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GENERALISED SIMSON'S THEOREM — A PLAGIOGONAL APPROACH

(GAK-NAG THEOREM)

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Abstract: In this work an attempt has been made with the help of PLAGIOGONAL approach to solve for a generalized Simson's theorem based on a problem proposed by Michael Metaxas in 2013 in Geometria Super Top – an international Math group of Brasil in Facebook. The theorem is proposed by Athanasios Gakopoulos and is investigated by Debabrata Nag using the concept of PLAGIOgonal analytical geometry. Here we have first presented the theorem and subsequently address the problem as mentioned above.

Keywords: **Simson's Theorem, PLAGIOGONAL, Analytical, Geometry, Circumcircle, Circumradius**

1.0 THE GENERALISED SIMPSON'S THEOREM:

Consider a triangle *ABC* with its *circumcircle* and another concentric circle having its radius more or less than the *circumradius R* of the triangle. Let P be any arbitrary point on the concentric circle of radius R' and three perpendiculars are dropped from it on the three sides of the triangle and thereby form a pedal triangle *DEF*. It is our claim that:

$$\frac{[\Delta DEF]}{[\Delta ABC]} = \frac{\rho^2}{4R^2} = \pm\frac{1}{4}\left[\left(\frac{R'}{R}\right)^2 - 1\right]$$

where term ρ in the numerator is zero when the arbitrary point P is lying on the *circumcircle* of the triangle (the Simson's theorem) and '±' occurs whether the radius of concentric circle is more or less than the *circumradius* of the given triangle. See the figure shown below for the two cases mentioned.

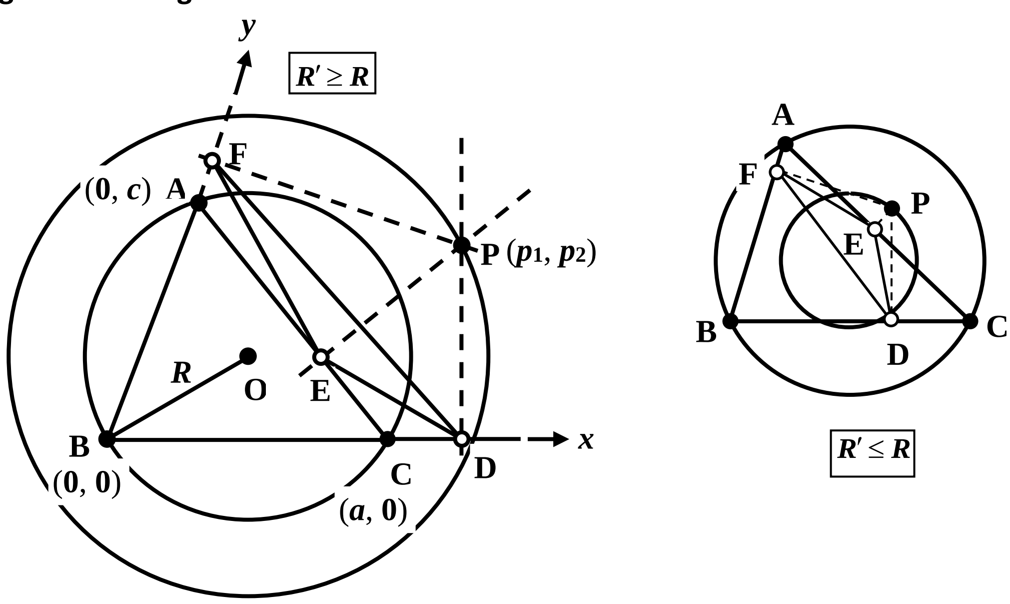

Figure 1: Generalised Simson's Theorem for a triangle *ABC*

2.0 MICHAEL METAXAS' PROBLEM: **It states that in the above figure if the radius of the concentric circle is $\sqrt{5}$ times the *circumradius* of the triangle *ABC* then area of the pedal triangle *DEF* equals to that of the triangle *ABC*.**

3.0 PROOF OF THE GENERALISED SIMPSON'S THEOREM – A PLAGIOGONAL APPROACH: **Let us consider the first case i.e., when $R' \geq R$ as shown in Figure 1. We assume the sides *BC* and *BA* of the triangle as the *x* and *y* – axes respectively with vertex B taken as the origin. Hence the angle between the oblique or PLAGIO axes is $\theta = \angle ABC = B$ also let us assume $\mu = \cos B$ and consequently, $b^2 = a^2 + c^2 - 2ca\mu$. Accordingly, co-ordinates of various points are marked in the above figure itself. If the co-ordinates of an arbitrary point on the bigger circle are assumed as shown as (p_1, p_2) then we know that the co-ordinates of the pedal points D and F are obviously $(p_1 + \mu p_2, 0)$ and $(0, \mu p_1 + p_2)$ respectively. Now, we know the equation of any line perpendicular to the side *AC* and passing though the point P is: $y = p_2 + \frac{a - \mu c}{c - a\mu} x - \frac{a - \mu c}{c - a\mu} p_1$ while equation of the side *AC* is $\frac{x}{a} + \frac{y}{c} = 1$. Thus solving these two equations we get the co-ordinates of E as:**

$$\left(\frac{a}{b^2}\left[(c - p_2)(c - a\mu) + p_1(a - \mu c)\right], \frac{c}{b^2}\left[(a - p_1)(a - c\mu) + p_2(c - a\mu)\right] \right)$$

Hence we can form the ratio of the areas of the pedal triangle to the main as follows:

$$\frac{[\Delta DEF]}{[\Delta ABC]} = \frac{\frac{\sin B}{2} \cdot |Det|}{\frac{ca \sin B}{2}} = \frac{|Det|}{ca} \quad \ldots\ldots\ldots (1)$$

where obviously:

$$Det = \begin{vmatrix} p_1 + \mu p_2 & 0 & 1 \\ \frac{a}{b^2}\left[(c - p_2)(c - a\mu) + p_1(a - \mu c)\right] & \frac{c}{b^2}\left[(a - p_1)(a - c\mu) + p_2(c - a\mu)\right] & 1 \\ 0 & \mu p_1 + p_2 & 1 \end{vmatrix}$$

Now expanding the determinant following the R_1 (row 1) expansion and simplifying the resulting algebra we obtain:

$$|Det| = \frac{ca\sin^2 B}{b^2}\left(p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2\right)$$

$$\Rightarrow |Det| = 2[\Delta ABC]\frac{\sin B}{b^2}\left(p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2\right)$$

Thus from (1) we obtain: $\dfrac{[\Delta DEF]}{[\Delta ABC]} = 2\dfrac{[\Delta ABC]}{ca}\dfrac{\sin B}{b^2}\left(p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2\right)$

Thus, finally: $\dfrac{[\Delta DEF]}{[\Delta ABC]} = \dfrac{\sin^2 B}{b^2}\left(p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2\right)$

$\Rightarrow \boxed{\dfrac{[\Delta DEF]}{[\Delta ABC]} = \dfrac{\rho^2}{4R^2}}$ **[QED] where of course:** $\rho^2 = p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2$

Similarly, for the second case when the radius of the concentric circle is less or equal to that of the *circumcircle*, then obviously we will have the ratio to be negative so we need a negative sign in front of the above expression.

4.0 PROOF OF MICHAEL METAXAS' PROBLEM:

First of all we observe that (refer Figure 1) equation of the *circumcircle* of the triangle *ABC* is $x^2 + y^2 + 2xy\mu - ax - cy = 0$ **… … … (2) and equation to its concentric circle will be:**

$x^2 + y^2 + 2xy\mu - ax - cy + \lambda^2 = 0$ **… … … (3) and since the point P is on this circle thus:**

$$p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2 + \lambda^2 = 0 \Rightarrow p_1^2 + p_2^2 + 2\mu p_1 p_2 - ap_1 - cp_2 = -\lambda \Rightarrow \rho^2 = -\lambda^2$$

Clearly the radius of the circle (3) is given by: $R' = \sqrt{R^2 - \lambda^2} = \sqrt{R^2 + \rho^2}$

Now from the previous article we can say that if

$$[\Delta DEF] = [\Delta ABC] \Rightarrow \rho^2 = 4R^2 \Rightarrow R'^2 = 5R^2 \Rightarrow \boxed{R' = \sqrt{5}\,R}$$ **[QED]**

5.0 CONCLUSION:

From our above discussions we conclude that the generalized Simson's theorem states:

$$\frac{[\Delta DEF]}{[\Delta ABC]} = \frac{\rho^2}{4R^2} = \pm\frac{1}{4}\left\{\left(\frac{R'}{R}\right)^2 - 1\right\} \quad R' \geq R$$
$$= 0 \quad if \;\; P \in (\Delta ABC)$$

where: $\rho^2 = p_1^2 + p_2^2 + 2p_1p_2\mu - ap_1 - cp_2$

Clearly if the arbitrary point P belongs to the *circumcircle* of the triangle the above area ratio becomes zero (as ρ becomes zero) indicating area of the triangle *DEF* is zero implying the fact that three pedal points D, E and F are all collinear (Simson's theorem) and the line known as Simson's line having its equation as:

$$xp_2 + (p_1 + p_2\cos B)y = p_2^2\cos B + p_1p_2 .$$

The above area ratio will be 1 if $R' = \sqrt{5}\,R$ and this result is true irrespective of whether the concentric circle is of larger radius or smaller than the *circumradius* of the given triangle. In the above area ratio, the positive sign occurs when the radius of the concentric circle is more than the *circumradius* of the given triangle and the sign will be negative otherwise.