5727

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5727. If $f : (0, \infty) \to (0, \infty)$ is a continuous function and $\int_a^b f(x)dx = 5(b-a)$ where $0 < a \le b$, then

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11}\Big)dx \le 9(b-a)$$

Solution 1 by Ivan Hadinata, Senior High School 1 Jember, Jember, Idonesia.
Note that

$$(1)\quad \frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11} = 18-2\Big(\frac{4^2}{f(x)+7}+\frac{5^2}{f(x)+9}+\frac{6^2}{f(x)+11}\Big)$$

By Titu's lemma,

$$(2)\qquad 2\Big(\frac{4^2}{f(x)+7} + \frac{5^2}{f(x)+9} + \frac{6^2}{f(x)+11}\Big) \ge \Big(\frac{15^2}{3f(x)+27}\Big) \ge \frac{144}{f(x)+9}$$

By AM-GM inequality,

$$(3)\qquad \frac{144}{f(x)+9} + f(x) + 9 \ge 2\sqrt{\Big(\frac{144}{f(x)+9}\Big)(f(x)+9)} = 24$$

Combining $(1), (2)$ and (3) gives us

$$\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11} \le 18 - (15 - f(x)) = f(x) + 3.$$

Then,

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx \le \int_a^b f(x)dx + \int_a^b 3dx = 8(b-a) \le 9(b-a).$$

proven. Equality holds if and only if $a = b$. $\square$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The function

$$g(x) := \frac{5x+3}{x+7} + \frac{6x+4}{x+9} + \frac{7x+5}{x+11} = 18 - \frac{32}{x+7} - \frac{50}{x+9} - \frac{72}{x+11}$$

is concave on $(0, \infty)$ since $g''(x) < 0$. Therefore, it can be estimated from above by its tangent in the point $(5, g(5))$, i.e., we have

$$g(x) \le g(5) + g'(5)(x-5).$$

We infer that

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11}\Big)dx = \int_a^b g(f(x))dx$$

$$\le \int_a^b \big(g(5)+g'(5)(f(x)-5)\big)dx = g(5)(b-a),$$

since by assumption $\int_a^b (f(x)-5)dx = 0$. Now the inequality follows since $g(5) = \frac{305}{42} \approx 7.2619 < 9$.
Remark: The inequality shown above is sharp. Equality occurs if $f(x) = 5$ on $(0,\infty)$. $\square$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.
We will prove the stronger inequality

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx \le \frac{305}{42}(b-a)$$

Clearly,

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx =$$

$$= \int_a^b \Big(5-\frac{32}{f(x)+7}+6-\frac{50}{f(x)+9}+7-\frac{72}{f(x)+11}\Big)dx =$$

$$= 18(b-a) - \int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx.$$

We need to prove that

$$(*) \qquad \frac{451}{42}(b-a) \le \int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx$$

Let $r > 0$. By the Cauchy-Schwarz inequality for integrals,

$$(b-a)^2 = \left(\int_a^b dx\right)^2 \le \int_a^b (f(x)+r)dx \int_a^b \Big(\frac{1}{f(x)+r}\Big)dx = (5+r)(b-a)\int_a^b \Big(\frac{1}{f(x)+r}\Big)dx$$

which implies

$$\int_a^b \Big(\frac{1}{f(x)+r}\Big)dx \ge \frac{b-a}{5+r}.$$

We conclude that

$$\int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx \ge (b-a)\Big(\frac{32}{5+7}+\frac{50}{5+9}+\frac{72}{5+11}\Big) = \frac{451}{42}(b-a)$$

which is (*). $\square$

Solution 4 by Michel Bataille, Rouen, France.
Let $g(x) = \frac{5x+3}{x+7}, h(x) = \frac{6x+4}{x+9}, k(x) = \frac{7x+5}{x+11}$. It is easily checked that g, h, k are non-decreasing and concave on $(0,\infty)$.
We want to prove that $\int_a^b \phi(x)dx \le 9(b-a)$ where $\phi(x) = g(f(x)) + h(f(x)) + k(f(x))$..
Let m and M be the minimum and the maximum of the continuous function f on the interval $[a,b]$.
Then, $0 < m \le M$ and since $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$, the hypothesis gives $m \le 5 \le M$.
From the concavity of g on the interval $[m,M]$, the curve $y = g(x)$ is under its tangent at $(5, g(5))$.
The equation of this tangent is $y - \frac{7}{3} = \frac{2}{9}(x-5)$ (note that $g'(x) = \frac{32}{(x+7)^2}$), that

is, $y = \frac{2x}{9} + \frac{11}{9}$ and therefore $g(f(x)) \le \frac{2f(x)}{9} + \frac{11}{9}$ for $x \in [a, b]$.
Similar calculations lead to $h(f(x)) \le \frac{25f(x)}{98} + \frac{113}{98}$ and $k(f(x)) \le \frac{9f(x)}{32} + \frac{35}{32}$ and we deduce that for $x \in [a, b]$,

$$\phi(x) \le \Big(\frac{2}{9} + \frac{25}{98} + \frac{9}{32}\Big) \cdot f(x) + \frac{11}{9} + \frac{113}{98} + \frac{35}{32} = \frac{10705}{14112} \cdot f(x) + \frac{48955}{14112}.$$

Integrating yields

$$\int_a^b \phi(x)dx \le \frac{10705}{14112} \int_a^b f(x)dx + \frac{48955}{14112}(b-a),$$

that is,

$$\int_a^b \phi(x)dx \le \Big(\frac{53525}{14112} + \frac{48955}{14112}\Big)(b-a) = \frac{6405}{882}(b-a).$$

Since $\frac{64405}{882} < 9$, we obtain a sharper result than the required one. □

Solution 5 by proposer.

$$\frac{5f(x)+3}{f(x)+7} \le \frac{f(x)+1}{2} \Leftrightarrow 10f(x) + 6 \le$$

$$\le f^2(x) + 8f(x) + 7 \Leftrightarrow f^2(x) - 2f(x) + 1 \ge 0$$

$$\Leftrightarrow (f(x)-1)^2 \ge 0$$

(1) $$\frac{5f(x)+3}{f(x)+7} \ge \frac{f(x)+1}{2}$$

$$\frac{6f(x)+4}{f(x)+9} \le \frac{f(x)+1}{2} \Leftrightarrow 12f(x) + 8 \le f^2(x) + 10f(x) + 9$$

$$\Leftrightarrow f^2(x) - 2f(x) + 1 \ge 0 \Leftrightarrow (f(x)-1)^2 \ge 0$$

(2) $$\frac{6f(x)+4}{f(x)+9} \le \frac{f(x)+1}{2}$$

$$\frac{7f(x)+5}{f(x)+11} \le \frac{f(x)+1}{2} \Leftrightarrow 14f(x) + 10 \le f^2(x) + 12(x) + 11$$

$$\Leftrightarrow f^2(x) - 2f(x) + 1 \ge 0 \Leftrightarrow (f(x)-1)^2 \ge 0$$

(3) $$\frac{7f(x)+5}{f(x)+11} \le \frac{f(x)+1}{2}$$

By adding (1);(2);(3):

$$\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11} \le \frac{3}{2}(f(x)+1)$$

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11}\Big)dx \le$$

$$\le \frac{3}{2}\Big(\int_a^b f(x)dx + \int_a^b dx\Big) = \frac{3}{2}(5(b-a) + (b-a)) =$$

$$= 9(b-a)$$

Equality holds for $a = b$. □

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