5757

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5757. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x)dx = \frac{1}{2}$$

Show that

$$2 + \int_0^1 f^2(x)dx \geq 6\int_0^1 xf(x)dx$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
We prove the more general inequality

$$\frac{1}{4}\int_0^1 f^2(x)dx + 2\left(\int_0^1 f(x)dx\right)^2 \geq 3\int_0^1 f(x)dx \cdot \int_0^1 xf(x)dx \tag{1}$$

Then substituting the given integral value and clearing fractions gives us the desired inequality.
Now set $\int_0^1 f(x)dx = t$ and consider the quadratic polynomial

$$t^2 - 3\left(\int_0^1 \left(x - \frac{1}{3}\right)f(x)\right)t + \frac{1}{4}\int_0^1 f^2(x)dx \tag{2}$$

The discriminant of this polynomial is

$$D = 9\left(\int_0^1 \left(x - \frac{1}{3}\right)f(x)dx\right)^2 - \int_0^1 f^2(x)dx$$

The CBS inequality yields

$$\begin{aligned} D &\leq 9 \cdot \int_0^1 \left(x - \frac{1}{3}\right)^2 dx \cdot \int_0^1 f^2(x)dx - \int_0^1 f^2(x)dx \\ &= \int_0^1 f^2(x)dx - \int_0^1 f^2(x)dx = 0. \end{aligned}$$

Since $D \leq 0$ and the coefficient of t^2 in (2) is positive, we see that the quadratic is nonnegative for all values of t. Therefore

$$\begin{aligned} \left(\int_0^1 f(x)dx\right)^2 + \frac{1}{4}\int_0^1 f^2(x)dx &\geq 3\left(\int_0^1 \left(x - \frac{1}{3}\right)f(x)dx\right) \cdot \int_0^1 f(x)dx \\ &= 3\left(\int_0^1 xf(x)dx - \frac{1}{3}\int_0^1 f(x)dx\right) \cdot \int_0^1 f(x)dx \\ &= 3\int_0^1 xf(x)dx \cdot \int_0^1 f(x)dx - \left(\int_0^1 f(x)dx\right)^2. \end{aligned}$$

which gives us (1). □

Solution 2 by Perfetti Paolo, dipartimento de matematica Universita di "Tor Vergata", Roma, Italy.

$$\int_0^1 (f-3x+a)^2 dx = \int_0^1 (f^2 - 6xf + 9x^2 + a^2 + 2af - 6xa) dx \geq 0$$

Thus

$$\int_0^1 (f^2 - 6xf) dx \geq -3 - a^2 - a + 3a \geq -2 \Leftrightarrow (a-1)^2 \leq 0 \Leftrightarrow a = 1$$

and this concludes the proof. □

Solution 3 by Albert Stadler, Herrliberg, Switzerland.
Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x)dx = \frac{1}{2}$. Show that

$$2 + \int_0^1 f^2(x)dx \geq 6 \int_0^1 xf(x)dx.$$

Solution of the problem
We have

$$0 \leq \int_0^1 (f(x) - 3x + 1)^2 dx = \int_0^1 (f^2(x) + 9x^2 + 1 - 6xf(x) + 2f(x) - 6x) dx =$$

$$= \int_0^1 f^2(x)dx + 3 + 1 - 6\int_0^1 xf(x)dx + 1 - 3$$

which implies

$$2 + \int_0^1 f^2(x)dx \geq 6 \int_0^1 xf(x)dx.$$

□

Solution 4 by Moti Levy, Rehovot, Israel.
Let $F(x) := \int_0^x f(t)dt$. After integrations by parts,

$$\int_0^1 xf(x)dx = xF(x)\Big]_0^1 - \int_0^1 F(x)dx = \frac{1}{2} - \int_0^1 F(x)dx \tag{3}$$

Substituting (3) in the original inequality we get

$$2 + \int_0^1 \Big(F'(x)\Big)^2 dx \geq 3 - 6\int_0^1 F(x)dx \int_0^1 xf(x)dx,$$

or,

$$\int_0^1 \Big(6F(x) + \Big(F'(x)\Big)^2\Big) dx \geq 1$$

Let

$$J(F) := \int_0^1 \Big(6F(x) + \Big(F'(x)\Big)^2\Big) dx \geq 1,$$

then the original inequality is equivalent to the statement that the functional $J(F)$ is grater than or equal to 1 for every differentiable function $F(x)$, which satisfies the boundary conditions $F(0) = 0$ and $F(1) = \frac{1}{2}$.
Every differentiable function $F(x)$, which satisfies the boundary conditions $F(0) = 0$ and $F(1) = \frac{1}{2}$ can be expressed as $F(x) = \frac{3}{2}x^2 - x + \eta(x)$, where $\eta(x)$ is

defferentiable function in the interval $(0,1)$ and $\eta(0)=\eta(1)=0$.
Then

$$J\Big(\frac{3}{2}x^2-x+\eta(x)\Big)=\int_0^1\Big(6\Big(\frac{3}{2}x^2-x+\eta(x)\Big)+\Big(3x-1+\eta(x)\Big)^2\Big)dx$$

$$=\int_0^1\Big(6\Big(\frac{3}{2}x^2-x\Big)+(3x-1)^2\Big)dx+\int_0^1 6\eta(x)+2(3x-1)\eta'(x)+(\eta'(x))^2dx$$

$$=1+\int_0^1 6\eta(x)+2(3x-1)\eta(x)+(\eta'(x))^2dx$$

Applying integration by parts, we obtain

$$J\Big(\frac{3}{2}x^2-x+\eta(x)\Big)=1+\int_0^1(\eta'(x))^2dx$$

It follows that $JF(x))\geq 1$ for every differentialbe function $F(x)$ which satisfies $F(0)=0$ and $F(1)=\frac{1}{2}$. The functional $J(F)$ attains its minimum when $\eta'(x)=0$ which implies (together with the boundary conditions $\eta(0)=\eta(1)=0$ that $\eta(x)=0$ in $(0,1)$. □

Solution 5 by Michel Bataille, Rouen, France.
Let $I=\int_0^1(3x-1)f(x)dx$. Then, we have

$$6\int_0^1 xf(x)dx=2I+2\int_0^1 f(x)dx=2I+1.$$

On the other hand, since $\int_0^1(3x-1)^2dx=\int_0^1(9x^2-6x+1)dx=1$, the Cauchy-Schwarz inequality gives

$$\int_0^1 f^2(x)dx=\left(\int_0^1(3x-1)^2dx\right)\left(\int_0^1 f^2(x)dx\right)\geq\left(\int_0^1(3x-1)f(x)dx\right)^2=I^2$$

As a result, we obtain

$$2+\int_0^1 f^2(x)dx-6\int_0^1 xf(x)dx\geq 2+I^2-2I-1=(I-1)^2\geq 0$$

and the desired inequality follows. □

Solution 6 by proposer.

$$2+\int_0^1 f^2(x)dx\geq 6\int_0^1 xf(x)dx$$

$$\int_0^1 f^2(x)dx-6\int_0^1 xf(x)dx+1+3-3+1\geq 0$$

$$\int_0^1 f^2(x)dx-6\int_0^1 xf(x)dx+2\cdot\frac{1}{2}+9\cdot\frac{1}{3}-6\cdot\frac{1}{2}+1\geq 0$$

$$\int_0^1 f^2(x)dx-6\int_0^1 xf(x)dx+2\int_0^1 f(x)dx+\int_0^1(9x^2-6x+1)dx\geq 0$$

$$\int_0^1 f^2(x)dx-2\int_0^1(3x-1)f(x)dx+\int_0^1(3x-1)^2dx\geq 0$$

$$\int_0^1(f(x)-(3x-1))^2dx\geq 0$$

Equality holds for $f(x) = 3x - 1$.

$$LHS = 2 + \int_0^1 (3x-1)^2 dx = 2 + 9 \cdot \frac{1}{3} - 6 \cdot \frac{1}{2} + 1 = 6 - 3 = 3$$

$$RHS = 6 \int_0^1 x(3x-1) dx = 18 \cdot \frac{1}{3} - 6 \cdot \frac{1}{2} = 6 - 3 = 3$$

□

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