PROPOSED PROBLEM

DANIEL SITARU - ROMANIA

5751. Show that if $0 < a \leq b < \frac{\pi}{2}$, then:

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2 \tan(x)dx \leq 0$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
It is sufficient to prove that:

$$\tan x \geq \frac{3x}{3 - x^2}, 0 \leq x < \frac{x}{2},$$

for then

$$0 \geq 2\int_a^b \left((x^2-3)\tan(x)+3x\right)dx = 6\log(\cos(x))\Bigg|_{x=a}^{x=b} +3x^2\Bigg|_{x=a}^{x=b} +2\int_a^b x^2 \tan(x)dx =$$

$$= 6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2 \tan(x)dx.$$

To prove initially stated inequality we start from the product representation of the cosine function:

$$\cos x = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{\pi^2(2n-1)^2}\right).$$

Logarithmic differentiation then gives

$$\tan x = -\frac{d}{dx}\log(\cos(x)) = 2\sum_{n=1}^{\infty}\frac{\frac{4x}{\pi^2(2n-1)^2}}{1 - \frac{4x^2}{\pi^2(2n-1)^2}} = 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}(2n-1)^{2k}} =$$

$$= 2\sum_{k=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}\right) = 2\sum_{k=1}^{\infty}\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}}x^{2k-1}.$$

Thus, if $0 \leq x < \frac{\pi}{2}$,

$$\tan x - \frac{3x}{3 - x^2} = 2\sum_{k=1}^{\infty}\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}}x^{2k-1} - \sum_{k=1}^{\infty}\frac{1}{3^{k-1}}x^{2k-1} =$$

$$= \sum_{k=3}^{\infty}\left(2\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}}\right)x^{2k-1} \geq 0,$$

taking into account that $\frac{\pi}{2} < \sqrt{3}, (2) = \frac{\pi^2}{6}, (4) = \frac{\pi^4}{90}, (2k) > 1, k \geq 1, (2\pi)^2 < 40$ so that

$$2\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}} > 2\frac{4^k - 1}{10^k} - \frac{1}{3^{k-1}} > 0, k \geq 3.$$

□

Solution 2 by Michel Bataille, Rouen, France.
The inequality is equivalent to

$$-3\int_a^b \tan(x)dx + 3\int_a^b xdx + \int_a^b x^2\tan(x)dx \leq 0,$$

that is, to $\int_a^b f(x)dx \geq 0$ where

$$f(x) = 3\tan(x) - 3x - x^2\tan(x).$$

Thus, it suffices to prove that $f(x) \geq 0$ for $x \in [0, \frac{\pi}{2})$. Since $f(0) = 0$, it is even sufficient to prove that $f'(x) \geq 0$.
A simple calculation gives $f'(x) = \frac{1}{\cos^2(x)}, g(x)$ where $g(x) = 3\sin^2(x) - x\sin(2x) - x^2$.
Now, for $x \in [0, \frac{\pi}{2})$, we obtain

$$g'(x) = 6\sin(x)\cos(x) - \sin(2x) - 2x\cos(2x) - 2x = 2\sin(2x) - 2x(1+\cos(2x)) = 4\cos^2(x)(\tan(x) - x);$$

since $\tan(x) \geq x$, we have $g'(x) \geq 0$, hence $g(x) \geq g(0)$ and consequently $f'(x) \geq 0$, as desired. □

Solution 3 by Moti Levy, Rehovot, Israel.
We rewrite the problem statement as follow:

$$\int_a^b x^2\tan(x)dx \leq -3\ln(\cos(b)) - \frac{3}{2}b^2 + 3\ln(\cos(a)) + \frac{3}{2}a^2 \tag{1}$$

Let

$$F(x) := -\left(3\ln(\cos(x)) + \frac{3}{2}x^2\right) \tag{2}$$

The inequality is equivalent to

$$\int_a^b x^2\tan(x)dx \leq F(b) - F(a), \tag{3}$$

but

$$F(b) - f(a) = \int_a^b 3(\tan(x) - x)dx.$$

Hence the original inequality is equivalent to

$$\int_a^b x^2\tan(x)dx \leq \int_a^b 3(\tan(x) - x)dx,$$

or to

$$\int_a^b \left((x^2 - 3)\tan(x) + x\right)dx \leq 0. \tag{4}$$

We now prove (4) by showing that the integrand is negative in (a, b) where $0 < a \leq b < \frac{\pi}{2}$

$$(x^2 - 3)\tan(x) + x \leq 0. \tag{5}$$

Inequality (5) is equivalent to

$$\frac{\tan(x)}{x} \geq \frac{1}{3 - x^2} \tag{6}$$

The series expansion of $\frac{\tan(x)}{x}$ implies that

$$\frac{\tan(x)}{x} \geq 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4. \tag{7}$$

One can check that

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3-x^2} \geq 0,$$

since the function $30 - 2x^4 - x^6$ is concave in $0 < x < \frac{\pi}{2}$ then

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3-x^2} = \frac{30 - 2x^4 - x^6}{15(3-x^2)} \geq 0 \text{ for } 0 < x < \frac{\pi}{2}, \tag{8}$$

It follows from (7) and (8) that the inequality $\frac{\tan(x)}{x} \geq \frac{1}{3-x^2}$ holds for $0 < x < \frac{\pi}{2}$. □

Solution 4 by Perfetti Paolo, dipatimento di matematica Universita di "Tor Vergata", Roma, Italy.

$$\frac{d}{da}\left(6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b x^2\tan(x)dx\right) = 2((3-a^2)\tan a - 3a)$$

$$3 - a^2 \geq 3 - \frac{\pi^2}{4} \text{ and } \tan a \geq a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^5}{315} \text{ thus}$$

$$(3-a^2)\tan a - 3a \geq (3-a^2)\left(a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^5}{315}\right) - 3a \geq$$

$$= \frac{a^5}{315}(21 + 9a^2 - 17a^4) \geq 0 \text{ for } a \leq \left(\frac{9+\sqrt{1509}}{34}\right)^{\frac{1}{2}} \sim 1.186$$

Thus for $a \leq 1.18$ the inequality is proved.
Now let's define $b = \frac{\pi}{2} - a$. The inequality $(3-a^2)\tan a - 3a$ becomes

$$\left(3 - \left(\frac{\pi}{2} - b\right)^2\right)\frac{\cos b}{\sin b} - 3\left(\frac{\pi}{2} - b\right) \geq \left(3 - \left(\frac{3}{2} - b4\right)^2\frac{1-\frac{b^2}{2}}{b} - 3\left(\frac{\pi}{2} - b\right)\right) \tag{1}$$

for $0 \leq b \leq \frac{\pi}{2} - 1.18 \sim 0.3907$ and $\cos b \geq 1 - \frac{b^2}{2}$, and $\sin b \leq b$. The r.h.s of (1)

$$\frac{24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4}{8b} \geq 0, \quad 0 \leq b \leq \frac{2}{5}$$

$$f(b) = 24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4 \text{ and}$$

$$f'(b) = 16b^3 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leq 16b^2 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leq 0$$

if and only if

$$2\pi - b(4+\pi^2) + (16\pi - 8)b^2 \geq 0 \text{ (true by) } (4+\pi^2)^2 - 8\pi(16\pi - 8) \sim -1011 < 0$$

This implies that $f(b)$ decreases and since

$$f\left(\frac{2}{5}\right) = \frac{15464}{625} - \frac{46\pi^2}{25} - \frac{232\pi}{125} \sim 0.75 \Rightarrow f(b) > 0$$

and this in turn implies that through (1) the inequality $(3-a^2)\tan a - 3a > 0$ also for $1.18 \leq a \leq \frac{\pi}{2}$. This implies that

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b \tan(x)dx$$

increases with a and then the maximum value is attained when $a = b$ thus proving the inequality. □

Solution 5 by proposed by G.C. Greubel, Newport News, VA.
Using the series

$$\ln(\cos(x)) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{a_{2k}x^{2k}}{k}$$

$$\tan(x) = \sum_{k=1}^{\infty} a_{2k}x^{2k-1},$$

and integral

$$\int x^2\tan(x)dx = \sum_{k=2}^{\infty}\frac{a_{2k-2}x^{2k}}{2k},$$

where

$$a_{2k} = \frac{4^k(4^k-1)|B_{2k}|}{(2k)!}$$

with B_n being the Bernoulli numbers, then

$$S = 6\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b x^2\tan(x)dx$$

$$= -3\sum_{k=1}^{\infty}\frac{a_{2k}}{k}(b^{2k}-a^{2k}) + 3(b^2-a^2) + 2\sum_{k=2}^{\infty}\frac{a_{2k-2}}{2k}(b^{2k}-a^{2k})$$

$$= -3\sum_{k=2}^{\infty}\frac{a_{2k}}{k}(b^{2k}-a^{2k}) + \sum_{k=2}^{\infty}\frac{a_{2k-2}}{k}(b^{2k}-a^{2k})$$

$$= -\sum_{k=2}^{\infty}\frac{3a_{2k}-a_{2k-2}}{k}(b^{2k}-a^{2k}).$$

Since $a_2 = 1$ and $a_4 = 1$ then:

$$S = -\sum_{k=3}^{\infty}\frac{3a_{2k}-a_{2k-2}}{k}(b^{2k}-a^{2k})$$

It is evident that $3a_{2n} > a_{2n-2}$ for $n \geq 3$ and leads to

$$6\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + \int_a^b x^2\tan(x)dx \leq 0$$

for $b \geq a$. Equality occurs when $b = a$. □

Solution 6 by proposer.

$$(1+x^2)(1+y^2) \overset{\text{AM-GM}}{\geq} 2x(1+y^2)$$

$$(1+x^2)(1+y^2) \overset{\text{AM-GM}}{\geq} 2y(1+x^2)$$

By adding:

$$2(1+x^2)(1+y^2) \geq 2x(1+y^2) + 2y(1+x^2)$$

$$(1+x^2)(1+y^2) \geq x(1+y^2) + y(1+x^2)$$

$$\frac{1}{x(1+y^2)+y(1+x^2)} \geq \frac{1}{(1+x^2)(1+y^2)}$$

$$\int_a^b\int_a^b \frac{dxdy}{x(1+y^2)+y(1+x^2)} \geq \int_a^b\int_a^b \frac{dxdy}{(1+x^2)(1+y^2)} =$$

$$= \left(\int_a^b \frac{dx}{1+x^2}\right)\left(\int_a^b \frac{dy}{1+y^2}\right) = (\arctan b - \arctan a)^2$$

Equality holds for $a = b$. □

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com