DANIEL SITARU MARIAN URSĂRESCU

OLYMPIAD PROBLEMS ALGEBRA VOLUME I

ABOUT AUTHORS

Daniel Sitaru, born on 9 August 1963 in Craiova, Romania, is a teacher at National Economic College "Theodor Costescu" in Drobeta Turnu Severin. He published 43 mathematical books, last five of these "Math Phenomenon", "Algebraic Phenomenon", "Analytical Phenomenon", "The Olympic Mathematical Marathon" and "699 Olympic Mathematical Challenges" (the last one with Mihály Bencze), were very appreciated world wide. He is the founding editor of "Romanian Mathematical Magazine", an Interactive Mathematical Journal with 5.600.000 visitors, in the last three years (www.ssmrmh.ro).Many problems from his books were published in famous journals such as "American Mathematical Monthly", "Crux Mathematicorum", "Math Problems Journal", "The Pentagon Journal", La Gaceta de la RSME", "SSMA Magazine". He also published an impressive number of original problems in all mathematical journals from Romania (GMB, Cardinal, Elipsa, Argument, Recreații Matematice). His articles from "Crux Mathematicorum" and "The Pentagon Journal" were also very appreciated.

Marian Ursărescu, was born on 1st of June 1965, in Focșani. He graduated from A.I. Cuza University, Faculty of Mathematics, in 1988. He is a teacher of mathematics from 1988 at "Roman Vodă" National Colledge in Roman. Starting from 1990 until now, he had 47 pupils that participated on the Mathematical National Olympiad, which from 28 had obtained prizes and Olympic mentions.

He published over 100 problems and articles in Mathematical National Gazette . Also, he published several problems and articles in mathematical magazines such as "Mathematical Recreations", "Romanian Mathematical Magazine", "Let's understand math." A lot of his proposed problems had been selected in various mathematical contests, olympiads and mathematical books. He co-authored "Functional Equations" together with M. O. Drâmbe and another 5 books with Mihaly Bencze and Daniel Sitaru.

FROM AUTHORS

In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal.

Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897).

In three years the website of RMM was visited by over 5,000,000 people from all over the world. With over 10,000 proposed problems posted, over 14,000 solutions and many math articles and math notes RMM is a big chance for young mathematicians from whole world to be known as great proposers and solvers. This book is a small part of RMM-Interactive Journal.

Many thanks to RMM-Team for proposed problems and solutions.

PREFACE

Solving problems is an integral and inseparable part of any Mathematical learning process. The present book 'Olympiad Problems ' is aimed to be a step in this direction. The book contains over 230 carefully crafted fully solved problems from Algebra. However, the Problems are neither calibrated nor arranged in any order of difficulty. The problems range from simple to very difficult.Some of these problems have already appeared in the online Romanian Mathematical Magazine (RMM). The RMM team consists of more than 9000 mathematics experts, lovers and enthusiasts. Whenever a problem is proposed in RMM, several group members put up their untiring efforts to provide different solutions to the problem. More than one solution to a problem shows the intrinsic beauty of mathematics - that we can reach the same result by following different approaches.The book 'Olympiad Problems' provides a good opportunity for Mathematical lovers to learn some of the new techniques to solve problems. How a simple substitution, use of an algebraic identity or geometric visualisation reduces a daunting problem to a simple problem are very well illustrated through solutions to the problems in the book.It is hoped that the readers will enrich their mathematical knowledge by using the book. Regarding the misprints and errors in the book, we hope there is none but the experience of last several years suggests otherwise. Whenever you come across an error or misprint in the book, you are requested to bring it to our notice.

Current Position: Retired after serving as an Associate Professor in the Department of Mathematics, Rajdhani college – University of Delhi Served in the University for 40 years Educational Qualification B.A. (Hons.) Mathematics, University of Delhi First Position in the University (Was awarded 2 Gold Medals) M.A. (Mathematics), University of Delhi First Position in the University (Was awarded 3 Gold Medals) M.Phil. (Computer Science), JNU Ph.D. (Mathematics), University of Delhi Project Udaan of CBSE for JEE (Main) Delivered several lectures in the Udaan project of CBSE. Associated with CBSE for other supports in the project. Books authored and co-authored Authored and co-authored several books published by McGraw Hill, Oxford University Press, Pearson and IGNOU Books Published by

McGraw Hill Educations 1. Complete Mathematics for JEE (Main) 3. Comprehensive Mathematics for IIT (Advanced) 4. Coordinate Geometry for Engineering Entrance Examinations 5. IIT Mathematics- Topic wise Solved Questions from 1978 5. Algebra I for JEE (Main) and JEE (Advanced) 6. Algebra II and Statistics for JEE (Main) and JEE (Advanced) 7. Trigonometry for JEE (Main) and JEE (Advanced). (Forthcoming) Books Published by Oxford University Press 1. Advantage Mathematics for Class 8 Books Published by Pearson 1. Mathematics for Class 9 (Forthcoming) 2. Mathematics for Class 10 (Forthcoming) IGONU Project Associated with IGNOU with development of course material Areas of Interest: Real analysis, Complex Analysis, Linear algebra, Probability and Statistics. Research Papers and Other Publications Published several research papers in reputed international Journals.

Dr. Ravi Prakash

CONTENT

EQUATIONS

1.1 Solve for real numbers:

$$\mathbf{4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} = 101}$$

Solution:

$$E = 4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} = 101$$

$$x < 0 \Rightarrow 4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} < 3 < 101: \textit{false}$$

If $x > 0$*, we notice that* $x = 2$ *and* $x = \log_4 5$ *satisfies the equation.*

Let $f: \mathbb{R}_+^* \to (0, +\infty), f(x) = 4^x + 25^{\frac{1}{x}}$

We prove that f *is strictly decreasing on* $(0, \sqrt{\alpha})$ *and strictly increasing on*

$(\sqrt{\alpha}, +\infty)$*, where* $\alpha = \log_4 25$, $4^{\log_4 25} = 25 \Rightarrow f(x) = 4^x + 4^{\frac{\alpha}{x}}$

Suppose that $\sqrt{\alpha} \le x \le y \Rightarrow f(y) - f(x) = (4^y - 4^x) + \left(4^{\frac{\alpha}{y}} - 4^{\frac{\alpha}{x}}\right) =$

$$= 4^x(4^{y-x} - 1) - 4^{\frac{\alpha}{y}}\left(4^{\frac{\alpha(y-x)}{y-1}}\right)$$

But $\alpha < xy \Rightarrow f(y) - f(x) > 4^x(4^{y-x} - 1) - 4^{\frac{\alpha}{y}}(4^{y-x} - 1) =$

$$\left.\begin{array}{l} = (4^{y-x} - 1)\left(4^x - 4^{\frac{\alpha}{y}}\right) \\ y > x \text{ and } \frac{\alpha}{y} < \sqrt{\alpha} < x \end{array}\right\} \Rightarrow f(y) - f(x) > 0 \Leftrightarrow f(y) > f(x) \Leftrightarrow f \textit{ is strictly}$$

increasing on $(\sqrt{\alpha}, +\infty)$ *. Similar for* $(0, \sqrt{\alpha}) \Rightarrow f(x) = 4^x + 25^{\frac{1}{x}}$ *is strictly convexe (1)*

Let $g: \mathbb{R}_+^* \to (0, +\infty), g(x) = 4^{x+\frac{\alpha}{x}},\ 4^x \cdot 25^{\frac{1}{x}} = 4^x \cdot 4^{\frac{\alpha}{x}} = 4^{x+\frac{\alpha}{x}}, \alpha = \log_4 25$

$$g(x) = 4^{x+\frac{\alpha}{x}} \Rightarrow \frac{d}{dx}f(x) = 4^{x+\frac{\alpha}{x}} \ln 4\left(1 - \frac{\alpha}{x^2}\right)$$

$$\frac{d}{dx^2}f(x) = \ln 4\left(1 - \frac{\alpha}{x^2}\right)\frac{d}{dx}f(x) = 4^{x+\frac{\alpha}{x}} \ln 4\alpha\,\frac{2x}{x^4} =$$

$$\left.\begin{array}{l} = \underbrace{4^{x+\frac{\alpha}{x}}\ln 4}_{>0}\underbrace{\left(1-\frac{\alpha}{x^2}+\alpha\frac{2}{x^3}\right)} \\ 1-\frac{\alpha}{x^2}+\alpha\frac{2}{x^3}=\frac{x^3-\alpha x+2x}{x^3}>0 \end{array}\right\} \Rightarrow g''(x)>0 \Rightarrow g \textit{ is strictly convexe (2)}$$

$E=f(x)+g(x)$, *which is a sum of 2 strictly convexe functions* $\Rightarrow E$ *has maximum 2 solutions which are* $x=2$ *and* $x=\log_4 5$

1.2 Find $(a_n)\subset\mathbb{N}$ such that:

$$\sum_{k=0}^{n} a_k\binom{n}{k}\binom{n+1}{k+1}=(n+1)\binom{2n}{n}, n\in\mathbb{N}$$

Solution:

$$\binom{n+1}{k+1}=\frac{(n+1)!}{(k+1)!\,(n-k)!}=\frac{n+1}{k+1}\cdot\frac{n!}{k!\,(n-k)!}=\frac{n+1}{k+1}\binom{n}{k}\Rightarrow$$

$$\Rightarrow\binom{n}{k}\binom{n+1}{k+1}=(n+1)\cdot\frac{1}{k+1}\binom{n}{k}\binom{n}{k}\Rightarrow\sum_{k=0}^{n}a_k\binom{n}{k}\binom{n+1}{k+1}$$

$$=(n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2$$

We know $\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}^2 \;\therefore\; \sum_{k=0}^{n}a_k\binom{n}{k}\binom{n+1}{k+1}=(n+1)\binom{2n}{n}\Rightarrow$

$$\Rightarrow(n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2=(n+1)\sum_{k=0}^{n}\binom{n}{k}^2$$

Thus a possible sequence is $a_k=k+1,\forall k\in\mathbb{N}$.

1.3 Solve for natural numbers:

$$\cos^4\left(\frac{\pi}{2n+1}\right)+\cos^4\left(\frac{2\pi}{2n+1}\right)+\cos^4\left(\frac{3\pi}{2n+1}\right)+\cdots+$$
$$+\cos^4\left(\frac{n\pi}{2n+1}\right)=\frac{55}{16}$$

Solution:

$$8\cos^4 x = 3 + 4\cos 2x + \cos 4x$$

$$\left.\begin{array}{l}8\cos^4\left(\frac{\pi}{2n+1}\right) = 3 + 4\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right)\\ 8\cos^4\left(\frac{2\pi}{2n+1}\right) = 3 + 4\cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{8\pi}{2n+1}\right)\\ 8\cos^4\left(\frac{3\pi}{2n+1}\right) = 3 + 4\cos\left(\frac{6\pi}{2n+1}\right) + \cos\left(\frac{12\pi}{2n+1}\right)\\ 8\cos^4\left(\frac{n\pi}{2n+1}\right) = 3 + 4\cos\left(\frac{2n\pi}{2n+1}\right) + \cos\left(\frac{4n\pi}{2n+1}\right)\end{array}\right\}$$

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{\cos\left(\frac{(2n+2)\pi}{2n+1}\right)\sin\left(\frac{2n\pi}{2n+1}\right)}{\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{-2\cos\left(\frac{\pi}{2n+1}\right)\sin\left(\frac{\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \frac{-\sin\left(\frac{2\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}$$

$$\therefore \frac{55}{2} = 3n + 4\left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) \to n = 10$$

1.4 Solve for natural numbers:

$$(\boldsymbol{x}+\boldsymbol{y})^{\boldsymbol{x}^{\boldsymbol{n}}+\boldsymbol{y}^{\boldsymbol{n}}} = (\boldsymbol{x}+\mathbf{1})^{\boldsymbol{x}^{\boldsymbol{n}}} \cdot (\boldsymbol{y}+\mathbf{1})^{\boldsymbol{y}^{\boldsymbol{n}}}, \boldsymbol{n} \in \mathbb{N}$$

Solution:

$$(x+y)^{x^n+y^n} \overset{(1)}{=} (x+1)^{x^n}(y+1)^{y^n}$$

(1)$\Leftrightarrow (x^n + y^n)\ln(x+y) = x^n\ln(x+1) + y^n\ln(y+1) \Leftrightarrow x^n\ln\left(\frac{x+y}{x+1}\right) +$

$$y^n\ln\left(\frac{x+y}{y+1}\right) \overset{(1)}{=} 0$$

$$\because x \geq 1 \therefore x + y \geq y + 1 \Rightarrow \frac{x+y}{y+1} \geq 1$$

$$\Rightarrow \ln\left(\frac{x+y}{y+1}\right) \ge 0 \Rightarrow y^n \ln\left(\frac{x+y}{y+1}\right) \overset{(i)}{\ge} 0\ (\because y^n \ge 1)$$

Also, $\because y \ge 1 \therefore x+y \ge x+1 \Rightarrow \frac{x+y}{x+1} \ge 1$

$$\Rightarrow \ln\left(\frac{x+y}{x+1}\right) \ge 0 \Rightarrow x^n \ln\left(\frac{x+y}{x+1}\right) \overset{(ii)}{\ge} 0\ \ (\because x^n \ge 1)$$

(i)+(ii)⇒ LHS of (1)≥ 0*, equality if* $x = y = 1$

and $\because$ *LHS* $= 0 \therefore x = y = 1$ *(Answer)*

1.5 Find all $m, n, p \in \mathbb{N}$ such that: $m^3 = np(n+p)$

Solution:

Rearranging we have: $n^2p + np^2 - m^3 = 0$ *(*)*

Solving quadratically we have: $n = \frac{-p^2 \pm \sqrt{p^4+4pm^3}}{2p} = -\frac{p}{2} \pm \frac{\sqrt{p^4+4m^3p}}{2p}$

Note that $m, n, p \in \mathbb{N}$ *which implies* p *must be an even integer. Set*$p = 2k$

gives us $n = -k \pm \frac{\sqrt{16k^4+8m^3k}}{4k}$*. It is worthy to note that:*

$$\frac{\sqrt{16k^4+8m^3k}}{4k} - k > 0 \Rightarrow \sqrt{16k^4 + 8m^3k} > 4k^2$$

Squaring on both sides yields $16k^4 + 8m^3k > 16k^4 \Rightarrow m^3k \ge 1$

For all $k > 1, 0 < m^3 < 1$ *which tells us that*

$m = 1$ *is the possible value. Plugging in (*) we get*

$$n^2p + np^2 = 1 \Rightarrow np(n+p) = 1$$

$$\begin{cases} np = 1 & (1) \\ n + p = 1 & (2) \end{cases}$$

Squaring on both sides yields $16k^4 + 8m^3k > 16k^4 \Rightarrow m^3k \ge 1$

For all $k > 1, 0 < m^3 < 1$ *which tells us that* $m = 1$ *is the possible value.*

Plugging in (*) we get $n^2p + np^2 = 1 \Rightarrow np(n+p) = 1$

$$\begin{cases} np = 1 & (1) \\ n + p = 1 & (2) \end{cases}$$

Further squaring in 2nd equation we observe

$$n^2+p^2+2np=1\Rightarrow n^2+p^2=-1$$

As $n^2>0,p^2>0\Rightarrow n^2+p^2>0$. *Hence* $n^2+p^2=-1$ *is impossible which proves there exists no solution for* m,n,p *in* $\mathbb{N}$.

1.6 Solve for natural numbers:

$$\boldsymbol{xy+yz+zx=2\sqrt{xyz}+4}$$

Solution:

$$xy+yz+xz=2\sqrt{xyz}+4,z(x+y)+xy\overset{Ma\geq Mg}{\geq}2\sqrt{z(x+y)\cdot xy}$$

$$\Rightarrow 2\left(\sqrt{xyz}+2\right)\geq 2\sqrt{xbz(x+b)},\sqrt{xyz}+2\geq\sqrt{xyz(x+y)}$$

$$\text{If } x=0\Rightarrow y\cdot z=4\Rightarrow y=4,z=1,y=1,z=4,y=z=2$$

$$\Rightarrow \text{sol. } (0,2,2),(2,0,2),(2,2,0),(0,1,4),(1,0,4),(1,4,0)$$

$$(4,1,0),(4,0,1),(0,4,1).\ P\uparrow\quad P=xyz\neq 0$$

$$\text{If } P=xyz=1\Rightarrow x=y=z=1\Rightarrow 3=2+4 \text{ false.}$$

$$\text{If } P=xyz=2\Rightarrow x=2,y=z=1\Rightarrow 2+2+1=2\sqrt{2}+4 \text{ false.}$$

$$\text{If } P=xyz=3\Rightarrow x=3,y=z=1\Rightarrow 3+3+1=2\sqrt{3}+4 \text{ false.}$$

$$\text{If } P=xyz=4\Rightarrow x=y=2,z=1\Rightarrow 4+2+2=4+4 \text{ true.}$$

$$x=4,y=z=1\Rightarrow 4+4+1=4+4 \text{ false}\Rightarrow \text{sol. } (2,2,1),(1,2,2),(2,1,2)$$

$$\sqrt{xyz}+2\geq\sqrt{xyz(x+y)}|:\sqrt{xyz},(P\neq 0),P>4\Rightarrow$$

$$\Rightarrow\sqrt{x+y}\leq 1+\frac{2}{\sqrt{xyz}}<1+\frac{2}{\sqrt{4}}=1+1=2$$

$$\Rightarrow\sqrt{x+y}<2,x+y<4\Rightarrow x+y\in\{1,2,3\}$$

$$x+y=0\Rightarrow x=y=0$$

$$\text{If } x+y=1\Rightarrow x=0 \text{ or } y=0\text{, see above}$$

$$\text{If } x+y=2\Rightarrow x=y=1,1+z+2=2\sqrt{z}+4,$$

$$2z=2\sqrt{z}+3,z=k^2,\sqrt{z}\in\mathbb{N},2k^2-2k=3$$

$$2\cdot(k^2-k)=3\Rightarrow\frac{2}{3}\textit{ false. If } x=0 \textit{ or } y=0 \textit{ see above.}$$

$$\textit{If } x+y=3\Rightarrow x=1, y=2, 2+z+2z=2\sqrt{2z}+4$$

$$6k^2-2-4k=0, 3k^2-2k-1=0, (k-1)(3k+1)=0,$$

$$\textit{If } 3k+1=0\Rightarrow k=-\frac{1}{3}\textit{ false. If } k-1=0\Rightarrow k=1\Rightarrow z=2, x=1, y=2$$

$$2+2+4=2\cdot 2+4 \textit{ true.}$$

1.7 Find all pairs (m,n) of positive integers for

$$\boldsymbol{8^m=2n^4+8n^3+12n^2+8n+5}$$

Solution:

Given: $2n^4+8n^3+12n^2+8n+5=8^m$

which further can be written as $2(n+1)^4=8^m-3$

shows that $8^m-3=2(2^{3m-1}-1)-1$

is always an odd integer where left hand expression is an even integer. Thus, there is no solution in $\mathbb{Z}^+$.

1.8 Find the number of ordered quadruples of positive integers (x,y,p,q) such that the following holds: $x^5y-xy^5=pq$, and p,q are primes.

Solution:

The given equation is equivalent to $xy(x-y)(x+y)(x^2+y^2)=pq$. *As* p *and* q *are primes, there exist a bijection such that*

$$(x;y;x-y;x+y;+x^2+y^2)\to(1;1;1;p;q).$$

For $x>y$, *the possible pair should be* $(x;y)=(p;1)$ *or*

$$(x;y)=(q;1).$$

Both cases result in $(x;y;x-y;x+y;x^2+y^2)=(2;1;1;3;5)$ *(contradictory to the bijection). In other words, the problem has no solution.*

1.9 Solve for natural numbers:

$$\boldsymbol{a+a^2+a^3+a^4+a^5+a^6=b^2}$$

Solution:

$$\boldsymbol{a}, b \in \mathbb{N}, a+a^2+a^3+a^4+a^5+a^6=b^2$$

If $a=0 \Rightarrow b=0$, *true.*

If $a \neq 0$. *Or* $p|a, p=$ *prime number,*

$$b^2=a(1+a+a^2)+a^4(1+a+a^2), b^2=(1+a+a^2)(a+a^4)$$

$$b^2=(1+a+a^2)a(1+a^3),$$

$$b^2=a(a+1)(a^2-a+1)(a^2+a+1)$$

If $p|a+1 \Rightarrow p|a \Rightarrow p|1$ *false* $\Rightarrow p \nmid (a+1)$

If $p|a^2+a+1, p|a \Rightarrow p|a^2 \Rightarrow p|a+1, p|a \Rightarrow \begin{matrix} p|1 \\ p=1 \end{matrix}$ *false*

$$\Rightarrow p \nmid (a^2+a+1)$$

If $p|a^2-a+1, p|a^2 \Rightarrow p|a-1, p|a \Rightarrow \begin{matrix} p|1 \\ p=1 \end{matrix}$ *false*

$\Rightarrow p \nmid (a^2-a+1) \Rightarrow$ *The number* a *is a perfect square,* $b^2=$ *perfect square*

$\Rightarrow (a+1)(a^2-a+1)(a^2+a+1)=$ *perfect square,*

$a=$ *perfect square* $=k^2, k \in \mathbb{N}^*$

If $q|a+1, q$ *prime,* $q \neq 3 \Rightarrow q|a^2+a$. *If* $q|a^2+a+1 \Rightarrow q|1, q=1$ *false*

$\Rightarrow q \nmid (a^2+a+1)$. *If* $q|a^2-a+1$ *and* $q|a^2+a \Rightarrow q|2a-1$ *and*

$$q|a+1 \Rightarrow q|2a+2 \Rightarrow q|3$$

$$\Rightarrow a+1=u^2, k^2+1=u^2, u \in \mathbb{N}, k \in \mathbb{N}^*, (u-k)(u+k)=1$$

$\begin{cases} u-k=1 \\ u+k=1 \end{cases} \Rightarrow 2u=2, u=1, k=0$ *false.*

If $q=3, 3|a+1 \Rightarrow a+1=M_3 \Rightarrow k^2+1=M_3 \Rightarrow k^2=M_3+2$ *false*

In conclusion, $a=b=0 \in \mathbb{N}$.

1.10 Solve for real numbers:

$$(\boldsymbol{x}+\sin \boldsymbol{x}+\cos \boldsymbol{x})^{3}=$$
$$=(\boldsymbol{x}+\sin \boldsymbol{x}-\cos \boldsymbol{x})^{3}+(\boldsymbol{x}+\cos \boldsymbol{x}-\sin \boldsymbol{x})^{3}+(\sin \boldsymbol{x}+\cos \boldsymbol{x}-\mathbf{3})^{3}$$

Solution:

$$(x+\sin x+\cos x)^{3}=(x+\sin x-\cos x)^{3}+$$
$$+(x+\cos x-\sin x)^{3}+(\sin x+\cos x-x)^{3}$$
$$(x+y+z)^{3}-x^{3}-y^{3}-z^{3}=3(x+y)(y+z)(z+x) \Rightarrow$$
$$\Rightarrow 3(x)(\cos x)(\sin x)=0.\ \ x=0, x=n\pi, \frac{(2n+1)\pi}{2}$$

Combining these values: $x=\frac{m\pi}{2}, m \in I$

1.11 Solve for real numbers:

$$\frac{\mathbf{1}}{\mathbf{1}+\mathbf{8}^{x}}+\frac{\mathbf{1}}{\mathbf{1}+\mathbf{27}^{x}}+\frac{\mathbf{1}}{\mathbf{1}+\mathbf{64}^{x}}=\frac{\mathbf{3}}{\mathbf{1}+\mathbf{24}^{x}}$$

Solution:

Let be $f:[0, \infty) \rightarrow \mathbb{R}, f(x)=\frac{1}{1+e^{x}}, f^{\prime \prime}(x)=\frac{e^{x}(e^{x}-1)}{(1+e^{x})^{3}} \geq 0,$

$f-$ *convexe*

If $u, v, w \geq 0$ *then by Jensen's inequality:*

$$f\left(\frac{u+v+w}{3}\right) \leq \frac{1}{3}(f(u)+f(v)+f(w))$$
$$\frac{1}{1+e^{\frac{u+v+w}{3}}} \leq \frac{1}{3}\left(\frac{1}{1+e^{u}}+\frac{1}{1+e^{v}}+\frac{1}{1+e^{w}}\right)$$

Denote $a=e^{u}, b=e^{v}, c=e^{w}$

$$\frac{1}{1+\sqrt[3]{abc}} \leq \frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)$$
$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \leq \frac{3}{1+\sqrt[3]{abc}}$$

Equality holds if $a=b=c$.

Denote $a=8^{x}, b=27^{x}, c=64^{x}$

$$\frac{1}{1+8^x}+\frac{1}{1+27^x}+\frac{1}{1+64^x}\le\frac{3}{1+\sqrt[3]{8^x\cdot 27^x\cdot 64^x}}=\frac{3}{1+24^x}$$

Equality holds for $8^x = 27^x = 64^x \to x = 0$

1.12 Solve in $\mathbb{R}$:

$$\log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)=\sqrt[3]{27-\cos x}$$

Solution:

If $0<\cos x<1$,

$$2^{\cos x}+1>2\Rightarrow \log_2(2^{\cos x}+1)>1$$

$$3^{\cos x}+2>3\Rightarrow \log_3(3^{\cos x}+2)>1$$

$$4^{\cos x}+3>4\Rightarrow \log_4(4^{\cos x}+3)>1$$

$$\Rightarrow \log_2(2^{\cos x}+1)+\log_3(3^{\cos x}+2)+\log_4(4^{\cos x}+3)>3$$

and $(27-\cos x)^{\frac{1}{3}}<3$. *Similarly, if* $-1<\cos x<0$*, then* $LHS<3$ *and* $RHS>3$. *Thus, only possible solution is*

$$\cos x=0\Rightarrow x=(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}$$

1.13 Find $x, y, z \ge 0$ such that:

$$\frac{2x^2+4}{z^2+2y+3}+\frac{2y^2+4}{x^2+2z+3}+\frac{2z^2+4}{y^2+2x+3}=3$$

Solution:

$$2y\le y^2+1\Rightarrow z^2+2y+3\le z^2+y^2+4\Rightarrow$$

$$E=\sum\frac{2x^2+4}{z^2+y^2+4}=3\Rightarrow E=\sum\frac{x^2+2}{y^2+2+z^2+2}=\frac{3}{2}\quad (1)$$

Let $x^2+2=a, y^2+2=b, z^2+2=c\Rightarrow$ *(1) becomes*

$$\sum\frac{a}{b+c}=\frac{3}{2}\quad (2)$$

But $\sum\frac{a}{b+c}\ge\frac{3}{2}$ *(3)*

From (2)+(3)$\Rightarrow a=b=c\Rightarrow x^2=y^2=z^2\Rightarrow x=y=z=1.$

1.14 Solve for real numbers:

$$\frac{1}{x+1}+\frac{1}{x(x+2)}+\frac{1}{x(x+1)(x+3)}+\dots+$$
$$+\frac{1}{x(x+1)\cdot\ldots\cdot(x+99)(x+101)}=$$
$$=\frac{1}{3}-\frac{1}{x(x+1)(x+2)\cdot\ldots\cdot(x+100)(x+101)}$$

Solution:

$$\overset{LHS}{=}\sum_{r=1}^{101}\frac{(x-1)!\,(x+r-1)}{(x+r)!}\Rightarrow(x-1)!\sum_{r=1}^{101}\left(\frac{1}{(x+r-1)!}-\frac{1}{(x+r)!}\right)\Rightarrow$$

$$\Rightarrow\frac{1}{x}-\frac{(x-1)!}{(x+101)!}\ \textit{(1).}\quad\overset{RHS}{=}\frac{1}{3}-\frac{(x-1)!}{(x+101)!}\ \textit{(2)}$$

From (1) and (2): $x=3$ *is the only solution.*

1.15 Solve for real numbers:

$$\frac{\left(e^{\pi x^{2018}}+1\right)\left(e^{2\pi x^{2018}}+1\right)\left(e^{4\pi x^{2018}}+1\right)\left(e^{8\pi x^{2018}}+1\right)\dots\left(e^{2^n\pi x^{2018}}+1\right)}{\left(\pi^{\frac{2e}{x}}+1\right)\left(\pi^{\frac{4e}{x}}+1\right)\left(\pi^{\frac{8e}{x}}+1\right)\left(\pi^{\frac{16e}{x}}+1\right)\dots\left(\pi^{\frac{2^{n+1}e}{x}}+1\right)}=\frac{e^{2^{n+1}\pi x^{2018}}-1}{\pi^{\frac{2^{n+2}e}{x}}-1}$$

Solution:

$$\textit{Put } e^{\pi x^{2018}}=t,\pi^{\frac{2e}{x}}=u$$

$$\textit{Numerator of LHS}=(t+1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right)$$

$$=\frac{1}{t-1}(t^2-1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right)=\dots=\frac{1}{t-1}\left(t^{2^{n+1}}-1\right)$$

Denominator of RHS

$$=(u+1)(u^2+1)\dots\left(u^{2^n}+1\right)=\frac{u^{2^{n+1}}-1}{u-1}\quad\therefore LHS=\frac{u-1}{t-1}\cdot\frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1}\quad\textit{(1)}$$

Also, RHS=$\frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1}$ *(2).From (1), (2), we get* $u-1=t-1\Rightarrow u=t$

$$\Rightarrow e^{\pi x^{2018}} = \pi^{\frac{2e}{x}} \Rightarrow \pi x^{2018} = \frac{2e}{x}\ln\pi \Rightarrow x^{2019} = \frac{2e\ln\pi}{\pi} \Rightarrow x = \left(\frac{2e\ln\pi}{\pi}\right)^{\frac{1}{2019}}$$

1.16 Solve for real numbers:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ e^{x} & e^{-x} & e & e^{-1} \\ e^{2x} & e^{-2x} & e^{2} & e^{-2} \\ e^{4x} & e^{-4x} & e^{4} & e^{-4} \end{vmatrix} \cdot \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ e^{x} & e^{-x} & e & e^{-1} \\ e^{3x} & e^{-3x} & e^{3} & e^{-3} \\ e^{4x} & e^{-4x} & e^{4} & e^{-4} \end{vmatrix} = \mathbf{0}$$

Solution:

$$Let\ a = e^{x}, b = e.\ Put\ \Delta_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & \frac{1}{a} & b & \frac{1}{b} \\ a^2 & \frac{1}{a^2} & b^2 & \frac{1}{b^2} \\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4} \end{vmatrix} = \frac{1}{a^4b^4}\Delta_2$$

$$\Delta_3 = \begin{vmatrix} 1 & a^4 & 1 & b^4 \\ a & a^3 & b & b^3 \\ a^2 & a^2 & b^2 & b^2 \\ a^4 & 1 & b^4 & 1 \end{vmatrix}$$

$$C_1 \to C_1 - C_2, C_3 \to C_3 - C_4$$

$$\Delta_2 = (1-a^2)(1-b^2)\Delta_3\ where\ \Delta_4 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ 0 & a^2 & 0 & b^2 \\ -(1+a^2) & 1 & -(1+b^2) & 1 \end{vmatrix}$$

$$Expand\ along\ R_3$$

$$\Delta_4 = -a^2\begin{vmatrix} 1+a^2 & 1+b^2 & b^4 \\ a & b & b^3 \\ -(1+a^2) & -(1+b^2) & 1 \end{vmatrix} - b^2\begin{vmatrix} 1+a^2 & a^4 & 1+b^2 \\ a & a^3 & b \\ -(1+a^2) & 1 & -(1+b^2) \end{vmatrix}$$

$$R_3 \to R_3 + R_1$$

$$\Delta_4 = -a^2\begin{vmatrix} 1+a^2 & 1+b^2 & b^4 \\ a & b & b^3 \\ 0 & 0 & 1+b^4 \end{vmatrix} - b^2\begin{vmatrix} 1+a^2 & a^4 & 1+b^2 \\ a & a^3 & b \\ 0 & 1+a^4 & 0 \end{vmatrix} =$$

$$= -a^2(1+b^4)[(1+a^2)b - (1+b^2)a] +$$

$$+b^2(1+a^4)[(1+a^2)b - (1+b^2)a]$$

$$= [(b-a) - ab(b-a)][b^2 - a^2 - a^2b^2(b^2 - a^2)] =$$

$$= (b-a)(1-ab)(b^2-a^2)(1-a^2b^2) =$$

$$= (b-a)^2(b+a)(1-ab)^2(1+ab)$$

Thus, $\Delta_1 = \frac{(1+a)}{(ab)^4}(1-b^2)(a+b)(1+ab)(1-a)(b-a)^2(1-ab)$

Next, put $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{a} & b & \frac{1}{b} \\ a^3 & \frac{1}{a^3} & b^3 & \frac{1}{b^3} \\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4} \end{vmatrix} = \frac{1}{a^4b^4}\Delta_5$ *where* $\Delta_5 = \begin{vmatrix} 1 & a^4 & 1 & b^4 \\ a & a^3 & b & b^3 \\ a^3 & a & b^3 & b \\ a^4 & 1 & b^4 & 1 \end{vmatrix}$

Use $C_1 \to C_1 - C_2, C_3 \to C_3 - C_4$

$$\Delta_5 = (1-a^2)(1-b^2)\Delta_6 \text{ where } \Delta_6 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ -a & a & -b & b \\ -(1+a^2) & 1 & -(1+b^2) & 1 \end{vmatrix}$$

$$R_4 \to R_4 + R_1, R_3 \to R_3 + R_2$$

$$\Delta_6 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ 0 & a+a^3 & 0 & b+b^3 \\ 0 & 1+a^4 & 0 & 1+b^4 \end{vmatrix} =$$

$$= (1+a^2)\begin{vmatrix} a^3 & b & b^3 \\ a+a^3 & 0 & b+b^3 \\ 1+a^4 & 0 & 1+b^4 \end{vmatrix} - a\begin{vmatrix} a^4 & 1+b^2 & b^4 \\ a+a^3 & 0 & b+b^3 \\ 1+a^4 & 0 & 1+b^4 \end{vmatrix} =$$

$$= -(1+a^2)b[(1+a^3)(1+b^4) - (1+a^4)(b+b^3)] +$$

$$+a(1+b^2)[(a+a^3)(1+b^4) - (1+a^4)(b+b^3)] =$$

$$= (a-b)(1-ab)[(a-b)(1-a^3b^3) + (1-ab)(a^3-b^3)]$$

$$= (a-b)^2(1-ab)^2[1+ab+a^2b^2+a^2+b^2+ab]$$

Thus,

$$\Delta = \frac{1}{(ab)^8}(1-a^2)^2(1-b^2)^2(b-a)^4(1-ab)^4(a+b)(1+ab)$$

$$(1+2ab+a^2b^2+a^2+b^2)$$

As $a, b > 0, b \neq 1$

$$\Delta = 0 \Leftrightarrow a^2 - 1 \text{ or } b = a \text{ or } ab = 1 \Leftrightarrow e^x = 1 \text{ or } e^x = e, e^{x+1} = 1 \Leftrightarrow$$
$$\Leftrightarrow x = 0, x = 1, x = -1.$$

1.17 Solve for real numbers:

$$\cos^{12} x + 4\cos^8 x \sin 2x + 2\sin^2 2x\,(3\cos^4 x - 4) +$$
$$+4\sin^3 2x - 3\cos x + 19 = 0$$

Solution:

$$\cos^{12} x + 4\cos^8 x \sin 2x + (3\cos^4 x - 4)(2\sin^2 2x) + 4\sin^3 2x -$$
$$-3\cos x + 19 = 0$$
$$\Rightarrow \cos^{12} x + 8\cos^9 x \sin x + 24\cos^6 x \sin^2 x + 32\cos^3 x \sin^3 x -$$
$$-32\cos^2 x \sin^2 x - 3\cos x + 19 = 0 \Rightarrow$$
$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16\sin^4 x - 32\cos^2 x \sin^2 x - 3\cos x + 19 = 0$$
$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16(\sin^2 x + \cos^2 x)^2 + 16\cos^4 x -$$
$$-3\cos x + 19 = 0$$
$$\Rightarrow (\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 3(\cos x - 1)$$
$$LHS \geq 0 \text{ and } RHS \leq 0$$
$$\text{Equality when } LHS = 0, RHS = 0$$
$$(\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 0, \cos x - 1 = 0$$
$$\Rightarrow \cos^3 x + 2\sin x = 0, \cos x = 0 \text{ and } \cos x = 1$$

Thus, no solution.

1.18 $A \in M_2(\mathbb{R}), \det A = tr\ A = 1$. Solve for real numbers:

$$\det(A^4 + I_2) + 10\det(A^2 + I_2) + x = 4\det(A^3 + I_2) + 16\det(A + I_2)$$

Solution:

$$pA(x) = x^2 - tr\ Ax + \det A = x^2 - x + 1, \text{ with } \begin{cases} \lambda_1 + \lambda_2 = 1 \\ \lambda_1\lambda_2 = 1 \end{cases}$$

$$\det(A + I_2) = (\lambda_1 + 1)(\lambda_2 + 1) = \lambda_1\lambda_2 + \lambda_1 + \lambda_2 + 1 = 3 \ (1)$$

$$\det(A^2+I_2)=(\lambda_1^2+1)(\lambda_2^2+1)=(\lambda_1\lambda_2)^2+\lambda_1^2+\lambda_2^2+1=2+$$
$$(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2=2+1-2=1 \quad \text{(2)}$$
$$\det(A^3+I_2)=(\lambda_1^3+1)(\lambda_2^3+1)=(\lambda_1\lambda_2)^2+\lambda_1^3+\lambda_2^3+1$$
$$=2+(\lambda_1+\lambda_2)(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2)$$
$$=2+\lambda_1^2+2\lambda_1\lambda_2+\lambda_2^2-3\lambda_1\lambda_2=2+(\lambda_1+\lambda_2)^2-3\lambda_1\lambda_2=2+1-3=0 \text{ (3)}$$
$$\det(A^4+I_2)=(\lambda_1^4+1)(\lambda_2^4+1)=(\lambda_1\lambda_2)^4+\lambda_1^4+\lambda_2^4+1=$$
$$=2+\lambda_1^4+\lambda_2^4=2+\lambda_1^4+\lambda_2^4+2\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2=$$
$$=2+(\lambda_1^2+\lambda_2^2)^2-2=((\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2)=1 \quad \text{(4)}$$

From (1)+(2)+(3)+(4)⇒

$$x+1+10=4\cdot 0+16\cdot 3\Rightarrow x+11=48\Rightarrow x=37$$

1.19 Solve for real numbers:

$$\frac{1}{(x+1)^3}+\frac{1}{(x+1)^2}+\frac{1}{x+1}=156+\log_5(x+1)$$

Solution:

Denote $x+1=t$, *then* $\frac{1}{t^3}+\frac{1}{t^2}+\frac{1}{t}=156+\log_5 t$ *(1)*

domain the equation (1) $t>0$:$f(x)=\frac{1}{t^3}+\frac{1}{t^2}+\frac{1}{t}\downarrow$ *in* $(0;+\infty)$

$g(x)=156+\log_5 t\uparrow$ *in* $(0;+\infty)$*and has at most one root*

$$t=\frac{1}{5}\Rightarrow x+1=\frac{1}{5}\Rightarrow x=-\frac{4}{5}$$

1.20 Solve for real numbers:

$$\left(x+\sqrt{x^2+1}\right)\left(x-[x]+\sqrt{(x-[x])^2+1}\right)=1,$$

$[*]$ − ***great integer function***

Solution:

$$\left(x+\sqrt{x^2+1}\right)\left(x-[x]+\sqrt{(x-[x])^2+1}\right)=1 \quad (*)$$

If $x>0$ *then* $x+\sqrt{x^2+1}>1; \{x\}=x-[x]\geq 0\Rightarrow \{x\}+\sqrt{\{x\}^2+1}>1$

$\Rightarrow$ *LHS (*)* $> 1 \Rightarrow$ *no solution. If* $x \le 0$ *then (*) becomes*

$$\{x\} + \sqrt{\{x\}^2 + 1} = -x + \sqrt{(-x)^2 + 1}$$

$$\text{Let } f(u) = u + \sqrt{u^2 + 1} \text{ with } u \ge 0$$

$$\Rightarrow f'(u) = 1 + \frac{u}{\sqrt{u^2 + 1}} > 0 (\forall u \ge 0) \Rightarrow f \nearrow [0, +\infty)$$

$$\Rightarrow f(\{x\}) = f(-x) \Leftrightarrow \{x\} = -x \;\Leftrightarrow x - [x] = -x \Leftrightarrow 2x = [x] \in \mathbb{Z}$$

$$\text{More, } 0 \le \{x\} < 1 \Rightarrow 0 \le -x < 1 \Leftrightarrow -1 < x \le 0$$

$$\Leftrightarrow -2 < 2x \le 0 \Leftrightarrow -2 < [x] \le 0 \Leftrightarrow [x] = -1 \text{ or } [x] = 0$$

$$\Leftrightarrow x = -\tfrac{1}{2} \text{ or } x = 0. \text{ Answer: } x = -\tfrac{1}{2} \text{ or } x = 0.$$

1.21 Solve for real numbers:

$$\frac{\left|\cos x \cdot \cos \frac{x}{2}\right|}{\sqrt{(2 - \cos^2 x)\left(2 - \cos^2 \frac{x}{2}\right)}} = \frac{\cos^2 x + \cos^2 \frac{x}{2}}{2 + \sin^2 x + \sin^2 \frac{x}{2}}$$

Solution:

$$\frac{\left|\cos x \cdot \cos \frac{x}{2}\right|}{\sqrt{(2 - \cos^2 x)\left(2 - \cos^2 \frac{x}{2}\right)}} = \frac{\cos^2 x + \cos^2 \frac{x}{2}}{2 + \sin^2 x + \sin^2 \frac{x}{2}}$$

$$\Leftrightarrow \frac{\left|\cos x \cdot \cos \frac{x}{2}\right|}{\sqrt{4 - 2\left(\cos^2 x + \cos^2 \frac{x}{2}\right) + \cos^2 x \cos^2 \frac{x}{2}}} = \frac{\cos^2 x + \cos^2 \frac{x}{2}}{4 - \left(\cos^2 x + \cos^2 \frac{x}{2}\right)} \quad (1)$$

$$\text{- Let } \begin{cases} \cos x \cdot \cos \frac{x}{2} = a \\ \cos^2 x + \cos^2 \frac{x}{2} = b \end{cases}; (1) \leftrightarrow \frac{|a|}{\sqrt{4 - 2b + a^2}} = \frac{b}{4 - b}$$

$$\Rightarrow \frac{a^2}{4 - 2b + a^2} = \frac{b^2}{(4 - b)^2} \leftrightarrow a^2(4 - b)^2 = b^2(4 - 2b + a^2)$$

$$\leftrightarrow a^2b^2 - 8a^2b + 16a^2 = a^2b^2 - 2b^3 + 4b^2 \leftrightarrow -4a^2b + 8a^2 = -b^3 + 2b^2$$

$$\leftrightarrow b^3 - 2b^2 - 4a^2b + 8a^2 = 0 \leftrightarrow b^2(b - 2) - 4a^2(b - 2) = 0$$

$$\leftrightarrow (b - 2)(b^2 - 4a^2) = 0 \leftrightarrow \begin{cases} b - 2 = 0 \\ b^2 - 4a^2 = 0 \end{cases} \rightarrow \begin{cases} b = 2 \\ b = 2a \\ b = -2a \end{cases}$$

$$b=2\to\cos^2 x+\cos^2\frac{x}{2}=2\to\cos^2 x=\cos^2\frac{x}{2}=1\Rightarrow x=2k+1\ (k\in\mathbb{Z})$$

$$b=2a\to\cos^2 x+\cos^2\frac{x}{2}=2\cos x\cos\frac{x}{2}\leftrightarrow$$

$$\left(\cos x-\cos\frac{x}{2}\right)^2=0\leftrightarrow\cos x=\cos\frac{x}{2}$$

$$\leftrightarrow -2\sin\frac{3x}{4}\cdot\sin\frac{x}{4}=0\leftrightarrow\begin{cases}\sin\frac{3x}{4}=0\\ \sin\frac{x}{4}=0\end{cases}\leftrightarrow\begin{cases}\frac{3x}{4}=k+1\\ \frac{x}{4}=k+1\end{cases}\leftrightarrow\begin{cases}x=\frac{4k+1}{3}\\ x=4k\pi\end{cases}$$

$$b=-2a\leftrightarrow\cos^2 x+\cos^2\frac{x}{2}=-2\cos x\cdot\cos\frac{x}{2}\leftrightarrow\left(\cos x+\cos\frac{x}{2}\right)^2=0$$

$$\leftrightarrow\cos x+\cos\frac{x}{2}=0\leftrightarrow 2\cos\frac{3x}{4}\cdot\cos\frac{x}{4}=0\leftrightarrow\begin{cases}\cos\frac{3x}{4}=0\\ \cos\frac{x}{4}=0\end{cases}\leftrightarrow$$

$$\leftrightarrow\begin{cases}\frac{3x}{4}=\frac{\pi}{2}+k2\pi\\ \frac{x}{4}=\frac{\pi}{2}+k2\pi\end{cases}\leftrightarrow\begin{cases}x=\frac{2\pi}{3}+\frac{8k\pi}{3}\\ x=2\pi+8k\pi\end{cases}(k\in\mathbb{Z})$$

1.22 Solve for real numbers:

$$\mathbf{(1+\sin x)\cdot(\sin x)^{\cos x}+(1+\cos x)\cdot(\cos x)^{\sin x}=1+\sin x+\cos x}$$

Solution:

$$\underbrace{1+\sin x+\cos x}_{\substack{\downarrow\\ LHS}}=\underbrace{(1+\sin x)(\sin x)^{\cos x}+(1+\cos x)(\cos x)^{\sin x}}_{\substack{\downarrow\\ RHS}}$$

$$RHS=(1+\sin x)\big(1+(\sin x-1)\big)^{\cos x}+(1+\cos x)\big(1+(\cos x-1)\big)^{\sin x}$$

$$\overset{Bernoulli}{\le}(1+\sin x)(1+\cos x\cdot\sin x-\cos x)+(1+\cos x)(1+\cos x\sin x-\sin x)$$

$$=1+\sin x-\cos^3 x+1+\cos x-\sin^3 x$$

$$=(1+\sin x+\cos x)-(\cos^3 x+\sin^3 x)+1$$

$$=LHS-(\cos^3 x+\sin^3 x)+1$$

So, $RHS=LHS$ *if-f* $\cos^3 x+\sin^3 x=1$

$$x = 2k\pi, k \in \mathbb{Z} \lor x = 2k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$$

1.23 Solve for real numbers:

$$\mathbf{2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} = \sqrt{6}\left(\sqrt{2} + \sqrt{3}\right)\left(5 - \sqrt{6}\right)}$$

Solution:

Equation $\Leftrightarrow 2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} = 4\sqrt{3} + 9\sqrt{2}$

If $x < 0 \Rightarrow 2^x \cdot 3^{\frac{1}{x}} < 1$ *and* $3^x \cdot 2^{\frac{1}{x}} < 1 \Rightarrow$

$2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} < 2 \Rightarrow$ *equation can't have negative solutions*

Let $x > 0; x = \frac{1}{2}$ *and* $x = 2$ *are solutions for this equation. We've proved that this are its only solutions.*

Let $p: (0, +\infty) \to \mathbb{R}; p(x) = a^x b^{\frac{1}{x}}, a, b > 1$

We show that p *is strictly increasing for* $\left(\sqrt{\log_a b}, +\infty\right)$

and strictly decreasing for $\left(0, \sqrt{\log_a b}\right)$ *(1)*

p *strictly increasing for* $\left(\sqrt{\log_a b}, +\infty\right) \Leftrightarrow \forall x_1, x_2 > \sqrt{\log_a b}$

Such that $x_1 < x_2 \Rightarrow p(x_1) < p(x_2) \Leftrightarrow$

$$a^{x_1} b^{\frac{1}{x_1}} < a^{x_2} b^{\frac{1}{x_2}} \Leftrightarrow b^{\frac{x_2 - x_1}{x_1 x_2}} < a^{x_2 - x_1} \Leftrightarrow$$

$b < a^{x_1 x_2}$ *(because* $a, b > 1$ *and* $x_1 < x_2$*)* $\Leftrightarrow$

$\log_a b < x_1 x_2$, *relation which is true because* $x_1, x_2 > \sqrt{\log_a b}$

Similarly, for $\left(0, \sqrt{\log_a b}\right)$

Let $p_1(x) = 2^x \cdot 3^{\frac{1}{x}}$ *and* $p_2(x) = 3^x \cdot 2^{\frac{1}{x}}$

For (1)$\Rightarrow p_1$ *it is increasing for* $\left(\sqrt{\log_2 3}, +\infty\right)$ *and strictly decreasing for* $\left(0, \sqrt{\log_2 3}\right)$ *(2)*

For (2)⇒ p_2 *it is strictly increasing for* $\left(\sqrt{\log_3 2}, +\infty\right)$ *and strictly decreasing for* $\left(0, \sqrt{\log_3 2}\right)$*. Because* $\log_3 2 < \log_2 3 \Rightarrow p_1(x) + p_2(x)$ *it is strictly decreasing for* $\left(0, \sqrt{\log_3 2}\right) \Rightarrow$ *for this interval the equation*

$p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ *has a unique solution* $x = \frac{1}{2}$.

$p_1(x) + p_2(x)$ *it is strictly increasing for* $\left(\sqrt{\log_2 3}, +\infty\right) \Rightarrow$ *for this interval the equation*

$p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ *has a unique solution* $x = 2$.

For internal $\left(\sqrt{\log_3 2}, \sqrt{\log_2 3}\right), p_1(x) + p_2(x) < 4\sqrt{3} + 9\sqrt{2} \Rightarrow$

the only solutions are $x = \frac{1}{2}, x = 2$

1.24 Solve for real numbers:

$$\sin(\pi \sin^2 x) + \sin(\pi \cos^2 x) = 2\sin^2(2x)$$

Solution:

Let $a = \pi \sin^2(x)$ *(1)*

$b = \pi \cos^2(x)$ *(2)*

then: $(a + b) = \pi \Rightarrow \sin(a + b) = 0 \Rightarrow \sin(a) \cdot \cos(b) + \cos(a)\sin(b) = 0$

$\Rightarrow \sin(a)\cos(b) = -\cos(a)\sin(b)$ *(3)*

$(a + b) = \pi \Rightarrow \cos(a + b) = -1 \Rightarrow \cos(a)\cos(b) - \sin(a)\sin(b) = -1$

Multiplying both sides by $\sin(a)$*:*

$\Rightarrow \sin(a)\ \cos(a)\cos(b) - \sin^2(a)\sin(b) = -\sin(a)$

From (3) $\Rightarrow -\cos^2(a)\sin(b) - \sin^2(a)\sin(b) = -\sin(a)$

$\Rightarrow -(\cos^2(a) + \sin^2(a))\sin(b) = -\sin(a)$

$\Rightarrow \sin(b) = \sin(a) \Rightarrow a = b$ *(4)*

If $a = 0$ *then from (1),* $x = n\pi$ *and this a solution of the original equation.*

Otherwise we have from (4): $\frac{a}{b} = 1$ *where* $b \neq 0 \Rightarrow \frac{1}{2} = 1 \Rightarrow \tan^2(x) = 1$

$\Rightarrow x = n\pi \pm \frac{\pi}{4}$ *or* $x = 2n\pi \pm \frac{\pi}{2}$

and these 4 solutions satisfy the original equation

Set of solutions: $x = n\pi$ *or* $x = \left(n\pi \pm \frac{\pi}{4}\right)$ *or* $x = \left(2n\pi \pm \frac{\pi}{2}\right)$

1.25 Solve for real numbers:

$$\sin^2 x \cdot \sin^{-1}(\cos^2 x) + \cos^2 x \cdot \sin^{-1}(\sin^2 x) = 1$$

Solution:

$$f(x) = \sin^2 x \sin^{-1}(\cos^2 x) + \cos^2(x)\cos^{-1}(\sin^2 x)$$

$$f'(x) = \sin 2x \sin^{-1}(\cos^2 x) - \left(\frac{\sin 2x}{\sqrt{1-\cos^4 x}}\right)\sin^2 x$$

$$-\sin 2x \sin^{-1}(\sin^2 x) + \frac{(\cos^4 x)\sin 2x}{\sqrt{1-\sin^4 x}}$$

$$= \sin(2x)\left(\sin^{-1}(\cos^2 x) - \sin^{-1}(\sin^2 x)\right)$$

$$+\sin 2x\left(\frac{\cos^2 x}{\sqrt{1-\sin^4 x}} - \frac{\sin^2 x}{\sqrt{2-\cos^4 x}}\right)$$

$$= \sin 2x\left(\sin^{-1}(\cos^2 x) - \sin^{-1}(\sin^2 x) + \frac{\cos^2 x}{\sqrt{1-\sin^4 x}} - \frac{\sin^2 x}{\sqrt{1-\cos^4 x}}\right)$$

Now, $f'(x) = 0$

(1) when $\sin 2x = 0 \Rightarrow 2x = n\pi$

$$x = \left(n\frac{\pi}{2}\right)$$

Which is not possible as $1 - \sin^4 x = 0 \Rightarrow f'(x) = \infty$

(2) $\sin^{-1}(\cos^2 x) + \frac{\cos^2 x}{\sqrt{1-\sin^4 x}} = \sin^{-1}(\sin^2 x) + \frac{\sin^2 x}{\sqrt{1-\cos^4 x}}$

which is clearly possible when $x = \frac{n\pi}{4}$ *where* $n = 2m+1$

$$x = (2m+1)\frac{\pi}{4}$$

also $f''(x) < 0$ *at* $x = (2n+1)\frac{\pi}{4} \Rightarrow f(x)$ *is maximum at* $x = \frac{\pi}{4} - \frac{\pi}{4}$

$$f\left(\frac{\pi}{4}\right) = f\left(-\frac{\pi}{4}\right) = \frac{1}{2}\left(2\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6} < 1$$

But RHS of $f(x) = 1$ *which is not possible. Hence no solution.*

1.26 Solve for real numbers:

$$\left\{\frac{1}{\sin^2 x}\right\} - \left\{\frac{1}{\cos^2 x}\right\} = [\cot^2 x] - [\tan^2 x]$$

$\{*\} = x - [x], [*]$ **- great integer function**

Solution:

$$\Leftrightarrow \left\{\frac{1}{\sin^2 x}\right\} + [\tan^2 x] = [\cot^2 x] + \left\{\frac{1}{\cos^2 x}\right\} \Leftrightarrow \left\{\frac{1}{\sin^2 x}\right\} = \left\{\frac{1}{\cos^2 x}\right\}$$

$$and\ [\tan^2 x] = [\cot^2 x], \left\{\frac{1}{\sin^2 x}\right\} = \left\{\frac{1}{\cos^2 x}\right\} \Rightarrow \left\{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right\} = \left\{\frac{1}{\cos^2 x}\right\} \Rightarrow$$

$$\Rightarrow \{1 + \cot^2 x\} = \left\{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}\right\} = \{1 + \tan^2 x\} \Rightarrow \{\cot^2 x\} = \{\tan^2 x\}, but$$

$$[\tan^2 x] = [\cot^2 x] \Rightarrow \tan^2 x = \cot^2 x \Rightarrow \tan^2 x = \frac{1}{\tan^2 x} \Rightarrow \tan^4 x = 1$$

$$\Rightarrow \tan x \in \{-1; 1\} \Rightarrow x \in \left\{\frac{\pi}{4} + k\pi | k \in \mathbb{Z}\right\} or\ x \in \left\{-\frac{\pi}{4} + k\pi | k \in \mathbb{Z}\right\}$$

1.27 Solve for real numbers:

$$x^6 - 3x^5 + x^4(\sin \pi x + \cos \pi x + 2) - 3x^3(\sin \pi x + \cos \pi x) +$$

$$+2x^2\left(\sin \pi x + \cos \pi x + \frac{1}{2}\right) - 3x + 2 = 0$$

Solution:

$$Let\ \sin \pi x + \cos \pi x = a$$

$$\Rightarrow x^6 - 3x^5 + x^4(a + 2) - 3x^3 a + 2x^2\left(a + \frac{1}{2}\right) - 3x + 2 = 0$$

$$\Rightarrow (x^6 - 3x^5 + 2x^4 + x^2 - 3x + 2) + a(x^4 - 3x^3 + 2x^2) = 0$$

$$\Rightarrow (x^4 + 1)(x - 1)(x - 2) + ax^2(x - 1)(x - 2) = 0$$

$$\Rightarrow (x - 1)(x - 2)[x^4 + 1 + ax^2] = 0 \quad (1)$$

$$\because a = \sin \pi x + \cos \pi x$$

$$\Rightarrow -\sqrt{2} \le a \le \sqrt{2} \Rightarrow 2 - \sqrt{2} \le a + 2 \le 2 + \sqrt{2}$$

$$\Rightarrow (a + 2) \in \left[2 - \sqrt{2}, 2 + \sqrt{2}\right] \quad \therefore (a + 2) > 0$$

$$\because x^4 + 1 + ax^2 = (x^2 - 1)^2 + (a + 2)x^2 \quad \because (a + 2) > 0$$

$$\therefore (x^4 + ax^2 + 1) > 0$$

From (1): $(x-1)(x-2)(x^4 + ax^2 + 1) = 0 \quad \because (x^4 + ax^2 + 1) > 0$

$$\Rightarrow (x-1) = 0 \text{ or } (x-2) = 0 \Rightarrow x = \{1,2\} \rightarrow \textit{real solutions}$$

1.28 Solve for real numbers:

$$\mathbf{\log_{\cos^{-1}x}(\sin^{-1}x) \cdot \log(1+\cos^{-1}x) == \log_{\sin^{-1}x}(\cos^{-1}x) \cdot \log(1+\sin^{-1}x)}$$

Solution:

$0 < \sin^{-1}x, \cos^{-1}x \neq 1$, *then we have:*

$$\log_{\cos^{-1}x}(\sin^{-1}x) \cdot \log(1+\cos^{-1}x) =$$

$$= \log_{\sin^{-1}x}(\cos^{-1}x) \cdot \log(1+\sin^{-1}x)$$

$$\Leftrightarrow \frac{\log(\sin^{-1}x)}{\log(\cos^{-1}x)}\log(1+\cos^{-1}x) = \frac{\log(\cos^{-1}x)}{\log(\sin^{-1}x)} \cdot \log(1+\sin^{-1}x)$$

$$\Leftrightarrow \frac{\log(1+\cos^{-1}x)}{\log^2(\cos^{-1}x)} = \frac{\log(1+\sin^{-1}x)}{\log^2(\sin^{-1}x)} \quad (*)$$

Let $f(t) = \frac{\log(1+t)}{\log^2 t}; 0 < t \neq 1 \Rightarrow f'(t) = \frac{\frac{1}{1+t}\cdot\log t - 2\frac{1}{t}\log(1+t)}{\log^3 t}$

$$f'(t) > 0 \Leftrightarrow 0 < t < 1 \Rightarrow f(t) \nearrow \textit{on } (0,1)$$

$$f'(t) < 0 \Leftrightarrow t > 1 \Rightarrow f(t) \searrow \textit{on } (1; +\infty)$$

$$\overset{(*)}{\Rightarrow} f(\cos^{-1}x) = f(\sin^{-1}x) \Leftrightarrow \cos^{-1}x = \sin^{-1}x \Leftrightarrow \sin(\cos^{-1}x) = \sin(\sin^{-1}x)$$

$$\Leftrightarrow x = \sqrt{1-x^2} \Leftrightarrow x^2 = \frac{1}{2} \overset{1\neq x>0}{\Leftrightarrow} x = \frac{\sqrt{2}}{2}$$

1.29 Solve for real numbers:

$$\frac{\left(\mathbf{2018}\sqrt[4n]{\boldsymbol{x}+\boldsymbol{b}} + \mathbf{2019}\sqrt[2n]{\boldsymbol{x}+\boldsymbol{b}} + \mathbf{1}\right)\left(\mathbf{2018}\sqrt[2n]{\boldsymbol{a}+\boldsymbol{b}} + \sqrt[4n]{\boldsymbol{a}+\boldsymbol{b}} + \mathbf{2019}\right)}{\left(\sqrt[4n]{\boldsymbol{x}+\boldsymbol{b}} + \mathbf{2019}\sqrt[2n]{\boldsymbol{x}+\boldsymbol{b}} + \mathbf{2018}\right)\left(\mathbf{2018}\sqrt[4n]{\boldsymbol{a}+\boldsymbol{b}} + \sqrt[2n]{\boldsymbol{a}+\boldsymbol{b}} + \mathbf{2019}\right)} =$$

$$= \sqrt[4n]{(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{x}+\boldsymbol{b})},\ \boldsymbol{n} \geq \mathbf{1}; \boldsymbol{a}, \boldsymbol{b} > 0$$

Solution:

$$\textit{Sea: } \begin{cases} \frac{1}{u} = \sqrt[4n]{x+b} \Rightarrow \frac{1}{u^2} = \sqrt[2n]{x+b} \\ p = \sqrt[4n]{a+b} \Rightarrow p^2 = \sqrt[2n]{a+b} \end{cases} \Rightarrow \frac{p}{u} = \sqrt[4n]{(a+b)(x+b)}$$

La ecuacion toma la forma:

$$\frac{(2018u+2019+u^2)(2018p^2+p+2019)}{(u+2019+2018u^2)(2018p+p^2+2019)}=\frac{p}{u}\Rightarrow$$

$$\Rightarrow\frac{u^3+2018u^2+2019u}{2018u^2+u+2019}=\frac{p^3+2018p^2+2019p}{2018p^2+p+2019}$$

Se observa lo siguiente:

Sea: $f_{(u)}=\frac{u^3+2018u^2+2019u}{2018u^2+u+2019}\Rightarrow f_{(p)}=\frac{p^3+2018p^2+2019p}{2018p^2+p+2019}$

Entonces en la ecuacion:

$$f_{(u)}=f_{(p)}\Rightarrow u=p\Rightarrow\frac{p}{u}=1\overset{Volviendo\ al\ CV}{\Rightarrow}\sqrt[4n]{(a+b)(x+b)}=1$$

Por lo tanto: $x=\frac{1}{a+b}-b$

Nota: Propiedad De Funciones:Si: $f_{(\lambda)}=f_{(\mu)}$ *Entonces:* $\lambda=u$

Si: $f_{(\lambda)}\cdot f_{(\mu)}=1$ *Entonces:* $\lambda\cdot\mu=1$

1.30 Solve for real numbers:

$$(\cos 2x)^{15}\cdot(\cos 4x)^6\cdot\cos 6x=\cos^{192}x$$

Solution:

$$(\cos 2x)^{15}\cdot(\cos 4x)^6\cdot\cos 6x=\cos^{192}x\quad (*)$$

We have: $\cos 2x=2\cos^2 x-1;\ \cos 4x=8\cos^4 x-8\cos^2 x+1$

$$\cos 6x=32\cos^6 x-48\cos^4 x+8\cos^2 x-1$$

Let $t=\cos^2 x\,;\,(0\le t\le 1)$

* *If* $0\le t<0{,}146447$ *we have:* $LHS_{(*)}<0\le\cos^{192}x\Rightarrow$ *(*) no roots.*

* *If* $1\ge t\ge 0{,}146447$ *we have:* $\cos 2x\le\cos^4 x\Leftrightarrow(\cos^2 x-1)^2\ge 0\Leftrightarrow$

$(t-1)^2\ge 0$ *(true)* $\Rightarrow(\cos 2x)^{15}\le(\cos^4 x)^{15}=\cos^{60}x$ *(1)*

$$\cos 4x\le\cos^{16}x\Leftrightarrow 8\cos^4 x-8\cos^2 x+1\le\cos^{16}x$$

$$\Leftrightarrow(t-1)^2(t^6+2t^5+3t^4+4t^3+5t^2+6t-1)\ge 0$$

(True because: $0{,}146447\le t\le 1$*)*

$$\Rightarrow |\cos 4x| \le \cos^{16} x \Rightarrow |\cos 4x|^6 \le (\cos^{16} x)^6 = (\cos x)^{96}$$
$$\cos 6x \le \cos^{36} x \ (\Rightarrow |\cos 6x| \le \cos^{36} x)$$
$$\Leftrightarrow 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1 \le \cos^{36} x$$
$$\Leftrightarrow t^{18} - 32t^3 + 48t^2 - 18t + 1 \ge 0$$
$$\Leftrightarrow (t-1)^2(t^{16} + 2t^{15} + 3t^{14} + 4t^{13} + 5t^{12} + 6t^{11} + 7t^{10} + 8t^9 +$$
$$+9t^8 + 10t^7 + 11t^6 + 12t^5 + 13t^4 + 14t^3 + 15t^2 - 16t + 1) \ge 0 \ \textit{(True)}$$
$$\Rightarrow LHS_{(*)} \le \cos^{60} x \cdot \cos^{90} x \cdot \cos^{36} x = \cos^{192} x$$
$$\textit{Equality} \Leftrightarrow x = k\pi \quad (k \in \mathbb{Z})$$

1.31 Solve for $x \in \left(0, \frac{\pi}{2}\right)$:

$$\sin x + \cos x + \tan x + \cot x + \frac{1}{2}(\sec x + \csc x) = 2\left(1 + \sqrt{2}\right)$$

Solution:

$$x \in \left(0; \frac{\pi}{2}\right) \Rightarrow \sin x ; \cos x > 0$$
$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \overset{(AM-GM)}{\ge} 2\sqrt{\frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x}} = 2$$
$$\sin x + \cos x + \frac{1}{2}(\sec x + \csc x) = \sin x + \cos x + \frac{1}{2}\left(\frac{1}{\cos x} + \frac{1}{\sin x}\right)$$
$$= (\sin x + \cos x)\left(1 + \frac{1}{2 \cdot \cos x \cdot \sin x}\right)$$
$$\overset{(AM-GM)}{\ge} 2\sqrt{\sin x \cdot \cos x}\left(1 + \frac{1}{2 \cdot \cos x \cdot \sin x}\right)$$
$$\overset{(t=\sqrt{\sin x \cdot \cos x})}{=} 2t + \frac{1}{t} \overset{(AM-GM)}{\ge} 2\sqrt{2}$$
$$\Rightarrow \sin x + \cos x + \tan x + \cot x + \frac{1}{2}(\sec x + \csc x) \ge 2\sqrt{2}$$
$$\textit{Equality} \Leftrightarrow \begin{cases} \tan x = \cot x \\ 2\sqrt{\sin x \cdot \cos x} = \frac{1}{\sqrt{\sin x \cdot \cos x}} \end{cases} \overset{0<x<\frac{\pi}{2}}{\Leftrightarrow} x = \frac{\pi}{4}$$

1.32 Solve for real numbers:

$$\sqrt[4]{x^4+16x^3+49x^2+81}+\sqrt[3]{x^3+25x^2+27}=\sqrt{4x^3+25x^2+100x+36}$$

Solution:

$$\sqrt[4]{x^4+16x^3+49x^2+81}+\sqrt[3]{x^3+25x^2+27}$$
$$=\sqrt{4x^3+25x^2+100x+36}$$

(1) can be written as $\sqrt[4]{A}+\sqrt[3]{B}\neq\sqrt{C}$ where A,B,C the respective polynomial roots.

1) Assume $x>0$ then all roots have meaning. We have $(A,B,C>0)$

$\sqrt[3]{B}>\sqrt[4]{A}\leftrightarrow B^4-A^3>0\leftrightarrow x^2\cdot f(x)>0$ *where* $f(x)=52x^9+2835x^8+53808x^7+$

$$+353647x^6+79476x^5+1488x^4-162324x^3+2130624x^2-236196x+1033833$$

$f(x)>0$ for $x>0$ as can be easily shown [if $x\geq 1$ obvious, if $x<1$ the constant overweight the negative terms]

$\sqrt{C}>2\sqrt[3]{B}\leftrightarrow C^3>64B^2\leftrightarrow C^3-64B^2>0$*, because* $C^3-64B^2=x\cdot g(x)$ *where*

$$g(x)=64x^8+1200x^7+12300x^6+77289x^5+325900x^4+863900x^3+$$
$$+1552096x^2+1090800x+388800>0$$

Therefore $\sqrt{C}>2\sqrt[3]{B}>\sqrt[3]{B}+\sqrt[3]{A}$ so, (1) has no solutions.

2) Assume $x<0$. The inequalities $A\geq 0, C\geq 0$ are true when $x\geq\vartheta$ where $\vartheta\simeq -0.4$ so $-|\vartheta|\leq x<0$ in which $B>0$ too. The above polynomial $f(x)$ as positive (easy as the negative terms – powers of 9,7,5 are smaller than the constant term).

We can also show that $\sqrt{C}<2\sqrt[4]{A}\leftrightarrow C^2-16A<0\leftrightarrow x\cdot h(x)<0$

$h(x)=16x^5+200x^4+1409x^3+5032x^2+11016x+7200$ as all negative terms are less than constant for $x=-0.4$ in which they become

maximal. Now, $\sqrt{C} < 2\sqrt[4]{A} < \sqrt[4]{A} + \sqrt[3]{B}$ *hence no solution. Therefore, the only solution is* $x = 0$. *Done.*

1.33 Solve for real numbers:

$$\mathbf{16x^4 - 16x^2 + 2 = \sqrt{1+x} + \sqrt{1-x}}$$

Solution:

$$\text{Equation is defined for } -1 \le x \le 1.$$

$$\text{Put } x = \cos(2\theta), 0 \le 2\theta \le \pi \text{ Or } 0 \le \theta \le \frac{\pi}{2}$$

$$16\cos^4 2\theta - 16\cos^2 2\theta + 2 = \sqrt{2}(\cos\theta + \sin\theta)$$

$$\Rightarrow 2\cos 8\theta = 2\cos\left(0 - \frac{\pi}{4}\right) \Rightarrow \cos 8\theta = \cos\left(\theta - \frac{\pi}{4}\right)$$

$$\Rightarrow 8\theta = 2n\pi \pm \left(\theta - \frac{\pi}{4}\right) \text{ (1). Taking + sign in (1)}$$

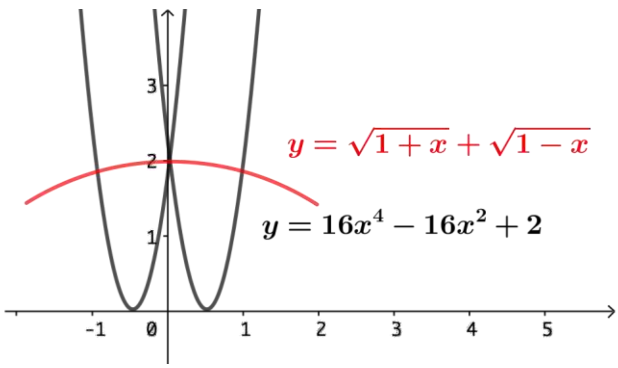
$$8\theta = 2n\pi + \theta - \frac{\pi}{4}, n \in \mathbb{Z} \Rightarrow 7\theta = (8n-1)\frac{\pi}{4} \Rightarrow \theta = (8n-1)\frac{\pi}{28}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{15\pi}{28} \leftarrow \text{Not possible. Taking – sign in (1)}$$

$$8\theta = 2n\pi - \theta + \frac{\pi}{4} \Rightarrow 9\theta = \frac{8n+1}{4}\pi \Rightarrow \theta = \frac{8n+1}{36}\pi \Rightarrow \theta = \frac{\pi}{36}, \frac{\pi}{4}, \frac{17}{36}\pi$$

$$\text{Thus, } x = \cos\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{18}\right), \cos\left(\frac{17}{18}\pi\right)$$

$$x = 0, \cos\left(\frac{\pi}{18}\right), -\cos\left(\frac{\pi}{18}\right)$$



1.34 Solve for real numbers:

$$(6x^2+1)^6+2(3x^2+1)^3+3(2x^2+1)^2=6(6x^2+1)(3x^2+1)(2x^2+1)$$

Solution:

$$Let\ a=6x^2+1, b=3x^2+1, c=2x^2+1$$

$$\frac{a^6}{6}+\frac{b^3}{3}+\frac{c^2}{2}=abc$$

$$\frac{a^6}{6}+\frac{b^3}{6}+\frac{b^3}{6}+\frac{c^2}{6}+\frac{c^2}{6}+\frac{c^2}{6}=abc$$

AM-GM

$$abc\geq 6\sqrt[6]{\frac{a^6}{6}\cdot\frac{b^6}{6}\cdot\frac{c^6}{6}}=abc.\ So,\ a=b=c$$

$$6x^2+1=3x^2+1=2x^2+1$$

$$\underbrace{3x^2=0\wedge x^2=0}_{\substack{\Downarrow\\ x=0}}$$

1.35 Solve for $x>0$:

$$e^2+\int_e^x\left(t^{\log t}(1+2\log t)\right)dt=x^4$$

Solution:

$$Let\ I=\int_e^x t^{\log t}(1+2\log t)dt=\int_e^x t^{\log t}\left(\frac{2\log t}{t}\right)t\,dt+\int_e^x t^{\log t}\,dt$$

$$Let\ t^{\log t}=y\Rightarrow \log y=(\log t)(\log t)$$

$$\frac{1}{y}\frac{dy}{dt}=\frac{2(\log t)}{t}\therefore\int t^{\log t}\frac{(2\log t)}{t}dt=\int dy=y=t^{\log t}$$

Thus,

$$I=t\cdot t^{\log t}\Big]_e^x-\int_e^x t^{\log t}\,dt+\int_e^x t^{\log t}\,dt$$

$$=x\cdot x^{\log x}-e\cdot e'=x^{\log x+1}-e^2$$

Thus, the given equation becomes $e^2+x^{1+\log x}-e^2=x^4$

$$\Rightarrow x\cdot x^{\log x}=x^4\Rightarrow x^{\log x}=x^3\Rightarrow x=1\ or\ \log x=3\Rightarrow x=1\ or\ x=e^3$$

1.36 Solve for real numbers:

$$\mathbf{3^{\sin^2 x+\sin x}+3^{\sin^2 y+\sin y}+3^{\sin^2 z+\sin z}=3^{\sin x+\sin y+\sin z}}$$

Solution:

$$\Leftrightarrow 3^{\sin^2 x+\sin x}+3^{\sin^2 y+\sin y}+3^{\sin^2 z+\sin z}\overset{AM-GM}{\geq}3\left(3^{\sum\sin^2 x+\sum\sin x}\right)^{\frac{1}{3}}$$

$$\Rightarrow\left(3^{\sin x+\sin y+\sin z}\right)^3\geq 27\cdot 3^{\sum\sin^2 x+\sum\sin x}$$

$$\Rightarrow 3^{3(\sin x+\sin y+\sin z)}\geq 3^{3+\sum\sin^2 x+\sum\sin x}$$

$$\Rightarrow 3\sum\sin x\geq 3+\sum\sin^2 x+\sum\sin x\Rightarrow\sum\sin^2 x-2\sum\sin x+1\leq 0$$

$$\Rightarrow(\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2\leq 0$$

But for any real

$$x,y,z\Rightarrow(\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2\geq 0$$

So, this is possible if and only if

$$\Leftrightarrow(\sin x-1)^2+(\sin y-1)^2+(\sin z-1)^2=0$$

$$(\sin x-1)^2=(\sin y-1)^2=(\sin z-1)^2=0\Rightarrow\sin x=\sin y=\sin z=1$$

$$\Leftrightarrow x=y=z=(4n+1)\frac{\pi}{2}[n\in\mathbb{Z}]\quad\textit{(Answer)}$$

1.37 Solve for real numbers:

$$\mathbf{(x^4-3x^2+1)\sqrt{x+2}=1}$$

Solution:

It is clear that $x>-2$ *and* $x<2$. *We can make the substitution:* $x=2\cos t$, *here*

$t\in(0;\pi)$. *We have:*

$$(16\cos^4 t-12\cos^2 t+1)\sqrt{2(1+\cot t)}=1$$

$$16\cos^4 t-12\cos^2 t+1=16\sin^4 t-20\sin^2 t+5=\frac{\sin 5t}{\sin t}$$

We have:

$$\frac{\sin 5t}{\sin t}\cdot 2\cos\frac{t}{2}=1\Rightarrow\sin 5t=\sin\frac{t}{2}\textit{ or }5t=\frac{t}{2}+2\pi k\Rightarrow$$

$$t = \frac{4\pi k}{9}; \text{ if 1) } k = 0; t = 0 \text{ and } x = 1.$$

$$x = 1 \text{ not root}$$

$$\text{2) } k = 1; x = 2\cos\frac{4\pi}{9}$$

$$\text{3) } k = 2; x = 2\cos\frac{8\pi}{9}$$

— — — — — — — — — — — — —

$$\sin 5t = \sin\frac{t}{2} \Rightarrow t = \frac{2\pi}{11} + \frac{4\pi k}{11}$$

$$x = 2\cos\frac{2\pi}{11}, x = 2\cos\frac{6\pi}{11}$$

$$root\left\{2\cos\frac{4\pi}{9}, 2\cos\frac{8\pi}{9}, 2\cos\frac{2\pi}{11}, 2\cos\frac{6\pi}{11}\right\}$$

1.38 Solve for real numbers:

$$\boldsymbol{a^{3x} + a^{2x} + b^{3x} + b^{2x} = a^x b^x (a^x + b^x + 2), a, b} > 0$$

Solution:

We denote:

$$a^x = u > 0, b^x = v > 0 \Rightarrow u^3 + u^2 + v^3 + v^2 = uv = (u + v + 2)$$

$$\text{But } u^3 + v^2 \geq av(u + v), (\forall) u > v > 0$$

$$(u + v)(u^2 - uv + v^2) - uv(u + v) \geq 0, (u + v)(u - v)^2 \geq 0 \text{ (true)}$$

$$\text{Equality for } u = v. \text{ And } u^2 + v^2 \geq 2uv, (\forall) u, v > 0, (u - v)^2 \geq 0$$

$$\text{Equality for } u = v.$$

Adding the two inequalities ⇒

$$(u^3 + v^3) + (u^2 + v^2) \geq uv(u + v) + 2uv = uv(u + v + 2)$$

$$\Rightarrow \text{Equality holds for } u = v \Rightarrow a^x = b^x \Rightarrow \left(\frac{b}{a}\right)^x = \left(\frac{b}{a}\right)^0 \Rightarrow x = 0, S = \{0\}$$

1.39 Solve for real numbers:

$$\int_1^x \left(\frac{\log t - 1}{t^2 - \log^2 t}\right) dt = \frac{1}{2}\log\left(\frac{e - 1}{e + 1}\right)$$

Solution:

$$I=\int_1^x \frac{\ln(t)-1}{t^2-\ln^2(t)}dt=\int_1^x \frac{\frac{\ln(t)-1}{t^2}}{1-\left(\frac{\ln(t)}{t}\right)^2}dt$$

Put: $z=\frac{\ln(t)}{t}$, *so* $dz=\frac{1-\ln(t)}{t^2}dt$

$$\therefore I=\int_0^{\frac{\ln(x)}{x}} \frac{dz}{z^2-1}=\frac{1}{2}\ln\left|\frac{\ln(x)-x}{\ln(x)+x}\right|$$

Then, if $\int_1^x \frac{\ln(t)-1}{t^2-\ln^2(t)}dt=\frac{1}{2}\ln\left(\frac{e-1}{e+1}\right)$ *(E)*

(E) $\Leftrightarrow \frac{\ln(x)-x}{\ln(x)+x}=\frac{e-1}{e+1}\vee\frac{x-\ln(x)}{x+\ln(x)}=\frac{e-1}{e+1}\Leftrightarrow \ln(x)=ex\vee\ln(x)=\frac{x}{e}$

Put: $f(x)=\frac{\ln(x)}{x}; x\in\,]0,+\infty[$

$$f'(x)=\frac{1-\ln(x)}{x^2}$$

So: $f'>0\Rightarrow 1>\ln(x)\Rightarrow x<e$

$f<0\Rightarrow 1-\ln(x)<0\Rightarrow x\geq e$

x	0	1	0	$+\infty$
	↗	↗	$\frac{1}{e}$	↘

$\therefore f \nearrow$ *for* $x<e$ *and* $f \searrow$ *for:* $x\geq e$

So: $\frac{\ln(x)}{x}-\frac{1}{e}\leq 0\ \forall x\in\,]0;+\infty[$

$y=\frac{1}{e}$ *is the maximum of* $x\to\frac{\ln(x)}{x}$

$f'(x)=0\Leftrightarrow x=e$ *unique value for real numbers*

$\therefore S(E)=\{x=e\}$

1.40 Solve for real numbers:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \log x & \log(ex) & \log(e^2x) & \log(e^3x) \\ \log^2 x & \log^2(ex) & \log^2(e^2x) & \log^2(e^3x) \\ \log^3 x & \log^3(ex) & \log^3(e^2x) & \log^3(e^3x) \end{vmatrix} = 7 + 2^{x-10} + \log_{12} x$$

Solution:

4th grade Vandermonde Determinant

$$(\ln ex - \ln x)\cdot(\ln e^2x - \ln x)(\ln e^3x - \ln x)(\ln e^2x - \ln ex)(\ln e^3x - \ln ex)\cdot$$
$$\cdot(\ln e^3x - \ln e^2x) = 7 + 2^{x-10} + \log_{12} x$$
$$\ln e \cdot \ln e^2 \cdot \ln e^3 \cdot \ln e \cdot \ln e^2 \cdot \ln e = 7 + 2^{x-10} + \log_{12} x$$
$$12 - 7 - 2^{x-10} = \log_{12} x\,, x > 0$$
$$5 - 2^{x-10} = \log_{12} x$$

So, $x = 12 \Rightarrow 5 - 4 = \log_{12} 12\,; 1 = 1 \Rightarrow x = 12$ *is a solution*

$f:(0,\infty) \to \mathbb{R}, f(x) = \log_{12} x \nearrow$ *on* $(0,\infty)$

$g:(0,\infty) \to \mathbb{R}, g(x) = 5 - 2^{x-10} \searrow$ *on* $(0,\infty)$

$\Rightarrow$ *Equation* $f(x) = g(x)$ *has an unique solution* $x = 12$.

$$S = \{12\}$$

1.41 Solve for real numbers:

$$\sqrt[5]{x^2 - 5x + 4} + \sqrt[5]{2 + x - x^2} = \sqrt[5]{6 - 4x}$$

Solution:

$$a = \sqrt[5]{x^2 - 5x + 4}, b = \sqrt[5]{2 + x - x^2}, a^5 + b^5 = 6 - 4x$$
$$\sqrt[5]{x^2 - 5x + 4} + \sqrt[5]{2 + x - x^2} = \sqrt[5]{6 - 4x}$$
$$a + b = \sqrt[5]{a^5 + b^5} \Rightarrow (a+b)^5 = a^5 + b^5 \Longrightarrow (a+b)^5 - a^5 - b^5 = 0$$
$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 - a^5 - b^5 = 0$$
$$5ab(a^3 + 2a^2b + 2ab^2 + b^3) = 0$$
$$5ab(a^2(a+b) + ab(a+b) + b^2(a+b)) = 0$$
$$5ab(a+b)(a^2 + ab + b^2) = 0$$

$$a = 0 \Longrightarrow x^2 - 5x + 4 = 0 \Longrightarrow x_1 = 1, x_2 = 4$$

$$b = 0 \Rightarrow 2 + x - x^2 = 0, x_3 = -1, x_4 = 2$$

$$a + b = 0 \Rightarrow x^2 - 5x + 4 = x^2 - x - 2 \Longrightarrow x_5 = \frac{3}{2}$$

$$a^2 + ab + b^2 = \left(a + \frac{b}{2}\right)^3 + \frac{3b^2}{4} \neq 0$$

1.42 Solve for real numbers:

$$\mathbf{3\sqrt[3]{e^{3x} - e^x} - 2\sqrt{e^{2x} - e^x} = e^x + 1}$$

Solution:

By Rado's inequality:

$$3\left(\frac{a + b + c}{3} - \sqrt[3]{abc}\right) \geq 2\left(\frac{a + b}{2} - \sqrt{ab}\right)$$

$$3\sqrt[3]{abc} - 2\sqrt{ab} - c \leq 0,\ \textit{equality for}\ \ a = b = c$$

$$a = e^x, b = e^x - 1, c = e^x + 1$$

$$3\sqrt[3]{e^{3x} - e^x} - 2\sqrt{e^{2x} - e^x} - (e^x + 1) = 0 \leftrightarrow$$

$$\leftrightarrow 3\sqrt[3]{e^x(e^x - 1)(e^x + 1)} - 2\sqrt{e^x(e^x - 1)} - (e^x + 1) = 0 \leftrightarrow$$

$$\leftrightarrow e^x = e^x - 1 = e^x + 1. \textit{ No solutions.}$$

1.43 Find $\boldsymbol{x, y, z \in \left(0, \frac{\pi}{2}\right)}$ such that

$$\mathbf{\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{z} = 3 + \frac{8 - 4\pi}{\pi^2}(x^2 + y^2 + z^2)}$$

Solution:

$$Let\ be\ f: \left(0, \frac{\pi}{2}\right] \to \mathbb{R}, f(x) = \sin x - x - \frac{8 - 4\pi}{\pi^3}x^3$$

$$f'(x) = \cos x - 1 - 3\frac{8 - 4\pi}{\pi^3}x^2, \qquad f''(x) = -\sin x - 6\frac{8 - 4\pi}{\pi^3}x$$

$$f'''(x) = -\cos x - 6\frac{8 - 4\pi}{\pi^3}, \qquad f^{IV} = \sin x > 0, \forall x \epsilon \left(0, \frac{\pi}{2}\right]$$

x	$0 \quad r_1 \quad r_2 \quad r_3 \quad \frac{\pi}{2}$
$f^{'V}$	+ + + + + + + +
f'''	- 0 + + + +
f''	0 - - $f''(?)$ - 0 + + +
f'	0 - - - - - 0 + +
f	0 ↘ ↘ — ↗ 0

$$f'''\left(\frac{\pi}{2}\right) = 6\frac{4\pi - 8}{\pi^3};\ \lim_{\substack{x\to 0\\ x>0}} f'''(x) = -1 + 6\frac{4\pi - 8}{\pi^3} = \frac{24\pi - 48 - \pi^3}{\pi^3} < 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f''(x) = 0;\quad f''\left(\frac{\pi}{2}\right) = -1 - 6^3\frac{8 - 4\pi}{\pi^3}\cdot\frac{\pi}{2} = \frac{-\pi^2 - 24 + 12\pi}{\pi^2} > 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f'(x) = 0;\quad f'\left(\frac{\pi}{2}\right) = -3\frac{8 - 4\pi}{\pi^3}\cdot\frac{\pi^2}{4} = -3\frac{2 - \pi}{\pi} = \frac{-6 + 3\pi}{\pi} > 0$$

$$\lim_{\substack{x\to 0\\ x>0}} f(x) = 0;\quad f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2} - \frac{8 - 4\pi}{\pi^3}\cdot\frac{\pi^3}{8} = 1 - \frac{\pi}{2} - \frac{2 - \pi}{2}$$

$$= \frac{2 - \pi + \pi - 2}{2} = 0$$

So $f(x) = 0.\ \forall x \in \left(0, \frac{\pi}{2}\right]$ *equality just for* $x = \frac{\pi}{2} \Longrightarrow$

$$\frac{\sin x}{x} + \frac{\sin y}{y} + \frac{\sin z}{3} \le 3 + \frac{8 - 4\pi}{\pi^3}(x^2 + y^2 + z^2)$$

$\forall\ x, y, z \in \left(0, \frac{\pi}{2}\right]$ *equality just for* $x = y = z = \frac{\pi}{2}$

1.44 Solve for real numbers:

$$\mathbf{32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0}$$

Solution:

$$32x^6 - 48x^4 + 36x^2 - 2 - \sqrt{3} = 0 \Leftrightarrow$$

$$(2x)^6 - 6(2x)^4 + 36(2x)^2 - 4 - 2\sqrt{3} = 0$$

$$Let\ 2x = \sqrt{t+2} \Rightarrow x = \frac{1}{2}\sqrt{t+2} \Rightarrow t^3 + \underbrace{6}_{p}t + \underbrace{(16 - 2\sqrt{3})}_{q} = 0$$

How

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} = 75 - 16\sqrt{3} > 0 \Rightarrow \exists t_1 \in \mathbb{R}, t_2, t_3 \in \mathbb{C}$$

Applying Cardano Theorem:

$$t = \sqrt[3]{\sqrt{3} - 8 + \sqrt{75 - 16\sqrt{3}}} + \sqrt[3]{\sqrt{3} - 8 - \sqrt{75 - 16\sqrt{3}}}$$

$$How\ t = \pm\frac{1}{2}\sqrt{t+2} \Rightarrow$$

$$x = \pm\sqrt[3]{\sqrt[3]{\sqrt{3} - 8 + \sqrt{75 - 16\sqrt{3}}} + \sqrt[3]{\sqrt{3} - 8 - \sqrt{75 - 16\sqrt{3}}} + 2}$$

1.45 Solve for real numbers:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^x & e^{2x} & e^{3x} & 2 \\ e^{3x} & e^{6x} & e^{9x} & 8 \\ e^{4x} & e^{8x} & e^{12x} & 16 \end{vmatrix} = 0$$

Solution:

$$Let:\ a = e^x; b = e^{2x}; c = e^{3x}\ and$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & 2 \\ a^3 & b^3 & c^3 & 8 \\ a^4 & b^4 & c^4 & 16 \end{vmatrix}$$

Using $c_1 \to c_1 - c_4;\ c_2 \to c_2 - c_4;\ c_3 \to c_3 - c_4$*. We get:*

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-2 & b-2 & c-2 & 2 \\ a^3 - 8 & b^3 - 8 & c^3 - 8 & 8 \\ a^4 - 16 & b^4 - 16 & c^4 - 16 & 16 \end{vmatrix} = (a-2)(b-2)(c-2)\Delta_1,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2+2a+4 & b^2+2b+4 & c^2+2c+4 \\ a^3+2a^2+4a+8 & b^3+2b^2+4b+8 & c^3+2c^2+4c+8 \end{vmatrix}$$

Using $R_3 \to R_3 - 2R_2;\ R_2 \to R_2 - 4R_1$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a^2+2a & b^2+2b & c^2+2c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Using $C_1 \to C_1 - 2C_2;\ C_2 \to C_2 - C_3$. *We get:* $\Delta_1 = (a-b)(b-c)\Delta_2$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & 1 \\ a+b+2 & b+c+2 & c^2+2c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Using $C_1 \to C_1 - C_2$ *,we get:*

$$\Delta_2 = \begin{vmatrix} a-c & b+c+2 \\ (a-c)(a+b+c) & b^2+bc+c^2 \end{vmatrix}$$

$$= -(a-c)(ab+bc+ca+2a+2b+2c)$$

Thus,

$$\Delta = (a-2)(b-2)(c-2)(a-b)(b-c)(c-a)(ab+bc+ca+2a+2b+2c)$$

$$a = 2 \text{ or } b = 2 \text{ or } c = 2 \text{ or } a = b \text{ or } c = a$$

$$x \in \left\{\log 2, \frac{1}{2}\log 2, \frac{1}{3}\log 3, 1\right\}$$

1.46 Solve for real numbers:

$$\sin 2x = \left(\sqrt{2}-1\right)(\sin x + \cos x + 1)$$

Solution:

$$\mathrm{sin2x} = \left(\sqrt{2}-1\right)(\mathrm{sinx} + \mathrm{cosx} + 1) \ldots (1)$$

Put $\mathrm{sinx} + \mathrm{cosx} = \mathrm{t}$. $\mathrm{sinx} = \mathrm{t}^2 - 1$

Now, (1) becomes: $\mathrm{t}^2 - 1 = \left(\sqrt{2}-1\right)(\mathrm{t}+1)$

$$\mathrm{t}+1 = 0 \text{ or } \mathrm{t}-1 = \sqrt{2}-1; \quad \mathrm{t} = -1 \text{ or } \mathrm{t} = \sqrt{2}$$

$$\mathrm{sinx} + \mathrm{cosx} = -1 \text{ or } \mathrm{sinx} + \mathrm{cosx} = \sqrt{2}$$

$$\cos\left(x-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}} \text{ or } \cos\left(x-\frac{\pi}{4}\right)=1$$

$$x-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4} \text{ or } x-\frac{\pi}{4}=2m\pi;\ m,n\in\mathbb{Z}$$

$$x\in\left\{2n\pi+\frac{\pi}{4};(2k+1)\pi;2m\pi-\frac{\pi}{2}/n,k,m\in\mathbb{Z}\right\}$$

1.47 Solve for real numbers:

$$\boldsymbol{(a-1)x+2=a+a^{\frac{x^2-1}{3}}, a}>1$$

Solution:

$$(a-1)x+2=a+a^{\frac{x^2-1}{3}}, a>1\ldots(*)$$

$$(a-1)x+2-a-a^{\frac{x^2-1}{3}}=0, a>1;\ a+a^{\frac{x^2-1}{3}}-(a-1)x-2=0$$

$$\textit{Let be the function: } f(x)=a+e^{\frac{1}{3}log(a)\cdot(x^2-1)}-(a-1)x-2$$

$$\lim_{x\to-\infty} f(x)=+\infty$$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left[x\left(\frac{a}{x}+\frac{e^{\frac{1}{3}log(a)\cdot(x^2-1)}}{x}-(a-1)-\frac{2}{x}\right)\right]=+\infty$$

$$f'(x)=\frac{1}{3}log(a)\cdot 2x\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}-(a-1)$$

$$f''(x)=\frac{2}{3}log(a)\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}+\frac{1}{3}log(a)\cdot 2x\cdot e^{\frac{1}{3}log(a)\cdot(x^2-1)}\cdot\frac{1}{3}log(a)\cdot 2x$$

$$f''(x)=\left(\frac{2}{3}log(a)+\frac{4}{9}log^2(a)\cdot x^2\right)e^{\frac{1}{3}log(a)\cdot(x^2-1)}>0$$

x	$-\infty$ 1 α 2 ∞
f''(x)	++++++++++++++++++++++++++++++
f'(x)	$-\infty$ ↗ ↗ ↗ ↗ ∞

Us prove $1 \overset{?}{<} \alpha \overset{?}{<} 2$. $f'(1) = \frac{2}{3}log(a) - (a-1) \overset{?}{<} 0$, *let us prove that:*

Let be the function $g(x) = \frac{2}{3}log(x) - (x-1)$

$$g'(x) = \frac{2}{3} \cdot \frac{1}{x} - 1 = \frac{2-3x}{3x};\ g'(x) = 0 \Leftrightarrow x = \frac{2}{3}$$

$$g\left(\frac{2}{3}\right) = \frac{2}{3}log\left(\frac{2}{3}\right) + \frac{1}{3} < 0$$

x	$0 \qquad \frac{2}{3} \qquad +\infty$
g'(x)	$+++++0-----$
g(x)	$-\infty \quad \nearrow\nearrow\nearrow \frac{2}{3}log\left(\frac{2}{3}\right) + \frac{1}{3} \quad \searrow\searrow\searrow -\infty$

$$\forall x \geq 1, g(x) \leq 0, so: g(1) = 1$$

Let us prove: $f'(2) = \frac{4}{3}alog(a) - (a-1) \overset{?}{>} 0$

Let be the function: $h(x) = \frac{4}{3}xlog(x) - x + 1$

$$h'(x) = \frac{1}{3}(4log(x) + 1)$$

$$h'(x) = 0 \Leftrightarrow x = \frac{1}{\sqrt[4]{e}}$$

$$h\left(\frac{1}{\sqrt[4]{e}}\right) = \frac{2}{3\sqrt[4]{e}} + 1$$

x	$0 \qquad \frac{1}{\sqrt[4]{e}} \qquad +\infty$
h'(x)	$-----0+++++$
h(x)	$\searrow\searrow\searrow \quad \frac{2}{3\sqrt[4]{e}} + 1 \quad \nearrow\nearrow\nearrow$

So, $h(x) > 0, \forall x > 0$

$$\frac{4}{3}alog(a) - (a-1) > 0, \forall x > 1 \text{ and hence } 1 < \alpha < 2$$

x	$-\infty$ 1 α 2 $+\infty$
$f'(x)$	$- - - - - -$ 0 $+ + + + +$
$f(x)$	$+\infty$ ↘↘ 0 ↘↘ $f(\alpha)$ ↗↗ 0 ↗↗ $+\infty$

Equation (*) has two solution: x=1, x=2.

1.48 Solve for real numbers:

$$\mathbf{5^{2x+1} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}}$$

Solution:

$$5^{2x+1} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

$$5 \cdot 5^{2x} + 20x^2 + 29x + 6 = 11 \cdot 5^x + x \cdot 5^{x+2}$$

$$5 \cdot 5^{-2x} - (25x + 11) \cdot 5^x + 20x^2 + 29x + 6 = 0$$

$$5^x = \frac{25x + 11 \pm \sqrt{(25x+11)^2 - 20(20x^2 + 29x + 6)}}{10}$$

$$= \frac{25x + 11 \pm \sqrt{625x^2 + 550x + 121 - 400x^2 - 580x - 120}}{10}$$

$$= \frac{25x + 11 \pm (15x - 1)}{10}$$

$$5^x = 4x + 1 \text{ or } 5^x = x + 2$$

$$i)\ 5^x = 4x + 1 \Leftrightarrow x \in \{0,1\}$$

$$ii)\ 5^x = x + 2 \Rightarrow x = -1 \textit{ one solution}$$

$$\textit{Let } f(x) =)\ 5^x - x - 2$$

$$f(0) = 1 - 1 \cdot 2 < 0 \textit{ and } f(1) = 5 - 1 - 1 \cdot 2 > 0 \Rightarrow \exists \alpha \in (0,1) \textit{ such that}$$

$$f(\alpha) = 0. \textit{ So: } x \in \{-1, 0, \alpha, 1\}$$

1.49 Solve for real numbers:

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \sqrt{x} & \sqrt[3]{x} & \sqrt[4]{x} & \mathbf{2} \\ x & \sqrt[3]{x^2} & \sqrt{x} & \mathbf{4} \\ x^2 & x\sqrt[3]{x} & x & \mathbf{16} \end{vmatrix} = \mathbf{0}$$

Solution:

$$Let\ \sqrt{x} = a, \sqrt[3]{x} = b, \sqrt[4]{x} = c$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & 2 \\ a^2 & b^2 & c^2 & 4 \\ a^4 & b^4 & c^4 & 16 \end{vmatrix}$$

$$Using\ c_1 \to c_1 - c_4, c_2 \to c_2 - c_4, c_3 \to c_4$$

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-2 & b-2 & c-2 & 2 \\ a^2-4 & b^2-4 & c^2-4 & 4 \\ a^4-16 & b^4-16 & c^4-16 & 16 \end{vmatrix} = -(a-2(b-2)(c-2)\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a+2 & b+2 & c+2 \\ a^3+2a^2+4a+8 & b^3+2b^2+4b+8 & c^3+2c^2+4c+8 \end{vmatrix}$$

$$Using\ c_1 \to c_1 - c_3, c_2 \to c_2 - c_3, c_2 \to c_3\ we\ get$$

$$\Delta_1 = (a-c)(b-c)\Delta_2$$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c+2 \\ a^2+c^2+ac+2(a+c)+4 & b^2+c^2+bc+2(b+c)+4 & c^3+2c^2+4c+8 \end{vmatrix}$$

$$\Delta_1 = (a-c)(b-c)[b^2+c^2+bc+2(b+c)+4-a^2-c^2-ac-2(a+c)-4]$$

$$= (a-c)(b-c)(b-a)(a+b+c+2)$$

$$\Delta = -(a-2(b-2)(c-2)(a-c)(b-c)(b-a)(a+b+c+2)$$

$$\Delta = 0 \Rightarrow a = 2\ or\ b = 2\ or\ c = 2\ or\ a = b\ or\ b = c\ or\ c = a$$

$$\Rightarrow x = 4\ or\ x = 8\ or\ x = 0\ or\ x = 1;\ So, x \in \{0,1,4,8\}$$

1.50 Solve for complex numbers:

$$\mathbf{3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0}$$

Solution:

$$3x^6 - 9x^5 + 18x^4 - 21x^3 + 15x^2 - 6x + 1 = 0$$

$$x^6 - 3x^5 + 6x^4 - 7x^3 + 5x^2 - 2x + \frac{1}{3} = 0$$

For: $(x^2 - x + a)(x^2 - x + b)(x^2 - x + c) = 0$ *we have:*

$$x^6 - 3x^5 + (a + b + c + 3)x^4 - (2a + 2b + 2c + 1)x^3 -$$

$$-(a + b + c + ab + bc + ca)x^2 + (ab + bc + ca)x + abc = 0$$

$$\Rightarrow \begin{cases} 3abc = 1 \\ a + b + c = 3 \\ ab + bc + ca = 2 \end{cases}$$

$$3a^3 - 9a^2 + 6a - 1 = 0 \Leftrightarrow 3(a - 1)^3 - 3(a - 1) - 1 = 0$$

$$\xRightarrow{a-1=w} 3w^3 - 3w - 1 = 0 \Leftrightarrow w^3 - w + \frac{1}{3} = 0 \xRightarrow{w=s+r}$$

$$\begin{cases} s^3 + r^3 + (3sr + 1)w + \frac{1}{3} = 0 \\ r^3 = \frac{1}{27s^3} \\ s^3 + r^3 - \frac{1}{3} = 0 \end{cases}$$

$$27s^6 - 9s^3 + 1 = 0 \Rightarrow s^3 = \frac{3 + \sqrt{-3}}{18} \xRightarrow{w=s+r}$$

$$\begin{cases} w_1 = \sqrt[3]{\frac{3+\sqrt{-3}}{18}} + +\sqrt[3]{\frac{3-\sqrt{-3}}{18}} \\ w_2 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} \\ w_3 = \left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{3-\sqrt{-3}}{18}\right)^{\frac{1}{3}} - \left(\frac{1+\sqrt{-3}}{2}\right)\left(\frac{3+\sqrt{-3}}{18}\right)^{\frac{1}{3}} \end{cases}$$

$$\begin{cases} a_1 = w_1 + 1 \\ a_2 = w_2 + 1 \\ a_3 = w_3 + 1 \end{cases}$$

Let: $a_1 = a;\ a_2 = b;\ a_3 = c \Rightarrow$

$$(x^2 - x + w_1 + 1)(x^2 - x + w_2 + 1)(x^2 - x + w_3 + 1) = 0$$

$$\begin{cases} a = 2.1371580426\ldots \\ b = 0.25777280103\ldots \\ c = 0.60506915636\ldots \end{cases}$$

$$(x^2 - x + 2.1371580426\ldots)(x^2 - x + .25777280103\ldots)(x^2 - x + c = .60506915636\ldots) = 0$$

FUNCTIONAL EQUATIONS

2.1 Determine all functions f *w*ith the following property: They are defined for all real numbers except $\frac{1}{3}$ and $-\frac{1}{3}$, and for each of those real numbers the equality

$$f\left(\frac{x+1}{1-3x}\right)+f(x)=x \text{ holds.}$$

Germany NMO

Solution:

$$f\left(\frac{x+1}{1-3x}\right)+f(x)=x_1 \quad \forall x\neq\frac{1}{3}, x\neq-\frac{1}{3}$$

$$x\to\frac{x+1}{1-3x}\Rightarrow f\left(\frac{\frac{x+1}{1-3x}+1}{1-3\frac{(x+1)}{1-3x}}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x}\Rightarrow$$

$$f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x} \quad (1)$$

$$x\to\frac{x-1}{3x+1}\Rightarrow f\left(\frac{\frac{x-1}{3x+1}+1}{1-3\left(\frac{x+1}{3x+1}\right)}\right)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1}\Rightarrow$$

$$\Rightarrow f(x)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1} \quad (2)$$

From hypothesis and (1) and (2) ⇒

$$\begin{cases} f\left(\frac{x+1}{1-3x}\right)+f(x)=x \\ f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x} \\ f(x)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1} \end{cases}$$

— — — — — — — — — — — — — — — — — —

$$\bigoplus 2\left(f(x)+f\left(\frac{x+1}{1-3x}\right)+f\left(\frac{x-1}{3x+1}\right)\right)=x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}$$

$$\Rightarrow f(x)+f\left(\frac{x+1}{1-3x}\right)+f\left(\frac{x-1}{3x+1}\right)=\frac{1}{2}\left(x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}\right)$$

$$f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x}$$

$$-------------------$$

$$\ominus\ f(t)=\frac{1}{2}\left(x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}\right)-\frac{x+1}{1-3x}\dots$$

2.2 Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$\boldsymbol{f(x)+f(3x)+f(9x)=91x^2+26x+3}$$

Solution:

$$f(x)+f(3x)+f(9x)=91x^2+26x+3 \quad \textit{(3)}$$

Put $g(x)=f(x)-x^2-2x-1$. *We have*

(1) $\Rightarrow g(x)+x^2+2x+1+g(3x)+9x^2+6x+1+g(9x)+81x^2+18x+1=$

$=91x^2+26x+3\Rightarrow g(x)+g(3x)+g(9x)=0$ *(2)*

Put $x\to 3x$, *we have (2)*$\Rightarrow g(3x)+g(9x)+g(27x)=0$ *(3)*

(2) and (3) $\Rightarrow g(x)=g(27x)$ *(4)*

Put $x\to\frac{x}{27}$, *we have (4)* $\Rightarrow g(x)=g\left(\frac{x}{27}\right)$ *(5)*

Put $x\to\frac{x}{27}$, *we have (5)* $\Rightarrow g\left(\frac{x}{27}\right)=g\left(\frac{x}{27^2}\right)$

Similarly, we have $g(x)=g\left(\frac{x}{27}\right)=g\left(\frac{x}{27^2}\right)=\dots=g\left(\frac{x}{27^n}\right)\ \forall n\in\mathbb{N}.$

The sequence (u_n) *such that* $u_0=x, u_{n+1}=\frac{x}{27^n}$.

We have $\lim_{n\to+\infty}u_n=0$

We have $g(u_0)=g(u_1)=\dots=g(u_n)=g(u_n+1)=\dots=g(\lim_{n\to\infty}u_n)=g(0)$

Put $x\to 0$, *we have (2)* $\Rightarrow 3g(0)=0\Rightarrow g(0)=0\Rightarrow g(x)=0\ \forall x\in\mathbb{R}$

So, $f(x)=x^2+2x+1\quad \forall x\in\mathbb{R}$

We have (1) $\Rightarrow x^2+2x+1+9x^2+6x+1+81x^2+18x+1=91x^2+$

$26x+3$ *(True)*

Therefore $f(x) = x^2 + 2x + 1 \ \forall x \in \mathbb{R}$

2.3 Find all continuous functions:

$$\boldsymbol{f}: \mathbb{R} \to \mathbb{R}, \boldsymbol{f(x^3) - f(y^3) = (x^2 + xy + y^2)f(x - y), \forall x, y} \in \mathbb{R}$$

Solution:

Consider a continuous function f *satisfying the proposed property. Let* $P(x, y)$ *be the property* $f(x^3) - f(y^3) = (x^2 + xy + y^2)f(x - y)$

From $P(1,1)$ *we conclude that* $f(0) = 0$.

From $P(x, 0)$ *we conclude that* $f(x^3) = x^2 f(x)$ *for every* x

From $P(tx, x)$ *for* $x \neq 0$ *we get*

$$t^2 f(tx) - f(x) = (t^2 + t + 1)f\big((t - 1)x\big) \qquad (1)$$

Which is also true when $x = 0$ *according to the first point.*

Setting $t = 0$ *in (1) we conclude that* f *is odd.*

Setting $t = 2$ *in (1) we conclude that* $f(2x) = 2f(x)$ *for all* x.

Now suppose that $f(nx) = nf(x)$ *for some positive integer* n *and for all* x.

Applying (1) with $t = n + 1$ *we get*

$$(n + 1)^2 f\big((n + 1)x\big) = f(x) + (n^2 + 3n + 3)nf(x) = (n + 1)^3 f(x)$$

that is $f\big((n + 1)x\big) = (n + 1)f(x)$ *for all* x. *Thus, since* f *is odd, we have proved that*

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, f(nx) = nf(x) \qquad (2)$$

Applying (2) with positive n *and* $\frac{x}{n}$ *instead of* x *we get also*

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*, f\left(\frac{x}{n}\right) = \frac{1}{n}f(x) \qquad (3)$$

Combining (2) and (3) we get for $n \in \mathbb{N}^*, m \in \mathbb{Z}$ *and* $x \in \mathbb{R}$ *the following*

$$f\left(\frac{m}{n}x\right) = \frac{1}{n}f(mx) = \frac{m}{n}f(x) \qquad (4)$$

Thus $f(r) = f(1)r$ *for all* $r \in \mathbb{Q}$. *Now, the continuity of* f *shows that* $f(x) = f(1)x$ *for all real* x.

Conversely, any function of the form $x \to ax$ satisfies the proposed functional equation.

2.4 Find all continuous functions $f: \mathbb{R} \to (0, \infty)$ such that:

$$f(x)f(2x)f(4x) = 2^x, \forall x \in \mathbb{R}$$

Solution:

$$f(x)f(2x)f(4x) = 2^x, \forall x \in \mathbb{R}$$

$$f(2x)f(4x)f(8x) = 2^{2x} \Rightarrow \frac{f(2x)f(4x)f(8x)}{f(x)f(2x)f(4x)} = \frac{2^{2x}}{2^x} = 2^x \Rightarrow f(8x) = 2^x f(x) \Rightarrow$$

$$\Rightarrow f(x) = 2^{\frac{x}{8}} f\left(\frac{x}{8}\right) = 2^{\frac{x}{8}} \cdot 2^{\frac{x}{8^2}} f\left(\frac{x}{8^2}\right) = 2^{\frac{x}{8}+\frac{x}{8^2}+\frac{x}{8^3}} f\left(\frac{x}{8^3}\right)$$

$f(x) = 2^{\frac{x}{8}+\frac{x}{8^2}+\frac{x}{8^3}+\cdots+\frac{x}{8^n}} f\left(\frac{x}{8^n}\right) = 2^{\frac{x}{7}\left(1-\left(\frac{1}{8}\right)^n\right)} f\left(\frac{x}{8^n}\right)$. *Taking limit as $n \to \infty$ we get* $f(x) = 2^{\frac{x}{7}} f(0)$ [$\because$ f *is continuous*].

Also, $f(x)f(2x)f(4x) = 2^x \Rightarrow f(0)f(0)f(0) = 1 \Rightarrow f(0) = 1$. *Thus,*

$$f(x) = 2^{\frac{x}{7}}$$

2.5 Find all continuous functions: $f, g, h: \mathbb{R} \to \mathbb{R}$ such that:

$$f\left(\frac{x+y}{2}\right) = \frac{g(x)+h(y)}{2}, \forall x, y \in \mathbb{R}$$

Solution:

Let's set $y = 0$: $f\left(\frac{x}{2}\right) = \frac{g(x)+h(0)}{2} \Rightarrow g(x) = 2f\left(\frac{x}{2}\right) = h(0)$ *(1)*

Set $x = 0$: $f\left(\frac{y}{2}\right) = \frac{g(0)+h(y)}{2} \Rightarrow h(y) = 2f\left(\frac{y}{2}\right) - g(0)$ *(2)*

Using (1), (2) we have: $f\left(\frac{x+y}{2}\right) = f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) - \frac{h(0)+g(0)}{2}$ *or*

$f(a+b) = f(a) + f(b) - \frac{h(0)+g(0)}{2}$ *where* $a: \frac{x}{2}, b: \frac{y}{2}, a, b \in \mathbb{R}$. *Now let's set* $k(a) = f(a) - \frac{h(0)+g(0)}{2}$. *Then* $k(a+b) = k(a) + k(b), \forall a, b \in \mathbb{R}$. *So k is a Cauchy function and continuos. So* $k(x) = cx, c \in \mathbb{R} \Rightarrow$

$$\Rightarrow f(x) = cx - \frac{h(0)+g(0)}{2}, \forall x \in \mathbb{R} \text{ and}$$

$$g(x) = cx - h(0) - \frac{h(0)+g(0)}{2} \Rightarrow$$

$$\Rightarrow g(x) = cx - \frac{3h(0)+g(0)}{2}, h(x) = cx - \frac{3g(0)-h(0)}{2};$$

and similarly these functions satisfy the equation.

2.6 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x) + f(y) + x^2y + xy^2 = f(x+y), \forall x, y \in \mathbb{R}$$

Solution:

$$f(x) + f(y) = f(x+y) - xy(x+y) \Rightarrow f(x) - \frac{x^3}{3} + f(y) - \frac{y^3}{3} = f(x+y) =$$

$$= \frac{x^3}{3} - \frac{y^3}{3} - xy(x+y) \Rightarrow f(x) - \frac{x^3}{3} + f(y) - \frac{y^3}{3} = f(x+y) - \frac{1}{3}(x+y)^3 \quad (1)$$

Now, let $g(x) = f(x) - \frac{x^3}{3}$, *g continuous (2)*

From (1)+(2) $\Rightarrow g(x) + g(y) = g(x+y) \Rightarrow g(x) = ax, a \in \mathbb{R}$ *(3) (from Cauchy equation). From (2)+(3)* $\Rightarrow f(x) - \frac{x^3}{3} = ax \Rightarrow f(x) = \frac{x^3}{3} + ax$

2.7 Find all function $f: \mathbb{R} \to \mathbb{R}$ satisfying:

$$f(x + ny^2) \geq (y+1)^n f(x), \forall x, y \in \mathbb{R}, 1 \leq n \in \mathbb{N}$$

Solution:

Set $x := x - n, y = 1 \Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$ *(*)*

Let $y = \frac{1}{n} \Rightarrow f\left(x + \frac{1}{n}\right) \geq \left(1 + \frac{1}{n}\right)^n f(x), \forall x \in \mathbb{R} \quad (1)$

Set: $x := x + \frac{1}{n} \Rightarrow f\left(x + \frac{2}{n}\right) \geq \left(1 + \frac{1}{n}\right)^n f\left(x + \frac{1}{n}\right), \forall x \in \mathbb{R} \quad (2)$

$$\overset{(1),(2)}{\Rightarrow} f\left(x + \frac{2}{n}\right) \geq \left(1 + \frac{1}{n}\right)^{2n} f(x), \forall x \in \mathbb{R}$$

By induction we have: $f\left(x+\frac{k}{n}\right) \geq \left(1+\frac{1}{n}\right)^{kn} f(x), \forall x \in \mathbb{R}, k \in \mathbb{N}$

Let $k = n \Rightarrow f(x+1) \geq \left(1+\frac{1}{n}\right)^{n^2} f(x), \forall x \in \mathbb{R}$ *(3)*

Suppose exists $x_0 \in \mathbb{R}$ *such that* $f(x_0) \neq 0 \Rightarrow f(x_0) > 0$ *(because (*)).*

From (3) we let n *from to* ∞

$$f(x_0+1) \geq \lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^{n^2} f(x_0) = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n}\right]^{n} f(x_0) = \lim_{n\to\infty} e^n = +\infty$$

But $f(t_0+1)$ *is real number* $\Rightarrow$ *contradiction* $\Rightarrow f(x) = 0, \forall x \in \mathbb{R}$.

2.8 Find all functions $f: \mathbb{R} \to \mathbb{R}$ continuous in $x = 0$ such that:

$$f(2018x) = f(2019)x + x^2$$

Solution:

More general: $1 < a < b \Rightarrow f(ax) = f(bx) + x^2$, *let* $bx = t \Rightarrow x = \frac{t}{b} \Rightarrow$

$$f\left(\frac{a}{b}t\right) = f(t) + \frac{1}{b^2}t^2, \text{ now } \frac{a}{b} = \alpha_1, \alpha \in (0,1) \Rightarrow$$

$$\left.\begin{aligned} f(\alpha t) - f(t) &= \frac{1}{b^2}t^2 \\ f(\alpha^2 t) - f(\alpha t) &= \frac{1}{b^2}\alpha^2 t^2 \\ &\vdots \\ f(\alpha^n t) - f(\alpha^{n-1} t) &= \frac{1}{b^2}\alpha^{2(n-1)} t^2 \end{aligned}\right\} \Rightarrow$$

$$f(\alpha^4 t) - f(t) = \frac{1}{b^2}t^2\left(1 + \alpha^2 + \dots + \alpha^{2(n-1)}\right) \Rightarrow$$

$$\lim_{n\to\infty} f(\alpha^n t) - f(t) = \lim_{n\to\infty} \frac{1}{b^2}t^2 \frac{\alpha^{2n}-1}{\alpha^2-1} \Rightarrow$$

$$f\left(\lim_{n\to\infty} \alpha^n t\right) - f(t) = \frac{1}{b^2}t^2 \frac{1}{1-\alpha^2} \Rightarrow$$

$$f(0) - f(t) = \frac{1}{b^2}\frac{t^2}{1-\frac{a^2}{b^2}} \Rightarrow f(0) - f(t) = \frac{t^2}{b^2-a^2}$$

$$\text{Let } f(0)=c \Rightarrow f(t)=c-\frac{t^2}{(b-a)(b+a)}$$

$$\text{In our case } a=2018, b=2019.\quad f(x)=c-\frac{x^2}{4037}$$

2.9 Find all ROLLE functions $f:[0,1]\to\mathbb{R}$ such that:

$$\begin{cases} f(0)=f(1)=\dfrac{2019}{2018} \\ 2017f'(x)+2018f(x)\le 2019, \forall x\in(0,1)\end{cases}$$

Solution:

$$2017f'(x)+2018f(x)\le 2019 \Rightarrow f'(x)+\frac{2018}{2017}f(x)\le\frac{2019}{2017}$$

$$\text{Multiplying both sides by } e^{\frac{2018x}{2017}} \text{ to obtain:}$$

$$\frac{d}{dx}\left[e^{\frac{2018x}{2017}}f(x)\right]\le\frac{2019}{2017}e^{\frac{2018x}{2017}}\Rightarrow\frac{d}{dx}\left[e^{\frac{2018x}{2017}}\left(f(x)-\frac{2019}{2018}\right)\right]\le 0$$

$$\Rightarrow F(x)=e^{\frac{2018x}{2017}}\left(f(x)-\frac{2019}{2018}\right)\text{ decreases on } [0,1]$$

$$\text{But } F(0)=F(1)=0$$

$$\therefore F(x) \text{ must be constant on } [0,1]$$

$$\Rightarrow F(x)=F(0)=0\Rightarrow f(x)=\frac{2019}{2018},\forall x\in[0,1]$$

2.10 Find all functions $f:\mathbb{R}\to(0,\infty)$ such that $\forall x,y\in\mathbb{R}$:

$$\mathbf{2(f(x)+f(y))\left(f^2(x)+f^2(y)+3f(x)+3f(y)\right)=}$$

$$\mathbf{=3(f(x)+3)(f(y)+3)(f(x)+f(y)-2)}$$

Solution:

$$\text{Let } x=y \text{ and put } f(x)=t>0. \text{ Equation gives us:}$$

$$2(t+t)(t^2+t^2+3t+3t)=3(t+3)(t+3)(t+t-2)$$

$$\Rightarrow 2(2t)(2t)(t+3)=3(t+3)^2(2t-2)\Rightarrow 4t^2=3(t+3)(t-1)$$

$$\Rightarrow 4t^2 = 3(t^2 + 2t - 3) \Rightarrow t^2 - 6t + 9 = 0$$

$$\Rightarrow (t-3)^2 = 0 \Rightarrow t = 3. \textit{Thus, } f(x) = 3\ \forall x \in \mathbb{R}$$

2.11 Let $\alpha, \beta > 0$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x)f(y) = f(x+y) + \left(\frac{\alpha}{\beta}\right)^2 \cdot xy, \forall x, y \in \mathbb{R}$$

Solution:

$$f(x)f(y) = f(x+y) + \left(\frac{\alpha}{\beta}\right)^2 \cdot xy \quad (*)$$

$$\textit{In (*), we put } x = y = 0 \to \big(f(0)\big)^2 = f(0) \leftrightarrow \left[\begin{matrix} f(0) = 0 \\ f(0) = 1 \end{matrix}\right.$$

$$f(0) = 0$$

$$\textit{Let } y = 0 \overset{(*)}{\to} f(x) \equiv 0 \to \left(\frac{\alpha}{\beta}\right)^2 xy \equiv 0 \ \textit{(contrary)}$$

$$f(0) = 1$$

$$\textit{Let } x = \frac{\beta}{\alpha}; y = -\frac{\beta}{\alpha} \overset{(*)}{\to} f\left(\frac{\beta}{\alpha}\right) f\left(-\frac{\beta}{\alpha}\right) = 0 \leftrightarrow \left[\begin{matrix} f\left(\frac{\beta}{\alpha}\right) = 0 \\ f\left(-\frac{\beta}{\alpha}\right) = 0 \end{matrix}\right.$$

$$\textit{With } f\left(\frac{\beta}{\alpha}\right) = 0, \textit{ let } y = \frac{\beta}{\alpha} \overset{(*)}{\to}: f\left(x + \frac{\beta}{\alpha}\right) = -\frac{\alpha}{\beta}x \leftrightarrow f(x) = -\frac{\alpha}{\beta}x + 1$$

$$\textit{With } f\left(-\frac{\beta}{\alpha}\right) = 0, \textit{ let } y = -\frac{\beta}{\alpha} \overset{(*)}{\to}: f\left(x - \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta}x \leftrightarrow f(x) = \frac{\alpha}{\beta}x + 1$$

$$\textit{Solution: } f(x) = -\frac{\alpha}{\beta}x + 1 \textit{ or } f(x) = \frac{\alpha}{\beta}x + 1$$

2.12 Find all functions $f: \mathbb{R} \to \mathbb{R}$ that satisfy:

$$f^2(x) \cdot f^2(y) = f(x+y) + \frac{2019}{2020}xy, \forall x, y \in \mathbb{R}$$

Solution:

$$\textit{Let } \alpha^2 = 2019, \beta^2 = 2020$$

$$f^2(x)\cdot f^2(y)=f(x+y)+\left(\frac{\alpha}{\beta}\right)^2\cdot xy \quad (1)$$

In (1) we put: $x=y=0\to f^4(0)=f(0)\to\left[\begin{array}{l}f(0)=0\\ f(0)=1\end{array}\right.$

Case: $f(0)=0$

In (1) we put: $x=0\to f(x)\equiv 0\to\left(\frac{\alpha}{\beta}\right)^2\cdot xy\equiv 0,\forall x$ *(contrary)*

Case: $f(0)=1$

In (1) we put: $x=\frac{\beta}{\alpha};y=-\frac{\beta}{\alpha}\to f^2\left(\frac{\beta}{\alpha}\right)\cdot f^2\left(-\frac{\beta}{\alpha}\right)=0\to\left[\begin{array}{l}f\left(\frac{\beta}{\alpha}\right)=0\\ f\left(-\frac{\beta}{\alpha}\right)=0\end{array}\right.$

$f\left(\frac{\beta}{\alpha}\right)=0$. *In (1) we put:* $y=\frac{\beta}{\alpha}\to f\left(x+\frac{\beta}{\alpha}\right)+\frac{\alpha}{\beta}x=0\to$

$$f(x)=\frac{\alpha}{\beta}\left(x-\frac{\beta}{\alpha}\right)=\frac{\alpha}{\beta}x-1=\sqrt{\frac{2019}{2020}}x-1$$

$f\left(-\frac{\beta}{\alpha}\right)=0$. *In (1) we put:* $y=-\frac{\beta}{\alpha}$

$$\to f\left(x-\frac{\beta}{\alpha}\right)-\frac{\alpha}{\beta}x=0\to f(x)=\frac{\alpha}{\beta}\left(x+\frac{\beta}{\alpha}\right)=\frac{\alpha}{\beta}x+1=\sqrt{\frac{2019}{2020}}x+1$$

We check two case |: don't satisfy. No solution.

2.13 Let $\alpha,\beta>0$. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)\cdot f(y)=f^2(x+y)+\frac{\alpha}{\beta}\cdot x^\alpha\cdot y^\beta,\forall x,y\in\mathbb{R}$$

Solution:

$$f(x)\cdot f(y)=f^2(x+y)+\frac{\alpha}{\beta}\cdot x^\alpha\cdot y^\beta \quad (*)$$

In (*), we put $y=0$: $f(x)\cdot f(0)=f^2(x)\leftrightarrow\left[\begin{array}{l}f(x)\equiv 0\\ f(x)\equiv f(0)\end{array}\right.$

Case: $f(x)\equiv 0$. *In (*), we let* $y=1$: $f^2(x+1)=-\frac{\alpha}{\beta}\cdot x^\alpha,\forall x\in\mathbb{R}$

It is false because we choose $x = 1 \to f^2(2) = -\frac{\alpha}{\beta}$ *(contrary)→ No solution.*

Case: $f(x) \equiv f(0) = c$ *(const)*

From (*) we have: $c \cdot c = c^2 + \frac{\alpha}{\beta} \cdot x^\alpha \cdot y^\beta \leftrightarrow \frac{\alpha}{\beta} \cdot x^\alpha \cdot y^\beta \equiv 0$ *(contrary)→*

No solution.

2.14 Let $\boldsymbol{\alpha}, \boldsymbol{\beta} > 0$. Find all functions $\boldsymbol{f}: \mathbb{R} \to \mathbb{R}$ satisfying the following relationship:

$$\boldsymbol{f(\alpha x + \beta y) \cdot f(\beta x + \alpha y) = f(x + y) + \alpha\beta \cdot xy, \forall x, y \in \mathbb{R}}$$

Solution:

$$f(\alpha x + \beta y) \cdot f(\beta x + \alpha y) = f(x + y) + \alpha\beta \cdot xy \text{ (*) } \forall x, y \in \mathbb{R}$$

$$\text{In (*) we put: } x = y = 0 \to f^2(0) = f(0) \leftrightarrow \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$$

Case: $f(0) = 0$. *In (*) we let:*

$$y = -\frac{\beta}{\alpha}x \to 0 = f\left(x - \frac{\beta}{\alpha}x\right) + \alpha\beta \cdot x \cdot \left(-\frac{\beta}{\alpha}x\right) \to f\left(\frac{\alpha - \beta}{\alpha}x\right) = \beta^2 x^2$$

If $\alpha = \beta$ *then* $f(0) = \alpha^2 x^2 \leftrightarrow 0 \equiv \alpha^2 x^2$ *(contrary)*

If $\alpha \neq \beta$ *then* $f\left(\frac{\alpha-\beta}{\alpha}x\right) = \beta^2 x^2$

$$\to f(x) = \beta^2 \cdot \frac{\alpha^2}{(\alpha - \beta)^2} x^2 = \left(\frac{\alpha\beta}{\alpha - \beta}\right)^2 x^2$$

But:

Deg of $LHS_{(*)} = 4 > 2 = \deg$ *of* $RHS_{(*)}$ *→ No solution.*

Case: $f(0) = 1$. *In (*) we let:*

$$y = -\frac{\beta}{\alpha}x \to f\left(\alpha x - \beta \cdot \frac{\beta}{\alpha}x\right) \cdot 1 = f\left(x - \frac{\beta}{\alpha}x\right) + \alpha\beta \cdot x \cdot \left(-\frac{\beta}{\alpha}x\right)$$

$$\to f\left(\frac{\alpha^2 - \beta^2}{\alpha}x\right) = f\left(\frac{\alpha - \beta}{\alpha}x\right) - \beta^2 x^2$$

If $\alpha = \beta$ *then* $1 = 1 - \beta^2 x^2 \leftrightarrow 0 \equiv \beta^2 x^2$ *(contrary)*

If $\alpha \neq \beta$ *then we put* $x := \frac{\alpha}{\alpha-\beta}x \rightarrow f\big((\alpha+\beta)x\big) = f(1) - \left(\frac{\alpha\beta}{\alpha-\beta}\right)^2 x^2$

$$\rightarrow f(x) = f(1) - \left(\frac{\alpha\beta}{\alpha-\beta}\right)^2\left(\frac{x}{\alpha+\beta}\right)^2 = f(1) - \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 x^2$$

But $f(0) = 1 \rightarrow f(0) = f(1) = 1$

Hence: $f(x) = 1 - \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 x^2 \rightarrow f(1) = 1 - \left(\frac{\alpha\beta}{\alpha^2-\beta^2}\right)^2 \neq 1 \rightarrow$ *No solution.*

Answer: No solution.

2.15 Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$\boldsymbol{f(xy+x)\cdot f(xy-x) = f(xy) + \frac{2019}{2020}xy, \forall x, y \in \mathbb{R}}$$

Solution:

$$f(xy+x)\cdot f(xy-x) = f(xy) + \frac{2019}{2020}xy \quad (*)$$

In (*) we let: $x = y = 0 \rightarrow f^2(0) = f(0) \rightarrow \left[\begin{array}{l} f(0) = 0 \\ f(0) = 1 \end{array}\right.$

Case: $f(0) = 0$

In (*) we let: $y = 1 \rightarrow f(2x)\cdot f(0) = f(x) + \frac{2019}{2020}x \rightarrow f(x) = -\frac{2019}{2020}x$

We check in (*): $\frac{2019}{2020}(xy+x)\cdot\frac{2019}{2020}\cdot(xy-x) = -\frac{2019}{2020}xy + \frac{2019}{2020}xy, \forall x, y$

$$\leftrightarrow \left(\frac{2019}{2020}\right)^2\cdot\left((xy)^2 - x^2\right) = 0, \forall x, y$$

(This is contrary) $\rightarrow$ *No solution*

Case: $f(0) = 1$

In (*) we let: $y = 1 \rightarrow f(2x)\cdot f(0) = f(x) + \frac{2019}{2020}x \rightarrow f(2x) = f(x) + \frac{2019}{2020}x$

$$\rightarrow f(2x) = f\left(\frac{x}{2^n}\right) + \frac{2019}{2020}\left(\frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n}\right)$$

$$= f\left(\frac{x}{2^n}\right) + \frac{2019}{2020}\cdot\frac{x}{2}\left(\frac{\left(\frac{x}{2}\right)^n - 1}{\frac{x}{2} - 1}\right) \quad (1)$$

Suppose: f *– continuous in* $x = 0$

In (1) we let $n \to +\infty$: $f(2x) = f(0) + \frac{2019}{2020} \cdot \frac{x}{2} \cdot \frac{0-1}{\frac{x}{2}-1} = 1 - \frac{x}{x-2}$

We also check in (*) → contrary → No solution.

2.16 Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\boldsymbol{f^2(x+y) = f(x) \cdot f(y) + 2019x^2y^2, \forall x, y \in \mathbb{R}}$$

Solution:

$$f\big(f(x)\big) = f\big(f(2x-x)\big) = f(2x)f(-x) + 8076x^4$$

Moreover $f\big(f(x)\big) = f(x)f(0)$

$$f\big(f(2x)\big) = f\big(f(x+x)\big) = f(x)^2 + 2019x^4 = f(2x)f(0)$$

and $f\big(f(0)\big) = f\big(f(x-x)\big) = f(x)f(-x) + 2019x^4$. *As* $f\big(f(x)\big) = f(x)f(0)$

hence $f(x)f(0) = f(2x)f(-x) + 8076x^4$ *multiplying each member by* $f(0)$ *we have:*

$$f(x)f(0)^2 = f(2x)f(0)f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = (f(x)^2 + 2019x^4)f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = f(x)f(x)f(-x) + 2019x^4f(-x) + 8076x^4f(0)$$

$$f(x)f(0)^2 = f(x)\big(f(f(0)\big) - 2019x^4 + 2019x^4f(-x) + 8076x^4f(0)$$

$f\big(f(0)\big) = f(0)^2$ *so* $2019x^4f(x) = 2019x^4f(-x) + 8076x^4f(0), \forall x \in \mathbb{R}$

and $f(x) = f(-x) + 4f(0), \forall x \in \mathbb{R}$ *we deduce* $f(-x) = f(x) + 4f(0)$ *then*

$f(0) = 0$. *Finally:* $0 = f\big(f(0)\big) = f(0)^2 = f(x)^2 + 2019x^4$. *No solution.*

2.17 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$\boldsymbol{f(0) = \frac{1}{4}, f(5x) - f(x) = x, \forall x \in \mathbb{R}}$$

Solution:

$$f(5x) - f(x) = x \Rightarrow f(x) - f\left(\frac{x}{5}\right) = \frac{x}{5} \Rightarrow f\left(\frac{x}{5}\right) - f\left(\frac{x}{5^2}\right) = \frac{x}{5^2} \Rightarrow \cdots$$

$$\Rightarrow \cdots f\left(\frac{x}{5^{n-1}}\right) - f\left(\frac{x}{5^n}\right) = \frac{x}{5^n}, \qquad n \in \mathbb{N} - \{0\}$$

$$f(x) - f\left(\frac{x}{5}\right) + f\left(\frac{x}{5}\right) - f\left(\frac{x}{5^2}\right) + \cdots + f\left(\frac{x}{5^{n-1}}\right) - f\left(\frac{x}{5^n}\right) = \sum_{k=1}^{n} \frac{x}{5^k}$$

$$f(x) - f\left(\frac{x}{5^n}\right) = x \cdot \frac{\frac{1}{5}\left(\frac{1}{5^n} - 1\right)}{\frac{1}{5} - 1}$$

$$\lim_{n\to\infty}\left(f(x) - f\left(\frac{x}{5^n}\right)\right) = \lim_{n\to\infty}\left(x \cdot \frac{\frac{1}{5}\left(\frac{1}{5^n} - 1\right)}{\frac{1}{5} - 1}\right)$$

$$f(x) - f\left(\lim_{n\to\infty}\frac{x}{5^n}\right) = x \cdot \frac{-\frac{1}{5}}{-\frac{4}{5}} \Rightarrow f(x) - f(0) = \frac{x}{4} \Longrightarrow f(x) = \frac{x+1}{4}$$

2.18 Let $\alpha > 0$. Find all functions $f: [0;1] \to \mathbb{R}$ such that:

$$\boldsymbol{f(\alpha x + y) = \alpha f^2(x) - 2020 \cdot x^{\alpha} \cdot t^{\sqrt{\alpha}}, \forall x, y \in [0,1]}$$

Solution:

$$f(\alpha x + y) = \alpha f^2(x) - 2020 \cdot x^{\alpha} \cdot t^{\sqrt{\alpha}} \ldots (*)$$

In (*) put $x = y = 0 \in [0;1] \Rightarrow f(0) = \alpha f^2(0) \Rightarrow \begin{cases} f(0) = 0 \\ f(0) = \frac{1}{\alpha} \end{cases}$

Case: $f(0) = 0$. *In (*) we let:*

$$x = 0 \Rightarrow f(y) = \alpha f^2(0) - 2020 \cdot 0^{\alpha} y^{\sqrt{\alpha}} \Rightarrow f(y) = 0 \Rightarrow f(x) = 0, \forall x, y \in [0,1]$$

Checking:

$$0 = \alpha \cdot 0 - 2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} \Rightarrow -2020 \cdot x^{\alpha} \cdot y^{\sqrt{\alpha}} = 0;\ \forall x, y \in [0,1]$$

$$if\ x = y = 0 \Rightarrow -2020 = 0\ which\ is\ contrary.$$

No solution.

$$Case: f(0) = \frac{1}{\alpha}$$

$$In\ (*)\ we\ let: x=0 \Rightarrow f(y)=\alpha f^2(0)=\alpha\cdot\frac{1}{\alpha^2}=\frac{1}{\alpha}$$

Checking:

$$\frac{1}{\alpha}=\alpha\cdot\frac{1}{\alpha}-2020\cdot x^{\alpha}\cdot y^{\sqrt{\alpha}} \Rightarrow -2020\cdot x^{\alpha}\cdot y^{\sqrt{\alpha}}=0, \forall x,y\in[0,1]$$

$if\ x=y=1 \Rightarrow -2020=0\ which\ is\ contrary$. *No solution.*

2.19 Let $a>b>0$. Find all functions $\varphi:\mathbb{R}\to\mathbb{R}$ such that:

$$\boldsymbol{a\cdot\varphi(bx^2)=b\cdot\varphi^2(ay)+\frac{a}{b}\cdot x^{2019}x^{2020}}, \forall \boldsymbol{x,y}\in \boldsymbol{R}$$

Solution:

$$a\cdot\varphi(bx^2)=b\cdot\varphi^2(ay)+\tfrac{a}{b}\cdot x^{2019}x^{2020} \ \ldots\ldots (*)$$

In (*) put $x=y=0\to a\cdot\varphi(0)=b\cdot\varphi^2(0)\to\begin{cases}\varphi(0)=0\\ \varphi(0)=\frac{a}{b}\end{cases}$

(i) Case: $\varphi(0)=0$, *in (*) put* $x=0\to b\cdot\varphi^2(y)=0$,

$\forall y\in R\to\varphi(y)=0,\forall x\in R$

Check: $a\cdot 0=b\cdot 0+\frac{a}{b}x^{2019}\cdot y^{2020}\to\frac{a}{b}\cdot x^{2019}y^{2020}=0$ *(contrary)* $\to$ *no solution.*

(ii) Case: $\varphi(0)=\frac{a}{b}$, *in (*) put* $x=0\to a\cdot\frac{a}{b}=b\cdot\varphi^2(ay)\to\varphi^2(ay)=\frac{a^2}{b^2}\to$

$$\varphi(ay)=\pm\frac{a}{b}\overset{x=ay}{\Rightarrow}\varphi(x)=\pm\frac{a}{b}$$

Check:

$\varphi(x)=\frac{a}{b}\to a\cdot\frac{a}{b}=b\cdot\frac{a^2}{b^2}+\frac{a}{b}x^{2019}x^{2020}\to\frac{a}{b}x^{2019}x^{2020}=0$ *(contrary)* $\to$ *no solution*

$$\varphi(x)=-\frac{a}{b}\to-\frac{a^2}{b}=b\cdot\frac{a^2}{b^2}+\frac{a}{b}\cdot x^{2019}x^{2020}\to$$

$\to 2\cdot\frac{a^2}{b}=\frac{a}{b}x^{2019}x^{2020}\to 2a=x^{2019}x^{2020}$. *contrary...no solution*

2.20 Find $m, n \in \mathbb{N}^*$ such that $x^2 - x + 3$ divide

$$(x+2)^m - (x^2+2)^n, x \in \mathbb{R}.$$

Solution:

We have $(x+2)^m - (x^2+2)^n = (x^2 - x + 3).Q(x)$ *(1)*

Put $x = \frac{1+i\sqrt{11}}{2}$*, we have (1)* $\Rightarrow \left(\frac{5+i\sqrt{11}}{2}\right)^m = \left(\frac{-1+i\sqrt{11}}{2}\right)^n$ *(2)*

Put $x = \frac{1-i\sqrt{11}}{2}$*, we have (1)* $\Rightarrow \left(\frac{5-i\sqrt{11}}{2}\right)^m = \left(\frac{-1-i\sqrt{11}}{2}\right)^n$ *(3)*

Put $\frac{(2)}{(3)}$*, we have* $\left(\frac{7+5i\sqrt{11}}{18}\right)^m = \left(\frac{-5-i\sqrt{11}}{6}\right)^n \Rightarrow \left(\frac{-5-i\sqrt{11}}{6}\right)^{2m} = \left(\frac{-5-i\sqrt{11}}{6}\right)^n$ *(4)*

Put α *is the angle satisfy* $\cos\alpha = \frac{-5}{6}$ *and* $\sin\alpha = \frac{-\sqrt{11}}{6}$

We have (4) $\Rightarrow \cos(2m\alpha) + i\cdot\sin(2m\alpha) = \cos(n\alpha) + i\cdot\sin(n\alpha)$

$$\Rightarrow \begin{cases}\cos(2m\alpha) = \cos(n\alpha) & (5)\\ \sin(2m\alpha) = \sin(n\alpha) & (6)\end{cases}$$

We have (5) $\Rightarrow \left[\begin{array}{ll}2m\alpha = n\alpha + k2\pi & (7)\\ 2m\alpha = -n\alpha + k2\pi & (8)\end{array}\right.$

Lemma: If $\frac{\pi}{\beta}$ *is a rational number, we have* $\cos\beta \in \left\{\pm 1; \pm\frac{1}{2}\right\}$

Prove

We have $\frac{\pi}{\beta}$ *is a rational number* $\Rightarrow \beta = r\pi \ (r \in Q)$

With De Moivre's Formula we deduce that $\cos r\pi + i\cdot\sin r\pi$ *and* $\cos r\pi - i\cdot\sin r\pi$ *are algebraic integers* $\Rightarrow 2\cos r\pi$ *is an algebraic integer.*

But $2\cos r\pi \in Q \Rightarrow 2\cos r\pi \in Z$

Now from $-2 \le 2\cos r\pi \le 2$ *so we have* $2\cos r\pi \in \{-2; -1; 0; 1; 2\}$

or $\cos\beta \in \left\{\pm 1; \pm\frac{1}{2}\right\}$

We have:

(7) $\Rightarrow (2m-n)\alpha = k2\pi \Rightarrow 2m - n = 0$ *(since* $\frac{\pi}{\alpha} \in I$*)* $\Rightarrow 2m = n$ *(9)*

We have:

(8) $\Rightarrow (2m+n)\alpha = k2\pi \Rightarrow 2m+n=0$ *(since* $\frac{\pi}{\alpha} \in I$*)* $\Rightarrow -2m = n$ *(10)*

We have:

(10) $\Rightarrow -\sin(n\alpha) = \sin(n\alpha) \Rightarrow \sin(n\alpha) = 0 \Rightarrow n\alpha = q2\pi \ (q \in Z) \Rightarrow n = 0$

(since $\frac{\pi}{\alpha} \in I$*) (Absurd)*

We have (9) $\Rightarrow \sin(n\alpha) = \sin(n\alpha)$ *(True)*

Therefore with $2m = n, x^2 - x + 3$ *divide* $(x+2)^m - (x^2+2)^n, x \in R$

SYSTEMS

3.1 Solve for real numbers:

$$\begin{cases} \boldsymbol{x^2 + \sqrt{y^2 + 12} = \sqrt{y^2 + 60}} \\ \boldsymbol{y^2 + \sqrt{z^2 + 12} = \sqrt{z^2 + 60}} \\ \boldsymbol{z^2 + \sqrt{x^2 + 12} = \sqrt{x^2 + 60}} \end{cases}$$

Proposed at Spanish-TST

Solution:

Find all $x, y, z \in \mathbb{R}$ satisfying:

$$x^2 + \sqrt{y^2 + 12} = \sqrt{y^2 + 60} \rightarrow \text{ (a)}$$

$$y^2 + \sqrt{z^2 + 12} = \sqrt{z^2 + 60} \rightarrow \text{ (b)}$$

$$z^2 + \sqrt{x^2 + 12} = \sqrt{x^2 + 60} \rightarrow \text{ (c)}$$

$$\sqrt{a^2 + 60} - \sqrt{a^2 + 12} > \text{or} < 4 \Leftrightarrow \sqrt{a^2 + 60} > \text{or} < 4 + \sqrt{a^2 + 12}$$

$$\Leftrightarrow a^2 + 60 > \text{or} < 16 + a^2 + 12 + 8\sqrt{a^2 + 12} \Leftrightarrow 4 > \text{or} < \sqrt{a^2 + 12}$$

$$\Leftrightarrow 16 > \text{or} < a^2 + 12 \Leftrightarrow a^2 < \text{or} > 4$$

$$\therefore \sqrt{a^2 + 60} - \sqrt{a^2 + 12} > 4 \Leftrightarrow a^2 < 4 \rightarrow \text{ (1)}$$

$$\text{and } \sqrt{a^2 + 60} - \sqrt{a^2 + 12} < 4 \Leftrightarrow a^2 > 4 \rightarrow \text{ (2)}$$

Let us assume $x^2 > 4 \therefore$ *(a)* $\Rightarrow \sqrt{y^2 + 60} - \sqrt{y^2 + 12} > 4 \Rightarrow y^2 < 4$ *(by (1))*

$\therefore$ *(b)* $\Rightarrow \sqrt{z^2 + 60} - \sqrt{z^2 + 12} < 4 \Rightarrow z^2 > 4$ *(by (2))*

$\therefore$ *(c)* $\Rightarrow \sqrt{x^2 + 60} - \sqrt{x^2 + 12} > 4 \Rightarrow x^2 < 4$ *(by (1)), thus leading to a condition. Hence,* $x^2 \ngtr 4 \rightarrow$ *(i)*

Similarly, if we assume $x^2 < 4$*, we shall obtain* $x^2 > 4$*, this again leading to a contradiction. Hence* $x^2 \nless 4 \rightarrow$ *(ii)*

(i), (ii) $\Rightarrow x^2 = 4 \therefore$ *(c)* $\Rightarrow z^2 = 4 \therefore$ *(b)* $\Rightarrow y^2 = 4$

$$\therefore \begin{pmatrix} x = 2 \\ y = 2 \\ z = 2 \end{pmatrix}, \begin{pmatrix} x = 2 \\ y = 2 \\ z = -2 \end{pmatrix}, \begin{pmatrix} x = 2 \\ y = -2 \\ z = 2 \end{pmatrix}, \begin{pmatrix} x = 2 \\ y = -2 \\ z = -2 \end{pmatrix}, \begin{pmatrix} x = -2 \\ y = 2 \\ z = 2 \end{pmatrix}, \begin{pmatrix} x = -2 \\ y = 2 \\ z = -2 \end{pmatrix}, \begin{pmatrix} x = -2 \\ y = -2 \\ z = 2 \end{pmatrix}, \begin{pmatrix} x = -2 \\ y = -2 \\ z = -2 \end{pmatrix}$$

are all possible solutions.

3.2 Solve for real numbers:

$$\begin{cases} \boldsymbol{a,b,c} > 0 \\ \boldsymbol{abc = 1} \\ \boldsymbol{a^4b + b^4c + c^4a = ab + bc + ca} \end{cases}$$

Solution:

$$a^4b + b^4c + c^4a = \frac{a^4}{ac} + \frac{b^4}{ab} + \frac{c^4}{bc} \overset{Bergström}{\geq} \frac{(a^2+b^2+c^2)}{ac+ab+bc} \geq$$

$$\geq \frac{(ac+bc+ab)^2}{ac+bc+ab} \geq ac + bc + ab \quad (1)$$

$$ab + bc + ca = a^4b + b^4c + c^4a \geq \frac{(a^2+b^2+c^2)}{ac+ab+bc} \Rightarrow$$

$$\Rightarrow ac + ab + bc \geq a^2 + b^2 + c^2, \text{ but } (1)$$

$$\Rightarrow a^2 + b^2 + c^2 \geq ac + ab + bc \Rightarrow$$

$$\Rightarrow a^2 + b^2 + c^2 = ab + ac + bc$$

$$\text{“}=\text{”}\ a = b = c \Rightarrow abc = 1 \Rightarrow a^3 = 1, a = 1, b = 1, c = 1$$

$$(a,b,c) = (1,1,1)$$

3.3 Solve for real numbers:

$$\begin{cases} \boldsymbol{x,y,z} > 0 \\ \boldsymbol{x + e^y + \ln z = 1 + e} \\ \boldsymbol{\ln^2\left(\frac{x}{y}\right) + \ln^2\left(\frac{y}{z}\right) + \ln^2\left(\frac{z}{x}\right) = \ln^2\left(\frac{xy}{z^2}\right) + \ln^2\left(\frac{yz}{x^2}\right) + \ln^2\left(\frac{zx}{y^2}\right)} \end{cases}$$

Solution:

$$\ln^2\left(\frac{x}{y}\right) - \ln^2\left(\frac{xy}{z^2}\right) + \ln^2\left(\frac{y}{x}\right) - \ln^2\left(\frac{yz}{x^2}\right) + \ln^2\left(\frac{z}{y}\right) - \ln^2\left(\frac{zx}{x^2}\right) = e$$

$$\left[\ln\left(\frac{x}{y}\right) - \ln\left(\frac{xy}{z^2}\right)\right]\left[\ln\left(\frac{x}{y}\right) + \ln\left(\frac{xy}{z^2}\right)\right] + \left[\ln\left(\frac{y}{z}\right) - \ln\left(\frac{yz}{x^2}\right)\right]\left[\ln\frac{y}{z} + \ln\frac{yz}{x^2}\right] +$$

$$+\left[\ln\left(\frac{z}{y}\right) - \ln\left(\frac{zx}{x^2}\right)\right]\left[\ln\left(\frac{z}{y}\right) + \ln\left(\frac{zx}{x^2}\right)\right] = 0$$

$$\Rightarrow \ln\frac{z^2}{y^2}\cdot\ln\frac{x^2}{z^2}+\ln\frac{x^2}{z^2}\cdot\ln\frac{y^2}{x^2}+\ln\frac{y^2}{x^2}\cdot\ln\frac{z^2}{y^2}=0$$

$$=ln1,4\left(\ln\frac{z}{y}\ln\frac{x}{z}+\ln\frac{x}{z}\cdot\ln\frac{y}{x}+\ln\frac{y}{x}\ln\frac{z}{y}\right)=0=ln1$$

$$\ln\frac{z}{y}\cdot\ln\frac{x}{z}+\ln\frac{x}{z}\cdot\ln\frac{y}{x}+\ln\frac{y}{x}\cdot\ln\frac{z}{y}=0$$

Let: $\ln\frac{y}{x}=\alpha,\ln\frac{z}{y}=\beta,\ln\frac{x}{z}=\gamma\Rightarrow\beta\gamma+\gamma\alpha+\alpha\beta=0$

but $a+\beta+\gamma=\ln\left(\frac{y}{x}+\ln\frac{z}{y}+\ln\frac{x}{z}\right)=\ln\frac{xyz}{xyz}=\ln 1=0$

$$\Rightarrow \alpha+\beta+\gamma=0|^2\Rightarrow a^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)=0=1$$

$$\alpha^2+\beta^2+\gamma^2=0\Rightarrow\alpha=\beta=\gamma=0\Rightarrow\ln\frac{y}{x}=\ln\frac{z}{y}=\ln\frac{x}{z}=0$$

$\Rightarrow x=y=z\Rightarrow x+e^x+\ln x=1+e$ *but* $e>1\Rightarrow f(x)=x+e^x+\ln x$

f *has unique solution* $x=y\Rightarrow x=y=z=1.$

3.4 Three different nonzero real numbers a,b,c satisfy the equations:

$$\boldsymbol{a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}=p,p\in\mathbb{R}}$$

***Prove that:* $\boldsymbol{abc+2p=0}$**

Proposed as subject- Argentina NMO

Solution:

$2\overset{(1)}{=}bp-ab,2\overset{(2)}{=}cp-bc\ \&\ 2\overset{(3)}{=}ap-ca,$ *(1) + (2) + (3)* $\Rightarrow 6=p(\sum a)-\sum ab$ *(a)*

Also, $\frac{2}{b}\overset{(4)}{=}p-a,\frac{2}{c}\overset{(5)}{=}p-b,\frac{2}{a}\overset{(6)}{=}p-c$

(4) × (5) × (6) $\Rightarrow\frac{8}{abc}=p^3-p^2(\sum a)+p(\sum ab)-abc=p^3-p(p\sum a-\sum ab)-abc$

$\underset{(b)}{\overset{by\ (a)}{=}}p^3-6p-abc,$ *(1) – (2)* $\Rightarrow 0=p(b-c)+b(c-a)\Rightarrow\frac{c-a}{b-c}=-\frac{p}{b}$ *(7)*

$$(2)-(3) \Rightarrow 0 = p(c-a) + c(a-b) \Rightarrow \frac{a-b}{c-a} \overset{(8)}{=} -\frac{p}{c}$$

$$(3)-(1) \Rightarrow 0 = p(a-b) + a(b-c) \Rightarrow \frac{b-c}{a-b} = \frac{-p}{a} \quad (9)$$

$$(7) \times (8) \times (9) \Rightarrow 1 = \frac{-p^3}{abc} \Rightarrow abc = -p^3 \quad (c)$$

$$(c) \Rightarrow (b) \text{ becomes: } \frac{8}{-p^3} = 2p^3 - 6p \Rightarrow p^6 - 3p^4 + 4 = 0. \text{ Let } p^2 = t$$

$$\text{Then, } t^3 - 3t^2 + 4 = 0 \Rightarrow (t-2)^2(t+1) = 0$$

$$\Rightarrow t = 2 \ \ (\because t \neq -1 \text{ as } t = p^2 \geq 0) \Rightarrow p^2 = 2 \Rightarrow p^2 \cdot p = 2p \Rightarrow p^3 = 2p \Rightarrow$$

$$p^3 + abc = 2p + abc \Rightarrow 0 = 2p + abc \quad (\text{by } (c))$$

3.5 Find $x \in \mathbb{R}$ such that:

$$\begin{cases} (\tan x)^{\cos x} \in \mathbb{Z} \\ (\cot x)^{\sin x} \in \mathbb{Z} \end{cases}$$

Solution:

As $(\tan x)^{\cos x} = e^{\cos x \ln(\tan x)}$ *and* $(\cot x)^{\sin x} = e^{\sin x \ln(\cot x)}$

We get $(\tan x)^{\cos x}$ *and* $(\cot x)^{\sin x}$ *are defined when*

$\tan x > 0$ *and* $\cot x > 0$ *i.e. when x lies in the first and the third quadrant.*

Also, for $0 < x < 1, 0 < a < 1, 0 < x^a < 1.$

$$\text{For } 0 < x < \frac{\pi}{4}, 0 < \tan x < 1, \frac{1}{\sqrt{2}} < \cos x < 1 \Rightarrow 0 < (\tan x)^{\cos x} < 1$$

$$\text{For } \frac{\pi}{4} < x < \frac{\pi}{2}, 0 < \cot x < 1, \frac{1}{\sqrt{2}} < \sin x < 1 \Rightarrow 0 < (\cot x)^{\sin x} < 1$$

$$\therefore \text{for } 0 < x < \frac{\pi}{2}, \text{ we have to just check up for } x = \frac{\pi}{4}.$$

$$\text{For } x = \frac{\pi}{4}, \tan x = \cot x = 1, (\tan x)^{\cos x} = (\cot x)^{\sin x} = 1^{\frac{1}{\sqrt{2}}} = 1.$$

$$\text{For } \pi < x < \frac{5\pi}{4}, 1 < \cot x < \infty \text{ and } -\frac{1}{\sqrt{2}} < \sin x < 0$$

$$\Rightarrow 0 < (\cot x)^{\sin x} = (\tan x)^{-\sin x} < 1. \text{ Similarly, for } \frac{5\pi}{4} < x < \frac{3\pi}{2}$$

$$0 < (\tan x)^{\cos x} < 1. \text{ For } x = \frac{5\pi}{4}, (\tan x)^{\cos x} = (\cot x)^{\sin x} = 1.$$

$$\text{Thus, general solution is } x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}.$$

3.6 For $a, b \in \mathbb{R}, a \neq b$ solve the system:

$$\begin{cases} 3x + z = 2y + a + b \\ 3x^2 + 3xz = y^2 + 2(a+b)y + ab \\ x^3 + 3x^2z = (a+b)y^2 + 2aby \end{cases}$$

Solution:

$$\begin{cases} 3x + z = 2y + a + b & (1) \\ 3x^2 + 3xz = y^2 + 2(a+b)y + ab & (2) \\ x^3 + 3x^2z = (a+b)y^2 + 2aby & (3) \end{cases}$$

We have (1) $\Rightarrow a + b = 3x + z - 2y$ *(4)*

Similarly, we have (2) $\Rightarrow ab = 3x^2 + 3xz - y^2 - 2(a+b)y$ *(5) and (3)* $\Rightarrow$

$\Rightarrow 2aby = x^3 + 3x^2 - (a+b)y^2$ *(6)*

(4) and (5) $\Rightarrow ab = 3x^2 + 3xz - y^2 - 2y(3x + z - 2y)$ *(7)*

(4) and (6) $\Rightarrow 2aby = x^3 + 3x^2 - y^2(3x + z - 2y)$ *(8)*

(7) and (8) $\Rightarrow 2y[3x^2 + 3xz - y^2 - 2y(3x + z - 2y)] = x^3 + 3x^2 -$

$y^2(3x + z - 2y) \Rightarrow$

$\Rightarrow x^3 + 3x^2z + 9xy^2 + 3y^2z - 6x^2y - 6xyz - 4y^3 = 0$

$\Rightarrow (x - y)^2(x - 4y + 3z) = 0 \Rightarrow x = y$ *or* $x - 4y + 3z = 0$

1) $x = y$

We have (1) $\Rightarrow 3x + z = 2x + a + b \Rightarrow x = a + b - z$

We have (2) $\Rightarrow 3x^2 + 3x(a + b - x) = x^2 + 2(a+b)x + ab \Rightarrow$

$\Rightarrow x^2 - (a+b)x + ab = 0 \Rightarrow (x - a)(x - b) = 0 \Rightarrow x = a$ *or* $x = b$

1.1) $x = a \Rightarrow x = y = z$

We have (1) $\Rightarrow z = a + b - x \Rightarrow z = a + b - a = b \Rightarrow (x, y, z) = (a, a, b)$

1.2) $x = b \Rightarrow x = y = b$

We have (1) $\Rightarrow z = a + b - x \Rightarrow z = a + b - b = a \Rightarrow (x, y, z) = (b, b, a)$

2) $x - 4y + 3z = 0 \Rightarrow z = \frac{4y - x}{3}$

We have (1) $\Rightarrow 3x + \frac{4y - x}{3} = 2y + a + b \Rightarrow x = \frac{2y + 3(a+b)}{8}$

We have (2) $\Rightarrow 3x^2+x(4y-x)=y^2+2(a+b)+ab \Rightarrow 2x^2+4xy=$

$$=y^2+2(a+b)y+ab \Rightarrow 2\left(\frac{2y+3(a+b)}{8}\right)^2+4y\cdot\frac{2y+3(a+b)}{8}=$$

$$=y^2+2(a+b)y+ab \Rightarrow \frac{1}{8}y^2-\frac{1}{8}y(a+b)+\frac{9}{32}(a+b)^2-ab=0$$

$$\Rightarrow \frac{1}{8}y^2-\frac{1}{8}y(a+b)+\frac{1}{32}(a+b)^2+\frac{1}{4}(a+b)^2-ab=0 \Rightarrow$$

$$\Rightarrow \frac{1}{8}\left(y-\frac{a+b}{2}\right)^2+\frac{(a-b)^2}{4}=0 \Rightarrow a=b \text{ (Absurd)}$$

So the system has 2 roots: $(x,y,z)=(a,a,b)$ *and* $(x,y,z)=(b,b,a)$

3.7 Solve for real positive numbers:

$$\begin{cases} 27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)}=8(x+y+z)^3 \\ x+y+z=\frac{1}{xyz} \end{cases}$$

Solution:

$$27\sqrt{\left(x^2+\frac{1}{y^2}\right)\left(y^2+\frac{1}{z^2}\right)\left(z^2+\frac{1}{x^2}\right)}\overset{(1)}{=}8(x+y+z)^3$$

$$x+y+z\overset{(2)}{=}\frac{1}{xyz}$$

$$\text{LHS of (1)}=\frac{27}{xyz}\sqrt{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)}$$

$$\overset{(a)}{=}\frac{27}{xyz}\sqrt{\{x^2y^2+xyz(x+y+z)\}\{y^2z^2+xyz(x+y+z)\}\{z^2x^2+xyz(x+y+z)\}}$$

$$(\because 1=xyz(x+y+z))$$

Now, $x^2y^2+xyz(x+y+z)=xy(xy+zx+yz+z^2)\overset{(b)}{=}xy(y+z)(z+x)$

Similarly, $y^2z^2+xyz(x+y+z)\overset{(c)}{=}yz(x+y)(z+x)$ &

$$z^2x^2 + xyz(x+y+z) \overset{(d)}{=} zx(x+y)(y+z)$$

(a), (b), (c), (d) $\Rightarrow LHS \overset{(i)}{=} 27(x+y)(y+z)(z+x)$

Now, $\sum x = \frac{1}{2}\{(x+y)+(y+z)+(z+x)\} \overset{A-G}{\geq} \frac{3}{2}\sqrt[3]{(x+y)(y+z)(z+x)}$

$$\Rightarrow \left(2\sum x\right)^3 \geq 27(x+y)(y+z)(z+x)$$

$$\Rightarrow 8\left(\sum x\right)^3 \overset{(ii)}{\geq} 27(x+y)(y+z)(z+x)$$

(i), (ii) $\Rightarrow$ *RHS of (1)* $\geq$ *LHS of (1), with equality occuring when* $x=y=z$.

But LHS of (1) = RHS of (1) $\therefore x=y=z$

$\therefore$ *using (2),* $3x = \frac{1}{x^3} \Rightarrow x^4 = \frac{1}{3} \Rightarrow x = \frac{1}{\sqrt{\sqrt{3}}}$

$\therefore$ *only possible solution is:* $(x,y,z) = \left(\frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}\right)$ *(answer)*

3.8 Solve for real numbers:

$$\begin{cases} \mathbf{2^y + 2^z + \tan^{-1} z = 9} \\ \mathbf{|3\sin x - 4\cos x| = y^2 - 6y + 14} \end{cases}$$

Solution:

$\rightarrow\because$ *we know that*

$$-\sqrt{a^2+b^2} \leq a\cos x + b\sin x \leq \sqrt{a^2+b^2} \Rightarrow -\sqrt{3^2+4^2}$$

$$\leq 3\sin x - 4\cos x \leq \sqrt{3^2+4^2}$$

$$\Rightarrow -5 \leq 3\sin x - 4\cos x \leq 5 \Rightarrow 0 \leq |3\sin x - 4\cos x| \leq 5$$

$$\because |3\sin x - 4\cos x| = y^2 - 6y + 14$$

$$|3\sin x - 4\cos x| = (y-3)^2 + 5 \Rightarrow LHS \leq 5, RHS \geq 5 \Rightarrow LHS = RHS = 5$$

$$\Rightarrow (y-3)^2 = 0 \Rightarrow y = 3$$

$$|3\sin x - 4\cos x| = 5 \Rightarrow 3\sin x - 4\cos x = \pm 5 \Rightarrow \frac{3}{5}\sin x - \frac{4}{5}\cos x = \pm 1$$

$$\frac{3}{5}\sin x - \frac{4}{5}\cos x = \pm 1$$

$$\sin(x-\alpha)=\sin\left(\pm\frac{\pi}{2}\right), \tan\alpha=\frac{4}{3}\Rightarrow x-\alpha=n\pi+(-1)^n\left(\pm\frac{\pi}{2}\right)$$

$$x=n\pi\pm(-1)^n\left(\frac{\pi}{2}\right)+\tan^{-1}\left(\frac{4}{3}\right), n\in I$$

Now, $2^y+2^z+\tan^{-1}z=9 \therefore y=3\Rightarrow 8+2^z+\tan^{-1}z=9\Rightarrow 2^z=$ $\tan^{-1}z=1$

Let $b(z)=2^z+\tan^{-1}z-1; b'(z)=2^z\ln 2+\frac{1}{1+z^2}>0\Rightarrow b'(z)>0\Rightarrow b(z)$ *is increasing function. So,* $b(z)$ *can have atmost one root* $\because b(0)=0\Rightarrow z=0$ *is the only possible solution.*

$$\begin{cases} x=n\pi\pm(-1)^n\left(\frac{\pi}{2}\right)+\tan^{-1}\left(\frac{4}{3}\right), n\in I \\ y=3 \\ z=0 \end{cases}$$

3.9 Solve the following system:

$$\begin{cases} \boldsymbol{x^e+y^e+z^e+u^e=\frac{56}{12\pi}} \\ \boldsymbol{(xy)^e+(xz)^e+(xu)^e+(yz)^e+(yu)^e+(zt)^e=\frac{89}{12\pi^2}} \\ \boldsymbol{(xyz)^e+(xyu)^e+(xzu)^e+(yzu)^e=\frac{56}{12\pi^3}} \\ \boldsymbol{(xyzu)^e=\frac{1}{\pi^4}} \end{cases}$$

Solution:

Let $\pi x^e=a, \pi y^e=b, \pi z^e=c, \pi u^e=d$

This system of equations reduces to

$$\begin{cases} \sum a=\frac{56}{12} \\ \sum ab=\frac{89}{12} \\ \sum abc=\frac{56}{12} \\ abcd=1 \end{cases}$$

Let us create a biquadratic equation in which a, b, c, d *are its roots.*

$$t^4-\left(\frac{56}{12}\right)t^3+\left(\frac{89}{12}\right)t^2-\left(\frac{56}{12}\right)t+1=0$$

$$12t^4-56t^3+89t^2-56t+12=0$$

Dividing throughout by t^2 $(\because\ t\neq 0)\Rightarrow 12t^2-56t+89-\frac{56}{t}+\frac{12}{t^2}=0$

$$\Rightarrow 12\left(t^2+\frac{1}{t^2}\right)-56\left(t+\frac{1}{t}\right)+89=0$$

$$\Rightarrow 12\left[\left(t+\frac{1}{t}\right)^2-2\right]-56\left(t+\frac{1}{t}\right)+89=0$$

$$\Rightarrow 12\left(t+\tfrac{1}{t}\right)^2-56\left(t+\tfrac{1}{t}\right)+65=0\ \ Let\ k=\left(t+\tfrac{1}{t}\right)$$

$$\Rightarrow 12k^2-56k+65=0,\ k=\tfrac{5}{2},\tfrac{13}{6}$$

$$t+\tfrac{1}{t}=\tfrac{5}{2},\ \ 2t^2-5t+2=0,\ t=2,\tfrac{1}{2}$$

$$Or\ t+\tfrac{1}{t}=\tfrac{13}{6},\ 6t^2-13t+6=0$$

$$t=\frac{3}{2},\frac{2}{3}\ \Rightarrow t=2,\frac{1}{2},\frac{3}{2},\frac{2}{3}\left|\left\{\begin{matrix}a\\b\\c\\d\end{matrix}\right\}\right.$$

$$a=\pi x^e=2\Rightarrow x=\left(\tfrac{2}{\pi}\right)^{\frac{1}{e}},b=\pi y^2=\tfrac{1}{2}\Rightarrow y=\left(\tfrac{1}{2\pi}\right)^{\frac{1}{e}}$$

$$c=\pi z^e=\tfrac{3}{2}\Rightarrow z=\left(\tfrac{3}{2\pi}\right)^{\frac{1}{e}},d=\pi u^e=\tfrac{2}{3}\Rightarrow u=\left(\tfrac{2}{3\pi}\right)^{\frac{1}{e}}$$

$$(a,b,c,d)\equiv(\pi x^e,\pi y^e,\pi z^e,\pi u^e)\equiv\left(2,\frac{1}{2},\frac{3}{2},\frac{2}{3}\right)$$

$$\begin{cases}x=\left(\dfrac{2}{\pi}\right)^{\frac{1}{e}}\\ y=\left(\dfrac{1}{2\pi}\right)^{\frac{1}{e}}\\ z=\left(\dfrac{3}{2\pi}\right)^{\frac{1}{e}}\\ u=\left(\dfrac{2}{3\pi}\right)^{\frac{1}{e}}\end{cases}$$

Note: Since (x, y, z, u) are symmetric in the given problem, so any combination of the above set is possible for (x, y, z, u).

3.10

$$\begin{cases} x^y = \sqrt[7]{\left(\tan\frac{3\pi}{7} - 4\sin\frac{\pi}{7}\right)^x} \\ \dfrac{10^{\frac{1}{y^2}} + 95}{x - \frac{1}{y}} = 2019 \end{cases}$$

Solution:

Denote: $z = \tan\frac{3\pi}{7} - 4\sin\frac{\pi}{7} = \tan\frac{3\pi}{7} - \frac{8\tan\frac{\pi}{14}}{1+\tan^2\frac{\pi}{14}}$

as $\tan\frac{\pi}{14} = \tan\left(\frac{\pi}{2} - \frac{3\pi}{7}\right) = \cot\frac{3\pi}{7}$, *thus*

$$z = \tan\frac{3\pi}{7} - \frac{8\cot\frac{3\pi}{7}}{1+\cot^2\frac{3\pi}{7}} = \frac{\tan\frac{3\pi}{7} - 7\cot\frac{3\pi}{7}}{1+\cot^2\frac{3\pi}{7}}$$

Let $\phi = \frac{3\pi}{7} \Rightarrow 3\pi = 7\phi = 3\phi + 4\phi$ *gives* $\tan 4\phi = -\tan 3\phi$

$$\frac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi} = -\tan(3\phi + \phi) = -\frac{4\tan\phi\,(1-\tan^2\phi)}{1 - 6\tan^2\phi + \tan^4\phi}$$

For the sake of convenience, let $\tan^2\phi = x$, *then:* $\frac{3-x}{1-3x} + \frac{4(1-x)}{1-6x+x^2} = 0$

simplifying yields $x^3 - 21x^2 + 35x - 7 = 0$, *since* $x = \tan^2\frac{3\pi}{7} \neq 0$, *so, cubic equation can be written as:* $x(x-7)^2 = 7(x+1)^2 \Rightarrow \frac{x(x-7)^2}{(x-1)^2} = 7$.

Replugging, we obtain that:

$z = \tan\frac{3\pi}{7} - \frac{8\cot\frac{3\pi}{7}}{1+\cot^2\frac{3\pi}{7}} = \frac{\tan\frac{3\pi}{7} - 7\cot\frac{3\pi}{7}}{1+\cot^2\frac{3\pi}{7}} = \sqrt{7}$. *Hence, we have:* $\begin{cases} x^y = \sqrt[7]{\left(\sqrt{7}\right)^x} \\ \frac{10^{\frac{1}{y^2}}+95}{x-\frac{1}{y}} = 2019 \end{cases}$

We note $y \neq 0; x^y = 7^{\frac{x}{14}} \Rightarrow x = \frac{14y\log x}{\log 7}$ *shows* $x > 0$ *putting in the second equation we have:* $\frac{\log 7 \cdot y\left(10^{\frac{1}{y^2}}+95\right)}{14y^2\log x - \log 7} = 2019$

2019 *is rational so,* $\log x$ *should be in the form of* $\log 7^a$ *for* $a \in \mathbb{N}$*. We have then:*

$$x = 7^a = 14ay, \forall a \geq 2, y = \frac{7^a}{14a} = \frac{7^{a-1}}{2a} > 1 \text{ (by induction) and hence}$$

$\frac{1}{y^2} = \frac{4a^2}{49^{a-1}} < 1$ *which implies* $10^{\frac{1}{y^2}} = 10^{\frac{4a^2}{49^{a-1}}} = 10^{\frac{p}{q}} \notin \mathbb{N}$, *where* $p, q \in \mathbb{Z}^+$ *and* $p < q$ *as*

$$4a^2 < 49^{a-1}, \forall a \geq 2 \text{ (by induction) and being the g.c.d } (p,q) = 1.$$

This shows that $10^{\frac{p}{q}} \in \mathbb{N}$ *if and only if* $q|p$ *but* $p < q$ *and* p, q *are co-primes integers so,* $q \nmid p$ *thus, only case we can have is* $q = 1 = 49^{a-1} \Rightarrow a = 1$, *which gives us*

$x = 7, y = \frac{1}{2}$. *The required answer therefore is* $(x,y) = \left(7, \frac{1}{2}\right)$.

3.11 Solve for real numbers:

$$\begin{cases} \sin x = \cos y \\ \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0 \end{cases}$$

Solution:

$$\Delta \to \begin{vmatrix} \sin(x+y) & \sin(y+\sqrt{xy}) & \sin(\sqrt{xy}+x) \\ \cos(x+y) & \cos(y+\sqrt{xy}) & \cos(\sqrt{xy}+x) \\ \cos(x-y) & \cos(y-\sqrt{xy}) & \cos(\sqrt{xy}-x) \end{vmatrix} = 0$$

$$\sin x = \cos y \Rightarrow \sin x = \sin\left(\frac{\pi}{2} - y\right)$$

$$\sin(x+y)=\sin x\cos y+\sin y\cos x=\cos^2 y+\sin^2 y=1 \quad \textit{(1)}$$

$$\therefore \sin^2 x=\cos^2 y$$

$$1-\cos^2 x=\cos^2 y\Rightarrow\cos^2 x=1-\cos^2 y\Rightarrow\cos x=\sin y$$

$$\cos(x+y)=\cos x\cos y-\sin x\sin y=\sin y\cos y-\cos y\sin y=0 \quad \textit{(2)}$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y=\sin y\cos y+\cos y\sin y=$$

$$2\sin y\cos y=\sin 2y \quad \textit{(3)}$$

$$\Delta=\begin{vmatrix}1 & \sin\left(y+\sqrt{xy}\right) & \sin\left(\sqrt{xy}+x\right)\\ 0 & \cos\left(y+\sqrt{xy}\right) & \cos\left(\sqrt{xy}+x\right)\\ \sin 2y & \cos\left(y-\sqrt{xy}\right) & \cos\left(\sqrt{xy}-x\right)\end{vmatrix}=0$$

We develop after the first column:

$$\cos\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}-x\right)-\cos\left(\sqrt{xy}+x\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}+x\right)-\sin\left(\sqrt{xy}+x\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0\Rightarrow$$

$$\Rightarrow\cos\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}-\frac{\pi}{2}+y\right)-\cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\cos\left(\sqrt{xy}+\frac{\pi}{2}-y\right)-\sin\left(\sqrt{xy}+\frac{\pi}{2}-y\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)$$
$$=0$$

$$\cos\left(y+\sqrt{xy}\right)\cdot\sin\left(y+\sqrt{xy}\right)-\sin\left(y-\sqrt{xy}\right)\cdot\cos\left(y-\sqrt{xy}\right)+$$

$$+\sin 2y\left(\sin\left(y+\sqrt{xy}\right)\cdot\sin\left(y-\sqrt{xy}\right)-\cos\left(y-\sqrt{xy}\right)\cdot\cos\left(y+\sqrt{xy}\right)\right)=0\Rightarrow$$

$$\Rightarrow\frac{\sin 2\left(y+\sqrt{xy}\right)}{2}-\frac{\sin 2\left(y-\sqrt{xy}\right)}{2}-\sin 2y\left(\cos 2y\right)=0\Rightarrow$$

$$\Rightarrow\frac{2\sin 2\sqrt{xy}\cos 2y}{2}-\sin 2y\cos 2y=0$$

$$\cos 2y\left(\sin 2\sqrt{xy}-\sin 2y\right)=0$$

Case I: $\cos 2y=0\Rightarrow 2y=\pm\frac{\pi}{2}+2k\pi\Rightarrow\begin{cases}y=\pm\frac{\pi}{4}+k\pi\Rightarrow\\ x=\frac{\pi}{2}-y=\frac{\pi}{2}\mp\frac{\pi}{4}-k\pi\end{cases};k\in\mathbb{Z}$

Case II: $\sin 2\sqrt{xy}-\sin 2y=0\Rightarrow\sin 2\sqrt{xy}=\sin 2y\Rightarrow\sin 2\sqrt{xy}=$

$\sin 2\sqrt{y\cdot y}\Rightarrow\begin{cases}y=0\\ x=\frac{\pi}{2}+2k\pi\end{cases};k\in\mathbb{N}$, *because* $x\neq y$; $\begin{matrix}\sin x>0\\ y>0\end{matrix}$

3.12 Solve in real numbers the system of equations:

$$\begin{cases} \boldsymbol{x+y+z+t=0,} \\ \boldsymbol{x^2+y^2+z^2+t^2=4,} \\ \boldsymbol{x^4+y^4+z^4+t^4=4.} \end{cases}$$

Solution:

$$\sum x \overset{(a)}{=} 0, \sum x^2 \overset{(b)}{=} 4, \sum x^4 \overset{(c)}{=} 4$$

Let $x+y+z=\sigma(x), x^2+y^2+z^2=\sigma(x^2)$ *and* $x^4+y^4+z^4=\sigma(x^4)$

Now, $\sigma(x^4) \overset{(1)}{=} 4-t^4$ *and* $\sigma(x^2) \overset{(2)}{=} 4-t^2$

$$\because \sigma(x^4) \geq \frac{1}{3}\{\sigma(x^2)\}^2$$

$$\therefore 4-t^2 \geq \frac{1}{3}(4-t^2)^2 \text{ (using (1), (2))} \Rightarrow 12-3t^4 \geq 16-8t^2+t^4$$

$$\Rightarrow 4t^4-8t^2+4 \leq 0 \Rightarrow 4(t^2-1)^2 \leq 0$$

$$\Rightarrow t^2-1=0(\because (t^2-1)^2 \geq 0 \text{ and } (t^2-1)^2 \leq 0) \Rightarrow t^2=t^4=1$$

$$\therefore \sigma(x^2) \underset{by\ (b)}{\overset{(3)}{=}} 3 \text{ and } \sigma(x^4) \overset{by\ (c)}{=} 3 \text{ (using (1), (2))}$$

$$\text{Again, } \sigma(x^4) \geq \frac{1}{3}\left(\sigma(x^2)\right)^2 \overset{by\ (3)}{=} (\because \sigma(x^2)=3)$$

$$\Rightarrow \sigma(x^4) \geq 3 \text{ with equality at } x=y=z \text{ and } \because \sigma(x^4)=3 \therefore \text{ equality occurs}$$

$$\Rightarrow x=y=z$$

Putting $x=y=z$ *in (3), we get* $3x^2=3$

$$\Rightarrow x^2=y^2=z^2=t^2=1 \text{ and } \because \sum x=0 \text{ all possible solutions are:}$$

$$\begin{pmatrix} x=1 \\ y=1 \\ z=-1 \\ t=-1 \end{pmatrix}, \begin{pmatrix} x=1 \\ y=-1 \\ z=-1 \\ t=-1 \end{pmatrix}, \begin{pmatrix} x=1 \\ y=-1 \\ z=-1 \\ t=-1 \end{pmatrix}, \begin{pmatrix} x=-1 \\ y=1 \\ z=1 \\ t=-1 \end{pmatrix}, \begin{pmatrix} x=-1 \\ y=1 \\ z=-1 \\ t=1 \end{pmatrix}, \begin{pmatrix} x=-1 \\ y=-1 \\ z=1 \\ t=1 \end{pmatrix}$$

3.13 Solve for real numbers:

$$\begin{cases} \mathbf{1 \le x, y, z \le 3} \\ (\boldsymbol{x+y+z})\left(\frac{\mathbf{1}}{\boldsymbol{x}}+\frac{\mathbf{1}}{\boldsymbol{y}}+\frac{\mathbf{1}}{\boldsymbol{z}}\right)=\frac{\mathbf{35}}{\mathbf{3}} \\ \mathbf{3}^{\boldsymbol{y}}+\mathbf{log}_{\mathbf{2}}\,\boldsymbol{z}=\mathbf{3} \end{cases}$$

Solution:

$$\textit{if } z \in (1,3] \textit{ then } \frac{\ln z}{\ln 2} > 0 \Leftrightarrow \log_2 z > 0$$

$$\textit{if } y \in (1,3] \textit{ then } 3^y > 3 \qquad (+)$$

$$- - - - - - - - - - - - - - - - - - - -$$

$$\textit{So: } 3^y + \log_2 z > 3$$

$$\textit{Likewise, if } z \in [1,3] \vee y \in (1,3] \textit{ and } z \in (1,3] \vee y \in [1,3]$$

$$3^y + \log_2 z = 3 \textit{ if-f } y = z = 1$$

$$\textit{Then } (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{35}{3} \textit{ because } (x+2)\left(\frac{1}{x}+2\right)=\frac{35}{3}$$

$$1+2x+\frac{2}{x}+4=\frac{35}{3} \Leftrightarrow 2\left(x+\frac{1}{x}\right)=\frac{\infty}{3} \Leftrightarrow x+\frac{1}{x}=\frac{10}{3} \overset{3x}{\Leftrightarrow}$$

$$\Leftrightarrow 3x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{100 - 4 \cdot 9}}{2 \cdot 3} = \frac{10 \pm 8}{6} \left\langle \begin{array}{l} \frac{18}{6} = 3 \\ \frac{2}{6} = \frac{1}{3} \end{array} \right.$$

$$(x, y, z) = (3,1,1)$$

3.14 Solve for real numbers:

$$\begin{cases} (\mathbf{log}(\boldsymbol{xy})+\boldsymbol{x})^{\mathbf{3}} = (\mathbf{log}(\boldsymbol{xy})-\boldsymbol{x})^{\mathbf{3}} + \left(\boldsymbol{x}-\mathbf{log}\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right)\right)^{\mathbf{3}} + \left(\boldsymbol{x}+\mathbf{log}\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right)\right)^{\mathbf{3}} \\ \boldsymbol{x}^{\boldsymbol{y}}+\boldsymbol{y}^{\boldsymbol{z}}+\boldsymbol{z}^{\boldsymbol{x}}=\mathbf{5} \\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} > 0 \end{cases}$$

Solution:

$$(a+b)^3 - (a-b)^3 = 2b(3a^2+b^2) \quad (*)$$

$$(a+b)^3 + (a-b)^3 = 2a(a^2+3b^2) \quad (**)$$

First equation is equivalent with:

$$\underbrace{(\log(xy)+x)^3-(\log(xy)-x)^3}_{(*)}=\underbrace{\left(x-\log\left(\frac{x}{y}\right)\right)^3+\left(x+\log\left(\frac{x}{y}\right)\right)^3}_{(**)}$$

$$2x(3\log^2(xy)+x^2)=2x\left(x^2+3\log^2\left(\frac{x}{y}\right)\right)$$

$\log^2(xy)=\log^2\left(\frac{x}{y}\right)$. $xy=\pm\frac{x}{y}$ *(sign "–" not accepted)*

$y=\frac{1}{y}\Leftrightarrow y=1$. *Second equation:* $x+1+z^x=5;\ x+z^x=4$

Let $x=a,a\in(0,4)\Rightarrow z=\sqrt[a]{4-a}$. *Conclusion:*

Solution is $\left(a,1,\sqrt[a]{4-a}\right),a\in(0,4)$

3.15 Solve for $x,y,z>0$:

$$\begin{cases}\boldsymbol{x-y+z=\frac{1}{2}}\\ \boldsymbol{3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=2(x+y+z)}\\ \boldsymbol{8xyz=1}\end{cases}$$

Solution:

Find all

$x,y,z>0|x-y+z\overset{(i)}{=}\frac{1}{2},3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)\overset{(ii)}{=}2\sum x$ *and*

$$8xyz\overset{(iii)}{=}1$$

$$\because 8xyz=1\therefore\frac{1}{1+2z+4zx}=\frac{1}{8xyz+2z+4zx}=\frac{1}{2z(1+2x+4xy)}\Rightarrow$$

$\Rightarrow\frac{1}{1+2z+4zx}\overset{(1)}{=}\frac{1}{2z(1+2x+4xy)}$ *and*

$$\frac{1}{1+2y+4yz}=\frac{1}{8xyz+2y+4yz}=\frac{1}{2y(1+2z+4zx)}\overset{by\ (1)}{=}\frac{1}{4yz(1+2x+4xy)}$$

$$\therefore\frac{1}{1+2y+4yz}\overset{(2)}{=}\frac{1}{4yz(1+2x+4xy)}$$

$\therefore$ *(1)+(2)*$\Rightarrow 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=$

$$=3\left(\frac{1}{1+2x+4xy}+\frac{1}{4yz(1+2x+4xy)}+\frac{1}{2z(1+2x+4xy)}\right)$$

$$=\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)\overset{\because 1=8xyz}{=}$$

$$=8xyz\left(\frac{3}{1+2x+4xy}\right)\left(1+\frac{1}{4yz}+\frac{1}{2z}\right)=$$

$$=\left(\frac{3}{1+2x+4xy}\right)(8xyz+2x+4xy)=$$

$$=\left(\frac{3}{1+2x+4xy}\right)(1+2x+4xy)=3$$

$$\therefore 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=3\overset{by\ (ii)}{=}$$

$$=2\sum x\Rightarrow x+y+z\overset{(iv)}{=}\frac{3}{2}$$

(iv)+(i)$\Rightarrow 2(z+x)=2\Rightarrow z+x\overset{(v)}{=}1\ \ (iv)\cdot(v)\Rightarrow y=\frac{1}{2}\overset{by\ (iii)}{=}4zx\overset{(vi)}{=}1$

(v)$\Rightarrow (z+x)^2=1\Rightarrow (z-x)^2+4zx=1\overset{by\ (iv)}{\Rightarrow}(z-x)^2+4zx=1\overset{by\ (iv)}{\Rightarrow}$

$$\Rightarrow (z-x)^2+1=1\Rightarrow (z-x)^2=0\Rightarrow z\overset{(vii)}{=}x$$

(vi), (vii)$\Rightarrow 4x^2=1\Rightarrow (2x+1)(2x-1)=0\Rightarrow$

$$x=\frac{1}{2}(\because x>0)\overset{by\ (vii)}{=}x=z=\frac{1}{2}$$

$\therefore x=y=z=\frac{1}{2}$ *(answer)*

3.16 Solve for integers:

$$\begin{cases} \boldsymbol{x+y+z=6} \\ \boldsymbol{x^2(y-z)+y^2(z-x)+z^2(x-y)=2} \end{cases}$$

Solution:

$x^2(y-z)+y^2(z-x)+z^2(x-y)=xy(x-y)-z(x+y)(x-y)+z^2(x-y)=(x-y)\big(x(y-z)-z(y-z)\big)=(x-y)(y-$

$z)(x - z) = 2$

$\because (x - y), (y - z) and\ (x - z) are\ integers\ whose\ product = 2,$

$\therefore\ the\ following\ are\ all\ possible\ cases:$

$\boxed{Case\ (1)}\ x - y = 1, y - z = 1\ and\ x - z = 2 \Rightarrow y = x - 1\ and\ z$

$= x - 2\ and\ \because x + y + z = 6, \therefore x + x - 1 + x - 2 = 6$

$\Rightarrow \boxed{x = 3 \therefore y = 2\ and\ z = 1}$

$\boxed{Case\ (2)}\ x - y = 1, y - z = 2\ and\ x - z = 1 \Rightarrow x - y + y - z = 3$

$\Rightarrow x - z = 3 \Rightarrow 3 = 1 \to impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (3)}\ x - y = 1, y - z = -1\ and\ x - z = -2 \Rightarrow x - y + y - z$

$= 0 \Rightarrow x - z = 0 \Rightarrow 0 = -2 \to impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (4)}\ x - y = 1, y - z = -2\ and\ x - z = -1 \Rightarrow y = x - 1\ and\ z$

$= x + 1\ and\ \because x + y + z = 6, \therefore x + x - 1 + x + 1 = 6$

$\Rightarrow \boxed{x = 2 \therefore y = 1\ and\ z = 3}$

$\boxed{Case\ (5)}\ x - y = 2, y - z = 1\ and\ x - z = 1 \Rightarrow x - y + y - z = 3$

$\Rightarrow x - z = 3 \Rightarrow 3 = 1 \to impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (6)}\ x - y = 2, y - z = -1\ and\ x - z = -1 \Rightarrow x - y + y - z$

$= 1 \Rightarrow x - z = 1 \Rightarrow 1 = -1 \to impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (7)}\ x - y = -1, y - z = 2\ and\ x - z = -1 \Rightarrow x - y + y - z$

$= 1 \Rightarrow x - z = 1 \Rightarrow 1 = -1 \to impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (8)}\ x - y = -1, y - z = -2\ and\ x - z = 1 \Rightarrow x - y + y - z$

$= -3 \Rightarrow x - z = -3 \Rightarrow -3 = 1 \rightarrow impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\boxed{Case\ (9)}\ x - y = -2, y - z = 1\ and\ x - z = -1 \Rightarrow y = x + 2\ and\ z$

$= x + 1\ and\ \because x + y + z = 6, \therefore x + x + 2 + x + 1 = 6$

$\Rightarrow \boxed{x = 1 \therefore y = 3\ and\ z = 2}$

$\boxed{Case\ (9)}\ x - y = -2, y - z = -1\ and\ x - z = 1 \Rightarrow x - y + y - z$

$= -3 \Rightarrow x - z = -3 \Rightarrow -3 = 1 \rightarrow impossible$

$\Rightarrow no\ solution\ exists\ under\ this\ case$

$\therefore all\ possible\ integer\ triplets\ of\ (x, y, z) satisfying$

$the\ 2\ given\ equations\ are\ \boxed{\boxed{(3,2,1), (2,1,3)\ and\ (1,3,2)}} (answer)$

3.17 Solve for $x, y, z > 0$:

$$\begin{cases} \boldsymbol{x - y + z = \frac{1}{2}} \\ \boldsymbol{3\left(\frac{1}{1 + 2x + 4xy} + \frac{1}{1 + 2y + 4yz} + \frac{1}{1 + 2z + 4zx}\right) = 2(x + y + z)} \\ \boldsymbol{8xyz = 1} \end{cases}$$

Solution:

$$3\sum \frac{1}{1 + 2x + 4xy} = 2(x + y + z) =$$

$$= 3\left(\frac{1}{1 + 2x + 4xy} + \frac{1}{1 + 2y + 4 \cdot \frac{1}{8x}} + \frac{1}{1 + 2 \cdot \frac{1}{8xy} + 4 \cdot \frac{1}{8y}}\right)$$

$$= 3\left(\frac{1}{1 + 2x + 4xy} + \frac{2x}{2x + 4xy + 1} + \frac{4xy}{4xy + 1 + 2x}\right) =$$

$$= 3\frac{1 + 2x + 4xy}{1 + 2x + 4xy} = 3 = 2(x + y + z) \Rightarrow x + y + z = \frac{3}{2}$$

$$x-y+z=\frac{1}{2}\Rightarrow 2y=1\Rightarrow y=\frac{1}{2}\Rightarrow x+z=1,8\cdot\frac{1}{2}xz=1$$

$$\Rightarrow xz=\frac{1}{4};t^2-t+\frac{1}{4}=0,\left(t-\frac{1}{2}\right)^2=0\Rightarrow t_1=t_2=\frac{1}{2}$$

$$\Rightarrow x=z=\frac{1}{2}.\ \textit{So, } x=y=z=\frac{1}{2}$$

3.18 Solve for $x,y,z>0$:

$$\begin{cases} 4(xy+yz+zx)=3 \\ 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4zx}\right)=2(x+y+z) \\ 8xyz=1 \end{cases}$$

Solution:

We know that $x+y+z\ge 3\sqrt[3]{xyz}\Leftrightarrow x+y+z\ge\frac{3}{2}$

We also know that: $\frac{3}{1+2x+4xy}=\frac{3}{1+2x+\frac{1}{2z}}=\frac{3}{\frac{1}{1}+\frac{1}{\frac{1}{2x}}+\frac{1}{2z}}\le\frac{1+\frac{1}{2x}+2z}{3}\Rightarrow$

$$\Rightarrow 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2+4yz}+\frac{1}{1+2z+4xz}\right)\le$$

$$\le\frac{3+\frac{1}{2x}+\frac{1}{2y}+\frac{1}{2z}+2(x+y+z)}{3}=\frac{3+3+2(x+y+z)}{3}=\frac{6+2(x+y+z)}{3}\ \textit{(1)}$$

$$x+y+z\ge\frac{3}{2}\Rightarrow 4(x+y+z)\ge 6\Rightarrow 6(x+y+z)\ge 6+2(x+y+z)\Rightarrow$$

$$\Rightarrow\frac{6+2(x+y+z)}{3}\le 2(x+y+z)\ \textit{(2)}$$

$$\overset{(1);(2)}{\Rightarrow} 3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4xz}\right)\le 2(x+y+z),$$

but we know that:

$$3\left(\frac{1}{1+2x+4xy}+\frac{1}{1+2y+4yz}+\frac{1}{1+2z+4xz}\right)=2(x+y+z)\Rightarrow$$

$$\Rightarrow 1=\frac{1}{2x}=2z=\frac{1}{2y}=2x=\frac{1}{2z}=2y\Rightarrow x=y=z=\frac{1}{2}$$

3.19 Solve for real numbers:

$$\begin{cases} \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} > 0 \\ \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} + \boldsymbol{xyz} \geq \boldsymbol{4} \\ \sqrt{\boldsymbol{x}} + \sqrt{\boldsymbol{y}} + \sqrt{\boldsymbol{z}} = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} \end{cases}$$

Solution:

With $x, y, z > 0$ *we have:*

$$x + y + z = 1 \cdot \sqrt{x} + 1 \cdot \sqrt{y} + 1 \cdot \sqrt{z} \overset{B.C.S}{\leq} \sqrt{3} \cdot \sqrt{x + y + z}$$

$$\Rightarrow \sqrt{x + y + z} \leq \sqrt{3} \Rightarrow x + y + z \leq 3 \quad (*)$$

$$xyz \leq \frac{(x + y + z)^3}{27} \Rightarrow 4 \leq xyz + x + y + z \leq \frac{(x + y + z)^3}{27} + (x + y + z)$$

$$\Leftrightarrow (x + y + z)^3 + 27(x + y + z) - 108 \geq 0$$

$$\overset{t=x+y+z>0}{\Leftrightarrow} t^3 + 27t - 108 \geq 0 \Leftrightarrow (t - 3)(t^2 + 3t + 36) \geq 0$$

$$\Leftrightarrow t \geq 3 \Leftrightarrow x + y + z \geq 3 \quad (**) \overset{(*),(**)}{\Rightarrow} x + y + z = 3 \Leftrightarrow x = y = z = 1.$$

$$\Rightarrow (x; y; z) = (1; 1; 1) \text{ (Answer)}$$

3.20 Solve for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t} > 0$:

$$\begin{cases} \boldsymbol{xt} = \boldsymbol{4e} \\ \dfrac{\boldsymbol{x}^2 + \boldsymbol{y}^2}{\boldsymbol{x} + \boldsymbol{y}} + \dfrac{\boldsymbol{y}^2 + \boldsymbol{z}^2}{\boldsymbol{y} + \boldsymbol{z}} + \dfrac{\boldsymbol{z}^2 + \boldsymbol{x}^2}{\boldsymbol{z} + \boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} \\ \boldsymbol{t}^{\log \boldsymbol{y}} = \boldsymbol{4} \end{cases}$$

Solution:

For all $a, b > 0$ *we have:* $\frac{a^2+b^2}{a+b} \geq \frac{a+b}{2}$ *(*)*

$$\Leftrightarrow 2(a^2 + b^2) \geq (a + b)^2 \Leftrightarrow a^2 - 2ab + b^2 \geq 0$$

$$\Leftrightarrow (a - b)^2 \geq 0 \quad \text{(true)}$$

Equality $\Leftrightarrow a = b$

So, $\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x} \geq \frac{x+y}{2} + \frac{y+z}{2} + \frac{z+x}{2} = x + y + z$

"=" $\Leftrightarrow x = y = z$

$$t^{\log y} = 4 \Rightarrow \log\left(t^{\log y}\right) = \log(4)$$

$$\Rightarrow (\log y)(\log t) = \log(4) \Rightarrow (\log x)(\log t) = \log(4)$$

$$\overset{xt=4e}{\Rightarrow} t = \frac{4e}{x} \Rightarrow (\log x)\left(\log\frac{4e}{x}\right) = \log(4)$$

$$\Leftrightarrow (\log x)(\log(4e) - \log x) = \log(4)$$

$$\Leftrightarrow (\log x)(1 + \log(4) - \log x) = \log(4)$$

$$\overset{\alpha=\log x}{\Leftrightarrow} \alpha(1 + \log(4) - \alpha) - \log(4) = 0$$

$$\Leftrightarrow -\alpha^2 + \alpha + (\alpha - 1)\log(4) = 0 \Leftrightarrow -\alpha(\alpha - 1) + (\alpha - 1)\log(4) = 0$$

$$\Leftrightarrow (\alpha - 1)(\log(4) - \alpha) = 0 \Leftrightarrow \alpha = 1 \text{ or } \alpha = \log(4)$$

(*) $\alpha = 1 \Rightarrow x = y = z = e \Rightarrow t = e$

(*) $\alpha = \log(4) \Rightarrow x = y = z = 4 \Rightarrow t = e$

3.21 Solve for real numbers:

$$\begin{cases} \boldsymbol{x \geq 0, y, z, t > 0, [*] - great\ integer\ function} \\ \boldsymbol{[x](x - [x]) + y + t = x^2 + 2z} \\ \boldsymbol{\frac{1}{y} + \frac{1}{z} + \frac{1}{t} = \frac{6}{(y+z)^3 + 2} + \frac{6}{(z+t)^3 + 2} + \frac{6}{(t+y)^3 + 2}} \end{cases}$$

Solution:

$$\therefore (y+z)^3 + 2 = (y+z)^3 + 1 + 1 \overset{AM-GM}{\geq} 3\sqrt[3]{(y+z)^3 \cdot 1 \cdot 1} = 3(y+z)$$

Similary:

$$(z+t)^3 + 2 \geq 3(z+t) \text{and } (t+y)^3 + 2 \geq 3(t+z)$$

$$\Omega = \frac{6}{(y+z)^3 + 2} + \frac{6}{(z+t)^3 + 2} + \frac{6}{(t+y)^3 + 2}$$

$$\leq \frac{2}{y+z} + \frac{2}{z+t} + \frac{2}{t+y} \leq \frac{1}{2}\left(\frac{1}{y} + \frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z} + \frac{1}{t}\right) + \frac{1}{2}\left(\frac{1}{t} + \frac{1}{y}\right)$$

$$= \frac{1}{y} + \frac{1}{z} + \frac{1}{t} = \Psi \left(\therefore We\ using: \frac{4}{\alpha + \beta} \leq \frac{1}{\alpha} + \frac{1}{\beta}, \forall \alpha, \beta > 0\right)$$

$$\Omega = \Psi \Leftrightarrow \begin{cases} y = z = t \\ y + z = z + t = t + y = 1 \end{cases} \Leftrightarrow y = z = t = \frac{1}{2}$$

$$[x](x - [x]) + y + t = x^2 + 2z \overset{y=z=t}{\Rightarrow} [x](x - [x]) = x^2 \ldots (1); (x \ge 0 \Rightarrow [x] \ge 0)$$

$$(1) \Leftrightarrow [x]\{x\} = (\{x\} + [x])^2 \Leftrightarrow [x]\{x\} = ([x])^2 + 2[x]\{x\} + (\{x\})^2$$

$$\Leftrightarrow ([x])^2 + [x]\{x\} + (\{x\})^2 = 0$$

But:

$$[x] \ge 0;\ \{x\} \ge 0, equality\ for\ [x] = \{x\} = 0 \Rightarrow x = 0$$

So,

$$(x, y, z, t) = \left(0; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}\right)$$

3.22 Solve for real numbers:

$$\begin{cases} \boldsymbol{x^3 + \log_2 x + \log_4 y = 67} \\ \boldsymbol{x^{\sqrt{xy}} \cdot y^{\frac{1}{\sqrt{xy}}} + x^{\frac{1}{\sqrt{xy}}} \cdot y^{\sqrt{xy}} = \sqrt{x^{x+y}} \cdot y^{\frac{2}{x+y}} + \sqrt{y^{x+y}} \cdot x^{\frac{2}{x+y}}} \end{cases}$$

Solution:

$$x^{\sqrt{xy}} \cdot y^{\frac{1}{\sqrt{xy}}} + x^{\frac{1}{\sqrt{xy}}} \cdot y^{\sqrt{xy}} = x^{\frac{x+y}{2}} \cdot y^{\frac{2}{x+y}} + y^{\frac{x+y}{2}} \cdot x^{\frac{2}{x+y}}$$

$$x^G \cdot y^{\frac{1}{G}} + x^{\frac{1}{G}} \cdot y^G = x^M \cdot y^{\frac{1}{M}} + y^M \cdot x^{\frac{1}{M}}, M = \frac{x+y}{2}, G = \sqrt{xy}, M \ge G$$

$$e^{G\ln(x) + \frac{1}{G}\ln(y)} + e^{\frac{1}{G}\ln(x) + G\ln(y)} = e^{M\ln(x) + \frac{1}{M}\ln(y)} + e^{M\ln(y) + \frac{1}{M}\ln(x)} \quad (*)$$

(*) satisfying when $G = M$

When $G = M = 4$ *and* $x = 4, y = 4$ *then:*

$$(u)^3 + \frac{\ln(4)}{\ln(2)} + \frac{\ln(4)}{\ln(4)} = 64 + 2 + 1 = 67$$

$$S = \{(x, y) = (4,4)\}$$

From (*):

$$e^{G\ln(x) + \frac{1}{G}\ln(y)} - e^{M\ln(x) + \frac{1}{M}\ln(y)} = e^{M\ln(x) + \frac{1}{M}\ln(y)} - e^{G\ln(y) + \frac{1}{G}\ln(x)}$$

$$Suppose: f(t) = e^{t\ln(x)+\frac{1}{t}\ln(y)}, t > 0$$

$$f'(t) = \left(\ln(x) - \frac{1}{t^2}\ln(y)\right) \cdot e^{t\ln(x)+\frac{1}{t}\ln(y)}$$

$$\exists c_1 \in]M, G[\text{ such that:}$$

$$\left.\begin{array}{c} f(G) - f(M) = \left(\ln(x) - \frac{1}{C_1^2}\ln(y)\right)(G - M) \\ \text{in a similarlly way:} \\ \exists c_2 \in]G, M[\text{ such that:} \\ f(M) - f(G) = \left(\ln(x) - \frac{1}{c_2^2}\ln(y)\right)(M - G) \end{array}\right\} \Rightarrow$$

$$\left(\ln(x) - \frac{1}{c_1^2}\ln(y)\right)(G - M) + \left(\ln(x) - \frac{1}{c_2^2}\ln(y)\right)(M - G) = 0$$

$$\left(\ln(x) - \frac{1}{c_1^2}\ln(y)\right)(G - M) - \left(\ln(x) - \frac{1}{c_2^2}\ln(y)\right)(G - M) = 0$$

$$\left[\ln(x) - \frac{1}{c_1^2}\ln(y) - \ln(x) + \frac{1}{c_2^2}\ln(y)\right](G - M) = 0$$

$$\underbrace{\left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right) \cdot \ln(y)}_{\neq 0}(G - M) = 0$$

$$\text{So: } G - M = 0 \Rightarrow G = M.$$

3.23 Solve for real numbers:

$$\begin{cases} \boldsymbol{x + y + z + u = 4} \\ \boldsymbol{x^2(x^2 - v^2) + y^2(y^2 - v^2) + v^4 = z^2(v^2 - z^2) + u^2(v^2 - u^2)} \end{cases}$$

Solution:

$$x^4 + y^4 + z^4 + u^4 + v^4 = (x^2 + y^2 + z^2 + u^2)v^2 \ldots (*)$$

$$\therefore x^4 + \frac{1}{4}v^4 \overset{Am-Gm}{\geq} 2\sqrt{x^4 \cdot \frac{1}{4}v^4} = x^2v^2$$

$$\therefore y^4 + \frac{1}{4}v^4 \overset{Am-Gm}{\geq} y^2v^2$$

$$\therefore z^4 + \frac{1}{4}v^4 \geq z^2v^2$$

$$\therefore u^4 + \frac{1}{4}v^4 \overset{Am-Gm}{\overbrace{\geq}} u^2v^2$$

$$x^4 + y^4 + z^4 + u^4 + \frac{4}{4}v^4 = (x^2 + y^2 + z^2 + u^2)v^2 \Leftrightarrow$$

$$x^4 + y^4 + z^4 + u^4 + v^4 = (x^2 + y^2 + z^2 + u^2)v^2 \Rightarrow (*)\text{true.}$$

$$\text{Equality} \Leftrightarrow x^2 = y^2 = z^2 = u^2 = \frac{1}{2}v^2 = \alpha^2 \ \ (\alpha > 0)$$

$$\Rightarrow |x| = |y| = |z| = |u| = \alpha;\ |v| = \alpha\sqrt{2}$$

$$\bullet\ If\ x;y;z;u \geq 0 \Rightarrow x + y + z + u = 4\alpha = 4 \Rightarrow \alpha = 1 \Rightarrow$$

$$\Rightarrow x = y = z = u = 1; v = \pm\sqrt{2}$$

$$\Rightarrow (x;y;z;u) = (1;1;1;1), v = \pm\sqrt{2}$$

$$\bullet\ If\ x;y;z;u < 0 \Rightarrow x + y + z + u < 0 < 4$$

No solution.

$$\bullet\ If\ \begin{cases} xy < 0 \\ zu < 0 \end{cases} or \begin{cases} xz < 0 \\ yu < 0 \end{cases} or \begin{cases} xu < 0 \\ yz < 0 \end{cases} then\ x + y + z + u = 0 < 4 \Rightarrow$$

No solution.

$$\bullet\ If\ x \geq 0; y = z = u \leq 0\ (and\ cyclic) then\ x + y + z + u = \alpha - \alpha - \alpha -$$

$\alpha = 4 \Rightarrow \alpha = -2 \Rightarrow$*no solution.*

$$\bullet\ If\ x < 0; y = z = u = 2 (and\ cyclic) then\ x + y + z + u = -\alpha + \alpha + \alpha +$$

$$\alpha = 2\alpha = 4 \Rightarrow$$

$$\alpha = 2 \Rightarrow x = -2; y = z = u = 2; v = \pm 2\sqrt{2}$$

$$\Rightarrow (x;y;z;u) = (-2;2;2;2)(and\ cyclic), v = \pm 2\sqrt{2}$$

3.24 Solve for complex numbers:

$$\begin{cases} \dfrac{x^7}{y^{30}} + \dfrac{y^7}{z^{30}} + \dfrac{z^7}{x^{30}} = \dfrac{(x+y+z)^7}{(x^5+y^5+z^5)^6} \\ x^4 - 3y^3 - 2z^2 - 3y + 1 = 0 \end{cases}$$

Solution:

$$\frac{x^7}{y^{30}}+\frac{y^7}{z^{30}}+\frac{z^7}{x^{30}}=\frac{x^7}{(y^5)^6}+\frac{y^7}{(z^5)^6}+\frac{z^7}{(x^5)^6}\overset{Radon}{\geq}\frac{(x+y+z)^7}{(x^5+y^5+z^5)^6}$$

Equality for

$$\frac{x}{y}=\frac{y}{z}=\frac{z}{x}\Leftrightarrow x=y=z$$

$$x\in\mathbb{Z}, x^4-3y^3-2z^2-3y+1=0\Leftrightarrow$$

$$(x^2-4x+1)(x^2+x+1)=0$$

$$x^2-4x+1=0\Rightarrow x_{1,2}=2\pm\sqrt{3}$$

$$x^2+x+1=0\Leftrightarrow x_{3,4}=\frac{-1\pm i\sqrt{3}}{2}$$

3.25 Solve for real numbers:

$$\begin{cases} \boldsymbol{x+y+z=11} \\ \boldsymbol{\frac{yz+36x}{x(y-x)(z-x)}+\frac{zx+36y}{y(x-y)(z-y)}+\frac{xy+36z}{z(x-z)(y-z)}=1} \\ \boldsymbol{xyz=36} \end{cases}$$

Solution:

$$Let \sum x=11 \quad (1);\ \prod x=36 \quad (3)$$

From second questions (+(1),(3)) we get:

$$-\sum\frac{y^2z^2+36^2}{36(x-y)(z-x)}=1$$

$$-36\prod(x-y)=y^3z^2-y^2z^3+36^2y-36^2z+z^3x^2-z^2x^3+36^2z-$$

$$-36^2x+x^3y^2-x^2y^3+36^2x-36^2y$$

$$-36\prod(x-y)=x^2z^2(z-x)+y^3(z+x)(z-x)-y^2(z^3-x^3)$$

$$-36\prod(x-y)=-(z-x)(y-z)(x-y)(xy+yz+zx)$$

$$-36\prod(x-y)=-\prod(x-y)\left(\sum xy\right)$$

If $x=y$ or $y=z$ or $z=x$ no real solution!

System become $\begin{cases} x+y+z=11 \\ xy+yz+zx=36 \\ xyz=36 \end{cases}$; x,y,z *solution of the equation:*

$$t^3-11t^2+36t-36=0$$

$$(t-2)(t-3)(t-6)=0$$

Solution are $(2,3,6)$ *and permutations.*

3.26 Solve for real numbers:

$$\begin{cases} \boldsymbol{xy+yz+zx=26} \\ \boldsymbol{\frac{48+yz(y+z)}{(x-y)(x-z)}+\frac{48+zx(z+x)}{(y-x)(y-z)}+\frac{48+xy(x+y)}{(z-x)(z-y)}=9} \\ \boldsymbol{xyz=24} \end{cases}$$

Solution:

$$x\neq y; y\neq z; z\neq x, x,y,z\neq 0$$

$$xy+yz+zx=26; xyz=24$$

$$\frac{48+yz(y+z)}{(x-y)(x-z)}+\frac{48+zx(z+x)}{(y-x)(y-z)}+\frac{48+xy(x+y)}{(z-x)(z-y)}=9 \Leftrightarrow$$

$$48\left(\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)}\right)+$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9 \Leftrightarrow$$

$$48\left(\frac{-(y-z)-(z-x)-(x-y)}{(x-y)(y-z)(z-x)}\right)+$$

$$+\left(\frac{yz(y+z)}{(x-y)(x-z)}+\frac{zx(z+x)}{(y-x)(y-z)}+\frac{xy(x+y)}{(z-x)(z-y)}\right)=9$$

$$\left(\frac{-yz(y+z)(y-z)-zx(z+x)(z-x)-xy(x+y)(x-y)}{(x-y)(y-z)(z-x)}\right)=9$$

$$yz(y^2-z^2)+zx(z^2-x^2)+xy(x^2-y^2)=-9(x-y)(y-z)(z-x)$$

$$x^3y+y^3z+z^3x-(xy^3+yz^3+zx^3)$$
$$+9(x^2z+z^2y+y^2x-xz^2-zy^2-yx^2)=0$$

$$(x-y)(y-z)(z-x)(x+y+z-9)=0 \overset{x\neq y;y\neq z;z\neq x}{\Longleftrightarrow} x+y+z-9=0$$

$$x+y+z=9$$

So, by Vieta's Theorem:

$$X^3-9X^2+26X-24=0$$

$$(x;y;z)=(2;3;4) \text{ and cyclic.}$$

3.27 Solve for real numbers:

$$\begin{cases} \boldsymbol{x,y\geq 0; [*]-great\ integer\ function} \\ \boldsymbol{(x+2)(y+3)=8} \\ \boldsymbol{\sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{xy}} \end{cases}$$

Solution:

Because:

$$x;y\geq 0 \ then\ [x];[y]\geq 0\ and\ x\geq [x]; y\geq [y]$$

$$\{x\}:=x-[x]; \{y\}:=y-[y]$$

Now,

$$\sqrt{[x]\cdot[y]}+\sqrt{(x-[x])(y-[y])}=\sqrt{[x]\cdot[y]}+\sqrt{\{x\}\cdot\{y\}}$$

$$\overset{BCS}{\leq} \sqrt{\left(\sqrt{[x]}^2+\sqrt{\{x\}}^2\right)\cdot\left(\sqrt{[y]}^2+\sqrt{\{y\}}^2\right)} \leq \sqrt{(\{x\}+[x])(\{y\}+[y])}=\sqrt{xy}$$

Equality for

$$x=y=\alpha>0\ or \begin{cases} x=0 \\ y\geq 0 \end{cases} or \begin{cases} y=0 \\ x\geq 0 \end{cases} or \begin{cases} x=[x] \\ y=[y] \end{cases}$$

$$\textit{If } x=0\ then\ (0+2)(y+3)=0\ and\ y=1\geq 0$$

$$\textit{If }\ y=0\ then\ (x+2)(0+3)=8\ and\ x=\tfrac{2}{3}\geq 0$$

$$(x+2)(y+3)=8 \overset{x=y=\alpha}{\Longrightarrow} (\alpha+2)(\alpha+3)=8$$

$$then\ \alpha^2+5\alpha-2=0 \Rightarrow \begin{cases} x=y=\dfrac{-5+\sqrt{33}}{2} \\ x=y=\dfrac{-5-\sqrt{33}}{2} \end{cases}$$

$$\text{But: } x,y\ge 0\Rightarrow x=y=\frac{-5+\sqrt{33}}{2}$$

$$x=[x]\in\mathbb{Z}^+;y=[y]\in\mathbb{Z}^+$$

$$(x+2)(y+3)=8\Rightarrow x=\frac{8}{y+3}-2\overset{y\in\mathbb{Z}^+}{\Longrightarrow}y+3\in\{4,8\}\overset{x,y\in\mathbb{Z}^+}{\Longrightarrow}y=1;x=0$$

So,

$$(x,y)\in\left\{(0;1);\left(\frac{2}{3};0\right);\left(\frac{-5+\sqrt{33}}{2};\frac{-5+\sqrt{33}}{2}\right)\right\}$$

3.28 Solve for real numbers:

$$\begin{cases}0\le x,y,z\le 2\\ \dfrac{x}{y+z+1}+\dfrac{y}{z+x+1}+\dfrac{z}{x+y+1}+xye^z=\dfrac{6}{5}+4e^2\end{cases}$$

Solution:

$$\text{Let: } f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z$$

$$f_2(x,y,z)=\frac{y}{z+x+1}+\frac{1}{3}xye^z$$

$$f_3(x,y,z)=\frac{z}{x+y+1}+\frac{1}{3}xye^z$$

$f(x,y,z)=f_1(x,y,z)+f_2(x,y,z)+f_3(x,y,z)$ *is convex function.*

$A=\{x,y,z\in\mathbb{R},0\le x,y,z\le 2\}$ *is a closed convex set*

$f(2,2,2)=\frac{6}{5}+4e^2$ *is the greatest value, then* $f(x,y,z)\le\frac{6}{5}+4e^2$ *is sum of there convex functions is convex.Let's prove* $f_1(x,y,z)=\frac{x}{y+z+1}+\frac{1}{3}xye^z$ *is convex.*$\nabla^2(f)$ *is a positive semi definite matrix.*

$$\begin{pmatrix}0 & e^z-\dfrac{1}{(y+z+1)^2} & ye^z-\dfrac{1}{(y+z+1)^2}\\ e^z-\dfrac{1}{(y+z+1)^2} & \dfrac{2x}{(y+z+1)^3} & xe^z+\dfrac{2x}{(y+z+1)^3}\\ ye^z-\dfrac{1}{(y+z+1)^2} & xe^z+\dfrac{2x}{(y+z+1)^3} & xye^z+\dfrac{2x}{(y+z+1)^3}\end{pmatrix}$$

Therefore, $x = y = z = 2$ is only solution for the given equation.

3.29 Solve for real numbers:

$$\begin{cases} \boldsymbol{x + y + [z] = 37} \\ \boldsymbol{x + 2[y] + 3[z] = 47} \end{cases}; [*] \text{ —great integer function.}$$

Solution:

$$\begin{cases} x + y + [z] = 37; \quad (1) \\ x + 2[y] + 3[z] = 47; (2) \end{cases}$$

$$\Rightarrow \begin{cases} x + y = 37 - [z] \in \mathbb{Z} \\ x = 47 - 2[y] - 3[z] \in \mathbb{Z} \end{cases} \Rightarrow \begin{cases} x = [x] \\ y = [y] \end{cases} \Rightarrow x, y \in \mathbb{Z}$$

Let: $[z] = k \in \mathbb{Z} \xRightarrow{(2)-(1)} y = 10 - k, x = k + 27$

$$(x, y, z) \in \{([\alpha] + 27, 10 - 2[\alpha], \alpha);\ \alpha \in \mathbb{R}\}$$

3.30 Solve for real numbers:

$$\begin{cases} \boldsymbol{6x + 3y + 2z = 18} \\ \boldsymbol{108(x + y + z)^{x+y+z} = xy^2z^3 \cdot 6^{x+y+z}} \end{cases}$$

Solution:

From second equation $x + y + z > 0$, since $6x + 3y + 2z > 0$ we conclude:

$$x, y, z > 0$$

$$\left(\frac{x+y+z}{6}\right)^{x+y+z} = x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3 \text{ ;(1)}$$

$$x\left(\frac{y}{2}\right)^2\left(\frac{z}{3}\right)^3 \overset{Am-Gm}{\leq} \left(\frac{x + y + z}{6}\right)^6$$

$$\therefore\ \frac{x+y+z}{6} = 1;\ t^{t-1} \geq 1 \text{ ;(2). From (1),(2) we have:}$$

$$\begin{cases} x = \dfrac{y}{2} = \dfrac{z}{3} \\ 6x + 3y + 2z = 18 \end{cases} \Rightarrow x = 1; y = 2; z = 3$$

COMPLEX NUMBERS

4.1 For $z_1, z_2 \in \mathbb{C}$, satisfy: $|z_1 + z_2| = |z_1| + |z_2|$. Prove:

$$|z_1 - z_2| = max\{|z_1|; |z_2|\} - min\{|z_1|; |z_2|\}$$

Solution:

$|z_1 + z_2| = |z_1| + |z_2| \Rightarrow z_1 = kz_2$ *for some* $k \geq 0$. *Now,*

$$|z_1 - z_2| = |(k-1)z_2| = |(k-1)z_2|$$

If $k \geq 1$, *then* $|z_1| = k|z_2| \geq |z_2|$,

and $|z_1 - z_2| = (k-1)|z_2| = k|z_2| - |z_2| = |z_1| - |z_2|$

$$= \max\{|z_1|, |z_2|\} - \min\{|z_1|, |z_2|\}$$

If $0 \leq k < 1, |z_1| = k|z_2| < |z_2|$ *and*

$$|z_1 - z_2| = |k-1||z_2| = (1-k)|z_2| = |z_2| - k|z_2|$$

$$= |z_2| - |z_1| = \max\{|z_1|, |z_2|\} - \min\{|z_1|, |z_2|\}$$

4.2 Solve for complex numbers:

$$\left|z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right|^2 + \left|z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right|^2 + |z-1|^2 - 3|z|^2 = z$$

Solution:

Using $\omega = \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right), \omega^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$

The given equation reduces to

$$|z - \omega^2|^2 + |z - \omega|^2 + |z-1|^2 - 3|z|^2 = z$$

We know that

$$|z|^2 = z\bar{z} \Rightarrow |z - \omega^2|^2 = (z - \omega^2)\left(\overline{z - \omega^2}\right) = (z - \omega^2)(\bar{z} - \omega)$$

$$= z\bar{z} - z\omega - \bar{z}\omega^2 + \omega^3 = |z|^2 - z\omega - \bar{z}\omega^2 + 1 \quad (1) \ \{\because \omega^3 = 1\}$$

Here, ω *is the cube root of unity*

$$|z-\omega|^2=(z-\omega)(\overline{z-\omega})=(z-\omega)(\bar{z}-\bar{\omega})=(z-\omega)(\bar{z}-\omega^2)=$$
$$=z\bar{z}-z\omega^2-\bar{z}\omega+\omega^3=|z|^2-z\omega^2-\bar{z}\omega+1 \quad (2)$$
$$|z-1|^2=(z-1)(\overline{z-1})=(z-1)(\bar{z}-1)=|z|^2-z-\bar{z}+1 \quad (3)$$

∴ *Adding (1); (2); (3):*

$$|z-\omega^2|^2+|z-\omega|^2+|z-1|^2$$
$$=3|z|^2-z(\omega+\omega^2+1)\cdot\bar{z}(\omega+\omega^2+1)+3$$
$$\{\because \omega+\omega^2+1=0\}\Rightarrow 3|z|^2+3$$
$$\Rightarrow |z-\omega^2|^2+|z-\omega|^2+|z-1|^2=3|z|^2+3$$
$$\Rightarrow |z-\omega^2|^2+|z-\omega|^2+|z-1|^2-3|z|^2=3 \quad (4)$$

But we have:

$$|z-\omega^2|^2+|z-\omega|^2+|z-1|^2-3|z|^2=z \quad (5)$$

So, from (4) and (5) $\Rightarrow z=3.$

4.3 If $z\in\mathbb{C}$, $\alpha\geq 2$ then:

$$|Rez|^\alpha+|Imz|^\alpha\geq 2^{1-\frac{\alpha}{2}}\cdot|z|^\alpha$$

Solution:

$$\sqrt[\alpha]{\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha}\overset{POWER\ MEANS}{\geq}\sqrt{\frac{1}{2}|Rez|^2+\frac{1}{2}|Imz|^2}, (\alpha\geq 2)$$
$$\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha\geq\frac{1}{2^{\frac{\alpha}{2}}}(|Rez|^2+|Imz|^2)^\alpha=\frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha$$
$$\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha\geq\frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha\to|Rez|^\alpha+|Imz|^\alpha\geq 2^{1-\frac{\alpha}{2}}|z|^\alpha$$

4.4 If $z\in\mathbb{C}$, $|z^2-2|=|4z+i|$ then: $|z|<2\sqrt{5}$

Solution:

$$Let\ z=x+iy, z^2=x^2-y^2+2ixy$$
Now, $|z^2-2|=|4z+i|\Rightarrow|(x^2-y^2-2)+2ixy|^2=|4x+(4y+1)i|^2$

$$\Rightarrow (x^2 - y^2 - 2)^2 + 4x^2y^2 = 16x^2 + (4y + 1)^2$$
$$\Rightarrow (x^2 - y^2)^2 + 4 - 4(x^2 - y^2) + 4x^2y = 16(x^2 + y^2) + 8y + 1$$
$$\Rightarrow (x^2 + y^2)^2 - 20(x^2 + y^2) + 3 = -8y^2 + 8y$$
$$\Rightarrow (x^2 + y^2 - 10)^2 = 97 - 8y^2 + 8y$$
$$= 97 - 8\left(\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}\right) = 99 - 8\left(y - \frac{1}{2}\right)^2 < 100$$
$$\Rightarrow |x^2 + y^2 - 10| < 10 \Rightarrow ||z|^2 - 10| < 10$$
$$\Rightarrow |z|^2 - 10 \leq ||z|^2 - 10| < 10 \Rightarrow |z|^2 < 20 \Rightarrow |z| < 2\sqrt{5}$$

4.5 $\boldsymbol{z_1, z_2, z_3 \in \mathbb{C}, different\ in\ pairs,}$

$\boldsymbol{|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3).}$ **If**

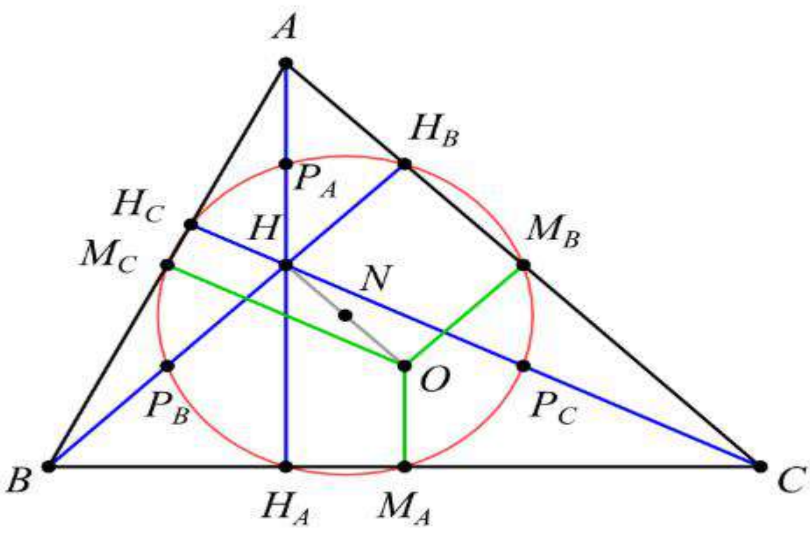
$$\boldsymbol{|z_2 + z_3 - z_1| + |z_3 + z_1 - z_2| + |z_1 + z_2 - z_3| = 6}$$

then $\boldsymbol{\Delta ABC}$ **is an equilateral one.**

Solution:

$$|z_2 + z_3 - z_1| = |z_1 - z_2 - z_3| = |2z_1 - (z_1 + z_2 + z_3)|$$
$$= 2\left|z_1 - \frac{z_1 + z_2 + z_3}{2}\right| = 2\left|z_1 - \frac{z_O + z_H}{2}\right| = 2AN,$$

$O - circumcentre, H - orthocentre,$ N $- ninepoint\ center$



$$|z_3+z_1-z_2|=2BN, |z_1+z_2-z_3|=2CN, \quad 2AN+2BN+2CN=6$$

$$AN+BN+CN=3 \ (1)$$

$$AN-median\ in\ \Delta AHO \to AN^2=\frac{2(AO^2+AH^2)-OH^2}{4}=$$

$$=\frac{2(R^2+4R^2-a^2)-9R^2+a^2+b^2+c^2}{4}=\frac{R^2+b^2+c^2-a^2}{4}=$$

$$=\frac{1+b^2+c^2-a^2}{4}\to\sum_{cyc}AN^2=\frac{3+b^2+c^2+a^2}{4}\overset{LEIBNIZ}{\leq}\frac{3+9}{4}=3$$

$$AN+BN+CN\overset{CBS}{\leq}\sqrt{3\sum_{cyc}AN^2}\leq\sqrt{3\cdot 3}=3 \quad (2)$$

By $(1),(2)\to\Delta ABC-equilateral$

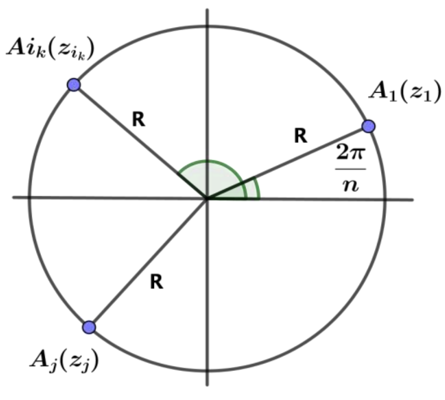
4.6 $A_1, A_2, \dots, A_n$ – regular polygon, $n\in\mathbb{N}, n\geq 3$,

$A_1(z_1), A_2(z_2), \dots, A_n(z_n), z_i\in\mathbb{C}, i\in\overline{1,n}$

If $O(0)$ – centre of polygon and exists $i,j\in\overline{1,n}, i\neq j$ such that

$z_i\cdot\overline{z_j}+\overline{z_i}\cdot z_j=0$ then n is divisible with 4

Solution:



In order to avoid confusion with imaginary number i, we use k instead of i, so that:

$z_j\overline{z_k}+z_k\overline{z_j}=0; j\neq k$ (1). We have $z_k=Re^{\frac{2\pi ik}{n}}, z_j=Re^{\frac{2\pi ij}{n}}$

Now, (1) gives $e^{\frac{2\pi ij}{n}}\cdot e^{-\frac{2\pi ik}{n}}+e^{\frac{2\pi ik}{n}}e^{-\frac{2\pi ij}{n}}=0$

$$\Rightarrow e^{\frac{2\pi i(j-k)}{n}}+e^{\frac{2\pi i(k-j)}{n}}=0\Rightarrow 2\cos\left(\frac{2\pi}{n}(j-k)\right)=0\Rightarrow \frac{2\pi}{n}(j-k)=\frac{\pi}{2},\frac{3\pi}{2}$$

$\Rightarrow 4|j-k|=n,3n$. If $n=4|j-k|$, we are done.

If $3n=4|j-k|$, then $3|4\,|j-k|\Rightarrow 3|\,|j-k|$, [$\because$ 3 is prime]

Let $|j-k|=3m$, where $m_1\in\mathbb{N}\therefore n=4m_1\Rightarrow n$ is a multiple of A.

4.7 $\boldsymbol{p\in\mathbb{R}, p\neq 1, A(z_1+pz_2+p^2z_3), B(z_2+pz_3+p^2z_1),}$
$\boldsymbol{C(z_3+pz_1+p^2z_2), M(z_1), N(z_2), P(z_3), z_1, z_2, z_3\in\mathbb{C}}$**. Prove that:**

$$\boldsymbol{AB=BC=CA\Rightarrow MN=NP=PM}$$

Solution:

Let $\omega_1=z_1+bz_2+p^2z_3, w_2=z_2+pz_3+p^2z_1, \omega_3=z_3+pz_1+p^2z_2$

Let $a=z_2-z_3, b=z_3-z_1, c=z_1-z_2$

ΔABC is equilateral$\Leftrightarrow(\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2-(\omega_1-\omega_2)^2=0$. Now,

$$\omega_2-\omega_3=a+pb+p^2c, \omega_3-\omega_1=b+pc+p^2a,$$

$$\omega_1-\omega_2=c+pa+p^2b$$

$$\Rightarrow(\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2+(\omega_1-\omega_2)^2$$

$$=a^2+p^2b^2+p^4c^2+2pab+2p^2ac+2p^3bc+$$

$$+b^2+p^2c^2+p^4a^2+2pbc+2p^2ab+2p^3ac+$$

$$+c^2+p^2a^2+p^4b^2+2pca+2p^2bc+2p^3ab$$

$$=(a^2+b^2+c^2)(1+p^2+p^4)+(2p+2p^2+2p^3)(ab+bc+ca)\ (1)$$

As $a+b+c=0\Rightarrow a^2+b^2+c^2=-2(bc+ca+ab)$ (2)

$\therefore$ From (1), (2), we get:

$$(\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2+(\omega_1-\omega_2)^2=(a^2+b^2+c^2)(1+p^2+p^4)-$$

$$-p(1+p+p^2)(a^2+b^2+c^2)=$$

$$= (a^2+b^2+c^2)(1+p^2+p^4-p-p^2-p^3)$$
$$= (a^2+b^2+c^2)(1-p+p^4-p^3)$$
$$= (a^2+b^2+c^2)[1-p-p^3(1-p)]$$
$$= (a^2+b^2+c^2)(1-p)(1-p)(1+p+p^2)$$
$$= (a^2+b^2+c^2)(1-p)^2(1+p+p^2)$$

Now, ΔABC *is equilateral*$\Leftrightarrow (\omega_2-\omega_3)^2+(\omega_3-\omega_1)^2+(\omega_1-\omega_2)^2=0$

$$\Rightarrow (a^2+b^2+c^2)(1-p)^2(1+p+p^2)=0$$

As $p\neq 1$ *and* $1+p+p^2\neq 0$, *we get* $a^2+b^2+c^2=0$

$\Rightarrow (z_2-z_3)^2+(z_3-z_1)^2+(z_1-z_2)^2=0 \Leftrightarrow \Delta MNP$ *is equilateral.*

4.8 $\boldsymbol{\Omega_1=|z_1+3i|+|z_2-i|+|z_3-2i|, z_1,z_2,z_3\in\mathbb{C}}$

$$\boldsymbol{\Omega_2=|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|}$$
$$\boldsymbol{+|-z_1+z_2+z_3-6i|}$$

Prove that: $\boldsymbol{\Omega_1\leq\Omega_2}$

Solution:

$$2\Omega_2=(|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|)$$
$$+(|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|)$$
$$+(|z_1+z_2-z_3+4i|+|-z_1+z_2+z_3-6i|)\geq$$
$$\geq |z_1+z_2-z_3+4i+z_1-z_2+z_3+2i|$$
$$+|z_1-z_2+z_3+2i-z_1+z_2+z_3-6i|$$
$$+|z_1+z_2-z_3+4i-z_1+z_2+z_3-6i|=$$
$$=|2z_1+6i|+|2z_3-4i|+|2z_2-2i|=2\Omega_1$$
$$\Omega_1\leq\Omega_2$$

4.9 $\boldsymbol{z_1,z_2,z_3\in\mathbb{C}-\{0\}}$ **different in pairs**

$$\boldsymbol{|z_1|=|z_2|=|z_3|=1, A(z_1), B(z_2), C(z_3)}$$

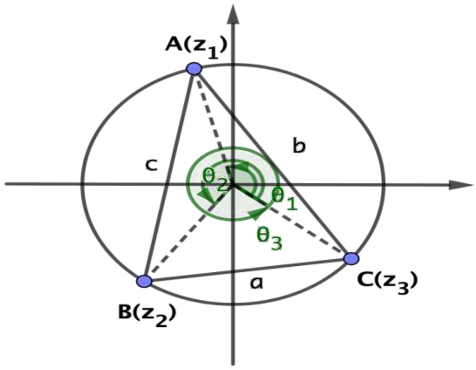
$$\boldsymbol{\frac{z_1(z_2+z_3)^2}{|z_2-z_3|^2}+\frac{z_2(z_3+z_1)^2}{|z_3-z_1|^2}+\frac{z_3(z_1+z_2)^2}{|z_1-z_2|^2}=z_1z_2z_3}$$

Prove that: $AB = BC = CA$.

Solution:

$$\textit{If}: |z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$$

$$\frac{z_1(z_2+z_3)^2}{|z_2-z_3|^2} + \frac{z_2(z_3+z_1)^2}{|z_3-z_1|^2} + \frac{z_3(z_1+z_2)^2}{|z_1-z_2|^2} = z_1z_2z_3$$



$$\frac{1}{a^2}z_1(z_2+z_3)^2 + \frac{1}{b^2}z_2(z_3+z_1)^2 + \frac{1}{c^2}z_3(z_1+z_2)^2 = z_1z_2z_3$$

$$\frac{1}{a^2}\frac{z_1(z_2^2+z_3^2+2z_2z_3)}{z_1z_2z_3} + \frac{1}{b^2}\frac{z_2(z_3^2+z_1^2+2z_3z_1)}{z_1z_2z_3} + \frac{1}{c^2}\frac{z_3(z_1^2+z_2^2+2z_1z_2)}{z_1z_2z_3} = 1$$

$$\frac{1}{a^2}\cdot\left(\frac{z_2}{z_3}+\frac{z_3}{z_2}+2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1}+\frac{z_1}{z_3}+2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2}+\frac{z_2}{z_1}+2\right) = 1$$

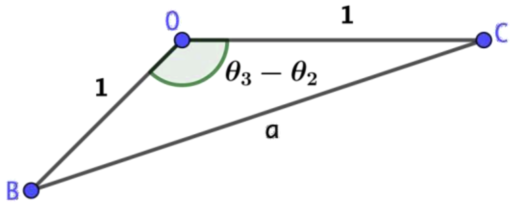
$$\frac{1}{a^2}\left(\frac{z_2}{z_3}+\frac{1}{\left(\frac{z_2}{z_3}\right)}+2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1}+\frac{1}{\left(\frac{z_3}{z_1}\right)}+2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2}+\frac{1}{\left(\frac{z_1}{z_2}\right)}+2\right) = 1$$

$$\left|\frac{z_2}{z_3}\right| = 1 \Rightarrow \frac{z_2}{z_3}\cdot\overline{\left(\frac{z_2}{z_3}\right)} = 1 \Rightarrow \frac{1}{\left(\frac{z_2}{z_3}\right)} = \overline{\left(\frac{z_2}{z_3}\right)}$$

$$\frac{1}{a^2}\left(\frac{z_2}{z_3}+\overline{\left(\frac{z_2}{z_3}\right)}+2\right) + \frac{1}{b^2}\left(\frac{z_3}{z_1}+\overline{\left(\frac{z_3}{z_1}\right)}+2\right) + \frac{1}{c^2}\left(\frac{z_1}{z_2}+\overline{\left(\frac{z_1}{z_2}\right)}+2\right) = 1$$

$$\frac{1}{a^2}\left(2\cos\left(\arg\left(\frac{z_2}{z_3}\right)\right)+2\right)+\frac{1}{b^2}\left(2\cos\left(\arg\left(\frac{z_3}{z_1}\right)\right)+2\right)$$
$$+\frac{1}{c^2}\left(2\cos\left(\arg\left(\frac{z_1}{z_2}\right)\right)+2\right)=1$$

$$\frac{2}{a^2}(\cos(\theta_2-\theta_3)+1)+\frac{2}{b^2}(\cos(\theta_3-\theta_1)+1)+\frac{2}{c^2}(\cos(\theta_1-\theta_2)+1)=1$$
$$\frac{2}{a^2}(\cos(\theta_3-\theta_2)+1)+\frac{2}{b^2}(\cos(\theta_3-\theta_1)+1)+\frac{2}{c^2}(\cos(\theta_2-\theta_1)+1)=1$$
$$\cos(\theta_3-\theta_2)=?$$



$$a^2=1+1-2(1)(1)\cos(\theta_3-\theta_2),\qquad a^2=2-2\cos(\theta_3-\theta_2)$$
$$2\cos(\theta_3-\theta_2)=2-a^2,\qquad \cos(\theta_3-\theta_2)=1-\frac{a^2}{2}$$
$$\frac{2}{a^2}\left(2-\frac{a^2}{2}\right)+\frac{2}{b^2}\left(2-\frac{b^2}{2}\right)+\frac{3}{c^2}\left(2-\frac{c^2}{2}\right)=1$$
$$\frac{4}{a^2}-1+\frac{4}{b^2}-1+\frac{4}{c^2}-1=1,\qquad \frac{4}{a^2}+\frac{4}{b^2}+\frac{4}{c^2}=4\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=1$$
$$\frac{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}{3}\ge\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{3},$$
$$\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2\ge\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$
$$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\le1\Rightarrow\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2\le3$$
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\sqrt{3},\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\ge\sqrt[3]{\frac{1}{abc}}\Rightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge3\sqrt[3]{\frac{1}{abc}}$$

$$\sqrt{3}\cdot 3\sqrt[3]{\frac{1}{abc}}\le\sqrt{3}\Rightarrow\frac{1}{\sqrt[3]{abc}}\le\frac{1}{\sqrt{3}}\Rightarrow\sqrt[3]{abc}\ge\sqrt{3}$$

$$abc\ge 3\sqrt{3}\Rightarrow 2R\sin A\cdot 2\sin B\cdot 2R\sin C\ge 3\sqrt{3}, R=1$$

$$8\sin A\sin B\sin C\ge 3\sqrt{3},\qquad \sin A\cdot\sin B\cdot\sin C\ge\frac{3\sqrt{3}}{8}$$

But as we know: $\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$. *So:* $\sin A\cdot\sin B\cdot\sin C=\frac{3\sqrt{3}}{8}$

Equality holds when $A=B=C=60°$

So: ΔABC *is equilateral.*

4.10 $\boldsymbol{\Omega_1=|z_1+z_2+z_3|, z_1,z_2,z_3\in\mathbb{C}}$

$$\boldsymbol{\Omega_2=|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|}$$
$$\boldsymbol{+|-z_1+z_2+z_3-6i|}$$

Prove that: $\boldsymbol{\Omega_1\le\Omega_2}$

Solution:

$$|z_1+z_2-z_3+4i|+|z_1-z_2+z_3+2i|+|-z_1+z_2+z_3-6i|\ge$$
$$\ge|z_1+z_2-z_3+z_1-z_2+z_3-z_1+z_2+z_3|=|z_1+z_2+z_3|\ \textit{(Q.E.D.)}$$

Let's prove that $|a|+|b|+|c|\ge|a+b+c|;\forall a,b,c\in\mathbb{C}$*, let* $a=x_1+y_1i$,

$b=x_2+y_2i$ *and* $c=x_3+y_3i\Leftrightarrow$

$$\Leftrightarrow\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}\ge$$
$$\sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}\ (?)$$

From Minkowski's inequality we have that

$$\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}\ge\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}|+\sqrt{x_3^2+y_3^2}\Rightarrow$$

$$\Rightarrow\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}+\sqrt{x_3^2+y_3^2}$$
$$\ge\sqrt{(x_1+x_2)^2+(y_2+y_1)^2}+\sqrt{x_3^2+y_3^2}\ge$$
$$\ge\sqrt{(x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2}\quad\textit{(Q.E.D.)}$$

4.11 If $z \in \mathbb{C} - \{0\}$ then:

$$|z-1|^4 + \left|z + \frac{1-i\sqrt{3}}{2}\right|^4 + \left|z + \frac{1+i\sqrt{3}}{2}\right|^4 \geq 3(1+2|z|^2+|z|^4)^2$$

Solution:

$$Let\ a = |z-1|^2 = |z|^2 - 2\operatorname{Re}(\bar{z}) + 1$$

$$b = \left|z + \frac{1+\sqrt{3}i}{2}\right|^2 = |z-w^2|^2 = |z|^2 - 2\operatorname{Re}(\bar{z}\omega^2) + 1$$

$$c = \left|z + \frac{1-\sqrt{3}i}{2}\right|^2 = |z-\omega|^2 = |z|^2 - 2\operatorname{Re}(\bar{z}\omega) + 1$$

$$a+b+c = 3|z|^2 - 2\operatorname{Re}\big(\bar{z}(1+\omega+\omega^2)\big) + 3 = 3|z|^2 + 3 = 3(|z|^2+1)$$

Now, $\frac{a^2+b^2+c^2}{3} \geq \left(\frac{a+b+c}{3}\right)^2 = (|z|^2+1)^2 \vDash |z-1|^4 + |z-\omega|^4 + |z-\omega^2|^4$

$$\geq 3(|z|^2+1)^2 = 3(|z|^4+2|z|^2+1)$$

4.12 $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1$

$A(z_1), B(z_2), C(z_3)$.Prove that:

$$\frac{1}{z_1z_2z_3} + \sum_{cyc} \frac{z_1}{(z_2-z_3)^2} = 0 \Rightarrow AB = BC = CA$$

Solution:

$$\frac{1}{z_1z_2z_3} + \frac{z_1}{(z_2-z_3)^2} + \frac{z_2}{(z_1-z_3)^2} + \frac{z_3}{(z_1-z_2)^2} = 0;\ z_1z_2z_3 \neq 0$$

$$1 + \frac{z_1^2z_2z_3}{(z_2-z_3)^2} + \frac{z_2^2z_1z_3}{(z_1-z_3)^2} + \frac{z_3^2z_1z_2}{(z_1-z_2)^2} = 0$$

$$1 + \frac{z_1^2z_2z_3}{z_2^2-2z_2z_3+z_3^2} + \frac{z_2^2z_1z_3}{z_1^2-2z_1z_3+z_3^2} + \frac{z_3^2z_1z_2}{z_1^2-2z_1z_2+z_2^2} = 0$$

$$1 + \frac{z_1^2}{\frac{z_2}{z_3}+\frac{z_3}{z_2}-2} + \frac{z_2^2}{\frac{z_1}{z_3}+\frac{z_3}{z_1}-2} + \frac{z_3^2}{\frac{z_1}{z_2}+\frac{z_2}{z_1}-2} = 0$$

$$\begin{cases}|z_1| = 1\\ |z_2| = 1\\ |z_3| = 1\end{cases} \Rightarrow \begin{cases}z_1\overline{z_1} = 1\\ z_2\overline{z_2} = 1\\ z_3\overline{z_3} = 1\end{cases} \Rightarrow \frac{z_2}{z_3}\cdot\frac{\overline{z_2}}{\overline{z_3}} = 1 \Rightarrow \frac{z_2}{z_3} = \frac{\overline{z_3}}{\overline{z_2}} = \overline{\left(\frac{z_3}{z_2}\right)}$$

$$\left(\frac{z_2}{z_3}\right)\overline{\left(\frac{z_2}{z_3}\right)} = 1 \Leftrightarrow \left|\frac{z_2}{z_3}\right| = 1$$

$$1 + \frac{z_1^2}{\overline{\left(\frac{z_3}{z_2}\right)} + \frac{z_3}{z_2} - 2} + \frac{z_2^2}{\overline{\left(\frac{z_3}{z_1}\right)} + \frac{z_3}{z_1} - 2} + \frac{z_3^2}{\overline{\left(\frac{z_1}{z_2}\right)} + \frac{z_2}{z_1} - 2} = 0$$

$$\begin{cases}\overline{\left(\frac{z_3}{z_2}\right)} + \frac{z_3}{z_2} - 2 = 2\cos\theta_1\\ \overline{\left(\frac{z_3}{z_1}\right)} + \frac{z_3}{z_1} - 2 = 2\cos\theta_2\\ \overline{\left(\frac{z_1}{z_2}\right)} + \frac{z_2}{z_3} - 2 = 2\cos\theta_3\end{cases} \Rightarrow \begin{cases}\theta_1 = \arg(z_3 - z_2)\\ \theta_2 = \arg(z_3 - z_1)\\ \theta_3 = \arg(z_2 - z_1)\end{cases}$$

$$1 + \frac{z_1^2}{2(\cos\theta_1 - 1)} + \frac{z_2^2}{2(\cos\theta_2 - 1)} + \frac{z_3^2}{2(\cos\theta_3 - 1)} = 0 \dots (i)$$

$$a^2 = 1 + 1 - 2\cdot 1\cdot 1\cdot\cos\theta_1$$

$$-a^2 = 2\cos\theta_1 - 2, \textit{analogs } -b^2 = 2\cos\theta_2 - 2; -c^2 = 2\cos\theta_3 - 2$$

Subbtituted in relation (i), we obtain:

$$1 + \frac{z_1^2}{-a^2} + \frac{z_2^2}{-b^2} + \frac{z_3^2}{-c^2} = 0 \dots (ii)$$

$$\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} + \frac{z_3^2}{c^2} = 1$$

$$1 = \left|\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2} + \frac{z_3^2}{c^2}\right| \le \frac{|z_1^2|}{a^2} + \frac{|z_2^2|}{b^2} + \frac{|z_3^2|}{c^2}$$

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge 1$$

$$a^2b^2 + b^2c^2 + c^2a^2 \ge a^2b^2c^2$$

$$a = 2R\sin A; R = 1;\ a^2 = 4\sin^2 A$$

$$\Rightarrow 16(\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A) \ge 64\sin^2 A\sin^2 B\sin^2 C$$

$$\sin^2 A\sin^2 B + \sin^2 B\sin^2 C + \sin^2 C\sin^2 A \ge 4\sin^2 A\sin^2 B\sin^2 C \dots (iii)$$

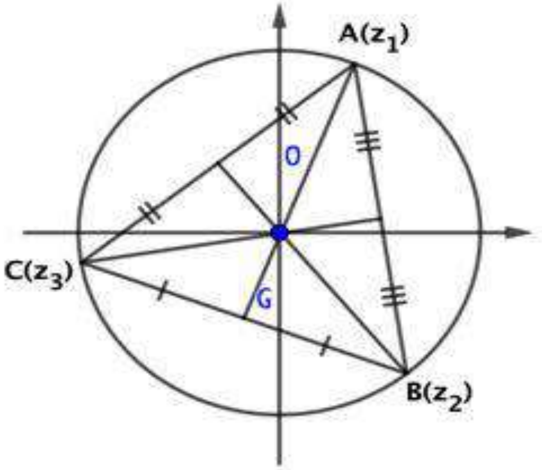
$$A = B = C = \frac{\pi}{3}$$

The triangle ABC is equilateral.

4.13 $\boldsymbol{z_1, z_2, z_3 \in \mathbb{C} - \{0\}}$**, different in pairs,** $\boldsymbol{A(z_1), B(z_2), C(z_3)}$**,**

$$|z_1| = |z_2| = |z_3| = 1. \text{ If } \frac{z_1}{z_2+z_3-z_1} + \frac{z_2}{z_3+z_1-z_2} + \frac{z_3}{z_1+z_2-z_3} + \frac{3}{2} = 0$$

then: $\boldsymbol{AB = BC = CA}$



Solution:

$$\left(\frac{z_1}{z_2+z_3-z_1} + \frac{1}{2}\right) + \left(\frac{z_2}{z_3+z_1-z_2} + \frac{1}{2}\right) + \left(\frac{z_3}{z_1+z_2-z_3} + \frac{1}{2}\right) = 0$$

$$\left(\frac{2z_1+z_2+z_3-z_1}{2(z_2+z_3-z_1)}\right) + \left(\frac{2z_2+z_3+z_1-z_2}{2(z_3+z_1-z_2)}\right) + \left(\frac{2z_3+z_1+z_2-z_3}{2(z_1+z_2-z_3)}\right) = 0$$

$$\frac{z_1+z_2+z_3}{2}\left(\frac{1}{z_2+z_3-z_1} + \frac{1}{z_3+z_1-z_2} + \frac{1}{z_1+z_2-z_3}\right) = 0$$

$$\frac{1}{2}z_G\left(\frac{1}{z_1+z_2+z_3-2z_1} + \frac{1}{z_1+z_2+z_3-2z_2} + \frac{1}{z_1+z_2+z_3-2z_3}\right) = 0$$

$$\frac{1}{2}z_G\left(\frac{1}{3z_G-2z_1} + \frac{1}{3z_G-2z_2} + \frac{1}{3z_G-2z_3}\right) = 0$$

$$\frac{1}{2}z_G\left[\frac{9z_G^2-6z_3z_G-6z_2z_G+4z_2z_3+9z_G^2-6z_Gz_3-6z_1z_G+4z_1z_3+9z_G^2-6z_Gz_2-6z_1z_G+4z_1z_2}{(3z_G-2z_1)(3z_G-2z_2)(3z_G-3z_3)}\right] = 0$$

$$\frac{1}{2}z_G\left[\frac{27z_G^2-12z_3z_6-12z_2z_G-12z_1z_G+4z_1z_3+4z_1z_2+4z_2z_3}{(3z_G-2z_1)(3z_G-2z_2)(3z_G-2z_3)}\right]=0$$

There are two cases: $z_G=0$

or: $27z_G^2-12(z_3+z_2+z_1)\cdot z_G+4z_1z_3+4z_1z_2+4z_2z_3=0$

$$27z_G^2-36z_G^2+4(z_1z_2+z_2z_3+z_1z_3)=0$$

$$9z_G^2=4(z_1z_2+z_2z_3+z_1z_3)\Rightarrow z_G^2=\frac{4}{9}(z_1z_2+z_2z_3+z_1z_3)$$

$$z_G^2=\frac{4}{9}(z_1z_2z_3)\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\Rightarrow z_G^2=\frac{4}{9}(z_1z_2z_3)(\overline{z_1}+\overline{z_2}+\overline{z_3})$$

$$|z_G|^2=\frac{4}{9}|\overline{z_1}+\overline{z_2}+\overline{z_3}|\Rightarrow|z_G|^2=\frac{4}{9}\cdot 3|\overline{z_G}|$$

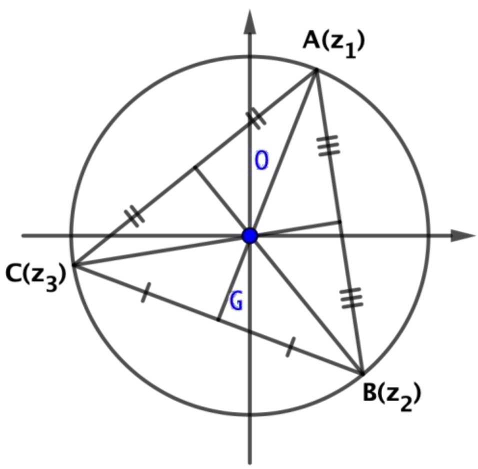
$$|z_G|^2=\frac{4}{3}|z_G|\Rightarrow|z_G|=\frac{4}{3}\Rightarrow\frac{4}{3}=\frac{|z_1+z_2+z_3|}{3}\leq|z_1|+|z_2|+|z_3|$$

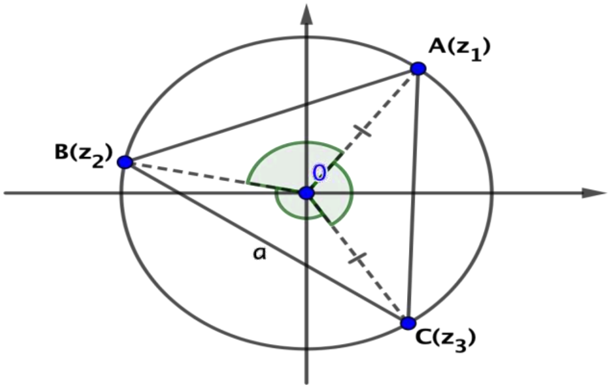
$4\leq 3$. *This is impossible, so:* $|z_G|=0\Rightarrow z_G=0\Rightarrow z_1+z_2+z_3=0$

$z_G=0$, *so:* $G\equiv 0$

Note: center of circle and centroid has the same point so the triangle ABC is equilateral triangle.

$$AB=BC=CA$$





4.14 If $z_1, z_2, z_3 \in \mathbb{C} - \{0\}$, different in pairs, $|z_1| = |z_2| = |z_3| = 1$,

$A(z_1), B(z_2), C(z_3)$,

$\sum_{cyc} z_1(z_2+z_3)^2\,|z_2+z_3|^2 = 3z_1z_2z_3$ then: $AB = BC = CA$.

Solution:

$$|z_1| = |z_2| = |z_3| = 1$$

$$\sum_{cyc} z_1(z_2+z_3)^2|z_2+z_3|^2 = 3z_1\cdot z_2\cdot z_3$$

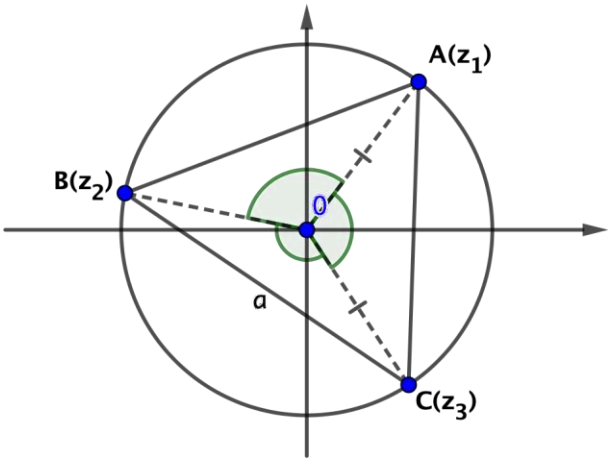
$$z_1(z_2+z_3)^2|z_2+z_3|^2 + z_2(z_1+z_3)^2\cdot|z_1+z_3|^2 + z_3(z_1+z_2)^2|z_1+z_2|^2$$
$$= 3z_1z_2z_3$$

$$|z_2+z_3|^2 = (z_2+z_3)\,\overline{(z_2+z_3)} = (z_2+z_3)(\overline{z_2}+\overline{z_3})$$
$$= z_2\cdot\overline{z_2} + z_2\cdot\overline{z_3} + z_3\cdot\overline{z_2} + z_3\cdot\overline{z_3}$$
$$(|z_1| = 1; z_1\cdot\overline{z_1} = 1)$$
$$|z_2+z_3|^2 = 2 + z_2\cdot\overline{z_3} + z_3\cdot\overline{z_2}$$
$$z_1\cdot(z_2+z_3)^2 = z_1(z_2^2 + 2z_2\cdot z_3 + z_3^2) = z_1\cdot z_2^2 + 2z_1\cdot z_2\cdot z_3 + z_1\cdot z_3^2$$
$$z_1(z_2+z_3)^2\cdot|z_2+z_3|^2 = (z_1\cdot z_2^2 + 2z_1z_2z_3 + z_1z_3^2)(2 + z_2\overline{z_3} + z_3\overline{z_2})$$
$$\frac{z_1(z_2+z_3)^2\cdot|z_2+z_3|^2}{z_1\cdot z_2\cdot z_3} = \left(\frac{z_1z_2^2}{z_1z_2z_3} + 2 + \frac{z_1z_3^2}{z_1z_2z_3}\right)(2 + z_1\overline{z_3} + z_3\overline{z_2})$$
$$= \left(\frac{z_2}{z_3} + 2 + \frac{z_3}{z_2}\right)(2 + z_2\overline{z_3} + z_3\overline{z_2})$$

$$z_1=\frac{1}{z_1};\ z_2=\frac{1}{\overline{z_2}},z_2=\frac{1}{\overline{z_2}};\ z_3=\frac{1}{\overline{z_3}},\overline{z_3}=\frac{1}{z_3};\ z_2\cdot\overline{z_3}=\frac{1}{\overline{z_2}\cdot z_3}$$

$$\frac{2}{1}+\frac{z_2}{z_3}+\frac{z_3}{z_2};\ 2z_3z_2+z_2+z_3;\ 2\cos^2\frac{\theta}{2}=1+\cos\theta$$

$$cos^2\theta=\frac{1+\cos 2\theta}{2}$$



$$\frac{z_1(z_2+z_3)^2|z_2+z_3|^2}{z_1z_2z_3}=(2+z_2\overline{z_3}+z_3\overline{z_2})(2+z_2\overline{z_3}+z_3\overline{z_2})$$

$$=(2+z_2\overline{z_3}+\overline{z_2\overline{z_3}})^2=(2+2\cos(\theta_2-\theta_3))^2=[2(1+\cos(\theta_2-\theta_3))]^2$$

$$=4\left[2\cos^2\left(\frac{\theta_2-\theta_3}{2}\right)\right]^2=16\cos^4\left(\frac{\theta_2-\theta_3}{2}\right)$$

$$16\left[\cos^4\left(\frac{\theta_2-\theta_3}{2}\right)+\cos^4\left(\frac{\theta_1-\theta_2}{2}\right)+\cos^4\left(\frac{\theta_1-\theta_3}{2}\right)\right]=3$$

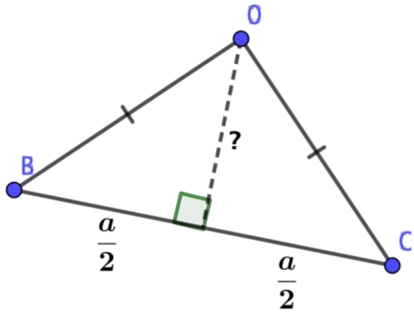
$$16\left[\frac{(4-a^2)^2}{16}+\frac{(4-b^2)^2}{16}+\frac{(4-c^2)^2}{16}\right]=3$$

$$(4-a^2)^2-(4-b^2)^2+(4-c^2)^2=3$$

$$0<a<2,0<a^2<4,0>-a^2>-4,\ 4>4-a^2>0$$

$$0<(4-a^2)^2<16$$

$\Rightarrow$ *This equation holds only when* $a=b=c=\sqrt{3}$



$$0 < a < 2, 0 > -a > -2, 4 > 4 - a > 2, 16 > (4-a)^2 > 4$$

$$1 = \frac{a^2}{4} + h^2, h^2 = 1 - \frac{a^2}{4}, h^2 = \frac{4-a^2}{4}, h = \frac{\sqrt{4-a^2}}{2}$$

$$\cos\left(\frac{\theta_2 - \theta_3}{2}\right) = \frac{\sqrt{4-a^2}}{2}$$

ABSTRACT ALGEBRA

5.1 Find $A=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}, t\in\mathbb{R}$, such that:

$$A^4-4A^3+6A^2-4A+I_2=O_2$$

Solution:

$$A=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$$

$$\det(A)=\cos^2 t+\sin^2 t=1\Rightarrow A^{-1} \textit{ exists and } A^{-1}=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

$$\textit{Let } B=A+A^{-1}=\begin{pmatrix}2\cos t & 0\\ 0 & 2\cos t\end{pmatrix}$$

$$\textit{Given equation is } A^4-4A^3+6A^2-4A+I_2=O_2$$

$$\Rightarrow A^2-4A+6I_2-4A^{-1}+A^{-2}=O_2\Rightarrow A^2+A^{-2}-4(A+A^{-1})+6I_2=O_2$$

$$\Rightarrow (A+A^{-1})^2-4(A+A^{-1})+4I_2=O_2\Rightarrow (A+A^{-1}-2I_2)^2=0$$

$$\Rightarrow (B-2I_2)^2=O_2$$

$$\therefore \begin{pmatrix}2\cos t-2 & 0\\ 0 & 2\cos t-2\end{pmatrix}^2=O_2\Rightarrow 4\begin{pmatrix}(\cot t-1)^2 & 0\\ 0 & (\cos t-1)^2\end{pmatrix}=0$$

$$\Rightarrow (\cot t-1)^2=0\Rightarrow \cot t=1,\sin t=0\ \therefore A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I_2$$

5.2 Find $A\in M_2((0,\infty)), A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ such that:

$$a^2+d^2=4,\frac{1}{b}+\frac{1}{c}=2,(\det A)^3-6(\det A)^2+11\det A=0$$

Solution:

$$(\det A)^3-6(\det A)^2+11\det A=0\Rightarrow (\det A)[(\det A-3)^2+2]=0$$

$$\textit{As, } \det(A) \textit{ is real, we get } \det(A)=0\Rightarrow ad-bc=0 \textit{ or } ad=bc$$

$$\textit{Now, } 2ad\le a^2+d^2=4\Rightarrow ad\le 2\quad \textit{(1)}$$

$$\text{Also, } \sqrt{bc} \geq \frac{1}{\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)} = \frac{1}{\frac{1}{2}} = 2 \Rightarrow bc \geq 4$$

$$\Rightarrow ad = bc \geq 4 \quad (2)$$

As (1) and (2) contradict each other, no such matrix exists.

5.3 Find $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}, a^2 + b^2 \neq 0$ such that:

$$A^4 - 3A^3 + 4A^2 - 3A + I_2 = O_2$$

Solution:

If we use $z = a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R}$ as field isomorphism of $\mathbb{C}$ onto $M = \left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R}\right\}$ then $z \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ will make the equation as:

$$z^4 - 3z^3 + 4z^2 - 3z + 1 = 0 \Rightarrow z^2 + \frac{1}{z^2} - 3\left(z + \frac{1}{z}\right) + 4 = 0$$

$$\Rightarrow \left(z + \frac{1}{z}\right)^2 - 3\left(z + \frac{1}{z}\right) + 2 = 0 \Rightarrow z + \frac{1}{z} = 1 \text{ or } z + \frac{1}{z} = 2$$

$$z = -\omega, -\omega^2 \text{ or } z = 1$$

In this case $|z| = 1$, and corresponding matrices became:

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ as identified earlier.}$$

5.4 If $A \in M_4(\mathbb{R}), \det A \neq 0, (Tr\ A)^2 = 3Tr(A^2)$ then:

$$Tr(A^3) = 3 \cdot \det A \cdot Tr(A^{-1})$$

Solution:

We will note the eigenvalues of A: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. We have:

$$\begin{matrix} Tr\ A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ Tr\ (A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \end{matrix} \Rightarrow (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2$$

$$= 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) \Rightarrow$$

$$\Rightarrow 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4) = 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)$$

$$\Rightarrow \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4$$
$$=\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2| \cdot (\lambda_1+\lambda_2+\lambda_3+\lambda_4)$$
$$\Rightarrow (\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4)$$
$$=\sum \lambda^2 \cdot \sum \lambda$$
$$\Rightarrow \lambda_1\lambda_2(\lambda_1+\lambda_2)+\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_1\lambda_3(\lambda_1+\lambda_3)+\lambda_1\lambda_3(\lambda_2+\lambda_4)$$
$$+\lambda_1\lambda_4(\lambda_1+\lambda_4)+$$
$$+\lambda_1\lambda_4(\lambda_2+\lambda_3)+\lambda_2\lambda_3(\lambda_2+\lambda_3)+\lambda_2\lambda_3(\lambda_1+\lambda_4)+\lambda_2\lambda_4(\lambda_2+\lambda_4)$$
$$+\lambda_2\lambda_4(\lambda_1+\lambda_3)+$$
$$+\lambda_3\lambda_4(\lambda_3+\lambda_4)+\lambda_3\lambda_4(\lambda_1+\lambda_2)$$
$$=\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3+\sum_{1\le i<j\le 4}\lambda_i\lambda_j(\lambda_i+\lambda_j)$$
$$\Rightarrow \lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3=3\lambda_1\lambda_2\lambda_3+3\lambda_1\lambda_2\lambda_4+3\lambda_1\lambda_3\lambda_4+3\lambda_2\lambda_3\lambda_4$$
$$\Rightarrow \begin{aligned} Tr\ A^3 &= \lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3 \\ \det A &= \lambda_1\lambda_2\lambda_3\lambda_4 \\ Tr\ A^{-1} &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}+\frac{1}{\lambda_4}\end{aligned} \Rightarrow (\det A)(Tr\ A^{-1})$$
$$=\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4$$
$$\Rightarrow Tr\ A^3=3(\det A)(Tr\ A^{-1})$$

5.5 Find $x, y, z, w \in \mathbb{R}$ such that:

$$\begin{pmatrix}\sin x & \cos y\\ \tan z & \cot w\end{pmatrix}^n=\begin{pmatrix}\sin^n x & \cos^n y\\ \tan^n z & \cot^n w\end{pmatrix}, \forall n\in\mathbb{N}-\{0\}$$

Solution:

For simplicity, we will note in $x=a, \cos y=b, \tan z=c, \cot w=d$. *Thus, the condition can be written as:* $\begin{pmatrix}a & b\\ c & d\end{pmatrix}^n=\begin{pmatrix}a^n & b^n\\ c^n & d^n\end{pmatrix}, \forall n\in\mathbb{N}\setminus\{0\}$

For $n=2$ *we have:*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a^2 & b^2\\ c^2 & d^2\end{pmatrix}\Leftrightarrow\begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & d^2+bc\end{pmatrix}=\begin{pmatrix}a^2 & b^2\\ c^2 & d^2\end{pmatrix}$$

$$\Rightarrow bc = 0 \Rightarrow b = 0 \text{ or } c = 0.$$

I. $b = 0$. *That means the only equality left is* $(ca + d) = c^2$

If $c = 0$ *then the matrix* $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$ *satisfies the identity in the hypothesis. (for any diagonal matrix* $\begin{pmatrix} 0^n & 0 \\ 0 & d^n \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^n$*)*

If $c \neq 0 \Rightarrow c = a + d$. *For* $n = 3$ *we have:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & b^3 \\ c^3 & d^3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & 0 \\ a+d & d \end{pmatrix}^3 = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a^3 & 0 \\ & d^3 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a^3 & 0 \\ a(a+d)^2 + d^2(a+d) & d^3 \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ (a+d)^3 & d^3 \end{pmatrix}$$

Thus, we have:

$$a(a+d)^2 + d^2(a+d) = (a+d)^3 |:(a+d) \Rightarrow a^2 + ad + d^2$$
$$= a^2 + 2ad + d^2$$
$$\Rightarrow ad = 0 \Rightarrow a = 0 \text{ or } d = 0$$

If $a = b = 0$ *or* $d = b = 0$ *then the matrices* $\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ *satisfy the identity in the hypothesis.*

Thus, in this case the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$ *and* $\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ *satisfy the identity.*

By applying the same algorithm we obtain the solutions:

$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *(duplicate),* $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$

Thus, the matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ *are the only ones which satisfy the identity above,* $a, b, c, d \in \mathbb{R}$. *Thus, the solutions for* x, y, z, w *are:*

I. $x, w \in \mathbb{R}, y = k\pi + \frac{\pi}{2} \wedge z = t\pi, k, z \in \mathbb{Z}, w \neq q\pi, \forall\, q \in \mathbb{Z}$

II. $z, w \in \mathbb{R}, x = k\pi \wedge y = t\pi + \frac{\pi}{2}, k, t \in \mathbb{Z}, w \neq q\pi$ *and* $z \neq p\pi + \frac{\pi}{2}, \forall q, p \in \mathbb{Z}$

$$III.\ x,z\in\mathbb{R}, y=t\pi+\frac{\pi}{2}, w=k\pi+\frac{\pi}{2}, k,t\in\mathbb{Z}, z\neq p\pi+\frac{\pi}{2}, \forall p\in\mathbb{Z}$$

$$IV.\ x,y\in\mathbb{R}, z=t\pi, w=k\pi+\frac{\pi}{2}, t,k\in\mathbb{Z}$$

$$V.\ y,w\in\mathbb{R}, x=k\pi, z=t\pi, t,k\in\mathbb{Z}, w\neq q\pi, \forall q\in\mathbb{Z}$$

5.6 If $A\in M_n(\mathbb{R}), n\geq 2, A^3+A^2+7A=9I_n$ then find:

$$\Omega=\det(A+2I_n)$$

Solution:

$$A^3+A^2+7A=9I_n\Rightarrow A^3+A^2+7A-9I_n=0$$

$$\Rightarrow \underbrace{\textit{the minimal polynom of } A}_{m_A}|x^3+x^2+7x-9$$

$$m_A|(x-1)\left(\underbrace{x^2+2x+9}_{\Delta<0}\right)$$

$$x^3+x^2+7x-9|\ \underline{x-1}$$

$$-x^3+x^2 \qquad\qquad x^2+2x+9$$

$$-----$$

$$/\ \ 2x^2+7x-9$$

$$-2x^2+2x$$

$$------$$

$$/\ 9x-9$$

$$-9x-9$$

$$-----$$

$$==$$

Then (Frobenius)

$P_A(x)=\det(xI_n-A)$ *and* m_A *have the same ireductible divisors.*

So, $P_A(x)=(x-1)^p(x^2+2x+9)^q=(-1)^n\det(A-xI_n)$

with $p+2q=n$

$$P(-2)=(-3)^p(4-4+9)^q=(-1)^n\det(A+2I_n)$$

$$(-1)^n \det(A+2I_n)=(-3)^p\cdot 3^{2q}$$
$$\det(A+2I_n)=(-1)^n\cdot(-3)^p\cdot(-3)^{2q};\ \det(A+2I_n)=(-1)^n(-3)^{2q+p}$$
$$\det(A+2I_n)=(-1)^n(-3)^n;\ \det(A+2I_n)=3^n$$

5.7 $\boldsymbol{X,Y\in M_2(\mathbb{R}), X^{19}+X^{17}=Y^{21}+Y^{19}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}$

Find:

$$\boldsymbol{\Omega=\lim_{n\to\infty}\sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}}$$

Solution:

$$X\in M_2(\mathbb{R});x^{19}+x^{17}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=A$$
$$X^{19}+X^{17}=A\Rightarrow \left.\begin{matrix}X^{20}+X^{18}=AX\\ X^{20}+X^{18}=XA\end{matrix}\right\}\Rightarrow AX=XA$$
$$X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow \left.\begin{matrix}AX=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}a-b & -a+b\\ c-d & -c+d\end{pmatrix}\\ XA=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a-c & b-d\\ -a+c & -b+d\end{pmatrix}\end{matrix}\right\}\Rightarrow$$
$$b=c;a=d\Rightarrow X=\begin{pmatrix}a & b\\ b & a\end{pmatrix}\Rightarrow$$
$$X^n=\begin{pmatrix}\frac{(a+b)^n+(a-b)^n}{2} & \frac{(a+b)^n-(a-b)^n}{2}\\ \frac{(a+b)^n-(a-b)^n}{2} & \frac{(a+b)^n+(a-b)^n}{2}\end{pmatrix}$$
$$\Rightarrow X^{19}+X^{17}=A\Rightarrow\begin{cases}\frac{(a+b)^{19}+(a-b)^{19}}{2}+\frac{(a+b)^{17}+(a-b)^{17}}{2}=1\\ \frac{(a+b)^{19}-(a-b)^{19}}{2}+\frac{(a+b)^{17}-(a-b)^{17}}{2}=-1\end{cases}$$
$$\Rightarrow (a+b)^{19}+(a+b)^{17}=0\Rightarrow a+b=0 \textit{ unique solution } b=-a$$
$$\Rightarrow (a-b)^{19}+(a-b)^{17}=2\Rightarrow a-b=1\Rightarrow a=\frac{1}{2},b=\frac{1}{2}$$

$$X=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}. \textit{ The same for } Y=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$

$$\sqrt[n]{\frac{Tr\,(X^{n+1})}{Tr\,(Y^{n+2})}}=\sqrt[n]{\frac{1}{1}}=1 \text{ constant sequence.}$$

$$\Omega=1$$

5.8 If $A\in M_n(\mathbb{Q})$, $\det A\neq 0$, $A^2+(A^{-1})^2=I_n$ then: $n\,\vdots\,4$

Solution:

$$A^2+(A^{-1})^2=I_n|\cdot A^2\Rightarrow A^4+I_n=A^2$$

$$A^4-A^2+I_n=0_n\Rightarrow m_a|x^4-x^2+1. \text{ But } x^4-x^2+1 \text{ etc}$$

$$\text{Irreducible in } \mathbb{Q}[x]$$

$$(x^4-x^2+1=\left(x^2+\sqrt{3}x+1\right)\left(x^2-\sqrt{3}x+1\right)=$$

$$=\left(x-\frac{-\sqrt{3}+i}{2}\right)\left(x-\frac{-\sqrt{3}-i}{2}\right)\left(x-\frac{\sqrt{3}+i}{2}\right)\left(x-\frac{\sqrt{3}-i}{2}\right))$$

$$\Rightarrow m_a=x^4-x^2+1$$

Then, according to Frobenius

$$\left.\begin{matrix}P_A=(x^4-x^2+1)^k\\ \text{But grade } P_A=n\end{matrix}\right|\Rightarrow 4k=n\Rightarrow n\,\vdots\,4$$

5.9 $A,B\in M_n(\mathbb{R})$,

$$\left(4+\sqrt{10+2\sqrt{5}}\right)(A^2+B^2)=\left(\sqrt{5}-1\right)(AB-BA)$$

If $\det(AB-BA)\neq 0$ then n is divisible with 20.

Solution:

$$\text{We know that } \sin\frac{\pi}{10}=\frac{\sqrt{5}-1}{4}$$

$$\sin^2\frac{\pi}{10}+\cos^2\frac{\pi}{10}=1\Rightarrow\left(\frac{\sqrt{5}-1}{4}\right)^2+\cos^2\frac{\pi}{10}=1\Rightarrow$$

$$\cos^2\frac{\pi}{10}=\frac{16-6+2\sqrt{5}}{16}$$

$$\cos^2\frac{\pi}{10}=\frac{10+2\sqrt{5}}{16}\Rightarrow\cos\frac{\pi}{10}=\frac{\sqrt{10+2\sqrt{5}}}{4}$$

Then $\left(1+\cos\frac{\pi}{10}\right)(A^2+B^2)=\sin\frac{\pi}{10}(AB-BA)$

$$2\cos^2\frac{\pi}{20}(A^2+B^2)=2\sin\frac{\pi}{20}\cos\frac{\pi}{20}(AB-BA)$$

$$\cos\frac{\pi}{20}(A^2+B^2)=\sin\frac{\pi}{20}(AB-BA)$$

$$A^2+B^2=\tan\frac{\pi}{20}(AB-BA)$$

$$A^2+B^2=(A+iB)(A-iB)+iAB-iBA$$

$$=(A+iB)(A-iB)+i(AB-BA)$$

So, $(A+iB)(A-iB)=A^2+B^2+i(BA-AB)$

$$=\tan\frac{\pi}{20}(AB-BA)-i(AB-BA)=\left(\tan\frac{\pi}{20}-i\right)(AB-BA)$$

$$\det[(A+iB)(A-iB)]=\left(\tan\frac{\pi}{20}-1\right)^n\underbrace{\det(AB-BA)}_{\substack{\nparallel\\0}}$$

$$\det(A+iB)\det(A-iB)=\left(\frac{\sin\frac{\pi}{20}-i\cos\frac{\pi}{20}}{\cos\frac{\pi}{20}}\right)^n\det(AB-BA)$$

$$\det(A+iB)\overline{\det(A+iB)}=\left(\frac{-i^2\sin\frac{\pi}{20}-i\cos\frac{\pi}{20}}{\cos\frac{\pi}{20}}\right)^n\det(AB-BA)$$

$$\Rightarrow\det(AB-BA)\cdot\frac{(-i)^n\cdot\left(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20}\right)^n}{\left(\cos\frac{\pi}{20}\right)^n}\in\mathbb{R}$$

$$\Rightarrow\sin\frac{n\pi}{20}=0\Rightarrow\frac{n\pi}{20}=n\pi\Rightarrow n=20k$$

5.10 $\boldsymbol{A,B\in M_3(\mathbb{R}),\det(I_3+(AB-BA)^2)=0}$**. Find:**

$$\boldsymbol{\Omega_1=\det(AB-BA),\Omega_2=Tr\,((AB-BA)^2)}$$

Solution:

$$Let\ be\ A=\left(a_{ij}\right)_{\substack{i=\overline{1,3}\\ j=1,3}};B=\left(b_{ij}\right)_{\substack{i=\overline{1,3}\\ j=1,3}}$$

$$Let\ be\ C=AB\Rightarrow c_{ij}=\sum_{k=1}^{3}a_{ik}b_{kj}\Rightarrow c_{ii}=\sum_{k=1}^{3}a_{ik}\,b_{kj}$$

$$Let\ be\ D=BA\Rightarrow d_{ij}=\sum_{k=1}^{3}b_{ik}\,a_{kj}\Rightarrow d_{ii}=\sum_{k=1}^{3}b_{ik}a_{ki}$$

$$Let\ be\ E=C-D(=AB-BC)\Rightarrow e_{ii}=\sum_{k=1}^{3}(a_{ik}b_{ki}-b_{ik}a_{ki})$$

$$\Rightarrow Tr(AB-BA)=Tr\ E=\sum_{i=1}^{3}\sum_{k=1}^{3}(a_{ik}b_{ki}-b_{ik}a_{ki})=$$

$$=\sum_{i=1}^{3}\sum_{k=1}^{3}a_{ik}b_{ki}-3\sum_{i=1}^{3}\sum_{k=1}^{3}b_{ik}a_{ki}=0$$

$$So,\ Tr\ E=0$$

$$\det(I_3+E^2)=\det(E^2+I_3)=\det(E^2-(iI_3)^2)=$$

$$=\det(E-iI_3)\det(E+iI_3)=0\Rightarrow P_E(x)=(x^2+1)(x-r)\ but$$

$$P_E(x)=x^3-TrEx^2+\frac{(Tr\ E)^2-Tr\ E^2}{2}x-\det E$$

$$P_E(x)=x^3-rx^2+x-r\Rightarrow Tr\ E=r=0$$

$$1=\frac{(Tr\ E)^2-Tr\ E^2}{2}\Rightarrow 2=-Tr\ E^2\Rightarrow Tr\ E^2=-2$$

$$\det E=r=0$$

$$\Omega_1=0\ and\ \Omega_2=-2$$

5.11 $\boldsymbol{A\in M_4(\mathbb{R}),\det A=-1,\det(A^2+I_4)=0}$**. Find:** $\boldsymbol{\Omega=Tr\ (A^*)}$

Solution:

$$\det(A^2+I_4)=\det(A-iI_4)\det(A+iI_4)\ =0$$

$\Rightarrow$ *i or – i root of* $P_a(x)$*, the characteristic polynomial of A. But* $P_a(x)\in R[x]$

Then, i and – i roots of $P_a(x)$

Let be λ_1 *and* λ_2 *the other roots of* $P_A(x)$

$$We\ have\ P_A(x) = x^4 + ax^3 + bx^2 + cx + d$$

$$-a = i - i + \lambda_1 + \lambda_2 = \lambda_1 + \lambda_2$$

$$b = i\lambda_1 + i\lambda_2 - i^2 - i\lambda_1 - i\lambda_2 + \lambda_1\lambda_2 = 1 + \lambda_1\lambda_2$$

$$-c = -i\lambda_1\lambda_2 + i\lambda_i\lambda_2 - i^2\lambda_1 - i^2\lambda_2 = \lambda_1 + \lambda_2$$

$$d = -i^2\lambda_1\lambda_2 = \lambda_1\lambda_2$$

$$P_A(x) = x^4 - (\lambda_1 + \lambda_2)x^3 + (1 + \lambda_1\lambda_2)x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$$

$$\left.\begin{matrix} P_A(0) = \lambda_1\lambda_2 \\ P_A(0) = \det A \end{matrix}\right\} \Rightarrow \lambda_1\lambda_2 = -1 \Rightarrow$$

$$P_A(x) = x^4 - (\lambda_1 + \lambda_2)x^3 - (\lambda_1 + \lambda_2)x - 1$$

$$P_A(x) = x^4 - Tr\,(A)x^3 + Tr(A^x)x^2 + cx + \det A$$

$$So,\ Tr\ A^* = 0 = \Omega$$

5.12 Find $X \in M_3(\mathbb{R})$ such that:

$$X^{2019} + X = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution:

$$X^{2020} + X^2 = X\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}X$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\Rightarrow \begin{cases} a = e = i \\ d = h = g = 0 \\ b = f \end{cases} \Rightarrow X = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}$$

So,

$$X = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = aI_3 + B,$$

$$\text{where } B=\begin{pmatrix} a & b & c\\ 0 & 0 & b\\ 0 & 0 & 0\end{pmatrix}, B^2=\begin{pmatrix} 0 & 0 & b^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=, B^3=O_3$$

$$\Rightarrow X^n=(aI_3+B)^n=\sum_{k=0}^{n}\binom{n}{k}(aI_3)^{n-k}B^k$$

$$=\binom{n}{0}a^nI_3^n+\binom{n}{1}a^{n-1}I_3^{n-1}B+\binom{n}{2}a^{n-2}I_3^{n-2}B^2$$

$$=a^{n-2}\begin{pmatrix} a^2 & nab & nac+\frac{n(n-1)}{2}ab^2\\ 0 & a^2 & nab\\ 0 & 0 & a^2\end{pmatrix}$$

$$X^{2019}+X=a^{2017}\begin{pmatrix} a^2 & 2019ab & 2019ac+\frac{2019\cdot 2018)}{2}ab^2\\ 0 & a^2 & 2019ab\\ 0 & 0 & a^2\end{pmatrix}$$

$$+\begin{pmatrix} a^2 & 2ab & 2ac+ab^2\\ 0 & a^2 & 2ab\\ 0 & 0 & a^2\end{pmatrix}=\begin{pmatrix} 2 & 2 & 0\\ 0 & 2 & 2\\ 0 & 0 & 2\end{pmatrix}$$

$$\Rightarrow\begin{cases} a^{2019}+a^2=2\\ 2019a^{2018}b+2ab=2\\ 2019a^{2018}c+2019\cdot 1009a^{2018}b^2+2ac+ab^2=0\end{cases}$$

We have:

$$a^{2019}+a^2-2=0$$

Let $f(a)=a^{2019}+a^2-2, f^{'}(x)=2019a^{2018}+2a$

$$a_1=0, a_2=-\sqrt[2017]{\frac{2}{2019}}$$

a	- ∞ a_2 0 ∞
f(a)	$-\infty$ $- - - - - - - - -(-2)+$ $+\infty$

$$\mathrm{a}=1\Rightarrow 2019\mathrm{b}+2\mathrm{b}=2\Rightarrow \mathrm{b}=\frac{2}{2021}$$

$$2019c+2019\cdot 1009\cdot\left(\frac{2}{2021}\right)^2+2c+\left(\frac{2}{2021}\right)^2=0$$

$$c=-\frac{4}{2021^3}(1+1009\cdot 2019)$$

So,

$$X=\begin{pmatrix}1 & \frac{2}{2021} & -\frac{4}{2021^3}(1+1009\cdot 2019)\\ 0 & 1 & \frac{2}{2021}\\ 0 & 0 & 1\end{pmatrix}$$

5.13 $A\in M_4(\mathbb{R}), A\cdot A^T=I_4, Tr\ A=0$. Find: $\Omega=Tr(A^{2019})$

Solution:

$$A\in M_4(\mathbb{R}); A\cdot {}^TA=I_4, Tr\ A=0, \Omega=Tr\ (A^{2019})$$

$$\det(A^TA)=1\Rightarrow(\det A)^2=1\Rightarrow\det A=\pm 1\Rightarrow\exists A^{-1}={}^TA$$

$$Tr\ {}^TA=Tr\ A=0\Rightarrow Tr\ A^{-1}=0, A^{-1}=\frac{1}{\det A}\cdot A^*\Rightarrow A^*=\det A\cdot A^{-1}$$

$$Tr\ A^*=\det A\, Tr\ A^{-1}=0, Tr\ A^*=0$$

1) $\det A=1$

$$A^4-OA^3+OA^2-XA+I_4=O_4, A^4-XA+I_4=O_4$$

2) *If* $\det A=-1$

$$A^4-XA-I_4=0_4$$

Let be the spectrum $A=\{A_1,A_2,A_3,A_4\}\not\ni 0$

Then the spectrum $A^{-1}=\left\{\frac{1}{\lambda_1};\frac{1}{\lambda_2};\frac{1}{\lambda_3};\frac{1}{\lambda_4}\right\}$

and as ${}^TA=A^{-1}\Rightarrow$ *the spectrum of* ${}^TA=\left\{\frac{1}{\lambda_1};\frac{1}{\lambda_2};\frac{1}{\lambda_3};\frac{1}{\lambda_4}\right\}$

But $Tr\ {}^TA=0\Rightarrow\frac{1}{\lambda_1};\frac{1}{\lambda_2};\frac{1}{\lambda_3};\frac{1}{\lambda_4}=0$

$$\Rightarrow\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_3=0\Rightarrow x=0$$

So, if 1) $\det A=1\quad A^4=-I_4$

$$\textit{if 2) } \det A = -1 \quad A^4 = I_4$$

$$\textit{1) } A^{2019} = (A^4)^{504} \cdot A^3 = (-I_4)^{504} \cdot A^3 = A^3$$

$$\textit{2) } A^{2019} = (A^4)^{504} A^3 = I_4^{504} \cdot A^3 = A^3$$

$$\textit{So, } Tr\, A^{2019} = Tr\, A^3$$

$$\textit{1) } \quad A^3 = -A^{-1} \Rightarrow A^3 = -{}^{T}A \Rightarrow Tr\, A^3 = -Tr\, {}^{T}A = 0$$

$$\textit{2) } \quad A^3 = A^{-1} \Rightarrow A^3 = {}^{T}A \Rightarrow Tr\, A^3 = Tr\, {}^{T}A = 0$$

3)

5.14 Let be $A \in M_3(\mathbb{R})$.Prove that:

$$\boldsymbol{det(A^2 + I_3) = 0 \Leftrightarrow detA = TrA} \textbf{ and } \boldsymbol{TrA^* = 1, A^*} \textbf{ −adjoint of } \boldsymbol{A}$$

Solution:

$$A \in M_3(\mathbb{R})$$

$$det(A^2 + I_3) = 0 \Leftrightarrow det(A^2 - i^2 I_3) = 0 \Leftrightarrow$$

$$det(A - iI_3)(A + iI_3) = 0 \Leftrightarrow det(A - iI_3) = 0 \textit{ or } det(A + iI_3) = 0$$

$$\Leftrightarrow i \textit{ or } -i \textit{ are roots for } P_A$$

$$\Leftrightarrow P_A(X) = X^3 - TrA \cdot X^2 + TrA^* - detA \cdot I_3$$

$$P_A(X) = (X - r)(X + i)(X - i), r \in \mathbb{R} \Leftrightarrow$$

$$P_A(X) = (X - r)(X^2 + 1) \Leftrightarrow P_A(X) = X^3 - rX^2 + X - r$$

$$\Leftrightarrow TrA = detA \textit{ and } TrA^* = 1$$

5.15 If $A, B \in M_n(\mathbb{R}), n \in \mathbb{N}, n \geq 2, A + B = AB, det(AB) \neq 0$ then:

$$\boldsymbol{det\Big((I_n - A^3 - B^3 + (AB)^3)(I_n - A^5 - B^5 + (AB)^5)(I_n - A^7 - B^7 + (AB)^7)\Big) \geq 0}$$

Solution:

$$A + B = AB \Leftrightarrow A + B - AB = O_4 \Leftrightarrow AB - A - B + I_n = I_n$$

$$\Leftrightarrow A(B - I_n) - (B - I_n) = I_n \Leftrightarrow (A - I_n)(B - I_n) = I_n \textit{ that mean}$$

$$XY = I_n \Leftrightarrow Y = X^{-1} \Rightarrow YX = I_n$$

$$(A - I_n)(B - I_n) = I_n \Rightarrow BA - B - A + I_n = I_n \Rightarrow BA = A + B \Rightarrow AB = BA$$

$$and\ (I_n - A)(I_n - B) = I_n; \quad (1)$$

$$I_n - A^3 - B^3 + (AB)^3 = I_n - A^3 - B^3 + A^3B^3 = I_n - A^3 - B^3(I_n - A^3)$$

$$= (I_n - A^3)(I_n - B^3) = (I_n - A)(I_n - B)(I_n + A + A^2)(I_n + B + B^2)$$

$$\overset{(1)}{\cong} (I_n + A + A^2)(I_n + B + B^2); \quad (2)$$

$$I_n - A^5 - B^5 + (AB)^5 = I_n - A^5 - B^5 + A^5B^5 = I_n - A^5 - B^5(I_n - A^5)$$

$$= (I_n - A^5)(I_n - B^5)$$

$$= (I_n - A)(I_n - B)(I_n + A + A^2 + A^3 + A^4)(I_n + B + B^2 + B^3 + B^4)$$

$$\overset{(1)}{\cong} (I_n + A + A^2 + A^3 + A^4)(I_n + B + B^2 + B^3 + B^4); \quad (3)$$

$$I_n - A^7 - B^7 + (AB)^7 = I_n - A^7 - B^7 + A^7B^7 = I_n - A^7 - B^7(I_n - A^7)$$

$$= (I_n - A^7)(I_n - B^7)$$

$$= (I_n - A)(I_n - B)(I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)(I_n + B + B^2 + B^3 + B^4 + B^5 + +B^6)$$

$$\overset{(1)}{\cong} (I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)(I_n + B + B^2 + B^3 + B^4 + B^5 + B^6); \quad (4)$$

From (2)+(3)+(4) we must show:

$$det(I_n + A + A^2)det(I_n + A + A^2 + A^3 + A^4)det(I_n + A + A^2 + A^3 + A^4 + A^5 + A^6)\cdot$$

$$\cdot det(I_n + B + B^2)det(I_n + B + B^2 + B^3 + B^4)det(I_n + B + B^2 + B^3 + B^4 + B^5 + B^6) \geq 0 \text{ true because}$$

$$det(I_n + X + X^2 + \dots + X^{2n}) \geq 0 \text{ (article-R.M.M.-22)}$$

http://www.ssmrmh.ro/2019/01/24/old-rmm-22/

5.16 $\boldsymbol{A, B \in M_4(\mathbb{C}), B^3 = I_4, A^3 = AB^2 + BA^2,}$

$$\boldsymbol{C = \begin{pmatrix} 28 & 18 & 36 & 723 \\ 120 & 121 & 45 & 891 \\ 330 & 27 & 151 & 210 \\ 450 & 150 & 180 & 181 \end{pmatrix}}$$

$$\textit{Prove that: } \det\left((CA - CB)(A^2 - B^2)\right) \neq 0$$

Solution:

$$\det\left((CA - CB)(A^2 - B^2)\right) = \det(CA^3 - CAB^2 - CBA^2 + CB^3)$$

$$= \det(CA^3 - C(AB^2 + BA^2) + CB^3) = \det(CA^3 - CA^3 + CB^3)$$

$$= \det(O_4 + CI_4) = \det(C)$$

I was working in $mod 10$

$$\hat{C} = \begin{pmatrix} \hat{8} & \hat{8} & \hat{6} & \hat{3} \\ \hat{0} & \hat{1} & \hat{5} & \hat{1} \\ \hat{0} & \hat{7} & \hat{1} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \end{pmatrix}$$

$$\det(\hat{C}) = \hat{8} \cdot \begin{vmatrix} \hat{1} & \hat{5} & \hat{1} \\ \hat{7} & \hat{1} & \hat{0} \\ \hat{0} & \hat{0} & \hat{1} \end{vmatrix} = \hat{8} \cdot \begin{vmatrix} \hat{1} & \hat{5} \\ \hat{7} & \hat{1} \end{vmatrix} = \hat{6}$$

$$U(\det(C)) = 8 \Rightarrow \det(C) \neq 0$$

.

5.17 $\boldsymbol{A(a,b,c), B(d,e,f), C(g,h,i)}$ ***belongs to*** $\boldsymbol{S: x^2 + y^2 + z^2 = R^2}$.

Prove that:

$$\begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{vmatrix}^2 \leq \boldsymbol{R^6}$$

Solution:

Let $\hat{a}$ *be unit vector along* $\overrightarrow{OA}$, $\hat{b}$ *along* $\overrightarrow{OB}$ *and* $\hat{c}$ *along* $\overrightarrow{OC}$, *then*

$$\overrightarrow{OA} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = R\hat{a};\ \overrightarrow{OB} = d\hat{\imath} + e\hat{\jmath} + f\hat{k} = R\hat{b};$$

$$\overrightarrow{OC} = g\hat{\imath} + h\hat{\jmath} + i\hat{k} = R\hat{c}$$

Now, $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 = \left[\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC}\right]^2 = \left[R\hat{a}\;R\hat{b}\;R\hat{c}\right]^2 = R^6\left[\hat{a}\hat{b}\hat{c}\right]^2$*. But*

$\left[\hat{a}\hat{b}\hat{c}\right] = \pm$ *volume of parallelipiped with sides* $\hat{a}, \hat{b}, \hat{c} \Rightarrow \left[\hat{a}\hat{b}\hat{c}\right]^2 \leq 1 \therefore$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 \leq R^6$$

5.18 In $\Delta\, ABC$ the following relationship holds:

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} \leq 4abcR\sqrt{\left(\sum \sin^2 A\right)\left(\sum \cos^2 A\right)}$$

Solution:

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & c^2-b^2 \\ 0 & b^2 & c^2-a^2 & -b^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & -a^2 & c^2-b^2 \\ b^2 & c^2-a^2 & -b^2 \end{vmatrix} =$$

$$a^2b^2 + a^2c^2 - a^4 + b^2c^2 - b^4 + a^2b^2 + a^2b^2 - (c^2-a^2)(c^2-b^2) =$$

$$= 2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \quad (1)$$

From (1) we must show this:

$$2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \leq 4abcR\sqrt{\left(\sum \sin 2A\right)\sum \cos^2 A} \quad (2)$$

From Cauchy inequality ⇒

$$\sqrt{\sum \sin^2 A} \geq \frac{1}{\sqrt{3}}\left(\sum \sin A\right) \text{ and } \sqrt{\sum \cos^2 A} \geq \frac{1}{\sqrt{3}}\left(\sum \cos A\right) \quad (3)$$

From (2)+(3) we must show this:

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \geq \frac{4}{3}abcR\left(\sum \sin A\right)\left(\sum \cos A\right) \; (4)$$

$$\text{But } \sum \sin A = \frac{a+b+C}{2R} \quad (5)$$

$$\sum \cos A = \sum \frac{b^2+c^2-a^2}{2bc} = \frac{\sum a(b^2+c^2-a^2)}{2abc} \quad (6)$$

From (4)+(5)+(6) we must show this:

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \geq$$

$$\geq \frac{1}{3}(a+b+c)(ab^2 + ac^2 - a^3 + ba^2 + bc^2 - b^3 + ca^2 + cb^2 - c^3) \Leftrightarrow$$

$$6(a^2b^2 + a^2c^2 + b^2c^2) - 3(a^4 + b^4 + c^4) \geq -a^4 - b^4 - c^4 + a^3(b+c) +$$

$$+b^3(a+c) + c^3(a+b) - a(b^3 + c^3) - b(a^3 + c^3) - c(a^3 + b^3) +$$

$$+a^2(b^2 + c^2) + b^2(a^2 + c^2) + c^2(a^2 + b^2) + abc(b+c) + abc(a+c)$$

$$+ abc(a+b)$$

$$\Leftrightarrow 2(a^4 + b^4 + c^4) - 4(a^2b^2 + b^2c^2 + a^2c^2)$$

$$+ 2abc(a+b+c) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + a^2c^2) + abc(a+b+c) \geq 0 \quad (7)$$

$$\Leftrightarrow a^4 + b^4 + c^4 + abc(a+b+c) \geq 2(a^2b^2 + b^2c^2 + a^2c^2) \quad (8)$$

By Schur's inequality we have:

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) +$$

$$ca(c^2 + a^2) \quad (9)$$

From (8)+(9) we must show:

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 2(a^2b^2 + b^2c^2 + a^2c^2) \quad (10)$$

But $ab(a^2 + b^2) \geq 2a^2b^2 \Leftrightarrow a^2 + b^2 \geq 2ab$ *which its true. Similarly:*

$bc(b^2 + c^2) \geq 2bc$ *and* $ac(a^2 + c^2) \geq 2a^2c^2 \Rightarrow$ *(10) its true.*

5.19 In $\Delta\ ABC$ the following relationship holds:

$$\begin{vmatrix} a & 0 & c & b \\ 0 & a & b & c \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} \geq 432r^4$$

Solution:

$$LHS = a \times \begin{vmatrix} a & b & c \\ c & 0 & a \\ b & a & 0 \end{vmatrix} + c \times \begin{vmatrix} 0 & a & c \\ b & c & a \\ c & b & 0 \end{vmatrix} - b \begin{vmatrix} 0 & a & b \\ b & c & 0 \\ c & b & a \end{vmatrix} =$$

$$= a\{a(-a^2) - b(-ab) + c \cdot ca\} + c\{-a(-ac) + c(b^2 - c^2)\} -$$

$$-b\{-a(ab) + b(b^2 - c^2)\} = a(-a^3 + ab^2 + ac^2) + c(a^2c + b^2c - c^3) +$$

$$+b(-a^2b + b^3 - bc^2)$$

$$= a^2(b^2 + c^2 - a^2) + c^2(a^2 + b^2 - c^2) + b^2(c^2 + a^2 - b^2)$$

$$= 2a^2bc\cos A + 2c^2abc\cos C + 2b^2ca\cos B = 2abc\left(\sum a\cos A\right) =$$

$$= 2Rabc(\sin 2A + \sin 2B + \sin 2C) = 2Rabc \cdot 4\sin A\sin B\sin C$$

$$= 2R \cdot 4Rrs\left(4\frac{abc}{8R^3}\right) = 16\frac{R^2rs \cdot Rrs}{R^3}$$

$$= 16r^2s^2 \overset{s\geq 3\sqrt{3}r}{\geq} 16 \cdot 27r^4 = 432r^4$$

5.20 If $\boldsymbol{a, b, c, d, e, f} > 0$ then:

$$64\begin{vmatrix} \mathbf{1} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} & \boldsymbol{b} \\ \boldsymbol{a} & \boldsymbol{c} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{a} & \mathbf{0} & \boldsymbol{d} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{a} & \mathbf{0} & \mathbf{0} & \boldsymbol{e} & \mathbf{0} \\ \boldsymbol{a} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{f} \end{vmatrix} \leq (\boldsymbol{a}+\boldsymbol{f})^2(\boldsymbol{b}+\boldsymbol{e})^2(\boldsymbol{c}+\boldsymbol{d})^2\left(\frac{\mathbf{1}}{\boldsymbol{ab}} - \frac{\mathbf{1}}{\boldsymbol{c}} - \frac{\mathbf{1}}{\boldsymbol{d}} - \frac{\mathbf{1}}{\boldsymbol{e}} - \frac{\mathbf{1}}{\boldsymbol{f}}\right)$$

Solution:

$$Let\ P = 64\begin{vmatrix} 1 & b & b & b & b \\ a & c & 0 & 0 & 0 \\ a & 0 & d & 0 & 0 \\ a & 0 & 0 & e & 0 \\ a & 0 & 0 & 0 & f \end{vmatrix}.\ Expanding\ this\ determinant,\ we\ get$$

$$P = 64(cdef - abdef - abcef - abcdf - abcde).$$

$$P = 64abcdef\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{f}\right)$$

$P=(4af)(4be)(4cd)\left(\frac{1}{ab}-\frac{1}{c}-\frac{1}{d}-\frac{1}{e}-\frac{1}{b}\right)$. By AM-GM: $\sqrt{af}\le\frac{a+f}{2}\Rightarrow$

$$4af\le(a+f)^2$$

$$\sqrt{bc}\le\frac{b+e}{2}\Rightarrow 4be\le(b+e)^2;\sqrt{cd}\le\frac{c+d}{2}\Rightarrow 4cd\le(c+d)^2\Rightarrow$$

$$\Rightarrow P\le(a+f)^2(b+e)^2(c+d)^2\left(\frac{1}{ab}-\frac{1}{c}-\frac{1}{d}-\frac{1}{e}-\frac{1}{f}\right)$$

5.21 If $A\in M_2(\mathbb{R})$ then:

$$\boldsymbol{det(A^2+2A+2I_2)\ge(2+Tr\,A)^2}$$

Solution:

$$\text{Let } A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, a,b,c,d\in\mathbb{R}$$

$$A^2+2A+2I_2=(A+I_2)^2+I_2=(A+I_2+iI_2)(A+I_2-iI_2)$$

$$=(A+I_2+iI_2)\overline{(A+I_2+iI_2)}$$

$$\det(A^2+2A+2I_2)=\det(A+(1+i)I_2)\left(\overline{A+(1+i)I_2}\right)$$

$$=\det(A+(1+i)I_2)\,\overline{\det(A+(1+i)I_2)}$$

$$=|\det(A+(1+i)I_2)|^2=$$

$$=\left|\begin{vmatrix}a+(1+i) & b\\ c & d+(1+i)\end{vmatrix}\right|^2=|(1+i)^2+(a+d)(1+i)+ad-bc|^2$$

$$=|(a+d+ad-bc)+(2+a+d)i|^2\ge\big(2+(a+d)\big)^2=(2+tr\,A)^2$$

5.22 $\boldsymbol{A,B\in M_2(\mathbb{R}),\det A\ne0,\det B\ne0,}$

$\boldsymbol{Tr(AB^{-1})=\det(AB^{-1})=1}$.Find: $\boldsymbol{\Omega=\det(I_2+A^{-1}B)}$

Solution:

$$\text{Let } X=AB^{-1}.\text{ As } tr(X)=1,\text{ we take } X=\begin{pmatrix}a & b\\ c & 1-a\end{pmatrix}$$

$$1=\det(X)=a(1-a)-bc$$

$$\det(I+AB^{-1})=\det(I+X)=\begin{vmatrix}a+1 & b\\ c & 2-a\end{vmatrix}=(a+1)(2-a)-bc$$

$$= 2 + a - a^2 - bc = 3$$

$$Now \det(I + A^{-1}B) = \det\{A^{-1}(AB^{-1} + I)B\} =$$

$$= \det(A^{-1}\det(AB^{-1} + I))\det(B) = (\det(A))^{-1}(\det B)\det(X)$$

$$= [\det(A)\,(\det(B))^{-1}]^{-1}(3) = (\det(AB^{-1}))^{-1}(3) = (1)(3) = 3$$

5.23 If $A, B \in M_5(\mathbb{R}), A^3 + 7I_5 = A^2, B^3 + 9I_5 = B^2$ then:

$$\det(AB) > 0$$

Solution:

$$A^2 - A^3 = 7I_5 \Rightarrow \det[A^2(I_5 - A)] = \det(7I_5) \Rightarrow (\det A)^2 \cdot \det(I_5 - A) =$$

$$7^5 \neq 0 \Rightarrow \det A \neq 0 \quad (1)$$

$$B^2 - B^3 = 9I_5 \Rightarrow \det[B^2(I_5 - B)] = \det(9I_5) \Rightarrow (\det B)^2 \cdot \det(I_5 - B) =$$

$$9^5 \neq 0 \Rightarrow \det B \neq 0 \quad (2)$$

$$Now,\ A^3 = A^2 - 7I_5 \Rightarrow A^4 = A^3 - 7A \Rightarrow$$

$$A^4 = A^2 - 7I_5 - 7A \Rightarrow A^4 = A^2 - 7A - 7I_5 \Rightarrow$$

$$\Rightarrow A^5 = A^3 - 7A^2 - 7A = A^2 - 7I_5 - 7A^2 - 7A \Rightarrow$$

$$\Rightarrow A^5 = -6A^2 - 7A - 7I_5 \Rightarrow A^5 = -6\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \Rightarrow$$

$$\det A^5 = \det\left[-6\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right)\right]$$

$$\Rightarrow \left.\begin{matrix} (\det A)^5 = (-6)^5 \cdot \det\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \\ But \det\left(A^2 + \frac{7}{6}A + \frac{7}{6}I_5\right) \geq 0 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow (\det A)^5 \leq 0 \Rightarrow \det A \leq 0 \quad (3).\ Now,\ B^3 = B^2 - 9I_5 \Rightarrow B^4 = B^3 - 9B \Rightarrow$$

$$(B^5 = B^4 - 9B^2) \quad B^4 = B^2 - 9B - 9I_5 \Rightarrow B^5 = B^3 - 9B^2 - 9B$$

$$= B^2 - 9I_5 - 9B^2 - 9B$$

$$\Rightarrow B^5 = -8B^2 - 9B - 9I_5 \Rightarrow B^5 = -8\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \Rightarrow\Rightarrow \det(B^5)$$

$$= \det\left[-8\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right)\right] \Rightarrow$$

$$\left.\begin{array}{l}(\det B)^5 = (-8)^5 \det\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \\ \det\left(B^2 + \frac{9}{8}B + \frac{9}{8}I_5\right) \geq 0\end{array}\right\} \Rightarrow (\det B)^5 \leq 0 \Rightarrow (\det B) \leq 0 \ \textit{(4)}$$

From (1)+(2)+(3)+(4) $\Rightarrow \det A < 0$ *and* $\det B < 0 \Rightarrow \det(AB) > 0.$

Observation: $A \in M_n(\mathbb{R}), p \in (0,4) \Rightarrow \det(A^2 + pA + pI_n) \geq 0$

because $\Leftrightarrow \det\left(A^2 + pA + \frac{p^2}{4}I_n - \frac{p^2}{4}I_n + pI_n\right) =$

$$= \det\left[\left(A + \frac{p}{2}I_n\right)^2 + \frac{-p^2 + 4p}{4}I_n\right] = \det\left[\left(A + \frac{p}{2}I_n\right)^2 + \left(\frac{\sqrt{4p - p^2}}{2}\right)^2 I_n^2\right]$$

$$= \det\left[\left(A + \frac{p}{2}I_n\right)^2 - i^2\left(\frac{\sqrt{4p - p^2}}{2}\right)^2 I_n^2\right] =$$

$$= \det\left[\left(A + \frac{p}{2}I_n + i\frac{\sqrt{4p - p^2}}{2}I_n\right)\overline{\left(A + \frac{p}{2} + i\frac{\sqrt{4p - p^2}}{2}I_n\right)}\right] \geq 0$$

5.24 If $A, B \in M_5(\mathbb{R}), A^3 - 2I_5 = A^2, B^3 - 3I_5 = B^2$ then:

$$\det(AB) > 0$$

Solution:

$$A^3 \cdot A^2 = 2I_5 \Rightarrow A^2(A - I_5) = 2I_5 \Rightarrow$$

$$\Rightarrow (\det A)^2 \cdot \det(A - I_5) = 2^5 \Rightarrow \det A \neq 0 \ \textit{(1)}$$

$$A^3 = A^2 + 2I_5 \Rightarrow \det A^3 = \det(A^2 + 2I_5) \Rightarrow$$

$$(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right)\left(A - \sqrt{2}iI_5\right) \Rightarrow$$

$$(\det A)^3 = \det\left(A + \sqrt{2}iI_5\right) \cdot \overline{\det\left(A + \sqrt{2}iI_5\right)} \geq 0 \ \textit{(2)}$$

From (1)+(2)$\Rightarrow \det A > 0$ *(3)*

$$B^3 - B^2 = 3I_5 \Rightarrow B^2(B - I_5) = 3I_5 \Rightarrow$$

$$\Rightarrow (\det B)^2 \cdot \det(B - I_5) = 3^5 \Rightarrow \det B \neq 0 \quad (4)$$

$$B^3 = B^2 + 3I_5 \Rightarrow \det B^3 = \det(B^2 + 3I_5) \Rightarrow$$

$$\Rightarrow (\det B)^3 = \det\left(B + \sqrt{3}iI_5\right)\left(B - \sqrt{3}iI_5\right) \Rightarrow$$

$$\Rightarrow (\det B)^3 = \det\left(B + \sqrt{3}iI_5\right) \cdot \overline{\det\left(B + \sqrt{3}iI_5\right)} \geq 0 \quad (5)$$

From (4)+(5)$\Rightarrow \det(B) > 0$ *(6)*

From (3)+(6) $\Rightarrow \det(AB) > 0$

5.25 Find $A, B \in M_2(\mathbb{R})$ such that:

$$\det A < 0, \det(A - B) > 0, \det(A + B) < 0, \det(2A + B) > 0$$

Solution:

Suppose that A and B satisfy the proposed conditions. Let $C = A^{-1}B$ and let

$$\chi(\lambda) = \det(\lambda I_2 - C) = \lambda^2 - tr\,(A)\lambda + \det(C)$$

be the characteristic polynomial of C. The proposed inequalities yields

$$\chi(1) = \frac{\det(A - B)}{\det A} < 0$$

$$\chi(-1) = \frac{\det(-A - B)}{\det A} = \frac{\det(A + B)}{\det A} > 0$$

$$\chi(-2) = \frac{\det(-2A - B)}{\det A} = \frac{\det(2A + B)}{\det A} < 0$$

But $\chi(\lambda)$ is positive for large $|\lambda|$, so the above conditions imply the second degree polynomial χ has at least 4 zeros and this is absurd. Thus, no such matrices exist.

5.26 If $A \in M_4(\mathbb{C}), det\ A \neq 0, Tr\ A = 0$ then:

$$Tr\,(A^3) = 3(det\,A)(Tr\,A^{-1})$$

Solution:

Let $A = \left(a_{ij}\right)_{4\times4} \in M_4(\mathbb{C})$ and $Tr\,(A) = 0, \det(A) \neq 0.$

Let $f(t) = t^4 - \alpha t^3 + \beta t^2 - \gamma t + \delta$ be the characteristic polynomial of A.

Then $\alpha = Tr\,(A) = 0$ *and* $\delta = \det(A) \neq 0.$

$$\therefore f(t) = t^4 + \beta t^2 - \gamma t + \delta$$

We have

$$A^4 = -\beta A^2 + \gamma A - \delta I_4 \quad (1)$$

$$\Rightarrow A^3 = -\beta A - \gamma I - \delta A^{-1}$$

$$Tr\,(A^3) = -\beta Tr(A) + 4\gamma - \delta Tr\,(A^{-1}) = 4\gamma - \delta Tr(A^{-1}) \quad (1)$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *be eigen values of* A*, then*

$$\sum \lambda_i = 0, \sum \lambda_i \lambda_j = B$$

Let λ *be an eigenvalue of* $A \Rightarrow \exists a\ x \neq 0$ *such that* $Ax = \lambda x \Rightarrow$

$$\Rightarrow A^2(x) = A(Ax) = A(\lambda x) = \lambda Ax = \lambda(\lambda x) = \lambda^2 x$$

Similarly, $A^3 = \lambda^3 x \Rightarrow \lambda^3$ *is an eigenvalue of* A^3*. If* A^{-1} *exists, then*

$$A^{-1}(Ax) = A^{-1}(\lambda x) \Rightarrow \lambda^{-1}x = A^{-1}x$$

$\therefore \lambda^{-1}$ *is an eigenvalue of* A^{-1}.

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *eigenvalues of* A*, then* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = Tr\,(A) = 0.$

Now, $Tr(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 = (\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1 + \lambda_2) +$

$$(\lambda_3 + \lambda_4)^3 - 3\lambda_3\lambda_4(\lambda_3 + \lambda_4)$$

$$= (-\lambda_3 - \lambda_4)^3 + 3\lambda_1\lambda_2(\lambda_3 + \lambda_4) + (\lambda_3 + \lambda_4)^3 + 3\lambda_3\lambda_4(\lambda_1 + \lambda_2)$$

$$[\because \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0]$$

$$= 3\lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right) = 3\det(A)\,Tr\,(A^{-1})$$

$$\left[\because \lambda_1\lambda_2\lambda_3\lambda_4 = \det(A) \text{ and } Tr(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right]$$

$$\sum \lambda_i\lambda_j\lambda_k = \gamma;\ \lambda_1\lambda_2\lambda_3\lambda_4 = \delta$$

Note

$$\gamma = \delta\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right) = \det(A)\,Tr\,(A^{-1}) \quad (2)$$

From (1), (2): $Tr(A^3) = 3\det(A)\,Tr\,(A^{-1})$

5.27 If $A, B \in M_2(\mathbb{C}), det(A+B)=1$ then:

$$\boldsymbol{det(A\cdot det\,B + B\cdot det\,A) = det(AB)}$$

Solution:

$$\textit{If } \det(A)=0 \textit{ or } \det(B)=0,$$

$$\textit{then } \det(\det(A)B+\det(B)A)=0=\det(A)\det(B).$$

$$\textit{Suppose } \det(A)\neq 0, \det(B)\neq 0. \textit{ Let } A=\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}, B=\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}$$

$$\textit{Let } \alpha=\det A, \beta=\det(B). \textit{ Now, } I=\det(A+B)=\det[A(B^{-1}+A^{-1})B]$$

$$=\det(A)\det(B)\det(B^{-1}+A^{-1}) \quad (1)$$

$$\textit{But } A^{-1}=\frac{1}{\alpha}\begin{pmatrix} d_1 & -b_1\\ -c_1 & a_1\end{pmatrix}, B^{-1}=\frac{1}{\beta}\begin{pmatrix} d_2 & -b_2\\ -c_2 & a_2\end{pmatrix}$$

$$\therefore B^{-1}+A^{-1}=\begin{bmatrix} \frac{d_1}{\alpha}+\frac{d_2}{\beta} & -\left(\frac{b_1}{\alpha}+\frac{b_2}{\beta}\right)\\ -\left(\frac{c_1}{\alpha}+\frac{c_2}{\beta}\right) & \frac{a_1}{\alpha}+\frac{a_2}{\beta}\end{bmatrix}$$

$$\textit{Now, note } \det(B^{-1}+A^{-1})=\det\begin{bmatrix} \frac{a_1}{\alpha}+\frac{a_2}{\beta} & \frac{b_1}{\alpha}+\frac{b_2}{\beta}\\ \frac{c_1}{\alpha}+\frac{c_2}{\beta} & \frac{d_1}{\alpha}+\frac{d_2}{\beta}\end{bmatrix}$$

$$\therefore \det(B^{-1}+A^{-1})=\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right) \quad (2)$$

$$\textit{Thus, from (1), (2): } 1=\alpha\beta\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right)=\frac{1}{\alpha\beta}\det\left[\frac{\alpha\beta}{\alpha}A+\frac{\alpha\beta}{\beta}B\right]$$

$$[\because A, B\ are\ 2\times\ 2\ matrices]\Rightarrow \det(\beta A+\alpha\beta)=\alpha\beta$$

$$\textit{or } \det[(\det B)A+(\det A)B]=\det A\det B=\det(AB)$$

5.28 If $A\in M_2(\mathbb{R}), B\in M_3(\mathbb{R}), C\in M_4(\mathbb{R})$,

$A^2-A=I_2, B^2-B=I_3, C^2-C=I_4$ then:

$$|\boldsymbol{det\,A+det\,B+det\,C}|<28$$

Solution:

Let $f(x)=x^2-x-1, f(x)=0\Rightarrow x_{1,2}=\frac{1\pm\sqrt{5}}{2}$. *Now the own values for* A *is* $\lambda_1,\lambda_2\Rightarrow$ *from McCoy theorem* $\Rightarrow\lambda_1,\lambda_2\in\left\{\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right\}\Rightarrow|\lambda_i|\le\frac{1+\sqrt{5}}{2}, i=1,2\Rightarrow$

$$|\det A|=|\lambda_1\lambda_2|=|\lambda_1|\cdot|\lambda_2|\le\left(\frac{1+\sqrt{5}}{2}\right)^2 \quad (1)$$

Let $\lambda_1,\lambda_2,\lambda_3$ *the own values for* $B\Rightarrow$ *from McCoy theorem* $\Rightarrow\{\lambda_1,\lambda_2,\lambda_3\}\in\left\{\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right\}$

$$\Rightarrow|\lambda_i|\le\frac{1+\sqrt{5}}{2}, i=1,2,3\Rightarrow$$

$$|\det B|=|\lambda_1||\lambda_2||\lambda_3|\le\left(\frac{1+\sqrt{5}}{2}\right)^3 \quad (2)$$

Let $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ *the own values for* $C\Rightarrow$

$$\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}\in\left\{\frac{1-\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right\}\Rightarrow|\lambda_i|\le\frac{1+\sqrt{5}}{2},$$

$$i=1,2,3,4\Rightarrow|\det C|=|\lambda_1|\cdot|\lambda_2|\cdot|\lambda_3|\cdot|\lambda_4|\le\left(\frac{1+\sqrt{5}}{2}\right)^4 \quad (3)$$

From (1)+(2)+(3)$\Rightarrow|\det A+\det B+\det C|\le|\det A|+|\det B|+|\det C|\le$

$$\le\left(\frac{1+\sqrt{5}}{2}\right)^2+\left(\frac{1+\sqrt{5}}{2}\right)^3+\left(\frac{1+\sqrt{5}}{2}\right)^4=7+3\sqrt{5}<28$$

5.29 If $A,B,C,D\in M_n(\mathbb{C}), n\in\mathbb{N}, n\ge 2, det(ABCD)\neq 0$ then:

$$\boldsymbol{rank(AB\cdot det(CD)+CD\cdot det(AB))=}$$

$$\boldsymbol{=rank\left(\frac{1}{det\,C\cdot det\,D}B^{-1}A^{-1}+\frac{1}{det\,A\cdot det\,B}D^{-1}C^{-1}\right)}$$

Solution:

We use two properties:

(1) $rank\ (\alpha \cdot A) = rank\ A, \forall \alpha \neq 0$ *(obvious)*

(2) $rank\ (A) = rank\ (A \cdot B^{-1}), \forall B =$ *invertible (from Sylvester)*

$$rank\ (AB \cdot \det(CD) + CD \cdot \det(AB)) = rank(B \det(CD) + A^{-1}C \cdot D \det(AB)) =$$

$$= rank\ (\det(CD)\, I_n + B^{-1}A^{-1}C \cdot D \det(AB))$$

$$= rank(\det(CD)\, D^{-1} + B^{-1}A^{-1}C \det(AB)) =$$

$$= rank(\det(CD)\, D^{-1} \cdot C^{-1} + B^{-1}A^{-1} \cdot \det(AB)) =$$

$$= rank(\det D \cdot D^{-1} \det C \cdot C^{-1} + \det B^{-1} \cdot \det A \cdot A^{-1})$$

$$= rank\ (D^*C^* + B^*A^*)\ \textit{(3)}$$

Now, $rank\ \left(\frac{1}{\det C \cdot \det D} B^{-1}A^{-1} + \frac{1}{\det A \cdot \det B} D^{-1}C^{-1}\right)$

$$= rank\left(\frac{1}{\det A \det B \det C \det D} B^*A^* + \frac{1}{\det A \det B \det C \det D} D^*C^*\right) =$$

$$= rank(B^*A^* + D^*C^*)\ \textit{(4)}$$

From (3) + (4) $\Rightarrow$ *relation from hypothesis.*

5.30 ***If*** $\boldsymbol{A, B \in M_2(\mathbb{C}), det(A + B) = 1}$ ***then:***

$$\boldsymbol{det(A \cdot det\, B + B \cdot det\, A) = det(AB)}$$

Solution:

If $\det(A) = 0$ *or* $\det(B) = 0$, *then* $\det(\det(A)\, B + \det(B) A) = 0 = \det(A) \det(B)$

Suppose $\det(A) \neq 0, \det(B) \neq 0$. *Let* $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

Let $\alpha = \det(A), \beta = \det(B)$. *Now,* $I = \det(A + B) = \det[A(B^{-1} + A^{-1})B]$

$$= \det(A) \det(B) \det(B^{-1} + A^{-1})\ \textit{(1)}$$

But $A^{-1} = \frac{1}{\alpha}\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}, B^{-1} = \frac{1}{\beta}\begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix}$

$$\therefore B^{-1}+A^{-1}=\begin{bmatrix}\frac{d_1}{\alpha}+\frac{d_2}{\beta} & -\left(\frac{b_1}{\alpha}+\frac{b_2}{\beta}\right)\\ -\left(\frac{c_1}{\alpha}+\frac{c_2}{\beta}\right) & \frac{a_1}{\alpha}+\frac{a_2}{\beta}\end{bmatrix}$$

Now, note $\det(B^{-1}+A^{-1})=\det\begin{bmatrix}\frac{a_1}{\alpha}+\frac{a_2}{\beta} & \frac{b_1}{\alpha}+\frac{b_2}{\beta}\\ \frac{c_1}{\alpha}+\frac{c_2}{\beta} & \frac{d_1}{\alpha}+\frac{d_2}{\beta}\end{bmatrix}$

$\therefore \det(B^{-1}+A^{-1})=\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right)$ *(2)*

Thus, from (1), (2):

$$I=\alpha\beta\det\left(\frac{1}{\alpha}A+\frac{1}{\beta}B\right)=\frac{1}{\alpha\beta}\det\left[\frac{\alpha\beta}{\alpha}A+\frac{\alpha\beta}{\beta}B\right]$$

[$\because A,B$ are 2×2 matrices] $\Rightarrow \det(\beta A+\alpha B)=\alpha\beta$

or $\det[(\det B)A+(\det A)B]=\det A\det B=\det(AB)$

5.31 $\boldsymbol{A\in M_n(\mathbb{R}),\det A\neq 0,\alpha\in(-1,1),A^2+A^{-2}=\alpha(A+A^{-1})}$

Find: $\boldsymbol{|\det A|}$

Solution:

Let $A\in M_n(\mathbb{R})$ be an invertible matrix with

$A^2+A^{-2}=\alpha(A+A^{-1})$*, for some* $\alpha\in(-1,1)$ *(H)*

Find $|\det(A)|$

Step 1. If $\alpha\in(-1,1)$ then all the complex roots of the polynomial

$$P(x)=X^4-\alpha X^3-\alpha X+1$$

belong to the unit circle.

Indeed, $P(z)=0$ is equivalent to $z^3=\frac{\alpha z-1}{z-\alpha}$ *thus*

$$|z|^6-1=\left|\frac{\alpha z-1}{z-\alpha}\right|^2-1=\frac{(1-\alpha^2)(1-|z|^2)}{|z-\alpha|^2}$$

and consequently

$$(|z|^2-1)\underbrace{\left[1-|z|^2+|z|^4+\frac{1-\alpha^2}{|z-\alpha|^2}\right]}_{positive}=0$$

Thus, $|z|=1$

Step 2 $|\det A|=1$

Consider A *as a complex matrix. If* $\lambda\in\mathbb{C}$ *is an eigenvalue of* A *then according to* $(H),\lambda$ *satisfies*

$$\lambda^2+\frac{1}{\lambda^2}=\alpha\left(\lambda+\frac{1}{\lambda}\right)$$

Equivalently $P(\lambda)=0$, *hence* $|\lambda|=1$ *according to Step 1. But* $\det A$ *is the product of all the eigenvalues of A, (each one is repeated according to its multiplicity), so* $|\det A|=1$.

5.32 Solve for real numbers:

$$\begin{vmatrix}\mathbf{1} & \mathbf{3+\sin x} & \mathbf{2+3\sin x} & \mathbf{2\sin x}\\ \mathbf{1} & \mathbf{2+\sin x+\cos x} & \mathbf{2\sin x+\sin x\cos x} & \mathbf{\sin 2x}\\ \mathbf{1} & \mathbf{1+\sin x+\cos x} & \mathbf{\sin x+\cos x+\sin x\cos x} & \mathbf{\sin x\cos x}\\ \mathbf{1} & \mathbf{3+\cos x} & \mathbf{2+3\cos x} & \mathbf{2\cos x}\end{vmatrix}=\mathbf{0}$$

Solution:

After simplification we have:

$$\left|\begin{pmatrix}1 & \sin x+3 & 3\sin x+2 & 2\sin x\\ 1 & \cos x+\sin x+2 & \sin x\cos x+2\cos x+2\sin x & \sin 2x\\ 1 & \cos x+\sin x+1 & \sin x\cos x+\cos x+\sin x & \sin x\cos x\\ 1 & \cos x+3 & 3\cos x+2 & 2\cos x\end{pmatrix}\right|=$$

$$-\frac{1}{4}(\sin x-2)(\sin x+\cos x-1)^2(4\sin x+\cos 2x-2(\sin x+2)\cos x+1).$$ *Solve for* x:

$$-\frac{1}{4}(\sin x-2)(-1+\cos x+\sin x)^2(1+\cos 2x+4\sin x-2\cos x)(\sin x+2)=0$$

Multiply both sides by a constant to simplify the equation.

Multiply both sides by -4*:*

$$(\sin x-2)(-1+\cos x+\sin x)^2(1+\cos 2x+4\sin x-2\cos x)(\sin x+2)=0$$

Find the roots of each term in the product separately. Split into three equations:

$$\sin x-2=0 \text{ or } (-1+\cos x+\sin x)^2=0 \text{ or}$$

$$1+\cos 2x+4\sin x-2\cos x\,(\sin x+2)=0$$

Isolate terms with x *to the left hand side. Add* 2 *to both sides:* $\sin x=2$ *or* $(-1+\cos x+\sin x)^2=0$ *or* $1+\cos 2x+4\sin x-2\cos x\,(\sin x+2)=0.$

After solving each equation separately and some calculations we have the following solutions

$$x=\pi\left(\frac{n-7}{4}\right); x=2\pi n; x=2\pi n+\frac{\pi}{2}; x=2\pi n+\frac{\pi}{4};\ x=2\pi n-\frac{3\pi}{4}$$

$$x=2\pi n-2i\tanh^{-1}\frac{1}{\sqrt{3}}; x=2\pi n+2i\tanh^{-1}\frac{1}{\sqrt{3}}; x=2\pi n+\pi-\sin^{-1}2$$

5.33 If $A,B,C\in M_n(\mathbb{Z}), n\ge 3, (A^*B^*)^*=BA, (B^*C^*)^*=CB$ then:

$$\det A+\det B+\det C<\sqrt{10}$$

Solution:

If $\det A=0$ *or* $\det B=0$ *or* $\det C=0$ *obvious.*

Let $\det A\neq 0, \det B\neq 0, \det C\neq 0.$

Lemma 1: $(AB)^*=B^*A^*$ *(1)*

Lemma 2: $(A^*)^*=(\det A)^{n-2}A$ *(2)*

From $(A^*B^*)^*=BA \overset{(1)}{\Rightarrow} ((BA)^*)^*=BA \overset{(2)}{\Rightarrow}$

$$\left.\begin{array}{c}(\det BA)^{n-2}BA=BA\\ BA\ invertible\end{array}\right\}\Rightarrow(\det BA)^{n-2}=1\Rightarrow$$

$$\Rightarrow\left.\begin{array}{c}\det BA=\pm1\Rightarrow\det A\cdot\det B=\pm1\\ but\ \det A\ \ and\ \det B\in\mathbb{Z}\end{array}\right\}\Rightarrow\det A,\det B\in\{-1,1\}\quad (3)$$

Similarly: $\det B, \det C \in \{-1,1\}$ *(4)*

From (3)+(4)$\Rightarrow \det A + \det B + \det C \leq 3 < \sqrt{10}$

5.34 If $X, Y, Z \in M_n(\mathbb{R}), n \geq 2, n \in \mathbb{N}, XY = YX, YZ = ZY, ZX = XZ$ then:

$$\det(9X^2 + 5Y^2 + 5Z^2 + 12XY + 6YZ + 12ZX) \geq 0$$

Solution:

We use: $\det(A \cdot \bar{A}) \geq 0, \forall A \in M_n(R)$ *(1)*

Because $XY = YX, YZ = ZY$ *and* $ZX = XZ$ *we can make algebraic calculus:*

$$\det\left[(3A + (2+i)B + (2-i)C)\overline{(3A + (2+\imath B + (2-\imath)C))}\right] \geq 0 \quad (2)$$

(From (1))

But $\det\left[(3A + (2+i)B + (2-i)C)\overline{(3A + (2+\imath)B + (2-\imath)C)}\right] =$

$= \det[(3A + (2+i)B + (2-i)C)(3A + (2-i)B + (2+i)C)] =$

$= \det(9X^2 + 5Y^2 + 5Z^2 + 12XY + 6YZ + 12ZX)$ *(3)*

From (2)+(3)$\Rightarrow \det(9X^2 + 5Y^2 + 5Z^2 + 12XY + 6YZ + 12XZ) \geq 0$

5.35 $A, B \in M_2(\mathbb{R}), Tr((AB)^2) = Tr(A^2B^2), n \in \mathbb{N}, n \geq 2$. Find:

$$\Omega = Tr[(AB - BA)^n]$$

Solution:

If X and Y are two $n \times n$ matrices, then: $Tr(XY) = Tr(YX)$

$Tr(X \pm Y) = Tr(X) \pm Tr(Y)$. *We are given:* $Tr((AB)^2) = Tr(A^2B^2) \Rightarrow$

$\Rightarrow Tr\{ABAB - AABB\} = 0 \Rightarrow Tr\{A(BA - AB)B\} = 0 \Rightarrow$

$\Rightarrow Tr\{BA(BA - AB)\} = 0$ *(1)*

$\Rightarrow Tr((BA)^2) = Tr(BA^2B) = Tr(BBA^2) = Tr(B^2A^2) \Rightarrow$

$\Rightarrow Tr\{BABA - BBAA\} = 0 \Rightarrow Tr\{B(AB - BA)A\} = 0 \Rightarrow$

$\Rightarrow Tr\{AB(AB - BA)\} = 0$ *(2)*

$$Now,\ Tr\{(AB-BA)^2\} = Tr\{AB(AB-BA)+BA(BA-AB)\} =$$

$$= Tr\big(AB(AB-BA)\big)+Tr\big(BA(BA-AB)\big) = 0+0 = 0\ [from\ (1),\ (2)]$$

$$Let\ x = AB-BA,\ then\ Tr(x) = Tr(AB)-Tr(BA) = 0.$$

$$Also,\ Tr(X^2) = 0\ [Prove\ above]$$

$$Let\ X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\ [\because Tr(X)=0]$$

$$X^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{pmatrix}$$

$$Tr(X^2) = 0 \Rightarrow 2(a^2+bc) = 0 \Rightarrow a^2+bc = 0$$

$$\therefore X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow Tr(X^n) = 0\ \ \forall n \geq 2$$

5.36 If $A \in M_2(\mathbb{Z})$ then:

$$\boldsymbol{\Omega = \det(A+A^T+A^*)+\det(-A+A^T+A^*)+\det(A-A^T+A^*)+}$$

$$\boldsymbol{\det(A+A^T-A^*)}\ \textbf{is divisible with 12.}$$

Solution:

$$Let\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, a,b,c,d \in \mathbb{Z}$$

$$A+A^T+A^* = \begin{pmatrix} 2a+d & c \\ b & a+2d \end{pmatrix} = B_1\ \ (say)$$

$$-A+A^T+A^* = \begin{pmatrix} d & c-2b \\ b-2c & a \end{pmatrix} = B_2\ \ (say)$$

$$A-A^T+A^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = B_2\ \ (say)$$

$$A+A^T-A^* = \begin{pmatrix} 2a-d & c+2b \\ 2c+b & 2d-a \end{pmatrix} = B_4\ \ (say)$$

$$\therefore \det(B_1)+\det(B_2)+\det(B_3)+\det(B_4)$$

$$= (2a+d)(a+2d)-bc+ad-(b-2c)(c-2b)+ad-bc$$

$$+(2a-d)(2d-a)-(c+2b)(2c+b) = 2a^2+5ad+2d^2-bc$$

$$+ad-(5bc-2c^2-2b^2)+ad-bc$$

$$+5ad-2d^2-2a^2-(2c^2+5bc+2b^2)$$

$$= 12(ad-bc)\ which\ is\ divisible\ by\ 12.$$

5.37 GENERALIZATION FOR A DAN RADU SECLEMAN'S INEQUALITY

If $A,B\in M_n(\mathbb{R}), n\geq 2, p\geq 1, n,p\in\mathbb{N}$,

$A^{2p+1}+B^{2p}=I_n, A^{4p+1}=A^{2p}$ then:

$$\det(I_n+A^{2p}+B^{2p})\geq 0$$

Solution:

$$A^{2p+1}+B^{2p}=I_n|\cdot A^{2p}\Rightarrow A^{4p+1}+B^{2p}\cdot A^{2p}=A^{2p}\Rightarrow$$

$$A^{2p}+B^{2p}A^{2p}=A^{2p}\Rightarrow B^{2p}A^{2p}=O_n \quad (1)$$

$$A^{2p}|A^{2p+1}+B^{2p}=I_n\Rightarrow A^{4p+1}+A^{2p}B^{2p}=A^{2p}\Rightarrow A^{2p}B^{2p}=O_n \quad (2)$$

From (1)+(2) we must show:

$$\det(I_n+A^{2p}+B^{2p}+A^{2p}\cdot B^{2p})\geq 0\Leftrightarrow$$

$$\det[(I_n+A^{2p})(I_n+B^{2p})]\geq 0\Leftrightarrow \det(I_n+A^{2p})\cdot\det(I_n+B^{2p})\geq 0 \quad (3)$$

But $\det(I+A^{2p})=\det(I_n^2-i^2A^{2p})=$

$$=\det[(I_n+iAP)(I_n-iAP)]=\det\left[(I_n+iA^p)\overline{(I_n+\imath A^p)}\right]\geq 0 \quad (4)$$

Similarly: $\det(I_n+B^{2p})\geq 0$ *(5)*

From (4)+(5)$\Rightarrow\det(I_n+A^{2p})\,(I_n+B^{2p})\geq 0\Rightarrow$ *(3) its true.*

5.38 If $A,B\in M_2(\mathbb{C}), \det A\neq 0, \det B\neq 0$ then:

$$\det(A\det B+B\det A)+\det\left(\frac{A}{\det A}+\frac{B}{\det B}\right)=$$

$$=\det(A+B)\left(\det(AB)+\frac{1}{\det(AB)}\right)$$

Solution:

First we prove this: Theorem (by Vasile Pop and Ovidiu Furdui)

If $A,B\in M_2(\mathbb{C})$ and $x,y\in\mathbb{C}$ then:

$$\det(xA+yB)=x^2\det A+y^2\det B+xy[\det(A+B)-\det A-\det B]$$

Demonstration: we use a determinant formula:

If $A, B \in M_2(\mathbb{C}) \wedge x \in \mathbb{C}$ *then:*

$$\det(A + xB) = \det A + (\det(A + B) - \det A - \det B)x + (\det B)x^2$$

For our theorem if $x = 0 \Rightarrow$ *then its trivial.*

$$\text{If } x \neq 0 \Rightarrow \det(xA + yB) = \det\left[x\left(A + \frac{y}{x}B\right)\right] =$$

$$= x^2 \det\left(A + \frac{y}{x}B\right)$$

$$= x^2[\det A + (\det(A + B) - \det A - \det B)]\frac{y}{x} + \det B\frac{y^2}{x^2}$$

$$= \det A\, x^2 + (\det(A + B) - \det A - \det B)xy + \det B y^2 \quad \text{(done)}$$

Now for our problem:

$$\text{Let } x = \det B, y = \det A \Rightarrow$$

$$\det(A\det B + B\det A) = (\det B)^2 + \det A + (\det A)^2 \cdot \det B +$$

$$+ \det(AB)\,(\det(A + B) - \det A - \det B) \quad (1)$$

$$\text{Let } x = \frac{1}{\det A}, y = \frac{1}{\det B} \Rightarrow$$

$$\det\left(\frac{A}{\det A} + \frac{B}{\det B}\right) = \frac{1}{\det A} + \frac{1}{\det B} + \frac{1}{\det(AB)}(\det(A + B) - \det A - \det B) \quad (2)$$

From (1) + (2) $\Rightarrow$

$$\det(A\det B + B\det A) + \det\left(\frac{A}{\det A} + \frac{B}{\det B}\right) =$$

$$= \det A\,(\det B)^2 + (\det A)^2 \det B + \det(AB)\det(A + B) -$$

$$-(\det A)^2 \cdot \det B - \det A \cdot (\det B)^2 + \frac{1}{\det A} + \frac{1}{\det B} + \frac{\det(A + B)}{\det AB} - \frac{1}{\det B}$$

$$-\frac{1}{\det A} =$$

$$= \det(AB) \cdot \det(A + B) + \frac{\det(A + B)}{\det AB} = \det(A + B)\left(\det(AB) + \frac{1}{\det(AB)}\right)$$

5.39

$$\boldsymbol{\Omega} = \begin{vmatrix} \sin^2 x & \sin^2 y \cdot \cos^2 x & \cos^2 y \cdot \cos^2 x \\ \cos^2 y \cdot \cos^2 x & \sin^2 x & \sin^2 y \cdot \cos^2 x \\ \sin^2 y \cdot \cos^2 x & \cos^2 y \cdot \cos^2 x & \sin^2 x \end{vmatrix}, x, y \in \mathbb{R}$$

Prove that: $|\Omega| \le 1$.

Solution:

$$\Omega = \begin{vmatrix} \sin^2 x & \sin^2 y \cos^2 x & \cos^2 y \cos^2 x \\ \cos^2 y \cos^2 x & \sin^2 x & \sin^2 y \cos^2 x \\ \sin^2 y \cos^2 x & \cos^2 y \cos^2 x & \sin^2 x \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$, *we get*

$$\Omega = \begin{vmatrix} 1 & 1 & 1 \\ \cos^2 y \cos^2 x & \sin^2 x & \sin^2 y \cos^2 x \\ \sin^2 y \cos^2 x & \cos^2 y \cos^2 x & \sin^2 x \end{vmatrix}$$

Using $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, *we get*

$$\Omega = \begin{vmatrix} 1 & 0 & 0 \\ \cos^2 y \cos^2 x & \sin^2 x - \cos^2 y \cos^2 x & \cos^2 x (\sin^2 y - \cos^2 y) \\ \sin^2 y \cos^2 x & (\cos^2 y - \sin^2 y) \cos^2 x & \sin^2 x - \sin^2 y \cos^2 x \end{vmatrix} =$$

$$= (\sin^2 x - \cos^2 y \cos^2 x)(\sin^2 x - \sin^2 y \cos^2 x)$$
$$+ \cos^4 x (\sin^2 y - \cos^2 y)^2$$
$$= \sin^4 x - \sin^2 x \cos^2 x (\cos^2 y + \sin^2 y) + \cos^4 x \sin^2 y \cos^2 y +$$
$$+ \cos^4 x (\cos^4 y + \sin^4 y - 2 \sin^2 y \cos^2 y)$$
$$= \sin^4 x - \sin^2 x \cos^2 x - 3 \cos^4 x \sin^2 y \cos^2 y + \cos^4 x (\cos^2 y + \sin^2 y)^2$$
$$= \sin^4 x + \cos^4 x - \sin^2 x \cos^2 x - 3 \cos^4 x \sin^2 y \cos^2 y$$
$$= (\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x - 3 \cos^4 x \sin^2 y \cos^2 y$$
$$= 1 - 3 \sin^2 x \cos^2 x - 3 \cos^4 x \sin^2 y \cos^2 y \le 1$$

Also, $3 \sin^2 x \cos^2 x + 3 \cos^4 x \sin^2 y \cos^2 y = \frac{3}{4} \sin^2 2x + \frac{3}{4} \cos^4 x \sin^2 2y \le$

$$\le \frac{3}{4} + \frac{3}{4} = \frac{3}{2} \Rightarrow 1 - \frac{3}{2} \le 1 - 3 \sin^2 x \cos^2 x - 3 \cos^4 x \sin^2 y \cos^2 y \le 1$$

$$\Rightarrow -\frac{1}{2} \leq \Omega \leq 1 \Rightarrow |\Omega| \leq 1$$

5.40 If $A, B, C \in M_n(\mathbb{R}), AB = BA, AC = CA, BC = CB, n \in \mathbb{N}, n \geq 2$ then:

$$\det(A^2 - 6AB + 10B^2 + 16BC + 10C^2 - 6AC) \geq 0$$

Solution:

We make a generalization:

Lemma 1: Let $P \in R[x], p(x) = x^2 + ax + b, \Delta = b^2 - 4b < 0$. *Then*

$\forall A, B \in M_n(\mathbb{R})$ *the following statement is true:*

$\det[(A + x_1B + x_2C)(A + x_2B + x_1C)] \geq 0, x_1, x_2$ *being the roots of* p

Demonstration: $If\ \Delta < 0 \Rightarrow x_1, x_2 \in \mathbb{C}, x_2 = x_1$ *and using* $\det(x \cdot \bar{x}) \geq 0$,

$$\forall x \in M_n(\mathbb{R}) \Rightarrow \det[(A + x_1B + x_2C)(A + x_2B + x_1C)]$$

$$= \det[(A + x_1B + x_2C)(\overline{A + x_1B + x_2C})] \geq 0$$

Lemma 2. If $AB = BA, AC = CA, BC = CB$ *then the conclusion of this theorem can be written this way:*

$$\det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 - 2b)BC] \geq 0$$

Demonstration: $\det[(A + x_1B + x_2C)(A + x_2B + x_1C)] =$

$$= \det[A^2 + x_1x_2(B^2 + C^2) + (x_1 + x_2)(AB + AC) + (x_1^2 + x_2^2)BC] =$$

$$= \det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 - 2b)BC] \geq 0$$

(we used $AB = BA, AC = CA, BC = CB$ *and Viéte relations)*

Now, in our case: $a = 6, b = 10$. *Done.*

5.41 If $A, B \in M_2(\mathbb{R}), AB = BA, \det A = \alpha > 0, \det(A + i\alpha B) = 0$ then find:

$$\Omega = \det(A^2 - \alpha AB + \alpha^2 B^2)$$

Solution:

If $AB = BA, \det(A) = \alpha > 0, \det(A + \alpha iB) = 0$, *find* $\det(A^2 - \alpha AB + \alpha^2 B^2)$.

As

$\det(A) > 0, A^{-1}$ *exists. Let* $C = A^{-1}B$.

Now, $\det(A + \alpha iB) = 0$ *(1)*

$$\Rightarrow \det\left[\alpha iA\left(-\frac{i}{\alpha}I + A^{-1}B\right)\right] = 0 \Rightarrow \det(\alpha iA)\det\left(C - \frac{i}{\alpha}I\right) = 0 \quad (2)$$

$$\Rightarrow \det\left(C - \frac{i}{\alpha}I\right) = 0$$

$$[\because \det(i\alpha A) = -\alpha^2(\alpha) \neq 0]$$

Characteristic equation of C *is*

$$t^2 - tr(C)t + \det(C) = 0 \quad (3)$$

In view of (2), $\frac{i}{\alpha}$ *satisfies (3)* $\Rightarrow -\frac{1}{\alpha^2} - \frac{i}{\alpha}tr(C) + \det(C) = 0$

$$\Rightarrow \det(C) = \frac{1}{\alpha^2} \text{ and } tr(C) = 0$$

As $\det(C) \neq 0$, *we get* $\det(A^{-1}B) \neq 0 \Rightarrow \det(B) \neq 0$ *and* $\det(B) = \frac{1}{\alpha} \Rightarrow$

$\Rightarrow \det(B) \neq 0$ *and* $\det(B) = \frac{1}{\alpha} \Rightarrow B^{-1}$ *exists. Let* $D = AB^{-1}$. *From (1):*

$$\det[(D + i\alpha)B] = 0 \Rightarrow \det(D + i\alpha)\det(B) = 0 \Rightarrow \det(D + i\alpha) = 0 \quad (4)$$

Characteristic equation of D *is* $t^2 - \big(tr(D)\big)t + \det(D) = 0$ *(5)*

In view of (4) $-i\alpha$ *satisfies (4)*

$$\therefore -a^2 + tr(D)(i\alpha) + \det(D) = 0 \Rightarrow \det(D) = a^2, tr(D) = 0$$

$\therefore$ *characteristic equation (5) becomes* $t^2 + \alpha^2 = 0$.

Now, $A^2 - \alpha AB + a^2B^2 = (A^2B^{-2} - \alpha AB^{-1} + a^2I)B^2 = (D^2 - \alpha D + a^2I)B^2 =$

$$= (0 - \alpha D)B^2 = -\alpha AB^{-1}B^2 = -\alpha AB$$

$$\det(A^2 - \alpha AB + \alpha^2B^2) = (-\alpha)^2\det(AB) = \alpha^2\det(A)\det(B) = \alpha^2$$

5.42 Find $A, B \in M_2(\mathbb{R})$ such that:

$$\det A < 0, \det(A - B) > 0, \det(A + B) < 0, \det(2A + B) > 0$$

Solution:

We will use the following formula:

$\det(A+xB)=ax^2+bx+c$, *when:* $a=\det B$, $b=tr(AB^*)$, $c=\det A$

We will note $p(x)=\det(A+xB)$. *Because* p *is a polygon of second degree, it's obvious that it can be at most two changes in the value of* $sgn\left(p(x)\right)$. *But:*

$p(-1)>0, p(0)<0, p\left(\frac{1}{2}\right)>0, p(1)<0 \Rightarrow 3$ *changes of sign. That means there are no matrices with the properties in the hypothesis.*

Observation:

$$\det(2A+B)=4\det\left(A+\frac{1}{2}B\right)=4p\left(\frac{1}{2}\right)>0\Rightarrow p\left(\frac{1}{2}\right)>0$$

5.43 $\boldsymbol{A\in M_3(\mathbb{R}), \det(A^2+2A+2I_3)=\det(A+I_3)=0}$

Find: $\boldsymbol{\Omega=\det A}$

Solution:

$A\in M_3(\mathbb{R})$ *then characteristics polynomial has highest degree* 3

$\therefore$ *We have to find a polynomial with their eigen values*

$\therefore \det(A^2+2A+2I_3)=0$ $\therefore$ *then polynomial is*

$x^2+2x+2=0$

It has two different eigen values $(-1+i)$ *and* $(-1-i)$

[by solving quadratic equation].

Here $|A+I|=0$

$\therefore$ *one eigen value of* A *is* -1 $\therefore$ *characteristic polynomial is*

$=(x+1)(x^2+2x+2)=x^3+2x^2+2x+x^2+2x+2=x^3+3x^2+4x+2$ $\therefore$ *then* $\det(A)=$ *product of eigen value* $=-2$

5.44 Solve for real numbers:

$$\begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+3} & \frac{1}{x+4} \\ \frac{1}{y+2} & \frac{1}{y+3} & \frac{1}{y+4} \\ \frac{1}{\sin x+2} & \frac{1}{\sin x+3} & \frac{1}{\sin x+4} \end{vmatrix} = 0$$

Solution:

We notice that it is determinant of Cauchy type:

$$D_3 = \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \frac{1}{a_1+b_3} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \frac{1}{a_2+b_3} \\ \frac{1}{a_3+b_1} & \frac{1}{a_3+b_2} & \frac{1}{a_3+b_3} \end{vmatrix} \text{ with } \begin{cases} a_1 = x \\ a_2 = y \\ a_3 = \sin x \end{cases} \text{ and } \begin{cases} b_1 = 2 \\ b_2 = 3 \\ b_3 = 4 \end{cases}$$

$$D_3 = \frac{D_2}{a_3+b_3} \cdot \prod_{k=1}^{2} \frac{(a_3-a_k)(b_3-b_k)}{(a_3+a_k)(b_3+b_k)}$$

$$\left.\begin{aligned} D_3 &= \frac{D_2}{\sin x+4} \cdot \frac{(\sin x - x)(4-2)}{(\sin x + x)(4+2)} \cdot \frac{(\sin x - y)(4-3)}{(\sin x + y)(4+3)} \\ D_2 &= \begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+3} \\ \frac{1}{y+2} & \frac{1}{y+3} \end{vmatrix} = \frac{y-x}{(x+2)(x+3)(y+2)(y+3)} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow D_3 = \frac{(y-x)(\sin x - x)\cdot 2\cdot(\sin x - y)}{6(\sin x+4)(\sin x+x)(\sin x+y)\cdot 7(x+2)(x+3)(y+2)(y+3)} = 0 \Rightarrow y = x \textit{ or}$$

$$\sin x = x \textit{ or } \sin x = y$$

5.45 $A \in M_2(\mathbb{R}), p \in \mathbb{R} - \{0\}, \det(A^2 + 2pA + 2p^2 I_2) = 0$. Find:

$$\mathbf{\Omega = \det A}$$

Solution:

$$A^2 + 2pA + 2p^2 I_2 = (A + pI_2)^2 + p^2 I_2 = [A + (p+ip)I_2][A + (p-ip)I_2]$$

$$0 = \det(A^2 + 2pA + 2p^2 I_2) = \det\{(A + (p+ip)I_2)(A + (p+ip)I_2)\}$$

$$= \det(A + (1 + i)pI_2)\det(A + (1 - i)pI_2) \Rightarrow \det(A + (1 + i)p) = 0$$

or $\det(A + (1 - i)p) = 0$. *Assume* $\det(A + (1 + i)p) = 0$.

$\Rightarrow -(1 + i)p$ *is eigenvalue of* A *another eigenvalue is* $-(1 - i)p$

$$\therefore \det(A) = (1 + i)(1 - i)p^2 = 2p^2$$

5.46

$$\boldsymbol{x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}, x \circ y = xy - 5x - 5y + 30, G = (5, \infty)}$$

Prove that $(\mathbb{R}, *) \cong (G, \circ)$ **as abelian groups.**

Solution:

We first show that $(\mathbb{R}, *)$, *where* $x * y = x\sqrt{1 + y^2} + y\sqrt{1 + x^2}$

is an abelian group. Clearly, $x * y \in \mathbb{R}, \forall x, y \in \mathbb{R}$

$*$ *is associative suppose* $x, y, z \in \mathbb{R}$. *Let* $x = \tan\alpha, y = \tan\beta, z = \tan\gamma$

$$-\frac{\pi}{2} < \alpha, \beta, \gamma < \frac{\pi}{2}$$

$$x * y = (\tan\alpha)\sqrt{1 + \tan^2\beta} + \tan\beta\sqrt{1 + \tan^2\alpha} = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}$$

$$(x * y) * z = \frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\sqrt{1 + \tan^2\gamma} + \tan\alpha\sqrt{1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2}$$

But $1 + \left(\frac{\sin\alpha + \sin\beta}{\cos\alpha\cos\beta}\right)^2 = \frac{(1 - \sin^2\alpha)(1 - \sin^2\beta) + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta}{\cos^2\alpha\cos^2\beta} =$

$$\frac{(1 + \sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}$$

$$\oplus$$

Thus, $(x * y) * z = \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta\cos\gamma} + \frac{\sin\gamma(1 + \sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma} =$

$$= \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$$

Similarly, $x * (y * z) = \frac{\sin\alpha + \sin\beta + \sin\gamma + \sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

Thus, $(x * y) * z = x * (y * z); \forall x, y, z \in \mathbb{R}$

$*$ *is commutative is obvious.*

Identity Element $= 0;\ x * 0 = x\sqrt{1+0^2} + 0\sqrt{1+x^2} = x; \forall x \in \mathbb{R}$

Inverse Element.For each $x \in \mathbb{R}, -x \in \mathbb{R}$ *is inverse of* x*.Indeed* $x * (-x) = 0$

$\therefore (\mathbb{R}, *)$ *is an abelian group. Next, we show that if* $G = (5, \infty)$*, and*

$a \circ b = ab - 5a - 5b + 30; \forall a, b \in \mathbb{G}$*, then* $(\mathbb{G}, \circ)$ *is an abelian group.*

Note $a \circ b = (a-5)(b-5) + 5$

$\circ$ *is commutative and its identity element is* 6*.*

$\circ$ *is associative*

Let $a, b, c \in \mathbb{G}$*,*

$$(a \circ b) \circ c = \big((a-5)(b-5)+5\big) \circ c$$

$$= \big((a-5)(b-5)+5-5\big)(c-5)+5$$

$$= (a-5)(b-5)(c-5)+5$$

Similarly, $a \circ (b \circ c) = (a-5)(b-5)(c-5)+5$

$$\therefore (a \circ b) \circ c = a \circ (b \circ c); \forall a, b, c \in \mathbb{G}$$

Finally, if $a \in 5$*, then* $a > 5$*, and* $b = 5 + \frac{1}{a-5}$ *is inverse of* a*. Indeed,*

$a \circ b = (a-5)(b-5)+5 = (a-5)\left(\frac{1}{a-5}\right) + 5 = 1 + 5 = 6 =$ *identity element.*

We now show that $\Phi: \mathbb{R} \to \mathbb{G}$ *defined by* $\Phi(x) = 5 + 5^{\sinh^{-1} x}$

is the required isomorphism of $\mathbb{R}$ *onto* $\mathbb{G}$

As $5^{\sinh^{-1} x} > 0, \forall x \in \mathbb{R}, \Phi(x) \in \mathbb{G}; \forall x \in \mathbb{R}$

For $x, y \in \mathbb{R}$

$$\Phi(x * y) = 5^{\sinh^{-1}\left(x\sqrt{1+y^2}+y\sqrt{1+x^2}\right)} + 5 \quad (1)$$

and $\Phi(x) \circ \Phi(y) = 5^{\sinh^{-1} x} \cdot 5^{\sinh^{-1} y} + 5 \quad (2)$

$$= 5^{\sinh^{-1} x + \sinh^{-1} y} + 5$$

But $\sinh^{-1} x + \sinh^{-1} y = \sinh^{-1}\left(x\sqrt{1+y^2} + y\sqrt{1+x^2}\right) \quad (3)$

$\therefore$ *from (1), (2), (3):* $\Phi(x * y) = \Phi(x) \circ \Phi(y)$

Thus, Φ *is a homomorphism from* $(\mathbb{R},*)$ *to* $(\mathbb{G},a)$

Φ *is one – to – one*

Let $x,y\in\mathbb{R}$ *and* $\Phi(x)=\Phi(y)$

$$\Rightarrow 5^{\sinh^{-1}x}+5=5^{\sinh^{-1}y}+5\Rightarrow \sinh^{-1}x=\sinh^{-1}y\Rightarrow x=y$$

$\therefore$ Φ *is one – to – one*

Φ *is onto*

Let $y\in\mathbb{G}\Rightarrow y>5\Rightarrow y-5>0$

Let $t=\log_5(y-5)\Rightarrow 5^t=y-5$

As $t\in\mathbb{R},\exists x\in\mathbb{R}$ *such that* $\sinh^{-1}x=t$ *or take* $x=\sinh t$.

Then $\Phi(x)=5^{\sinh^{-1}x}+5=5^t+5=y-5+5=y$

$\therefore$ Φ *is onto.*

Hence, $(\mathbb{R},*)\cong(\mathbb{G},\circ)$ *as abelian groups.*

5.47 Let A be a ring with identity. For each $a\in A$ we define

$$E_a:=\{x\in A: xa=1\}$$

Show that if $c\in E_a$ and $|E_a|\geq 2$ then the function $\varphi_a: E_a\to E_a$ defined by

$\varphi_a(x)=ax+c-1$ is injective but not surjective.

Solution:

To show injective, suppose $\varphi_a(x)=\varphi_a(y)$*, then:*

$$ax+c-1=ay+c-1\Rightarrow ax=ay\Rightarrow cax=cay\Rightarrow x=y.$$

To show it is not surjective, we argue by contradiction. So, we suppose it is surjective, there is $x\in E_a$ *such that:* $ax+c-1=c\in E_a$ *therefore* $ax=1$ *since* $|E_a|\geq 2$*, we can choose* $y\in E_a, y\neq x$ *such that:* $ya=1$ *multiplying* x *on the right on both sides we have:* $yax=x\Rightarrow y=x$ *which is a contradiction. Hence the mapping is injective but not surjective.*

5.48 Find the last 3 digits of:

$$\Omega = 2019^{\underbrace{201920192019\ldots 201953}_{50\ times\ "2019"}}$$

Solution:

$$\underbrace{201920192019\ldots 2019}_{\text{2019 times}}53 =$$

$$= 2 \times 10^{201} + 0 \times 10^{200} + 1 \times 10^{199} + 9 \times 10^{198} + \cdots +$$

$$+2 \times 10^5 + 0 \times 10^4 + 1 \times 10^3 + 9 \times 10^2 + 5 \times 10 + 3 \times 10^\circ$$

We can use Euler's quotient function and Euler's theorem:

Since $1000 = 8 \times 125$. *We evaluate* $\Omega(mod\ 1000)$

Evaluating $\Omega(mod\ 8)$; $\phi(8) = 4$; $2019 = 3(mod\ 8)$

$$20192019\ldots 201953 = 53 = 1(mod\ 4)$$

Since all other terms are multiples of 4.

So, $\Omega = 3^{'} = 3(mod\ 8)$;*Evaluating* $\Omega(mod\ 125)$

$$\phi(125) = 4 \times 25 = 100$$

$$2019 = 19\ mod\ 125$$

$$20192019\ldots 201953 = 53\ (mod\ 100)$$

Since all other terms are multiples of 100, *so* $\Omega = 19^{53}(mod\ 125)$

$$= 19^{52} \times 19 = (19^4)^{13} \times 19 = 11^{13} \times 19$$

$$= (71^4)^3 \times 71 \times 19 = 56^3 \times 71 \times 19$$

$$= 116 \times 71 \times 19 = 109(mod\ 125)$$

So, $\Omega = 3\ mod\ 8 = 3 + 8k$ *and* $\Omega = 109\ mod\ 125$

$3 + 8k = 109$ *mod* 125; $k = 107$ *mod* 125

So, $\Omega = 859\ mod\ 1000$; *The last three digits are* 859.

5.49 F_n: n^{th} Fibonacci Number.Prove that:

$$F_{2n+1} = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!\,(2k)!}$$

Solution:

Let $S(n)=\sum_{k=0}^{n}\binom{n+k}{2k}$. *We call* $S(n)$ *the main sum.*

Let $P(n)=\sum_{k=0}^{n}\binom{n+k}{2k-1}$, *where* P *is called auxiliary sum.*

We use the well-known Pascal's identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ *(*)*

Let us find the recurrence between S *and* P*!*

Let $n\in\mathbb{N}$

$$S(n)\overset{(*)}{=}\sum_{k=0}^{n}\left[\binom{n+k-1}{2k}+\binom{n+k-1}{2k-1}\right]$$

$$=\sum_{k=0}^{n}\binom{n-1+k}{2k}+\sum_{k=0}^{n}\binom{n-1+k}{2k-1}$$

$$=\sum_{k=0}^{n}\binom{n-1+k}{2k}+\sum_{k=0}^{n}\binom{n-1+k}{2k-1}=S(n-1)+\sum_{k=0}^{n-1}\binom{n-1+k}{2k-1}+\binom{2n-1}{2n-1}\quad (**)$$

By (**) $\Rightarrow S(n)=S(n-1)+P(n-1)+1$ *(***)*

On the other side: $n\in\mathbb{N}$

$$P(n)=\sum_{k=0}^{n}\begin{vmatrix}n+k\\2k-1\end{vmatrix}\overset{(*)}{=}\sum_{k=0}^{n}\begin{vmatrix}n-1+k\\2k-1\end{vmatrix}+\sum_{k=0}^{n}\begin{vmatrix}n-1+k\\2k-2\end{vmatrix}$$

$$=\sum_{k=0}^{n-1}\begin{vmatrix}n-1+k\\2k-1\end{vmatrix}+\begin{vmatrix}2n-1\\2n-1\end{vmatrix}+$$

$$+\sum_{k=1}^{n}\binom{n+(k-1)}{2(k-1)};t=k-1$$

$$=P(n-1)+1+\sum_{t=0}^{n-1}\begin{vmatrix}n+t\\2t\end{vmatrix}=P(n)+\binom{2n}{2n}+\sum_{t=0}^{n-1}\begin{vmatrix}n+t\\2t\end{vmatrix}\Rightarrow P(n)=P(n-1)+S(n)\ (****)$$

By (***)$\Rightarrow P(n-1)=S(n)-S(n-1)-1;n\in\mathbb{N}$ *(*****)*

By (*****)$\Rightarrow P(n) = S(n+1) - S(n) - 1$ *(******)*

By (****)$\Rightarrow P(n) - P(n-1) = S(n)$

By (*****) and (******)$\Rightarrow S(n+1) - S(n) - 1 - (S(n) - S(n-1) - 1) = S(n)$

$S(n+1) - 2S(n) + S(n-1) = S(n)$, *so, we obtain:*

$$S(n+1) = 3S(n) - S(n-1); n \in \mathbb{N} \Rightarrow \lambda^2 = 3\lambda - 1 \Leftrightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \wedge \lambda_2 = \frac{3-\sqrt{5}}{2}$$

$$S(n) = c_1\left(\frac{3+\sqrt{5}}{2}\right)^n + c_2\left(\frac{3-\sqrt{5}}{2}\right)^n \quad (VII)$$

Obviously, $S(0) = 1$ *and* $S(1) = 2$

We have: $c_1 + c_2 = 1$ *and* $c_1\left(\frac{3+\sqrt{5}}{2}\right) + c_2\left(\frac{3-\sqrt{5}}{2}\right) = 2$

$$c_2 = 1 - c_1$$

$$c_1\left(\frac{3+\sqrt{5}}{2}\right) + (1-c_1)\left(\frac{3-\sqrt{5}}{2}\right) = 2$$

$$c_1\left(\frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2}\right) + \frac{3-\sqrt{5}}{2} = 2$$

$$c_1\sqrt{5} = 2 - \frac{3-\sqrt{5}}{2}$$

$$c_1\sqrt{5} = \frac{1+\sqrt{5}}{2} \Leftrightarrow c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$c_2 = 1 - c_1$; *so,* $c_2 = \frac{2\sqrt{5}-1-\sqrt{5}}{2\sqrt{5}}$

$$c_2 = \frac{\sqrt{5}-1}{2\sqrt{5}} = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

Now, $S(n) = c_1\left(\frac{3+\sqrt{5}}{2}\right)^n + c_2\left(\frac{3-\sqrt{5}}{2}\right)^n$

$$= \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n + (-1)\cdot\frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^n$$

$$S(n)=\frac{1}{\sqrt{5}}\left(\left(\left(\frac{1+\sqrt{5}}{2}\right)\cdot\left(\frac{1+\sqrt{5}}{2}\right)^2\right)^n-\frac{1-\sqrt{5}}{2}\left(\left(\left(\left(\frac{1-\sqrt{5}}{2}\right)^2\right)\right)^n\right)\right)$$

$$S(n)=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}\right)$$

By Binet's formula $\Rightarrow S(n)=F_{2n+1}$

Moreover, by (******): $P(n)=S(n+1)-S(n)-1$

$$P(n)=F_{2(n+1)+1}-F_{2n+1}-1$$

$$P(n)=F_{2n+3}-F_{2n+1}-1$$

$$P(n)=F_{2n+2}-1$$

Therefore,

$P(n)=\sum_{k=0}^{n}\binom{n+k}{2k-1}=F_{2n+2}-1$ *and*

$$S(n)=\sum_{k=0}^{n}\binom{n+k}{2k}=F_{2n+1}$$

5.50

$$\boldsymbol{x^9+1=(x+1)(x^2+ax+1)(x^2+bx+1)(x^2+cx+1)(x^2+dx+1)}$$

$\forall \boldsymbol{x}\in\mathbb{C}$. **Find:** $\boldsymbol{\Omega=a^6+b^6+c^6+d^6}$

Solution:

Buscamos ias 9 raices de x^9+1

$\Rightarrow \cos\left(\frac{k\pi}{9}\right)+i\sin\left(\frac{k\pi}{9}\right)$*, para* $k=\pm1,\pm3,\pm5,\pm7,9$

Para encontrar factores con coeficientes reaies, multiplicamos ios pares conjugados

$$\left(x-\cos\left(\frac{k\pi}{9}\right)-i\sin\left(\frac{k\pi}{9}\right)\right)\left(x-\cos\left(-\frac{k\pi}{9}\right)-i\sin\left(-\frac{k\pi}{9}\right)\right)=$$

$$=\left(x-\cos\left(\frac{k\pi}{9}\right)\right)^2-\left(i\sin\left(\frac{k\pi}{9}\right)\right)^2=x^2-2\cos\left(\frac{k\pi}{9}\right)x+1$$

Entonces

$$x^9+1=(x+1)\left(x^2-2\cos\left(\frac{\pi}{9}\right)x+1\right)\left(x^2-2\cos\left(\frac{3\pi}{9}\right)x+1\right)\cdot$$

$$\cdot\left(x^2-2\cos\left(\frac{5\pi}{9}\right)x+1\right)\left(x^2-2\cos\left(\frac{7\pi}{9}\right)x+1\right)$$

$$\Omega=\left(-2\cos\left(\frac{\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{3\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{5\pi}{9}\right)\right)^6+\left(-2\cos\left(\frac{7\pi}{9}\right)\right)^6$$

$$\Omega=2^6(\cos^6(20^\circ)+\cos^6(100^\circ)+\cos^6(140^\circ)+\cos^6(60^\circ))$$

Sabemos que si:

i) $a+b+c=0$*, se cumple:*

$$a^6+b^6+c^6=3\left(\frac{a^3+b^3+c^3}{3}\right)^2+2\left(\frac{a^2+b^2+c^2}{2}\right)^3$$

ii) $\cos^2 x+\cos^2(120^\circ+x)+\cos^2(120^\circ-x)=\frac{3}{2}$

iii) $\cos^3 x+\cos^3(120^\circ+x)+\cos^3(120^\circ-x)=\frac{3}{4}\cos 3x$

Entonces como

$\cos x+\cos(120^\circ-x)+\cos(120^\circ+x)=0$*, se cumple i)*

Reemplazando ii) y iii) en i)

$$\cos^6 x+\cos^6(120^\circ-x)+\cos^6(120^\circ+x)=3\left(\frac{3}{4}\cdot\frac{\cos 3x}{3}\right)^2+2\left(\frac{3}{4}\right)^3$$

Si $x=20^\circ$

$$\cos^6(20^\circ)+\cos^6(100^\circ)+\cos^6(140^\circ)=\frac{3}{16}\cdot\cos^2(60^\circ)+\frac{27}{32}$$

Finalmente

$$\Omega=2^6\left[\cos^6(60^\circ)+\frac{3}{16}\cdot\cos^2(60^\circ)+\frac{27}{32}\right]$$

$$\Omega=64\left[\frac{1}{64}+\frac{3}{16}\cdot\frac{1}{4}+\frac{27}{32}\right]$$

$$\Omega=1+3+54=58$$

5.51 Let $\begin{cases} x_0 = 2020 \\ x_{n+1} = 2x_n - n^2 + 2n + 2019 \cdot 2020^n, n = 0, 1, 2, \dots \end{cases}$

Find: $\Omega = x_{2030}$

Solution:

Using the formulas: $S_1 = x + 2^2x^2 + \dots + nx^n = \frac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$,

$$S_2 = x + 2^2x^2 + \dots + n^2x^n$$

$$= \frac{n^2x^{n+3} + x^{n+2}(-2n^2 - 2n + 1) + (n+1)^2x^{n+1} - x^2 - x}{(x-1)^3}$$

We denote $\alpha = 2019 \cdot 2020, b_n = -n^2 + 2n + \alpha^n, x_{n+1} = 2x_n + b_n$

$$x_1 = 2x_0 + b_0, x_2 = 2x_1 + b_1 = 2^2x_0 + 2b_0 + b_1, x_3 = x_2 + b_2$$

$$= 2^3x_0 + 2^2b_0 + x_n =$$

$$= 2^n x_0 + \sum_{k=0}^{n-1} 2^{n-k-1}\, b_k = 2^n x_0 + \sum_{k=0}^{n-1} 2^{n-k-1}\left(-k^2 + 2k + \alpha^k\right)$$

$$x_n = 2^n x_0 + 2^{n-1} \cdot \sum_{k=0}^{n-1} \frac{-k^2 + 2k + \alpha^k}{2^k} =$$

$$= 2^n x_0 + 2^{n-1} \cdot \left[-\sum_{k=0}^{n-1} k^2 \left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{n-1} k \cdot \left(\frac{1}{2}\right)^k + \sum_{k=0}^{n-1} \left(\frac{\alpha}{2}\right)^k\right]$$

$$\Omega = x_{2030} = 2^{2030}x_0 + 2^{2029}\left[-\sum_{k=0}^{2029} k^2 \left(\frac{1}{2}\right)^k + 2\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k + \sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k\right]$$

$$\sum_{k=0}^{2029}\left(\frac{\alpha}{2}\right)^k = 1 + \left(\frac{\alpha}{2}\right) + \cdots \left(\frac{\alpha}{2}\right)^{2029} = \frac{\left(\frac{\alpha}{2}\right)^{2030} - 1}{\frac{\alpha}{2} - 1} = \frac{\alpha^{2030} - 2^{2030}}{2^{2029}(\alpha - 2)}$$

$$\sum_{k=0}^{2029} k\left(\frac{1}{2}\right)^k = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + \cdots + 2029\left(\frac{1}{2}\right)^{2029}$$

$$= \frac{2029\left(\frac{1}{2}\right)^{2031} - 2030\left(\frac{1}{2}\right)^{2030} + \frac{1}{2}}{\frac{1}{4}} =$$

$$=\frac{\frac{2029}{2^{2031}}-\frac{2030}{2^{2030}}+\frac{1}{2}}{\frac{1}{4}}=\frac{2029-4060+2^{2030}}{2^{2029}}=\frac{2^{2030}-2031}{2^{2029}}$$

$$\sum_{k=0}^{2029}k^2\left(\frac{1}{2}\right)^k=$$

$$=\frac{2029\left(\frac{1}{2}\right)^{2032}+(-2\cdot 2029^2-2\cdot 2029+1)\left(\frac{1}{2}\right)^{2031}+2030^2\left(\frac{1}{2}\right)^{2030}-\frac{3}{4}}{-\frac{1}{8}}$$

$$=\left(\frac{2029}{2^{2032}}-\frac{8237739}{2^{2031}}+\frac{4120900}{2^{2030}}-\frac{3}{4}\right)\cdot\left(-\frac{8}{1}\right)=$$

$$=\frac{2029-16475478+16483600-3\cdot 2^{2030}}{2^{2032}}\cdot\left(-\frac{8}{1}\right)=\frac{3\cdot 2^{2030}-10\cdot 151}{2^{2029}}$$

$$\Omega=2^{2030}\cdot 2020+2^{2029}\left[\frac{3\cdot 2^{2030}-10151}{2^{2029}}+\frac{2^{2030}-2031}{2^{2028}}+\frac{\alpha^{2030}-2^{2030}}{2^{2029}(\alpha-2)}\right]=$$

$$=2^{2030}\cdot 2020-3\cdot 2^{2030}+10151+2^{2031}-4062+\frac{\alpha^{2030}-2^{2030}}{\alpha-2}=$$

$$=2^{2030}(2020-3+2)+6089+\frac{\alpha^{2030}-2^{2030}}{\alpha-2}=$$

$$\Omega=2019\cdot 2^{2030}+6089+\frac{20192020^{2030}-2^{2030}}{20192018}$$

$$\Omega=\frac{40767684341\cdot 2^{2030}+122949197602+20192020^{2030}}{20192018}$$

5.52

$$\boldsymbol{x^{2n+1}+1=(x+1)\prod_{k=1}^{n}(x^2+a_ix+1),\forall x\in\mathbb{C}}$$

Find:

$$\boldsymbol{\Omega = \sum_{i=1}^{n} a_i^4 , a_i \in \mathbb{C}, i \in \overline{1, n}}$$

Solution:

$$8\cos^4 x = 3 + 4\cos(2x) + \cos(4x)$$

$$a_1^4 = \left(-2\cos\left(\frac{\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{\pi}{2n+1}\right)$$

$$= 6 + 8\cos\left(\frac{2\pi}{2n+1}\right) + 2\cos\left(\frac{4\pi}{2n+1}\right)$$

$$a_2^4 = \left(-2\cos\left(\frac{3\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{3\pi}{2n+1}\right)$$

$$= 6 + 8\cos\left(\frac{6\pi}{2n+1}\right) + 2\cos\left(\frac{12\pi}{2n+1}\right)$$

$$a_3^4 = \left(-2\cos\left(\frac{5\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{5\pi}{2n+1}\right)$$

$$= 6 + 8\cos\left(\frac{10\pi}{2n+1}\right) + 2\cos\left(\frac{20\pi}{2n+1}\right)$$

$$a_{n-1}^4 = \left(-2\cos\left(\frac{(2n-3)\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{(2n-3)\pi}{2n+1}\right) =$$

$$= 6 + 8\cos\left(\frac{2(2n-3)\pi}{2n+1}\right) + 2\cos\left(\frac{4(2n-3)\pi}{2n+1}\right)$$

$$a_n^4 = \left(-2\cos\left(\frac{(2n-1)\pi}{2n+1}\right)\right)^4 = 16\cos^4\left(\frac{(2n-1)\pi}{2n+1}\right) =$$

$$= 6 + 8\cos\left(\frac{2(2n-1)\pi}{2n+1}\right) + 2\cos\left(\frac{4(2n-1)\pi}{2n+1}\right)$$

$$a_n^4 = 6n + 8\left[\frac{2\cos\left(\frac{2n\pi}{2n+1}\right)\sin\left(\frac{2n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right] + 2\left[\frac{2\cos\left(\frac{4n\pi}{2n+1}\right)\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right]$$

$$a_n^4 = 6n + 8\left[\frac{\sin\left(\frac{4n\pi}{2n+1}\right)}{2\sin\left(\frac{2\pi}{2n+1}\right)}\right] + 2\left[\frac{\sin\left(\frac{8n\pi}{2n+1}\right)}{2\sin\left(\frac{4\pi}{2n+1}\right)}\right] \rightarrow a_n^4 = 6n + 4(-1) + (-1);$$

$$\therefore a_n^4 = 6n - 5$$

5.53 If $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, m, n, p, q \in \mathbb{N}$ then:

$$\frac{F_m^2}{(F_qF_n + F_{q+1}F_p)^2} + \frac{F_n^2}{(F_qF_p + F_{q+1}F_m)^2} + \frac{F_p^2}{(F_qF_m + F_{q+1}F_n)^2} \geq \frac{3}{F_{q+2}^2}$$

Solution:

From Cauchy's inequality

$$\Rightarrow \left(\frac{F_m}{F_2F_n+F_{q+1}F_p}\right)^2 + \left(\frac{F_n}{F_qF_p+F_{q+1}F_m}\right)^2 + \left(\frac{F_p^2}{F_qF_m+F_{q+1}F_n}\right)^2 \geq$$

$$\geq \frac{1}{3}\left(\frac{F_m}{F_qF_n + F_{q+1}F_p} + \frac{F_n}{F_qF_p + F_{q+1}F_m} + \frac{F_p}{F_qF_m + F_{q+1}F_n}\right)^2$$

Then we must show this: $\left(\frac{F_m}{F_qF_n+F_{q+1}F_p} + \frac{F_n}{F_qF_p+F_{q+1}F_m} + \frac{F_p}{F_qF_m+F_{q+1}F_n}\right)^2 \geq \frac{9}{F_{q+2}^2} \Leftrightarrow$

$$\Leftrightarrow \frac{F_m}{F_qF_n+F_{q+1}F_p} + \frac{F_n}{F_qF_p+F_{q+1}F_m} + \frac{F_p}{F_qF_m+F_{q+1}F_n} \geq \frac{3}{F_{q+2}} \quad (1)$$

But from Cauchy's inequality we have $\frac{F_m}{F_qF_n+F_{q+1}F_p} + \frac{F_n}{F_qF_p+F_{q+1}F_m} +$

$$\frac{F_p}{F_qF_m+F_{q+1}F_m} =$$

$$= \frac{F_m^2}{F_mF_qF_n + F_mF_{q+1}F_p} + \frac{F_n^2}{F_nF_qF_p + F_nF_{q+1}F_m} + \frac{F_p^2}{F_pF_qF_m + F_pF_{q+1}F_m} \geq$$

$$\geq \frac{(F_m + F_n + F_p)^2}{F_q(F_mF_n + F_nF_p + F_pF_m) + F_{q+1}(F_mF_p + F_nF_m + F_pF_m)} =$$

$$= \frac{(F_m+F_n+F_p)^2}{(F_mF_n+F_nF_p+F_pF_m)(F_q+F_{q+1})} = \frac{(F_m+F_n+F_p)^2}{(F_mF_n\cdot F_nF_p+F_pF_m)\cdot F_{q+2}} \quad (2)$$

From (1)+(2) we must show: $\frac{(F_m+F_n+F_p)^2}{(F_mF_n+F_nF_p+F_pF_m)F_{q+2}} \geq \frac{3}{F_{q+2}} \Leftrightarrow$

$$\Leftrightarrow (F_m + F_n + F_p)^2 \geq 3(F_mF_n + F_nF_p + F_pF_m) \Leftrightarrow (F_m^2 + F_n^2 + F_p^2)$$

$$\geq F_mF_n + F_nF_p + F_pF_m$$

5.54 $F_0 = 0, F_1 = 1, F_n + F_{n+1} = F_{n+2}, n \in \mathbb{N}$. ***Prove that:***

$$\frac{sin^3 t}{sin\, t \cdot F_n + cos\, t \cdot F_{n+1}} + \frac{cos^3 t}{cos\, t \cdot F_n + sin\, t \cdot F_{n+1}} \geq \frac{1}{F_{n+2}},$$

$$n \in \mathbb{N}^*, t \in \left(0, \frac{\pi}{2}\right)$$

Solution:

$$\sum_{cyc} \frac{\sin^3 t}{\sin t \cdot F_n + \cos t \cdot F_{n+1}} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(\sin^2 t + \cos^2 t)^2}{\sin t\,(\sin t \cdot F_n + \cos t \cdot F_{n+1}) + \cos t\,(\cos t \cdot F_n + \sin t \cdot F_{n+1})} =$$

$$= \frac{1}{F_n + 2 \sin t \cos t \cdot F_{n+1}} \geq \frac{1}{F_n + (\sin^2 t + \cos^2 t) F_{n+1}} = \frac{1}{F_{n+2}}$$

MISCELLANEOUS INEQUALITIES

6.1 Let $a,b,c>0$ *and* $a+b+c=3$. Prove that:

$$a\cdot \arcsin\left(\frac{b}{b+1}\right)+b\cdot \arcsin\left(\frac{c}{c+1}\right)+c\cdot \arcsin\left(\frac{a}{a+1}\right)\le\frac{\pi}{2}$$

Solution:

Given inequality can be written as:

$$\left(\frac{a}{\sum a}\right)\sin^{-1}\left(\frac{b}{b+1}\right)+\left(\frac{b}{\sum a}\right)\sin^{-1}\left(\frac{c}{c+1}\right)+\left(\frac{c}{\sum a}\right)\sin^{-1}\left(\frac{a}{a+1}\right)\overset{(1)}{\le}\frac{\pi}{6}$$

$$Let\,\frac{a}{\sum a}=p_1,\frac{b}{\sum a}=p_2,\frac{c}{\sum a}=p_3.\ Then\ p_1+p_2+p_3=1.\ Now,$$

$$\because f''(x)=-\frac{(3x+2)}{(x+1)^5\left(\frac{2x+1}{(x+1)^2}\right)^{\frac{3}{2}}}<0,\forall x>0\ \because f(x)=\sin^{-1}\left(\frac{x}{x+1}\right),\forall x>0$$

is concave, ∴ by Jensen,

$$\text{LHS of }(1)=p_1f(b)+p_2f(c)+p_3f(a)\overset{(2)}{\le}f(p_1b+p_2c+p_3a)=$$

$$=\sin^{-1}\left(\frac{\frac{ab+bc+bc}{\sum a}}{\frac{\sum ab}{\sum a}+1}\right)=\sin^{-1}\left(\frac{\sum ab}{\sum ab+3}\right)\because 3\left(\sum ab\right)\le\left(\sum a\right)^2=9\therefore\sum ab\le 3$$

$$\therefore 1-\frac{3}{\sum ab+3}\le 1-\frac{3}{3+3}=\frac{1}{2}\Rightarrow\frac{\sum ab}{\sum ab+3}\overset{(3)}{\le}\frac{1}{2}$$

$$(2),(3)\Rightarrow \text{LHS of }(1)\le\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}=\text{RHS of }(1)$$

6.2 Let $n\in\mathbb{N}^*\wedge n\ge 2$ and $x_1,x_2,\dots,x_n\in(0;+\infty)$. Prove:

$$e^n x_1^{\frac{1}{x_1}}x_2^{\frac{1}{x_2}}\dots x_n^{\frac{1}{x_n}}\le e^{x_1+x_2+\dots+x_n}$$

Solution:

$$ex^{\frac{1}{x}}\le e^x,x\in(0,\infty)$$

$$ex^{\frac{1}{x}} \le e^x \Leftrightarrow 1 + \frac{1}{x}\log x \le x \Leftrightarrow x + \log x - x^2 \le 0, x \in (0,\infty)$$

$$f(x) = x + \log x - x^2, x \in (0,\infty), f'(x) = 1 + \frac{1}{x} - 2x, x \in (0,\infty)$$

$$f'(x) = 0 \Rightarrow 1 + \frac{1}{x} - 2x = 0 \Rightarrow x = 1$$

$$f''(x) = -\frac{1}{x^2} - 2 < 0, x \in (0,\infty) \Rightarrow \max\{f(x)|0 < x < \infty\} = f(1) = 0$$

$$\Rightarrow f(x) \le f(1) = 0 \Rightarrow x \le x + \log x - x^2 \le 0, x \in (0,\infty) \Rightarrow$$

$$ex^{\frac{1}{x}} < e^x, x \in (0,\infty) \Rightarrow e^n x_1^{\frac{1}{x_1}} \dots x_n^{\frac{1}{x_n}} \le e^{x_1+\dots+x_n}$$

6.3 If $x, y \ge 0$ then:

$$(e^x + 1)\sqrt{e^y} + (e^y + 1)\sqrt{e^x} \le (e^x + 1)(e^y + 1)$$

Solution:

$$e^x + 1 > 2e^{\frac{x}{2}}, \forall x > 0 \Rightarrow e^x + 1 - e^{\frac{x}{2}} > e^{\frac{x}{2}}, \forall x > 0 \Rightarrow$$

$$\Rightarrow \left(e^x + 1 - e^{\frac{x}{2}}\right)\left(e^y + 1 - e^{\frac{y}{2}}\right) > e^{\frac{x}{2}}e^{\frac{y}{2}}, \forall x, y, > 0 \Rightarrow$$

$$\Rightarrow (e^x + 1)(e^y + 1) > \sqrt{e^y}(e^x + 1) + \sqrt{e^x}(e^y + 1), \forall x, y > 0$$

6.4 If $a, b, c > 0, abc = 1$ then:

$$e^{a^{3a^3}} + e^{b^{3b^3}} + e^{c^{3c^3}} \ge 3e$$

Solution:

for $x > 0$, we get $x^{3x^3} \ge x^{3^{3x^2}} \ge x^{3x} \ge x^3$. Hence for $a, b, c > 0$ and

$abc = 1$, we have:

$$a^{3a^3}b^{3b^3}e^{3c^3} \geq a^3b^3c^3 = (abc)^3 = 1 \Rightarrow e^{a^{3a^3}b^{3b^3}c^{3c^3}} \geq e^{(abc)^3} = e^1$$

$$\Rightarrow e^{\sqrt[3]{a^{3a^3}b^{3b^3}c^{3c^3}}} \geq e^1 \Rightarrow e^{3\sqrt{a^{3a^3}b^{3b^3}c^{3c^3}}} \geq e^1$$

$$\Rightarrow e^{\left(a^{3a^3}+b^{3b^3}+c^{3c^3}\right)} \geq e^3 \Rightarrow$$

$$\Rightarrow \sqrt[3]{e^{\left(a^{3a^3}+b^{3b^3}+c^{3c^3}\right)}} \geq e \Rightarrow e^{a^{3a^3}} + e^{b^{3b^3}} + e^{b^{3b^3}} \geq 3e$$

6.5 $\boldsymbol{A = (a_{ij})_{\substack{1\leq i\leq n,\\ 1\leq j\leq n}}, a_{ij} = 10i + j, n \geq 2, n \in \mathbb{N}^*}$***. Find*** $\boldsymbol{X, Y \in M_n(\mathbb{R})}$

such that: $\boldsymbol{\det X} < 0, \boldsymbol{\det Y} < 0, A + Y = X.$

Solution:

$A = (a_{ij})_{n\times n}$, *where* $a_{ij} = 10i + j$. *Let* $x = (x_{ij})_{n\times n}$, *where* $x_{ij} = a_{ij}$

$$if\, i > j = 0\ if\, i < j, x_{11} = -1 and\, x_{ii} = a_{ii} + 1, \forall i \geq 2$$

$$Let\ Y = (y_{ij})_{n\times n},\ where\ y_{ij} = 0\ if\, i > j = -a_{ij}\ if\, i < j$$

$$y_{11} = -12 = -(a_{11} + 1), y_{ii} = 1\ \forall i \geq 2$$

$$Note\ that\ A + Y = X\ and\ \det(Y) = -12 < 0\ and$$

$$\det(X) = -(23)(34) \dots (10n + n + 1) < 0$$

6.6 ***If*** $\boldsymbol{n \in \mathbb{N}, n \geq 2}$ ***then:***

$$\boldsymbol{\log(n!) + 1 - n < \sum_{k=2}^{n}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{k}\right) < \log(n!)}$$

Solution:

$$For\ k = 2, \ln k - 1 < 0 = \ln 2 - \ln 2 < \tfrac{1}{2} \quad (1)$$

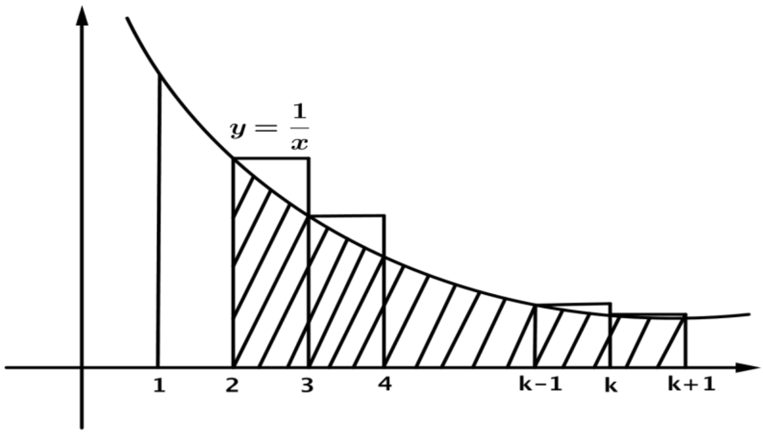


Fig. 1

For $k \geq 3, \ln(k) - 1 < \ln(k) - \ln(2) = \int_2^k \frac{1}{x} dx < \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1}$ *[see Fig. 1]*$< \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k}$ *(2)*

$$\Rightarrow \sum_{k=2}^{n} (\ln k - 1) < \sum_{k=2}^{n} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) \text{ [using (1), (2)]}$$

$$\Rightarrow \ln(n!) - (n-1) < \sum_{k=2}^{n} \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) \quad (3)$$

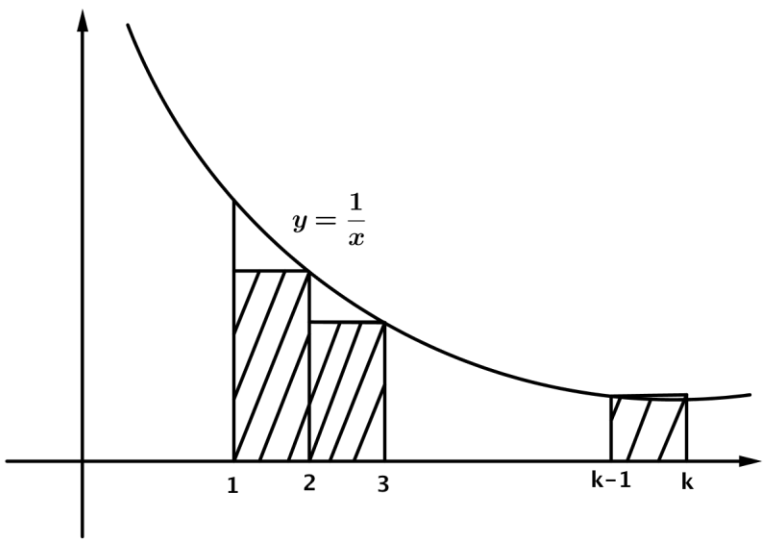


Fig. 2

$$For\ k \geq 2, \ln k = \int_1^k \frac{1}{x} dx > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \quad [see\ Fig.\ 2]$$

$$\Rightarrow \ln(n!) = \sum_{k=2}^{n} \ln k > \sum_{k=2}^{n} \left(\frac{1}{2} + \cdots + \frac{1}{k}\right) \quad (4)$$

From (3), (4) the inequality follows.

6.7 If $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1$ then:

$$\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}} \cdot \tan^{-1}\left(\sqrt[3]{abc}\right)$$

Solution:

Let $f(x) = \frac{\tan^{-1} x}{x}, 0 < x \leq 1$

$$f'(x) = \left(\frac{x}{1+x^2} - \tan^{-1} x\right)\frac{1}{x^2}, 0 < x < 1 = \frac{x - (1+x^2)\tan^{-1} x}{(1+x^2)x^2},$$

$$0 < x < 1.\ Let\ g(x) = x - (1+x^2)\tan^{-1} x, 0 \leq x \leq 1$$

$$g'(x) = 1 - (1+x^2)\frac{1}{1+x^2} - 2x\tan^{-1} x = -2x\tan^{-1} x < 0\ for$$

$$0 < x < 1 \Rightarrow g(x)\ is\ strictly\ decreasing\ on\ [0,1].$$

$$\therefore g(x) < g(0)\ \forall x \in (0,1) \Rightarrow x - (1+x^2)\tan^{-1} x < 0; \forall x \in (0,1)$$

$$Thus,\ f'(x) < 0\ for\ 0 < x < 1 \Rightarrow f(x)\ is\ strictly\ decreasing\ on\ (0,1]$$

$$Now, \frac{\sqrt{3}}{3} \leq a, b, c \leq 1 \Rightarrow \frac{ab+bc+ca}{3} \geq (abc)^{\frac{2}{3}} \Rightarrow$$

$$\sqrt[3]{abc} \leq \left[\frac{1}{2}(ab+bc+ca)\right]^{\frac{1}{2}} \Rightarrow$$

$$\Rightarrow f\left((abc)^{\frac{1}{3}}\right) \geq f\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \Rightarrow \frac{\tan^{-1}(abc)^{\frac{1}{3}}}{(abc)^{\frac{1}{3}}} \geq \frac{\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\Rightarrow \sqrt[3]{abc}\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \le \sqrt{\frac{ab+bc+ca}{3}}\tan^{-1}\left(3\sqrt[3]{abc}\right)$$

6.8 Let $x, y \in (0; +\infty) \wedge x + y = 1$ and $n \in \mathbb{N}^*$.Prove*:*

$$(xy)^n \ge \frac{16^n + 1}{4^n} - \frac{1}{x^n y^n}$$

Solution:

$$Put\ x = \cos^2\theta\,, y = \sin^2\theta\,, 0 < \theta < \frac{\pi}{2}$$

$$P = (xy)^n + (xy)^{-n} = (\cos\theta\sin\theta)^{2n} + (\cos\theta\sin\theta)^{-2n}$$

$$\frac{dp}{d\theta} = (2n)(\cos\theta\sin\theta)^{2n-1}(\cos 2\theta) - 2n(\cos\theta\sin\theta)^{-2n-1}(\cos 2\theta)$$

$$= 2n(\cos 2\theta)(\cos\theta\sin\theta)^{-2n-1}[(\cos\theta\sin\theta)^{4n} - 1]$$

$$As\ \cos\theta\sin\theta > 0, 0 < \cos\theta\sin\theta < 1,$$

$$\frac{dp}{d\theta} < 0\ if\ 0 < \theta < \frac{\pi}{4}$$

$$= 0\ if\ \theta = \frac{\pi}{4}$$

$$> 0\ if\ \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow P\ is\ least\ when\ \theta = \frac{\pi}{4}$$

$$Thus,\ P \ge P\left(\frac{\pi}{4}\right) = \frac{1}{2^{2n}} + 2^{2n} = \frac{16^n+1}{4^n}$$

6.9 If $a \ge 4, b, c \ge 0, a + c \le 2b, x, y, z \in \mathbb{R}$ then:

$$(a-3)(c - x^2 - y^2 - z^2) \le (b - x - y - z)^2$$

Solution:

$$(a-3)(c - x^2 - y^2 - z^2) \overset{(1)}{\le} (b - x - y - z)^2$$

$$(1) \Leftrightarrow c(a-3) - (a-3)\left(\sum x^2\right)$$

$$\leq b^2 + \left(\sum x\right)^2 - 2b\left(\sum x\right)$$

$$\Leftrightarrow (a-3)\left(\sum x^2\right) + \left(\sum x\right)^2 - 2b\left(\sum x\right) + b^2 - c(a-3) \overset{(2)}{\geq} 0$$

$$\because \sum x^2 \geq \frac{(\sum x)^2}{3} \;\&\; a-3 \geq 1 > 0,$$

$$\therefore \textit{LHS of (2)} \geq \left(\frac{a-3}{3}+1\right)\left(\sum x\right)^2 - 2b\left(\sum x\right)$$

$$+b^2 - c(a-3) = \frac{a}{3}\left(\sum x\right)^2 - 2b\left(\sum x\right) + b^2 - c(a-3)$$

$$\overset{(?)}{\geq} 0 \Leftrightarrow a\left(\sum x\right)^2 - 6b\left(\sum x\right) + 3\{b^2 - c(a-3)\} \underset{(3)}{\overset{?}{\geq}} 0$$

$\because a \geq 4 > 0$ & LHS of (3) is a quadratic in $(\sum x)$ & $\because \sum x \in \mathbb{R}$ (as $x, y, z \in \mathbb{R}$), $\therefore$ it suffices to prove that the discriminant is ≤ 0 that is, it suffices to prove:

$$36b^2 - 4a \cdot 3\{b^2 - c(a-3)\} \leq 0 \Leftrightarrow 3b^2 - a\{b^2 - c(a-3)\} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow ac(a-3) - b^2(a-3) \leq 0 \Leftrightarrow (a-3)(ac-b^2) \leq 0$$

$\because a - 3 \geq 1 > 0, \therefore$ *it suffices to prove:* $ac - b^2 \leq 0 \Leftrightarrow 4b^2 \overset{(4)}{\geq} 4ac$

But LHS of (4) $\geq (a+c)^2 (\because 2b \geq a+c; b \geq 0; a+c \geq 4 > 0)$

$\overset{?}{\geq} 4ac \Leftrightarrow (a-c)^2 \geq 0 \rightarrow$ *true* $\Rightarrow$ *(4) is true (proved)*

6.10 If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{(\tan x + \cot x)(\tan y + \cot y)(\tan z + \cot z)}{(\tan x + \cot y)(\tan y + \cot z)(\tan z + \cot x)} \geq 1$$

Solution:

Let $\tan x = a, \tan y = b, \tan z = c \because x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow a, b, c > 0$

So, to prove $\frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)} \geq 1$ *or*

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)$$

$$\Rightarrow abc+\frac{ac}{b}+\frac{bc}{a}+\frac{ab}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{1}{abc} \geq abc+a+c+\frac{1}{b}+b+\frac{1}{c}+\frac{1}{a}+\frac{1}{abc} \Rightarrow$$

$$\Rightarrow \frac{ac}{b}+\frac{bc}{a}+\frac{ab}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab} \geq (a+b+c)+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \Rightarrow$$

$$\Rightarrow \left(\frac{a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2}{abc}\right) \geq \left(\frac{a^2bc+b^2ac+c^2ab+ab+bc+ac}{abc}\right)$$

or $(a^2b^2+b^2c^2+c^2a^2+a^2+b^2+c^2) \geq (a^2bc+b^2ac+c^2ab+$

$ab+bc+ac)$ *(1)* $\because$ *we know that*

$$p^2+q^2+r^2 \geq pq+qr+pr$$

Taking $p=ab, q=bc, r=ac$, *we get*

$$a^2b^2+b^2c^2+c^2a^2 \geq a^2bc+b^2ac+c^2ab \quad (2)$$

Taking $p=a, q=b, r=c$

$$a^2+b^2+c^2 \geq ab+bc+ac \quad (3)$$

Adding (2) & (3), we get (1)⇒ (2)+(3)⇒ (1)

So, (1)⇒ $\left(\sum a^2b^2+\sum a^2\right) \geq \left(\sum a^2bc+\sum ab\right)$

This is true

and hence $\frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)} \geq 1$ *or* $\frac{(\tan x+\cot x)(\tan y+\cot y)(\tan z+\cot z)}{(\tan x+\cot y)(\tan y+\cot z)(\tan z+\cot x)} \geq 1$

6.11

$$\boldsymbol{\Omega(x,y)=\sum_{n=1}^{\infty}\frac{2n^2+(2x+2y+5)n+2xy+6x-y}{3^n(n+y)(n+y+1)(n+y+2)}}, \boldsymbol{x,y}>0$$

Prove that:

$$\boldsymbol{\Omega(x,y)\cdot\Omega(y,x)\le\frac{1}{9\sqrt[3]{xy}}}$$

Solution:

$$\Omega(x,y)=\sum_{n}^{\infty}\frac{2n^2+(2x+2y+5)n+2xy+6x-y}{3^n(n+y)(n+y+1)(n+y+2)}=$$

$$=\sum_{n=1}^{\infty}\left(\frac{n+x}{3^{n-1}(n+y)(n+y+1)}-\frac{n+x+1}{3^n(n+y+1)(n+y+2)}\right)$$

$$=\frac{x+1}{(y+1)(y+2)}\Rightarrow$$

$$\Rightarrow\Omega(x,y)\cdot\Omega(y,x)=\frac{1}{x+1+1}\cdot\frac{1}{y+1+1}\le\frac{1}{3\sqrt[3]{x}}\cdot\frac{1}{3\sqrt[3]{y}}$$

$$\Omega(x,y)\cdot\Omega(y,x)\le\frac{1}{9\sqrt[3]{xy}}$$

6.12 If $x>0$ then:

$$\left(e^{x^2}+e^{(x+3)^2}\right)\left(\frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\right)>\left(e^{(x+1)^2}+e^{(x+2)^2}\right)\left(\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}\right)$$

Solution:

$$Let\ f(x)=e^{x^2}-e^{(x+1)^2}\ \forall x>0$$

$$f'(x)\overset{(1)}{=}-2\left((x+1)e^{(x+1)^2}-xe^{x^2}\right)$$

$$Now,\ (x+1)^2(\ln e)>x^2(\ln e)(\because 2x+1>0\ as\ x>0)$$

$$\Rightarrow e^{(x+1)^2}\overset{(i)}{>}e^{x^2}$$

$$Also,\ x+1\overset{(ii)}{>}x\ \&\because x>0\ \therefore (i).(ii)\Rightarrow(x+1)e^{(x+1)^2}-xe^{x^2}>0\Rightarrow$$

$$\Rightarrow f'(x)<0\ (by\ (1))\ \therefore f(x)\downarrow\therefore e^{x^2}-e^{(x+1)^2}<e^{(x+2)^2}-e^{(x+3)^2}\Rightarrow$$

$$\Rightarrow e^{x^2}+e^{(x+3)^2}\overset{(a)}{>}e^{(x+1)^2}+e^{(x+2)^2}$$

$$Now,\ let\ g(x)=\frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}\ \forall x>0$$

$$g'(x)=\frac{e^{x+1}(e^x+1)^2-e^x(e^{x+1}+1)^2}{(e^{x+1}+1)^2(e^x+1)^2}=\frac{et(t+1)^2-t(et+1)^2}{(et+1)^2(t+1)^2}\ (t=e^x)$$

$$=\frac{et(t^2+2t+1)-t(e^2t^2+2et+1)}{(et+1)^2(t+1)^2}=\frac{t(1-e)(et^2-1)}{(et+1)^2(t+1)^2}<0$$

$$\left(\because et^2>1\ as\ t=e^x>1\ (\because x>0)\right)\therefore g(x)\downarrow$$

$$\therefore \frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}>\frac{1}{1+e^{x+2}}-\frac{1}{1+e^{x+3}}\Rightarrow$$

$$\Rightarrow \frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\overset{(b)}{>}\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}$$

(a).(b) ⇒ given inequality is true (proved)

6.13 $\boldsymbol{\Omega(x)=-\frac{1}{2}+4\sum_{n=1}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)},x\in\mathbb{R}}$**. If** $\boldsymbol{a\in(0,1),b>1}$

then: $\boldsymbol{\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<1+\Omega(a)\cdot\Omega(b)}$

Solution:

$$\Omega(x)=-1+4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)},x\in\mathbb{R}$$

$$\Rightarrow\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<\Omega(a)\Omega(b)+1,0<a<1,b>1$$

$$\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{n+2}\cdot\frac{1}{(n+1)(n+3)}$$

$$=\frac{1}{n+2}\left(\frac{1}{2(n+1)}-\frac{1}{2(n+3)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\right)$$

$$S_1=\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{2}\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}\right)$$

$$=\frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{(N+2)(N+3)}\right)=\frac{1}{4}$$

$$\frac{n}{(n+1)(n+2)(n+3)}=\frac{1}{(n+2)(n+3)}-\frac{1}{(n+1)(n+2)(n+3)}$$

$$S_2=\sum_{n=0}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}=\sum_{n=0}^{\infty}\frac{1}{(n+2)(n+3)}-S_1$$

$$=\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-S_1=\lim_{N\to\infty}\left(\frac{1}{2}-\frac{1}{N+3}\right)-\frac{1}{4}=\frac{1}{4}$$

$$\Omega(x)=-1+4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)}=-1+4(S_2\cdot x+S_1)=$$

$$=-1+4\left(\frac{1}{4}+\frac{1}{4}x\right)=x$$

$$\left(\Omega(a)\right)^{\Omega(b)}+\left(\Omega(b)\right)^{\Omega(a)}<\Omega(a)\Omega(b)+1$$

$$\Leftrightarrow a^b+b^a-ab-1<0, 0<a<1, b>1$$

$$Let\ f(b)=a^b+b^a-ab-1, 0<a<1, b>1$$

$$f'(b)=a^b\log(a)+ab^{a-1}-a=a^b\log(a)-a(1-b^{a-1})<0\ \forall b>1\Rightarrow f\searrow(1,\infty)$$

$$For\ b>1\Leftrightarrow f(b)<f(1)=0\Leftrightarrow a^b+b^a<1+ab, 0<a<1, b>1$$

6.14 ***Let x, y, z be positive real numbers such that: $x^2+y^2+z^2=3$.***

Find the minimum of value:

$$\boldsymbol{P=\frac{x}{\sqrt{y}+\sqrt{z}}+\frac{y}{\sqrt{z}+\sqrt{x}}+\frac{z}{\sqrt{x}+\sqrt{y}}}$$

Solution:

Let $x, y, z>0$ *such that* $x^2+y^2+z^2=3$. *Find Min*: $P=\sum\frac{x}{\sqrt{y}+\sqrt{z}}$

By Cauchy-Schwarz we have: $P=\sum\frac{x^2}{x\sqrt{y}+x\sqrt{z}}\geq\frac{(x+y+z)^2}{\sum x\sqrt{y}+\sum y\sqrt{x}}\geq$

$$\frac{(x+y+z)^2}{2\sqrt{(x+y+z)(xy+yz+zx)}}$$

Let $t=x+y+z$ *then* $0<t\leq 3$ *and* $xy+yz+zx=\frac{t^2-3}{2}$. *We will prove that:*

$$\frac{t^2}{2\sqrt{t\cdot\frac{t^2-3}{2}}}\geq\frac{3}{2}\Leftrightarrow t^4\geq\frac{9(t^3-3t)}{2}\Leftrightarrow t(2t^3-9t^2+27)\geq 0\Leftrightarrow$$

$$t(t-3)^2(2t+3)\geq 0\ \text{(true)}$$

$$\text{So, } P\geq\frac{3}{2}\Rightarrow P_{Min}=\frac{3}{2}\Leftrightarrow x=y=z=1.$$

6.15 If $x,y\in\mathbb{R}$ then:

$$\frac{5\sin^2 x}{1+\cos^2 x}+\frac{5\cos^2 x\cdot\sin^2 y}{1+\sin^2 x+\cos^2 x\cdot\cos^2 y}+\frac{5\cos^2 x\cdot\cos^2 y}{1+\sin^2 x+\cos^2 x\cdot\sin^2 y}\geq 3$$

Solution:

$$\frac{5\sin^2 x}{1+1-\sin^2 x}=\frac{5\sin^2 x}{2-\sin^2 x};$$

$$\frac{5\cos^2 x\cdot\sin^2 y}{1+\sin^2 x+\cos^2 x\cdot(1-\sin^2 y)}=\frac{5\cos^2 x\cdot\sin^2 y}{2-\cos^2 x\cdot\sin^2 y}$$

$$\frac{5\cos^2 x\cdot\cos^2 y}{1+\sin^2 x+\cos^2 x\cdot(1-\cos^2 y)}=\frac{5\cos^2 x\cdot\cos^2 y}{2-\cos^2 x\cdot\cos^2 y}$$

We take the function $f(x)=\frac{5x}{2-x}$, *this function is convex,*

$f''(x)=\frac{20}{(2-x)^3}>0$ *then by Jensen's inequality, we have*

$$\frac{f(\sin^2 x) + f(\cos^2 x \sin^2 y) + f(\cos^2 x \cos^2 y)}{3} \geq f\left(\frac{\sin^2 x + \cos^2 x \sin^2 y + \cos^2 x \cos^2 y}{3}\right)$$

or $f(\sin^2 x) + f(\cos^2 x \sin^2 y) + f(\cos^2 x \cos^2 y) \geq 3 \cdot f\left(\frac{1}{3}\right)$

(since $\sin^2 x + \cos^2 x \sin^2 y + \cos^2 x \cdot \cos^2 y = 1$*)*

$f\left(\frac{1}{3}\right) = \frac{5 \cdot \frac{1}{3}}{2 - \frac{1}{3}} = 1$*, we have:*

$$f(\sin^2 x) + f(\cos^2 x \cdot \sin^2 y) + f(\cos^2 x \cdot \cos^2 y) \geq 3$$

6.16 If $a, b, c > 0$ then:

$$\frac{9 + 4a + 4a^2}{1 + a} + \frac{9 + 4b + 4b^2}{1 + b} + \frac{9 + 4c + 4c^2}{1 + c} \geq 24$$

Solution:

$$f: (0, \infty) \to \mathbb{R}, f(a) = \frac{9 + 4a + 4a^2}{1 + a}, f'(a) = \frac{(2a + 5)(2a - 1)}{(1 + a)^2}$$

$$min(f(a)) = f\left(\frac{1}{2}\right) = 8 \to f(a) \geq 8$$

$$f(a) + f(b) + f(c) \geq 8 + 8 + 8 = 24$$

6.17 If $a, b, c, d \in \mathbb{N} - \{0\}, a > b > c > d$ then:

$$bd(2^a - 1)(2^c - 1) > ac(2^b - 1)(2^d - 1)$$

Solution:

$$bd(2^a - 1)(2^c - 1) > ac(2^b - 1)(2^d - 1) \quad (1)$$

$$(1) \Rightarrow \frac{2^a - 1}{a} \cdot \frac{2^c - 1}{c} > \frac{2^b - 1}{b} \cdot \frac{2^d - 1}{d}$$

$$denote\ f(x)=\frac{2^x-1}{x}$$

$$we\ prove\ that\ f\ increasing\ function$$

$$f'(x)=\frac{2^x\cdot\ln 2\cdot x-2^x+1}{x}=\frac{2^x(\ln 2^x-1)+1}{x^2}>0\Rightarrow f\uparrow$$

$$then\ we\ have\ \otimes\begin{cases}\frac{2^a-1}{a}>\frac{2^b-1}{b} & (2)\\ \frac{2^c-1}{c}>\frac{2^d-1}{d} & (3)\end{cases}\Rightarrow f(a)\cdot f(c)>f(b)\cdot f(d)$$

6.18 If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{x(\cos x+\cos z)+y(\cos y+\cos x)+z(\cos z+\cos y)}{x(\cos x+\cos y)+y(\cos y+\cos z)+z(\cos z+\cos x)}\geq 1$$

Solution:

$$x\cos x+x\cos z+y\cos y+y\cos x+z\cos z+z\cos y$$

$$\geq x\cos x+x\cos y+y\cos y+$$

$$+y\cos z+z\cos z+z\cos x$$

$$x\cos z+y\cos x+z\cos y\geq x\cos y+y\cos z+z\cos x$$

$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos y)\geq 0$$

$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos z+\cos z-\cos y)\geq 0$$

$$x(\cos z-\cos b)+b(\cos x-\cos z)-z(\cos x-\cos z)-z(\cos z-\cos y)\geq 0$$

$$(y-z)(\cos x-\cos z)+(x-z)(\cos z-\cos y)\geq 0$$

$$(x-z)(\cos z-\cos y)\geq(z-y)(\cos x-\cos z)$$

$$If\ x=z\Rightarrow 0\geq 0\ true.\ If\ z=y\Rightarrow 0\geq 0\ true.$$

$$If\ x\neq z,z\neq y,x-z>0\ and\ z-y>0\Rightarrow y<z<x\Rightarrow$$

$$\cos z<\cos y,\cos x<\cos z$$

$$\Rightarrow (x-z)(\cos y-\cos z)\le (z-y)(\cos z-\cos x)$$

$$\frac{\cos y-\cos z}{z-y}\le \frac{\cos z-\cos x}{x-z}\Big|\cdot(-1)$$

$$\frac{\cos y-\cos z}{y-z}\ge \frac{\cos z-\cos x}{z-x};\quad f(x)=\cos x$$

$$T.\ Lagrange\ [x,z],[y,z],f'(x)=-\sin x$$

$$-\sin c_1\ge -\sin c_2,\sin c_1\le \sin c_2$$

$$(\exists)c_1\in(y,z),(\exists)c_2\in(z,x),y<z<x\Rightarrow c_1<c_2\Rightarrow \sin c_1<\sin c_2$$

true.

6.19 For $0<a<b<1\wedge m,n\in\mathbb{N}\wedge \boldsymbol{m}\ge\boldsymbol{n}\ge 2$. Prove:

$$\frac{\boldsymbol{b}\sqrt[m]{\boldsymbol{b}}-\boldsymbol{a}\sqrt[m]{\boldsymbol{b}}}{\boldsymbol{b}\sqrt[n]{\boldsymbol{b}}-\boldsymbol{a}\sqrt[n]{\boldsymbol{a}}}\ge\frac{\boldsymbol{mn}+\boldsymbol{n}}{\boldsymbol{mn}+\boldsymbol{m}}$$

Solution:

$$\frac{b\sqrt[m]{b}-a\sqrt[m]{a}}{b\sqrt[n]{b}-a\sqrt[n]{a}}\ge\frac{mn+n}{mn+m}\Leftrightarrow$$

$$\frac{m}{m+1}\left(b\sqrt[m]{b}-a\sqrt[m]{a}\right)\ge\frac{n}{n+1}\left(b\sqrt[n]{b}-a\sqrt[n]{a}\right)$$

$$\Leftrightarrow \int_a^b\sqrt[m]{x}\,dx\ge\int_a^b\sqrt[n]{x}\,dx\Leftrightarrow x^n\ge x^m\Leftrightarrow\left(\frac{1}{x}\right)^{m-n}\ge 1, \textit{which is true}$$

$$\because 1\ge x>0$$

6.20 If $\boldsymbol{x}\in\left(0,\frac{\pi}{2}\right),\boldsymbol{n}\in\mathbb{N},\boldsymbol{n}\ge 3$ then:

$$\prod_{k=3}^{n}\sqrt[k]{\boldsymbol{\sin^k x+\cos^k x}}\ge 2^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$$

Solution:

$$For\ k \geq 3.\ Let\ f_k(x) = (\sin^k x + \cos^k x)^{\frac{1}{k}}, 0 < x < \frac{\pi}{2}$$

$$\ln f_k(x) = \frac{1}{k}\ln(\sin^k x + \cos^k x)$$

$$\frac{1}{f_k(x)}f_k'(x) = \frac{1}{k}\cdot\frac{k[\sin^{k-1} x\cos x - \cos^{k-1} x\sin x]}{\sin^k x + \cos^k x} \Rightarrow$$

$$\Rightarrow f_k'(x) = \frac{(\sin x\cos x)(\sin^{k-2} x - \cos^{k-2} x)}{\sin^k x + \cos^k x}f_k(x)$$

$$f_k'(x) < 0\ for\ 0 < x < \frac{\pi}{4}$$

$$= 0\ for\ x = \frac{\pi}{4}$$

$$> 0\ for\ \frac{\pi}{4} < x < \frac{\pi}{2}$$

$$\therefore f_k(x)\ attains\ its\ minimum\ value\ at\ x = \frac{\pi}{4} \Rightarrow f_k(x) \geq \left(\frac{2}{2^{\frac{k}{2}}}\right)^{\frac{1}{k}} =$$

$$2^{\frac{1}{k}-\frac{1}{2}} \Rightarrow$$

$$\Rightarrow \prod_{k=3}^n f_k(x) \geq 2^{a_n}\ where\ a_n = \frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\frac{n-2}{2} = 1+\frac{1}{2}+\frac{1}{3}+$$

$$\cdots - \frac{1}{n} - \frac{n+1}{2}$$

$$Thus\ \prod_{k=3}^n(\sin^k x + \cos^k x)^{\frac{1}{k}} \geq 2^{1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$$

6.21 If $x, y, z \in \mathbb{R}$ then:

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 x}}+\frac{1}{e^{\cos^2 y}}+\frac{1}{e^{\cos^2 z}} > 3\left(\frac{1}{2}+\frac{\sqrt{e}}{e}\right)$$

Solution:

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}} \geq 2\sqrt{\frac{1}{e^{\sin^2 x+\cos^2 x}}} = \frac{2}{\sqrt{e}} \Rightarrow \frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}} \geq \frac{2}{\sqrt{e}} > \frac{1}{2}+\frac{1}{\sqrt{e}}$$

$$(because\ \frac{1}{\sqrt{e}} > \frac{1}{2} \Leftrightarrow$$

$2 > \sqrt{e} \Leftrightarrow 4 > e)$; $\frac{1}{e^{\sin^2 y}} + \frac{1}{e^{\cos^2 y}} > \frac{1}{2} + \frac{1}{\sqrt{e}}$; $\frac{1}{e^{\sin^2 t}} + \frac{1}{e^{\cos^2 z}} > \frac{1}{2} + \frac{1}{\sqrt{e}} \Rightarrow$

$$\sum\left(\frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\cos^2 x}}\right) > 3\left(\frac{1}{2} + \frac{1}{\sqrt{e}}\right)$$

6.22 If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}, \boldsymbol{a}, \boldsymbol{b} \geq 2$ then:

$$(2a-1)(3a-1)\cdot \ldots \cdot (a^2-1) + (2b-1)\cdot(3b-1)\cdot \ldots \cdot (b^2-1) > 2\sqrt{\frac{a!\cdot b!\cdot a^2\cdot b^b}{ab\cdot \sqrt[ab]{a^b\cdot b^a}}}$$

Solution:

Consider $f(x) = \ln(1-x) + x$. *Clearly* $f''(x) = -\frac{1}{(x-1)^2}$ *so* f *is concave. Thus the function*

$x \to \frac{(f(x)-f(0))}{(x-0)}$ *is decreasing on* $(0,1)$. *Thus, for* $x \in (0,1)$ *and* $n \geq 2$ *we have:* $\frac{f\left(\frac{x}{n}\right)}{\frac{x}{n}} > \frac{f(x)}{x}$.

Consequently $f\left(\frac{x}{n}\right) - \frac{f(x)}{n} > 0$. *Applying this to* $x = \frac{1}{k}$ *and adding we get:*

$$0 < \sum_{k=2}^{m} f\left(\frac{1}{kn}\right) - \frac{1}{n}\sum_{k=2}^{m} f\left(\frac{1}{k}\right) = \sum_{k=2}^{m} \ln\left(1 - \frac{1}{kn}\right) + \frac{1}{n}\sum_{k=2}^{m}\frac{1}{k} - \frac{1}{n}\sum_{k=2}^{m}\ln\frac{k-1}{k} - \frac{1}{n}\sum_{k=2}^{m}\frac{1}{k}$$

$$= \frac{\ln m}{n} + \ln\frac{\prod_{k=2}^{m}(kn-1)}{n^{m-1}m!} = \ln\left(\frac{m^{\frac{1}{n}}}{n^{m-1}m!}\prod_{k=2}^{m}(kn-1)\right)$$

So, we have proved that for integers $n, m \geq 2$ *the next inequality holds:*

$$\prod_{k=2}^{m}(kn-1) > \frac{n^m m!}{n\cdot m^{\frac{1}{m}}} \quad (1)$$

Applying (1) with $n = m = a$ *and* $n = m = b$ *and using the AM-GM inequality we get*

$$\prod_{k=2}^{a}(ka-1)+\prod_{k=2}^{b}(kb-1)\geq 2\sqrt{\prod_{k=2}^{a}(ka-1)\cdot\prod_{k=2}^{b}(kb-1)}$$

$$>2\sqrt{\frac{a^a a!}{a\cdot a^{\frac{1}{a}}}\cdot\frac{b^b b!}{b\cdot b^{\frac{1}{b}}}}$$

Which is equivalent to the proposed inequality.

6.23 If $\boldsymbol{m,n\in\mathbb{N},a,b,c>0,u\geq 0}$ – fixed then:

$$\sum(\boldsymbol{m+a^{m+1}})\left(\boldsymbol{n+\frac{1}{(b+c+u)^{m+1}}}\right)\geq\frac{\boldsymbol{3(m+1)(n+1)(a+b+c)}}{\boldsymbol{2(a+b+c)+3u}}$$

Solution:

$$m+a^{m+1}=1+1+\cdots+1+a^{m+1}\geq(m+1)\sqrt[m+1]{1\cdot 1\cdot\ldots\cdot 1\cdot a^{m+1}}$$

$$m+a^{m+1}\geq(m+1)\,a\quad (1)$$

$$n+\frac{1}{(b+c+u)^{n+1}}=1+1+\cdots+1+\frac{1}{(b+c+n)^{n+1}}\geq$$

$$(n+1)\sqrt[n+1]{\frac{1\cdot 1\cdot\ldots\cdot 1}{(b+c+u)^{n+1}}}\Rightarrow n+\frac{1}{(b+c+u)^{n+1}}\geq\frac{n+1}{(b+c+u)}\quad (2)$$

From (1) and (2) ineqality becomes:

$$\sum(m+a^{m+1})\left(n+\frac{1}{(b+c+u)^{n+1}}\right)\geq(m+1)(n+1)\sum\frac{a}{b+c+u}.$$

We must show this: $\sum\frac{a}{b+c+u}\geq\frac{3(a+b+c)}{2(a+b+c)+3u}$ *(3). From Cauchy's inequality* $\Rightarrow\sum\frac{a}{b+c+u}=\sum\frac{a^2}{a(b+c+u)}\cdot\sum(ab+ac+au)\geq(a+b+c)^2\Rightarrow$

$$\Rightarrow\sum\frac{a}{b+c+u}\geq\frac{(a+b+c)^2}{2(ab+bc+ac)+(a+b+c)^u}\quad (4)$$

From (3)+(4) we must show: $\frac{(a+b+c)^2}{2(ab+ac+bc)+(a+b+c)u} \geq \frac{3(a+b+c)}{2(a+b+c)+3u} \Leftrightarrow$

$$\Leftrightarrow \frac{(a+b+c)}{2(ab+ac+bc)+(a+b+c)u} \geq \frac{3}{2(a+b+c)+3u} \Leftrightarrow$$

$$\Leftrightarrow 2(a+b+c)^2 + 3u(a+b+c)$$

$$\geq 6(ab+ac+bc) + 3u(a+b+c) \Leftrightarrow$$

$$\Leftrightarrow (a+b+c)^2 \geq 3(ab+ac+bc) \Leftrightarrow a^2+b^2+c^2 \geq ab+ac+bc$$

(true)

6.24 If $a, b, c > 1$ then:

$$\frac{1}{\log_a c + 2\log_a b} + \frac{1}{\log_b a + 2\log_b c} + \frac{1}{\log_c b + 2\log_c a} \geq 1$$

Solution:

$$\sum_{cyc} \frac{1}{\log_a c + 2\log_a b} = \sum_{cyc} \frac{\log a}{\log c + 2\log b} = \sum_{cyc} \frac{(\log a)^2}{\log a \log c + 2\log a \log b} \geq$$

$$\geq \frac{\left(\sum_{cyc} \log a\right)^2}{3\sum_{cyc} \log a \log b} \geq 1$$

6.25 If $x, y, z \in \mathbb{R}, x + y + z = 0$ then:

$$\frac{|2x+3| + |2y+3| + |2z+3| + 9}{2} \geq |x-3| + |y-3| + |z-3|$$

Solution:

$$|2x+3+2y+3+2z+3| + \sum_{cyc(x,y,z)} |2x+3| \overset{HLAWKA}{\geq} \sum_{cyc(x,y,z)} |2x+3+2y+3|$$

$$|2(x+y+z)+9| + \sum_{cyc(x,y,z)} |2x+3| \geq 2\sum_{cyc(x,y,z)} |x+y+3|$$

$$\frac{1}{2}\left(\sum_{cyc(x,y,z)}|2x+3|+9\right)\geq\sum_{cyc(x,y,z)}|-z+3|=\sum_{cyc(x,y,z)}|x-3|$$

6.26 If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(1-\tan x)}{1+\tan x}\right|\leq\sqrt{2}$$

Solution:

$$Let\ P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(1-\tan x)}{1+\tan x}\right|$$

$$P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(\cos x-\sin x)}{\cos x\frac{(\sin x+\cos x)}{\cos x}}\right|$$

$$P=\left|\frac{2-(\cos x-\sin x)^2}{(\sin x+\cos x)}\right|;\ P=\left|\frac{2-1+\sin 2x}{\sin x+\cos x}\right|$$

$$P=\left|\frac{1+\sin 2x}{\sin x+\cos x}\right|;P=\left|\frac{(\sin x+\cos x)^2}{\sin x+\cos x}\right|P=|\sin x+\cos x|$$

$$P=\sqrt{2}\left|\sin\left(x+\frac{\pi}{4}\right)\right|\leq\sqrt{2}$$

6.27 If $x,y,z>0$ then:

$$\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}}+\frac{3\sqrt{3}}{\sqrt{2(x+y+z)}}\geq 2\sqrt{2}\left(\frac{1}{\sqrt{x+2y+z}}+\frac{1}{\sqrt{y+2x+z}}+\frac{1}{\sqrt{x+2z+y}}\right)$$

Solution:

$$f:(0,\infty)\to(0,\infty),f(a)=a^{-\frac{1}{2}},f'(a)=-\frac{1}{2}a^{-\frac{3}{2}},f''(a)=\frac{3}{4}a^{-\frac{5}{2}}>0,f-convexe$$

$$\frac{1}{3}\sum f(a)+f\left(\frac{a+b+c}{3}\right)\overset{POPOVICIU}{\geq}\frac{2}{3}\sum f\left(\frac{a+b}{2}\right)$$

$$a=x+y, b=y+z, c=z+x$$

$$\frac{1}{3}\sum f(x+y)+f\left(\frac{2x+2y+2z}{3}\right)\overset{POPOVICIU}{\geq}\frac{2}{3}\sum f\left(\frac{x+2y+z}{2}\right)$$

$$\frac{1}{3}\sum\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{\frac{2(x+y+z)}{3}}}\geq\frac{2}{3}\sum\frac{1}{\sqrt{\frac{x+2y+z}{2}}}$$

$$\sum\frac{1}{\sqrt{x+y}}+\frac{3\sqrt{3}}{\sqrt{2(x+y+z)}}\geq 2\sqrt{2}\sum\frac{1}{\sqrt{x+2y+z}}$$

6.28 If $a<b<c<d<e<f<g<h, a,b,c,d,e,f,g,h\in\mathbb{R}$ then:

$$(a+b+c+d+e+f+g+h)^2\geq 16(ah+bg+cf+de)$$

Solution:

$\because a<b<c<d<e<f<g<h$, *we can consider* $b=a+x, c=a+x+y$,

$d=a+x+y+z, e=a+x+y+z+u, f=a+x+y+z+u+v$,

$g=a+x+y+z+u+v+w, h=a+x+y+z+u+v+w+t$,

where $x,y,z,u,v,w,t>0$ $\therefore$ *by these substitutions, given inequality transforms into:*

$$(8a+7x+6y+5z+4u+3v+2w+t)^2$$
$$-16a(a+x+y+z+u+v+w+t)-$$
$$-16(a+x)(a+x+y+z+u+v+w)$$
$$-16(a+x+y)(a+x+y+z+u+v)-$$

$$-16(a+x+y+z)(a+x+y+z+u)\geq 0$$
$$\Leftrightarrow t^2+8tu+6tv+4tw+14tx+12ty+$$
$$+10tz+16u^2+24uv+16uw+8ux+16uy+24uz+9v^2+12vw$$
$$+10vx+20vy+$$
$$+30vz+4w^2+12wx+24wy+20wz+x^2+4xy+6xz+4y^2+$$
$$12yz+9z^2>0\to true \because x,y,z,u,v,w,t>0 \quad \textit{(proved)}$$

6.29 If $b>a\geq e$ then :

$$\frac{\pi^b-\pi^a}{e\cdot log\frac{b}{a}}>\pi^e$$

Solution:

$$f:[a,b]\to\mathbb{R}, f(x)=\pi^x, f'(x)=\pi^x\cdot log\pi\ ,$$

$$g:[a,b]\to\mathbb{R}, g(x)=logx, g'(x)=\frac{1}{x}$$

$$\frac{\pi^b-\pi^a}{log\frac{b}{a}}=\frac{f(b)-f(a)}{g(b)-g(a)}\overset{CAUCHY}{\triangleq}\frac{f'(c)}{g'(c)}=\frac{\pi^c log\pi}{\frac{1}{c}}>c\cdot\pi^c>e\cdot\pi^e\ ,$$

$$b>c>a\geq e$$

6.30 If $x,y,z\geq 0, x+y+z=\frac{\pi}{4}$ then:

$$\sum tan\,x\,(1+tan\,y)\geq 2\sqrt{tan\,x\cdot tan\,y\cdot tan\,z}$$

Solution:

$$\to x+y+z=\frac{\pi}{4}; x+y=\frac{\pi}{4}-2; \tan(x+y)=\tan\left(\frac{\pi}{4}-z\right);$$

$$\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{1-\tan z}{1+\tan z}$$

$$(\tan x+\tan y)(1+\tan z)=(1-\tan z)(1-\tan x\tan y)$$

$$\Rightarrow \tan x+\tan y+\tan z+$$

$$+\tan x\tan y+\tan y\tan z+\tan x\tan z=1+\tan x\tan y\tan z\Rightarrow$$

$$\Rightarrow \sum \tan x\,(1+\tan y)=1+\tan x\tan y\tan z.$$

Using AM-GM: $\frac{1+\tan x\tan y\tan z}{2}\geq\sqrt{\tan x\cdot\tan y\cdot\tan z}\Rightarrow 1+\tan x+$

$$\tan y\tan z\geq 2\sqrt{\tan x\tan y\tan z}\Rightarrow$$

$$\sum \tan x\,(1+\tan y)\geq 2\sqrt{\tan x\tan y\tan z}$$

6.31 $-1<a,b,c<1, \Omega(a)=\int_0^{\pi}\frac{\log(1+a\cos x)}{\cos x}dx$. ***Prove that:***

$$\frac{1}{\pi^2}\left(\Omega^2(a)+\Omega^2(b)+\Omega^2(c)\right)\geq\sum(\sin^{-1}a\cdot\sin^{-1}b)$$

Solution:

Let $f(a)=\frac{\ln(1+a\cos x)}{\cos x}$ *is a continuous function in a* $\Rightarrow \Omega'(a)=$

$$\int_0^{\pi}\frac{1}{1+a\cos x}dx$$

$$\left.\begin{array}{c} Let\ \tan\frac{x}{2}=t\Rightarrow x=2\arctan t\Rightarrow dx=\frac{2}{1+t^2}dt \\ x=0\Rightarrow t=0; x=\pi\Rightarrow t=\infty \end{array}\right\}\Rightarrow$$

$$\Omega'(a)=\int_0^{\infty}\frac{1}{1+a\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$= 2\int_0^\infty \frac{1}{1+t^2+a-at^2}dt = 2\int_0^\infty \frac{1}{(1-a)t^2+1+a}dt$$

$$= \frac{2}{1-a}\int_0^\infty \frac{1}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2}dt =$$

$$= \frac{2}{1-a}\cdot\frac{1}{\sqrt{\frac{1+a}{1-a}}}\arctan\frac{t}{\sqrt{\frac{1+a}{1-a}}}\Bigg|_0^\infty = \frac{\pi}{\sqrt{1-a^2}} \Rightarrow$$

$$\left.\begin{matrix}\Omega(a) = \pi\int\frac{1}{\sqrt{1-a^2}}da = \pi\arcsin a + c\\ But\ \Omega(a) = 0 \Rightarrow c = 0\end{matrix}\right\} \Rightarrow \Omega(a) = \pi\arcsin a \Rightarrow we$$

must show:

$\sum(\arcsin a)^2 \ge \sum\arcsin a\cdot\arcsin b$, *which its true because*

$$\sum x^2 \ge \sum xy$$

6.32 If $x, y, z \in \mathbb{R}, x+y+z=0$ then:

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)} \ge \left(1+\frac{1}{\sqrt{e^x}}\right)\left(1+\frac{1}{\sqrt{e^y}}\right)\left(1+\frac{1}{\sqrt{e^z}}\right)$$

Solution:

$$\sqrt{1+e^x} \overset{QM-AM}{\gtreqqless} \frac{1}{\sqrt{2}}\left(1+\sqrt{e^x}\right) \to \prod\sqrt{1+e^x} \ge \frac{1}{2\sqrt{2}}\prod\left(1+\sqrt{e^x}\right) \leftrightarrow$$

$$2\sqrt{2}\prod\sqrt{1+e^x} \ge \frac{1}{\sqrt{e^{x+y+z}}}\cdot\prod\left(1+\sqrt{e^x}\right), (x+y+z=0) \leftrightarrow$$

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)} \ge \prod\left(1+\frac{1}{\sqrt{e^x}}\right)$$

6.33 If $x, y, z > 0$ then:

$$\frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

Solution:

$$a = y+z, b = z+x, c = x+y, s = x+y+z, S = \sqrt{xyx(x+y+z)}$$

$$s \overset{MITRINOVIC}{\leq} \frac{3\sqrt{3}R}{2} \leftrightarrow \frac{sS}{4RS} \leq \frac{3\sqrt{3}}{8} \leftrightarrow \frac{sS}{abc} \leq \frac{3\sqrt{3}}{8}$$
$$\leftrightarrow \frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

6.34 If $x, y, z \in \mathbb{R}, x+y+z = 0$ then:

$$4^x + 4^y + 4^z \geq 2(2^{x+y} + 2^{y+z} + 2^{z+x}) - 3$$

Solution:

$$f: \mathbb{R} \to \mathbb{R}, f(x) = 4^x, f''(x) = 4^x log^2 4 > 0, f - convexe$$

By Popoviciu's inequality:

$$\frac{1}{3}\sum f(x) + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\sum f\left(\frac{x+y}{2}\right) \leftrightarrow$$

$$\leftrightarrow \frac{1}{3}\sum 4^x + 4^0 \geq \frac{2}{3}\sum 4^{\frac{x+y}{2}} \leftrightarrow \sum 4^x \geq 2\sum 2^{x+y} - 3$$

6.35 If $a, b, c > 0, a+b+c = 3, 0 \leq x \leq 1$ then:

$$a\left(\frac{b}{a}\right)^x + b\left(\frac{c}{b}\right)^x + c\left(\frac{a}{c}\right)^x + b\left(\frac{a}{b}\right)^x + c\left(\frac{b}{c}\right)^x + a\left(\frac{c}{a}\right)^x \leq 6$$

Solution:

Because $a+b+c = 3 \Rightarrow \exists m, n, p > 0$ *such that:* $a = \frac{3m}{m+n+p}, b = \frac{3n}{m+n+p}, c = \frac{3p}{m+n+p}$. *Inequality becomes:*

$$\frac{m}{m+n+p}\cdot\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\cdot\left(\frac{m}{p}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{m}{n}\right)^x+$$
$$\frac{p}{m+n+p}\cdot\left(\frac{n}{p}\right)^x+\frac{m}{m+n+p}\cdot\left(\frac{p}{m}\right)^x\le 2 \qquad (1)$$

Let $f:(0,+\infty)\to\mathbb{R}, f(\alpha)=\alpha^x; f'(\alpha)=x\alpha^{x-1}, f''(\alpha)=$
$x(x-1)\alpha^{x-2}\Rightarrow f''(x)<0$, *we use Jensen's generalization:*
$p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\le f(p_1x_1+p_2x_2+p_3x_3)$ *with*
$p_1,p_2,p_3>0\wedge p_1+p_2+p_3=1$. *Let* $p_1=\frac{m}{m+n+p}, p_2=\frac{n}{m+n+p}, p_3=$
$\frac{p}{m+n+p}, x_1=\frac{n}{m}$,

$$x_2=\frac{p}{n}, x_3=\frac{m}{p}\Rightarrow\frac{m}{m+n+p}\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{m}{p}\right)^x\le$$
$$\left(\frac{n+p+m}{m+n+p}\right)^x=1 \qquad (2)$$

Let $p_1=\frac{n}{m+n+p}, p_2=\frac{p}{m+n+p}, p_3=\frac{m}{m+n+p}, x_1=\frac{m}{n}, x_2=\frac{n}{p}, x_3=\frac{p}{m}\Rightarrow$

$$\Rightarrow\frac{n}{m+n+p}\left(\frac{m}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{n}{p}\right)^x+\frac{m}{\frac{m}{+n+p}}\cdot\left(\frac{p}{m}\right)^x\le\left(\frac{m+n+p}{m+n+p}\right)^x=1 \qquad (3)$$

From (2)+(3) $\Rightarrow$ *(1) its true.*

6.36 If $x,y\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{(\sin^2x+\sin^2y)^{\sin^2x+\sin^2y}\cdot(\cos^2x+\cos^2y)^{\cos^2x+\cos^2y}}{(\sin x)^{2\sin^2x}\cdot(\sin y)^{2\sin^2y}\cdot(\cos x)^{2\cos^2x}\cdot(\cos y)^{2\cos^2y}}\le 4$$

Solution:

Let $\sin^2x=a,\sin^2y=b,\cos^2x=c,\cos^2y=d$

Then, given inequality $\Leftrightarrow\frac{(a+b)^{a+b}(c+d)^{c+d}}{a^ab^bc^cd^d}\overset{(1)}{\le}4$

$$Now,\ \sqrt[a+b]{a^a b^b} \overset{weighted\ GM-HM}{\geq} \frac{a+b}{\frac{a}{a}+\frac{b}{b}} = \frac{a+b}{2} \Rightarrow a^a b^b \overset{(a)}{\geq} \frac{(a+b)^{a+b}}{2^{a+b}}.$$

$$Similarly, c^c d^d \overset{(b)}{\geq} \frac{(c+d)^{c+d}}{2^{c+d}}$$

$$(a).(b) \Rightarrow a^a b^b c^c d^d \geq \frac{(a+b)^{a+b}\cdot(c+d)^{c+d}}{2^{a+b+c+d}} = \frac{(a+b)^{a+b}(c+d)^{c+d}}{2^{(\sin^2 x+\cos^2 x)+(\sin^2 y+\cos^2 y)}} =$$

$$\frac{(a+b)^{a+b}(c+d)^{c+d}}{4} \Rightarrow \frac{(a+b)^{a+b}(c+d)^{c+d}}{a^a b^b c^c d^d} \leq 4 \Rightarrow (1)\ is\ true\ (Proved)$$

6.37 If $x, y > 0, x + 2y \leq 5, 3x + y \geq 7, (x + 2y)(3x + y) \geq 20$ then: $4x + 3y \geq 9$

Solution:

$$Let\ y = tx;\ t > 0. We\ have:$$

$$\left.\begin{array}{l} x + 2y \leq 5 \Leftrightarrow x \leq \dfrac{5}{1+2t} \\ 3x + y \geq 7 \Leftrightarrow x \geq \dfrac{7}{3+t} \end{array}\right\} \Rightarrow \frac{7}{3+t} \leq x \leq \frac{5}{1+27} \Rightarrow 9t \leq 8 \Rightarrow t \leq \frac{8}{9}$$

$$(x + 2y)(3x + y) \geq 20 \Leftrightarrow x^2 \geq \frac{20}{(1+2t)(3+2t)} \Rightarrow$$

$$x^2 \geq \frac{324}{175};\ \forall t \leq \frac{8}{9} \Rightarrow x \geq \frac{18\sqrt{7}}{35}\ (1)$$

$$We\ need\ to\ prove:\ 4x + 3y \geq 9 \Leftrightarrow x \geq \frac{9}{4+3t}.\ In\ fact: \frac{7}{3+t} \geq \frac{9}{4+3t} \Leftrightarrow$$

$$\frac{12t+1}{(3+t)(4+3t)} > 0\ \ (true)$$

$$\Rightarrow x \geq \frac{7}{3+t} \geq \frac{9}{4+3t};\ \forall t \in \left(0, \frac{8}{9}\right]\ and\ t \leq \frac{8}{9} \Rightarrow \frac{7}{3+t} \geq \frac{9}{5} > \frac{18\sqrt{7}}{35}\ \ (true)$$

6.38 If $0 < x < \frac{\pi}{2}$ then:

$$\pi \cdot e^{\sum_{k=1}^{n} \log\left(\cos\left(\frac{x}{2^k}\right)\right)} > 2$$

Solution:

$$For\ 0 < x < \frac{\pi}{2}; 0 < \cos\left(\frac{x}{2^k}\right) < 1, \forall k \in \mathbb{N}.\ Let$$

$$a_n = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right)\dots\cos\left(\frac{x}{2^n}\right)$$

$$Note\ a_{n+1} < a_n \Rightarrow < a_n > is\ a\ strictly\ decreasing\ sequence.\ Also$$

$$2^n \sin\left(\frac{x}{2^n}\right)a_n = 2^{n-1}\left[2\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^n}\right)\right]\cos\left(\frac{x}{2^{n-1}}\right)\dots\cos\left(\frac{x}{2}\right) =$$

$$= 2^{n-2}\left[2\sin\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^{n-1}}\right)\right]\dots\cos\left(\frac{x}{2}\right)$$

$$= \dots = \sin x \Rightarrow a_n = \frac{\sin(x)}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin x}{x} \cdot \frac{\frac{n}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}(1) = \frac{\sin x}{x}$$

$$As < a_n > is\ strictly\ increasing\ and\ \lim_{n\to\infty} a_n = \frac{\sin x}{x}$$

$$a_n > \frac{\sin x}{x}; \forall n \in \mathbb{N}\quad (1)$$

$$\left[\frac{\sin x}{x} = g/b(a_n)\right]$$

$$Also, for\ 0 < x < \frac{\pi}{2}$$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x) < 0 \Rightarrow \frac{\sin x}{x}\ is\ strictly$$

$$decreasing\ on\ \left(0, \frac{\pi}{2}\right] \Rightarrow$$

$$\Rightarrow \frac{\sin x}{x} > \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}\ for\ 0 < x < \frac{\pi}{2}\ (2)$$

$$\text{From (1), (2): } a_n > \frac{2}{\pi}, \forall n \in \mathbb{N}. \text{ Now,}$$

$$\sum_{k=1}^{n} \log\left(\cos\frac{x}{2^k}\right) = \log a_n > \log\left(\frac{2}{\pi}\right) \Rightarrow \prod e^{\sum_{k=1}^{n} \log\cos\left(\frac{x}{2^k}\right)}$$

$$> \prod e^{\log\left(\frac{2}{\pi}\right)} = 2$$

6.39 For $b > a \geq 1 \wedge n \in \mathbb{N} \wedge n \geq 2$. Prove:

$$\prod_{k=1}^{n} \frac{b^{2k+1} - a^{2k+1}}{b^{2k} - a^{2k}} \geq \frac{(2n+1)!}{4^n (n!)^2}$$

Solution:

$$\text{We know } x^{2k} \geq x^{2k-1} \text{ for all } x \geq 1$$

$$\int_a^b x^{2k} \geq \int_a^b x^{2k-1}\, dx \Rightarrow \frac{b^{2k+1} - a^{2k+1}}{b^{2k} - a^{2k}} \geq \frac{2k+1}{2k}$$

$$\Rightarrow \prod_{k=1}^{n}\left(\frac{b^{2k+1} - a^{2k+1}}{b^{2k} - a^{2k}}\right) \geq \prod_{k=1}^{n}\left(\frac{2k+1}{2k}\right) = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)}{2^n \cdot n!}$$

$$= \frac{1\cdot 2\cdot 3\cdot\ldots\cdot 2n\cdot(2n+1)}{2^n\cdot n!(2\cdot 4\cdot 6\cdot\ldots\cdot 2n)} = \frac{(2n+1)!}{4^n (n!)^2} \text{ (proved)}$$

6.40 In acute ΔABC the following relationship holds:

$$\frac{1}{\pi}(A\tan^\alpha A + B\tan^\alpha B + C\tan^\alpha C) \geq \sqrt{3^\alpha}$$

Solution:

$$\text{WLOG: } A \leq B \leq C \to$$

$$\tan A \leq \tan B \leq \tan C \to \tan^\alpha A \leq \tan^\alpha B \leq \tan^\alpha C$$

$$\sum A tan^{\alpha} A \overset{CEBYSHEV}{\geq} \frac{1}{3}\sum A \sum tan^{\alpha} A = \frac{\pi}{3}\sum tan^{\alpha} A \leftrightarrow$$

$$\leftrightarrow \frac{1}{\pi}\sum A tan^{\alpha} A \geq \frac{1}{3}\sum tan^{\alpha} A \overset{JENSEN}{\geq} \frac{1}{3}\cdot 3tan^{\alpha}\left(\frac{A+B+C}{3}\right)$$
$$= tan^{\alpha}\frac{\pi}{3} = 3^{\frac{\alpha}{2}}$$

6.41 If $a, b, c \in (0, 1], x, y > 0$ then:

$$\frac{3}{2}\log(x^2+y^2) > (a+b+c)\log x + (3-a-b-c)\log y$$

Solution:

If $a, b, c \in (0; 1], x, y > 0$ *then* $\frac{3}{2}\log(x^2+y^2) > (a+b+c)\log x +$

$$(3-a-b-c)\log y \quad (1)$$

$$\textit{Case 1. } \log\left(\frac{x}{y}\right) > 0$$

We have

(1) $\Rightarrow (a+b+c-3)\cdot(\log x - \log y) + 3\log x < \frac{3}{2}\log(x^2+y^2) \Rightarrow$

$$\Rightarrow (a+b+c-3)\cdot\log\left(\frac{x}{y}\right) + 3\log x < \frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right) > 0$ *and* $a+b+c-3 \leq 0$ *so*

$$(a+b+c-3)\cdot\log\left(\frac{x}{y}\right) \leq 0$$

$$\Rightarrow (a+b+c-3)\cdot\log\left(\frac{x}{y}\right) + 3\log x \leq 3\log x$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2) > \frac{3}{2}\log(x^2) = 3\log x$. *So,*

$$(a+b+c-3)\cdot\log\left(\frac{x}{y}\right) + 3\log x < \frac{3}{2}\log(x^2+y^2) \Rightarrow \textit{(1) true}$$

$$\textit{Case 2. } \log\left(\frac{x}{y}\right) < 0$$

We have (1)

$$\Rightarrow (a+b+c)\cdot(\log x-\log y)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$$

$$\Rightarrow (a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)$$

We have $\log\left(\frac{x}{y}\right)<0$ *and* $a+b+c>0$ *so,*

$$(a+b+c)\cdot\log\left(\frac{x}{y}\right)<0\Rightarrow(a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<3\log y$$

On the other hand, we have $\frac{3}{2}\log(x^2+y^2)>\frac{3}{2}\log(y^2)=3\log y$

So $(a+b+c)\cdot\log\left(\frac{x}{y}\right)+3\log y<\frac{3}{2}\log(x^2+y^2)\Rightarrow$ *(1) true*

Therefore, we have QED.

6.42 If $\alpha\geq 2$ *then* $\sum_{k=1}^{\infty}(\xi(\alpha k)-1)\leq\frac{3}{4}$ where ξ denote the Riemann function.

Solution:

For $\alpha\geq 2$ *prove that* $\sum_{k\geq1}(\zeta(\alpha k)-1)\leq\frac{3}{4}$ *where* ζ *is the Riemann zeta function.*

Clearly the function $\alpha\to\sum_{k\geq1}(\zeta(\alpha k)-1)$ *is decreasing on* $[2,\infty)$ *so*

$$\sum_{k\geq1}(\zeta(\alpha k)-1)\leq\sum_{k\geq1}(\zeta(2k)-1)=\sum_{k\geq1}\left(\sum_{j\geq2}\frac{1}{j^{2k}}\right)=\sum_{j\geq2}\left(\sum_{k\geq1}\frac{1}{j^{2k}}\right)$$

$$=\sum_{j\geq2}\frac{1}{j^2-1}=\frac{1}{2}\sum_{j\geq2}\left(\frac{2j-1}{j(j-1)}-\frac{2j+1}{(j+1)j}\right)=\frac{3}{4}$$

6.43 ***If*** $\boldsymbol{n}\in\mathbb{N}^*,\boldsymbol{m}\in\mathbb{R}^*,\boldsymbol{x_k}>0,k\in\overline{\boldsymbol{1},\boldsymbol{n}}$ ***then:***

$$\sum_{k=1}^{n}\left((\tan^{-1}x_k)^{m+1}+\left(\tan^{-1}\frac{1}{x_k}\right)^{m+1}\right)\geq\frac{n\cdot\pi^{m+1}}{2^{2m+1}}$$

Solution:

$a^{m+1}+b^{m+1}\geq\frac{(a+b)^{m+1}}{2^m},\forall a,b>0,n\in\mathbb{N}^*$ *(demonstration by induction)*

$$\Rightarrow(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\geq\frac{\left(\arctan x_k+\arctan\frac{1}{x_k}\right)^{m+1}}{2^m}\quad (1)$$

But $\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$ *(2)*, $\forall x>0$, *because function* $f:(0,+\infty)\to\mathbb{R}$,

$f(x)=\arctan x+\arctan\frac{1}{x},f'(x)=0\Rightarrow f(x)=const=k$, *but* $f(1)=\frac{\pi}{2}\Rightarrow k=\frac{\pi}{2}$

From (1)+(2)$\Rightarrow(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\geq\frac{\left(\frac{\pi}{2}\right)^{m+1}}{2^m}\Rightarrow$

$$(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\geq\frac{\pi^{m+1}}{2^{2m+1}}\quad (3)$$

From (3) $\Rightarrow\sum_{k=1}^{n}\left[(\arctan x_k)^{m+1}+\left(\arctan\frac{1}{x_k}\right)^{m+1}\right]\geq\frac{n\pi^{m+1}}{2^{2m+1}}$

6.44 For $\boldsymbol{a},\boldsymbol{b}\in[\mathbf{1};+\infty)\wedge\boldsymbol{m},\boldsymbol{n}\in\mathbb{N}^*\wedge\boldsymbol{m}\geq\boldsymbol{n}\geq\mathbf{2}$**. Prove:**

$$\frac{\sum_{k=0}^{m}a^{m-k}b^k}{\sum_{l=0}^{n}a^{n-l}b^l}\geq\frac{m+1}{n+1}$$

Solution:

If $a=b$

$$\sum_{k=0}^{m} a^{m-k} b^k = (m+1)a^m$$

and

$$\sum_{k=0}^{n} a^{n-k} b^k = (n+1)a^n \therefore \frac{\sum_{k=0}^{m} a^{m-k} b^k}{\sum_{k=0}^{n} a^{n-k} b^k} = \frac{m+1}{n+1} a^{m-n} \geq \frac{m+1}{n+1}$$

$$[\because a \geq 1, m-n \geq 0]$$

If $a \neq b$,

$$\sum_{k=0}^{m} a^{m-k} b^k = \frac{a^m\left(\left(\frac{b}{a}\right)^{m+1} - 1\right)}{\frac{b}{a} - 1} = \frac{b^{m+1} - a^{m+1}}{b-a}$$

and

$$\sum_{k=0}^{n} a^{n-k}\, b^k = \frac{b^{n+1} - a^{n+1}}{b-a}$$

$$\therefore S = \frac{\sum_{k=0}^{m} a^{m-k} b^k}{\sum_{k=0}^{n} a^{n-k} b^k} = \frac{b^{m+1} - a^{m+1}}{b^{n+1} - a^{n+1}} = \frac{(m+1)c^m}{(n+1)c^n}$$

[By Cauchy's Mean Value Theorem] for some cycling between a and b.

$$\Rightarrow S = \frac{m+1}{n+1} c^{m-n} \geq \frac{m+1}{n+1} \text{ as } 1 \leq a < c < b \text{ or } 1 \leq b < c < a.$$

6.45 If $a, b, c, d, e, f > 0$ then:

$$\frac{a+b+c}{\sqrt[3]{abc}\left(\frac{d}{e} + \frac{e}{f} + \frac{f}{d}\right)} \leq \frac{\sqrt[3]{def}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)}{d+e+f}$$

Solution:

For all $a, b, c, d, e, f > 0$*, we let* $a = m^3, b = n^3, c = p^3, d = x^3, e = y^3, f = z^3$*. Consider*

$$\frac{(a+b+c)}{\sqrt[3]{abc}\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)} \le \frac{\sqrt[3]{def}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{(d+e+f)}.$$

$$Iff \frac{(a+b+c)(d+e+f)}{\sqrt[3]{abc}\sqrt[3]{def}} \le \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right).$$

$$\left(\sqrt[3]{\frac{a^2}{bc}}+\sqrt[3]{\frac{b^2}{ca}}+\sqrt[3]{\frac{c^2}{ab}}\right)\left(\sqrt[3]{\frac{d^2}{ef}}+\sqrt[3]{\frac{e^2}{fd}}+\sqrt[3]{\frac{f^2}{de}}\right) \le \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right).$$

$$Iff \left(\frac{m^2}{np}+\frac{n^2}{pm}+\frac{p^2}{mn}\right)\left(\frac{x^2}{yz}+\frac{y^2}{zx}+\frac{z^2}{xy}\right) \le \left(\frac{m^3}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3}\right)\left(\frac{x^3}{y^3}+\frac{y^2}{z^2}+\frac{z^2}{x^3}\right) \text{ and}$$

it is to be true because $\frac{x^3}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^3} \ge \frac{x^2}{yz}+\frac{y^2}{xz}+\frac{z^2}{xy}$ *and* $\frac{m^2}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3} \ge$

$\frac{m^2}{np}+\frac{n^2}{pm}+\frac{p^2}{mn}$. *Therefore it is to be true.*

6.46 In ΔABC the following relationship holds:

$$\frac{\left((\mathbf{a}+\mathbf{1})(\mathbf{b}+\mathbf{1})(\mathbf{c}+\mathbf{1})\right)^{\frac{1}{2}}}{\mathbf{e}^{\mathbf{a}+\mathbf{b}+\mathbf{c}}} < \mathbf{1}$$

Solution:

$$\begin{cases} e^{b+c} > e^a > a+1 \\ e^{c+a} > e^b > b+1 \\ e^{a+b} > e^c > c+1 \end{cases}$$

$$\to \prod e^{b+c} > \prod (a+1) \to e^{2a+2b+2c} > \prod (a+1) \to$$

$$\to e^{a+b+c} > \sqrt{(a+1)(b+1)(c+1)} \to \frac{\left((a+1)(b+1)(c+1)\right)^{\frac{1}{2}}}{e^{a+b+c}} < 1$$

6.47 If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\prod \ln(1+\tan^2 x) \cdot \prod \ln(1+\cot^2 y) \le \prod \ln^2\left(\frac{2}{\sin 2z}\right)$$

Solution:

$$For\ 0<x<\frac{\pi}{2};\ln(1+\tan^2 x)\ln(1+\cot^2 x)\le$$

$$\le\left\{\frac{\ln(1+\tan^2 x)+\ln(1+\cot^2 x)}{2}\right\}^2=\left\{\frac{1}{2}\ln(\sec^2 x\csc^2 x)\right\}^2=$$

$$=\left(\ln\left(\frac{2}{\sin 2x}\right)\right)^2$$

$$Now,\ 0<x,y,z<\frac{\pi}{2};\prod\ln(1+\tan^2 x)\prod\ln(1+\cot^2 y)=$$

$$=\prod\ln(1+\tan^2 x)(1+\cot^2 x)\le\prod\left[\ln\left(\frac{2}{\sin 2x}\right)\right]^2$$

6.48 If $x,y\ge 0,n\ge 1,n\in\mathbb{Q}, AM=\frac{x+y}{2}, GM=\sqrt{xy}$ then :

$$\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2\ge AM^{2n}+GM^{2n}$$

Solution:

The power means inequality gives us:

$$\sqrt[n]{\frac{x^n+y^n}{2}}\ge AM\ge GM\leftrightarrow\left(\frac{x^n+y^n}{2}\right)^2\ge AM^{2n}\ge GM^{2n}\rightarrow$$

$$\rightarrow 2\left(\frac{x^n+y^n}{2}\right)^2\ge AM^{2n}+GM^{2n}\rightarrow\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2\ge AM^{2n}+GM^{2n}$$

6.49 If $x,y\in\mathbb{R},\Omega=\begin{vmatrix} sinxsiny & sinxcosy & cosx\\ cosx & sinxsiny & sinxcosy\\ sinxcosy & cosx & sinxsiny\end{vmatrix}$ then:

$$|\Omega|\le 1$$

Solution:

$$\overrightarrow{OA} = sinxsiny\vec{i} + sinxcosy\vec{j} + cosx\vec{k}$$

$$\overrightarrow{OB} = cosx\vec{i} + sinxsiny\vec{j} + sinxcosy\vec{k}$$

$$\overrightarrow{OC} = sinxcosy\vec{i} + cosx\vec{j} + sinxsiny\vec{k}$$

$$\left|\overrightarrow{OA}\right|^2 = \left|\overrightarrow{OB}\right|^2 = \left|\overrightarrow{OC}\right|^2 = sin^2xsin^2y + sin^2xcos^2y + cos^2x =$$

$$= sin^2x(sin^2y + cos^2y) + cos^2x = 1 \rightarrow$$

$$\left|\overrightarrow{OA}\right| = \left|\overrightarrow{OB}\right| = \left|\overrightarrow{OC}\right| = 1$$

$$|\Omega| = \left|\overrightarrow{OA} \cdot \left(\overrightarrow{OB} \text{X} \overrightarrow{OC}\right)\right| \overset{HADAMARD}{\leqq} \left|\overrightarrow{OA}\right| \cdot \left|\overrightarrow{OB}\right| \cdot \left|\overrightarrow{OC}\right| = 1$$

6.50 If $0 < a < b$ then:

$$e^{\frac{1}{b}} < \left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}} < e^{\frac{1}{a}}$$

Solution:

$$Suppose\ 0 < a < b,\ then\ a < \sqrt{ab} < \tfrac{a+b}{2} < b$$

$$Let\ f(x) = \ln x\,,\ x \in \left[\sqrt{ab}, \tfrac{a+b}{2}\right]$$

By the first mean value theorem, there exists $c \in \left(\sqrt{ab}, \frac{a+b}{2}\right)$ *such that*

$$\frac{\ln\left(\frac{a+b}{2}\right) - \ln\sqrt{ab}}{\frac{a+b}{2} - \sqrt{ab}} = \frac{1}{c} \Rightarrow \frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}\ln\left(\frac{a+b}{2\sqrt{ab}}\right) = \frac{1}{c}$$

$$\Rightarrow \left(\tfrac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}} = e^{\frac{1}{c}}\ \ (1)$$

$$But\ a < \sqrt{ab} < c < \tfrac{a+b}{2} < b \Rightarrow \tfrac{1}{b} < \tfrac{1}{c} < \tfrac{1}{a}\ \ (2)$$

From (1), (2), we get

$$e^{\frac{1}{b}} < \left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{(\sqrt{b}-\sqrt{a})^2}} < e^{\frac{1}{a}}$$

6.51 If $P \in \mathbb{R}[x]$ with distinct roots $x_1, x_2, \dots, x_n \in \mathbb{R}, n \in \mathbb{N}^*$ then:

$$\frac{P''(x)}{P(x)} < \left(\frac{P'(x)}{P(x)}\right)^2 + \sum_{k=1}^{n} \frac{P''(x_k)}{P'(x_k)}, \forall x \in \mathbb{R} - \{x_1, x_2, \dots, x_n\}$$

Solution:

$$Let\ P(x) = A(x-x_1)(x-x_2)\dots(x-x_n)$$

$$P'(x) = A(x-x_2)(x-x_3)\dots(x-x_n) + A(x-x_1)(x-x_3)\dots(x-x_n) + \dots + A(x-x_1)(x-x_2)\dots(x-x_{n-1})$$

$$P''(x) = \begin{bmatrix} A(x-x_3)(x-x_4)\dots(x-x_n) \\ +A(x-x_2)(x-x_4)\dots(x-x_n) \\ +\dots+A(x-x_2)\dots(x-x_{n-1}) \end{bmatrix} + \begin{bmatrix} A(x-x_3)(x-x_4)\dots(x-x_n) \\ +A(x-x_1)(x-x_4)\dots(x-x_n) \\ +\dots+A(x-x_1)\dots(x-x_{n-1}) \end{bmatrix} + \dots + \begin{bmatrix} A(x-x_2)\dots(x-x_{n-1}) \\ +A(x-x_1)\dots(x-x_{n-1}) \\ +\dots+A(x-x_1)\dots(x-x_{n-2}) \end{bmatrix}$$

$$\frac{P''(x_1)}{P'(x_1)} = \frac{2}{x_1-x_2} + \frac{2}{x_1-x_3} + \dots + \frac{2}{x_1-x_n}$$

Similarly,

$$\frac{P''(x_r)}{P'(x_r)} = 2\sum_{\substack{j=1 \\ j\neq r}}^{n} \frac{1}{x_r-x_j} \Rightarrow \sum_{r=1}^{n} \frac{P''(x_r)}{P'(x_r)} = 0$$

$$Also,\ \frac{P''(x)}{P(x)} - \left(\frac{P'(x)}{P(x)}\right)^2 = \frac{d}{dx}\left[\frac{P'(x)}{P(x)}\right] = \frac{d}{dx}\left[\frac{d}{dx}\left(\ln\left(P(x)\right)\right)\right] = \frac{d^2}{dx^2}[\ln|A| + \ln|x-x_1| + \dots + \ln|x-x_n|]$$

$$= \frac{d}{dx}\left[\frac{1}{x-x_1}+\frac{1}{x-x_2}+\cdots+\frac{1}{x-x_n}\right]$$
$$= -\left[\frac{1}{(x-x_1)^2}+\frac{1}{(x-x_2)^2}+\cdots+\frac{1}{(x-x_n)^2}\right]<0$$

$$\textit{Hence, } \frac{P''(x)}{P(x)} < \left(\frac{P'(x)}{P(x)}\right)^2 + \sum_{r=1}^{n} \frac{P''(x_r)}{P'(x_r)}$$

6.52 If $a,b,c>0, a+b+c=3, x\in\mathbb{R}$ then:

$$\left(\sqrt[3]{a\sin^2 x}+\sqrt[3]{b\cos^2 x}\right)\left(\sqrt[3]{b\sin^2 x}+\sqrt[3]{c\cos^2 c}\right)\left(\sqrt[3]{c\sin^2 x}+\sqrt[3]{a\cos^2 x}\right)\le 4$$

Solution:

$$\sqrt[3]{a\cdot\sin^2 x\cdot 1}+\sqrt[3]{b\cdot\cos^2 x\cdot 1}\le\left(\sqrt[3]{a}^3+\sqrt[3]{b}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{\sin^2 x}^3+\sqrt[3]{\cos^2 x}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{1}^3+\sqrt[3]{1}^3\right)^{\frac{1}{3}}, \textit{(Holder)}$$

$$\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\le\sqrt[3]{2(a+b)}, \sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le\sqrt[3]{2(a+b)2(b+c)2(a+c)},$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le 2\sqrt[3]{(a+b)(b+c)(a+c)}$$

$$2\sqrt[3]{(a+b)(b+c)(a+c)}\le 2\,\frac{a+b+b+c+c+a}{3}=2\cdot\frac{6}{3}$$
$$=4, \left(M_a\ge M_g\right)$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le 4.$$

6.53 If $a,b,c,d\in\mathbb{R}$ then:

$$\boldsymbol{a+b+c+d \leq \frac{1}{2} + (a+b)(c+d) + a^2 + b^2 + c^2 + d^2}$$

Solution:

$$(a+b+c+d-1)^2 + (a-b)^2 + (c-d)^2 \geq 0$$

$$\Rightarrow a^2 + b^2 + c^2 + d^2 - 2(a+b+c+d) + 1$$

$$+2(ab+bc+cd+ad+ac+bd) + a^2 + b^2 - 2ab + c^2 + d^2 - 2cd \geq 0$$

$$\Rightarrow 2(a^2+b^2+c^2+d^2) - 2(a+b+c+d) + 2(a+b)(c+d) + 1 \geq 0$$

$$\Rightarrow a+b+c+d \leq \frac{1}{2} + (a+b)(c+d) + a^2 + b^2 + c^2 + d^2$$

6.54 If $0 < a \leq b < \frac{\pi}{2}$ then:

$$\boldsymbol{\frac{tanb}{tana} \geq e^{2(b-a)}}$$

Solution:

$$f:[a,b] \to \mathbb{R}, f(x) = ln(tanx)$$

$$f(b) - f(a) \overset{LAGRANGE}{\triangleq} f'(c)(b-a), c \in (a,b) \to ln(tanb) - ln(tana) = \frac{1}{sinccosc}(b-a)$$

$$ln\left(\frac{tana}{tanb}\right) = \frac{2(b-a)}{sin2c} \geq 2(b-a) \to ln\left(\frac{tana}{tanb}\right) \geq lne^{2(b-a)} \to$$

$$\to \frac{tana}{tanb} \geq e^{2(b-a)}$$

6.55 Prove that:

$$\mathbf{2^x + 3^x + 4^x \geq x\ln 24 + 3, \forall x \in \mathbb{R}}$$

Solution:

$$\begin{cases} f(x) = 2^x - x\ln 2 \\ g(x) = 3^x - x\ln 3 \\ h(x) = 4^x - x\ln 4 \end{cases} \to \begin{cases} f'(x) = (2^x - 1)\ln 2 \\ g'(x) = (3^x - 1)\ln 3 \\ h'(x) = (4^x - 1)\ln 4 \end{cases} \to \begin{cases} f(x) \geq f(0) = 1 \\ g(x) \geq g(0) = 1 \\ h(x) \geq h(0) = 1 \end{cases} \to$$

$$f(x) + g(x) + h(x) \geq 3 \to 2^x - x\ln 2 + 3^x - x\ln 3 + 4^x - x\ln 4 \geq 3$$

$$2^x + 3^x + 4^x \geq x\ln 24 + 3, \forall x \in \mathbb{R}$$

6.56 If $\boldsymbol{a, b \in \mathbb{R}, A, B \in M_n(\mathbb{R}), AB = BA}$ then:

$$\mathbf{det(I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB) \geq 0}$$

Solution:

We first show that if $x, y \in M_n(\mathbb{R})$ and $xy = yx$, then

$$\det(x^2 + y^2) \geq 0.$$

Note that $x^2 + y^2 = (x + iy)(x - iy)\ \ [\because xy = yx]$

$$\det(x^2 + y^2) = \det\big((x + iy)(x - iy)\big) = \det(x + iy)\det(x - iy)$$

$$= \det(x + iy)\det(\overline{x + iy}) = \det(x + iy)\overline{\det(x + iy)} = |\det(x + iy)|^2 \geq 0$$

Now, for $a, b \in R, A, B \in M_n(\mathbb{R}), AB = BA$, we have

$$I_n + 2(a^2 + b^2)(A^2 + B^2) + 2(a + b)(A + B) + 8abAB =$$

$$= I_n + [(a + b)^2 + (a - b)^2](A^2 + B^2) + 2(a + b)(A + B)$$

$$+ [(a + b)^2 - (a - b)^2](2AB)$$

$$= I_n + (a + b)^2(A^2 + B^2 + 2AB) + 2(a + b)(A + B)$$

$$+ (a - b)^2(A^2 + B^2 - 2AB)$$

$$= I_n + (a + b)^2(A + B)^2 + 2(a + b)(A + B) + (a - b)^2(A - B)^2$$

$$= [I_n + (a+b)(A+B)]^2 + \left((a-b)(A-B)\right)^2 = x^2 + y^2$$

$$where\ x = I_n + (a+b)(A+B) \in M_n(\mathbb{R})\ and$$

$$y = (a-b)(A-B) \in M_n(\mathbb{R})$$

$$Thus,$$

$$\det\{I_n + 2(a^2+b^2)(A^2+B^2) + 2(a+b)(A+B) + 8abAB\} =$$

$$= \det(x^2+y^2) \geq 0$$

6.57 For $0 < a < b \wedge x_1, x_2, \dots, x_n \in [a;b] \wedge \alpha > 0$. Prove:

$$\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}} \leq a^\alpha + b^\alpha$$

Solution:

$$a \leq x_1, x_2, \dots, x_n \leq b \to a^n \leq \prod_{k=1}^{n} x_k \leq b^n \to a^\alpha \leq \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \leq b^\alpha \to$$

$$\left(a^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)\left(b^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right) \leq 0$$

$$\to (ab)^\alpha - (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 \leq 0 \to$$

$$\left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 + (ab)^\alpha \leq (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \to \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}}$$

$$\leq a^\alpha + b^\alpha$$

6.58 $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{R}[X], n \geq 2$

If $a_0, a_1, \dots, a_n > 0$ then: $P\left(1 + \frac{1}{n}\right) \geq P(1) + \frac{1}{n}P'(1)$

Solution:

$$For\ k\in\mathbb{N}, n\in\mathbb{N}, \left(1+\frac{1}{n}\right)^k \geq 1+\frac{k}{n}$$

$$\therefore P\left(1+\frac{1}{n}\right)=\sum_{k=0}^{n} a_k\left(1+\frac{1}{n}\right)^k \geq \sum_{k=0}^{n} a_k\left(1+\frac{k}{n}\right) \quad [\because a_k>0]$$

$$=\sum_{k=0}^{n} a_k+\frac{1}{n}\sum_{k=1}^{n} k a_k = P(1)+\frac{1}{n}P'(1)$$

6.59 If $a,b,c>0$ then:

$$\boldsymbol{a^a\cdot b^b\cdot c^c \geq \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}\left(\frac{b+c}{2}\right)^{\frac{b+c}{2}}\left(\frac{c+a}{2}\right)^{\frac{c+a}{2}} \geq (abc)^{\frac{a+b+c}{3}}}$$

USA-TST

Solution:

Applying Weighted AM≥GM;

$$\sqrt[a+b]{a^a b^b} \geq \frac{a+b}{2}, \sqrt[b+c]{b^b c^c} \geq \frac{b+c}{2}\ and\ \sqrt[c+a]{c^c a^a} \geq \frac{c+a}{2}$$

$$\Rightarrow \prod_{cyc} a^{2a} \geq \prod_{cyc}\left(\frac{a+b}{2}\right)^{a+b} \Rightarrow \prod_{cyc} a^a \geq \prod_{cyc}\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}$$

Again applying Weighted AM ≥ GM;

$$\prod_{cyc}\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \geq \left(\frac{\sum_{cyc}\left(\frac{a+b}{2}\right)}{\frac{(a+b)/2}{(a+b)/2}+\frac{(b+c)/2}{(b+c)/2}+\frac{(c+a)/2}{(c+a)/2}}\right)^{a+b+c} = \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

$$\geq (abc)^{\frac{a+b+c}{3}}$$

6.60 If $0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$ is an arithmetical progression with common difference d then:

$$\tan^{-1}\frac{d}{1+x_1x_2}+\tan^{-1}\frac{d}{1+x_2x_3}+\cdots+\tan^{-1}\frac{d}{1+x_{n-1}x_n}\leq \ln\sqrt{\frac{x_n}{x_1}}$$

Solution:

$$f(x)=\tan^{-1}x-\frac{\ln x}{2}\to f'(x)=\frac{1}{1+x^2}-\frac{1}{2x}=-\frac{(x-1)^2}{2x(1+x^2)}\leq 0$$
$$\to f-decreasing;\quad x_1\leq x_n\to f(x_1)\geq f(x_n)$$

$$\sum \tan^{-1}\frac{d}{1+x_{k-1}x_k}=\sum \tan^{-1}\frac{x_k-x_{k-1}}{1+x_{k-1}x_k}=\sum(\tan^{-1}x_k-\tan^{-1}x_{k-1})$$

$$=\tan^{-1}x_n-\tan^{-1}x_1\leq \ln\sqrt{\frac{x_n}{x_1}}\leftrightarrow \tan^{-1}x_1-\frac{1}{2}\ln x_1 \tan^{-1}x_n-\frac{1}{2}\ln x_n$$
$$\leftrightarrow f(x_1)\geq f(x_n)$$

6.61 For $a, b \in (0;+\infty) \wedge 0 \leq \theta \leq \pi$. Prove:

$$\frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})}\leq 1+\sin\theta$$

Solution:

$$Consider$$
$$(a^3+b^3)(a^6+b^6)(a^8+b^8)-(a+b)(a^5+b^5)(a^{11}+b^{11})$$
$$=(a^3+b^3)(a^{14}+a^8b^6+a^6b^8+b^{14})$$
$$-(a+b)(a^{16}+a^5b^{11}+a^{11}b^5+b^{16})$$
$$=a^{17}+a^{11}b^6+a^9b^8+a^3b^{14}+b^{17}+a^6b^{11}+a^8b^9+a^{14}b^3-$$
$$-[a^{17}+a^6b^{11}+a^{12}b^5+ab^{16}+b^{17}+a^{11}b^6+b^{12}a^5+a^{16}b]$$
$$=a^9b^8+a^8b^9+a^3b^{14}+a^{14}b^3-a^{12}b^5-a^5b^{12}-ab^{16}-a^{16}b$$

$$= a^9b^5(b^3-a^3)+a^5b^9(a^3-b^3)+ab^{14}(a^2-b^2)+a^{14}b(b^2-a^2)$$

$$= a^5b^5(a^3-b^3)(b^3-a^3)+ab(b^{13}-a^{13})(a^2-b^2) \le 0$$

$$\Rightarrow (a^3+b^3)(a^6+b^6)(a^8+b^8) \le (a+b)(a^5+b^5)(a^{11}+b^{11})$$

$$\Rightarrow \frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})} \le 1 \le 1+\sin\theta$$

$$(0 \le \theta \le \pi)$$

6.62 If $a, b > 0, a \neq b$ then:

$$\mathbf{0} < \frac{\frac{a-b}{\ln a - \ln b} - \sqrt{ab}}{\frac{a+b}{2} - \sqrt{ab}} < \frac{\mathbf{1}}{\mathbf{3}}$$

Solution:

$$Put\ A = \frac{\frac{a-b}{\ln a-\ln b}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}.\ We\ need\ to\ prove\ that\ 0 < A < \frac{1}{3}$$

$$1)\ LEMMA: \frac{a-b}{\ln a-\ln b} > \sqrt{ab}\ when\ a,b>0\ and\ a \neq b$$

$$We\ have\ \frac{a-b}{\ln a-\ln b} > \sqrt{ab} \Rightarrow \frac{\ln a-\ln b}{a-b} < \frac{1}{\sqrt{ab}} \Rightarrow \frac{\ln\left(\frac{a}{b}\right)}{\frac{a}{b}-1} < \sqrt{\frac{b}{a}}\ \ (1)$$

$$Put\ \frac{a}{b} = t\ (t>0, t \neq 1),\ we\ have\ (1) \Rightarrow \frac{\ln t}{t-1} < \frac{1}{\sqrt{t}}\ \ (2)$$

$$Put\ f(t) = \ln t - \frac{t-1}{\sqrt{t}}$$

$$f'(t) = \frac{-\left(\sqrt{t}-1\right)^2}{2\sqrt{t^3}} < 0 \Rightarrow f(t)\ is\ decreasing\ function \Rightarrow f(t) < f(1)$$

$$when\ t > 1\ and$$

$$f(t) > f(1)\ when\ t<1 \Rightarrow f(t) < 0\ when\ t>1\ and\ f(t)>0\ when$$

$$t < 1.$$

$$1.1.)\ If\ t>1.\ We\ have\ (2) \Rightarrow \ln t < \frac{t-1}{\sqrt{t}}\ \ (True)$$

1.2) If $t < 1$. *We have (2)* $\Rightarrow \ln t > \frac{t-1}{\sqrt{t}}$ *(True)*

$$\Rightarrow (1)\ true \Rightarrow \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$$

Applying the lemma $\Rightarrow \frac{a-b}{\ln a - \ln b} > \sqrt{ab}$ *(since* $0 < \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$*)*

On the other hand, by AM-GM inequality, we have $\frac{a+b}{2} - \sqrt{ab} > 0$

(since $a \neq b$*)*

2) We need to prove that $A < \frac{1}{3} \Rightarrow$

$$\Rightarrow \frac{3(a-b)}{\ln a - \ln b} - 3\sqrt{ab} < \frac{a+b}{2} - \sqrt{ab} \Rightarrow \frac{3(a-b)}{\ln a - \ln b} < \frac{a+b}{2} + 2\sqrt{ab}$$

$$\Rightarrow \frac{3\left(\frac{a}{b}-1\right)}{\ln\left(\frac{a}{b}\right)} < \frac{\frac{a}{b}+1}{2} + 2\sqrt{\frac{a}{b}} \quad (3)$$

Put $\frac{a}{b} = t\ (t > 0, t \neq 1)$, *we have (3)* $\Rightarrow \frac{3(t-1)}{\ln t} < \frac{t+1}{2} + 2\sqrt{t}$ *(4)*

Put $g(t) = \frac{t+1}{2} + 2\sqrt{t} - \frac{3(t-1)}{\ln t}$

$$g'(t) = \frac{1}{\sqrt{t}} + \frac{1}{2} + \frac{3(t-1) - 3t \cdot \ln t}{t \cdot \ln^2 t}$$

$$= \frac{2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t}{2t \cdot \ln^2 t}$$

Put $h(t) = 2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t$

$$h'(t) = \frac{\ln t \cdot \left(-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t\right)}{\sqrt{t}}$$

$h'(t) = 0 \Rightarrow \ln t = 0$ *(5) or* $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t = 0$ *(6)*

(5): $\ln t = 0 \Rightarrow t = 1$

(6): $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t = 0 \Rightarrow \ln t = \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

Put $y(t) = \ln t - \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

$y'(t) = \frac{(\sqrt{t}-1)^2}{t(\sqrt{t}+1)^2} > 0 \Rightarrow y(x)$ *is increasing function* $\Rightarrow y(x) = 0$ *has at most 1 root*

On the other hand, we have $y(1) = 0 \Rightarrow t = 1$ *is the root of (6)*

So $h'(t) = 0 \Rightarrow t = 1$

So we have

2.1) $g'(t) < 0$ *when* $t < 1$

So when $t < 1 \Rightarrow g(t)$ *is decreasing function*

$\Rightarrow g(t) > \lim_{t\to 1^+} g(t) \Rightarrow g(t) > 0$

2.2) $g'(t) > 0$ *when* $t > 1$

So when $t < 1 \Rightarrow g(t)$ *is an increasing function* $\Rightarrow g(t) > \lim_{t\to 1^+} g(t)$

So, $g(t) > 0\ \forall t > 0$

$\Rightarrow$ *(4) true* $\Rightarrow$ *(3)* $\Rightarrow A < \frac{1}{3} \Rightarrow$ *Q.E.D*

t	0		1		$+\infty$
$g(t)$	-6	↗	0	↗	$+\infty$

6.63 If $x, y, z > 0$ then:

$$x + y + z \geq \ln\left(\frac{z+2}{(x-1)^2 - 2x + 5}\right) + \ln\left(\frac{y+2}{(z-1)^2 - 2z + 5}\right) + \ln\left(\frac{x+2}{(y-1)^2 - 2y + 5}\right) + 3$$

Solution:

$$f(x)=e^{x-1}(x^2-4x+6) \text{ is convex because } f''(x)=e^{x-1}x^2>0, \forall x>0$$

$$y=f'(1)(x-1)+f(1) \Leftrightarrow y=x+2 \text{ is the tangent line at } \big(1,f(1)\big)$$

so we have: $e^{x-1}(x^2-4x+6)\geq x+2 \overset{x^2-4x+6>0}{\Leftrightarrow} e^{x-1}\geq \frac{x+2}{(x-1)^2-2x+5}$

(1)

Likewise we have $e^{y-1}\geq \frac{y+2}{(y-1)^2-2y+5}$ *(2) and* $e^{z-1}\geq \frac{z+2}{(z-1)^2-2z+5}$ *(3)*

$$\overset{(1).(2).(3)}{\Rightarrow} e^{x-1}y^{y-1}e^{z-1}\geq \frac{(x+2)}{(y-1)^2-2y+5}\cdot\frac{(y+2)}{(z-1)^2-2z+5}\cdot\frac{(z+2)}{(x-1)^2-2x+5} \overset{\ln x\nearrow}{\Leftrightarrow}$$

$$(x-1)+(y-1)+(z-1)\geq \sum_{cyc}\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right) \Leftrightarrow$$

$$x+y+z\geq \textstyle\sum_{cyc}\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3 \ \textit{(proved)}$$

equality holds when $x=y=z=1$.

6.64 If $a,b,c>0, x,y,z>1$ then:

$$\boldsymbol{\log_{y^bz^c} x^a+\log_{z^bx^c} y^a+\log_{x^by^c} z^a\geq \frac{3a}{b+c}}$$

Solution:

$$\textit{Inequality} \Leftrightarrow a\big(\log_{y^bz^c}x+\log_{z^bx^c}y+\log_{x^by^c}z\big)\geq \frac{3a}{b+c} \Leftrightarrow$$

$$\frac{1}{\log_x y^bz^c}+\frac{1}{\log_y z^bx^c}+\frac{1}{\log_z x^by^c}\geq\frac{3}{b+c}$$

$$\Leftrightarrow \frac{1}{b\log_x y+c\log_x z}+\frac{1}{b\log_y z+c\log_y x}+$$

$$+\frac{1}{b\log_z x+c\log_z y}\geq\frac{3}{b+c}\Leftrightarrow$$

$$\frac{\ln x}{b\ln y+c\ln z}+\frac{\ln y}{b\ln z+c\ln x}+\frac{\ln z}{b\ln x+c\ln y}\geq\frac{3}{b+c} \quad (1)$$

$$Let \ln x = m, \ln y = n, \ln z = p, m, n, p > 0$$

$$(1) \Leftrightarrow \frac{m}{bn+cp} + \frac{n}{bp+cm} + \frac{p}{bm+cn} \geq \frac{3}{b+c} \quad (2)$$

Inequality (2) is a generalization of Nesbitt inequality (to prove let

$$bn + cp = x_1,$$

$$bp + cm = x_2 \text{ and } bm + cn = x_3 \text{ and use } x + \frac{1}{\alpha} \geq 2, \forall \alpha > 0$$

6.65 *For* $0 < a < b$. *Prove:* $\frac{e^{b^2} - e^{a^2}}{b-a} \geq (a+b)(ab+1)$.

Solution:

$$Let\ f(x) = 2xe^{x^2}\ for\ all\ x \geq 0$$

$$f'(x) = 2e^{x^2} + 4x^2e^{x^2}, f''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2} \geq 0\ for\ all$$

$x \geq 0$.*Hence* f *is convex* $\therefore$ *applying Hermite – Hadamard Inequality.*

$$\frac{f(a) + f(b)}{2} \geq \frac{1}{b-a}\int_a^b f(x)\,dx \geq f\left(\frac{a+b}{2}\right) \Rightarrow$$

$$\Rightarrow \frac{1}{b-a}\int_a^b 2xe^{x^2}\,dx \geq 2\left(\frac{a+b}{2}\right)e^{\left(\frac{a+b}{2}\right)^2}$$

$$\Rightarrow \frac{e^{b^2} - e^{a^2}}{b-a} \geq (a+b)\left(1 + \left(\frac{a+b}{2}\right)^2\right) \because e^x \geq 1 + x$$

$$\therefore \frac{e^{b^2} - e^{a^2}}{b-a} \geq (a+b)(1+ab) \quad (proved)$$

6.66 *For* $a \geq 1 \wedge b \geq 1$. *Prove:*

$$\frac{\sum_{k=0}^{8} b^{8-k}a^k}{\sum_{k=0}^{7} a^{7-k}b^k} \geq \frac{9}{8}.$$

Solution:

Let $c > 1$. *By the Cauchy's mean value theorem, there exists* $\alpha \in (1, c)$ *such that*

$$\frac{c^9-1}{c^8-1} = \frac{9\alpha^8}{8\alpha^7} = \frac{9}{8}\alpha > \frac{9}{8} \ (1)$$

Case 1 $a = b = 1$, *then*

$$\frac{\sum_{k=0}^{8} b^{8-k}a^k}{\sum_{k=0}^{7} b^k a^{7-k}} = \frac{9}{8}.$$

Case 2 $a \neq b$. *Let* $a > b \geq 1$. *Put* $\frac{a}{b} = c > 1$. *Now,*

$$\sum_{k=0}^{8} b^{8-k}a^k = \frac{b^8(c^9-1)}{c-1} \text{ and } \sum_{k=0}^{7} a^{7-k}\, b^k = \frac{b^7(c^8-1)}{c-1}$$

Thus,

$$\frac{\sum_{k=0}^{8} b^{8-k}\, a^k}{\sum_{k=0}^{7} b^k a^{7-k}} = \frac{b(c^9-1)}{c^8-1} > \frac{9}{8}b \geq \frac{9}{8}$$

$$[\because b \geq 1]$$

6.67 ***For*** $\boldsymbol{a, b, c \in (0; +\infty)}$***. Prove:***

$$\frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}} \geq e^6$$

Solution:

From the graphs of $y = e^x$ *and* $y = x + 1$, *it is clear that:*

$$\forall x, e^x \geq x + 1 \rightarrow \ (1)$$

Choosing $x = a^b - 1$ *in (1), we get:* $e^{a^b} - 1 \geq a^b \Rightarrow \frac{e^{a^b}}{a^b} \overset{(a)}{\geq} e$

Similarly, $\frac{e^{b^c}}{b^c} \overset{(b)}{\geq} e, \frac{e^{c^a}}{c^a} \overset{(c)}{\geq} e, \frac{e^{a^c}}{a^c} \overset{(d)}{\geq} e, \frac{e^{c^b}}{c^b} \overset{(e)}{\geq} e, \frac{e^{b^a}}{b^a} \overset{(f)}{\geq} e$

$$(a)\cdot(b)\cdot(c)\cdot(d)\cdot(e)\cdot(f) \Rightarrow \frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}} \geq e^6$$

6.68 If $a, b, c > 0$ then:

$$\frac{e^a + e^b + e^c}{\sqrt{a} + \sqrt{b} + \sqrt{c}} > 2$$

Solution:

$$f(x) = e^x - 2\sqrt{x}, f'(x) = e^x - \frac{1}{\sqrt{x}}, f''(x) = e^x + \frac{1}{2x\sqrt{x}} > 0$$

$$f(a) + f(b) + f(c) \overset{Jensen}{\geq} 3f\left(\frac{a+b+c}{3}\right) \leftrightarrow$$

$$\leftrightarrow e^a - 2\sqrt{a} + e^b - 2\sqrt{b} + e^c - 2\sqrt{c} \geq 3e^{\frac{a+b+c}{3}} - 6\sqrt{\frac{a+b+c}{3}} >$$

$$> 3\left(\frac{a+b+c}{3} + 1\right) - 6\sqrt{\frac{a+b+c}{3}} = 3\left(\sqrt{\frac{a+b+c}{3}} - 1\right)^2 \geq 0 \leftrightarrow$$

$$e^a - 2\sqrt{a} + e^b - 2\sqrt{b} + e^c - 2\sqrt{c} > 0 \rightarrow \frac{e^a + e^b + e^c}{\sqrt{a} + \sqrt{b} + \sqrt{c}} > 2$$

6.69 ***For Δ ABC have $\widehat{BAC} = \frac{\pi}{2}$, put $\widehat{ABC} = \alpha, \widehat{ACB} = \beta$ and $\theta \geq 2$***

Prove:

$$\frac{2}{(\sqrt{2})^\theta} \leq \sin^\theta \alpha + \sin^\theta \beta \leq 1$$

Solution:

$$\frac{2}{(\sqrt{2})^\theta} \overset{(i)}{\leq} \sin^\theta \alpha + \sin^\theta \beta \overset{(ii)}{\leq} 1$$

$$A = \frac{\pi}{2} \Rightarrow B + C = \frac{\pi}{2} \Rightarrow \alpha + \beta = \frac{\pi}{2} \Rightarrow \sin\beta = \cos\alpha \quad (1)$$

$$\because \alpha+\beta=\frac{\pi}{2}, \therefore 0<\alpha,\beta<\frac{\pi}{2} \Rightarrow 0<\sin\alpha, \sin\beta<1 \because \theta\geq 2$$

$$\therefore \sin^{\theta}\alpha \overset{(a)}{\leq} \sin^2\alpha \ \ \& \ \ \sin^{\theta}\beta \overset{(b)}{\leq} \sin^2\beta=\cos^2\alpha$$

(a)+(b)$\Rightarrow \sin^{\alpha}\alpha+\sin^{\theta}\beta\leq \sin^2\alpha+\cos^2\alpha=1 \Rightarrow$ *(ii) is true (*)*

Let $\alpha=\frac{\pi}{4}+x \ \ \& \ \ \beta=\frac{\pi}{4}-x; \ -\frac{\pi}{4}<x<\frac{\pi}{4}$

$$\therefore \sin\alpha \overset{(2)}{=} \sin\left(\frac{\pi}{4}+x\right)=\frac{\cos x+\sin x}{\sqrt{2}} \quad \&$$

$$\sin\beta \overset{(3)}{=} \cos\left(\frac{\pi}{4}+x\right)=\frac{\cos x-\sin x}{\sqrt{2}} (2), (3)$$

$$\Rightarrow \sin^{\theta}\alpha+\sin^{\theta}\beta=\frac{1}{(\sqrt{2})^{\theta}}\left[\{(\cos x+\sin x)^2\}^{\frac{\theta}{2}}+\{(\cos x-\sin x)^2\}^{\frac{\theta}{2}}\right]$$

$$\overset{(4)}{=} \frac{1}{(\sqrt{2})^{\theta}}\left[(1+\sin 2x)^{\frac{\theta}{2}}+\big(1+(-\sin 2x)\big)^{\frac{\theta}{2}}\right]$$

From Bernoulli's inequality, we have,

$\forall r\geq 1 \ \& \ \forall t>-1, (1+t)^r\geq 1+rt$ *(5)*

$$\because -\frac{\pi}{2}<2x<\frac{\pi}{2}, \therefore -1<\sin 2x<1$$

So, $\because \sin 2x>-1 \ \& \frac{\theta}{2}\geq 1,$

$\therefore (1+\sin 2x)^{\frac{\theta}{2}}\geq 1+\frac{\theta}{2}\cdot\sin 2x$ *(5)*

Again, $\because -\sin 2x>-1 \ \ \& \ \frac{\theta}{2}\geq 1,$

$\therefore \big(1+(-\sin 2x)\big)^{\frac{\theta}{2}}\geq 1+\frac{\theta}{2}(-\sin 2x)$ *(6)*

(5) + (6) along with (4) $\Rightarrow \sin^{\theta}\alpha+\sin^{\theta}\beta\geq \frac{2+\frac{\theta}{2}\sin 2x-\frac{\theta}{2}\sin 2x}{(\sqrt{2})^{\theta}}=$

$\frac{2}{(\sqrt{2})^{\theta}} \Rightarrow$ *(i) is true (*)*

6.70 ***For*** $\mathbf{0} < a < b < 1$***. Prove:***

$$\frac{b\sqrt[3]{b}-a\sqrt[3]{a}}{b\sqrt{b}-a\sqrt{a}} \geq \frac{8}{9}$$

Solution:

$$Let\ f(x)=x^{\frac{4}{3}};\ g(x)=x^{\frac{3}{2}}, a\leq x\leq b.$$

$$By\ the\ Cauchy's\ mean\ value\ theorem\ \exists c\in(a,b),\ s.t$$

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{b^{\frac{4}{3}}-a^{\frac{4}{3}}}{b^{\frac{3}{2}}-a^{\frac{3}{2}}}=\frac{4}{3}\cdot\frac{2}{3}\cdot\frac{c^{\frac{1}{3}}}{c^{\frac{1}{2}}}=\frac{8}{9}\left(\frac{1}{c^{\frac{1}{6}}}\right)$$

$$>\frac{8}{9}\quad\left[\because c^{\frac{1}{6}}<b^{\frac{1}{6}}<1\right]$$

INDEX OF AUTHORS

PROPOSERS

ROVSEN PIRGULIEV	1.4,1.6,1.17,1.21,1.22,1.24,1.25,1.26 1.27,1.28,3.5
SEYRAN IBRAHIMOV	1.5,1.9,5.2,6.29
PEDRO PANTOJA	1.7
NARENDRA BHANDARI	1.8,1.32,5.48
NGUYEN VAN NHO	1.11,1.12,4.1,4.3,6.2,6.8,6.19,6.25 6.27,6.32,6.33,6.34,6.35,6.39,6.40 6.44,6.46,6.54,6.55,6.57,6.61,6.65 6.66,6.67,6.65,6.66,6.67,6.68,6.69, 6.70
.JHOAW CARLOS	1.15,1.29,3.9
DAN NEDEIANU	1.20
VASILE MIRCEA POPA	1.33
JALIL HAJIMIR	1.34,1.42,2.17,3.29
MARIN CHIRCIU	1.38,1.47
NGUYEN VAN CANH	2.7,2.8,2.9,2.10,2.11,2.12,2.13 2.14,2.15,2.16,2.18,2.19,5.51
CHIRFOT CARMEN	3.2
BOGDAN RUBLYOV	3.6
URFAN ALIYEV	3.10
HUNG NGUYEN VIET	3.12
OLEG TURCAN	5.47
ALPASLAN CERAN	5.49
D.M.BĂTINEȚU-GIURGIU	5.53,6.23,6.43,6.64

NECULAI STANCIU	5.54,6.23,6.43,6.64
DIMITRIS KASTRIOTIS	6.1
LAZAROS ZACHARIADIS	6.4,6.63
HOANG LE NHAT TUNG	6.14
ELIEZER OKEKE	6.16,6.48,6.53
MIHALY BENCZE	6.42,6.60
B.G.CARLSON	6.62

SOLVERS

DANIEL SITARU	1.11,1.41,1.42,2.17,4.3,6.25,6.29 6.32,6.33,6.34,6.40,6.46,6.54,6.55 6.57,6.60,6.68
MARIAN URSĂRESCU	1.13,1.18,1.23,2.1,2.6,2.9,3.3 4.5,5.7,5.15,5.18,5.23,5.24,5.28 5.29,5.33,5.34,5.37,5.38,5.40,5.54 6.21,6.23,6.31,6.35,6.43,6.58 6.64
RADU BUTELCĂ	1.1
RAVI PRAKASH	1.2,1.12,1.15,1.16,1.17,1.33,1.35 1.45,1.46,1.49,2.4,2.8,2.10,3.5 4.1,4.4,4.6,4.7,4.8,4.11,5.1,5.2 5.3,5.17,5.21,5.22,5.26,5.27,5.30 5.35,5.36,5.39,5.41,5.45,5.46,6.3

	6.5,6.6,6.7,6.8,6.20,6.38,6.44,6.47 6.49,6.50,6.51,6.53,6.56,6.61,6.66 6.70
GABRIEL RUDDY CRUZ MENDEZ	1.3,5.52
SOUMAVA CHAKRABORTY	1.4,3.1,3.4,3.7,3.12,3.15,3.16,5.19 6.1,6.9,6.12,6.28,6.36,6.67,6.69
NARENDRA BHANDARI	1.5,1.7,3.10
GEORGE FLORIN ȘERBAN	1.6,1.9,1.38,1.40,3.17,5.51,6.18 6.52
BENNY LE VAN	1.8
SAGAR KUMAR	1.10,1.14,1.25
ROVSEN PIRGULIYEV	1.19,6.15,6.17
TRAN HONG	1.20,1.28,1.30,1.31,2.7,2.11,2.12 2.13,2.14,2.18,2.19,3.19,3.20,3.21 3.23,3.26,3.27,6.37
HOANG LE NHAT TUNG	1.21
LAZAROS ZACHARIADIS	1.22,3.13,6.4,6.63
SANTOS MARINS JUNIOR	1.24
REMUS FLORIN STANCA	1.26,3.18,4.10
AMIT DUTTA	1.27,3.8,3.9,4.2,5.20,6.10,6.26 6.30
JHOAW CARLOS	1.29
MICHAEL STERGHIOU	1.32
BEDRI HAJRIZI	1.34,1.48,3.14,3.25,3.29

AVISHEK MITRA	1.36
RAHIM SHAHBAZOV	1.37
KAMEL BENAICHA	1.39
FLORENTIN VIȘESCU	1.43,5.6,5.8,5.9,5.10,5.11,5.12 5.13,5.14,5.16
ORLANDO IRAHOLA ORTEGA	1.44
KHALED ABD IMOUTI	1.47,3.22,4.9,4.12,4.13,4.14
LETY SAUCEDA	1.50
KHANH HUNG VU	2.2,2.20,6.41,6.62
OMRAN KOUBA	2.3,5.25,5.31,6.22,6.42
CHRIS KYRIAZIS	2.5
BADO IDRISS OLIVIER	2.16
SEYRAN IBRAHIMOV	3.2
KHANH HUNG VU	3.6
ADRIAN POPA	3.11
PETRE DANIEL ALEXANDRU	3.24
JALIL HAJIMIR	3.28,3.30
ANDREW OKUKURA	5.4,5.5,5.42
SRINIVASA RAGHAVA	5.32
DJEERAJ BADERA	5.43
DANIEL BACIU	5.44
KELVIN HONG	5.47
AJAO YINKA	5.49
JOVICA MIKIC	5.49

SERGIO ESTEBAN	5.50
SOUMITRA MANDAL	5.54,6.19.,6.24,6.39,6.58,6.65
DIMITRIS KASTRIOTIS	6.2,6.13
SHAFIQUR RAHMAN	6.12
DO HUU DUC THINH	6.14
SANONG HUAYRERAI	6.45
HENRY RICARDO	6.48

REFFERENCES

1.Romanian Mathematical Magazine-Interactive Journal-www.ssmrmh.ro

2.Mihaly Bencze,Daniel Sitaru,Marian Ursărescu:”*Olympic Mathematical Energy***”-*Studis-Publishing House*-Iași-2018**

3.Daniel Sitaru, George Apostolopoulos:”*The Olympic Mathematical Marathon***”-Publishing House- Cartea Românească-Pitești-2018**

4. Mihaly Bencze,Daniel Sitaru:”*Quantum Mathematical Power***” -Publishing House Studis, Iasi-2018**

5.Daniel Sitaru,Marian Ursărescu:”*Calculus Marathon***”-*Studis-Publishing House*-Iasi-2018**

6.Daniel Sitaru, Mihaly Bencze:”*699 Olympic Mathematical Challenges***”-Publishing House Studis, Iasi-2017**

7.Daniel Sitaru,Marian Ursărescu:”*Ice Math-Contests Problems***”-*Studis-Publishing House*-Iasi-2019**