5739

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5739. Prove that for any triangle ΔABC:

$$\frac{h_a^3}{h_b^3}+\frac{h_b^3}{h_c^3}+\frac{h_c^2}{h_a^3}\geq\frac{\sin^2 B}{\sin^2 A}+\frac{\sin^2 C}{\sin^2 B}+\frac{\sin^2 A}{\sin^2 C}$$

where h_a, h_b, h_c are the altitudes respectively issued from the vertices A, B, C.

Solution 1 by Michel Bataille, Rouen, France.
Let F and R be the area and the circumradius of the triangle. Let $a = BC$, $b = CA, c = AB$. Since $ah_a = bh_b = ch_c = 2F$ and $2R\sin A = a$, $2R\sin B = b, 2R\sin C = c$, the inequality is equivalent to

$$\frac{b^3}{a^3}+\frac{c^3}{b^3}+\frac{a^3}{c^3}\geq\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\tag{1}$$

From an inequality of means, we have

$$\frac{b^3}{a^3}+\frac{c^3}{b^3}+\frac{a^3}{c^3}\geq\frac{1}{\sqrt{3}}\Big(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\Big)^{\frac{3}{2}}\tag{2}$$

and from AM-GM, we have

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\geq 3\Big(\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}\cdot\frac{a^2}{c^2}\Big)^{\frac{1}{3}}=3$$

hence

$$\Big(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\Big)^{\frac{3}{2}}=\Big(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\Big)^{\frac{1}{2}}\Big(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\Big)\geq\sqrt{3}\Big(\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\Big).$$

Combining with (2), the desired inequality (1) follows. □

Solution 2 by Albert Stadler, Herrliberg, Switzerland.
By law of sines,

$$\frac{h_a}{h_b}=\frac{b}{a}=\frac{\sin B}{\sin A},\frac{h_b}{h_c}=\frac{c}{b}=\frac{\sin C}{\sin B},\frac{h_c}{h_a}=\frac{a}{c}=\frac{\sin A}{\sin C}.$$

So

$$\frac{h_a^3}{h_b^3}+\frac{h_b^3}{h_c^3}+\frac{h_c^3}{h_a^3}=\frac{\sin^3 B}{\sin^3 A}+\frac{\sin^3 C}{\sin^3 B}+\frac{\sin^3 A}{\sin^3 C}$$

By Hölder's inequality,

$$\frac{\sin^2 B}{\sin^2 A}+\frac{\sin^2 C}{\sin^2 B}+\frac{\sin^2 A}{\sin^2 C}\leq\Big(\frac{\sin^3 B}{\sin^3 A}+\frac{\sin^3 C}{\sin^3 B}+\frac{\sin^3 A}{\sin^3 C}\Big)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}}$$

It remains to prove that

$$\Big(\frac{\sin^3 B}{\sin^3 A}+\frac{\sin^3 C}{\sin^3 B}+\frac{\sin^3 A}{\sin^3 C}\Big)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}}\leq\frac{\sin^3 B}{\sin^3 A}+\frac{\sin^3 C}{\sin^3 B}+\frac{\sin^3 A}{\sin^3 C}$$

which is equivalent to $\frac{sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C} \geq 3$. However this inequality follows from the AM-GM inequality:

$$\frac{1}{3}\Big(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\Big) \geq \frac{\sin B}{\sin A} \cdot \frac{\sin C}{\sin B} \cdot \frac{\sin A}{\sin C} = 1.$$

□

Solution 3 by proposer.
First we prove that:

$$\sum_{cyc} \frac{h_a^3}{h_b^3} \geq \sum_{cyc} \frac{h_a^2}{h_b^2} \tag{1}$$

By multiplying (1) with $(h_a h_b h_c)^3$:

$$\sum_{cyc} h_a^6 h_c^3 \geq \sum_{cyc} h_a^5 h_b h_c^3 \tag{2}$$

We prove (2):

$$\sum_{cyc} h_a^6 h_c^3 = \sum_{cyc} \frac{9h_a^6 h_c^3}{9} =$$

$$= \sum_{cyc} \frac{7h_a^6 h_c^3 + h_a^6 h_c^3 + h_a^6 h_c^3}{9} =$$

$$= \frac{1}{9} \sum_{cyc} (7h_a^6 h_c^3 + h_b^6 h_a^3 + h_c^6 h_b^3) \overset{\text{AM-GM}}{\geq}$$

$$\geq \frac{1}{9} \cdot 9 \sum_{cyc} \sqrt[9]{(h_a^6 h_c^3)^7 \cdot h_b^6 h_a^3 \cdot h_c^6 h_b^3} =$$

$$= \sum_{cyc} \sqrt[9]{h_a^{45} \cdot h_b^9 \cdot h_c^{27}} = \sum_{cyc} h_a^5 h_b h_c^3$$

Result (2) is true. Result (1) is true.

$$\sum_{cyc} \frac{h_a^3}{h_b^3} \geq \sum_{cyc} \frac{h_a^2}{h_b^2} = \sum_{cyc} \frac{(\frac{2F}{a})^2}{(\frac{2F}{b})^2} =$$

$$= \sum_{cyc} \Big(\frac{4F^2}{a^2} \cdot \frac{b^2}{4F^2}\Big) = \sum_{cyc} \frac{b^2}{a^2} =$$

$$\overset{\text{sine-law}}{=} \sum_{cyc} \frac{(2R \sin B)^2}{(2R \sin A)^2} = \sum_{cyc} \frac{4R^2 \sin^2 B}{4R^2 \sin^2 A} =$$

$$= \sum_{cyc} \frac{\sin^2 B}{\sin^2 A} = \frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C}$$

Equality holds for an equilateral triangle: $a = b = c$.

□

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