)}\left(\frac{1}{4}\right)+\psi^{(2)}\left(\frac{1}{2}\right)\right)$$

$$\{\therefore\psi^{(k)}(n)=(-1)^{k+1}\Gamma(k+1)\,\zeta(k+1,n)\}$$

$$=\frac{1}{16}\left(-(\pi^2-8G)+\frac{\pi^2}{6}\right)+\frac{1}{256}\left(-\left(-2\pi^3-56\zeta(3)\right)-14\zeta(3)\right)$$

$$=\frac{1}{16}\left(8G-\frac{5\pi^2}{6}\right)+\frac{1}{256}(42\zeta(3)+2\pi^3)$$

$$\int_0^1\frac{x^2\ln(x)+ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx=\frac{G}{2}+\frac{21\zeta(3)}{128}+\frac{\pi^3}{128}-\frac{5\pi^2}{96}$$

Solution 2 by Cosghun Memmedov-Azerbaijan

$$\Omega=\int_0^1\frac{x^2\ln(x)+ln^2(\sqrt{x})}{(x+1)(x^2+1)}dx$$

$$=\int_0^1\frac{x^2\ln(x)}{(x+1)(x^2+1)}dx+\int_0^1\frac{\ln(\sqrt{x})}{(x+1)(x^2+1)}dx=\Omega_1+\Omega_2$$

$$\Omega_1=\int_0^1\frac{x^2\ln(x)}{(x+1)(x^2+1)}dx=\frac{1}{2}\int_0^1\frac{xln(x)}{x^2+1}dx-\frac{1}{2}\int_0^1\frac{\ln(x)}{x^2+1}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{x+1}dx=$$

$$\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}\ln(x)\,dx+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n}\ln(x)\,dx=$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+2)^2}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=\frac{G}{2}-\frac{5}{8}\eta(2)=\frac{G}{2}-\frac{5\pi^2}{96}$$

$$\Omega_2=\int_0^1\frac{\ln(\sqrt{x})}{(x+1)(x^2+1)}dx$$

$$=\frac{1}{4}\int_0^1\frac{\ln(x)}{(x+1)(x^2+1)}dx=\frac{1}{8}\int_0^1\frac{ln^2(x)}{x^2+1}dx-\frac{1}{8}\int_0^1\frac{xln^2(x)}{x^2+1}dx+$$

$$\frac{1}{8}\int_0^1\frac{ln^2(x)}{x+1}dx=\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n}ln^2(x)dx-\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{2n+1}\ln(x)\,dx+$$

$$\frac{1}{8}\sum_{n=1}^{\infty}(-1)^n\int_0^1 x^{n}ln^2(x)dx=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}-\frac{1}{32}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}+\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3}=$$

$$=\frac{1}{4}\cdot\frac{\pi^3}{32}+\frac{7}{32}\cdot\frac{3}{4}\zeta(3)=\frac{\pi^3}{128}+\frac{21}{128}\zeta(3)$$

$$\Omega=\Omega_1+\Omega_2=\frac{G}{2}+\frac{\pi^3}{128}+\frac{21}{128}\zeta(3)-\frac{5\pi^2}{96}$$

2495. Find a closed form:

$$\int_0^1\frac{x^3\ln(x)}{x^4+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Shirvan Tahirov-Azerbaijan

$$\int_0^1\frac{x^3\ln(x)}{x^4+1}dx\overbrace{\underbrace{x}_{4x^3dx=du}}^{x^4=u}\frac{1}{4}\int_0^1\frac{\ln(u)}{1+u}\frac{du}{4}$$

$$=\frac{1}{16}\int_0^1\frac{\ln(u)}{1+u}du=\frac{1}{16}\int_0^1\ln(u)\sum_{k=0}^{\infty}(-1)^k u^k du=$$

$$=\frac{1}{16}\int_0^1\sum_{k=0}^{\infty}(-1)^k u^k\ln(u)du=\frac{1}{16}\sum_{k=0}^{\infty}(-1)^k\int_0^1 u^k\ln(u)du\overset{I.B.P}{=}$$

$$=\frac{1}{16}\sum_{k=0}^{\infty}(-1)^k\left[\frac{\ln(u)\,.\,u^{k+1}}{k+1}\bigg|_0^1-\int_0^1\frac{u^{k+1}}{k+1}\frac{du}{u}\right]=\frac{1}{16}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\int_0^1 u^k du=$$

$$=\frac{1}{16}\cdot\left(-\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left(\frac{u^{k+1}}{k+1}\Big|_0^1\right)\right)=\frac{1}{16}\cdot\left(-\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2}\right)=\frac{1}{16}\cdot(-\eta(2))=$$

$$=\frac{1}{16}\cdot\left(-\frac{\zeta(2)}{2}\right)=\frac{1}{16}\cdot\left(-\frac{\pi^2}{12}\right)=-\frac{\pi^2}{192}$$

$$Note:\ \boxed{\eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s},\ \eta(s)=(1-2^{1-s})\zeta(s)}$$

2496. Find a closed form:

$$\int_0^1\frac{x^3\ln(x)}{x^2+1}dx$$

Proposed by Vasile Mircea Popa-Romania

Solution by Arowolo Isaiah-Nigeria

$$\int_0^1\frac{x^3\ln(x)}{x^2+1}dx\overset{x\to x^2}{=}\frac{1}{4}\int_0^1\frac{x\ln\left(x^{\frac{1}{2}}\right)}{x+1}dx=\frac{1}{4}\int_0^1\frac{x\ln(x)}{x+1}dx=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\int_0^1x^{n+1}\ln(x)\,dx\overset{x=n+1}{=}$$

$$=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial x}\int_0^1x^ndx=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{\partial}{\partial x}\left(\frac{1}{x+1}\right)=\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\cdot\left(-\frac{1}{(n+2)^2}\right)=$$

$$=-\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+2)^2}=-\frac{1}{4}\left(\frac{1}{4}\left(\psi^{(1)}(1)-\psi^{(1)}\left(\frac{3}{2}\right)\right)\right)=$$

$$\frac{1}{16}\left(\psi^{(1)}(1)-\psi^{(1)}\left(\frac{3}{2}\right)\right)=-\frac{1}{16}\left(4-\frac{\pi^2}{3}\right)=\frac{\pi^2}{48}-\frac{1}{4}$$

2497. Find a closed form:

$$\int_0^1\int_0^1\frac{\arcsin(x)+\arccos(y)}{\sqrt{xy}}dxdy$$

Proposed by Ankush Kumar Parcha-India

Solution by Amin Hajiyev-Azerbaijan

$$\int_0^1\int_0^1\frac{\arcsin(x)+\arccos(y)}{\sqrt{xy}}dxdy=\int_0^1\int_0^1\frac{\arcsin(x)}{\sqrt{xy}}dxdy+\int_0^1\int_0^1\frac{\arccos(y)}{\sqrt{xy}}dxdy=M+K$$

$$M=\int_0^1\int_0^1\frac{\arcsin(x)}{\sqrt{xy}}dxdy=\int_0^1\frac{1}{\sqrt{y}}dy\int_0^1\frac{\arcsin(x)}{\sqrt{x}}dx=2\int_0^1\frac{\arcsin(x)}{\sqrt{x}}dx\overset{I.B.P}{=}$$

$$2|_0^1\,2\sqrt{x}\arcsin(x)-4\int_0^1\frac{\sqrt{x}}{\sqrt{1-x^2}}dx$$

$$K=\int_0^1\int_0^1\frac{\arccos(y)}{\sqrt{xy}}dxdy=\int_0^1\frac{\arccos(y)}{\sqrt{y}}dy\int_0^1\frac{1}{\sqrt{x}}dx=2\int_0^1\frac{\arccos(y)}{\sqrt{y}}dx\overset{I.B.P}{\cong}$$

$$2\Big|_0^1 2\sqrt{x}\arccos(y)+4\int_0^1\frac{\sqrt{y}}{\sqrt{1-y^2}}dy=4\int_0^1\frac{\sqrt{y}}{\sqrt{1-y^2}}dy$$

$$M+K=2\pi-4\int_0^1\frac{\sqrt{x}}{\sqrt{1-x^2}}dx+4\int_0^1\frac{\sqrt{y}}{\sqrt{1-y^2}}dy=2\pi$$

2498. Find a closed form:

$$\int_{e^2}^{e^3}\frac{\ln(x)-4}{(1-ln^2(x))}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Arowolo Isaiah-Nigeria

$$\int_{e^2}^{e^3}\frac{\ln(x)-4}{(1-ln^2(x))}dx\overset{x=e^x}{\cong}\int_2^3\frac{x-4}{(1-x^2)}dx=\frac{1}{2}\int_2^3\frac{x-4}{x+1}dx-\frac{1}{2}\int_2^3\frac{x-4}{x-1}dx=$$

$$\frac{1}{2}\int_2^3\frac{x+1-5}{x+1}dx$$

$$-\frac{1}{2}\int_2^3\frac{x-1-3}{x-1}dx=\frac{1}{2}\int_2^3dx-\frac{5}{2}\int_2^3\frac{dx}{x+1}-\frac{1}{2}\int_2^3dx+\frac{3}{2}\int_2^3\frac{1}{x-1}dx$$

$$=$$

$$-\frac{5}{2}\int_2^3\frac{dx}{x+1}+\frac{3}{2}\int_2^3\frac{1}{x-1}dx=-\frac{5}{2}(\ln(x+1))\Big|_2^3+\frac{3}{2}(\ln(x-1))\Big|_2^3=$$

$$-\frac{5}{2}(\ln(4)-\ln(3))+\frac{3}{2}(\ln(2)-\ln(1))=-\frac{5}{2}\ln(4)+\frac{5}{2}\ln(3)+\frac{3}{2}\ln(2)=$$

$$-5\ln(2)+\frac{3}{2}\ln(2)+\frac{5}{2}\ln(3)=\frac{5}{2}\ln(3)-\frac{7}{2}\ln(2)$$

$$\int_{e^2}^{e^3}\frac{\ln(x)-4}{(1-ln^2(x))}dx=\frac{5}{2}\ln(3)-\frac{7}{2}\ln(2)$$

2499. Find a closed form:

$$\int_0^{\ln(2)}\frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\int_0^{\ln(2)} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}dx \overset{e^x\to t}{\cong} \int_1^2 \frac{2t^3+t^2-1}{t(t^3+t^2-t+1)}dt$$

$$\frac{2t^3+t^2-1}{t(t^3+t^2-t+1)} = \frac{A}{t} + \frac{Bt^2+Ct+D}{t^3+t^2-t+1}$$

$$\begin{cases} A+B=2 \\ D-A=0 \\ A+C=1 \\ A=-1 \end{cases} -->> \begin{cases} A=-1 \\ B=3 \\ C=2 \\ D=-1 \end{cases}$$

$$\frac{2t^3+t^2-1}{t(t^3+t^2-t+1)} = -\frac{1}{t} + \frac{3t^2+2t-1}{t^3+t^2-t+1}$$

$$\int_1^2 \frac{2t^3+t^2-1}{t(t^3+t^2-t+1)}dt = -\int_1^2 \frac{1}{t}dt + \int_1^2 \frac{3t^2+2t-1}{t^3+t^2-t+1}dt =$$

$$-\ln(t)\Big|_1^2 + \int_1^2 \frac{d(t^3+t^2-t+1)}{t^3+t^2-t+1} = -\ln(2) + \ln(t^3+t^2-t+1)\Big|_1^2 =$$

$$-\ln(2)+\ln(11)-\ln(2) = \ln\left(\frac{11}{4}\right)$$

$$\int_0^{\ln(2)} \frac{2e^{3x}+e^{2x}-1}{e^{3x}+e^{2x}-e^x+1}dx = \ln\left(\frac{11}{4}\right)$$

2500. Find a closed form:

$$\int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz$$

Proposed by Ankush Kumar Parcha-India

Solution by Rana Ranino-Algeria

$$Useful\ identity: \int_0^1\int_0^1\int_0^1 f(xyz)dxdydz = \frac{1}{2}\int_0^1 f(x)ln^2(x)dx$$

$$\Omega = \int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz = \frac{1}{2}\int_0^1 \frac{(1+x)(3+x)}{(2+x)(4+x)}ln^2(x)dx =$$

$$\underbrace{\frac{1}{2}\int_0^1 ln^2(x)dx}_{2} - \frac{1}{4}\int_0^1 \frac{ln^2(x)}{2+x}dx - \frac{3}{4}\int_0^1 \frac{ln^2(x)}{4+x}dx$$

$$\int_0^1 \frac{ln^n(x)}{a+x}dx = -(-1)^n n!\,Li_{n+1}\left(-\frac{1}{a}\right)$$

$$\Omega = 2 - \frac{1}{4}\left(-Li_3\left(-\frac{1}{2}\right)\right) - \frac{3}{4}\left(-2Li_3\left(-\frac{1}{4}\right)\right)$$

$$\int_0^1\int_0^1\int_0^1 \frac{(1+xyz)(3+xyz)}{(2+xyz)(4+xyz)}dxdydz = 2+\frac{1}{2}Li_3\left(-\frac{1}{2}\right)+\frac{3}{2}Li_3\left(-\frac{1}{4}\right)$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru