5716

DANIEL SITARU - ROMANIA

5716. Prove:

$$\text{If } x, y \in \mathbb{R}, \text{ then } |\cos x \cos y \sin(x+y)| \leq \frac{3\sqrt{3}}{8}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
We need to prove that $f(x,y) := \cos^2 x \cos^2 y \sin^2(x+y) \leq \frac{27}{64}$, $f(x,y)$ is periodic with respect to x and y. Hence the extrema of $f(x,y)$ are assumed at points where the partial derivatives with respect to x and y vanish. We find

$$\frac{\partial}{\partial x} f(x,y) = -2\sin x \cos x \cos^y \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$\frac{\partial}{\partial y} f(x,y) = -2\sin y \cos y \cos^2 x \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$0 = \frac{\partial}{\partial x} f(x,y) - \frac{\partial}{\partial y} f(x,y) = 2\cos x \cos y \sin(y-x)\sin^2(x+y)$$

So either $x \equiv \frac{\pi}{2} (\mod \pi)$, or $y \equiv \frac{\pi}{2} (\mod \pi)$, or $x \equiv -y (\mod \pi)$, or $x \equiv y (\mod \pi)$.
The first three alternatives lead to $f(x,y) = 0$, while the last one leads to

$$0 = -2\sin x \cos^3 x \sin^2(2x) + 2\cos^4 x \sin(2x)\cos(2x) =$$

$$= 2\cos^3 x \Big(-\sin x(4\sin^2 x \cos^2 x) + \cos x(2\sin x \cos x)\cos(2x) \Big) =$$

$$= 4\cos^5 x \sin x (2\cos(2x) - 1).$$

So either $x \equiv \frac{\pi}{2} (\mod \pi)$, or $x \equiv 0 (\mod \pi)$, or $x \equiv \pm\frac{\pi}{6} (\mod \pi)$. When combined with $y \equiv x (\mod \pi)$ we get indeed $f(x,y) \leq \cos^2(\frac{\pi}{6})\cos^2(\frac{\pi}{6})\sin^2(\frac{\pi}{3}) = \frac{27}{64}$. □

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Using well-known trigonometric formulas we obtain

$$f(x,y) := |\cos x \cos y \sin(x+y)| = \frac{1}{2}|\cos(x+y) + \cos(x-y)|\sqrt{1-\cos^2(x+y)}$$

$$\leq \frac{1}{2}|z+1|\sqrt{1-z^2}$$

where $z = |\cos(x+y)|$. Hence,

$$f^2(x,y) \leq \frac{1}{4}(z+1)^2(1-z^2) = \frac{27}{64} - \left(z - \frac{1}{2}\right)^2 \left(\frac{11}{16} + \frac{3}{4}z + \frac{1}{4}z^2\right)$$

Since $0 \leq z \leq 1$, we conclude that

$$f(x,y) \leq \sqrt{\frac{27}{64}} = \frac{3\sqrt{3}}{8}$$

Remark: The inequality is sharp. Equality occurs if $x = y = \frac{\pi}{6}$ $\square$

Solution 3 by Brian Bradie, Department of Mathematics, Chrisopher Newport University, Newport News, VA.

Let $f(x,y) = \cos x \cos y \sin(x+y)$. Then

$$\frac{\partial f}{\partial x} = -\sin x \cos y \sin(x+y) + \cos x \cos y \cos(x+y)$$

$$= \cos y(\cos x \cos(x+y) - \sin x \sin(x+y)) = \cos y \cos(2x+y)$$

which is equal to 0 when

$$y = \left(n + \frac{1}{2}\right)\pi \text{ or } 2x + y = \left(n + \frac{1}{2}\right)\pi,$$

for some integer n. Similarly,

$$\frac{\partial f}{\partial y} = \cos x \cos(x+2y),$$

which is equal to 0 when

$$x = \left(m + \frac{1}{2}\right)\pi \text{ or } x + 2y = \left(m + \frac{1}{2}\right)\pi,$$

for some integer m. It follows that f has four categories of critical points:
1. $((m+\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ for any integers m and n
2. $((m+\frac{1}{2})\pi, (n-2m-\frac{1}{2})\pi)$, for any integers m and n
3. $((m-2n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$, for any integers m and n
4. $(\frac{1}{3}(2n-m+\frac{1}{2})\pi, \frac{1}{3}(2m-n+\frac{1}{2})\pi)$, for any integers m and n

When evaluate at any critical point from the first three categories, f is equal to 0. For the critical points in the fourth category, note

$$2m - n = 2n - m + 3(m-n) \Rightarrow 2m - n \equiv 2n - m (\mod 3).$$

This leads to three cases to consider:
Case 1: $2n - m \equiv 0(\mod 3)$
Then

$$x = j\pi + \frac{\pi}{6}, \quad y = k\pi + \frac{\pi}{6}, \quad \text{and} \quad x + y = (j+k)\pi + \frac{\pi}{3}$$

for some integers j and k, and

$$f(x,y) = \pm\frac{3\sqrt{3}}{8}.$$

Case 2: $2n - m \equiv (\mod 3)$
Then

$$x = j\pi + \frac{\pi}{2}, \quad y = k\pi + \frac{\pi}{2}, \quad \text{and} \quad x + y = (j+k+1)\pi$$

for some integers j and k, and

$$f(x,y) = 0.$$

Case 3: $2n - m \equiv 2(\mod 3)$
Then

$$x = j\pi + \frac{5\pi}{6}, \quad y = k\pi + \frac{5\pi}{6}, \quad \text{and} \quad x + y = (j+k)\pi + \frac{5\pi}{3}$$

for some integers j and k, and

$$f(x,y) = \pm\frac{3\sqrt{3}}{8}$$

Thus, for all $x, y \in \mathbb{R}$,

$$-\frac{3\sqrt{3}}{8} \le f(x,y) \le \frac{3\sqrt{3}}{8},$$

or

$$|f(x,y)| \le \frac{3\sqrt{3}}{8}.$$

$\square$

Solution 4 by David Huckaby, Angelo State University, San Angelo, TX.

Let $f(x,y) = \cos x \cos y \sin(x+y)$. Note that $f(x+\pi, y) = \cos(x+\pi)\cos y \sin(x+\pi+y) = -\cos x \cos y[-\sin(x+y)] = f(x,y)$. Similarly, $f(x, y+\pi) = f(x,y)$. So we need only consider the square $[-\frac{\pi}{2}, -\frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{\pi}{2}]$.

We first note that since $\cos(-\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$, $f(x,y) = 0$ for every point on the boundary of $[-\frac{\pi}{2}, -\frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{\pi}{2}]$.

To find extrema for f in the interior of $[-\frac{\pi}{2}, -\frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{\pi}{2}]$, we compute $\frac{\partial f}{\partial x} = -\sin x \cos y \sin(x+y) + \cos x \cos y \cos(x+y) = \cos y[\cos x \cos(x+y) - \sin x \sin(x+y)]$. From the symmetry of $f(x,y)$ in x and y, $\frac{\partial f}{\partial y} = \cos x[\cos y \cos(x+y) - \sin y \sin(x+y)]$. Setting $\frac{\partial f}{\partial x} = 0$ gives $\cos y = 0$ or $\cos x \cos(x+y) - \sin x \sin(x+y) = 0$. Since $\cos y \neq 0$ in the interior of $[-\frac{\pi}{2}, -\frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\cos x \cos(x+y) - \sin x \sin(x+y) = 0$. Now

$$\cos x \cos(x+y) - \sin x \sin(x+y)$$

$$= \cos x[\cos x \cos y - \sin x \sin y] - \sin x[\sin x \cos y + \cos x \sin y]$$

$$= \cos^2 x \cos y - \sin^2 x \cos y - 2\cos x \sin x \sin y$$

$$= \cos 2x \cos y - \sin 2x \sin y$$

$$= \cos(2x+y).$$

So $\frac{\partial f}{\partial x} = 0$ implies $\cos(2x+y) = 0$. By Symmetry, $\frac{\partial f}{\partial y} = 0$ implies $\cos(x+2y) = 0$. Now $\cos(2x+y) = 0$ when $2x+y = \frac{\pi}{2} + \pi n$ for any integer n. Solving for y gives $y = -2x + \frac{\pi}{2} + \pi n$. Similarly, $\cos(x+2y) = 0$ when $x + 2y = \frac{\pi}{2} + \pi n$ for some integer n. Solving for y gives $y = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi n}{2}$. Setting these two values of y equal to each other yields $-2x + \frac{\pi}{2} + \pi n = -\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi n}{2}$, whence $x = \frac{\pi}{6} + \frac{\pi n}{3}$.

The only values of $x = \frac{\pi}{6} + \frac{\pi n}{3}$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ are $x = \pm\frac{\pi}{6}$. So any point (x,y) yielding an extremum of f in the interior of $[-\frac{\pi}{2}, -\frac{\pi}{2}] \times [\frac{\pi}{2}, \frac{\pi}{2}]$ must lie on $(\frac{\pi}{6}, y)$ or $(-\frac{\pi}{6}, y)$. By symmetry, any extremum must also lie on $(x, \frac{\pi}{6})$ or $(x, -\frac{\pi}{6})$. So there are only four possible points that could yield an extremum.

Note that if $x + y = 0$, then $\sin(x+y) = 0$ so that $f(x,y) = 0$. So we need only check two points: $f(\frac{\pi}{6}, \frac{\pi}{6}) = \frac{3\sqrt{3}}{8}$ and $f(-\frac{\pi}{6}, -\frac{\pi}{6}) = -\frac{3\sqrt{3}}{8}$. (Note that rather than using direct calculation, the latter can be obtained from the former by noting that $f(-x,-y) = \cos(-x)\cos(-y)\sin(-(x+y)) = \cos x \cos y[-\sin(x+y)] = -f(x,y)$.)

So f attains a maximum value of $\frac{3\sqrt{3}}{8}$ and a minimum value of $-\frac{3\sqrt{3}}{8}$. Thus $|\cos x \cos y \sin(x+y)| = |f(x,y)| \le \frac{3\sqrt{3}}{8}$. $\square$

Solution 5 by Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $f(x,y) = \cos x \cos y \sin(x+y)$, and consider $g(x) = f(x,x) = \cos^2 x \sin(2x)$, which has period π. Since $g'(x) = (2\cos(2x) - 1)(\cos(2x) + 1)$, by the first derivative test we see that g achieves its maximum value of $\frac{3\sqrt{3}}{8}$ at $x = \frac{\pi}{6} + n\pi$ and its minimum value of $-\frac{3\sqrt{3}}{8}$ at $x = -\frac{\pi}{6} + n\pi$, where n is an integer. Thus

$$f\Big(\frac{\pi}{6} + n\pi, \frac{\pi}{6} + n\pi\Big) = \frac{3\sqrt{3}}{8} \text{ and } f\Big(-\frac{\pi}{6} + n\pi, -\frac{\pi}{6} + n\pi\Big) = -\frac{3\sqrt{3}}{8}.$$

Since $f(x, y)$ attains the two values above, in searching for absolute extreme values of $f(x, y)$ we may assume $f(x, y) \neq 0$; that is, we assume $\cos x, \cos y$ and $\sin(x + y)$ are all nonzero.
Since the partial derivatives of $f(x, y) = \cos x \cos y \sin(x + y)$ are

$$f_x(x, y) = \cos y(\cos x \cos(x + y) - \sin x \sin(x + y)) \text{ and}$$

$$f_y(x, y) = \cos x(\cos y \cos(x + y) - \sin y \sin(x + y)),$$

then any critical points with $f(x, y) \neq 0$ must satisfy

$$\sin x \cos y \sin(x + y) = \cos x \cos y \cos(x + y) = \cos x \sin y \sin(x + y),$$

and $\tan x = \tan y$. Thus, $y = x + n\pi$, where n is an integer, and since $\cos^2(n\pi) = 1$, then

$$f(x, y) = \cos x \cos(x+n\pi) \sin(2x+n\pi) = \cos^2 x \cos^2(n\pi) \sin(2x) = \cos^2 x \sin(2x) = g(x).$$

From the analysis of $g(x)$ above, $f(x, y)$ must achieve its maximum at $\frac{3\sqrt{3}}{8}$ and its minimum at $-\frac{3\sqrt{3}}{8}$. □

Solution 6 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.
Note that

$$|\cos x \cos y \sin(x + y)| \leq \frac{3\sqrt{3}}{8} \Leftrightarrow (\cos x \cos y \sin(x + y))^2 \leq \frac{27}{64}$$

$$(\sin(x + y) + \sin x + \sin y)^2 \leq \frac{27}{4},$$

which must be proved.
Let $f(x, y) = \sin(x + y) + \sin x + \sin y$, over $x, y \in \mathbb{R}$. It is enough to show that $f(x, y)^2 \leq \frac{27}{4}$.
Observe that $f(x, y) = f(2a\pi + x, 2b\pi + y), \forall a, b \in \mathbb{Z}$; so, WLOG, $x, y \in [0, 2\pi]$.
CASE 1: If $x, y \in [0, \pi]$.
We have

$$(1) \qquad -1 \leq \sin(x + y) \leq f(x, y) \leq \sin(x + y) + 2\sin\Big(\frac{x + y}{2}\Big)$$

Consider the function $f_1(x) = \sin 2x+2 \sin x, \forall x \in [0, \pi]$. Then, $f_1'(x) = 2(2\cos x1)(\cos x+1)$; f_1 is increasing when $x \in [0, \frac{\pi}{3}]$ and decreasing when $x \in [\frac{\pi}{3}, \pi]$. Therefore, $\sin(x + y) + 2\sin(\frac{x+y}{2}) \leq f(\frac{2\pi}{3}) = \frac{3\sqrt{3}}{2}$. By (1), we get $-1 \leq f(x, y) \leq \frac{3\sqrt{3}}{2}$ and thus, $f(x, y)^2 \leq \frac{27}{4}$.
CASE 2: If $x, y \in [\pi, 2\pi]$.
Let $x = \pi + x_1$ and $y = \pi + y_1$ where $x_1, y_1 \in [0, \pi]$. Then, $f(x, y) = \sin(x_1 + y_1) - \sin x_1 - \sin y_1$.
We have

$$(2) \qquad \sin(x_1 + y_1) - 2\sin\Big(\frac{x_1 + y_1}{2}\Big) \leq f(x, y) \leq \sin(x_1 + y_1) \leq 2$$

Consider the function $f_2(x) = \sin 2x - 2\sin x, \forall x \in [0, \pi]$. Then, $f_2'(x) = 2(2\cos x + 1)(\cos x - 1)$; f_2 is decreasing for $x \in [0, \frac{2\pi}{3}]$ and increasing for $x \in [\frac{2\pi}{3}, \pi]$. Therefore, $\sin(x_1 + y_1) - 2\sin(\frac{x_1+y_1}{2}) \geq f_2(\frac{2\pi}{3}) = \frac{-3\sqrt{3}}{2}$. By (2), we get $\frac{-3\sqrt{3}}{2} \leq f(x,y) \leq 1$ and thus, $f(x,y)^2 \leq \frac{27}{4}$.
CASE 3: If one of x and y is in $[0,\pi]$ while another one is in $[\pi, 2\pi]$. By symmetry, WLOG $x \in [0,\pi]$ and $y \in [\pi, 2\pi]$.
We have $-1 \leq \sin(x+y) \leq 1, 0 \leq \sin x \leq 1$, and $-1 \leq \sin y \leq 0$. Summing up these 3 inequalities give us $-2 \leq f(x,y) \leq 2$, so $f(x,y)^2 \leq 4 < \frac{27}{4}$.
All 3 cases above yield that $f(x,y)^2 \leq \frac{27}{4}$ and the result follows. □

Solution 7 by Michael C. Fleski, Delta College, University Center, MI.
Let P be the product in question. We want to maximize the quantity
$P = \cos(x)\cos(y)\sin(x+y)$. So, we take derivatives of the expression finding

$$\frac{\partial P}{\partial y} = -\cos(x)\sin(y)\sin(x+y) + \cos(x)\cos(y)\cos(x+y) = 0$$

$$\cos(x)(-\sin(y)\sin(x+y) + \cos(y)\cos(x+y)) = 0$$

$$\cos(x)\cos(x+2y) = 0 \to x = \frac{(2p+1)\pi}{2}; x + 2y = \frac{(2n+1)\pi}{2}$$

and

$$\frac{\partial P}{\partial x} = -\sin(x)\cos(y)\sin(x+y) + \cos(x)\cos(y)\cos(x+y) = 0$$

$$\cos(y)(-\sin(x)\sin(x+y) + \cos(x)\cos(x+y)) = 0$$

$$\cos(y)\cos(2x+y) = 0 \to y = \frac{(2q+1)\pi}{2}; 2x + y = \frac{(2m+1)\pi}{2}$$

with $m, n, p, q \in \mathbb{Z}$
We analyze the results by cases.
CASE 1: $\cos(x) = 0$ or $\cos(y) = 0$
Arbitrarily choosing the case of $\cos(x) = 0$ leads to

$$P = (1)\cos(y)\sin(y) = \frac{1}{2}\sin(2y)$$

The maximum value of $\sin(2y) = 1$ leading to $|P| = \frac{1}{2} < \frac{3\sqrt{3}}{8}$.
For the other conditions, by taking the difference in the equations gives

$$y - x = (n-m)\pi = r\pi \to y = x + r\pi \qquad r \in \mathbb{Z}$$

Because of the periodicty involved with the proble, we can restrict $r = 0, 1$.
By adding the expressions, one find

$$y + x = \frac{1}{3}(n+m)\pi + \frac{\pi}{3}$$

Combining our relations together allows for solutions to the angles of x and y as

$$y = \frac{\pi}{6} + \frac{\pi}{3}(2n-m) \qquad x = \frac{\pi}{6} + \frac{\pi}{3}(2m-n)$$

CASE 2: $n-m=r=0$

This restriction makes $x=y=\frac{\pi}{6}+\frac{n\pi}{3}$. Hence,

n	$x=y$	$\|P\|$
0	$\frac{\pi}{6}$	$\|\cos(\frac{\pi}{6})\cos(\frac{\pi}{6})\sin(\frac{2\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
1	$\frac{3\pi}{6}$	$\|\cos(\frac{3\pi}{6})\cos(\frac{3\pi}{6})\sin(\frac{6\pi}{6})\| = \|(0)(0)(0)\| = 0$
2	$\frac{5\pi}{6}$	$\|\cos(\frac{5\pi}{6})\cos(\frac{5\pi}{6})\sin(\frac{10\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
3	$\frac{7\pi}{6}$	$\|\cos(\frac{7\pi}{6})\cos(\frac{7\pi}{6})\sin(\frac{14\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
4	$\frac{9\pi}{6}$	$\|\cos(\frac{9\pi}{6})\cos(\frac{9\pi}{6})\sin(\frac{18\pi}{6})\| = \|(0)(0)(0)\| = 0$
5	$\frac{11\pi}{6}$	$\|\cos(\frac{11\pi}{6})\cos(\frac{11\pi}{6})\sin(\frac{22\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$

CASE 3: $n-m=r=1$

Since $x+y=\frac{1}{3}(m+n)\pi+\frac{\pi}{3}$ and $m+n$ must be odd, we restrict $m+n=1,3,5$ as $\sin(2\pi+x)=\sin x$.

Therefore, we have cases: $n=1, m=0; n=2, m=1$; and $n=3, m=2$ to consider.

n	m	x	y	$\|P\|$
1	0	$\frac{-\pi}{6}$	$\frac{5\pi}{6}$	$\|\cos(\frac{-\pi}{6})\cos(\frac{5\pi}{6})\sin(\frac{4\pi}{6})\| = \frac{3\sqrt{3}}{8}$
2	1	$\frac{\pi}{6}$	$\frac{7\pi}{6}$	$\|\cos(\frac{\pi}{6})\cos(\frac{7\pi}{6})\sin(\frac{8\pi}{6})\| = \frac{3\sqrt{3}}{8}$
3	2	$\frac{3\pi}{6}$	$\frac{9\pi}{6}$	$\|\cos(\frac{3\pi}{6})\cos(\frac{9\pi}{6})\sin(\frac{12\pi}{6})\| = 0$

Consequently, there is no value of $|P|>\frac{3\sqrt{3}}{8}$. This means that

If $x,y\in\mathbb{R}$, then $||\cos(x)\cos(y)\sin(x+y)||\le\frac{3\sqrt{3}}{8}$. □

Solution 8 by Michel Bataille, Rouen, France.

We have

$$\cos x\cos y\sin(x+y)=\sin x\cos x\cos^2 y+\sin y\cos y\cos^2 x$$

$$=\frac{1}{4}((1+\cos 2y)\sin 2x+(1+\cos 2x)\sin 2y)$$

$$=\frac{1}{4}(\sin 2x+\sin 2y+\sin(2x+2y)),$$

hence the problem boils down to proving that $|f(x,y)|\le\frac{3\sqrt{3}}{2}$ for all $x,y\in\mathbb{R}$ where

$$f(x,y)=\sin x+\sin y+\sin(x+y).$$

Note that due to periodicity it suffices to prove the inequality for $(x,y)\in[-\pi,\pi]\times[-\pi,\pi]$.

Now, if $(u,v)\in\mathbb{R}^2$ and $f(u,v)$ is a local extremum of f, we must have $\frac{\partial f}{\partial x}(u,v)=\frac{\partial f}{\partial y}(u,v)=0$, that is, $\cos u+\cos(u+v)=\cos v+\cos(u+v)=0$ or equivalently: ($u=v(\mod 2\pi)$ and $\cos 2u+\cos u=0$) or ($u=-v(\mod 2\pi)$ and $\cos u=-1$). Thus, the candidates for and extremum in $[-\pi,\pi]\times[-\pi,\pi]$ are $(\frac{\pi}{3},\frac{\pi}{3}),(-\frac{\pi}{3},-\frac{\pi}{3}),(\pi,\pi),(-\pi,-\pi),(-\pi,\pi),(\pi,-\pi)$. Being continuous on the compact set $[-\pi,\pi]\times[-\pi,\pi]$, the function f attains its (absolute) maximum and minimum on this set (and or $\mathbb{R}^2$) at one of these pairs. However, we have $f(\pi,\pi)=f(-\pi,-\pi)=f(-\pi,\pi)=f(\pi,-\pi)=0$ while $f(\frac{\pi}{4},\frac{\pi}{4})>0$ and $f(-\frac{\pi}{4},-\frac{\pi}{4})<0$, hence no extremum is attained at $(\pi,\pi),(-\pi,-\pi),(-\pi,\pi)$ or $(\pi,-\pi)$. It follows that the maximum and the minimum of f are $f(\frac{\pi}{3}, \frac{\pi}{3}) = \frac{3\sqrt{3}}{2}$ and $f(-\frac{\pi}{3}, -\frac{\pi}{3}) = -\frac{3\sqrt{3}}{2}$. Thus we have

$$-\frac{3\sqrt{3}}{2} \le f(x,y) \le \frac{3\sqrt{3}}{2}$$

for all $(x,y) \in [-\pi,\pi] \times [-\pi,\pi]$ (and all $(x, y \in \mathbb{R}^2)$. The result follows. □

Solution 9 by Moti Levy, Rehovot, Israel.

Since

$$|\cos(x)| = \left|\sin\left(\frac{\pi}{2} - x\right)\right| = \left|\sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right)\right| = \sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right),$$
$$|\sin(x)| = \sin(x \mod \pi),$$

the original inequality can be rewritten as follows:

$$\sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right) \sin((x+y) \mod \pi) \le \frac{3\sqrt{3}}{8}.$$

By AM-GM inequality:

$$\sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right) \sin\left(\left(\frac{\pi}{2} - y\right) \mod \pi\right) \sin((x+y) \mod \pi)$$
$$\le \left(\frac{\sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right) + \sin\left(\left(\frac{\pi}{2} - y\right) \mod \pi\right) + \sin((x+y) \mod \pi)}{3}\right)^3.$$

By Jensen's inequality

$$\left(\frac{\sin\left(\left(\frac{\pi}{2} - x\right) \mod \pi\right) + \sin\left(\left(\frac{\pi}{2} - y\right) \mod \pi\right) + \sin((x+y) \mod \pi)}{3}\right)^3$$
$$\le \left(\sin\left(\frac{\left(\frac{\pi}{2} - x\right) \mod \pi + \left(\frac{\pi}{2} - y\right) \mod \pi + \sin((x+y) \mod \pi)}{3}\right)\right)^3$$
$$= \sin^3\left(\frac{\left(\left(\frac{\pi}{2} - x\right) + \left(\frac{\pi}{2} - y\right) + (x+y)\right) \mod \pi}{3}\right)$$
$$= \sin^3\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{8}.$$

□

Solution 10 by Perfetti Paolo, dipatimento di matematica, Universita de "Tor Vergata", Roma, Italy.

It is equivalent

$$F(x,y) = (\cos x)^2 (\cos y)^2 (\sin(x+y))^2 \frac{27}{64}$$

$F(x,y)$ is π - periodic both in x and y.

We search the maximum of $F(x,y)$ which exists because $F(x,y)$ is continuous and periodic hence it suffices to search the maximum in $[0,\pi] \times [0,\pi]$ which is compact. Let's observe that $F(0,y) \equiv F(x,0) = 0$ and $F(\pi,y) = \frac{(\sin(2y))^2}{4}, F(x,\pi) = \frac{(\sin(2x))^2}{4}$ thus on the boundary of the square $[0,\pi] \times [0,\pi]$ the functions does not exceed the value $\frac{1}{4}$.

$$F_x = (-2\sin(2x)(\sin(x+y))^2 + (\cos x)^2 \sin 2(x+y))(\cos y)^2 = 0$$
$$F_y = (-2\sin(2x)(\sin(x+y))^2 + (\cos y)^2 \sin 2(x+y))(\cos x)^2 = 0$$

$$F_x = (-2\sin x\sin(x+y) + 2\cos x\cos(x+y))\cos x(\cos y)^2\sin(x+y) = 0$$
$$F_y = (= 2\sin y\sin(x+y) + 2\cos y\cos(x+y))\cos y(\cos x)^2\sin(x+y) = 0$$

$(x,y) = (\frac{\pi}{2}, y), y \in \mathbb{R}$ and $(x,y) = (x, \frac{\pi}{2}), x \in \mathbb{R}$ all are critical points .Moreover $\{(x,y) \in [0,\pi] \times [0,\pi] : x+y = k\pi, k = 0,1,2,\}$ also are critical points. Since $F(x,y)$ annihilates on each of the above points, no one of them can be point of maximum. Actually the are all point of minimum.
Based on that we can write

$$(1) \qquad F_x = -\sin x\sin(x+y) + \cos x\cos(x+y) = 0 \Rightarrow \operatorname{cotg}(x+y) = \tan x$$

$$(2) \qquad F_y = -\sin y\sin(x+y) + \cos y\cos(x+y) \Rightarrow \operatorname{cotg}(x+y) = \tan y$$

hence $\tan x = \tan y, y = x$. It follows

$$\tan x = \frac{1}{\tan(2x)} \Leftrightarrow \tan x = \frac{1-(\tan x)^2}{2\tan x} \Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6} + k\pi$$

Clearly by periodicity of $F(x,y)$ it suffices to consider $x = \frac{\pi}{6}$ and then $y = \frac{\pi}{6}$

$$F\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{27}{64} > \frac{1}{4}$$

and then $(\frac{\pi}{6}, \frac{\pi}{6})$ is the point of the searched maximum. □

Solution 11 by proposer. First, we prove that for $x, y \in \mathbb{R}$

$$(1) \qquad \cos^2 x + \cos^2 y + \sin^2(x+y) \le \frac{9}{4}$$

$$\frac{1+\cos 2x}{2} + \frac{1+\cos 2y}{2} + 1 - \cos^2(x+y) \le \frac{9}{4}$$
$$2 + 2\cos 2x + 2 + 2\cos 2y + 4 - 4\cos^2(x+y) \le 9$$
$$2(\cos 2x + \cos 2y) - 4\cos^2(x+y) \le 1$$
$$2 \cdot 2\cos\frac{2x+2y}{2}\cos\frac{2x-2y}{2} - 4\cos^2(x+y) \le 1$$
$$4\cos(x+y)\cos(x-y) - 4\cos^2(x+y) \le 1$$
$$4\cos(x+y)[\cos(x-y) - \cos(x+y)] \le 1$$

Denote $x + y = u; x - y = v$

$$4\cos u(\cos v - \cos u) \le 1$$
$$4\cos u\cos v - 4\cos^2 u \le 1$$
$$4\cos^2 u - 4\cos u\cos v + \cos^2 v + \sin^2 v \ge 0$$
$$(2\cos u - \cos v)^2 + \sin^2 v \ge 0$$

By AM-GM:

$$\sqrt[3]{\cos^2 x\cos^2 y\sin^2(x+y)} \le \frac{\cos^2 x + \cos^2 y + \sin^2(x+y)}{3} \overset{(1)}{\le} \frac{\frac{9}{4}}{3} = \frac{3}{4}$$
$$\cos^2 x\cos^2 y\sin^2(x+y) \le \frac{27}{64}$$
$$|\cos x\cos y\sin(x+y)| \le \frac{3\sqrt{3}}{8}$$

Equality holds for $x = y = \frac{\pi}{6}$. □

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com