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CONTENT

ABOUT THE FAMOUS BĂTINEȚU – GIURGIU'S SEQUENCES

By Daniel Sitaru, Claudia Nănuți – Romania

Abstract: In this paper we will give the definition of Bătinețu – Giurgiu's sequences, a few properties of these and some applications.

The original Bătinețu – Giurgiu's sequence is defined as:

$$(B-G)_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}};n\geq 2$$

We will prove that:

$$\lim_{n\to\infty}(B-G)_n=e$$

using only elementary methods. (without Stirling).

Main result:

$$\lim_{n\to\infty}(B-G)_n=e$$

Proof: Let be: $v_n=\left(\frac{n+1}{n}\right)^2\cdot\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}};n\geq 2$

$$\lim_{n\to\infty}v_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{\sqrt[n]{n!}}{n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\cdot\lim_{n\to\infty}\sqrt[n+1]{\frac{(n+1)^{n+1}}{(n+1)!}}=$$

$$\overset{CAUCHY-D'ALEMBERT}{=}1\cdot\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}\cdot\lim_{n\to\infty}\frac{\frac{(n+2)^{n+2}}{(n+2)!}}{\frac{(n+1)^{n+1}}{(n+1)!}}=$$

$$=\lim_{n\to\infty}\frac{n!\,(n+1)}{(n+1)^n(n+1)}\cdot\frac{n^n}{n!}\cdot\lim_{n\to\infty}\frac{(n+2)^{n+1}\cdot(n+2)}{(n+1)!\cdot(n+2)}\cdot\frac{(n+1)!}{(n+1)^{n+1}}=$$

$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{n+2}{n+1}-1\right)^{n+1}=$$

$$=\lim_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}=e^{-1}\cdot e^1=1$$

$$\lim_{n\to\infty}v_n=1$$

$$\lim_{n\to\infty}v_n^n=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{n!}{(n+1)!}\cdot\sqrt[n+1]{(n+1)!}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\cdot\lim_{n\to\infty}\frac{n!\cdot n}{(n+1)!}\cdot\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\left(\frac{n+1}{n}\right)=$$

$$= e^2 \cdot \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n+1} = e^2 \cdot 1 \cdot 1 \cdot \lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}} =$$

$$= e^2 \cdot \lim_{n\to\infty} \frac{(n+2)!}{(n+2)^{n+2}} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = e^2 \cdot \lim_{n\to\infty} \frac{(n+1)!\,(n+2)}{(n+2)(n+1)!} \cdot \left(\frac{n+1}{n+2}\right)^{n+1} =$$

$$= e^2 \cdot \lim_{n\to\infty} \left(1 + \frac{n+1}{n+2} - 1\right)^{n+1} = e^2 \cdot \lim_{n\to\infty} \left(1 + \frac{-1}{n+2}\right)^{n+1} = e^2 \cdot e^{-1} = e$$

$$\lim_{n\to\infty} (B-G)_n = \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = \lim_{n\to\infty} \frac{n^2}{\sqrt[n]{n!}}(v_n - 1) =$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{\sqrt[n]{n!}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n\right) = \lim_{n\to\infty} \left(\frac{n}{\sqrt[n]{n!}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n\right) = e \cdot 1 \cdot \ln e = e$$

Definition 1: Let be $t > 0$ and $(a_n)_{n\ge1}; a_n > 0$ a sequence. The sequence $(a_n)_{n\ge1}$ has the Bătinețu – Giurgiu's property if exists:

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} = a > 0$$

The couple t and a_n define a (t, B) – sequence .

Property 1: If $t > 0$ and $(a_n)_{n\ge1}$ is a (t, B) – sequence then:

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} = \frac{a}{e^t} \qquad \textbf{(1)}$$

Proof: $(a_n)_{n\ge1}$ is a (t, B) – sequence hence exists:

$$\lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} = a > 0$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^t} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^{nt}}} \overset{CAUCHY'S-D'ALEMBERT}{=} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{(n+1)t}} \cdot \frac{n^{nt}}{a_n} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} \cdot \left(\frac{n}{n+1}\right)^{(n+1)t} = a \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{(n+1)t} = \frac{a}{e^t}$$

Property 2: If $t > 0$ and $(a_n)_{n\ge1}$ is a (t, B) – sequence and $u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}; n \ge 2$ then:

$$\lim_{n\to\infty} u_n^n = e^t \qquad \textbf{(2)}$$

Proof:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \frac{n^t}{\sqrt[n]{a_n}} \cdot \left(\frac{n+1}{n}\right)^t \overset{(1)}{=} \frac{a}{e^t} \cdot \frac{e^t}{a} \cdot 1 = 1. \text{ Hence:}$$

$$\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1 \text{ and:}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n\to\infty} \frac{a_{n+1}}{n^t \cdot a_n} \cdot \frac{(n+1)^t}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^t = a \cdot \frac{e^t}{a} \cdot 1 = e^t$$

Definition 2: If $t > 0$ and $(a_n)_{n\geq 1}$ is a (t, B) – sequence we define:

$$B_n = \frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^{t+1}}{\sqrt[n]{a_n}}; n \geq 2$$

and the name of this sequence is the t - Bătinețu – Giurgiu's sequence noted as $(t, B - G)$.

Theorem 1: If $(t, B - G)$ is a Bătinețu – Giurgiu's sequence then:

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^t}{\sqrt[n]{a_n}}\right) = \frac{e^t}{a} \quad (4)$$

Proof: Let be: $v_n = \left(\frac{n+1}{n}\right)^{t+1} \cdot \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}; n \geq 2$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\frac{n+1}{n} \cdot \frac{(n+1)^t}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{\sqrt[n]{a_n}}{n^t}\right) = 1 \cdot \frac{e^t}{a} \cdot \frac{a}{e^t} = 1$$

$$\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$$

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{n(t+1)} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} = e^{t+1} \cdot \lim_{n\to\infty} \frac{a_n \cdot n^t}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} \cdot \left(\frac{n+1}{n}\right)^t =$$

$$= e^{t+1} \cdot \frac{1}{a} \cdot \frac{a}{e^t} \cdot 1 = e$$

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{n^{t+1}}{\sqrt[n]{a_n}}\right) = \lim_{n\to\infty} \frac{n^{t+1}}{\sqrt[n]{a_n}}(v_n - 1) = \lim_{n\to\infty} \left(\frac{n^{t+1}}{\sqrt[n]{a_n}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n\right) =$$

$$= \lim_{n\to\infty} \left(\frac{n^t}{\sqrt[n]{a_n}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n\right) = \frac{e^t}{a} \cdot 1 \cdot \ln e = \frac{e^t}{a}$$

Definition 3: If $t > 0$ and $(a_n)_{n\geq 1}$ is a $(t+1, B)$ sequence we define:

$$G_n = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t}; n \geq 2$$

and the name of this sequence is the $t+1$ - Bătinețu – Giurgiu's sequence noted $(t+1, B - G)$.

Theorem 2:

If $(t+1, B - G)$ is a Bătinețu – Giurgiu's sequence then:

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t}\right) = \frac{a}{e^{t+1}} \quad (5)$$

Proof: Let be: $w_n=\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t}\cdot\frac{n^t}{\sqrt[n]{a_n}}$; $n\ge 2$. By (1): $\lim\limits_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{t+1}}=\frac{a}{e^{t+1}}$

$$\lim_{n\to\infty} w_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t}\cdot\frac{n^t}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\cdot\left(\frac{n}{n+1}\right)^t=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^{t+1}}\cdot\frac{n^{t+1}}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n}=\frac{a}{e^{t+1}}\cdot\frac{e^{t+1}}{a}\cdot 1=1$$

$$\lim_{n\to\infty}\frac{w_n-1}{\ln w_n}=1$$

$$\lim_{n\to\infty} w_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\left(\frac{n}{n+1}\right)^{nt}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}}=$$

$$=\lim_{n\to\infty}\frac{a_{n+1}}{n^{t+1}a_n}\cdot\frac{(n+1)^{t+1}}{\sqrt[n+1]{a_{n+1}}}\cdot\left(\frac{n}{n+1}\right)^{nt}\cdot\left(\frac{n}{n+1}\right)^{t+1}=a\cdot\frac{1}{e^t}\cdot\frac{e^{t+1}}{a}\cdot 1=e$$

$$\lim_{n\to\infty} G_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_n}}{n^t}\right)=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}\cdot(w_n-1)=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^t}\cdot\frac{w_n-1}{\ln w_n}\cdot\ln w_n=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^{t+1}}\cdot\frac{w_n-1}{\ln w_n}\cdot\ln w_n^n=\frac{a}{t+1}\cdot 1\cdot\ln e=\frac{a}{e^{t+1}}$$

Theorem 3:

If $t\ge 0$; $(a_n)_{n\ge 1}$; $a_n>0$; $n\in\mathbb{N}^*$ and $\lim\limits_{n\to\infty}\frac{n^t\cdot a_{n+1}}{a_n}=a>0$ then:

$$\lim_{n\to\infty} n^{t+1}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)=-ate^t\quad\textbf{(6)}$$

Proof: Denote $d_n=\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$; $n\ge 2$

$$\lim_{n\to\infty} n^t\cdot\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{a_n\cdot n^{nt}}\overset{CAUCHY-D'ALEMBERT}{=}$$

$$=\lim_{n\to\infty}\frac{a_{n+1}\cdot(n+1)^{t(n+1)}}{a_n\cdot n^{tn}}=\lim_{n\to\infty}\frac{a_{n+1}\cdot n^t}{a_n}\cdot\left(\frac{n+1}{n}\right)^{t(n+1)}=a\cdot e^t$$

$$\lim_{n\to\infty} n^t\cdot\sqrt[n]{a_n}=a\cdot e^t\qquad(7)$$

$$\lim_{n\to\infty} d_n=\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}\cdot(n+1)^t}{\sqrt[n]{a_n}\cdot n^t}\cdot\left(\frac{n}{n+1}\right)^t=\frac{a\cdot e^{t+1}}{a\cdot e^t}\cdot\frac{1}{e}=1$$

$$\lim_{n\to\infty}\frac{d_n-1}{\ln d_n}=1$$

$$\lim_{n\to\infty} d_n^n=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n]{a_{n+1}}}=$$

$$= \lim_{n\to\infty} \frac{n^t \cdot a_{n+1}}{a_n} \cdot \frac{1}{(n+1)^t \cdot \sqrt[n]{a_{n+1}}} \cdot \left(\frac{n+1}{n}\right)^t = a \cdot \frac{1}{a \cdot e^t} = e^{-t}$$

$$\lim_{n\to\infty} n^{t+1}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \lim_{n\to\infty} n^{t+1} \cdot \sqrt[n]{a_n} \cdot (d_n - 1) =$$

$$= \lim_{n\to\infty} n^{t+1} \cdot \sqrt[n]{a_n} \cdot \frac{d_n - 1}{\ln d_n} \cdot \ln d_n = \lim_{n\to\infty} n^t \cdot \sqrt[n]{a_n} \cdot \frac{d_n - 1}{\ln d_n} \cdot d_n^n \overset{(7)}{=} a \cdot e^t \cdot 1 \cdot \ln e^{-1} = -a \cdot t \cdot e^t$$

Application 1:

For $t = 1$ in (4):

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{e}{a} \quad (8)$$

For $a_n = n!$ in (8):

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}\right) = e$$

Application 2:

For $t = 1$ in (6):

$$\lim_{n\to\infty} n^2\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = -a \cdot e \quad (9)$$

For $a_n = \frac{1}{n!}$ in (9):

$$\lim_{n\to\infty} n^2 \left(\frac{1}{\sqrt[n+1]{(n+1)!}} - \frac{1}{\sqrt[n]{n!}}\right) = -e$$

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A SIMPLE PROOF FOR BERGSTROM'S REVERSED INEQUALITY AND APPLICATIONS

By Daniel Sitaru, Claudia Nănuți-Romania

Abstract: In this paper we will give a simple proof for Bergstrom's reversed inequality and a few applications.

BERGSTROM'S REVERSED INEQUALITY

If $a,b,c,x,y,z>0$ then:

$$\frac{x^2}{b^2+c^2}+\frac{y^2}{c^2+a^2}+\frac{z^2}{a^2+b^2}\le\frac{x^2+y^2}{4c^2}+\frac{y^2+z^2}{4a^2}+\frac{z^2+x^2}{4b^2}\quad (1)$$

Proof: Inequality (1) can be written:

$$\frac{x^2+y^2}{4c^2}+\frac{y^2+z^2}{4a^2}+\frac{z^2+x^2}{4b^2}-\frac{x^2}{b^2+c^2}-\frac{y^2}{c^2+a^2}-\frac{z^2}{a^2+b^2}\ge 0$$

$$x^2\left(\frac{1}{4b^2}+\frac{1}{4c^2}-\frac{1}{b^2+c^2}\right)+y^2\left(\frac{1}{4a^2}+\frac{1}{4c^2}-\frac{1}{a^2+c^2}\right)+z^2\left(\frac{1}{4b^2}+\frac{1}{4a^2}-\frac{1}{a^2+b^2}\right)\ge 0$$

$$x^2\cdot\frac{(b^2+c^2)^2-4b^2c^2}{4b^2c^2(b^2+c^2)}+y^2\cdot\frac{(c^2+a^2)^2-4a^2c^2}{4a^2c^2(a^2+c^2)}+z^2\cdot\frac{(a^2+b^2)-4a^2b^2}{4a^2b^2(a^2+b^2)}\ge 0$$

$$x^2\cdot\frac{(b^2-c^2)^2}{4b^2c^2(b^2+c^2)}+y^2\cdot\frac{(c^2-a^2)^2}{4a^2c^2(a^2+c^2)}+z^2\cdot\frac{(a^2-b^2)^2}{4a^2b^2(a^2+b^2)}\ge 0$$

Equality holds for $a=b=c$.

Corollary 1: If a,b,c are sides in ΔABC then:

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\le\frac{a^2+b^2}{4c^2}+\frac{b^2+c^2}{4a^2}+\frac{c^2+a^2}{4b^2}$$

Proof: We take in (1): $x=a;y=b;z=c$.

Corollary 2: If $a,b,c>0$ then:

$$\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{1}{a^2+b^2}\le\frac{1}{2a^2}+\frac{1}{2b^2}+\frac{1}{2c^2}$$

Proof: We take in (1): $x=y=z=1$.

Application 1: In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a^2}{b^2+c^2}\le\frac{3}{4}+\frac{1}{16}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)$$

Proof: $m_a^2+m_c^2=\frac{1}{2}(b^2+c^2)-\frac{1}{4}a^2+\frac{1}{2}(a^2+b^2)-\frac{1}{4}c^2=$

$$=\frac{2b^2+2c^2-a^2+2a^2+2b^2-c^2}{4}=\frac{a^2+4b^2+c^2}{4},\qquad \frac{m_a^2+m_c^2}{4b^2}=\frac{a^2+4b^2+c^2}{16b^2}$$

$$\sum_{cyc}\frac{m_a^2}{b^2+c^2}\overset{(1)}{\le}\sum_{cyc}\frac{m_a^2+m_c^2}{4b^2}=\sum_{cyc}\frac{a^2+4b^2+c^2}{16b^2}=\sum_{cyc}\frac{a^2}{16b^2}+\sum_{cyc}\frac{4b^2}{16b^2}+\sum_{cyc}\frac{c^2}{16b^2}=$$

$$=\frac{3}{4}+\frac{1}{16}\left(\sum_{cyc}\frac{a^2}{b^2}+\sum_{cyc}\frac{c^2}{b^2}\right)=\frac{3}{4}+\frac{1}{16}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}\right)$$

Equality holds for $a=b=c$.

Application 2: If $x,y,z>0$ then:

$$4x^2y^2z^2\left(\frac{1}{x^2+y^2}+\frac{1}{y^2+z^2}+\frac{1}{z^2+x^2}\right)\le\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+\frac{y^2}{x^2}+\frac{z^2}{y^2}+\frac{x^2}{z^2}$$

Proof: We take in (1):

$$a=\frac{1}{x};b=\frac{1}{y};c=\frac{1}{z}$$

$$\sum_{cyc}\frac{x^2}{\frac{1}{y^2}+\frac{1}{z^2}}\le\frac{1}{4}\sum_{cyc}\frac{\frac{1}{x^2}+\frac{1}{y^2}}{\frac{1}{z^2}},\qquad 4\sum_{cyc}\frac{x^2y^2z^2}{y^2+z^2}\le\sum_{cyc}\frac{z^2(x^2+y^2)}{x^2y^2}$$

$$4x^2y^2z^2\sum_{cyc}\frac{1}{y^2+z^2}\le\sum_{cyc}z^2\left(\frac{1}{y^2}+\frac{1}{x^2}\right),\qquad 4x^2y^2z^2\sum_{cyc}\frac{1}{y^2+z^2}\le\sum_{cyc}\frac{z^2}{x^2}+\sum_{cyc}\frac{z^2}{y^2}$$

Equality holds for $x=y=z$.

Application 3: In ΔABC the following relationship holds:

$$\frac{w_a^2}{b^2+c^2}+\frac{w_b^2}{c^2+a^2}+\frac{w_c^2}{a^2+b^2}\le\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proof: We take $x=w_a,y=w_b,z=w_c$ in (1):

$$\sum_{cyc}\frac{w_a^2}{b^2+c^2}\le\sum_{cyc}\frac{w_a^2+w_b^2}{4c^2}\overset{GUGGENHEMER}{\le}\le\sum_{cyc}\frac{s(s-a)+s(s-b)}{4c^2}=\sum_{cyc}\frac{s(2s-a-b)}{4c^2}=$$

$$=\frac{s}{4}\sum_{cyc}\frac{a+b+c-a-b}{c^2}=\frac{s}{4}\sum_{cyc}\frac{1}{c}=\frac{s}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Reference:

[1] Romanian Mathematical Magazine – www.ssmrmh.ro

A SIMPLE PROOF FOR ROGER'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru, Claudia Nănuți-Romania

Abstract: In this paper we will give a simple proof for Roger's inequality and a few applications.

ROGER'S INEQUALITY

If $\boldsymbol{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n} > 0; n \in \mathbb{N}^*; \boldsymbol{s \in (0,1)}$ **then:**

$$\sum_{i=1}^{n} a_i^s \cdot b_i^{1-s} \le \left(\sum_{i=1}^{n} a_i\right)^s \cdot \left(\sum_{i=1}^{n} b_i\right)^{1-s} \quad (1)$$

Proof: Let be $f:(0,\infty) \to \mathbb{R}; f(x) = x^\alpha - \alpha x; \alpha \in (0,1)$

$$f'(x) = \alpha x^{\alpha-1} - \alpha = \alpha(x^{\alpha-1} - 1)$$

$$f'(x) = 0 \Rightarrow x^{\alpha-1} - 1 = 0 \Rightarrow x^{\alpha-1} = 1 \Rightarrow x = 1$$

f increasing on $(0,1)$, f decreasing on $(1,\infty)$

$$\max_{x>0} f(x) = f(1) = 1 - \alpha$$

$$f(x) \le f(1) \Rightarrow x^\alpha - \alpha x \le 1 - \alpha \quad (2)$$

Leg be $p = \frac{1}{\alpha}; q = \frac{p}{p-1} = \frac{1}{1-\alpha}, \ \frac{1}{p} + \frac{1}{q} = \alpha + 1 - \alpha = 1$

$$\alpha \in (0,1) \Rightarrow p > 1$$

For $A, B > 0$ let be $x = \frac{A}{B}$ in (2):

$$\left(\frac{A}{B}\right)^\alpha - \alpha\left(\frac{A}{B}\right) \le 1 - \alpha, \qquad \left(\frac{A}{B}\right)^{\frac{1}{p}} - \frac{1}{p} \cdot \left(\frac{A}{B}\right) \le 1 - \frac{1}{p} = \frac{1}{q}$$

$$\left(\frac{A}{B}\right)^{\frac{1}{p}} - \frac{1}{p}\left(\frac{A}{B}\right) \le \frac{1}{q}, \qquad \frac{A^{\frac{1}{p}}}{B^{\frac{1}{p}}} - \frac{A}{pB} \le \frac{1}{q}$$

$$\frac{A^{\frac{1}{p}} \cdot B^{\frac{1}{q}}}{B^{\frac{1}{p}}} - \frac{A}{p} \cdot B^{\frac{1}{q}-1} \le \frac{1}{q} \cdot B^{\frac{1}{q}}, \qquad A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} - \frac{A}{p} \cdot B^{\frac{1}{p}+\frac{1}{q}-1} \le \frac{1}{q} \cdot B^{\frac{1}{p}+\frac{1}{q}}$$

$$A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} - \frac{A}{p} \cdot B^{1-1} \le \frac{1}{q} \cdot B, \ A^{\frac{1}{p}} \cdot B^{\frac{1}{q}} \le \frac{A}{p} + \frac{B}{q} \quad (3)$$

Let be:

$$A = \frac{x_i^p}{\sum_{i=1}^{n} x_i^p}; B = \frac{y_i^q}{\sum_{i=1}^{n} y_i^q} \quad \text{in (3)}$$

$$\left(\frac{x_i^p}{\sum_{i=1}^n x_i^p}\right)^{\frac{1}{p}} \cdot \left(\frac{y_i^q}{\sum_{i=1}^n y_i^q}\right)^{\frac{1}{q}} \le \frac{1}{p} \cdot \frac{x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \cdot \frac{y_i^2}{\sum_{i=1}^n y_i^2}$$

$$\frac{x_i \cdot y_i}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}} \le \frac{1}{p\sum_{i=1}^n x_i^p} \cdot x_i^p + \frac{1}{q \cdot \sum_{i=1}^n y_i^q} \cdot y_i^q$$

By summing:

$$\frac{\sum_{i=1}^n x_i y_i}{\left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}}} \le \frac{1}{p} \cdot \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n x_i^p} + \frac{1}{q} \cdot \frac{\sum_{i=1}^n y_i^q}{\sum_{i=1}^n y_i^q} = \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = \frac{1}{p} + \frac{1}{q} = 1$$

$$\sum_{i=1}^n x_i y_i \le \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} \quad \text{(Holder)}$$

In Holder's inequality we take:

$$x_i = a_i^s; y_i = b_i^{1-s}, \ \ p = \frac{1}{s}; q = \frac{1}{1-s}$$

$$\sum_{i=1}^n a_i^s b_i^{1-s} \le \left(\sum_{i=1}^n (a_i^s)^{\frac{1}{s}}\right)^s \cdot \left(\sum_{i=1}^n \left(b_i^{\frac{1}{1-s}}\right)^{\frac{1}{1-s}}\right)^{1-s}$$

$$\sum_{i=1}^n a_i^s b_i^{1-s} \le \left(\sum_{i=1}^n a_i\right)^s \cdot \left(\sum_{i=1}^n b_i\right)^{1-s}$$

Application 1 (Huygens' inequality): If $a_1, a_2, b_1, b_2 > 0$ then:

$$(a_1 + a_2)(b_1 + b_2) \ge \left(\sqrt{a_1 b_1} + \sqrt{a_2 b_2}\right)^2$$

Proof: We take in (1): $n = 2; s = \frac{1}{2}$

$$a_1^{\frac{1}{2}} \cdot b_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} \cdot b_2^{\frac{1}{2}} \le (a_1 + a_2)^{\frac{1}{2}}(b_1 + b_2)^{1-\frac{1}{2}}$$

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} \le \sqrt{(a_1 + a_2)(b_1 + b_2)}, \qquad (a_1 + a_2)(b_1 + b_2) \ge \left(\sqrt{a_1 b_1} + \sqrt{a_2 b_2}\right)$$

Application 2: If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \ge \left(\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3}\right)^2$$

Proof: We take in (1): $n = 3; s = \frac{1}{2}$

$$a_1^{\frac{1}{2}} \cdot b_1^{\frac{1}{2}} + a_2^{\frac{1}{2}} \cdot b_2^{\frac{1}{2}} + a_3^{\frac{1}{2}} \cdot b_3^{\frac{1}{2}} \le (a_1 + a_2 + a_3)^{\frac{1}{2}} \cdot (b_1 + b_2 + b_3)^{1-\frac{1}{2}}$$

$$\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3} \le \sqrt{(a_1 + a_2 + a_3)(b_1 + b_2 + b_3)}$$

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \ge \left(\sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \sqrt{a_3 b_3}\right)^2$$

Application 3: If $a_1, a_2, b_1, b_2 > 0$ then:

$$\boldsymbol{(a_1+a_2)(b_1+b_2)^2 \geq \left(\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2}\right)^3}$$

Proof: We take in (1): $n=2; s=\frac{1}{3}$

$$a_1^{\frac{1}{3}}\cdot b_1^{\frac{2}{3}}+a_2^{\frac{1}{3}}\cdot b_2^{\frac{2}{3}} \leq (a_1+a_2)^{\frac{1}{3}}\cdot(b_1+b_2)^{\frac{2}{3}}$$

$$\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2} \leq \sqrt[3]{(a_1+a_2)(b_1+b_2)^2}, \qquad (a_1+a_2)(b_1+b_2)^2 \geq \left(\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2}\right)^3$$

Application 4: If $a_1, a_2, a_3, b_1, b_2, b_3 > 0$ then:

$$\boldsymbol{(a_1+a_2+a_3)(b_1+b_2+b_3)^2 \geq \left(\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2}+\sqrt[3]{a_3b_3^2}\right)^3}$$

Proof: We take in (1): $n=3; s=\frac{1}{3}$

$$a_1^{\frac{1}{3}}b_1^{\frac{2}{3}}+a_2^{\frac{1}{3}}b_2^{\frac{2}{3}}+a_3^{\frac{1}{3}}\cdot b_3^{\frac{2}{3}} \leq (a_1+a_2+a_3)^{\frac{1}{3}}\cdot(b_1+b_2+b_3)^{\frac{2}{3}}$$

$$\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2}+\sqrt[3]{a_3b_3^2} \leq \sqrt[3]{(a_1+a_2+a_3)(b_1+b_2+b_3)^2}$$

$$(a_1+a_2+a_3)(b_1+b_2+b_3)^2 \geq \left(\sqrt[3]{a_1b_1^2}+\sqrt[3]{a_2b_2^2}+\sqrt[3]{a_3b_3^2}\right)^3$$

Application 5: If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n > 0; n \in \mathbb{N}^*; s \in (0,1)$ then:

$$\boldsymbol{\sum_{i=1}^n a_i^s\cdot\left(\frac{1}{b_i}\right)^{1-s} \leq \left(\sum_{i=1}^n a_i\right)^s\cdot\left(\sum_{i=1}^n \frac{1}{b_i}\right)^{1-s}} \qquad \textbf{(4)}$$

Proof: We replace in (1): $b_i \to \frac{1}{b_i}$

Application 6: If $a_1, a_2, b_1, b_2 > 0$ then:

$$\boldsymbol{(a_1+a_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right) \geq \left(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}\right)^2}$$

Proof: We take in (4): $n=2; s=\frac{1}{2}$

$$a_1^{\frac{1}{2}}\cdot\left(\frac{1}{b_1}\right)^{1-\frac{1}{2}}+a_2^{\frac{1}{2}}\cdot\left(\frac{1}{b_2}\right)^{1-\frac{1}{2}} \leq (a_1+a_2)^{\frac{1}{2}}\cdot\left(\frac{1}{b_1}+\frac{1}{b_2}\right)^{1-\frac{1}{2}}$$

$$\frac{a_1^{\frac{1}{2}}}{b_1^{\frac{1}{2}}}+\frac{a_2^{\frac{1}{2}}}{b_2^{\frac{1}{2}}}\le (a_1+a_2)^{\frac{1}{2}}\cdot\left(\frac{1}{b_1}+\frac{1}{b_2}\right)^{\frac{1}{2}},\quad \sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}\le\sqrt{(a_1+a_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)}$$

$$(a_1+a_2)\left(\frac{1}{b_1}+\frac{1}{b_2}\right)\ge\left(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}\right)^2$$

Application 7: If $a_1,a_2,a_3,b_1,b_2,b_3>0$ then:

$$\boldsymbol{(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)\ge\left(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}}\right)^2}$$

Proof: We take in (4): $n=3;s=\frac{1}{2}$

$$a_1^{\frac{1}{2}}\cdot\left(\frac{1}{b_1}\right)^{1-\frac{1}{2}}+a_2^{\frac{1}{2}}\cdot\left(\frac{1}{b_2}\right)^{1-\frac{1}{2}}+a_3^{\frac{1}{2}}\cdot\left(\frac{1}{b_3}\right)^{1-\frac{1}{2}}\le(a_1+a_2+a_3)^{\frac{1}{2}}\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^{1-\frac{1}{2}}$$

$$\frac{a_1^{\frac{1}{2}}}{b_1^{\frac{1}{2}}}+\frac{a_2^{\frac{1}{2}}}{b_2^{\frac{1}{2}}}+\frac{a_3^{\frac{1}{2}}}{b_3^{\frac{1}{2}}}\le(a_1+a_2+a_3)^{\frac{1}{2}}\cdot\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^{\frac{1}{2}}$$

$$\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}}\le\sqrt{(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)}$$

$$(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)\ge\left(\sqrt{\frac{a_1}{b_1}}+\sqrt{\frac{a_2}{b_2}}+\sqrt{\frac{a_3}{b_3}}\right)^2$$

Application 8: If $a_1,a_2,a_3,b_1,b_2,b_3>0$ then:

$$\boldsymbol{(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^2\ge\left(\sqrt[3]{\frac{a_1}{b_1^2}}+\sqrt[3]{\frac{a_2}{b_2^2}}+\sqrt[3]{\frac{a_3}{b_3^2}}\right)^3}$$

Proof: We take in (4): $n=3;s=\frac{1}{3}$

$$a_1^{\frac{1}{3}}\cdot\left(\frac{1}{b_1}\right)^{1-\frac{1}{3}}+a_2^{\frac{1}{3}}\cdot\left(\frac{1}{b_2}\right)^{1-\frac{1}{3}}+a_3^{\frac{1}{3}}\cdot\left(\frac{1}{b_3}\right)^{\frac{1}{3}}\le(a_1+a_2+a_3)^{\frac{1}{3}}\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^{1-\frac{1}{3}}$$

$$\sqrt[3]{\frac{a_1}{b_1^2}}+\sqrt[3]{\frac{a_2}{b_2^2}}+\sqrt[3]{\frac{a_3}{b_3^2}}\le\sqrt[3]{(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^2}$$

$$(a_1+a_2+a_3)\left(\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}\right)^2 \geq \left(\sqrt[3]{\frac{a_1}{b_1}}+\sqrt[3]{\frac{a_2}{b_2}}+\sqrt[3]{\frac{a_3}{b_3}}\right)^3$$

Application 9: If $\boldsymbol{x}, \boldsymbol{y} \in \left(\mathbf{0}, \frac{\pi}{2}\right)$ then:

$$\sqrt{\sin 2x}+\sqrt{\sin 2y} \leq \sqrt{2(\sin x+\sin y)(\cos x+\cos y)}$$

Proof: We take in (1): $n=2; s=\frac{1}{2}$;

$$a_1=\sin x\,; a_2=\sin y\,; b_1=\cos x\,; b_2=\cos y$$

$$\sqrt{a_1b_1}+\sqrt{a_2b_2} \leq \sqrt{(a_1+a_2)(b_1+b_2)}$$

$$\sqrt{\sin x\cos x}+\sqrt{\sin y\cos y} \leq \sqrt{(\sin x+\sin y)(\cos x+\cos y)}$$

$$\sqrt{2\sin x\cos x}+\sqrt{2\sin y\cos y} \leq \sqrt{2(\sin x+\sin y)(\cos x+\cos y)}$$

$$\sqrt{\sin 2x}+\sqrt{\sin 2y} \leq \sqrt{2(\sin x+\sin y)(\cos x+\cos y)}$$

Application 10: If $\boldsymbol{x}, \boldsymbol{y} \in \left(\mathbf{0}, \frac{\pi}{2}\right)$ then:

$$\sqrt{\sin 2x}+\sqrt{\sin 2y}+\sqrt{\sin 2z} \leq \sqrt{2(\sin x+\sin y+\sin z)(\cos x+\cos y+\cos z)}$$

Proof: We take in (1): $n=3; s=\frac{1}{2}$

$$a_1=\sin x\,; a_2=\sin y\,; a_3=\sin z\,;\ \ b_1=\cos x\,; b=\cos y\,; b_3=\cos z$$

$$\sqrt{\sin x\cos x}+\sqrt{\sin y\cos y}+\sqrt{\sin z\cos z} \leq \sqrt{(\sin x+\sin y+\sin z)(\cos x+\cos y+\cos z)}$$

$$\sqrt{2\sin x\cos x}+\sqrt{2\sin y\cos y}+\sqrt{2\sin z\cos z} \leq$$

$$\leq \sqrt{2(\sin x+\sin y+\sin z)(\cos x+\cos y+\cos z)}$$

$$\sqrt{\sin 2x}+\sqrt{\sin 2y}+\sqrt{\sin 2z} \leq \sqrt{2(\sin x+\sin y+\sin z)(\cos x+\cos y+\cos z)}$$

Application 11: If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{t} \in \mathbb{R}$ then:

$$\sqrt{e^{x+y}}+\sqrt{e^{z+t}} \leq \sqrt{(e^x+e^z)(e^y+e^t)}$$

Proof: We take in (1): $n=2; s=\frac{1}{2}$

$$a_1=e^x; a_2=e^z;\ b_1=e^y; b_2=e^t$$

$$\sqrt{a_1b_1}+\sqrt{a_2b_2} \leq \sqrt{(a_1+a_2)(b_1+b_2)}, \qquad \sqrt{e^x\cdot e^y}+\sqrt{e^z\cdot e^t} \leq \sqrt{(e^x+e^z)(e^y+e^t)}$$

$$\sqrt{e^{x+y}}+\sqrt{e^{z+t}} \leq \sqrt{(e^x+e^z)(e^y+e^t)}$$

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

A SIMPLE PROOF FOR JANIC&VASIC'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru, Claudia Nănuți -Romania

Abstract: In this paper we will give a simple proof for Janic&Vasic's inequality and a few applications.

JANIC&VASIC'S INEQUALITY

If $\mathbf{0} < k \leq n; n \in \mathbb{N}^*; \boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n} > 0$ **then:**

$$\sum_{k=1}^{n} \frac{a_1+a_2+\dots+a_k}{a_{k+1}+a_{k+2}+\dots+a_n} \geq \frac{nk}{n-k} \quad (1)$$

Proof:

$$\sum_{k=1}^{n} \frac{a_1+a_2+\dots+a_k}{a_{k+1}+a_{k+2}+\dots+a_n} = \sum_{k=1}^{n} \left(\frac{a_1+a_2+\dots+a_k}{a_{k+1}+a_{k+2}+\dots+a_n}+1\right) - n =$$

$$= \sum_{k=1}^{n} \frac{a_1+a_2+\dots+a_k+a_{k+1}+\dots+a_n}{a_{k+1}+a_{k+2}+\dots+a_n} - n =$$

$$= (a_1+a_2+\dots+a_n)\sum_{k=1}^{n} \frac{1}{a_{k+1}+a_{k+2}+\dots a_n} - n \geq$$

$$\overset{BERGSTROM}{\geq} (a_1+a_2+\dots+a_n) \cdot \frac{\left(\underbrace{1+1+\dots+1}_{for\ "n"\ times}\right)^2}{(n-k)(a_1+a_2+..+a_n)} - n =$$

$$= \frac{n^2}{n-k} - n = \frac{n^2-n^2+nk}{n-k} = \frac{nk}{n-k}$$

Application 1: If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0$ **then:**

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq 6 \quad (2)$$

Proof: We take in (1): $n = 3; k = 2$. Equality holds for $a = b = c$.

Application 2: If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} > 1$ **then:**

$$\log_z(xy) + \log_x(yz) + \log_y(zx) \geq 6$$

Proof: We take in (2): $a = \ln x\,; b = \ln y\,; c = \ln z$

$$\frac{\ln x + \ln y}{\ln z} + \frac{\ln y + \ln z}{\ln x} + \frac{\ln z + \ln x}{\ln y} \geq 6$$

$$\frac{\ln(xy)}{\ln z}+\frac{\ln(yz)}{\ln x}+\frac{\ln(zx)}{\ln y}\geq 6,\qquad \log_z(xy)+\log_x(yz)+\log_y(zx)\geq 6$$

Equality holds for: $x=y=z$.

Application 3: If $a,b,c,d>0$ then:

$$\frac{a+b+c}{d}+\frac{b+c+d}{a}+\frac{c+d+a}{b}+\frac{d+a+b}{c}\geq 12\qquad (3)$$

Proof: We take in (1): $n=4;k=3$. Equality holds for $a=b=c=d$.

Application 4: If $x,y,z,t>1$ then:

$$\log_t(xyz)+\log_x(yzt)+\log_y(ztx)+\log_z(txy)\geq 12$$

Proof: We take in (3): $a=\ln x\,;b=\ln y\,;c=\ln z\,;d=\ln t$

$$\frac{\ln x+\ln y+\ln z}{\ln t}+\frac{\ln y+\ln z+\ln t}{\ln x}+\frac{\ln z+\ln t+\ln x}{\ln y}+\frac{\ln t+\ln x+\ln y}{\ln z}\geq 12$$

$$\frac{\ln(xyz)}{\ln t}+\frac{\ln(yzt)}{\ln x}+\frac{\ln(ztx)}{\ln y}+\frac{\ln(txy)}{\ln z}\geq 12$$

$$\log_t(xyz)+\log_x(yzt)+\log_y(ztx)+\log_z(txy)\geq 12$$

Equality holds for $x=y=z=t$.

Application 5:

If $a,b,c,d,e>0$ then:

$$\frac{a+b+c}{d+e}+\frac{b+c+d}{e+a}+\frac{c+d+a}{a+b}+\frac{d+a+b}{b+c}+\frac{a+b+c}{c+d}\geq\frac{15}{2}\qquad (4)$$

Proof: We take in (1): $n=5;k=3$

Application 6:

If $x,y,z,t,u>1$ then:

$$\log_{tu}(xyz)+\log_{ux}(yzt)+\log_{xy}(ztu)+\log_{yz}(tux)+\log_{zt}(uxy)\geq\frac{15}{2}$$

Proof: We take in (4):

$$a=\ln x\,;b=\ln y\,;c=\ln z\,;d=\ln t\,;e=\ln u\,,\qquad a,b,c,d,e>0$$

$$\sum_{cyc}\frac{\ln x+\ln y+\ln z}{\ln t+\ln u}\geq\frac{15}{2};\ \sum_{cyc}\frac{\ln(xyz)}{\ln(tu)}\geq\frac{15}{2};\ \sum_{cyc}\log_{tu}(xyz)\geq\frac{15}{2}$$

Equality holds for $x=y=z=t=u$.

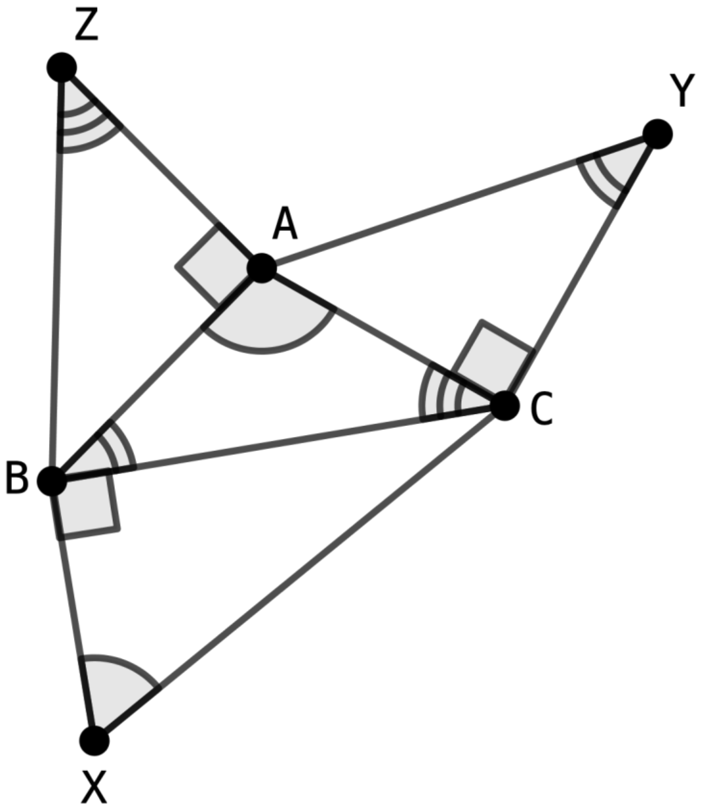
Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

A NEW MEHMET ȘAHIN'S CONFIGURATION

By Daniel Sitaru, Claudia Nănuți-Romania

Abstract: Let be the triangle ABC with sides a, b, c. Let be the points $X, Y, Z \notin Int(\Delta ABC)$ such that $m(\sphericalangle BAZ) = m(\sphericalangle CBX) = m(\sphericalangle ACY) = 90°$ and $m(\sphericalangle BXC) = m(\sphericalangle A)$;

$m(\sphericalangle CYA) = m(\sphericalangle B); m(\sphericalangle AZB) = m(\sphericalangle C)$. In this article we will study the properties of this configuration.



Property 1: $[BXC] + [CYA] + [AZB] = 2[ABC]$

Proof: Lemma: In any triangle ABC with area F holds:

$$16F^2 = 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4$$

Proof of lemma: By Heron's formula:

$$F^2 = s(s-a)(s-b)(s-c)$$

$$F^2 = \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2}$$

$$16F^2 = ((b+c)^2 - a^2)(a^2 - (b-c)^2)$$

$$16F^2 = (b^2 + c^2 - a^2 + 2bc)(a^2 - b^2 - c^2 + 2bc)$$

$$16F^2 = 4b^2c^2 - (b^2 + c^2 - a^2)^2$$

$$16F^2 = 4b^2c^2 - b^4 - c^4 - a^4 + 2a^2b^2 + 2b^2c^2 - 2b^2c^2$$

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$$

$$16F^2 = 2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4 \qquad (1)$$

Back to the main proof: In ΔCBX:

$$\cot A = \frac{BX}{BC} \Rightarrow BX = BC\cot A = a\cot A$$

Analogous: $CY = b\cot B\,;AZ = c\cot C$

$$[BXC] + [CYA] + [AZB] = \frac{BC\cdot BX}{2} + \frac{CA\cdot CY}{2} + \frac{AB\cdot AZ}{2} = \frac{a^2\cot A}{2} + \frac{b^2\cot B}{2} + \frac{c^2\cot C}{2} =$$

$$= \sum_{cyc}\frac{a^2\cot A}{2} = \frac{1}{2}\sum_{cyc}\frac{a^2\cos A}{\sin A} = \frac{1}{2}\cdot\sum_{cyc}\frac{a^2(b^2+c^2-a^2)}{2bc\sin A} = \frac{1}{2}\sum_{cyc}\frac{a^2(b^2+c^2-a^2)}{4F} =$$

$$= \frac{1}{8F}\sum_{cyc}(a^2b^2 + a^2c^2 - a^4) = \frac{1}{8F}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right) \overset{(1)}{=} \frac{1}{8F}\cdot 16F^2 = 2F = 2[ABC]$$

Property 2: $AZ + ZB + BX + XC + CY + YA = 2(r_a + r_b + r_c)$

Proof: Lemma: In any triangle ABC with r – inradii; R – circumradii; r_a, r_b, r_c – exradies holds:

$$\boldsymbol{r_a + r_b + r_c = 4R + r} \qquad \textbf{(2)}$$

Proof of lemma:

$$r_a + r_b + r_c = \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} =$$

$$= F\sum_{cyc}\frac{1}{s-a} = \frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}(s-b)(s-c) =$$

$$= \frac{Fs}{s(s-a)(s-b)(s-c)}\cdot\sum_{cyc}(s^2 - s(b+c) + bc) = \frac{Fs}{F^2}\sum_{cyc}(s^2 - s(2s-a) + bc) =$$

$$= \frac{s}{F}\sum_{cyc}(s^2 - 2s^2 + sa + bc) = \frac{s}{rs}\left(-3s^2 + s\sum_{cyc}a + \sum_{cyc}bc\right) =$$

$$= \frac{1}{r}(-3s^2 + s\cdot 2s + s^2 + r^2 + 4Rr) = \frac{1}{r}(4Rr + r^2) = 4R + r$$

Back to the main proof: In ΔBXC :

$$XC^2 = BX^2 + BC^2 = a^2\cot^2 A + a^2 = a^2(1 + \cot^2 A) =$$

$$= a^2\left(1+\frac{\cos^2 A}{\sin^2 A}\right) = a^2\cdot\frac{\sin^2 A+\cos^2 A}{\sin^2 A} = \frac{a^2}{\sin^2 A}, \qquad XC = \frac{a}{\sin A} = \frac{2R\sin A}{\sin A} = 2R$$

Analogous: $YA = ZB = 2R$

$$AZ + ZB + BX + XC + CY + YA = c\cot C + 2R + a\cot A + 2R + b\cot B + 2R =$$

$$= 6R + \sum_{cyc} a\cot A = 6R + \sum_{cyc} a\cdot\frac{\cos A}{\sin A} = 6R + \sum_{cyc}\frac{a}{\sin A}\cdot\cos A = 6R + \sum_{cyc} 2R\cos A =$$

$$= 6R + 2R\sum_{cyc}\cos A = 6R + 2R\left(1+\frac{r}{R}\right) = 6R + 2R + 2r = 8R + 2r = 2(4R + r) =$$

$$\stackrel{(2)}{=} 2(r_a + r_b + r_c)$$

Property 3: $[AZBXCY] = 3[ABC]$

Proof: Lemma:In ΔABC the following relationship holds:

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C \quad (3)$$

Proof of lemma:

$$\sin 2A + \sin 2B + \sin 2C = 2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2} + \sin 2C =$$

$$= 2\sin(A+B)\cos(A-B) + \sin 2C = 2\sin(\pi - c)\cos(A-B) + \sin 2C =$$

$$= 2\sin C\cos(A-B) + 2\sin C\cos C = 2\sin C\left(\cos(A-B) + \cos C\right) =$$

$$= 2\sin C\cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2} = 4\sin C\cos\frac{\pi-2B}{2}\cos\frac{A-(\pi-A)}{2} =$$

$$= 4\sin C\cos\left(\frac{\pi}{2}-B\right)\cos\left(A-\frac{\pi}{2}\right) = 4\sin C\sin B\sin A$$

Back to the main proof: $[AZBXCY] = [ABC] + [ABZ] + [BCX] + [CAY] =$

$$= F + \sum_{cyc}\frac{AB\cdot AZ}{2} = F + \sum_{cyc}\frac{c\cdot\cot C}{2} = F + \frac{1}{2}\sum_{cyc} c^2\cdot\frac{\cos C}{\sin C} = F + \frac{1}{2}\sum_{cyc}\frac{c}{\sin C}\cdot c\cos C =$$

$$= F + \frac{1}{2}\sum_{cyc} 2R\cdot 2R\sin C\cos C = F + \frac{1}{2}\cdot 2R^2\sum_{cyc} 2\sin C\cos C = F + R^2\sum_{cyc}\sin 2C \stackrel{(3)}{=}$$

$$= F + R^2\cdot 4\sin A\sin B\sin C = F + 4R^2\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R} = F + \frac{abc}{2R} = F + \frac{4RF}{2R} =$$

$$= F + 2F = 3F = 3[ABC]$$

Reference:

[1] Romanian Mathematical Magazine, www.ssmrmh.ro

NEW INEQUALITIES WITH FIBONACCI AND LUCAS NUMBERS

By D.M. Bătineţu-Giurgiu, Mihaly Bencze, Daniel Sitaru and Neculai Stanciu

ABSTRACT. In this paper we present new limits of sequences and functions.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in N$ **.**
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in N$ **.**

Theorem 1: In any triangle ABC with usual notations and the area F holds the following inequality

$$\frac{a^{F_n} b^{F_{n+2}}}{h_a^{F_{n+1}}} + \frac{b^{F_n} c^{F_{n+2}}}{h_b^{F_{n+1}}} + \frac{c^{F_n} a^{F_{n+2}}}{h_c^{F_{n+1}}} \geq 2^{F_n + F_{n+2}} (\sqrt{3})^{2-F_{n+2}} F^{F_n}.$$

Proof: $$\sum_{cyc} \frac{a^{F_n} b^{F_{n+2}}}{h_a^{F_{n+1}}} = \sum_{cyc} \frac{a^{F_n + F_{n+1}} b^{F_{n+2}}}{(ah_a)^{F_{n+1}}} = \frac{1}{(2F)^{F_{n+1}}} \sum_{cyc} (ab)^{F_{n+2}} \overset{Radon}{\geq}$$

$$\overset{Radon}{\geq} \frac{1}{2^{F_{n+1}} F^{F_{n+1}}} \cdot \frac{1}{3^{F_{n+2}-1}} \cdot (\sum_{cyc} ab)^{F_{n+2}} \overset{Gordon}{\geq} \frac{1}{2^{F_{n+1}} F^{F_{n+1}} 3^{F_{n+2}-1}} \cdot (4F\sqrt{3})^{F_{n+2}} =$$

$$= 2^{F_n + F_{n+2}} (\sqrt{3})^{2-F_{n+2}} F^{F_n}$$, Q.E.D.

Theorem 2: In any triangle ABC with usual notations and the area F holds the following inequality

$$\frac{a^{L_n} b^{L_{n+2}}}{h_a^{L_{n+1}}} + \frac{b^{L_n} c^{L_{n+2}}}{h_b^{L_{n+1}}} + \frac{c^{L_n} a^{L_{n+2}}}{h_c^{L_{n+1}}} \geq 2^{L_n + L_{n+2}} (\sqrt{3})^{2-L_{n+2}} F^{L_n}$$

Proof: $$\sum_{cyc} \frac{a^{L_n} b^{L_{n+2}}}{h_a^{L_{n+1}}} = \sum_{cyc} \frac{a^{L_n + L_{n+1}} b^{L_{n+2}}}{(ah_a)^{L_{n+1}}} = \frac{1}{(2F)^{L_{n+1}}} \sum_{cyc} (ab)^{L_{n+2}} \overset{Radon}{\geq}$$

$$\overset{Radon}{\geq} \frac{1}{2^{L_{n+1}} F^{L_{n+1}}} \cdot \frac{1}{3^{L_{n+2}-1}} \cdot (\sum_{cyc} ab)^{L_{n+2}} \overset{Gordon}{\geq} \frac{1}{2^{L_{n+1}} F^{L_{n+1}} 3^{L_{n+2}-1}} \cdot (4F\sqrt{3})^{L_{n+2}} =$$

$$= 2^{L_n + L_{n+2}} (\sqrt{3})^{2-L_{n+2}} F^{L_n}$$, Q.E.D.

Theorem 3: In any triangle ABC with usual notations and the area F holds the following inequality

$$\frac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}}+\frac{b^{F_n^2}c^{F_{2n+1}}}{h_b^{F_{n+1}^2}}+\frac{c^{F_n^2}a^{F_{2n+1}}}{h_c^{F_{n+1}^2}}\geq 2^{F_{2n+1}+F_n^2}(\sqrt{3})^{2-F_{2n+1}}F^{F_n^2}.$$

Proof: $$\sum_{cyc}\frac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}}=\sum_{cyc}\frac{a^{F_n^2+F_{n+1}^2}b^{F_{2n+1}}}{(ah_a)^{F_{n+1}^2}}=\frac{1}{(2F)^{F_{n+1}^2}}\sum_{cyc}(ab)^{F_{2n+1}}\overset{Radon}{\geq}$$

$$\overset{Radon}{\geq}\frac{1}{2^{F_{n+1}^2}F^{F_{n+1}^2}}\cdot\frac{1}{3^{F_{2n+1}-1}}\cdot\left(\sum_{cyc}ab\right)^{F_{2n+1}}\overset{Gordon}{\geq}\frac{1}{2^{F_{n+1}^2}F^{F_{n+1}^2}3^{F_{2n+1}-1}}\cdot(4F\sqrt{3})^{F_{2n+1}}=$$

$$=2^{F_{2n+1}+F_n^2}(\sqrt{3})^{2-F_{2n+1}}F^{F_n^2}$$, Q.E.D.

Theorem 4: (a) If $a,b,c\in R_+^*$ such that $abc=1$, then the following inequality is true

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}};$$

(b) If $a,b,c\in R_+^*$ such that $ab+bc+ca=3$, then is true

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}};$$

(c) If $a,b,c\in R_+^*$ such that $a+b+c=3$, then is true

$$\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}+\frac{1}{b^3(F_n^2c+F_{n+1}^2a)}+\frac{1}{c^3(F_n^2a+F_{n+1}^2b)}\geq\frac{3}{F_{2n+1}}.$$

Proof:

(a). $$\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$$

$$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)}=\frac{ab+bc+ca}{F_{n+2}}\geq\frac{3\cdot\sqrt[3]{(abc)^2}}{F_{n+2}}=\frac{3}{F_{n+2}}.$$

(b). $\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$

$$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)}=\frac{3}{(abc)^2F_{n+2}}\text{, (1).}$$

But, $ab+bc+ca\geq 3\cdot\sqrt[3]{(abc)^2}\Leftrightarrow(ab+bc+ca)^3\geq 27(abc)^2\Leftrightarrow(abc)^2\leq 1\Leftrightarrow\frac{1}{abc}\geq 1$, (2).

From (1) and (2) yields the desired inequality.

c). $\sum_{cyc}\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n^2+acF_{n+1}^2}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n^2+acF_{n+1}^2)}=$

$$=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(F_n^2+F_{n+1}^2)(ab+bc+ca)}=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{F_{2n+1}(ab+bc+ca)}\overset{Bergstrom}{\geq}\frac{\left(\frac{9}{a+b+c}\right)^2}{F_{2n+1}(ab+bc+ca)}\geq$$

$$\geq\frac{9}{F_{2n+1}\cdot\frac{(a+b+c)^2}{3}}=\frac{3}{F_{2n+1}}\text{, Q.E.D.}$$

Remarks:

- **If** $a,b,c\in R_+^*$ **such that** $abc=1$**, then**

$$\sum_{cyc}\frac{1}{a^3(b+c)}\geq\frac{3}{2}$$ **, (IMO – 1995 – Toronto – Canada).**

- **The problem from above show that the inequality also occurs if** $a,b,c\in R_+^*$ **such that** $a+b+c=3$ **and if** $a,b,c\in R_+^*$ **such that** $ab+bc+ca=3$**.**

Theorem 5: If $a,b,c\in R_+^*$ **such that** $abc=1$**, then holds**

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}}.$$

Proof: $\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$

$$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)}=\frac{ab+bc+ca}{F_{n+2}}\geq\frac{3\cdot\sqrt[3]{(abc)^2}}{F_{n+2}}=\frac{3}{F_{n+2}}.$$

Theorem 6: If $a,b,c\in R_+^*$ **such that** $ab+bc+ca=3$**, then is true**

$$\frac{1}{a^3(F_nb+F_{n+1}c)}+\frac{1}{b^3(F_nc+F_{n+1}a)}+\frac{1}{c^3(F_na+F_{n+1}b)}\geq\frac{3}{F_{n+2}}.$$

Proof. $\sum_{cyc}\frac{1}{a^3(F_nb+F_{n+1}c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n+acF_{n+1}}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n+acF_{n+1})}=$

$$=\frac{(ab+bc+ca)^2}{(abc)^2(F_n+F_{n+1})(ab+bc+ca)}=\frac{3}{(abc)^2F_{n+2}}, (1).$$

But, $ab+bc+ca\geq 3\cdot\sqrt[3]{(abc)^2}\Leftrightarrow(ab+bc+ca)^3\geq 27(abc)^2\Leftrightarrow(abc)^2\leq 1\Leftrightarrow\frac{1}{abc}\geq 1$, (2).

From (1) and (2) yields the desired inequality.

Theorem 7: If $a,b,c\in R_+^*$ **such that** $a+b+c=3$**, then**

$$\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}+\frac{1}{b^3(F_n^2c+F_{n+1}^2a)}+\frac{1}{c^3(F_n^2a+F_{n+1}^2b)}\geq\frac{3}{F_{2n+1}}.$$

Proof: $\sum_{cyc}\frac{1}{a^3(F_n^2b+F_{n+1}^2c)}=\sum_{cyc}\frac{\frac{1}{a^2}}{abF_n^2+acF_{n+1}^2}\overset{Bergstrom}{\geq}\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{\sum_{cyc}(abF_n^2+acF_{n+1}^2)}=$

$$=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{(F_n^2+F_{n+1}^2)(ab+bc+ca)}=\frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{F_{2n+1}(ab+bc+ca)}\overset{Bergstrom}{\geq}\frac{\left(\frac{9}{a+b+c}\right)^2}{F_{2n+1}(ab+bc+ca)}\geq$$

$$\geq\frac{9}{F_{2n+1}\cdot\frac{(a+b+c)^2}{3}}=\frac{3}{F_{2n+1}}, \text{Q.E.D.}$$

Remark: If $a,b,c\in R_+^*$ **such that** $abc=1$**,then** $\sum_{cyc}\frac{1}{a^3(b+c)}\geq\frac{3}{2}$**, (IMO – 1995 – Toronto – Canada).** The inequality also occurs if $a,b,c\in R_+^*$ such that $a+b+c=3$ and if $a,b,c\in R_+^*$ such that $ab+bc+ca=3$.

Theorem 8. If $m>1$**,** $n\in N^*$ **then** $\sum_{k=1}^{n}(1+L_k)^{2(m+1)}\geq\frac{(m+1)^{2(m+1)}}{m^{2m}}(L_nL_{n+1}-2)$**.**

Proof : We consider the function $f:R_+^*\to R_+^*$, $f(x)=m^m(1+x)^{m+1}-(m+1)^{m+1}x$ with $f'(x)=(m+1)\left((m+mx)^m-(m+1)^{m+1}\right)$ and $f''(x)=(m+1)\cdot m^{m+1}(1+x)^{m-1}>0$, $\forall x\in R_+^*$.

Therefore, f is convex and it has the minimum point $x_0 = \frac{1}{m}$. So, $f(x) \geq f\left(\frac{1}{m}\right) = 0$, i.e.

$$m^m (1+x)^{m+1} \geq (m+1)^{m+1} x \Leftrightarrow (1+x)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} x^2, \quad (1).$$

From (1) we get $(1+L_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} L_k^2$, $\forall k \in N^*$, so we obtain that

$$\sum_{k=1}^{n} (1+L_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \sum_{k=1}^{n} L_k^2,$$

and taking account by $\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$ we obtain the desired inequality.

Theorem 9: If $m > 1$**,** $n \in N^*$ **then** $\sum_{k=1}^{n} (1+F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} F_n F_{n+1}$**.**

Proof: We consider the function $f : R_+^* \to R_+^*$, $f(x) = m^m (1+x)^{m+1} - (m+1)^{m+1} x$ with $f'(x) = (m+1)\left((m+mx)^m - (m+1)^{m+1}\right)$ and $f''(x) = (m+1) \cdot m^{m+1} (1+x)^{m-1} > 0$, $\forall x \in R_+^*$.

Therefore, f is convex and it has the minimum point $x_0 = \frac{1}{m}$. So, $f(x) \geq f\left(\frac{1}{m}\right) = 0$, i.e.

$$m^m (1+x)^{m+1} \geq (m+1)^{m+1} x \Leftrightarrow (1+x)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} x^2, \quad (1).$$

From (1) we get $(1+F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} F_k^2$, $\forall k \in N^*$, so we obtain that

$\sum_{k=1}^{n} (1+F_k)^{2(m+1)} \geq \frac{(m+1)^{2(m+1)}}{m^{2m}} \sum_{k=1}^{n} F_k^2$, and taking account by $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$

we obtain the desired inequality

Theorem 10: If $x, y, z > 0$**, then**

$$\frac{x^2}{(5F_{2n}^2 y + 2z)(5F_{2n}^2 z + 2y)} + \frac{y^2}{(5F_{2n}^2 z + 2x)(5F_{2n}^2 x + 2z)} + \frac{z^2}{(5F_{2n}^2 x + 2y)(5F_{2n}^2 y + 2x)} \geq \frac{3}{L_{4n}^2}.$$

Proof:

$$5F_{2n}^2 + 2 = 5 \cdot \frac{1}{5}\left(\alpha^{2n} - \beta^{2n}\right)^2 + 2 = \alpha^{4n} + \beta^{4n} - 2(\alpha\beta)^{2n} + 2 = \alpha^{4n} + \beta^{4n} = L_{4n}, \quad (1).$$

$$\frac{x^2}{(5F_{2n}^2 y + 2z)(5F_{2n}^2 z + 2y)} = \frac{x^2}{25F_{2n}^4 yz + 10F_{2n}^2 y^2 + 10F_{2n}^2 z^2 + 4yz} =$$

$$= \frac{x^2}{(25F_{2n}^4 + 4)yz + 10F_{2n}^2 (y^2 + z^2)} \overset{PM-GM}{\geq} \frac{x^2}{(25F_{2n}^4 + 4) \cdot \frac{y^2 + z^2}{2} + 10F_{2n}^2 (y^2 + z^2)} =$$

$$= \frac{2x^2}{y^2 + z^2} \cdot \frac{1}{25F_{2n}^4 + 10F_{2n}^2 + 4} == \frac{2x^2}{y^2 + z^2} \cdot \frac{1}{(5F_{2n}^2 + 2)^2} \overset{(1)}{=} \frac{2x^2}{y^2 + z^2} \cdot \frac{1}{L_{4n}^2}, \quad (2).$$

And other two similar inequality, i.e.

$$\frac{y^2}{(5F_{2n}^2 z+2x)(5F_{2n}^2 x+2z)} \geq \frac{2y^2}{z^2+x^2}\cdot\frac{1}{L_{4n}^2}, \quad (3).$$

$$\frac{z^2}{(5F_{2n}^2 x+2y)(5F_{2n}^2 y+2x)} \geq \frac{2z^2}{x^2+y^2}\cdot\frac{1}{L_{4n}^2}, \quad (4).$$

Adding up the inequalities (2), (3) and (4) and taking account by Nesbitt-Ionescu inequality (i.e. $\sum_{cyclic}\frac{a}{b+c} \geq \frac{3}{2}$ for any $a,b,c>0$) we obtain:

$$\frac{x^2}{(5F_{2n}^2 y+2z)(5F_{2n}^2 z+2y)}+\frac{y^2}{(5F_{2n}^2 z+2x)(5F_{2n}^2 x+2z)}+\frac{z^2}{(5F_{2n}^2 x+2y)(5F_{2n}^2 y+2x)} \geq$$

$$\geq \frac{2x^2}{y^2+z^2}\cdot\frac{1}{L_{4n}^2}+\frac{2y^2}{z^2+x^2}\cdot\frac{1}{L_{4n}^2}+\frac{2z^2}{x^2+y^2}\cdot\frac{1}{L_{4n}^2} \overset{Nesbitt-Ionescu}{\geq} 2\cdot\frac{3}{2}\cdot\frac{1}{L_{4n}^2}=\frac{3}{L_{4n}^2}, \text{ Q.E.D.}$$

Theorem 11: If ABC is a triangle with a,b,c the lengths of the sides, R the lengths of circumradius, r the length of the inradius and s the semiperimeter, then

$$\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2+\left(\frac{F_n^2b^2+F_{n+1}^2c^2}{a}\right)^2+\left(\frac{F_n^2c^2+F_{n+1}^2a^2}{b}\right)^2 \geq 2F_{2n+1}^2(s^2-r^2-4Rr),$$

$\forall n\in N^*$.

Proof: By Bergström's inequality and the formula $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ we have

$$\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2+\left(\frac{F_n^2b^2+F_{n+1}^2c^2}{a}\right)^2+\left(\frac{F_n^2c^2+F_{n+1}^2a^2}{b}\right)^2=\sum_{cyclic}\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2=$$

$$=\sum_{cyclic}\frac{(F_n^2a^2+F_{n+1}^2b^2)^2}{c^2} \overset{Bergstrom}{\geq} \frac{\left(\sum_{cyclic}(F_n^2a^2+F_{n+1}^2b^2\right)^2}{a^2+b^2+c^2}=\frac{(F_n^2+F_{n+1}^2)^2(a^2+b^2+c^2)^2}{a^2+b^2+c^2}=$$

$=2F_{2n+1}^2(s^2-r^2-4Rr)$, q.e.d.

Theorem 12: If ABC be a triangle with a,b,c the lengths of the sides, R the lengths of circumradius, r the length of the inradius and s the semiperimeter, then

$$\left(\frac{F_na^2+F_{n+1}b^2}{c}\right)^2+\left(\frac{F_nb^2+F_{n+1}c^2}{a}\right)^2+\left(\frac{F_nc^2+F_{n+1}a^2}{b}\right)^2 \geq 2F_{n+2}^2(s^2-r^2-4Rr),\ \forall n\in N^*.$$

Proof: By Bergström's inequality and the formula $a^2+b^2+c^2=2(s^2-r^2-4Rr)$ we have

$$\left(\frac{F_na^2+F_{n+1}b^2}{c}\right)^2+\left(\frac{F_nb^2+F_{n+1}c^2}{a}\right)^2+\left(\frac{F_nc^2+F_{n+1}a^2}{b}\right)^2=\sum_{cyclic}\left(\frac{F_na^2+F_{n+1}b^2}{c}\right)^2=$$

$$=\sum_{cyclic}\frac{(F_na^2+F_{n+1}b^2)^2}{c^2}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyclic}(F_na^2+F_{n+1}b^2)\right)^2}{a^2+b^2+c^2}=\frac{(F_n+F_{n+1})^2(a^2+b^2+c^2)^2}{a^2+b^2+c^2}=$$

$=2F_{n+2}^2(s^2-r^2-4Rr)$, q.e.d.

Theorem 13: If $a,b,c\in\left(0,\frac{\pi}{2}\right)$ **, then**

$$\frac{\tan a}{F_n\sin 2b+F_{n+1}\sin 2c}+\frac{\tan b}{F_n\sin 2c+F_{n+1}\sin 2a}+\frac{\tan c}{F_n\sin 2a+F_{n+1}\sin 2b}>\frac{3}{2F_{n+2}},$$

$\forall n\in N^*$

Proof. From $a,b,c\in\left(0,\frac{\pi}{2}\right)$ yields that $\tan a>a,\tan b>b,\tan c>c$ and

$\sin 2a=2\sin a\cos a<2\sin a<2a$, similarly $\sin 2b<2b$, $\sin 2c<2c$.

Hence:

$$\frac{\tan a}{F_n\sin 2b+F_{n+1}\sin 2c}+\frac{\tan b}{F_n\sin 2c+F_{n+1}\sin 2a}+\frac{\tan c}{F_n\sin 2a+F_{n+1}\sin 2b}=$$

$$=\sum_{cyclic}\frac{\tan a}{F_n\sin 2b+F_{n+1}\sin 2c}>\sum_{cyclic}\frac{a}{2bF_n+2cF_{n+1}}=\frac{1}{2}\sum_{cyclic}\frac{a^2}{abF_n+acF_{n+1}}\overset{Bergstrom}{\geq}\frac{1}{2}\cdot\frac{(a+b+c)^2}{\sum_{cyclic}(abF_n+acF_{n+1})}=$$

$$=\frac{1}{2}\cdot\frac{(a+b+c)^2}{(F_n+F_{n+1})(ab+bc+ca)}=\frac{(a+b+c)^2}{2F_{n+2}(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2F_{n+2}(ab+bc+ca)}=\frac{3}{2F_{n+2}}$$, q.e.d.

Theorem 14: If $a,b,c\in\left(0,\frac{\pi}{2}\right)$ **, then**

$$\frac{\tan a}{F_n^2\sin 2b+F_{n+1}^2\sin 2c}+\frac{\tan b}{F_n^2\sin 2c+F_{n+1}^2\sin 2a}+\frac{\tan c}{F_n^2\sin 2a+F_{n+1}^2\sin 2b}>\frac{3}{2F_{2n+1}},\forall n\in N^*$$

Proof: From $a,b,c\in\left(0,\frac{\pi}{2}\right)$ yields that $\tan a>a,\tan b>b,\tan c>c$ and

$\sin 2a=2\sin a\cos a<2\sin a<2a$, similarly $\sin 2b<2b$, $\sin 2c<2c$.

Hence: $\frac{\tan a}{F_n^2\sin 2b+F_{n+1}^2\sin 2c}+\frac{\tan b}{F_n^2\sin 2c+F_{n+1}^2\sin 2a}+\frac{\tan c}{F_n^2\sin 2a+F_{n+1}^2\sin 2b}=$

$$=\sum_{cyclic}\frac{\tan a}{F_n^2\sin 2b+F_{n+1}^2\sin 2c}>\sum_{cyclic}\frac{a}{2bF_n^2+2cF_{n+1}^2}=\frac{1}{2}\sum_{cyclic}\frac{a^2}{abF_n^2+acF_{n+1}^2}\overset{Bergstrom}{\geq}\frac{1}{2}\cdot\frac{(a+b+c)^2}{\sum_{cyclic}(abF_n^2+acF_{n+1}^2)}=$$

$$=\frac{1}{2}\cdot\frac{(a+b+c)^2}{(F_n^2+F_{n+1}^2)(ab+bc+ca)}=\frac{(a+b+c)^2}{2F_{2n+1}(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{2F_{2n+1}(ab+bc+ca)}=\frac{3}{2F_{2n+1}}$$, q.e.d.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro

CERTAIN LIMITS OF FIBONACCI AND LUCAS' SEQUENCES AND FUNCTIONS

By D.M. Bătineţu-Giurgiu, Mihaly Bencze, Daniel Sitaru and Neculai Stanciu

ABSTRACT: In this paper we present new limits of sequences and functions.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n, \forall n \in N$ **.**
Lucas sequence: $(L_n)_{n\geq 0}, L_0 = 2, L_1 = 1, L_{n+2} = L_{n+1} + L_n, \forall n \in N$ **.**

Theorem 1: $\lim_{n\to\infty}(\sqrt[n]{n!})^2\left(\frac{\sqrt[n]{n!L_n}}{n^2} - \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\right) = \frac{\alpha}{e^3}\left(1+\frac{1}{2}\ln 5\right)$

Proof: We have $L_n = \alpha^n + \beta^n$, $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

$\lim_{n\to\infty}\frac{\sqrt[n]{n!L_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!L_n}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!L_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!L_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{L_{n+1}}{L_n} = \frac{1}{e}\cdot\alpha = \frac{\alpha}{e}$,and analogous

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!F_n}}{n} = \frac{\alpha}{e}.$$

If we denote $u_n = \frac{\sqrt[n]{n!L_n}}{\sqrt[n+1]{(n+1)!F_{n+1}}}\left(\frac{n+1}{n}\right)^2$, we deduce $\lim_{n\to\infty}u_n = 1$; $\lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1$ and

$$\lim_{n\to\infty}u_n^n = \lim_{n\to\infty}\frac{n!L_n}{(n+1)!F_{n+1}}\left(\frac{n+1}{n}\right)^{2n}\sqrt[n+1]{(n+1)!F_{n+1}} = e^2\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1}\cdot\frac{L_n}{F_{n+1}} =$$

$$= e^2\cdot\frac{\alpha}{e}\cdot\lim_{n\to\infty}\frac{\sqrt{5}(\alpha^n+\beta^n)}{\alpha^{n+1}-\beta^{n+1}} = \alpha e\cdot\frac{\sqrt{5}}{\alpha} = e\sqrt{5}.$$

Hence,

$$\lim_{n\to\infty}(\sqrt[n]{n!})^2\left(\frac{\sqrt[n]{n!L_n}}{n^2} - \frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\right) = \lim_{n\to\infty}(\sqrt[n]{n!})^2\cdot\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}(u_n - 1) =$$

$$= \lim_{n\to\infty}\frac{(\sqrt[n]{n!})^2\sqrt[n+1]{(n+1)!F_{n+1}}}{(n+1)^2}\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^2\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1}\cdot\frac{n}{n+1}\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n^n = \frac{1}{e^2}\cdot\frac{\alpha}{e}\cdot 1\cdot 1\cdot\ln(e\sqrt{5}) = \frac{\alpha}{e^3}\left(1+\frac{1}{2}\ln 5\right).$$

Theorem 2: If $m \in N$, then $\lim_{n\to\infty}n^{\cos^2 F_m}\left((\sqrt[n+1]{(n+1)!})^{\sin^2 F_m} - (\sqrt[n]{n!})^{\sin^2 F_m}\right) = \frac{\sin^2 F_m}{e^{\sin^2 F_m}}$.

Proof: We have well-known $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, so $\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m} = e^{-\sin^2 F_m}$.

We denote $u_n = \left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{\sin^2 F_m}$, $\forall n \geq 2$, so $\lim_{n\to\infty} u_n = 1$ and $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$.

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\right)^{\sin^2 F_m} \frac{1}{\left(\sqrt[n+1]{(n+1)!}\right)^{\sin^2 F_m}} = \lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{\sin^2 F_m} = e^{\sin^2 F_m}.$$

We have $x_n \overset{denote}{=} n^{\cos^2 F_m}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{\sin^2 F_m} - \left(\sqrt[n]{n!}\right)^{\sin^2 F_m}\right) = n^{\cos^2 F_m}(u_n - 1)\left(\sqrt[n]{n!}\right)^{\sin^2 F_m} =$

$$= n^{\cos^2 F_m + \sin^2 F_m}\left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m}(u_n - 1) = n\left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m}\frac{u_n - 1}{\ln u_n}\ln u_n =$$

$$= \left(\frac{\sqrt[n]{n!}}{n}\right)^{\sin^2 F_m}\frac{u_n - 1}{\ln u_n}\ln u_n^n.$$

Hence,

$$\lim_{n\to\infty} n^{\cos^2 F_m}\left(\left(\sqrt[n+1]{(n+1)!}\right)^{\sin^2 F_m} - \left(\sqrt[n]{n!}\right)^{\sin^2 F_m}\right) = \lim_{n\to\infty} x_n = e^{-\sin^2 F_m}\cdot 1\cdot \ln e^{\sin^2 F_m} = \frac{\sin^2 F_m}{e^{\sin^2 F_m}}.$$

Theorem 3: $\lim_{n\to\infty}\left(\int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} \sqrt{\frac{\sin^4 x + \cos^4 x}{8}} dx\right) \geq \frac{\alpha}{4e}$.

Proof: By Bergström's inequality we have $\sin^4 x + \cos^4 x \geq \frac{(\sin^2 x + \cos^2 x)^2}{2} = \frac{1}{2}$, (1).

From (1) we deduce that

$$I_n = \int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} \sqrt{\frac{\sin^4 x + \cos^4 x}{8}} dx \geq \frac{1}{4}\int_{\sqrt[n]{n!F_n}}^{\sqrt[n+1]{(n+1)!F_{n+1}}} dx = \frac{1}{4}\left(\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n}\right), (2).$$

Next, we have:

- $F_{n+2} - F_{n+1} - F_n = 0$, $\forall n \in N^* \Leftrightarrow \frac{F_{n+2}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Leftrightarrow \frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1 = 0 \Rightarrow$

$\Rightarrow \lim_{n\to\infty}\left(\frac{F_{n+2}}{F_{n+1}}\cdot\frac{F_{n+1}}{F_n} - \frac{F_{n+1}}{F_n} - 1\right) = 0 \Leftrightarrow x^2 - x - 1 = 0$, where $x = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}$, so $x = \frac{1 \pm \sqrt{5}}{2}$.

Since $\frac{F_{n+1}}{F_n} > 0$, yields $x = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \frac{\sqrt{5}+1}{2} = \alpha$.

- $\lim_{n\to\infty}\frac{\sqrt[n]{n!F_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n = \frac{\alpha}{e}$.

Therefore

$$\sqrt[n+1]{(n+1)!F_{n+1}} - \sqrt[n]{n!F_n} = \sqrt[n]{n!F_n}(u_n - 1) = \sqrt[n]{n!F_n}\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n =$$

$$=\frac{\sqrt[n]{n!F_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\ \forall n\geq 2,$$ (3), where $u_n=\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{\sqrt[n]{n!F_n}}$.

- $\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!F_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{n!F_n}}\cdot\frac{n+1}{n}\right)=\frac{\alpha}{e}\cdot\frac{e}{\alpha}\cdot 1=1$, so $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$, and
- $\lim\limits_{n\to\infty}u_n^n=\lim\limits_{n\to\infty}\frac{(n+1)!F_{n+1}}{n!F_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!F_{n+1}}}=\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!F_{n+1}}}=\alpha\cdot\frac{e}{\alpha}=e$.

By (3) we obtain

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!F_{n+1}}-\sqrt[n]{n!F_n}\right)=\frac{\alpha}{e}\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\frac{\alpha}{e}\cdot\ln e=\frac{\alpha}{e}\cdot 1=\frac{\alpha}{e},\ (4).$$

From (2) and (4) yields the desired conclusion.

Theorem 4: $\lim\limits_{n\to\infty}\left(\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}}\sqrt[3]{\frac{\sin^6 x+\cos^6 x}{16}}dx\right)\geq\frac{\alpha}{4e}$.

Proof: By Radon's inequality we have $\sin^6 x+\cos^6 x\geq\frac{(\sin^2 x+\cos^2 x)^3}{2^2}=\frac{1}{4}$, (1).

From (1) we deduce that

$$I_n=\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}}\sqrt[3]{\frac{\sin^6 x+\cos^6 x}{16}}dx\geq\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}}\sqrt[3]{\frac{1}{4^3}}dx=\frac{1}{4}\int_{\sqrt[n]{n!L_n}}^{\sqrt[n+1]{(n+1)!L_{n+1}}}dx=$$

$$=\frac{1}{4}\left(\sqrt[n+1]{(n+1)!L_{n+1}}-\sqrt[n]{n!L_n}\right),\ (2).$$

Now,

$$L_{n+2}-L_{n+1}-L_n=0,\ \forall n\in N^*\Leftrightarrow\frac{L_{n+2}}{L_n}-\frac{L_{n+1}}{L_n}-1=0\Leftrightarrow\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1=0\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\left(\frac{L_{n+2}}{L_{n+1}}\cdot\frac{L_{n+1}}{L_n}-\frac{L_{n+1}}{L_n}-1\right)=0\Leftrightarrow x^2-x-1=0,$$ where $x=\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}$, so $x=\frac{1\pm\sqrt{5}}{2}$.

Since $\frac{L_{n+1}}{L_n}>0$, yields $x=\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}=\frac{\sqrt{5}+1}{2}=\alpha$.

We deduce: $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!L_n}}{n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n!L_n}{n^n}}=\lim\limits_{n\to\infty}\frac{(n+1)!L_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!L_n}=\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}$.

$$\sqrt[n+1]{(n+1)!L_{n+1}}-\sqrt[n]{n!L_n}=\sqrt[n]{n!L_n}(u_n-1)=\sqrt[n]{n!L_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=$$

$$=\frac{\sqrt[n]{n!L_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n,\ \forall n\ge 2,\ (3),$$

$$\text{where } u_n=\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{\sqrt[n]{n!L_n}}.$$

$\lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!L_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{n!L_n}}\cdot\frac{n+1}{n}\right)=\frac{\alpha}{e}\cdot\frac{e}{\alpha}\cdot 1=1$, so $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$, and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!L_{n+1}}{n!L_n}\cdot\frac{1}{\sqrt[n+1]{(n+1)!L_{n+1}}}=\lim_{n\to\infty}\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{(n+1)!L_{n+1}}}=\alpha\cdot\frac{e}{\alpha}=e.$$

By (3) we obtain

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!L_{n+1}}-\sqrt[n]{n!L_n}\right)=\frac{\alpha}{e}\cdot 1\cdot\ln\left(\lim_{n\to\infty}u_n^n\right)=\frac{\alpha}{e}\cdot\ln e=\frac{\alpha}{e}\cdot 1=\frac{\alpha}{e},\ (4).$$

From (2) and (4) yields the desired conclusion.

Theorem 5: If $a>0$, then $\lim\limits_{n\to\infty}\sqrt[n]{n!F_n}(\sqrt[n]{a}-1)=\frac{\alpha}{e}\ln a$.

Proof: Since $\sqrt[n]{n!F_n}(\sqrt[n]{a}-1)=\sqrt[n]{n!F_n}\left(e^{\frac{\ln a}{n}}-1\right)=\sqrt[n]{n!F_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n}=$

$=\sqrt[n]{\frac{n!F_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a$, we have $\lim\limits_{n\to\infty}\sqrt[n]{n!F_n}(\sqrt[n]{a}-1)=\ln a\cdot 1\cdot\lim\limits_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}}=$

$$=\ln a\cdot\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n}=\ln a\cdot\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}\ln a.$$

Theorem 6: If $a>0$, then $\lim\limits_{n\to\infty}\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1)=\frac{2\alpha}{e}\ln a$.

Proof: Since $\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1)=\sqrt[n]{(2n-1)!!F_n}\left(e^{\frac{\ln a}{n}}-1\right)=\sqrt[n]{(2n-1)!!F_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n}=$

$$=\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a$$, we have

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!F_n}(\sqrt[n]{a}-1)=\ln a\cdot 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}=$$

$$=\ln a\cdot\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!F_n}=\ln a\cdot\lim_{n\to\infty}\frac{2n+1}{n+1}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=\frac{2\alpha}{e}\ln a.$$

Theorem 7: If $a>0$ and $(b_n)_{n\geq 1}$ is a positive real sequence with $\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b>0$, then

$$\lim_{n\to\infty}\sqrt[n]{b_nF_n}(\sqrt[n]{a}-1)=\frac{\alpha b}{e}\ln a.$$

Proof: Since $\sqrt[n]{b_nF_n}(\sqrt[n]{a}-1)=\sqrt[n]{b_nF_n}\left(e^{\frac{\ln a}{n}}-1\right)=\sqrt[n]{b_nF_n}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\frac{\ln a}{n}=$

$=\sqrt[n]{\frac{b_nF_n}{n^n}}\cdot\frac{e^{\frac{\ln a}{n}}-1}{\frac{\ln a}{n}}\cdot\ln a$, we have $\lim_{n\to\infty}\sqrt[n]{n!F_n}(\sqrt[n]{a}-1)=\ln a\cdot 1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{b_nF_n}{n^n}}=$

$$=\ln a\cdot\lim_{n\to\infty}\frac{b_{n+1}F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_nF_n}=\ln a\cdot\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\cdot\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{\alpha b}{e}\ln a.$$

Theorem 8: $\lim_{n\to\infty}n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3}=\frac{\alpha}{e}$.

Proof: Since $n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3}=\frac{\sqrt[n]{n!F_n}}{n}\cdot n^3\cdot\sin\frac{1}{n^3}=\frac{\sin\frac{1}{n^3}}{\frac{1}{n^3}}\cdot\sqrt[n]{\frac{n!F_n}{n^n}}$, then

$$\lim_{n\to\infty}n^2\sqrt[n]{n!F_n}\sin\frac{1}{n^3}=1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{n!F_n}{n^n}}=\lim_{n\to\infty}\frac{(n+1)!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{n!F_n}=\lim_{n\to\infty}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=\frac{\alpha}{e}.$$

Theorem 9: $\lim_{n\to\infty}n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2}=\frac{2\alpha}{e}$.

Proof. Since $n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2}=\frac{\sqrt[n]{(2n-1)!!F_n}}{n}\cdot n^2\cdot\sin\frac{1}{n^2}=\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}\cdot\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}$, then

$$\lim_{n\to\infty} n\sqrt[n]{n!F_n}\sin\frac{1}{n^2}=1\cdot\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!F_n}{n^n}}=\lim_{n\to\infty}\frac{(2n+1)!!F_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!F_n}=$$

$$=\lim_{n\to\infty}\frac{2n+1}{n+1}\frac{F_{n+1}}{F_n}\left(\frac{n}{n+1}\right)^n=\frac{2\alpha}{e}.$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT THE PROBLEM PP37343-OCTOGON MATHEMATICAL MAGAZINE

By Marin Chirciu-Romania

In ΔABC the following relationship holds:

$$\sum\frac{y+z}{x}\cdot\frac{c^2}{h_b^2}\ge 8$$

By D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

Solution:

$$\sum\frac{y+z}{x}\cdot\frac{c^2}{h_b^2}=\sum\frac{y+z}{x}\cdot\frac{b^2c^2}{b^2h_b^2}=\sum\frac{y+z}{x}\cdot\frac{b^2c^2}{4S^2}=\sum\frac{y+z}{x}\cdot\frac{a^2b^2c^2}{4S^2a^2}=4R^2\sum\frac{y+z}{x}\cdot\frac{1}{a^2}$$

Lemma: Let be $x,y,z>0$ and $f:D\to\mathbb{R}$ a function. The following relationship holds:

$$\sum\frac{y+z}{x}f^2(a)\ge 2\sum f(b)\,f(c)$$

Proof:

We have $\sum\frac{y+z}{x}f^2(a)=\sum\left(\frac{y+z}{x}+1-1\right)f^2(a)=\sum\frac{x+y+z}{x}f^2(a)-\sum f^2(a)\overset{CS}{\ge}$

$$\overset{CS}{\ge}(x+y+z)\frac{(\sum f(a))^2}{x+y+z}-\sum f^2(a)=(x+y+z)\frac{(\sum f(a))^2}{(x+y+z)}-\sum f^2(a)=$$

$$=\left(\sum f(a)\right)^2-\sum f^2(a)=\sum f^2(a)+2\sum f(b)\,f(c)-\sum f^2(a)=2\sum f(b)f(c).$$

Using Lemma, with $f(a)=\frac{1}{a}$ we obtain

$$\sum\frac{y+z}{x}\cdot\frac{1}{a^2}\ge 2\sum f(b)f(c)=2\sum\frac{1}{b}\cdot\frac{1}{c}=2\frac{\sum a}{abc}=2\cdot\frac{2p}{4Rrp}=\frac{1}{Rr}$$

We obtain

$$LHS = \sum \frac{y+z}{x} \cdot \frac{c^2}{h_b^2} = 4R^2 \sum \frac{y+z}{x} \cdot \frac{1}{a^2} \overset{Lemma}{\geq} 4R^2 \cdot \frac{1}{Rr} = \frac{4R}{r} \overset{Euler}{\geq} 8 = RHS$$

Equality holds if and only if the triangle is equilateral. **Remark:**The inequality can be strengthened.

In ΔABC holds:

$$\sum \frac{y+z}{x} \cdot \frac{c^2}{h_b^2} \geq \frac{4R}{r}$$

Marin Chirciu-Romania

Solution:

$$\sum \frac{y+z}{x} \cdot \frac{c^2}{h_b^2} = \sum \frac{y+z}{x} \cdot \frac{b^2c^2}{b^2h_b^2} = \sum \frac{y+z}{x} \cdot \frac{b^2c^2}{4S^2} = \sum \frac{y+z}{x} \cdot \frac{a^2b^2c^2}{4S^2a^2} = 4R^2 \sum \frac{y+z}{x} \cdot \frac{1}{a^2}$$

Lemma: Let be $x, y, z > 0$ and $f: D \to \mathbb{R}$ a function. The following relationship holds:

$$\sum \frac{y+z}{x} f^2(a) \geq 2 \sum f(b) f(c)$$

Proof: We have $\sum \frac{y+z}{x} f^2(a) = \sum \left(\frac{y+z}{x} + 1 - 1\right) f^2(a) = \sum \frac{x+y+z}{x} f^2(a) - \sum f^2(a) \overset{CS}{\geq}$

$$\overset{CS}{\geq} (x+y+z) \frac{(\sum f(a))^2}{x+y+z} - \sum f^2(a) = (x+y+z) \frac{(\sum f(a))^2}{(x+y+z)} - \sum f^2(a) =$$

$$= \left(\sum f(a)\right)^2 - \sum f^2(a) = \sum f^2(a) + 2 \sum f(b) f(c) - \sum f^2(a) = 2 \sum f(b) f(c)$$

Using Lemma, with $f(a) = \frac{1}{a}$ we obtain:

$$\sum \frac{y+z}{x} \cdot \frac{1}{a^2} \geq 2 \sum f(b) f(c) = 2 \sum \frac{1}{b} \cdot \frac{1}{c} = 2 \frac{\sum a}{abc} = 2 \cdot \frac{2p}{4Rrp} = \frac{1}{Rr}$$

We obtain

$$LHS = \sum \frac{y+z}{x} \cdot \frac{c^2}{h_b^2} = 4R^2 \sum \frac{y+z}{x} \cdot \frac{1}{a^2} \overset{Lemma}{\geq} 4R^2 \cdot \frac{1}{Rr} = \frac{4R}{r} \overset{Euler}{\geq} 8 = RHS$$

Equality holds if and only if the triangle is equilateral.

Note: The inequality strengthens Problem PP37343 from Octogon Magazine.

In ΔABC holds:

$$\sum \frac{y+z}{x} \cdot \frac{c^2}{h_b^2} \geq 8$$

D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

USAMO INEQUALITIES – GENERALIZATIONS

By Hüseyin Yigit Emekçi – Izmir – Turkey

1. Introduction

Generalization 1: Let a,b,c,d,n,k be nonnegative reals, $n>0$ and $a+b+c+d=n^k$. Then prove that

$$\frac{a}{\frac{kb^{k+1}}{n}+n^k}+\frac{b}{\frac{kc^{k+1}}{n}+n^k}+\frac{c}{\frac{kd^{k+1}}{n}+n^k}+\frac{d}{\frac{ka^{k+1}}{n}+n^k}\ge\frac{k\left(4-n^{k-1}\right)+4}{4k+4}$$

Solution:

$$\frac{a}{\frac{kb^{k+1}}{n}+n^k}+\frac{b}{\frac{kc^{k+1}}{n}+n^k}+\frac{c}{\frac{kd^{k+1}}{n}+n^k}+\frac{d}{\frac{ka^{k+1}}{n}+n^k}\ge p$$

$$\frac{a}{\frac{kb^{k+1}}{n}+n^k}-\frac{a}{n^k}+\frac{b}{\frac{kc^{k+1}}{n}+n^k}-\frac{b}{n^k}+\frac{c}{\frac{kd^{k+1}}{n}+n^k}-\frac{c}{n^k}+\frac{d}{\frac{ka^{k+1}}{n}+n^k}-\frac{d}{n^k}\ge$$

$$\ge p-\frac{a+b+c+d}{n^k}=p-1$$

$$a\left(\frac{1}{\frac{kb^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+b\left(\frac{1}{\frac{kc^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+c\left(\frac{1}{\frac{kd^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)+d\left(\frac{1}{\frac{ka^{k+1}}{n}+n^k}-\frac{1}{n^k}\right)$$

$$=-\left(\sum_{cyc}\frac{akb^{k+1}}{n^{k+1}\left(n^k+\frac{kb^{k+1}}{n}\right)}\right)\ge p-1$$

$$\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\frac{kb^{k+1}}{n}\right)}\right)\le 1-p$$

$$\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\frac{kb^{k+1}}{n}\right)}\right)=\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}\left(n^k+\underbrace{\frac{b^{k+1}}{n}+\frac{b^{k+1}}{n}+\cdots+\frac{b^{k+1}}{n}}_{k}\right)}\right)\le$$

$$\overset{AM-GM}{\le}\sum_{cyc}\left(\frac{akb^{k+1}}{n^{k+1}(k+1)b^k}\right)$$

$$=\sum_{cyc}\frac{akb}{n^{k+1}(k+1)}=\frac{k}{n^{k+1}(k+1)}(ab+bc+ca+ad)=\frac{k}{n^{k+1}(k+1)}(a+c)(b+d)$$

$$\overset{AM-GM}{\leq} \frac{k\left(\frac{a+b+c+d}{2}\right)^2}{n^{k+1}(k+1)} = \frac{kn^{2k}}{4n^{k+1}(k+1)} = \frac{kn^{k-1}}{4(k+1)} \leq 1-p$$

$$\rightarrow p \leq \frac{4k+4-kn^{k-1}}{4k+4} = \frac{k\left(4-n^{k-1}\right)+4}{4k+4}$$

Application on USAMO 2017/ 6

Let a, b, c, d be nonnegative reals such that $a+b+c+d=4$. Find the minimum value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

Solution: This problem is a special case of *Generalization 1.* The problem denotes case $n=k=2$. By that

$$\sum_{cyc} \frac{a}{\frac{kb^{k+1}}{n}+n^k} \geq \frac{k\left(4-n^{k-1}\right)+4}{4k+4} = \frac{2}{3}$$

which finishes the problem directly.

Generalization 2: Let a, b, c be positive reals. Then prove that

$$\sum_{cyc} \frac{\left((k+1)a+kb+kc\right)^2}{(k+1)a^2+kb^2+kc^2} \leq 6k+\frac{3}{2k+1}+1$$

Solution: Let us see the inequality is homogenous. WLOG Assume that $a+b+c=1$.

$$\sum_{cyc} \frac{\left((k+1)a+kb+kc\right)^2}{(k+1)a^2+kb^2+kc^2} = \sum_{cyc} \frac{(a+k)^2}{(k+1)a^2+k(1-a)^2} = \sum_{cyc} \frac{(a+k)^2}{(2k+1)a^2-2ak+k}$$

$$= \sum_{cyc} \left(\frac{1}{2k+1}+\frac{2ak+\frac{2ak}{2k+1}+k^2-\frac{k}{2k+1}}{(2k+1)a^2-2ak+k}\right) = S$$

By making some changes on denominator.

$$(2k+1)a^2-2ak+k = (2k+1)a^2-2ak+\frac{k^2}{2k+1}+k-\frac{k^2}{2k+1} \overset{AGO}{\geq}$$

$$\geq 2ak-2ak+k-\frac{k^2}{2k+1} = k-\frac{k^2}{2k+1}$$

$$S \leq \sum_{cyc} \left(\frac{1}{2k+1}+\frac{2ak+\frac{2ak}{2k+1}+k^2-\frac{k}{2k+1}}{k-\frac{k^2}{2k+1}}\right)$$

$$= \frac{3}{2k+1}+\frac{2k(a+b+c)+\frac{2k}{2k+1}(a+b+c)+3k^2-\frac{3k}{2k+1}}{k-\frac{k^2}{2k+1}}$$

$$=\frac{3}{2k+1}+\frac{2k+\frac{2k}{2k+1}+3k^2-\frac{3k}{2k+1}}{k-\frac{k^2}{2k+1}}=\frac{3}{2k+1}+\frac{(2k+1)\left(3k^2+2k-\frac{k}{2k+1}\right)}{k^2+k}$$

$$=\frac{3}{2k+1}+\frac{(2k+1)k\left(3k+2-\frac{1}{2k+1}\right)}{k(k+1)}=\frac{3}{2k+1}+\frac{(2k+1)\left(3k+2-\frac{1}{2k+1}\right)}{k+1}$$

$$=\frac{3}{2k+1}+\frac{(2k+1)\left(\frac{6k^2+7k+1}{2k+1}\right)}{k+1}=\frac{3}{2k+1}+\frac{6k^2+7k+1}{k+1}=\frac{3}{2k+1}+6k+1$$

Application on USAMO 2003 5

Let a, b, c be positive reals. Then prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+\frac{(2b+c+a)^2}{2b^2+(c+a)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\le 8$$

Solution: Problem is specialized use of *Generalization 2.* which $k=1$. By replacing it

$$\sum_{cyc}\frac{\left((k+1)a+kb+kc\right)^2}{(k+1)a^2+kb^2+kc^2}\le 6k+\frac{3}{2k+1}+1=8$$

which is the conclusion of the problem.

REFERENCES: Mathematical Association of America – United States of America Mathematical Olympiad.

THREE REFINEMENTS OF A WELL-KNOWN INEQUALITY:

$$\sum a^2\ge\sum ab,\ (a,b,c>0)$$

By Neculai Stanciu, Romania

If $a,b,c>0$, then are true the following inequalities:

(i) $\sum a^2\ge\sum ab+\frac{3}{16}\sum|a-b|^2$;

(ii) $\sum a^2\ge\sum ab+\frac{\sqrt{3}}{3}\sum|a-b||a-c|$;

(iii) $\sum a^2\ge\sum ab+\frac{1}{8}\sum\left(|a-c|+|b-c|\right)^2$.

(i) WLOG we can assume that $a\le b\le c$. Let $x,y\ge 0$ such that $b=a+x, c=a+x+y$.

$$\sum a^2-\sum ab=\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)=\frac{1}{2}\left(x^2+y^2+(x+y)^2\right)=x^2+xy+y^2,$$

then the inequality becomes:

$$16(x^2+xy+y^2)\geq 3(x+y+x+y)^2 \Leftrightarrow 4(x-y)^2\geq 0, \text{ true.}$$

(ii) Analogously as above, let $a\leq b\leq c$ şi $x,y\geq 0$ such that $b=a+x,\ c=a+x+y$.

Inequality becomes: $x^2+xy+y^2\geq \frac{1}{\sqrt{3}}\left(x(x+y)+xy+y(x+y)\right)$

$$\Leftrightarrow (\sqrt{3}-1)x^2+(\sqrt{3}-3)xy+(\sqrt{3}-1)y^2\geq 0,$$

which is true because the discriminant of equation $(\sqrt{3}-1)t^2+(\sqrt{3}-3)t+\sqrt{3}-1\geq 0$ is $\Delta=(\sqrt{3}-3)^2-4(\sqrt{3}-1)^2=2\sqrt{3}-4<0$.

(iii). Analogously as in (i) and (ii), let $a\leq b\leq c$ and $b=a+x$, $c=a+x+y$ with $x,y\geq 0$. Then, the inequality becomes:

$$x^2+xy+y^2\geq \frac{1}{8}\left((x+2y)^2+(2x+y)^2+(x+y)^2\right)$$

$$\Leftrightarrow 8x^2+8xy+8y^2\geq 6x^2+10xy+6y^2 \Leftrightarrow x^2-xy+y^2\geq 0 \Leftrightarrow \left(x-\frac{y}{2}\right)^2+\frac{3y^2}{4}\geq 0, \text{ true.}$$

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

EXTENSIONS AND REFINEMENTS FOR NESBITT AND CÎRTOAJE'S INEQUALITIES

By Dorin Mărghidanu-Romania

In this short note, are presented some possibilities for extending and refining Nesbitt's inequality, using the monotony of a function associated with this inequality. Some consequences and applications are also presented.

Keywords: Nesbitt's inequality, inequality of Vasile Cîrtoaje, refinement, monotony

2020 Mathematics Subject Classification: 26D15

It is very well known in mathematical literature and mathematical practice – the Nesbitt's famous inequality , [1],

If $a, b, c > 0$, then, $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$, **(1)**

For this wonderful inequality, there are dozens of proofs, extensions, generalizations, refinements. Expressions similar to the one in *Nesbitt's inequality* also appear in a very nice *inequality* by *Vasile Cîrtoaje* in *Gazeta Matematică*, [2]:

Let a, b, c be strictly positive real numbers. Prove that,

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \ge \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \quad \textbf{(2)}$$

Inequality (2) is extended by *Titu Zvonaru* in a problem proposed in the Canadian magazine *Crux mathematicorum,* [3]:

If m and n are positive integers such that $m \ge n$ and $a, b, c > 0$, then

$$\frac{a^m}{b^m+c^m} + \frac{b^m}{c^m+a^m} + \frac{c^m}{a^m+b^m} \ge \frac{a^n}{b^n+c^n} + \frac{b^n}{c^n+a^n} + \frac{c^n}{a^n+b^n} \quad \textbf{(3)}$$

We will note, for $t \in \mathbb{R}_+$ and $a, b, c > 0$,

$$\mathcal{N}(t; a, b, c) \coloneqq \frac{a^t}{b^t+c^t} + \frac{b^t}{c^t+a^t} + \frac{c^t}{a^t+b^t} \quad (4)$$

and we will call it a Nesbitt expression of type t, in the variables a, b, c.

When the variables a, b, c are implied – as below, we will write $\mathcal{N}(t)$ more briefly.

For *Nesbitt expression of type t* we will prove – in two ways – the following monotonicity result:

1. Proposition (of the monotony of Nesbitt's expressions)

The function $\mathcal{N}: \mathbb{R}_+ \to \mathbb{R}, \mathcal{N}(t) = \frac{a^t}{b^t+c^t} + \frac{b^t}{c^t+a^t} + \frac{c^t}{a^t+b^t}$ is a monotonically increasing function.

Proof 1 (algebraic)

We will prove that for any two types $x, y \in \mathbb{R}_+$, we have the implication,

$$x > y \Rightarrow \mathcal{N}(x) \ge \mathcal{N}(y).$$

Indeed, after some calculations, we have:

$$\mathcal{N}(x) - \mathcal{N}(y) = \sum_{cyc} \left(\frac{a^x}{b^x + c^x} - \frac{a^y}{b^y + c^y} \right) =$$

$$= \sum_{cyc} \frac{a^y b^y (a^{x-y} - b^{x-y}) - c^y a^y (c^{x-y} - a^{x-y})}{(b^x + c^x)(b^y + c^y)} =$$

$$= \sum_{cyc} a^y b^y \left(a^{x-y} - b^{x-y}\right) \cdot \left(\frac{1}{(b^x+c^x)(b^y+c^y)} - \frac{1}{(a^x+c^x)(a^y+c^y)} \right) = \quad (5)$$

$$= \sum_{cyc} a^y b^y (a^{x-y} - b^{x-y}) \cdot \frac{(a^{x+y} - b^{x+y}) + (a^x - b^x)c^y + (a^y - b^y)c^x}{(b^x + c^x)(b^y + c^y)(a^x + c^x)(a^y + c^y)} \ge 0$$

which justifies the assertion in the statement. The last inequality occurs because the parenthesis $(a^{x-y} - b^{x-y})$ and the brackets from the numerator of the fraction below the sum have the same sign. Equality holds if $a = b = c$.

Proof 2 (analytical)

Relative to some type, $x \geq 0$ – after some routine calculations, we obtain:

$$\mathcal{N}'(x) = \sum_{cyc} \frac{a^x b^x (a^x + b^x + 2c^x)}{(b^x + c^x)^2 (a^x + c^x)^2} (a^x - b^x)(\ln a - \ln b)$$

Let's note that for $x \geq 0$ and for the function $f: \mathbb{R}_+ \to \mathbb{R}, f(t) = t^x$, for any $a, b > 0$, we have with the *mean value theorem (Lagrange)*, $a^x - b^x = x \cdot c^{x-1} \cdot (a - b)$, (with c between a and b), so $(a^x - b^x)$ and $(a - b)$ have the same sigh.

Also $(\ln a - \ln b)$ and $(a - b)$ have the same sign.

It turns out that the parentheses of the type $(a^x - b^x)$, $(\ln a - \ln b)$ have the same sign, for any $a, b > 0$, hence, $\mathcal{N}'(x) \geq 0$, which means that the function $\mathcal{N}$ is monotonically increasing on $\mathbb{R}_+$.

2. Remark: The sign of $\mathcal{N}'(x)$ can also be determined with the help of the *Cauchy mean value theorem*, considering the functions: $f, g: \mathbb{R}_{>0} \to \mathbb{R}, f(t) = t^x, g(t) = \ln t$, where for any $a, b > 0$, there is c between a and b for which we have: $\frac{a^x - b^x}{\ln a - \ln b} = \frac{x \cdot c^{x-1}}{\frac{1}{c}} \Leftrightarrow \frac{a^x - b^x}{\ln a - \ln b} = x \cdot c^x > 0$, so $(a^x - b^x)$, $(\ln a - \ln b)$ have the same sign.

The previous proposition provides very simple (almost instantaneous) proofs for the inequalities mentioned at the beginning of this note.

3. Corollary (Nesbitt's inequality)

Nesbitt's inequality (1) is a consequence of *Proposition* 1.

Proof: Indeed, considering the types $1 > 0$, with *Proposition* 1, it follows $\mathcal{N}(1) \geq \mathcal{N}(0)$. Equality occurs if $a = b = c$. In addition to dozens of proofs of *Nesbitt's inequality*, this constitutes even a new proof for this well known inequality.

4. Corollary (the inequality of Vasile Cîrtoaje)

Vasile Cîrtoaje's inequality (2) is a consequence of *Proposition* 1.

Proof: Indeed, considering the types $2 > 1$, with *Proposition* 1, it follows $\mathcal{N}(2) \geq \mathcal{N}(1)$. Equality occurs if $a = b = c$.

5. Corollary

Inequality (3) from *Titu Zvonaru's problem* is a consequence of *Proposition* 1.

Proof: Taking in *Proposition* 1, the types: $m \geq n$, (m, n – natural numbers), it follows

$\mathcal{N}(m) \geq \mathcal{N}(n)$.

Based on the above *monotonicity theorem*, numerous refinements of the *inequality of Nesbitt* can be imagined.We only illustrate here with the following multiple inequality, which is also a very beautiful one inequality, and which would otherwise be very difficult to demonstrate in the absence of a monotonicity result.

6. Corollary (a multiple refinement of Nesbitt's inequality, [4])

If a, b, c are strictly positive real numbers, then,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{\sqrt{a}}{\sqrt{b}+\sqrt{c}}+\frac{\sqrt{b}}{\sqrt{c}+\sqrt{a}}+\frac{\sqrt{c}}{\sqrt{a}+\sqrt{b}} \geq$$

$$\geq \frac{\sqrt[3]{a}}{\sqrt[3]{b}+\sqrt[3]{c}}+\frac{\sqrt[3]{b}}{\sqrt[3]{c}+\sqrt[3]{a}}+\frac{\sqrt[3]{c}}{\sqrt[3]{a}+\sqrt[3]{b}} \geq \cdots \geq$$

$$\geq \frac{\sqrt[n]{a}}{\sqrt[n]{b}+\sqrt[n]{c}}+\frac{\sqrt[n]{b}}{\sqrt[n]{c}+\sqrt[n]{a}}+\frac{\sqrt[n]{c}}{\sqrt[n]{a}+\sqrt[n]{b}} \geq \cdots \geq \frac{3}{2}.$$

The *Proof* immediately follows from the monotonicity proposition for types:

$1 > \frac{1}{2} > \frac{1}{3} > \cdots > \frac{1}{n} > \cdots > 0$, for which we obviously have,

$$\mathcal{N}(1) \geq \mathcal{N}\left(\frac{1}{2}\right) \geq \cdots \geq \mathcal{N}\left(\frac{1}{n}\right) \geq \cdots . \geq \mathcal{N}(0).$$

Equality occurs iff $a = b = c$.

7. Remark: An immediate consequence of the above result is that,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}+\sqrt[n]{c}}+\frac{\sqrt[n]{b}}{\sqrt[n]{c}+\sqrt[n]{a}}+\frac{\sqrt[n]{c}}{\sqrt[n]{a}+\sqrt[n]{b}}\right) = \frac{3}{2},$$

which can also be considered as a new demonstration of *Nesbitt's inequality*.

On the same idea, we can obtain an extension for any natural powers of the variables a, b, c from Vasile Cîrtoaje's inequality:

8. Corollary (an extension of Cîrtoaje's inequality)

If a, b, c are positive real numbers and $p \in \mathbb{N}_{>2}$, then,

$$\frac{a^p}{b^p+c^p}+\frac{b^p}{c^p+a^p}+\frac{c^p}{a^p+b^p} \geq \frac{a^{p-1}}{b^{p-1}+c^{p-1}}+\frac{b^{p-1}}{c^{p-1}+a^{p-1}}+\frac{c^{p-1}}{a^{p-1}+b^{p-1}} \geq$$

$$\geq \cdots \geq \frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

The Proof also immediately follows from the monotonicity proposition for the types:

$p > p-1 > \cdots > 2 > 1$, for which we obviously have,

$$\mathcal{N}(p) \geq \mathcal{N}(p-1) \geq \cdots \geq \mathcal{N}(2) \geq \mathcal{N}(1).$$

Obviously, expansions can also be formulated for real powers of the variables (types).

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[1] Nesbitt, A.M., Problem 15114, *Educational Times,* 55, 1902.

[2] Cîrtoaje Vasile, *Proposed problem, Gazeta Matematica, 2002.*

[3] Zvonaru Titu, Proposed problem 2970, *Crux Mathematicorum, Proposed in* v.30, n.6, pp. 3, Oct. 2004, *Solved in* Vol. 31, No. 6, pp. 414, 415, Oct. 2005.

[4] Marghidanu Dorin, (*A multiple refinement of Nesbitt's inequality*), *Proposed problem, Mathematical Inequalities*, 29 Sept., 2023.

WEIGHTED NESBITT'S INEQUALITY

By Dorin Mărghidanu -Romania

Abstract: In this short note, a weighted version is presented and at in the same time an extension of Nesbitt's inequality. Consequences of this inequality are also presented

Key words* : *Nesbitt's inequality , convex function , Jensen's inequality , weights

2020 *Mathematics Subject Classification* : *26D15*

It is known in mathematical practice and in mathematical literature - *Nesbitt's* famous and beautiful *inequality* , [1] :

- if $a, b, c > 0$, then , $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \quad , \qquad (N)$$

For this famous inequality, there are dozens of proofs , extensions , generalizations and various refinements . Our intention is to obtain an inequality, when in the left member of inequality (N) weights appear. For this we will use *Jensen's weighted inequality* :

- if $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function , I – interval , then for any $x_k \in I$

and any weights $w_k > 0$, $k \in \{1, 2, \cdots, n\}$, for which we have $\sum_{k=1}^{n} w_k x_k \in I$, $\sum_{k=1}^{n} w_k = 1$,

we have the inequality $$\sum_{k=1}^{n} w_k f(x_k) \geq f\left(\sum_{k=1}^{n} w_k x_k\right) \quad , \qquad (J)$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$. We will thus have the following statement ,

1. *Proposition* (***weighted Nesbitt's inequality***)

For any $a, b, c > 0$ and any weights $m, n, p > 0$ with $m+n+p=1$ holds the inequality ,

$$m\times\frac{a}{b+c} + n\times\frac{b}{c+a} + p\times\frac{c}{a+b} \geq \frac{ma+nb+pc}{(n+p)a+(p+m)b+(m+n)c} \quad , \qquad (w\mathbf{N})$$

with equality if and only if $a = b = c$.

Proof: With the notation $S = a+b+c$ and remarking that $\frac{a}{b+c} = \frac{a}{S-a}$, etc.

let be the function $f : (0, S) \longrightarrow \mathbb{R}$, $f(x) = \frac{x}{S-x}$, for which we have

$$f'(x) = \frac{S}{(S-x)^2} \quad , \quad f''(x) = \frac{2S}{(S-x)^3} \geq 0 ,$$ so the function is convex .

After a slight preparation , and then with an application of *Jensen's weighted inequality* for case $n=3$ and weights $m, n, p > 0$, we have :

$$m\times\frac{a}{b+c} + n\times\frac{b}{c+a} + p\times\frac{c}{a+b} = m\times f(a) + n\times f(b) + p\times f(c) \overset{Jensen}{\geq}$$

$$\overset{Jensen}{\geq} f(ma+nb+pc) = \frac{ma+nb+pc}{S-(ma+nb+pc)} = \frac{ma+nb+pc}{(1-m)a+(1-n)b+(1-p)c} =$$

$$= \frac{ma+nb+pc}{(n+p)a+(p+m)b+(m+n)c} \quad \times \qquad \mathbf{W}$$

2. *Remark*

Taking $m = n = p \ (= 1/3)$ in *Nesbitt's weighted inequality* (w**N**) we get *Nesbitt's classical inequality* (N) .We exemplify with the following simple application ,

3. *Corolar*

For any $a, b, c > 0$ holds the inequality ,

$$2\times\frac{a}{b+c} + 3\times\frac{b}{c+a} + 4\times\frac{c}{a+b} \geq \frac{2a+3b+4c}{7a+6b+5c} \quad ,$$

with equality if and only if $a = b = c$.

Proof Choosing in *Nesbitt's weighted inequality* **(wN)** the weights, $m = 2/9$, $n = 3/9$, $p = 4/9$,

for which we obviously have $m + n + p = 1$, we obtain the inequality from corollary .

4. *Proposition* ***(generalization of the weighted Nesbitt's inequality)*, [4], b)**

For any $a_1, a_2, \ldots, a_n > 0$ and any weights $w_1, w_2, \ldots, w_n > 0$,

with $w_1 + w_2 + \ldots + w_n = 1$, holds the inequality ,

$$w_1 \times \frac{a_1}{a_2+a_3+\ldots+a_n} + w_2 \times \frac{a_2}{a_1+a_3+\ldots+a_n} + \ldots + w_n \times \frac{a_n}{a_1+a_2+\ldots+a_{n-1}} \geq$$
$$\geq \frac{w_1a_1+w_2a_2+\ldots+w_na_n}{(1-w_1)a_1+(1-w_2)a_2+\ldots+(1-w_n)a_n} \times \qquad \textbf{(gwN)}$$

with equality if and only if , $a_1 = a_2 = \ldots = a_n$.

Proof With the notation $S = a_1 + a_2 + \ldots + a_n$, we also consider here the function

$f : (0, S) \longrightarrow \mathbb{R}$, $f(x) = \dfrac{x}{S-x}$, which (as we saw in the proof of *Proposition* 1)

is a *convex function* on **(0, S)**. After an easy preparation , and then with the application of *Jensen's weighted inequality* (J) , we have :

$$w_1 \times \frac{a_1}{a_2+a_3+\ldots+a_n} + w_2 \times \frac{a_2}{a_1+a_3+\ldots+a_n} + \ldots + w_n \times \frac{a_n}{a_1+a_2+\ldots+a_{n-1}} =$$
$$= w_1 \times \frac{a_1}{S-a_1} + w_2 \times \frac{a_2}{S-a_2} + \ldots + w_n \times \frac{a_n}{S-a_n} =$$
$$= w_1 \times f(a_1) + w_2 \times f(a_2) + \ldots + w_n \times f(a_n) \overset{Jensen}{\geq}$$
$$\overset{Jensen}{\geq} f(w_1a_1+w_2a_2+\ldots+w_na_n) = \frac{w_1a_1+w_2a_2+\ldots+w_na_n}{S-(w_1a_1+w_2a_2+\ldots+w_na_n)} =$$
$$= \frac{w_1a_1+w_2a_2+\ldots+w_na_n}{(1-w_1)a_1+(1-w_2)a_2+\ldots+(1-w_n)a_n} \times$$

5. *Remark* Taking $w_1 = w_2 = \ldots = w_n = 1/n$ in the generalization of *Nesbitt's weighted inequality* **(gwN)** the *generalization of Nesbitt's classical inequality* (gN) is obtained

$$\frac{a_1}{a_2+a_3+\ldots+a_n} + \frac{a_2}{a_1+a_3+\ldots+a_n} + \ldots + \frac{a_n}{a_1+a_2+\ldots+a_{n-1}} \geq \frac{n}{n-1} \times \qquad \textbf{(gN)}$$

(*Olympiad , German Democratic Republic ,1967*) ..

For $n = 3$, the *classical Nesbitt's inequality* (N) is obtained .

By customizing the weights in *Proposition* 4 , numerous inequalities can be obtained . Here is an example :

6. *Corolar* , **[4], a)**

For any $a, b, c, d > 0$, holds the inequality ,

$$\frac{a}{b+c+d} + 2\times\frac{b}{c+d+a} + 3\times\frac{c}{d+a+b} + 4\times\frac{d}{a+b+c} \geq 10\times\frac{a+2b+3c+4d}{9a+8b+7c+6d} ,$$

with equality if and only if $a = b = c = d$.

Proof Taking in **(gwN)** , $n = 4$, the weights : $w_1 = 1/10$, $w_2 = 2/10$, $w_3 = 3/10$, $w_4 = 4/10$, for which we obviously have $w_1 + w_2 + w_3 + w_4 = 1$, the inequality from the statement is obtained .

References **:**

[**1**] **Nesbitt, A.M.** , Problem 15114 , "***Educational Times***" , **55** , **1902**

[**2**] **Jensen, J. L. W. V.** , ***"Sur les fonctions convexes et les inégalités entre les valeurs moyennes"***, Acta Mathematica. **30** (1) , pp. 175–193 , **1906** .

A NEW PROOF FOR EULER'S INEQUALITY

By Neculai Stanciu-Romania

If ABC **is a nonisosceles triangle, then** $R\left(\sum\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}\right)\left(\sum\frac{\cos\frac{A}{2}}{\sin\frac{B-C}{2}}\right) = R - 2r > 0$ **.**

Proof. $\sum a^2 = 2(p^2 - r^2 - 4Rr)$; $abc = 4pRr$; $ab + bc + ca = p^2 + r^2 + 4Rr$;

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)\left[(a+b+c)^2 - 3(ab+bc+ca)\right] = 2p(p^2 - 3r^2 - 12Rr) ;$$

$$\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2} ; \sum\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}} = \sum\frac{b-c}{a} = -\frac{(a-b)(b-c)(c-a)}{abc} ;$$

$$\sum\frac{\cos\frac{A}{2}}{\sin\frac{B-C}{2}}=\sum\frac{a}{b-c}=\frac{\sum(a^2b+a^2c-a^3-abc)}{(a-b)(b-c)(c-a)};$$

$$\sum(a^2b+a^2c-a^3-abc)=\sum a^2(2p-a)-\sum a^3-3abc=$$

$$=2p\sum a^2-2\left(\sum a^3-3abc\right)-9abc=$$

$$=4p(p^2-r^2-4Rr)-2(2p)(2p^2-2r^2-8Rr-p^2-r^2-4Rr)-36Rrp=$$

$$=4p(p^2-r^2-4Rr-p^2+3r^2+12Rr-9Rr)=4p(2r^2-Rr).$$

$$R\left(\sum\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}\right)\left(\sum\frac{\cos\frac{A}{2}}{\sin\frac{B-C}{2}}\right)=R\cdot\frac{(a-b)(b-c)(c-a)}{abc}\cdot\frac{4rp(R-2r)}{(a-b)(b-c)(c-a)}=$$

$$=R\cdot\frac{R-2r}{R}=R-2r.$$

Since, $R\left(\sum\frac{\sin\frac{B-C}{2}}{\cos\frac{A}{2}}\right)\left(\sum\frac{\cos\frac{A}{2}}{\sin\frac{B-C}{2}}\right)>0$, then we obtain the famous Euler's inequality.

RMM SOLVED PROBLEMS

By Marin Chirciu – Romania

S.2400. If I – incenter in ΔABC then:

$$IA^4+IB^4+IC^4\le\frac{(a^2+b^2+c^2)^2}{27}$$

Daniel Sitaru, Claudia Nănuți-Romania

Remark: The problem can be developed.

In ΔABC holds: $48r^4\le IA^4+IB^4+IC^4\le 4(5R^4-68r^4)$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC holds:

$$\sum \boldsymbol{IA^4 = s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2)}$$

Proof:

$$\sum IA^4 = \sum\left(\frac{r}{\sin\frac{A}{2}}\right)^4 = r^4\sum\left(\frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}}\right)^4 = r^4\sum\frac{b^2c^2}{(s-b)^2(s-c)^2} =$$

$$= r^4 \cdot \frac{s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2)}{r^4} = s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2)$$

We've used above:

$$\sum\frac{b^2c^2}{(s-b)^2(s-c)^2} = \frac{s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2)}{r^4}$$

$$\sum b^2c^2(s-a)^2 = s^2[s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2)]$$

Let's get back to the main problem.Using the Lemma we obtain:

$$\sum IA^4 = s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2) \overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq} (4R^2 + 4Rr + 3r^2)(4R^2 + 4Rr + 3r^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2) =$$

$$= 16(R^4 - 2R^3r + R^2r^2 - Rr^3 + r^4) \overset{Euler}{\leq} 16\left(\frac{5R^4}{4} - 17r^4\right) = 4(5R^4 - 68r^4)$$

Equality holds if and only if the triangle is equilateral.

$$\sum IA^4 = s^2(s^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2) \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} (16Rr - 5r^2)(16Rr - 5r^2 + 2r^2 - 16Rr) + r^2(32R^2 + r^2) =$$

$$= 16r^2(2R^2 - 3Rr + r^2) \overset{Euler}{\geq} 16r^2 \cdot 3r^2 = 48r^4$$

Equality holds if and only if the triangle is equilateral

J.2105. In ΔABC, G – centroid, $d_a = d(G,BC), d_b = d(G,CA), d_c = d(G,AB)$, then

$$\sum \frac{a^4 + b^4}{d_a d_b} \geq 96\sqrt{3}F$$

D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

Solution: We have $d_a = d(G,BC) = \frac{h_a}{3}$

$$LHS=\sum\frac{a^4+b^4}{d_ad_b}=\sum\frac{a^4+b^4}{\frac{h_a}{3}\frac{h_b}{3}}=9\sum\frac{a^4+b^4}{h_ah_b}=9\sum\frac{a^4+b^4}{\frac{2F}{a}\cdot\frac{2F}{b}}=$$

$$=\frac{9}{4F^2}\sum ab(a^4+b^4)\overset{AGM}{\geq}$$

$$=\frac{9}{4F^2}\sum ab\cdot 2a^2b^2=\frac{9}{2F^2}\sum(ab)^3\overset{Holder}{\geq}\frac{9}{2F^2}\frac{(\sum ab)^3}{9}=\frac{(\sum ab)^3}{2F^2}\overset{Gordon}{\geq}\frac{\left(4\sqrt{3}F\right)^3}{2F^2}=$$

$$=96\sqrt{3}F=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

In ΔABC, G – centroid, $d_a=d(G,BC),d_b=d(G,CA),d_c=d(G,AB)$, then:

$$\sum\frac{a^{2n}+b^{2n}}{d_ad_b}\geq 72\left(\frac{4F}{\sqrt{3}}\right)^{n-1},n\in\mathbb{N}$$

Marin Chirciu-Romania

Solution: We have $d_a=d(G,BC)=\frac{h_a}{3}$.

For $n=0$ the inequality can be written $\sum\frac{2}{d_ad_b}\geq 72\left(\frac{4F}{\sqrt{3}}\right)^{-1}\Leftrightarrow\sum\frac{1}{d_ad_b}\geq 36\frac{\sqrt{3}}{4F}\Leftrightarrow$

$\Leftrightarrow\sum\frac{1}{d_ad_b}\geq\frac{9\sqrt{3}}{F}$, which follows from:

$$LHS=\sum\frac{1}{d_ad_b}=\sum\frac{1}{\frac{h_a}{3}\frac{h_b}{3}}=9\sum\frac{1}{h_ah_b}=9\sum\frac{1}{\frac{2F}{a}\cdot\frac{2F}{b}}=\frac{9}{4F^2}\sum ab\overset{Gordon}{\geq}$$

$$\overset{Gordon}{\geq}\frac{9}{4F^2}4\sqrt{3}F=\frac{9\sqrt{3}}{F}=RHS$$

For $n=1$ inequality can be written $\sum\frac{a^2+b^2}{d_ad_b}\geq 72$, which follows from:

$$LHS=\sum\frac{a^2+b^2}{d_ad_b}\geq 72=\sum\frac{a^2+b^2}{\frac{h_a}{3}\frac{h_b}{3}}=9\sum\frac{a^2+b^2}{h_ah_b}=9\sum\frac{a^2+b^2}{\frac{2F}{a}\cdot\frac{2F}{b}}=$$

$$=\frac{9}{4F^2}\sum ab(a^2+b^2)\overset{AGM}{\geq}$$

$$=\frac{9}{2F^2}\sum(ab)^2\overset{CS}{\geq}\frac{9}{2F^2}\frac{(\sum a)^2}{3}\overset{Gordon}{\geq}\frac{9}{2F^2}\frac{\left(4\sqrt{3}F\right)^2}{3}=72=RHS$$

For $n\geq 3$ we use Holder's inequality.

$$LHS=\sum\frac{a^{2n}+b^{2n}}{d_ad_b}=\sum\frac{a^{2n}+b^{2n}}{\frac{h_a}{3}\frac{h_b}{3}}=9\sum\frac{a^{2n}+b^{2n}}{h_ah_b}=9\sum\frac{a^{2n}+b^{2n}}{\frac{2F}{a}\cdot\frac{2F}{b}}=$$

$$=\frac{9}{4F^2}\sum ab\,(a^{2n}+b^{2n})\overset{AGM}{\geq}$$

$$=\frac{9}{4F^2}\sum ab\cdot 2a^nb^n=\frac{9}{2F^2}\sum (ab)^{n+1}\overset{Holder}{\geq}\frac{9}{2F^2}\frac{(\sum ab)^{n+1}}{3^n}=\frac{(\sum ab)^{n+1}}{2F^2\cdot 3^{n-2}}\overset{Gordon}{\geq}$$

$$\overset{Gordon}{\geq}\frac{\left(4\sqrt{3}F\right)^{n+1}}{2F^2\cdot 3^{n-2}}=\frac{2^{2n+2}\cdot 3^{\frac{n+1}{2}}F^{n+1}}{2F^2\cdot 3^{n-2}}=$$

$$=2^{2n+1}\cdot 3^{\frac{n+1}{2}-n+2}F^{n+1-2}=2^{2n+1}\cdot 3^{\frac{-n+5}{2}}F^{n-1}=72\left(\frac{4F}{\sqrt{3}}\right)^{n-1}=RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain Problem J.21015 from RMM – 40 Spring Edition 2024, proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți.

In ΔABC, G – centroid, $d_a=d(G,BC), d_b=d(G,CA), d_c=d(G,AB)$, then

$$\sum\frac{a^4+b^4}{d_ad_b}\geq 96\sqrt{3}F$$

D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

J.2205. In $a,b,c>0$ then:

$$\sum\frac{a^2(a+b+c)}{c(b+c)^2}\geq\frac{9}{4}$$

Neculai Stanciu – Romania

Solution:

$$LHS=\sum\frac{a^2(a+b+c)}{c(b+c)^2}=(a+b+c)\sum\frac{\left(\frac{a}{b+c}\right)^2}{c}\overset{CS}{\geq}(a+b+c)\frac{\left(\sum\frac{a}{b+c}\right)^2}{a+b+c}=$$

$$=\left(\sum\frac{a}{b+c}\right)^2\overset{Nesbitt}{\geq}\left(\frac{3}{2}\right)^2=\frac{9}{4}=RHS.$$

Equality holds if and only if $a=b=c$. **Remark:** The problem can be developed: **If $a,b,c>0$ and $\lambda\geq 0$ then:**

$$\sum\frac{a^2(a+b+c)}{c(b+\lambda c)^2}\geq\frac{9}{(\lambda+1)^2}$$

Marin Chirciu-Romania

Solution:

$$LHS=\sum\frac{a^2(a+b+c)}{c(b+\lambda c)^2}=(a+b+c)\sum\frac{\left(\frac{a}{b+\lambda c}\right)^2}{c}\overset{CS}{\geq}(a+b+c)\frac{\left(\sum\frac{a}{b+\lambda c}\right)^2}{a+b+c}=$$

$$=\left(\sum\frac{a}{b+\lambda c}\right)^2\overset{Nesbitt}{\geq}\left(\frac{3}{\lambda+1}\right)^2=\frac{9}{(\lambda+1)^2}=RHS$$

Equality holds if and only if $a=b=c$.

Note: For $\lambda=1$ we obtain Proposed problem by Neculai Stanciu in RMM 41, Summer Edition 2024. If $a,b,c>0$ then:

$$\sum\frac{a^2(a+b+c)}{c(b+c)^2}\geq\frac{9}{4}$$

Neculai Stanciu – Romania

Remark: The problem can be developed. **If $a,b,c>0$ and $\lambda\geq 0, n\in\mathbb{N}, n\geq 2$ then:**

$$\sum\frac{a^n(a+b+c)}{c(b+\lambda c)^n}\geq\frac{9}{(\lambda+1)^n}$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{a^n(a+b+c)}{c(b+\lambda c)^n}=(a+b+c)\sum\frac{\left(\frac{a}{b+\lambda c}\right)^n}{c}\overset{Holder}{\geq}(a+b+c)\frac{\left(\sum\frac{a}{b+\lambda c}\right)^n}{3^{n-2}(a+b+c)}$$

$$=\frac{1}{3^{n-2}}\left(\sum\frac{a}{b+\lambda c}\right)^n\overset{Nesbitt}{\geq}\frac{1}{3^{n-2}}\left(\frac{3}{\lambda+1}\right)^n=\frac{9}{(\lambda+1)^n}=RHS.$$

Equality holds if and only if $a=b=c$.

Note: For $\lambda=1$ and $n=2$ we obtain Proposed problem by Neculai Stanciu in RMM 41 Summer Edition 2024. If $a,b,c>0$ then:

$$\sum\frac{a^2(a+b+c)}{c(b+c)^2}\geq\frac{9}{4}$$

J.2204. In ΔABC holds:

$$\sum\frac{m_b^2+m_c^2}{h_b^2+h_c^2}\leq 6\left(\frac{R}{2r}\right)^3-3$$

George Apostolopoulos – Greece

Solution: Lemma:

$$\frac{m_b^2+m_c^2}{h_b^2+h_c^2}\leq\left(\frac{R}{2r}\right)^2$$

Proof:

Using Panaitopol's inequality $m_a\leq\frac{Rs}{a}$ and $h_a=\frac{2S}{a}$ we obtain:

$$\frac{m_b^2+m_c^2}{h_b^2+h_c^2} \le \frac{\left(\frac{Rs}{b}\right)^2+\left(\frac{Rs}{c}\right)^2}{\left(\frac{2S}{b}\right)^2+\left(\frac{2S}{b}\right)^2} = \frac{(Rs)^2}{(2S)^2} = \frac{R^2s^2}{4r^2s^2} = \frac{R^2}{4r^2} = \left(\frac{R}{2r}\right)^2.$$

Let's get back to the main problem. Using the Lemma we obtain:

$$LHS = \sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \overset{Lemma}{\le} \sum \left(\frac{R}{2r}\right)^2 \overset{Euler}{\le} 6\left(\frac{R}{2r}\right)^3 - 3 = RHS.$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed and strengthened.

In ΔABC holds:

$$\mathbf{3 \le \sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \le 3\left(\frac{R}{2r}\right)^2}$$

Marin Chirciu-Romania

Solution: Using the Lemma we obtain:

$$\sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \overset{Lemma}{\le} \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2 \overset{Euler}{\le} 6\left(\frac{R}{2r}\right)^3 - 3$$

Equality holds if and only if the triangle is equilateral. Using $m_a \ge h_a$ we obtain

$$\frac{m_b^2+m_c^2}{h_b^2+h_c^2} \ge 1 \Rightarrow \sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \ge 3$$

Equality holds if and only if the triangle is equilateral.

Remark: RHS inequality strengthen the proposed problem by George Apostolopoulos in RMM 41 Summer Edition 2024

In ΔABC holds:

$$\mathbf{\sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \le 6\left(\frac{R}{2r}\right)^3 - 3}$$

George Apostolopoulos – Greece

Remark: In the same way.

In ΔABC holds:

$$\mathbf{3 \le \sum \frac{w_b^2+w_c^2}{h_b^2+h_c^2} \le 3\left(\frac{R}{2r}\right)^2}$$

Marin Chirciu-Romania

Solution: RHS

Lemma: In ΔABC holds:

$$\frac{w_b^2+w_c^2}{h_b^2+h_c^2} \le \left(\frac{R}{2r}\right)^2$$

Solution: Using Panaitopol's inequality $m_a \le \frac{Rs}{a}$ and $h_a = \frac{2S}{a}$ we obtain:

$$\frac{w_b^2+w_c^2}{h_b^2+h_c^2} \overset{w_a \le m_a}{\le} \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \le \frac{\left(\frac{Rs}{b}\right)^2+\left(\frac{Rs}{c}\right)^2}{\left(\frac{2S}{b}\right)^2+\left(\frac{2S}{b}\right)^2} = \frac{(Rs)^2}{(2S)^2} = \frac{R^2s^2}{4r^2s^2} = \frac{R^2}{4r^2} = \left(\frac{R}{2r}\right)^2$$

Let's solve RHS inequality. Using the Lemma we obtain:

$$\sum \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \overset{Lemma}{\le} \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2$$

Equality holds if and only if the triangle is equilateral.

LHS: Using $w_a \ge h_a$ we obtain $\frac{w_b^2+w_c^2}{h_b^2+h_c^2} \ge 1 \Rightarrow \sum \frac{w_b^2+w_c^2}{h_b^2+h_c^2} \ge 3$. Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC holds:**

$$3 \le \sum \frac{s_b^2+s_c^2}{h_b^2+h_c^2} \le 3\left(\frac{R}{2r}\right)^2$$

Marin Chirciu-Romania

Solution: RHS.

Lemma: In ΔABC holds:

$$\frac{s_b^2+s_c^2}{h_b^2+h_c^2} \le \left(\frac{R}{2r}\right)^2$$

Solution: Using Panaitopol's inequality $m_a \le \frac{Rs}{a}$ and $h_a = \frac{2S}{a}$ we obtain:

$$\frac{s_b^2+s_c^2}{h_b^2+h_c^2} \overset{s_a \le m_a}{\le} \frac{m_b^2+m_c^2}{h_b^2+h_c^2} \le \frac{\left(\frac{Rs}{b}\right)^2+\left(\frac{Rs}{c}\right)^2}{\left(\frac{2S}{b}\right)^2+\left(\frac{2S}{b}\right)^2} = \frac{(Rs)^2}{(2S)^2} = \frac{R^2s^2}{4r^2s^2} = \frac{R^2}{4r^2} = \left(\frac{R}{2r}\right)^2$$

Let's solve RHS inequality. Using the Lemma we obtain:

$$\sum \frac{s_b^2+s_c^2}{h_b^2+h_{c^2}} \overset{Lemma}{\le} \sum \left(\frac{R}{2r}\right)^2 = 3\left(\frac{R}{2r}\right)^2$$

Equality holds if and only if the triangle is equilateral.

LHS: Using $s_a \ge h_a$ we obtain $\frac{s_b^2+s_c^2}{h_b^2+h_c^2} \ge 1 \Rightarrow \sum \frac{s_b^2+s_c^2}{h_b^2+h_c^2} \ge 3$.

Equality holds if and only if the triangle is equilateral.

J.2158. In ΔABC holds:

$$\sum \frac{ar_a}{r_b + r_c} \geq \frac{F}{r}$$

Ertan Yldirim – Turkiye

Solution: Lemma: In ΔABC holds:

$$\sum \frac{ar_a}{r_b + r_c} = \frac{(4R + r)^2 - 2s^2}{s}$$

Proof: Using $r_a = \frac{s}{s-a}$ we obtain:

$$\sum \frac{ar_a}{r_b + r_c} = \sum \frac{a\frac{s}{s-a}}{\frac{s}{s-b} + \frac{s}{s-c}} = \sum \frac{(s-b)(s-c)}{s-a} = \frac{(4R + r)^2 - 2s^2}{s}$$

Let's get back to the main problem. Using the Lemma we obtain:

$$\sum \frac{ar_a}{r_b + r_c} = \frac{(4R + r)^2 - 2s^2}{s} \overset{Doucet}{\geq} s = \frac{F}{r}$$

Equality holds if and only if the triangle is equilateral. **Remark:** Let's find an inequality with opposite sense.

In ΔABC holds:

$$\sum \frac{ar_a}{r_b + r_c} \leq \frac{27R^3}{8F}$$

Marin Chirciu-Romania

Solution: Using the Lemma we obtain:

$$\sum \frac{ar_a}{r_b + r_c} = \frac{(4R + r)^2 - 2s^2}{s} \overset{Gerretsen}{\leq} \frac{(4R + r)^2 - 2(16Rr - 5r^2)}{s} =$$

$$= \frac{16R^2 - 24Rr + 11r^2}{s} \overset{Euler}{\leq} \frac{\frac{27R^3}{8r}}{s} = \frac{27R^3}{8F}$$

Remark: We can write the double inequality:

In ΔABC holds:

$$\frac{F}{r} \leq \sum \frac{ar_a}{r_b + r_c} \leq \frac{27R^3}{8F}$$

See above. Equality holds if and only if the triangle is equilateral.

J.2106. In $\Delta ABC, G$ – centroid, $d_a = d(G,BC), d_b = d(G,CA), d_c = d(G,AB), x,y > 0$ then

$$\sum \frac{x^2a^4 + y^2b^4}{d_bd_c} \ge 24\sqrt{3}(x+y)^2F$$

D.M. Bătinețu – Giurgiu – Romania

Solution: We have $d_a = d(G,BC) = \frac{h_a}{3}$.

$$LHS = \sum \frac{x^2a^4 + y^2b^4}{d_bd_c} = \sum \frac{x^2a^4 + y^2b^4}{\frac{h_b}{3}\frac{h_c}{3}} = 9\sum \frac{x^2a^4 + y^2b^4}{h_bh_c} = 9\sum \frac{x^2a^4 + y^2b^4}{\frac{2F}{b}\cdot\frac{2F}{c}} =$$

$$= \frac{9}{4F^2}\sum \frac{x^2a^4 + y^2b^4}{\frac{1}{bc}} \overset{CS}{\ge} \frac{9}{4F^2}\frac{(\sum xa^2)^2 + (\sum yb^2)^2}{\sum \frac{1}{bc}} = \frac{9}{4F^2}\frac{(x^2+y^2)(\sum a^2)^2}{\sum \frac{1}{bc}} \overset{CS}{\ge}$$

$$\overset{CS}{\ge} \frac{9}{4F^2}\frac{\frac{1}{2}(x+y)^2(\sum a^2)^2}{\frac{1}{2Rr}} \overset{IW}{\ge} \frac{9}{4F^2}\frac{\frac{1}{2}(x+y)^2\left(4\sqrt{3}F\right)^2}{\frac{1}{2Rr}} = 54(x+y)^2 2Rr \overset{Mitrinovic}{\ge}$$

$$\overset{Mitrinovic}{\ge} 24\sqrt{3}(x+y)^2F = RHS.$$

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

In $\Delta ABC, G$ – centroid, $d_a = d(G,BC), d_b = D(G,CA), d_c = d(G,AB), x,y > 0$ then

$$\sum \frac{x^2a^{2n} + y^2b^{2n}}{d_bd_c} \ge 108(x+y)^nRr\left(\frac{2F}{\sqrt{3}}\right)^{n-2}, n \in \mathbb{N}, n \ge 2$$

Marin Chirciu-Romania

Solution: We have $d_a = d(G,BC) = \frac{h_a}{3}$.

$$LHS = \sum \frac{x^2a^{2n} + y^2b^{2n}}{d_bd_c} = \sum \frac{x^2a^{2n} + y^2b^{2n}}{\frac{h_b}{3}\frac{h_c}{3}} = 9\sum \frac{x^2a^{2n} + y^2b^{2n}}{h_bh_c} =$$

$$= 9\sum \frac{x^2a^{2n} + y^2b^{2n}}{\frac{2F}{b}\cdot\frac{2F}{c}} = \frac{9}{4F^2}\sum \frac{x^2a^{2n} + y^2b^{2n}}{\frac{1}{bc}} \overset{CS}{\ge} \frac{9}{4F^2}\frac{(\sum xa^2)^n + (\sum yb^2)^n}{3^{n-2}\sum \frac{1}{bc}} =$$

$$= \frac{9}{4F^2}\frac{(x^n+y^n)(\sum a^2)^n}{3^{n-2}\sum \frac{1}{bc}} \overset{CS}{\ge} \frac{9}{4F^2}\frac{\frac{1}{2^{n-1}}(x+y)^n(\sum a^2)^n}{3^{n-2}\frac{1}{2Rr}} \overset{IW}{\ge}$$

$$\overset{IW}{\ge} \frac{9}{4F^2}\frac{\frac{1}{2^{n-1}}(x+y)^n\left(4\sqrt{3}F\right)^n}{3^{n-2}\frac{1}{2Rr}} = 54(x+y)^2 2Rr \overset{Mitrinovic}{\ge} 24\sqrt{3}(x+y)^2F = RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain Problem J.2106 from RMM 40 Spring Edition 2024, proposed by D.M. Bătinețu – Giurgiu.

In ΔABC, G – centroid, $d_a=d(G,BC), d_b=d(G,CA), d_c=d(G,AB), x,y>0$ then

$$\sum \frac{x^2a^4+y^2b^4}{d_bd_c} \ge 24\sqrt{3}(x+y)^2F$$

J.2117. If $x,y,z\ge 0$ then in ΔABC

$$\sum \frac{e^x}{y+z+2}\cdot\frac{1}{h_a^2}\ge\frac{\sqrt{3}}{2F}$$

D.M. Bătinețu – Giurgiu – Romania

Solution: Using the inequality $e^x\ge x+1, x\in\mathbb{R}$, with equality for $x=0$ we obtain:

$$LHS=\sum\frac{e^x}{y+z+2}\cdot\frac{1}{h_a^2}\ge\sum\frac{x+1}{y+z+2}\cdot\frac{a^2}{a^2h_a^2}=\sum\frac{x+1}{y+z+2}\cdot\frac{a^2}{4F^2}=$$

$$=\frac{1}{4F^2}\sum\frac{x+1}{y+z+2}\cdot a^2=\frac{1}{4F^2}\sum\frac{x+1}{(y+1)+(z+1)}\cdot a^2=\frac{1}{4F^2}\sum\frac{m}{n+p}\cdot a^2\overset{Tsintsifas}{\ge}$$

$$\overset{Tsintsifas}{\ge}\frac{1}{4F^2}\cdot 2\sqrt{3}F=\frac{\sqrt{3}}{2F}=RHS.$$

Lemma (G. Tsintsifas).

In ΔABC holds:

$$\frac{x}{y+z}a^2+\frac{y}{z+x}b^2+\frac{z}{x+y}c^2\ge 2\sqrt{3}S, \text{ where } x,y,z>0$$

G. Tsintsifas-Greece

Solution: We have $\sum\frac{x}{y+z}a^2=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\frac{x+y+z}{y+z}a^2-\sum a^2\overset{Bergstrom}{\ge}$

$$\ge(x+y+z)\frac{(\sum a)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{(2s)^2}{2(x+y+z)}-2(s^2-r^2-4Rr)=$$

$$=2s^2-2(s^2-r^2-4Rr)=2(r^2+4Rr).$$

Above we have used the known inequalities in triangle:

$$\sum a=2s \text{ and } \sum a^2=2(s^2-r^2-4Rr)$$

It remains to prove that $2(r^2+4Rr)\ge 2\sqrt{3}S\Leftrightarrow r^2+4Rr\ge\sqrt{3}rs\Leftrightarrow 4R+r\ge s\sqrt{3}$, with is Doucet's inequality.

Equality holds if and only if $a=b=c$ and $x=y=z=0$.

SP.505. If $x, y, z > 0, \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} = 3$ then:

$$x^2\sqrt{x} + y^2\sqrt{y} + z^2\sqrt{z} \geq x^2 + y^2 + z^2$$

Daniel Sitaru – Romania

Solution: With the substitution $(\sqrt{x}, \sqrt{y}, \sqrt{z}) = (a, b, c)$ the problem can be reformulated.

If $a, b, c > 0, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$ then: $a^5 + b^5 + c^5 \geq a^2 + b^4 + c^4$

Proof:

$$\sum a^5 = \sum a^4 \cdot a \overset{Chebyshev}{\geq} \frac{1}{3}\sum a^4 \sum a \overset{(1)}{\geq} \sum a^4,$$

where (1) $\Leftrightarrow \frac{1}{3}\sum a^4 \sum a \geq \sum a^4 \Leftrightarrow \sum a \geq 3$, witch follows from $\sum a + \sum \frac{1}{a} \geq 9$ and $\sum \frac{1}{a} = 3$.

Equality holds if and only if $a = b = c = 1$. Going back to the notation it follows that the inequality from the statement holds, with equality for $x = y = z = 1$.

Remark: The problem can be developed.

If $x, y, z > 0, \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{z}} = 3$ and $n \in \mathbb{N}$ then

$$x^n\sqrt{x} + y^n\sqrt{y} + z^n\sqrt{z} \geq x^n + y^n + z^n$$

Marin Chirciu-Romania

Solution: For $n = 0$, we have $\sqrt{x} + \sqrt{y} + \sqrt{z} \geq 3$, see $\sum \sqrt{x} \cdot \sum \frac{1}{\sqrt{x}} \geq 9$ and $\sum \frac{1}{\sqrt{x}} = 3$.

Next we take $n \in \mathbb{N}^*$. With the substitution $(\sqrt{x}, \sqrt{y}, \sqrt{z}) = (a, b, c)$ the problem can be reformulated.

If $a, b, c > 0, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$ then

$$a^{n+1} + b^{n+1} + c^{n+1} \geq a^n + b^n + c^n.$$

Proof:

$$\sum a^{n+1} = \sum a^n \cdot a \overset{Chebyshev}{\geq} \frac{1}{3}\sum a^n \sum a \overset{(1)}{\geq} \sum a^n,$$

where (1) $\Leftrightarrow \frac{1}{3}\sum a^n \sum a \geq \sum a^n \Leftrightarrow \sum a \geq 3$, which follows from $\sum a \sum \frac{1}{a} \geq 9$ and $\sum \frac{1}{a} = 3$.

Equality holds if and only if $a = b = c = 1$.

Going back to the notation it follows that the inequality from the statement holds, with equality for $x = y = z = 1$.

Note: For $n = 2$ we obtain Problem SP.505 from RMM – 42.

SP.505. If $x,y,z>0,\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=3$ then

$$x^2\sqrt{x}+x^2\sqrt{x}+x^2\sqrt{x}\geq x^2+y^2+z^2$$

Daniel Sitaru – Romania

S.2466. In ΔABC

$$\sum h_a^2\cot\frac{A}{2}\geq 9F$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

Solution: Lemma.

In ΔABC holds:

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}.$$

Proof:

$$\sum h_a^2\cot\frac{A}{2}=\sum\left(\frac{2S}{a}\right)^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=4S^2\sum\frac{1}{a^2}\cdot\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{4S^2\cdot s}{S}\sum\frac{s-a}{a^2}=4sr\cdot s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}=$$

$$=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}$$

We have used above: $\sum\frac{s-a}{a^2}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}$.

$$\sum b^2c^2(s-a)=s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]$$

Let's get back to the main problem. Using the lemma the inequality can be written:

$$\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}\geq 9F\Leftrightarrow$$

$$\Leftrightarrow\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}\geq 9pr\Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-12Rr)+r^3(4R+r)\geq 36R^2r^2$, which follows from Gerretsen's inequality $s^2\geq 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)\geq 36R^2r^2\Leftrightarrow$$

$$\Leftrightarrow(16R-5r)(4R-3r)+r(4R+r)\geq 36R^2\Leftrightarrow 7R^2-16Rr+4r^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(7R-2r)\geq 0, \text{ obviously from Euler's inequality } R\geq 2r.$$

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

In ΔABC holds:

$$F\left(4-\frac{2r}{R}\right)^2\leq \sum h_a^2\cot\frac{A}{2}\leq 9F\left(\frac{R}{2r}\right)^3$$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC holds:

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}$$

Proof:

$$\sum h_a^2\cot\frac{A}{2}=\sum\left(\frac{2S}{a}\right)^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=4S^2\sum\frac{1}{a^2}\cdot\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{4S^2\cdot s}{S}\sum\frac{s-a}{a^2}=4sr\cdot s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}=$$

$$=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}$$

We've used above: $\sum\frac{s-a}{a^2}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}$.

$$\sum b^2c^2\,(s-a)=s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]$$

Let's get back to the main problem. Using the Lemma we obtain:

RHS:

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{s[(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}=$$

$$=\frac{s(16R^4-16R^3r+16r^4)}{4R^2r}=\frac{4s(R^4-R^3r+r^4)}{R^2r}\overset{Euler}{\leq}\frac{4s\cdot\frac{9R^5}{32r}}{R^2r}=s\cdot\frac{9R^3}{8r^2}=9F\left(\frac{R}{2r}\right)^3$$

LHS: Using the Lemma the inequality can be written:

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{s[(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}=$$

$$=\frac{sr(64R^2-64Rr+16r^2)}{4R^2}=\frac{4sr(4R^2-4Rr+r^2)}{R^2}=F\left(4-\frac{2r}{R}\right)^2$$

Equality holds if and only if the triangle is equilateral.

Remark: LHS inequality strengthen Problem S.2466 from RMM – 42: $\sum h_a^2 \cot\frac{A}{2} \geq 9F$

We can write the inequalities:

In ΔABC holds:

$$\boldsymbol{9F \leq F\left(4-\frac{2r}{R}\right)^2 \leq \sum h_a^2 \cot\frac{A}{2} \leq 9F\left(\frac{R}{2r}\right)^3}$$

Solution:

See $F\left(4-\frac{2r}{R}\right)^2 \leq \sum h_a^2 \cot\frac{A}{2} \leq 9F\left(\frac{R}{2r}\right)^3$ and $9F \leq F\left(4-\frac{2r}{R}\right)^2 \Leftrightarrow R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilatearal.

Remark: In the same way:

In ΔABC holds:

$$\boldsymbol{9F \leq \sum r_a^2 \cot\frac{A}{2} \leq 9F\left(\frac{R}{2r}\right)}$$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC holds:

$$\boldsymbol{\sum r_a^2 \cot\frac{A}{2} = s(4R+r)}$$

Proof:

$$\sum r_a^2 \cot\frac{A}{2}=\sum\left(\frac{S}{s-a}\right)^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=S^2\sum\frac{1}{(s-a)^2}\cdot\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{S^2\cdot s}{S}\sum\frac{1}{s-a}=sr\cdot s\cdot\frac{4R+r}{sr}=s(4R+r).$$

We've used above: $\sum\frac{1}{s-a}=\frac{4R+r}{sr}$. Let's get back to the main problem. Using the Lemma we obtain:

RHS:

$$\sum r_a^2 \cot\frac{A}{2}=s(4R+r)\overset{Euler}{\leq} s\left(4R+\frac{R}{2}\right)=s\cdot\frac{9R}{2}=sr\cdot\frac{9R}{2r}=9F\cdot\frac{R}{2r}.$$

Equality holds if and only if the triangle is equilateral.

LHS:

$$\sum r_a^2 \cot\frac{A}{2} = s(4R+r) \overset{Euler}{\geq} s(4\cdot 2r + r) = s\cdot 9r = 9F$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

In ΔABC holds:

$$\sum h_a^2 \cot\frac{A}{2} \geq \frac{2r}{R}\sum r_a^2 \cot\frac{A}{2}$$

Marin Chirciu-Romania

Solution: Lemma 1: In ΔABC holds:

$$\sum h_a^2 \cot\frac{A}{2} = \frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}$$

Proof:

$$\sum h_a^2 \cot\frac{A}{2} = \sum\left(\frac{2S}{a}\right)^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = 4S^2\sum\frac{1}{a^2}\cdot\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{4S^2\cdot s}{S}\sum\frac{s-a}{a^2} = 4sr\cdot s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s} =$$

$$= \frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{16R^2r}$$

We've used above: $\sum\frac{s-a}{a^2} = \frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}$.

$$\sum b^2c^2\,(s-a) = s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]$$

Lemma 2: In ΔABC holds:

$$\sum r_a^2 \cot\frac{A}{2} = s(4R+r)$$

Proof:

$$\sum r_a^2 \cot\frac{A}{2} = \sum\left(\frac{S}{s-a}\right)^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = S^2\sum\frac{1}{(s-a)^2}\cdot\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{S^2\cdot s}{S}\sum\frac{1}{s-a} = sr\cdot s\cdot\frac{4R+r}{sr} = s(4R+r)$$

We've used above: $\sum\frac{1}{s-a}=\frac{4R+r}{sr}$. Let's get back to the main problem.

Using the above lemmas we have the sums:

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}\text{ and }\sum r_a^2\cot\frac{A}{2}=s(4R+r)$$

Inequality can be written:

$$\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}\ge\frac{2r}{R}\cdot s(4R+r)\Leftrightarrow$$

$$s^2(s^2+2r^2-12Rr)+r^3(4R+r)\ge 8Rr^2(4R+r),$$

which follows from Gerretsen's inequality $s^2\ge 16Rr-5r^2$.

It remains to prove that:

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)\ge 8Rr^2(4R+r)\Leftrightarrow$$

$$\Leftrightarrow(16R-5r)(4R-3r)+r(4R+r)\ge 8R(4R+r)\Leftrightarrow 4R^2-9Rr+2r^2\ge 0\Leftrightarrow$$

$(R-2r)(4R-r)\ge 0$, obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

In ΔABC holds:

$$\mathbf{3F\cdot\frac{2r}{R}\le\sum h_a^2\tan\frac{A}{2}\le 3F}$$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC holds:

$$\sum h_a^2\tan\frac{A}{2}=\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}$$

Proof:

$$\sum h_a^2\tan\frac{A}{2}=\sum\left(\frac{2S}{a}\right)^2\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=4S^2\sum\frac{1}{a^2}\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{4S^2}{S}\sum\frac{(s-b)(s-c)}{a^2}=4sr\cdot\frac{s^2(s^2+2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}=$$

$$=\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}.$$

We've used above: $\sum\frac{(s-b)(s-c)}{a^2}=\frac{s^2(s^2+2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}$.

$$\sum b^2c^2\,(s-b)(s-c)=r^2[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]$$

Let's get back to the main problem. Using the Lemma we obtain:

RHS

$$\sum h_a^2\tan\frac{A}{2}=\frac{rs[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s^2}=$$

$$=\frac{F}{4R^2}\left[s^2+2r^2-4Rr+\frac{r(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{F}{4R^2}\left[4R^2+4Rr+3r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$=\frac{F}{4R^2}[4R^2+5r^2+(4R+r)(R+r)]=\frac{F(8R^2+5Rr+6r^2)}{4R^2}\overset{Euler}{\leq}\frac{F\cdot 12R^2}{4R^2}=3F$$

LHS

$$\sum h_a^2\cot\frac{A}{2}=\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4R^2r}=$$

$$=\frac{F}{4R^2}\left[s^2+2r^2-4Rr+\frac{r(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{F}{4R^2}\left[16Rr-5r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{F}{4R^2}\left[12Rr-3r^2+\frac{2r(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{Fr(28R^2-7Rr-2r^2)}{4R^3}=\frac{Fr\cdot 24R^2}{4R^3}=\frac{6F}{R}=3F\cdot\frac{2r}{R}.$$

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way: **In ΔABC holds:**

$$\frac{81R^3}{8s}\leq\sum r_a^2\tan\frac{A}{2}\leq\frac{82R^3-575r^3}{s}$$

Marin Chirciu-Romania

Solution: Lemma:

In ΔABC holds:

$$\sum r_a^2\tan\frac{A}{2}=\frac{(4R+r)^3-12Rs^2}{s}$$

Proof:

$$\sum r_a^2\tan\frac{A}{2}=\sum\left(\frac{S}{s-a}\right)^2\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=S^2\sum\frac{1}{(s-a)^2}\cdot\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{S^2}{S}\sum\frac{(s-b)(s-c)}{(s-a)^2}=sr\cdot\frac{(4R+r)^3-12Rrs^2}{rs^2}=\frac{(4R+r)^3-12Rs^2}{s}$$

We've used above: $\sum\frac{(s-b)(s-c)}{(s-a)^2}=\frac{(4R+r)^3-12Rs^2}{rs^2}$.

$$\sum(s-b)^3(s-c)^3=r^3[(4R+r)^3-12Rs^2]$$

Let's get back to the main problem. Using the Lemma we obtain:

RHS

$$\sum r_a^2\tan\frac{A}{2}=\frac{(4R+r)^3-12Rs^2}{s}\overset{Gerretsen}{\leq}\frac{(4R+r)^3-12R(16Rr-5r^2)}{s}=$$

$$=\frac{64R^3-144R^2r+72Rr^2+r^3}{s}\overset{Euler}{\leq}\frac{82R^3-575r^3}{s}$$

Equality holds if and only if the triangle is equilateral.

LHS

$$\sum r_a^2\tan\frac{A}{2}=\frac{(4R+r)^3-12Rs^2}{s}\overset{Gerretsen}{\geq}\frac{(4R+r)^3-12R(4R^2+4Rr+3r^2)}{s}=$$

$$=\frac{16R^3-20Rr^2+r^3}{s}\overset{Euler}{\geq}\frac{\frac{81R^3}{8}}{s}=\frac{81R^3}{8s}$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC holds:

$$\sum h_a^2\tan\frac{A}{2}\leq\sum r_a^2\tan\frac{A}{2}$$

Marin Chirciu-Romania

Solution: Lemma 1: In ΔABC holds:

$$\sum h_a^2\tan\frac{A}{2}=\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}$$

Proof:

$$\sum h_a^2\tan\frac{A}{2}=\sum\left(\frac{2S}{a}\right)^2\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=4S^2\sum\frac{1}{a^2}\cdot\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{4S^2}{S}\sum\frac{(s-b)(s-c)}{a^2}=4sr\cdot\frac{s^2(s^2+2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}=$$

$$=\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}.$$

We've used above: $\sum\frac{(s-b)(s-c)}{a^2}=\frac{s^2(s^2+2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}$.

$$\sum b^2c^2\,(s-b)(s-c)=r^2[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]$$

Lemma 2: In ΔABC holds:

$$\sum r_a^2\tan\frac{A}{2}=\frac{(4R+r)^3-12Rs^2}{s}$$

Proof:

$$\sum r_a^2\tan\frac{A}{2}=\sum\left(\frac{S}{s-a}\right)^2\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=S^2\sum\frac{1}{(s-a)^2}\cdot\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{S^2}{S}\sum\frac{(s-b)(s-c)}{(s-a)^2}=sr\cdot\frac{(4R+r)^3-12Rs^2}{rs^2}=\frac{(4R+r)^3-12Rs^2}{s}.$$

We've used above: $\sum\frac{(s-b)(s-c)}{(s-a)^2}=\frac{(4R+r)^3-12Rs^2}{rs^2}$.

$$\sum(s-b)^3(s-c)^3=r^3[(4R+r)^3-12Rs^2]$$

Let's get back to the main problem. Using the above Lemmas we have the sums:

$\sum h_a^2\tan\frac{A}{2}=\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}$ and $\sum r_a^2\tan\frac{A}{2}=\frac{(4R+r)^3-12Rs^2}{s}$.

The inequality can be written:

$$\frac{r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]}{4R^2s}\le\frac{(4R+r)^3-12Rs^2}{s}\Leftrightarrow$$

$$\Leftrightarrow r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]\le 4R^2[(4R+r)^3-12Rs^2]\Leftrightarrow$$

$$\Leftrightarrow r[s^2(s^2+2r^2-4Rr)+r(4R+r)^3]\le 4R^2[(4R+r)^3-12Rs^2]\Leftrightarrow$$

$$s^2(rs^2+2r^3-4Rr^2+48R^3)\le(4R+r)^3(4R^2-r^2),$$

which follows from Gerretsen's inequality $s^2\le\frac{R(4R+r)^2}{2(2R-r)}\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$\frac{R(4R+r)^2}{2(2R-r)}(r(4R^2+4Rr+3r^2)+2r^3-4Rr^2+48R^3)\le(4R+r)^3(4R^2-r^2)\Leftrightarrow$$

$$\Leftrightarrow R(r(4R^2+4Rr+3r^2)+2r^3-4Rr^2+48R^3)\le 2(2R-r)(4R+r)(4R^2-r^2)\Leftrightarrow$$

$$\Leftrightarrow R(48R^3+4R^2r+5r^3)\le 64R^4-16R^3r-24R^2r^2+4Rr^3+2r^3\Leftrightarrow$$

$$\Leftrightarrow 16R^4 - 20R^3r - 24R^2r^2 - Rr^3 + 2r^4 \geq 0 \Leftrightarrow (R - 2r)(16R^3 + 12R^2r - r^3) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$. Equality holds if and only if the triangle is equilateral.

JP.503. In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot h_c \geq \frac{a+b+c}{R}$$

Marian Ursărescu-Romania

Solution: Lemma: In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot h_c = \frac{2s}{R}$$

Proof.

$$\sum \frac{a+b}{ab} \cdot h_c = \sum \frac{a+b}{ab} \cdot \frac{2S}{c} = \frac{2S}{abc} \sum (a+b) = \frac{2S}{4RS} \cdot 2 \sum a = \frac{1}{R} \cdot 2s = \frac{2s}{R}$$

Let's get back to the main problem.

Using the Lemma the inequality can be written:

$\frac{2s}{R} \geq \frac{a+b+c}{R}$, obviously, with equality.

SP.507. In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot r_c \geq \frac{a+b+c}{R}$$

Marian Ursărescu-Romania

Solution: Lemma: In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot r_c = \frac{s}{r}$$

Proof:

$$\sum \frac{a+b}{ab} \cdot r_c = \sum \frac{a+b}{ab} \cdot \frac{S}{s-c} = S \sum \frac{a+b}{ab(s-c)} = sr \cdot \frac{1}{r^2} = \frac{s}{r}$$

We've used above: $\sum \frac{a+b}{ab(s-c)} = \frac{1}{r^2}$. Let's get back to the main problem. Using the Lemma the inequality can be written:

$\frac{s}{r} \geq \frac{a+b+c}{R} \Leftrightarrow \frac{s}{r} \geq \frac{2s}{R} \Leftrightarrow R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot r_c \geq \sum \frac{a+b}{ab} \cdot h_c$$

Marin Chirciu-Romania

Solution: Lemma 1: In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot r_c = \frac{s}{r}$$

Proof:

$$\sum \frac{a+b}{ab} \cdot r_c = \sum \frac{a+b}{ab} \cdot \frac{S}{s-c} S \sum \frac{a+b}{ab(s-c)} = sr \cdot \frac{1}{r^2} = \frac{s}{r}.$$

We've used above: $\sum \frac{a+b}{ab(s-c)} = \frac{1}{r^2}$.

Lemma 2: In ΔABC holds:

$$\sum \frac{a+b}{ab} \cdot h_c = \frac{2s}{R}$$

Proof:

$$\sum \frac{a+b}{ab} \cdot h_c = \sum \frac{a+b}{ab} \cdot \frac{2S}{c} = \frac{2S}{abc} \sum (a+b) = \frac{2S}{4Rs} \cdot 2 \sum a = \frac{1}{R} \cdot 2s = \frac{2s}{R}.$$

Using the above Lemmas we have the sums:

$\sum \frac{a+b}{ab} \cdot r_c = \frac{s}{r}$ and $\sum \frac{a+b}{ab} \cdot h_c = \frac{2s}{R}$

The inequality can be written: $\frac{s}{r} \geq \frac{2s}{R} \Leftrightarrow R \geq 2r$, (Euler). Equality holds if and only if the triangle is equilateral.

JP.509. In ΔABC holds:

$$\prod \left(1 + \cot \frac{A}{2}\right) \geq \left(1 + \sqrt{3} \cdot \frac{2r}{R}\right)^3$$

George Apostolopoulos – Greece

Solution: Lemma: In ΔABC holds:

$$\prod \left(1 + cot \frac{A}{2}\right) = \frac{2(s+r+2R)}{r}$$

Proof:

$$\prod \left(1 + \cot \frac{A}{2}\right) = \prod \left(1 + \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\right) = \prod \left(1 + \frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}\right) =$$

$$=\prod\left(1+\frac{s(s-a)}{S}\right)=\prod\left(1+\frac{s(s-a)}{sr}\right)=\prod\left(1+\frac{s-a}{r}\right)=\frac{2(s+r+2R)}{r}.$$

Let's get back to the main problem. Using the Lemma the inequality can be written:

$\frac{2(s+r+2R)}{r}\geq\left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3$, which follows from Mitrinovic's inequality $s\geq 3\sqrt{3}r$.

It remains to prove that:

$$\frac{2(3\sqrt{3}+r+2R)}{r}\geq\left(1+\sqrt{3}\cdot\frac{2r}{R}\right)^3\Leftrightarrow 2R^3(3\sqrt{3}r+r+2R)\geq r(R+\sqrt{3}\cdot 2r)^3\Leftrightarrow$$

$$\Leftrightarrow 4R^4+(6\sqrt{3}+1)R^3r-6\sqrt{3}R^2r^2-36Rr^3-24\sqrt{3}r^4\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)\left[4R^3+(9+6\sqrt{3})R^2r+(18+6\sqrt{3})Rr^2+12\sqrt{3}r^3\right]\geq 0,$$

obviously from Euler's inequality $R\geq 2r$. Equality holds if and only if the triangle is equilateral.

Remark: In the same way. In ΔABC holds:

$$\prod\left(\mathbf{1}+\tan\frac{\boldsymbol{A}}{\mathbf{2}}\right)\geq\left(\mathbf{1}+\frac{\sqrt{\mathbf{3}}}{\mathbf{3}}\cdot\frac{\mathbf{2}\boldsymbol{r}}{\boldsymbol{R}}\right)^{\mathbf{3}}$$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC holds:

$$\prod\left(\mathbf{1}+\tan\frac{\boldsymbol{A}}{\mathbf{2}}\right)=\frac{\mathbf{2}(\boldsymbol{s}+\boldsymbol{r}+\mathbf{2}\boldsymbol{R})}{\boldsymbol{s}}$$

Proof:

$$\prod\left(1+\tan\frac{A}{2}\right)=\prod\left(1+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\right)=\prod\left(1+\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}\right)=$$

$$=\prod\left(1+\frac{(s-b)(s-c)}{S}\right)=\prod\left(1+\frac{(s-b)(s-c)}{sr}\right)=\prod\left(1+\frac{(s-b)(s-c)}{sr}\right)=$$

$$=\frac{2(s+r+2R)}{s}.$$

Let's get back to the main problem. Using the Lemma the inequality can be written:

$$\frac{2(s+r+2R)}{s}\geq\left(1+\frac{\sqrt{3}}{3}\cdot\frac{2R}{r}\right)^3\Leftrightarrow 2\left(1+\frac{r+2R}{s}\right)\geq\left(1+\frac{\sqrt{3}}{3}\cdot\frac{2r}{R}\right)^3$$

which follows from Mitrinovic's inequality $s\leq\frac{3\sqrt{3}R}{2}$. It remains to prove tha:

$$2\left(1+\frac{r+2R}{\frac{3\sqrt{3}R}{2}}\right)\geq\left(1+\frac{\sqrt{3}}{3}\cdot\frac{2r}{R}\right)^3\Leftrightarrow 18R^2\left(4R+3\sqrt{3}R+2r\right)\geq\sqrt{3}\left(3R+2\sqrt{3}\cdot r\right)^3\Leftrightarrow$$

$$\Leftrightarrow\left(8+3\sqrt{3}\right)R^3-14R^2r-12Rr^2-8r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)\left[\left(8+3\sqrt{3}\right)R^2+\left(2+6\sqrt{3}\right)Rr+4r^2\right]\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed. **In ΔABC holds:**

$$\prod\left(\mathbf{1}+\tan\frac{A}{2}\right)\leq\frac{1}{3\sqrt{3}}\prod\left(\mathbf{1}+\cot\frac{A}{2}\right)$$

Solution: Lemma 1: In ΔABC holds:

$$\prod\left(\mathbf{1}+\tan\frac{A}{2}\right)=\frac{2(s+r+2R)}{s}$$

Proof:

$$\prod\left(1+\tan\frac{A}{2}\right)=\prod\left(1+\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\right)=\prod\left(1+\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}\right)=$$

$$=\prod\left(1+\frac{(s-b)(s-c)}{S}\right)=\prod\left(1+\frac{(s-b)(s-c)}{sr}\right)=\prod\left(1+\frac{(s-b)(s-c)}{sr}\right)=$$

$$=\frac{2(s+r+2R)}{s}$$

Lemma 2: In ΔABC holds:

$$\prod\left(\mathbf{1}+\cot\frac{A}{2}\right)=\frac{2(s+r+2R)}{r}$$

Proof:

$$\prod\left(1+\cot\frac{A}{2}\right)=\prod\left(1+\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\right)=\prod\left(1+\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}\right)=$$

$$=\prod\left(1+\frac{s(s-a)}{S}\right)=\prod\left(1+\frac{s(s-a)}{sr}\right)=\prod\left(1+\frac{s-a}{r}\right)=\frac{2(s+r+2R)}{r}$$

Let's get back to the main problem. Using the above Lemmas we have the sums:

$$\prod\left(1+\tan\frac{A}{2}\right)=\frac{2(s+r+2R)}{s}\text{ and }\prod\left(1+\cot\frac{A}{2}\right)=\frac{2(s+r+2R)}{r}$$

The inequality can be written:

$$\frac{2(s+r+2R)}{s} \le \frac{1}{3\sqrt{3}} \cdot \frac{2(s+r+2R)}{r} \Leftrightarrow s \ge 3\sqrt{3}r, \text{ (Mitrinovic)}$$

Equality holds if and only if the triangle is equilateral.

J.2116. If $x, y, z \ge 0$ then in ΔABC

$$\sum \frac{e^x}{y+z+2} \cdot a^2 \ge 2\sqrt{3}F$$

D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution: Using the inequality $e^x \ge x+1, x \in \mathbb{R}$, with equality for $x = 0$ we obtain:

$$LHS = \sum \frac{e^x}{y+z+2} \cdot a^2 \ge \sum \frac{x+1}{y+z+2} \cdot a^2 = \sum \frac{x+1}{y+z+2} \cdot a^2 = \sum \frac{x+1}{y+z+2} \cdot a^2 =$$

$$= \sum \frac{x+1}{(y+1)+(z+1)} \cdot a^2 = \sum \frac{m}{n+p} \cdot a^2 \overset{Tsintsifas}{\ge} 2\sqrt{3}F = RHS$$

Lemma (G. Tsintsifas): In ΔABC holds:

$$\frac{x}{y+z} a^2 + \frac{y}{z+x} b^2 + \frac{z}{x+y} c^2 \ge 2\sqrt{3}S, \text{ where } x, y, z > 0$$

G. Tsintsifas-Greece

Solution:

We have $\sum \frac{x}{y+z} a^2 = \sum \left(\frac{x}{y+z} + 1 - 1\right) a^2 = \sum \frac{x+y+z}{y+z} a^2 - \sum a^2 \overset{Bergstrom}{\ge}$

$$\ge (x+y+z) \frac{(\sum a)^2}{\sum (y+z)} - \sum a^2 = (x+y+z) \frac{(2s)^2}{2(x+y+z)} - 2(s^2 - r^2 - 4Rr) =$$

$$= 2s^2 - 2(s^2 - r^2 - 4Rr) = 2(r^2 + 4Rr)$$

Above we've used the known identities in triangle $\sum a = 2s$ and $\sum a^2 = 2(s^2 - r^2 - 4Rr)$

It remains to prove that $2(r^2 + 4Rr) \ge 2\sqrt{3}S \Leftrightarrow r^2 + 4Rr \ge \sqrt{3}rs \Leftrightarrow 4R + r \ge s\sqrt{3}$,

which is Doucet's inequality .Equality holds if and only if $a = b = c$ and $x = y = z = 0$.

S.2410. In ΔABC:

$$\sum \frac{r_a}{\sqrt{r_b r_c}} bc \ge 4\sqrt{3}F$$

D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

Solution:

$$LHS = \sum \frac{r_a}{\sqrt{r_b r_c}} bc \overset{AM-GM}{\ge} 3\sqrt[3]{\prod \frac{r_a}{\sqrt{r_b r_c}} bc} = 3\sqrt[3]{\prod bc} = 3\sqrt[3]{(abc)^2} \overset{Carlitz}{\ge}$$

$$\geq 3 \cdot \frac{4F}{\sqrt{3}} = 4\sqrt{3}F = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: In ΔABC holds:

$$\sum \frac{h_a}{\sqrt{h_b h_c}} bc \geq 4\sqrt{3}F$$

Marin Chirciu-Romania

Solution:

$$LHS = \sum \frac{h_a}{\sqrt{h_b h_c}} bc \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{h_a}{\sqrt{h_b h_c}} bc} = 3\sqrt[3]{\prod bc} = 3\sqrt[3]{(abc)^2} \overset{Carlitz}{\geq}$$

$$\geq 3 \cdot \frac{4F}{\sqrt{3}} = 4\sqrt{3}F = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark:In the same way: In ΔABC holds:

$$\sum \frac{w_a}{\sqrt{w_b w_c}} bc \geq 4\sqrt{3}F$$

Marin Chirciu-Romania

Solution:

$$LHS = \sum \frac{w_a}{\sqrt{w_b w_c}} bc \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{w_a}{\sqrt{w_b w_c}} bc} = 3\sqrt[3]{\prod bc} = 3\sqrt[3]{(abc)^2} \overset{Carlitz}{\geq}$$

$$\geq 3 \cdot \frac{4F}{\sqrt{3}} = 4\sqrt{3}F = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: In ΔABC holds:

$$\sum \frac{m_a}{\sqrt{m_b m_c}} bc \geq 4\sqrt{3}F$$

Marin Chirciu-Romania

Solution:

$$LHS = \sum \frac{m_a}{\sqrt{m_b m_c}} bc \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{m_a}{\sqrt{m_b m_c}} bc} = 3\sqrt[3]{\prod bc} = 3\sqrt[3]{(abc)^2} \overset{Carlitz}{\geq}$$

$$\geq 3\cdot\frac{4F}{\sqrt{3}}=4\sqrt{3}F=RHS$$

Equality holds if and only if is equilateral.

Remark: In the same way: In ΔABC holds:

$$\sum\frac{\boldsymbol{s_a}}{\sqrt{\boldsymbol{s_b s_c}}}\boldsymbol{bc}\geq\boldsymbol{4\sqrt{3}F}$$

Marin Chirciu-Romania

Solution:

$$LHS=\sum\frac{s_a}{\sqrt{s_bs_c}}bc\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{s_a}{\sqrt{s_bs_c}}bc}=3\sqrt[3]{\prod bc}=3\sqrt[3]{(abc)^2}\overset{Carlitz}{\geq}$$

$$\geq 3\cdot\frac{4F}{\sqrt{3}}=4\sqrt{3}F=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: If $x,y,z>0$, in ΔABC holds:

$$\sum\frac{\boldsymbol{x}}{\sqrt{\boldsymbol{yz}}}\boldsymbol{bc}\geq\boldsymbol{4\sqrt{3}F}$$

Marin Chirciu-Romania

Solution:

$$LHS=\sum\frac{x}{\sqrt{yz}}bc\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{x}{\sqrt{yz}}bc}=3\sqrt[3]{\prod bc}=3\sqrt[3]{(abc)^2}\overset{Carlitz}{\geq}$$

$$\geq 3\cdot\frac{4F}{\sqrt{3}}=4\sqrt{3}F=RHS.$$

Equality holds if and only if the triangle is equilateral.

S.2367. Solve for real numbers: $\boldsymbol{x^3(x+1)^3+8x^3+8=12x^2(x+1)}$

Daniel Sitaru, Luiza Dumitrescu – Romania

Solution: Lemma: If $x>0$ then:

$$\boldsymbol{x^3(x+1)^3+8x^3+8\geq 12x^2(x+1)}$$

Proof:

$$x^3(x+1)^3+8x^3+8\overset{AM-GM}{\geq}3\sqrt[3]{x^3(x+1)^3\cdot 8x^3\cdot 8}=12x^2(x+1)$$

with equality for $x^3(x+1)^3=8x^3=8\Leftrightarrow x=1$.Let’s get back to the main problem.

Using the Lemma we deduce that the equation admits the only positive solution $x = 1$.

Remark: The problem can be developed.Let $a \geq 0, \lambda = a + 1$. Sove for positive real numbers:

$$x^3(x+a)^3 + \lambda^3 x^3 + \lambda^3 = 3\lambda^2 x^2(x+a)$$

Marin Chirciu-Romania

Solution: Lemma: If $x > 0$ then:

$$\boldsymbol{x^3(x+a)^3 + \lambda^3 x^3 + \lambda^3 \geq 3\lambda^2 x^2(x+a)}$$

Proof:

$$x^3(x+a)^3 + \lambda^3 x^3 + \lambda^3 \overset{AM-GM}{\geq} 3\sqrt[3]{x^3(x+a)^3 \cdot \lambda^3 x^3 \cdot \lambda^3} = 3\lambda^2 x^2(x+a),$$

with equality for $x^3(x+a)^3 = \lambda^3 x^3 = \lambda^3 \Leftrightarrow x = 1$.

Let's get back to the main problem. Using the Lemma, we deduce that the equation admits the only positive solution $x = 1$.

Note: For $a = 1$ we obtain Problem S.2367 from RMM – 42:

S.2367. Solve for real numbers:

$$\boldsymbol{x^3(x+1)^3 + 8x^3 + 8 = 12x^2(x+1)}$$

Daniel Sitaru, Luiza Dumitrescu – Romania

SP. 497. In ΔABC:

$$\boldsymbol{\sum \frac{r_a^4}{\sin 2A} \geq \frac{81\sqrt{3}}{8} R^4}$$

George Apostolopoulos – Greece

Solution:

$$LHS = \sum \frac{r_a^4}{\sin 2A} \overset{CS}{\geq} \frac{(\sum r_a^2)^2}{\sum \sin 2A} = \frac{[(4R+r)^2 - 2s^2]^2}{\frac{2rs}{R^2}} \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{R^2}{2rs}[(4R+r)^2 - 2(4R^2 + 4Rr + 3r^2)]^2 = \frac{R^2}{2rs}(8R^2 - 5r^2) \overset{Euler}{\geq} \frac{R^2}{2rs}\left(\frac{27}{4}R^2\right)^2$$

$$= \frac{27^2 R^6}{32rs} \overset{Mitrinovic}{\geq} \frac{27^2 R^6}{32r \cdot \frac{3\sqrt{3}R}{2}} \overset{Euler}{\geq} \frac{81\sqrt{3}}{8} R^4 = RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way:

In ΔABC holds:

$$\sum \frac{r_a^{2n}}{\sin 2A} \ge 2\sqrt{3}\left(\frac{3R}{2}\right)^{2n}, n \in \mathbb{N}$$

Marin Chirciu-Romania

Solution: For $n = 0$ and $n = 1$ we use CS inequality. For $n \ge 2$ we use Holder's inequality.

$$LHS = \sum \frac{r_a^{2n}}{\sin 2A} \overset{Holder}{\ge} \frac{(\sum r_a^2)^n}{3^{n-2}\sum \sin 2A} = \frac{[(4R+r)^2 - 2s^2]^n}{3^{n-2}\cdot\frac{2rs}{R^2}} \overset{Gerretsen}{\ge}$$

$$\overset{Gerretsen}{\ge} \frac{R^2}{2rs}\cdot\frac{[(4R+r)^2-2s^2]^n}{3^{n-2}} = \frac{R^2}{2rs}\cdot\frac{(8R^2-5r^2)^n}{3^{n-2}} \overset{Euler}{\ge} \frac{R^2}{2rs}\cdot\frac{1}{3^{n-2}}\left(\frac{27}{4}R^2\right)^n =$$

$$= \frac{R^2}{2rs}\cdot\frac{1}{3^{n-2}}\left(\frac{27}{4}R^2\right)^n \overset{Mitrinovic}{\ge} \frac{R^2}{2r\cdot\frac{3\sqrt{3}R}{2}}\cdot\frac{1}{3^{n-2}}\cdot\frac{3^{3n}}{2^{2n}} \overset{Euler}{\ge} \frac{R^2}{R\cdot\frac{3\sqrt{3}R}{2}}\cdot\frac{1}{3^{n-2}}\cdot\frac{3^{3n}}{2^{2n}}\cdot R^{2n} =$$

$$= 2\sqrt{3}\left(\frac{3R}{2}\right)^{2n} = RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $n = 2$ we obtain Problem SP.497 from RMM – 42.

SP.497. In ΔABC holds:

$$\sum \frac{r_a^4}{\sin 2A} \ge \frac{81\sqrt{3}}{8}R^4$$

George Apostolopoulos – Greece

J.2113. If $x, y, z > 0$ then in ΔABC:

$$\sum \frac{x}{y+z}\cdot\frac{a}{h_a} \ge \sqrt{3}$$

D.M. Bătinețu – Giurgiu – Romania

Solution:

$$LHS = \sum \frac{x}{y+z}\cdot\frac{a}{h_a} = \sum \frac{x}{y+z}\cdot\frac{a^2}{ah_a} = \sum \frac{x}{y+z}\cdot\frac{a^2}{2F} = \frac{1}{2F}\sum \frac{x}{y+z}\cdot a^2 \overset{Tsintsifas}{\ge}$$

$$\ge \frac{1}{2F}\cdot 2\sqrt{3}F = \sqrt{3} = RHS$$

Lemma (G. Tsintsifas): In ΔABC holds:

$$\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2 \ge 2\sqrt{3}S, \text{ where } x, y, z > 0.$$

G. Tsintsifas-Greece

Solution: We have $\sum\frac{x}{y+z}a^2=\sum\left(\frac{x}{y+z}+1-1\right)a^2=\sum\frac{x+y+z}{y+z}a^2-\sum a^2\overset{Bergstrom}{\geq}$

$$\geq(x+y+z)\frac{(\sum a)^2}{\sum(y+z)}-\sum a^2=(x+y+z)\frac{(2s)^2}{2(x+y+z)}-2(s^2-r^2-4Rr)=$$

$$=2s^2-2(s^2-r^2-4Rr)=2(r^2+4Rr).$$

Above we've used the known inequalities in triangle: $\sum a=2s$ and $\sum a^2=2(s^2-r^2-4Rr)$.

It remains to prove that $2(r^2+4Rr)\geq 2\sqrt{3}S\Leftrightarrow r^2+4Rr\geq\sqrt{3}rs\Leftrightarrow 4R+r\geq s\sqrt{3}$, which is Doucet's inequality.

Equality holds if and only if $a=b=c$ and $x=y=z$.

J.2107. In ΔABC holds:

$$\sum\frac{a^4+b^4}{h_ah_b}\geq\frac{32}{\sqrt{3}}F$$

D.M. Bătinețu – Giurgiu – Romania

Solution:

$$LHS=\sum\frac{a^4+b^4}{h_ah_b}=\sum\frac{a^4+b^4}{h_ah_b}=\sum\frac{a^4+b^4}{\frac{2F}{a}\cdot\frac{2F}{b}}=\frac{1}{4F^2}\sum ab(a^4+b^3)\overset{AGM}{\geq}$$

$$=\frac{1}{4F^2}\sum ab\cdot 2a^2b^2=\frac{1}{2F^2}\sum(ab)^3\overset{Holder}{\geq}\frac{1}{2F^2}\frac{(\sum ab)^3}{9}=\frac{(\sum ab)^3}{18F^2}\overset{Gordon}{\geq}\frac{\left(4\sqrt{3}F\right)^3}{18F^2}=$$

$$=\frac{32}{\sqrt{3}}F=RHS$$

Equality holds if and only if the triangle is equilateral. **Remark: The problem can be developed.**

In ΔABC holds:

$$\sum\frac{a^{2n}+b^{2n}}{h_ah_b}\geq 8\left(\frac{4F}{\sqrt{3}}\right)^{n-1}$$

Marin Chirciu-Romania

Solution: For $n=0$ inequality can be written $\sum\frac{2}{h_ah_b}\geq 8\left(\frac{4F}{\sqrt{3}}\right)^{-1}\Leftrightarrow\sum\frac{1}{h_ah_b}\geq\frac{\sqrt{3}}{F}$, which follows from:

$$LHS=\sum\frac{1}{h_ah_b}=\sum\frac{1}{\frac{2F}{a}\cdot\frac{2F}{b}}=\frac{1}{4F^2}\sum ab\overset{Gordon}{\geq}\frac{1}{4F^2}\cdot 4\sqrt{3}F=\frac{\sqrt{3}}{F}=RHS$$

For $n=1$ the inequality can be written $\sum\frac{a^2+b^2}{d_ad_b}\geq 8$, which follows from:

$$LHS=\sum\frac{a^2+b^2}{h_ah_b}=\sum\frac{a^2+b^2}{\frac{2F}{a}\cdot\frac{2F}{b}}=\frac{1}{4F^2}\sum ab(a^2+b^2)\overset{AGM}{\geq}\frac{1}{4F^2}\sum ab\cdot 2ab=$$

$$=\frac{1}{2F^2}\sum(ab)^2\overset{CS}{\geq}\frac{1}{2F^2}\cdot\frac{(\sum a)^2}{3}\overset{Gordon}{\geq}\frac{1}{2F^2}\cdot\frac{\left(4\sqrt{3F}\right)^2}{3}=8=RHS.$$

For $n\geq 3$ we use Holder's inequality.

$$LHS=\sum\frac{a^{2n}+b^{2n}}{h_ah_b}=\sum\frac{a^{2n}+b^{2n}}{\frac{2F}{a}\cdot\frac{2F}{b}}=\frac{1}{4F^2}\sum ab(a^{2n}+b^{2n})\overset{AGM}{\geq}$$

$$=\frac{1}{4F^2}\sum ab\cdot 2a^nb^n=\frac{1}{2F^2}\sum(ab)^{n+1}\overset{Holder}{\geq}\frac{1}{2F^2}\cdot\frac{(\sum ab)^{n+1}}{3^n}=\frac{(\sum ab)^{n+1}}{2F^2\cdot 3^n}\overset{Gordon}{\geq}$$

$$\geq\frac{\left(4\sqrt{3}F\right)^{n+1}}{2F^2\cdot 3^n}=\frac{2^{2n+2}\cdot 3^{\frac{n+1}{2}}F^{n+1}}{2F^2\cdot 3^n}=2^{2n+1}\cdot 3^{\frac{n+1}{2}-n}F^{n+1-2}=$$

$$=2^{2n+1}\cdot 3^{\frac{-n+1}{2}}F^{n-1}=8\left(\frac{4F}{\sqrt{3}}\right)^{n-1}=RHS$$

Equality holds if and only if the triangle is equilateral.

Note: For $n=2$ we obtain Problem J.2107 from RMM – 40 Spring Edition 2024, proposed by D.M. Bătinețu – Giurgiu.

In ΔABC holds:

$$\sum\frac{a^4+b^4}{h_ah_b}\geq\frac{32}{\sqrt{3}}F$$

D.M. Bătinețu – Giurgiu – Romania

J.2434. If $a,b,c>0, abc=1$ then:

$$\prod(a^3+1)\geq\prod\left(a+\frac{1}{a}\right)$$

Ilir Demiri – Azerbaijan

Solution:

$$\prod(a^3+1)\geq\prod\left(a+\frac{1}{a}\right)\overset{abc=1}{\Leftrightarrow}\prod(a^3+1)\geq\prod(a^2+1)\Leftrightarrow\prod\frac{a^3+1}{a^2+1}\geq 1$$

Lemma: If $a>0$ then:

$$\frac{a^3+1}{a^2+1}\geq\sqrt{a}$$

Proof:

$$\frac{a^3+1}{a^2+1} \geq \frac{a+1}{2} \geq \sqrt{a}$$

Indeed: $\frac{a^3+1}{a^2+1} \geq \frac{a+1}{2} \Leftrightarrow a^3 - a^2 - a + 1 \geq 0 \Leftrightarrow (a-1)^2(a+1) \geq 0$, with equality for $a = 1$.

Then $\frac{a+1}{2} \geq \sqrt{a} \Leftrightarrow \left(\sqrt{a}-1\right)^2 \geq 0$,with equality for $a = 1$. Let's get back to the main problem.

$$\prod \frac{a^3+1}{a^2+1} \overset{Lemma}{\geq} \prod \sqrt{a} = \sqrt{abc} \overset{abc=1}{=} 1.$$

Equality holds if and only if $a = b = c = 1$.

Remark: The problem can be developed: **If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0, abc = 1$ and $\boldsymbol{n} \in \mathbb{N}$ then:**

$$\prod (\boldsymbol{a}^{n+2} + \mathbf{1}) \geq \prod \left(\boldsymbol{a}^{n} + \frac{\mathbf{1}}{\boldsymbol{a}}\right)$$

Marin Chirciu-Romania

Solution:

$$\prod (a^{n+2}+1) \geq \prod \left(a^n + \frac{1}{a}\right) \overset{abc=1}{\Leftrightarrow} \prod (a^{n+2}+1) \geq \prod (a^{n+1}+1) \Leftrightarrow$$

$$\Leftrightarrow \prod \frac{a^{n+2}+1}{a^{n+1}+1} \geq 1$$

Lemma: If $\boldsymbol{a} > 0$ and $\boldsymbol{n} \in \mathbb{N}$ then:

$$\frac{\boldsymbol{a}^{n+2}+\mathbf{1}}{\boldsymbol{a}^{n+1}+\mathbf{1}} \geq \sqrt{\boldsymbol{a}}$$

Proof: We have $\frac{a^{n+2}+1}{a^{n+1}+1} \geq \frac{a+1}{2} \geq \sqrt{a}$. Indeed:

$\frac{a^{n+2}+1}{a^{n+1}+1} \geq \frac{a+1}{2} \Leftrightarrow a^{n+2} - a^{n+1} - a + 1 \geq 0 \Leftrightarrow (a^{n+1}-1)(a-1) \geq 0$, because both factors have the same sign, with equality with $a = 1$. Then $\frac{a+1}{2} \geq \sqrt{a} \Leftrightarrow \left(\sqrt{a}-1\right)^2 \geq 0$, with equality for $a = 1$.

Let's get back to the main problem. Using the Lemma we obtain

$$\prod \frac{a^{n+2}+1}{a^{n+1}+1} \overset{Lemma}{\geq} \prod \sqrt{a} = \sqrt{abc} \overset{abc=1}{=} 1$$

Equality holds if and only if $a = b = c = 1$.

Note: For $n = 1$ we obtain Problem J.2434 from RMM – 42.

J.2434. If $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} > 0, abc = 1$ then:

$$\prod (\boldsymbol{a}^3 + \mathbf{1}) \geq \prod \left(\boldsymbol{a} + \frac{\mathbf{1}}{\boldsymbol{a}}\right)$$

Reference: [1] Romanian Mathematical Magazine – www.ssmrmh.ro

PROPOSED PROBLEMS

PROBLEMS FOR JUNIORS

J.2631 Solve the equation:

$$\frac{\sin^4 x + \cos^4 x}{4} = \frac{\sin^6 x + \cos^6 x}{4\cos^2 2x + \sin^2 2x}$$

Proposed by Gilena Dobrică –Romania

J.2632 Let $a > 0, a \neq 1$. Solve the equation $x^{3+\log_a x} < a^2x^2$

Proposed by Meda Iacob, Carmen Vlad – Romania

J.2633 Let be the function $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x^2 + x + 1$. Solve the inequality $f\big(f(x)\big) \leq 0$.

Proposed by Lavinia Trincu, Monica Matei – Romania

J.2634 Find the prime numbers p, q such that $2p^2 + 1 = q^5$

Proposed by Gigi Zaharia – Romania

J.2635 Solve in real numbers set the system:

$$\begin{cases} x^5 - y^4 = 2022 \cdot 2023^4 \\ y^5 - x^4 = 2022 \cdot 2023^4 \end{cases}$$

Proposed by Carina Maria Viespescu, Caterina Zetu – Romania

J.2636 If $m \geq 0$ and $a, b > 0$, then:

$$(a^{2m+2} + 1)(b^{2m+2} + 1) \geq \frac{3^{m+1}}{4^{2m+1}} \cdot ((a+b)^2 + 1)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2637 If $t, x, y, z > 0$ and $x + y + z = a$, then:

$$(x^2 + t^2)(y^2 + t^2)(z^2 + t^2) \geq \frac{3}{4}t^4a^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2638 If $x, y, z > 0$ then in ΔABC with the area F the following inequality holds:

$$\left(\frac{x^2a^2}{(y+z)^2h_a^2}+2\right)\left(\frac{y^2b^2}{(z+x)^2h_b^2}+2\right)\left(\frac{z^2c^2}{(x+y)^2h_c^2}+2\right) \geq 9$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2639 If $a, b, c > 0, x \in \mathbb{R}$ and $t \geq 0, u \geq 1$, then:

$$\left(\frac{a}{b\sin^2 x + c\cdot\cos^2 x}\right)^{t+u} + \left(\frac{b}{c\cdot\sin^2 x + a\cdot\cos^2 x}\right)^{t+u} + \left(\frac{c}{a\cdot\sin^2 x + b\cdot\cos^2 x}\right)^{t+u} \geq 3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2640 If $a, b, c > 0, x \in \mathbb{R}, t \in \mathbb{R}_+ = [0,\infty), u \in [1,\infty)$, then:

$$\left(\left(\frac{a}{b\sin^2 x + c\cos^2 x}\right)^{2t+2u} + 2\right)\cdot\left(\left(\frac{b}{c\sin^2 x + a\cos^2 x}\right)^{2t+2u} + 2\right)\cdot$$

$$\cdot\left(\left(\frac{c}{a\sin^2 x + b\cos^2 x}\right)^{2t+2u} + 2\right) \geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2641 If $a, b, c > 0, m \geq 0$ and $x \in \mathbb{R}$ then:

$$\left(\frac{a}{b\sin^2 x + c\cos^2 x}\right)^{m+1} + \left(\frac{b}{c\sin^2 x + a\cos^2 x}\right)^{m+1} + \left(\frac{c}{a\sin^2 x + b\cos^2 x}\right)^{m+1} \geq 3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2642 In any ΔABC with the area F the following inequality holds: $(r_a^4+2)(r_b^4+2)(r_c^4+2) \geq 81F^2$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2643 If $x, y, z \geq 0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{a^4\cdot e^{2x}}{(y+z+2)^2+2}\right)\cdot\left(\frac{b^4e^{2y}}{(z+x+2)^2}+2\right)\cdot\left(\frac{c^4e^{2z}}{(x+y+2)^2}+2\right) \geq 36\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2644 In any ΔABC with the area F the following inequality holds:

$$(r_a^2r_c^2+2)(r_b^2r_c^2+2)(r_c^2r_a^2+2) \geq 81F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2645 Let be $t, u, x, y, z > 0$, then:

$$(x^2+t^2u)(y^2+t^2u)(z^2+t^2u) \geq \frac{3}{4}\cdot t^4\cdot u^2\cdot(x+y+z)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2646 If $m \geq 0$ and $a, b > 0$, then:

$$(a^{2m+2}+1)(b^{2m+2}+1) \geq \frac{(a+b)^{2m+2}}{4^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2647 In any ΔABC with the area F the following inequality holds:

$$\left(\left(\frac{a}{r_a}\right)^4+2\right)\cdot\left(\left(\frac{b}{r_b}\right)^4+2\right)\cdot\left(\left(\frac{c}{r_c}\right)^4+2\right) \geq 144\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2648 If $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{x^2a^4}{y+z}+\frac{y^2b^4}{z+x}+\frac{z^2c^4}{x+y} \geq \frac{6\cdot F^2}{x+y+z}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2649 If H – orthocenter in acute $\Delta ABC, AD, BE, CF$ – altitudes, $HD = x, HE = y, HF = z$ then:

$$yz\cdot\sec A + xz\cdot\sec B + xy\cdot\sec C \leq 6r^2$$

Proposed by Ertan Yildirim-Turkiye

J.2650 In ABC prove that:

$$\frac{1}{2(R-r)} \leq \frac{a^2}{b^2r_b+c^2r_c}+\frac{b^2}{c^2r_c+a^2r_a}+\frac{c^2}{a^2r_a+b^2r_b} \leq \frac{a^4+b^4+c^4}{16F^2\cdot R}$$

Proposed by Mehmet Șahin –Turkiye

J.2651 In ΔABC

$$\frac{4}{R} \leq \sum\frac{h_b+h_c}{r_a^2} \leq \frac{R}{r^2}$$

Proposed by Marin Chirciu – Romania

J.2652 If $a_i > 0, i \in \{1,2,\dots,n\}, n \in \mathbb{N}_{\geq 1}$ then:

$$2\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) \leq \frac{n}{n+1}+a_1+\frac{a_2}{4}+\dots+\frac{a_n}{n^2}+\frac{1}{n+1}\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots\frac{1}{a_n}\right)^{n+1}$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

J.2653 If $a_i, b_i > 0, i \in \{1,2,\dots,n\}, n \in \mathbb{N}$ then:

$$\sum_{i=1}^{n}\left(\frac{1}{a_i}+\frac{1}{b_i}\right)^{a_i+b_i} \geq \sum_{i=1}^{n}\left(\frac{a_i^2(1-b_i)+b_i^2(1-a_i)}{a_ib_i}\right)+3n$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

J.2654 If $a_i > 0, i \in \{1,2,\dots,n+1\}, n \in \mathbb{N}_{\geq 1}$ such that $(a_1 \cdot a_2 \dots a_n \geq 1)$ then:

$$\frac{a_1 - n + 1}{\sqrt[n]{a_2 + \dots + a_{n+1}}} + \frac{a_2 - n + 1}{\sqrt[n]{a_2 + \dots + a_1}} + \dots + \frac{a_{n+1} - n + 1}{\sqrt[n]{a_1 + \dots + a_n}} \geq \frac{n(n+1)(2-n)}{2n-1}$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

J.2655 In ΔABC

$$\sum \frac{a^3}{h_b + h_c} \leq \sum \frac{a^3}{r_b + r_c}$$

Proposed by Marin Chirciu – Romania

J.2656 If $a, b, c > 0, ab + bc + ca + abc = 4$ then:

$$\sum \frac{a}{\sqrt{3(a^2 + 2)}} \leq 1$$

Proposed by Marin Chirciu – Romania

J.2657 Find all values of $\alpha \in \mathbb{R}$ such that $\left(1 - \frac{2023}{n}\right)^n \leq \alpha \leq \left(1 + \frac{2023}{n}\right)^{n+1}, \forall n \in \mathbb{N}, n \geq 1$

Proposed by Nguyen Van Canh-Vietnam

J.2658 Let $a, b, c \geq 0: ab + bc + ca = 3$. Prove that:

$$\sqrt{\frac{b+c}{bc+1}} + \sqrt{\frac{c+a}{ca+1}} + \sqrt{\frac{a+b}{ab+1}} \geq \frac{a+b+c+3}{\sqrt{a+b+c+abc}}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2659 In ΔABC holds:

$$\sum \frac{1 + \cot^4 A}{1 + \cot^2 A} \geq \sqrt{3}$$

Proposed by Marin Chirciu – Romania

J.2660 If $a, b, c > 0, a + b + c = 3$ then find min of

$$P = 2(a^2 + b^2 + c^2) + 3abc$$

Proposed by Marin Chirciu – Romania

J.2661 In ΔABC holds:

$$\sum \frac{2 + \sqrt{3}\tan\frac{B}{2}}{1 + 3\tan^2\frac{A}{2}} \geq \frac{9}{2}$$

Proposed by Marin Chirciu – Romania

J.2662 If $a,b,c>0, ab+bc+ca=3$ then:

$$\sqrt[3]{\frac{\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}}{3}}\ge\frac{abc+2}{3}$$

Proposed by Marin Chirciu – Romania

J.2663 In ΔABC the following relationship holds:

$$\frac{1}{3}\left(\frac{m_a}{h_c}+\frac{w_a}{h_b}+\frac{r_a}{h_a}\right)\ge$$

$$\ge\sqrt[3]{\frac{R}{2r}}+\frac{1}{3}\max\left(\left(\sqrt{\frac{m_a}{h_c}}-\sqrt{\frac{w_a}{h_b}}\right)^2,\left(\sqrt{\frac{w_a}{h_b}}-\sqrt{\frac{r_a}{h_a}}\right)^2,\left(\sqrt{\frac{r_a}{h_a}}-\sqrt{\frac{m_a}{h_c}}\right)^2\right)$$

Proposed by Bogdan Fuștei – Romania

J.2664 In ΔABC the following relationship holds:

$$\frac{(r_aw_b)^5}{r_a^5+w_b^5}+\frac{(r_bw_c)^5}{r_b^5+w_c^5}+\frac{(r_cw_a)^5}{r_c^5+w_a^5}\le\left(\frac{3}{2}\right)^6\frac{(81R^5-2560)^2}{32r^5}$$

Proposed by Zaza Mzhavanadze – Georgia

J.2665 In ΔABC the following relationship holds:

$$\frac{2}{3}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\cdot\frac{\sqrt{a^2+b^2+c^2}}{R}\le\frac{h_a+h_b-h_c}{h_c}+\frac{2n_c}{\sqrt{4r^2+(a-b)^2}}$$

Proposed by Bogdan Fuștei – Romania

J.2666 Find $n\in\mathbb{N}$ such that $A=n!-39$ to be a perfect square ($n!=1\cdot2\cdot\ldots\cdot n$ for $n\in\mathbb{N}^*$ and $0!=1$)

Proposed by Elena Alexie-Romania

J.2667 Find the prime numbers p such that the number

$$A=5^{p^3}+7^{p^3}+9^{p^3}+11^{p^3}+13^{p^3}+15^{p^3}+17^{p^3}$$

can be divided with p.

Proposed by Grigorie Dan-Romania

J.2668 Solve the following equation:

$$\sin 3x+2\cos 2x+3\sin x+4=0$$

Proposed by Delia Popescu-Romania

J.2669 Prove that it doesn't exist $n \in \mathbb{N}$ such that the number $A = 28^n + 19^n + 10^n + 3^{n+1} + 5$ to be a perfect square.

Proposed by Veronica Popescu-Romania

J.2670 If $a, b, c, x, y, z > 0$, then: $(a^4 + x^2)(b^4 + y^2)(c^4 + z^2) \geq \frac{9}{8} t^4 (a + b + c)^4$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2671 If $m \geq 0$ and $a, b, c > 0$ then:

$$(a^{2m+2} + 2^{m+1})(b^{2m+2} + 2^{m+1})(c^{2m+2} + 2^{m+1}) \geq \frac{3^{m+1}}{8^m}(a + b + c)^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2672 If $m \geq 0$ and $a, b, c, x, y, z > 0$ then:

$$(a^{2m+2} + x^{m+1})(b^{2m+2} + y^{m+1})(c^{2m+2} + z^{m+1}) \geq \frac{3^{m+1}}{2^{5m+2}}\left(a\sqrt{yz} + b\sqrt{zx} + c\sqrt{xy}\right)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2673 If $m \geq 0$ and $x, y, z, t > 0$ then:

$$(x^{2m+2} + t^{2m+2})(y^{2m+2} + t^{2m+2})(z^{2m+2} + t^{2m+2}) \geq \frac{3^{m+1} t^{4m+4}}{2^{5m+2}}(x + y + z)^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2674 In any ΔABC the following inequality holds:

$$\left(\left(\frac{a^2 + bc}{b + c}\right)^2 + 2\right)\left(\left(\frac{b^2 + ca}{c + a}\right)^2 + 2\right)\left(\left(\frac{c^2 + ab}{a + b}\right)^2 + 2\right) \geq 36\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2675 In any ΔABC with the semiperimeter s the following inequality holds:

$$\left(\left(\frac{s + a}{b + c}\right)^2 + 2\right)\left(\left(\frac{s + b}{c + a}\right)^2 + 2\right)\left(\left(\frac{s + c}{a + b}\right)^2 + 2\right) \geq \frac{675}{16}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2676 If $t, u, v, w, x, y, z > 0$ then:

$$\prod_{cyc}(x^2 + t^2 v) + \prod_{cyc}(x^2 + u^2 w) \geq \frac{3}{4}(t^4 v^2 + u^4 w^2)(x + y + z)^2$$

(a generalization of the problem 3326 from CRUX MATHEMATICORUM)

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2678 If $u, v, x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x^2}{(uy+vz)^2}a^8+2\right)\left(\frac{y^2}{(uz+vx)^2}b^8+2\right)\left(\frac{z^2}{(ux+vy)^2}c^8+2\right) \geq \frac{768}{(u+v)^2}\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2679 If $x \in \mathbb{R}$, then in any ΔABC with the area F the following inequality holds:

$$((a^2\sin^2 x + bc\cos^2 x)^2+2)((b^2\sin^2 x + ca\cos^2 x)^2+2)\cdot((c^2\sin^2 x + ab\cos^2 x)^2+2\,) \geq 144F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2680 If $u, v > 0$ and $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{\tan x}{u\sin y + v\sin z} + \frac{\tan y}{u\sin z + v\sin x} + \frac{\tan z}{u\sin x + v\sin y} > \frac{3}{u+v}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2681 If $m \geq 0$, then in any ΔABC with the area F the following inequality holds:

$$(a^{2m+2}+2)(b^{2m+2}+2)(c^{2m+2}+2) \geq 4^{m+1}\left(\sqrt{3}\right)^{5-m}F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2682 If $m \geq 0$, then in any ΔABC with the area F the following inequality holds:

$$((a^2+bc)^{2m+2}+2)((b^2+ca)^{2m+2}+2)((c^2+ab)^{2m+2}) \geq \frac{64^{m+1}\cdot F^{2m+2}}{3^{m-2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2683 In any ΔABC with the area F the following inequality holds:

$$(m_a^4+2)\left(m_b^4+2\right)(m_c^4+2) \geq 81F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2684 If $a, b, c, d, m, t > 0$ then:

$$(a^2+mt^2)(b^2+mt^2)(c^2+mt^2)(d^2+mt^2) \geq \frac{9}{16}\cdot m^3t^6(a+b+c+d)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2685 If $m \geq 0, n, p, x, y, z > 0$ then:

$$\frac{x^{m+1}}{ny+pz} + \frac{y^{m+1}}{nz+px} + \frac{z^{m+1}}{nx+py} \geq \frac{3^{m+1}}{(n+p)^{2m+1}(x+y+z)^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2686 If $x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$(x^2a^4 + 2)(y^2b^4 + 2)(z^2c^4 + 2) \geq 12\left(\frac{xy}{\sin^2\frac{C}{2}} + \frac{yz}{\sin^2\frac{A}{2}} + \frac{zx}{\sin^2\frac{B}{2}}\right) \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2687 If $m \geq 0, x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\frac{(x^2a^4)^{m+1}}{y+z} + \frac{(y^2b)^{m+1}}{z+x} + \frac{(z^2c^4)^{m+1}}{x+y} \geq \frac{\left(\sqrt{3}\right)^{m+1}F^{2m+2}}{2^m(x+y+z)^{2m+2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2688 If $x, y, z > 0$ and $A_1B_1C_1, A_2B_2C_2$ are triangles with areas F_1, F_2 then:

$$\frac{x+y}{z}a_1a_2 + \frac{y+z}{x}b_1b_2 + \frac{z+x}{y}c_1c_2 \geq 8\sqrt{3} \cdot \sqrt{F_1} \cdot \sqrt{F_2}$$

(a generalization of Tsintsifas' inequality)

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2689 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$p_a + p_b + p_c \leq \frac{14R - r}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2690 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \leq \frac{4R + r}{3r}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2691 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$\frac{p_a}{w_a} + \frac{p_b}{w_b} + \frac{p_c}{w_c} \leq 3 + \frac{4\sqrt{2}}{3}\left(\frac{R}{2r} - 1\right)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2692 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$\frac{1}{p_a} + \frac{1}{p_b} + \frac{1}{p_c} \geq \frac{2}{R}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2693 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$\frac{p_a p_b p_c}{r_a r_b r_c} \leq \frac{8R - 7r}{9r}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2694 Let $a, b, c \geq 0: ab + bc + ca = 3$. Prove that:

$$\sqrt{a+1} + \sqrt{b+1} + \sqrt{c+1} \geq 3\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2695 If $x, y, z > 0$ then:

$$\sum \frac{x^2}{\left(x + \sqrt{(x+y)(x+z)}\right)^2} \leq \frac{\sum x^2 y^2}{(\sum xy)^2}$$

Proposed by Marin Chirciu – Romania

J.2696 Let $a, b, c \geq 0: a + b + c = 3$. Prove that:

$$\sqrt{a^2 + a} + \sqrt{b^2 + b} + \sqrt{c^2 + c} \leq \sqrt{\frac{3\sqrt[3]{abc} + 33}{2}}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2697 Let $\lambda \geq 0$ fixed. If $a, b, c > 0, abc = 1$ then:

$$\sum \frac{a^2}{(ab + \lambda)(\lambda ab + 1)} \geq \frac{3}{(\lambda + 1)^2}$$

Proposed by Marin Chirciu – Romania

J.2698 Let $a, b, c \geq 0: 2(ab + bc + ca) = a + b + c$. Prove that:

$$\frac{a(b+c-1)}{a+1} + \frac{b(c+a-1)}{b+1} + \frac{c(a+b-1)}{c+1} \geq 0$$

Proposed by Phan Ngoc Chau-Vietnam

J.2699 If $x_1 \leq x_2 \leq \cdots \leq x_n$ are positive numbers such that: $x_1 x_2 \dots x_n = 10^n$ then:

$$(1 + 10^{x_1})(1 + 20^{2x_2})(1 + 10^{3x_3}) \dots (1 + 10^{nx_n}) \geq \left(1 + \sqrt{10^{10(n+1)}}\right)^n$$

Proposed by Marin Chirciu – Romania

J.2700 Let $a_1, a_2, \dots, a_n > 0$ such that $\sqrt{\frac{a_1}{2}} + \sqrt{\frac{a_2}{2}} + \cdots + \sqrt{\frac{a_n}{2}} = \lambda$. Find the minimum of expression

$$\sqrt{\frac{a_1 + a_2}{2}} + \sqrt{\frac{a_2 + a_3}{2}} + \cdots + \sqrt{\frac{a_n + a_1}{2}}$$

Proposed by Marin Chirciu – Romania

J.2701 If $a, b, c > 0: ab + bc + ca = 3$ then:

$$\frac{1}{\sqrt[3]{a^2 + 10ab + b^2}} + \frac{1}{\sqrt[3]{b^2 + 10bc + c^2}} + \frac{1}{\sqrt[3]{c^2 + 10ca + a^2}} \geq \sqrt[3]{\frac{9}{4}}$$

Proposed by Nguyen Thai An-Vietnam

J.2702 Prove that in all triangles ABC holds the following inequalities:

a) $\sum \sqrt{\frac{h_a h_b}{(r-h_a)(r-h_b)}} \geq \frac{9}{2}$, b) $\sum \sqrt{\frac{r_a r_b}{(r-r_a)(r-r_b)}} \geq \frac{9}{2}$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.2703 $a, b, n, m > 0$ are such that $(na + m)(b + 1) = m + 1, m, n$ fixed.

Find minimum of $nab + \frac{1}{ab}$.

Proposed by Marin Chirciu – Romania

J.2704 If $x_1 \leq x_2 \leq \cdots \leq x_n$ are positive numbers such that $x_1 x_2 \dots x_n = a^n, a > 1$, then:

$$(1 + a^{x_1})(1 + a^{2x_2})(1 + a^{3x_3}) \dots (1 + a^{nx_n}) \geq \left(1 + \sqrt{a^{a(n+1)}}\right)^n$$

Proposed by Marin Chirciu – Romania

J.2705 In any ΔABC the following relationship holds:

$$\frac{w_a^3(m_b^3 + r_c^3)}{w_a^3 + m_b^3 + r_c^3} + \frac{w_b^3(m_c^3 + r_a^3)}{w_b^3 + m_c^3 + r_a^3} + \frac{w_c^3(m_a^3 + r_b^3)}{w_c^3 + m_a^3 + r_b^3} \leq \frac{27(9R^3 - 64r^3)^2}{32r^3}$$

Proposed by Zaza Mzhavanadze – Georgia

J.2706 Given positive real numbers such that $ab + bc + ca = abc$. Prove that:

$$2\sqrt{\frac{1}{a + bc} + \frac{1}{b + ac} + \frac{1}{c + ab}} \leq \frac{1}{4} + \frac{a}{b + ca} + \frac{b}{c + ab} + \frac{c}{a + bc}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2707 Given non-negative real numbers such that: $a + b + c = a^2 + b^2 + c^2 > 0$. Prove that:

$$a\sqrt{\frac{1 + b^2 + c^2}{b^2 + c^2}} + b\sqrt{\frac{1 + c^2 + a^2}{c^2 + a^2}} + c\sqrt{\frac{1 + a^2 + b^2}{a^2 + b^2}} \geq 2\sqrt{a + b + c}$$

Proposed by Nguyen Thai An, Thai Na Nhat Minh – Vietnam

J.2708 In $\Delta ABC, \omega$ – Brocard's angle holds: $\frac{1}{\sin \omega} \geq 2 + \frac{2(n_b + m_b - g_b - s_b)^2}{9R(h_a + h_b + h_c)}$

Proposed by Bogdan Fuștei – Romania

J.2709 In ΔABC the following relationship holds:

$$\frac{1}{2}\sum_{cyc}\frac{n_a}{h_a}+\sum_{cyc}\frac{r_a}{s+n_a}\geq\sum_{cyc}\frac{m_a}{a}$$

Proposed by Bogdan Fuștei – Romania

J.2710 Let $a,b,c\geq 0: a+b+c=3$. Prove that:

$$\sqrt{a\left(8a+8-\sqrt[3]{abc}\right)}+\sqrt{b\left(8b+8-\sqrt[3]{abc}\right)}+\sqrt{c\left(8c+8-\sqrt[3]{abc}\right)}\leq 3\sqrt{15}$$

Proposed by Phan Ngoc Chau-Vietnam

J.2711 Solve for real numbers: $\left\{1+x^2=4+y^2, 2+xy=\sqrt{(4+x^2-2x)(1+x^2)}\right\}$

Proposed by Carlos Paiva-Brazil

J.2712 If $a_1,a_2,\dots,a_n>0$ and $n\in\mathbb{N}, (n\geq 2, m\geq 0)$ then:

1. $(a_1)^2+(a_2)^2+\cdots+(a_n)^2\geq\frac{2}{n-1}\times S$, 2. $(a_1+a_2+\cdots+a_n)^2\geq\frac{2n}{n-1}\times S$

3. $(a_1)^m+(a_2)^m+\cdots+(a_n)^m\geq\left(\frac{2n\times S}{(n-1)\left(n^{2-\frac{2}{m}}\right)}\right)^{\frac{m}{2}}$

$$(S=a_1a_2+a_1a_3+\cdots+a_1a_n+a_2a_3+\cdots+a_2a_n+\cdots+a_{n-1}a_n)$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

J.2713 If $a_i\in\mathbb{R}, i\in\overline{1,n}$ and $k>1$ then:

$$\sum_{i=1}^{n}a_i^2\geq(k^2-1)\left(\sum_{i=1}^{n}\frac{a_i}{k^i}\right)^2$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

J.2714 If $a,b,c,d>0, a+b+c+d=1$ then:

$$\frac{1}{2+4a}+\frac{1}{4+16b}+\frac{1}{6+36c}+\frac{1}{12+144d}\geq\frac{1}{2}$$

Proposed by Marin Chirciu – Romania

J.2715 In all nonisosceles triangle holds:

$$\left(\sum\frac{\sin\frac{A-B}{2}\sin^2\frac{C}{2}}{\cos^2\frac{C}{2}}\right)\left(\sum\frac{\cos^2\frac{C}{2}}{\sin\frac{A-B}{2}\sin^2\frac{C}{2}}\right)=5-\frac{16R}{r}+\left(\frac{s}{r}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.2716 In ΔABC holds: $(w_a + w_b + w_c)^2 \leq 4p^2 + 5r^2 - 16Rr$

Proposed by Marin Chirciu – Romania

J.2717 In ΔABC holds:

$$r\left(5 - \frac{r}{R}\right) \leq \sum \frac{a}{b+c} m_a \leq \frac{R}{4r}(4R + r)$$

Proposed by Marin Chirciu – Romania

J.2718 In all nonisosceles triangle holds:

$$\left(\sum \frac{\sin\frac{C-B}{2}\tan\frac{A}{2}}{\sin\frac{C-B}{2} - \sin\frac{A}{2}}\right)\left(\sum \frac{\cos\frac{C-B}{2} - \sin\frac{A}{2}}{\sin\frac{C-B}{2}\tan\frac{A}{2}}\right) = 5 - \frac{16R}{r} + \left(\frac{s}{r}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.2719 In any ΔABC with the area F the following inequality holds:

$$\frac{b^2 + c^2}{(s-a)^2} + \frac{c^2 + a^2}{(s-b)^2} + \frac{a^2 + b^2}{(s-c)^2} \geq 24$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2720 In ΔABC with the area F and the semiperimeter s the following inequality holds:

$$\frac{a^{87} + b^{87}}{(s-c)^{87}} + \frac{b^{87} + c^{87}}{(s-a)^{87}} + \frac{c^{87} + a^{87}}{(s-b)^{87}} \geq 3 \cdot 2^{88}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2721 Let be $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{x^2 \cdot a^3}{(y+z)^2 \cdot h_a} + \frac{y^2 \cdot b^3}{(z+x)^2 \cdot h_b} + \frac{z^2 \cdot c^3}{(x+y)^2 \cdot h_c} \geq 2F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2722 If $t, u, x, y, z > 0$ then:

$$(x^2 + tu^2)(y^2 + tu^2)(z^2 + tu^2) \geq \frac{9}{4} t^2 u^4 (xy + yz + zx)$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți-Romania

J.2723 If $t, u, x, y, z > 0$ then:

$$(x^2 + tu^2)(y^2 + tu^2)(z^2 + tu^2) \geq \frac{9}{4} t^2 u^4 (xy + yz + zx)$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

J.2724 In any ΔABC the following inequality holds:

$$\left(\frac{a^4}{r_b^2\cdot r_c^2}+2\right)\cdot\left(\frac{b^4}{r_c^2\cdot r_a^2}+2\right)\cdot\left(\frac{c^4}{r_a^2\cdot r_b^2}+2\right)\geq 48$$

Proposed by D.M. Bătinețu – Giurgiu, Cătălin Pană-Romania

J.2725 In any ΔABC with the area F the following inequality holds:

$$\frac{a^{2023}\cdot b^{87}}{h_b^{1936}}+\frac{b^{2023}\cdot c^{87}}{h_c^{1936}}+\frac{c^{2023}\cdot a^{87}}{h_b^{1936}}\geq\frac{2^{2110}}{\left(\sqrt{3}\right)^{2021}}\cdot F^{87}$$

Proposed by D.M. Bătinețu – Giurgiu, Sorin Pîrlea-Romania

J.2726 If $a,b,c>0$, then:

$$\frac{1}{a^2+b^2+c^2}+\frac{1}{b(a+c)}+\frac{1}{c(b+a)}+\frac{1}{a(c+b)}+(a+b+c)^2\geq 8$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2728 If $x,y>0$, then in any ΔABC with the area F the following inequality holds:

$$(ax+by)^2+(bx+cy)^2+(cx+ay)^2\geq 4(x+y)^2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2729 In a rectangular parallelepiped having the dimensions a,b,c and the diagonal d the following inequality holds:

$$(a^2+b^2+c^2)\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)\geq 20$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2730 If $m,x,y,z>0$ then:

$$\left(\frac{mx^2}{(y+z)^2}+1\right)\cdot\left(\frac{my^2}{(z+x)^2}+1\right)\cdot\left(\frac{mz^2}{(x+y)^2}+1\right)\geq\frac{27m}{8}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2731 If $a,b,c>0$, then:

$$\left(\frac{a^2}{(b+c)^2}+1\right)\cdot\left(\frac{b^2}{(c+a)^2}+1\right)\cdot\left(\frac{c^2}{(a+b)^2}+1\right)\geq\frac{27}{8}$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

J.2732 Let a,b and c be the lengths of the hypotenuse and respectively the legs of right triangle ABC. Prove that:

$$(a^4+1)\left(\frac{1}{a^4}+1\right)\cdot(b^4+1)\left(\frac{1}{b^4}+1\right)\cdot(c^4+1)\cdot\left(\frac{1}{c^4}+1\right)\geq\frac{225}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

J.2733 If $x, y, z > 0$ then:

$$\left(\frac{x^4}{(y+z)^4}+1\right)\cdot\left(\frac{y^4}{(z+x)^4}+1\right)\cdot\left(\frac{z^4}{(x+y)^4}+1\right)\geq\frac{27}{64}$$

Proposed by D.M. Bătinețu – Giurgiu, Alecu Orlando-Romania

J.2734 If $n_a, n_b, n_c -$ Nagel's cevians in ΔABC then:

$$min\left\{\frac{n_a n_b}{h_a h_b},\frac{n_b n_c}{h_b h_c},\frac{n_c n_a}{h_c h_a}\right\}\leq\frac{R}{r}-1\leq max\left\{\frac{n_a n_b}{h_a h_b},\frac{n_b n_c}{h_b h_c},\frac{n_c n_a}{h_c h_a}\right\}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2735 If $n_a, n_b, n_c -$ Nagel's cevians in ΔABC then:

$$max\left\{\frac{n_a n_b}{h_a h_b},\frac{n_b n_c}{h_b h_c},\frac{n_c n_a}{h_c h_a}\right\}+min\left\{\frac{n_a}{h_a},\frac{n_b}{h_b},\frac{n_c}{h_c}\right\}\geq\frac{R}{r}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2736 If $n_a, n_b, n_c -$ Nagel's cevians in ΔABC then:

$$\frac{n_a n_b}{h_a h_b}+\frac{n_b n_c}{h_b h_c}+\frac{n_c n_a}{h_c h_a}\geq\frac{R}{r}+1$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2737 If $n_a, n_b, n_c -$ Nagel's cevians in ΔABC then:

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\geq\sqrt{\frac{s^2-12Rr}{r^2}+6}\geq\sqrt{\frac{4R}{r}+1}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2738 If $n_a, n_b, n_c -$ Nagel's cevians in ΔABC then:

$$\frac{n_a+n_b+n_c}{h_a+h_b+h_c}\geq\frac{3R}{R+4r}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

J.2739 In any triangle ABC the following inequality holds:

$$(r_a^2\cdot r_b^2+2)(r_b^2 r_c^2+2)(r_c^2 r_a^2+2)\left(\frac{1}{(r_a+r_b)^2}+\frac{1}{(r_b+r_c)^2}+\frac{1}{(r_c^2+r_a)^2}\right)\geq\frac{243}{16}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniela Barbu-Romania

J.2740 In triangle ABC the following inequality holds:

$$(r_a^2+2)\cdot(r_b^2+2)\cdot(r_c^2+2)\cdot\left(\frac{1}{(r_a+r_b)^2}+\frac{1}{(r_b+r_c)^2}+\frac{1}{(r_c+r_a)^2}\right)\geq\frac{81}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaela Stăncele-Romania

J.2741 If $a,b,c,x,y>0$, then:

$$\left(\frac{a^2}{(bx+cy)^2}+2\right)\cdot\left(\frac{b^2}{(cx+ay)^2}+2\right)\cdot\left(\frac{c^2}{(a+by)^2}+2\right)\geq\frac{27}{(x+y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2742 If $x,y,z=1$ then:

$$(x^2+2)(y^2+2)(z^2+2)\geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2743 If $m,n\geq 0,m+n,x,y,z>0$ and ABC is a triangle with the area F, then:

$$\left(\left(\frac{mx}{y+z}a^2+\frac{ny}{z+x}b^2\right)^2+2\right)\cdot\left(\left(\frac{my}{z+x}b^2+\frac{nz}{x+y}c^2\right)^2+2\right)\cdot$$

$$\cdot\left(\left(\frac{mz}{x+y}c^2+\frac{nx}{y+z}a^2\right)^2+2\right)\geq 36(m+n)^2\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2744 Let be $x,y,z>0$, then in triangle ABC with the area F the following inequality holds:

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}\geq 6\cdot F\cdot r$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2745 If $m,n\geq 0,x,y,m+n>0$ then in any triangle ABC with the area F the following inequality holds:

$$m(ab+x)(bc+x)(ca+z)ca+n(ab+y)(bc+y)(ca+y)\geq 9(mx^2+ny^2)\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Cătălina Stan-Romania

J.2746 In any triangle ABC with the area F the following inequality holds:

$$(a^2+2b)(b^2+2c)(c^2+2a)\geq 144F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Ionuț Ivănescu-Romania

J.2747 Let be $A_1A_2\ldots A_{12}$ a convex polygon having the sides with the lengths $a_k=A_kA_{k+1}$,

$k=\overline{1,12},A_{13}=A_1$ and the area F, then:

$$((a_1^2+a_2^2+a_3^2)^2+2+2)\left(\left(a_4^2+a_5^2+a_6^2+a_7^2\right)^2+2\right)\cdot$$

$$\cdot\left(\left(a_8^2+a_9^2+a_{10}^2+a_{11}^2+a_{12}^2\right)^2+2\right)\geq 48F^2\cdot\tan^2\frac{\pi}{12}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2748 Let be $x, y > 0$ and ABC a triangle with the area F, then:

$$(xr_a+yr_b)^2+(xr_b+yr_c)^2+(xr_c+yr_a)^2\geq 3(x+y)^2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2749 Let be $x, y, z > 0$ and triangle ABC, then:

$$\frac{x^2a^3}{(y+z)^2}+\frac{y^2b^3}{(z+x)^2}+\frac{z^2c^3}{(x+y)^2}\geq 18\sqrt{3}r^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

J.2750 Let $A_1B_1C_1, A_2B_2C_2$ two triangles with the area F_1 respectively F_2, then:

$$(a_1^2+b_2^2+c_2^2)(b_1^2+c_2^2+a_2^2)(c_1^2+a_2^2+b_2^2)\geq 192\sqrt{3}F_1F_2^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.2751 Find all functions $f, g: \mathbb{R}\to\mathbb{R}$: $yf(x)-xf(y)=g(x^2-y^2), \forall x, y\in\mathbb{R}$

Proposed by Jalil Hajimir-Canada

J.2752 If $f(x+[x])=\frac{[x]}{x}$, find $f(x)$. $[*]$ is the greatest integer part of $*$

Proposed by Jalil Hajimir-Canada

J.2753 Solve for real numbers: $\begin{cases}[2x]=2[y]\\ [2y]=2[x]\end{cases}$, $[*]$ – GIF

Proposed by Jalil Hajimir-Canada

J.2754 Minimize: $f(x,y,z)=\frac{1}{2x^2+3}+\frac{1}{2y^2+3}+\frac{1}{2z^2+3}$ Subject to: $x+y+z=6,\ x,y,z>1$

Proposed by Jalil Hajimir-Canada

J.2755 $A_1A_2\dots A_p$ – convex polygon, $p\in\mathbb{N}, p\geq 3, a_1, a_2, \dots, a_p$ – sides,

$a_1+a_2+\dots+a_p=2s, m, n, k>0, m+n=k^4$. Prove that:

$$\left(\sum_{cyc}\sqrt[4]{na_1^4+ma_2^4}\right)\left(\sum_{cyc}\sqrt[4]{n\hat{A}_1^4+m\hat{A}_2^4}\right)\geq 2k^2s(p-2)\pi$$

Proposed by Radu Diaconu – Romania

J.2756 Find $x, y, z \in \mathbb{Z}$ such that: $3^x + 3^{2y} = z^2$

Proposed by Kerimov Elsen-Azerbaijan

J.2757 Find $x, y, z \in \mathbb{Z}$ such that:

$$x^4 + 2xy^3 + 2xz^3 + 2y^3z + 2yz^3 + 3 = y^4 + z^4 + x^2y^2 + 2x^2yz + x^2z^2 + 3y^2z^2$$

Proposed by Kerimov Elsen, Rustemov Kenan-Azerbaijan

J.2758 Solve in $\mathbb{N}$: $x^5 + y^5 - 2z^5 = 10(3x + 3y + 5z + 5t + 8t^3)$

Proposed by Toubal Fethi-Algeria

J.2759 (r_a, r_b, r_c) – triliniar coordinates of $P \in Int(\Delta ABC)$, $R_a = PA, R_b = PB, R_c = PC$. Prove that:

$$\sum_{cyc} R_a \geq \sum_{cyc} r_a\left(\frac{b}{c} + \frac{c}{b}\right) \geq 2\sum_{cyc} \frac{r_a(n_a g_a + rr_a)}{bc} \geq 2\sum_{cyc} \frac{r_a(m_a w_a + rr_a)}{bc} \geq 2\sum_{cyc} r_a$$

Proposed by Bogdan Fuștei – Romania

J.2760 If $x, y > 0$ and ABC is a triangle with the area F, then:

$$\frac{a^3}{bx + cy} + \frac{b^3}{cx + ay} + \frac{c^3}{ax + by} \geq \frac{4\sqrt{3}}{x + y} F$$

Proposed by D.M. Bătinețu – Giurgiu,Alina Tigae – Romania

J.2761 If $m, x, y, z, t, u > 0$ then in ΔABC with the area F the following inequality holds:

$$\left(\frac{mx^2 \cdot a^8}{(ty + uz)^2} + 1\right) \cdot \left(\frac{my^2 \cdot b^8}{(tz + ux)^2} + 1\right) \cdot \left(\frac{mz^2 \cdot c^8}{(tx + uy)^2} + 1\right) \geq \frac{192}{(t + u)^2} \cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu,Camelia Dană – Romania

J.2762 If $x, y, z > 0$ and $x^2 + y^2 = z^2$, then: $(x^2 + y^2 + z^2) \cdot \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) \geq 10$

Proposed by D.M. Bătinețu – Giurgiu,Carmen Vlad– Romania

J.2763 If $m \geq 0, x, y, z > 0$ and $x^2 + y^2 = z^2$ then:

$$(x^{2m+2} + y^{2m+2} + z^{2m+2}) \cdot \left(\frac{1}{x^{2m+2}} + \frac{1}{y^{2m+2}} + \frac{1}{z^{2m+2}}\right) \geq 10^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu,Daniela Dîrnu– Romania

J.2764 In triangle ABC with the semiperimeter s, we have:

$$\frac{b^{2023} + c^{2023}}{(s - a)^{2023}} + \frac{c^{2023} + a^{2023}}{(s - b)^{2023}} + \frac{a^{2023} + b^{2023}}{(s - c)^{2023}} \geq 3 \cdot 2^{2024}$$

Proposed by D.M. Bătinețu – Giurgiu,Mihaela Mirea– Romania

J.2765 If $m \geq 0$ and $t, x, y, z > 0$ then:

$$\left(\left(\frac{x}{y+z}\right)^{2m+2} + t^{2m+2}\right) \cdot \left(\left(\frac{y}{x+x}\right)^{2m+2} + t^{2m+2}\right) \cdot \left(\left(\frac{z}{x+y}\right)^{2m+2} + t^{2m+2}\right) \geq \frac{27^{m+1}}{2^{7m+4}} \cdot t^{4m+4}$$

Proposed by D.M. Bătinețu – Giurgiu,Sebastian Ilinca– Romania

J.2766 Let be $ABC, A_1B_1C_1$ two triangles with the area F, respectively F_1, then:

$$(a^2 + 2b_1^2)(b^2 + 2c_1^2)(c^2 + 2a_1^2) \geq 192\sqrt{3}F_1F_2^2$$

Proposed by D.M. Bătinețu – Giurgiu,Cristina Ene – Romania

J.2767 If $t, x, y, z > 0$, then:

$$\left(\frac{x^2}{(y+z)^2} + t\right)\left(\frac{y^2}{(z+x)^2} + t\right)\left(\frac{z^2}{(x+y)^2} + t\right) \geq \frac{27}{16} \cdot t^2$$

Proposed by D.M. Bătinețu – Giurgiu,Oana Simona Dascălu– Romania

J.2768 Let be $n \in \mathbb{N}, n \geq 2$ and $a_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}$, then:

$$\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} \frac{1}{a_k^3} \geq 3n$$

Proposed by D.M. Bătinețu – Giurgiu,Ramona Nălbaru– Romania

J.2769 Let be $m \geq 0, x, y, z > 0$ and ABC a triangle with the area F, then:

$$\frac{x^{m+1} \cdot a^{m+2}}{(y+z)^{m+1}} + \frac{y^{m+1} \cdot b^{m+2}}{(z+x)^{m+1}} + \frac{z^{m+1} \cdot c^{m+2}}{(x+y)^{m+1}} \geq 2\left(\sqrt{3}\right)^{m+1} F \cdot r^m$$

Proposed by D.M. Bătinețu – Giurgiu,Ileana Stanciu– Romania

J.2770 In any triangle ABC with the area F the following inequality holds:

$$(a^2 + r)(b^2 + r)(c^2 + r) \geq 3F^2$$

Proposed by D.M. Bătinețu – Giurgiu,Mihaela Nascu– Romania

J.2771 If $m \geq 0$ and ABC is a triangle with the area F, then:

$$a^{m+4} + b^{m+4} + c^{m+4} + \frac{1}{a^m} + \frac{1}{b^m} + \frac{1}{c^m} \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu,Elena Grigore– Romania

J.2772 In any ΔABC with the area F the following inequality holds:

$$((a^2 + b^2)^2 + 2) \cdot ((b^2 + c^2)^2 + 2) \cdot ((c^2 + a^2)^2 + 2) \geq 576 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu,Simona Radu– Romania

J.2773 Let be $a,b,c,d>0$ then: $\left((a+b)(c+d)+cd\right)\cdot\left(\frac{1}{(a+b+c)^2}+\frac{1}{(c+d)^2}+\frac{1}{(a+b+d)^2}\right)\geq\frac{9}{4}$

Proposed by D.M. Bătinețu – Giurgiu,Iulia Sanda – Romania

J.2774 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$((ax+cy)^2b^2+2)((bx+ay)^2c^2+2)((cx+by)^2a^2+2)\geq 144(x+y)^2F^2$$

Proposed by D.M. Bătinețu – Giurgiu,Sorina Tudor– Romania

J.2775 If $m\geq 0$ and $x,y>0$ then in triangle ABC with the area F the following inequality holds:

$$\left(\frac{a^{2m+4}}{(bx+cy)^{2m}}+2\right)\left(\frac{b^{2m+4}}{(cx+ay)^{2m}}+2\right)\left(\frac{c^{2m+4}}{(ax+by)^{2m}}+2\right)\geq\frac{144}{(x+y)^{2m}}F^2$$

Proposed by D.M. Bătinețu – Giurgiu,Nicolae Radu– Romania

J.2776 Let be $x,y>0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{m_a^3}{xRm_b+yrm_c}+\frac{m_b^3}{xRm_c+yrm_a}+\frac{m_c^3}{xRm_a+yrm_b}\geq\frac{3\sqrt{3}}{xr+yr}F$$

Proposed by D.M. Bătinețu – Giurgiu,Cristian Catană– Romania

J.2777 If $t\geq 0$ and $x,y>0$, then in triangle ABC with the area F, the following inequality holds:

$$\frac{m_a^{t+2}}{(xRm_b+yrm_c)^t}+\frac{m^{t+2}}{(xRm_c+yrm_a)^t}+\frac{m_c^{t+2}}{(xRm_a+yrm_b)^t}\geq\frac{3\sqrt{3}}{(xR+yr)^t}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu,Dan Mitricoiu– Romania

J.2778 If $m\geq 0,x,y>0$ and ABC is a triangle with the area F, then:

$$(x^2+y^2)(a^{2m+2}+b^{2m+2}+c^{2m+2})\geq 2^{2m+3}xy\left(\sqrt{3}\right)^{1-m}F^{m+1}+$$

$$+(xa^{m+1}-yb^{m+1})^2+(xb^{m+1}-yc^{m+1})^2+(xc^{m+1}-ya^{m+1})$$

Proposed by D.M. Bătinețu – Giurgiu,Claudiu Ciulcu– Romania

J.2779 If $x,y,z>0$, then:

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+2\right)\left(\frac{1}{(y+z)^4}+2\right)\left(\frac{1}{(z+x)^4}+2\right)\geq\frac{721}{16}$$

Proposed by D.M. Bătinețu – Giurgiu,Monica Velea– Romania

J.2780 In any triangle ABC the following inequality holds:

$$\left(\frac{s_a^4}{(w_a+m_a)^2}+2\right)\cdot\left(\frac{s_b^4}{(w_b+m_b)^2}+2\right)\cdot\left(\frac{s_c^4}{(w_c+m_b)^2}+2\right)\geq\frac{3r^4}{R^2}$$

Proposed by D.M. Bătinețu – Giurgiu,Nicolae Mușuroia– Romania

J.2781 If $t,x,y,z>0$ then:

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\cdot\left(\frac{1}{(x+y)^4}+t^2\right)\left(\frac{1}{(y+z)^4}+t^2\right)\left(\frac{1}{(z+x)^4}+t^2\right)\geq\frac{729}{64}t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Dorina Goiceanu– Romania

J.2782 In triangle ABC with the area F the following inequality holds:

$$(r_a^4+2)(r_b^4+2)(r_c^4+2)\geq 81F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Ileana Duma– Romania

J.2783 If $m\geq 0$ and $a,b,c>0$, then:

$$(a^{2m+2}(a+b)^{2m+2}+2)((ab)^{2m+2}+2)(b^{4m+4}+2)\geq\frac{1}{3^{2m-1}}(a+b)^{4m+4}$$

Proposed by D.M. Bătinețu – Giurgiu, Roxana Vasile– Romania

J.2784 If $u,v,x,y,z>0$ and $u+v=8$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{xa^u}{y+z}+\frac{yb^v}{z+x}+\frac{zc^u}{x+y}\right)\cdot\left(\frac{xa^v}{y+z}+\frac{yb^u}{z+x}+\frac{zc^v}{x+y}\right)\geq 64F^4$$

Proposed by D.M. Bătinețu – Giurgiu,Meda Iacob– Romania

J.2785 If s is the semiperimeter of ΔABC, then:

$$\frac{b^{2023}+c^{2023}}{(b+c)^{87}\cdot(s-a)^{1936}}+\frac{c^{2023}+a^{2023}}{(c+a)^{87}\cdot(s-b)^{1936}}+\frac{a^{2023}+b^{2023}}{(a+b)^{87}\cdot(s-c)^{1936}}\geq 3\cdot 2^{1850}$$

Proposed by D.M. Bătinețu – Giurgiu,Carmen Terheci– Romania

J.2786 If $x,y>0$ then in ΔABC with the area F the following inequality holds:

$$\frac{a^4}{xb^2+ym_c^2}+\frac{b^4}{xc^2+ym_a^2}+\frac{c^4}{xa^2+ym_b^2}\geq\frac{16\sqrt{3}}{4x+3y}F$$

Proposed by D.M. Bătinețu – Giurgiu,Corina Ionescu– Romania

J.2787 If $x,y>0$ and $x+y=2$ then in ΔABC with the area F the following inequality holds:

$$(a^x+b^y+c^x)(a^y+b^x+c^y)\geq 12\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu,Maria Lavinia Popa– Romania

J.2788 Let $m\geq 0$ and a,b,c be the dimensions of a rectangular parallelepiped with diagonal d, then:

$$(a^{2m+2}+b^{2m+2}+c^{2m+2}+d^{2m+2})\cdot\left(\frac{1}{a^{2m+2}}+\frac{1}{b^{2m+2}}+\frac{1}{c^{2m+2}}+\frac{1}{d^{2m+2}}\right)\geq\frac{5^{m+1}}{2^{3m-1}}$$

Proposed by D.M. Bătinețu – Giurgiu,Mihaela Duță– Romania

J.2789 If $m, n \geq 0$ and ABC is a triangle with the semiperimeter s, then:

$$\frac{(a^{m+1}+b^{m+1})(a^{n+1}+b^{n+1})}{(s-c)^{m+n+2}}+\frac{(b^{m+1}+c^{m+1})(b^{n+1}+c^{n+1})}{(s-a)^{m+n+2}}+\frac{(c^{m+1}+a^{m+1})(c^{n+1}+a^{n+1})}{(s-b)^{m+n+2}} \geq 3\cdot 2^{m+n+4}$$

Proposed by D.M. Bătinețu – Giurgiu, Cătălin Nicola– Romania

J.2790. Let be $m \geq 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+1}}{(r_b r_c)^{m+1}\cdot h_a^{m+1}}+\frac{b^{m+1}}{(r_c r_a)^{m+1} h_b^{m+1}}+\frac{c^{m+1}}{(r_a r_c)^{m+1} h_c^{n+1}} \geq \frac{2^{m+1}}{3^m\cdot F^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Laura Zaharia– Romania

J.2791 Let be $t, u, x, y \geq 0, t+u, x+y > 0$ then in any ΔABC the following inequality holds:

$$\frac{(ta+ub)^2}{r_b(xr_c+yr_a)}+\frac{(tb+uc)^2}{r_c(xr_a+yr_b)}+\frac{(tc+r_a)^2}{r_a(xr_b+yr_c)} \geq \frac{4(t+u)^2}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Gheorghe Boroica– Romania

J.2792 If $m \geq 0$ and ABC is a triangle with the area F, then:

$$\frac{a^{3m+3}}{(b+c)^{m+1}}+\frac{b^{3m+3}}{(c+a)^{m+1}}+\frac{c^{3m+3}}{(a+b)^{m+1}} \geq 2^{m+1}\left(\sqrt{3}\right)^{1-m} F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2793 If $m \geq 0$ and ABC is a triangle with the semiperimeter s and the area F, then:

$$\frac{a^{m+1}+b^{m+1}}{(s-c)^{m+1}}+\frac{b^{m+1}+c^{m+1}}{(s-a)^{m+1}}+\frac{c^{m+1}+a^{m+1}}{(s-b)^{m+1}} \geq 3\cdot 2^{m+2}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2794 If $x, y > 0$, then in ΔABC with the area F the following inequality holds:

$$(xr_a+yr_b)^2+(xr_b+yr_c)^2+(xr_c+yr_a)^2 \geq 12\sqrt{3}xy\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2795 If $m \geq 0$ then in ΔABC with the area F the following inequality holds:

$$\frac{a^{m+1}\cdot b}{h_b^m}+\frac{b^{m+1}\cdot c}{h_c^m}+\frac{c^{m+1}\cdot a}{h_c^m} \geq 2^{m+2}\left(\sqrt{3}\right)^{1-m}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2796 If $t, u, x, y, z > 0$ then:

$$u^4(x^2+t^2)(y^2+t^2)(z^2+t^2)+t^4(x^2+u^2)(y^2+u^2)(z^2+u^2) \geq \frac{3}{4}t^4u^4(x+y+z)^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2797 Let be $m \geq 0, x, y > 0$ then in triangle ABC with the area F the following inequality holds:

$$(xr_a + yr_b)^{2m+2}(xr_b + yr_c)^{2m+2} + (xr_c + yr_a)^{2m+2} \geq \left(\sqrt{3}\right)^{m+3} \cdot (x+y)^{2m+2} \cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2798 If $m \geq 0, x, y > 0$ then in triangle ABC with the area F the following inequality holds:

$$(ax + by)^{2m+2} + (bx + cy)^{2m+2} + (cx + ay)^{2m+2} \geq 4^{m+1} \cdot (x+y)^{2m+2} \cdot \left(\sqrt{3}\right)^{1-m} \cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2799 Let be $x, y > 0$ and ABC a triangle having the area F, then:

$$((xa^2 + ybc)^2 + 2) \cdot ((xb^2 + yca)^2 + 2) \cdot ((xc^2 + yab)^2 + 2) \geq 144(x+y)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2800 Let be $x, y > 0$ and ABC a triangle with the area F, then:

$$((xab + yr_br_c)^2 + 2)((xbc + yr_cr_a)^2 + 2)((xca + yr_ar_b)^2 + 2) \geq 9(4x + 3y)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.2801 If $x, y > 0$ then in triangle ABC:

$$\left(\frac{ax + by}{h_c}\right)^2 + \left(\frac{bx + cy}{h_a}\right)^2 + \left(\frac{cx + ay}{h_b}\right)^2 \geq 4(x+y)^2$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2802 In triangle ABC with the area F the following inequality holds:

$$\frac{a^4}{b^2 + m_c^2} + \frac{b^4}{c^2 + m_a^2} + \frac{c^4}{a^2 + m_b^2} \geq \frac{16}{7}\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2803 In any triangle ABC the following inequality holds:

$$\frac{a^3}{(b^2 + h_c^2)h_a} + \frac{b^3}{(c^2 + h_a^2)h_b} + \frac{c^3}{(a^2 + h_b^2)h_c} \geq \frac{8\sqrt{3}}{7}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2804 In any triangle ABC with the area F the following inequality holds:

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

J.2805 If $a, b, c, d > 0$ and $(a+b+c)^4 + (b+c+d)^4 + (c+d+a)^4 + (d+a+b)^4 = 324$

then: $(a+b+c)(a^3+b^3+c^3)+(b+c+d)(b^3+c^3+d^3)+(c+d+a)(c^3+d^3+a^3)+$

$$+(d+a+b)(d^3+a^3+b^3)\geq 36$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2643 If $m\geq 0, a, t, x, y, z>0$ and $x+y+z\geq a$, then:

$$(x^{2m+2}+t^{2m+2})(y^{2m+2}+t^{2m+2})(z^{2m+2}+t^{2m+2})\geq \frac{3^{m+1}t^{4m+4}a^{2m+2}}{2^{5m+2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2644 Let be $a, b, c, x, y\in\mathbb{R}_+^*$ and $m\geq 0$, then:

$$\left(\frac{a}{xb+yc}\right)^{m+1}+\left(\frac{b}{xc+ya}\right)^{m+1}+\left(\frac{c}{xa+yb}\right)^{m+1}\geq\frac{3}{(x+y)^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2645 If $m\geq 0$ and $a, b, c>0$, then:

$$(a^{2m+2}+1)(b^{2m+2}+1)(c^{2m+2}+1)\geq\frac{3^{m+1}}{4^{3m+1}}(a+b+c)^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Elena Nedelcu– Romania

S.2646 In any ΔABC the following inequality holds:

$$\left(\frac{a^2}{h_a^2}+2\right)\left(\frac{b^2}{h_b^2}+2\right)\left(\frac{c^2}{h_c^2}+2\right)\geq 9$$

Proposed by D.M. Bătinețu – Giurgiu, Daniela Iancu– Romania

S.2647 If $t \geq 0$ and $x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{(x+t)^2a^4}{(y+z+2t)^2}+2\right)\left(\frac{(y+t)^2b^4}{(z+x+2t)^2}+2\right)\left(\frac{(z+t)^2c^4}{(x+y+2t)^2}+2\right) \geq 36F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Carmen Vlad– Romania

S.2648 In any ΔABC the following inequality holds:

$$\left(\frac{a^2}{(b+c)^4}+2\right)\left(\frac{b^2}{(c+a)^4}+2\right)\left(\frac{c^2}{(a+b)^4}+2\right) \geq \frac{9}{16R^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Simona Chiriță– Romania

S.2649 If $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x^2a^3}{(y+z)^2h_a}+2\right)\left(\frac{y^2b^3}{(z+x)^2h_b}+2\right)\left(\frac{z^2c^3}{(x+y)^2h_c}+2\right) \geq 18F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2650 In any ΔABC with the area F the following inequality holds:

$$((a^2+bc)^2+2)((b^2+ca)^2+2)((c^2+ab)^2+2) \geq 576F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2651 In any ΔABC with the area F the following inequality holds:

$$\left(r_a^2r_b^2+r^2\right)\left(r_b^2r_c^2+r^2\right)\left(r_c^2r_a^2+r^2\right) \geq \frac{3}{4}F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2652 In any ΔABC with the area F the following inequality holds:

$$\left(\frac{1}{h_a^2}+2\right)\left(\frac{1}{h_b^2}+2\right)\left(\frac{1}{h_c^2}+2\right) \geq \frac{3\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Gabriela Militaru– Romania

S.2653 If $x \in \mathbb{R}$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{a^6}{(a\sin^2 x+b\cos^2 x)^2}+2\right)\left(\frac{b^6}{(b\sin^2 x+c\cos^2 x)^2}+2\right)\left(\frac{c^6}{(c\sin^2 x+a\cos^2 x)^2}+2\right) \geq 144F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihnea Alexie– Romania

S.2654 If $t, u > 0$, then in any ΔABC with the semiperimeter s the following inequality holds:

$$\left(\left(\frac{ts+ua}{b+c}\right)^2+2\right)\left(\left(\frac{ts+ub}{c+a}\right)^2+2\right)\left(\left(\frac{ts+uc}{a+b}\right)^2+2\right) \geq \frac{27}{16}(3t+2u)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Horia Mușat– Romania

S.2655 If $x, y > 0$, then in any ΔABC the following inequality holds:

$$\left(\frac{a^2}{(bx+cy)^4}+2\right)\left(\frac{b^2}{(cx+ay)^4}+2\right)\left(\frac{c^2}{(ax+by)^4}+2\right) \geq \frac{9}{(x+y)^4R^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Vlad Oroviceanu– Romania

S.2656 If $x, y, z, t, u > 0$, then: $(x^2+ut^2)(y^2+ut^2)(z^2+ut^2) \geq \frac{3}{4}t^4u^2(x+y+z)^2$

(a generalization of Arkady M. Alt inequality)

Proposed by D.M. Bătinețu – Giurgiu, Alexandra Mitroi– Romania

S.2657 If ABC is a triangle, then prove that:

$$\sum \frac{2}{\sqrt{3}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)} \leq 1$$

Proposed by Neculai Stanciu – Romania

S.2658 If $a, b, c > 0$ and $abc \geq 1, n \in \mathbb{N}^*$ then:

$$\sum \frac{a^{12n}}{abc}+\frac{3}{a^2b^2c^2} \geq \frac{a^{6n}+b^{6n}+c^{6n}+9}{2}$$

Proposed by Marin Chirciu – Romania

S.2659 In ΔABC holds:

$$\frac{2}{r} \leq \sum \frac{r_b+r_c}{h_a^2} \leq \frac{R}{r^2}$$

Proposed by Marin Chirciu – Romania

S.2660 Compare:

a. 2008^{2023} with 2007^{2024}

b. $\sin(\cos\sqrt{x})$ with $\cos(\sin\sqrt{x})$ where $x \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$.

Proposed by Nguyen Van Canh – Vietnam

S.2661 Let $\lambda > 0$ fixed. If $a, b, c > 0$ then find min of

$$P = \sum \left(\frac{b+c}{\lambda a}+1\right)^3$$

Proposed by Marin Chirciu – Romania

S.2662 In ΔABC holds:

$$\frac{27r}{2Rp} \leq \sum \frac{\sin^3 B+\sin^3 C}{r_a} \leq \frac{27R^2}{16r^2p}$$

Proposed by Marin Chirciu – Romania

S.2663 In any ΔABC and $\forall n,m \in \mathbb{N}: n \geq m+1$; the following relationship holds:

$$\frac{h_a^n}{(r_a^3+h_a^3)^m}+\frac{w_b^n}{(r_b^3+w_b^3)^m}+\frac{m_c^n}{(r_c^3+m_c^3)^m} \geq \frac{2^{2m}\cdot 3^{n-3m+1}\cdot r^n}{(9R^3-64r^3)^m}$$

Proposed by Zaza Mzhavanadadze – Georgia

S.2664 In ΔABC:

$$\sum \frac{\sin^3 B+\sin^3 C}{h_a} \leq \sum \frac{\sin^3 B+\sin^3 C}{r_a}$$

Proposed by Marin Chirciu – Romania

S.2665 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sqrt{\frac{2R}{r}}\cdot \sum_{cyc} \frac{n_a}{\sqrt{(b-c)^2+4R^2}} \geq \frac{1}{\sqrt{2}}\cdot \sum_{cyc} \sqrt{\frac{m_b}{h_c}+\frac{m_c}{h_b}}+\frac{1}{r}\sum_{cyc} AI$$

Proposed by Bogdan Fuștei – Romania

S.2666 If $n \geq 1$ then:

$$(n+1)^2+\frac{1}{12} \geq 3\left(\frac{1}{6}+(n!)^{\frac{1}{n}}\right)^2$$

Proposed by Khaled Abd Imouti-Syria

S.2667 In ΔABC the following relationship holds:

$$\prod_{cyc}\left(1+\frac{\sin A}{\sin B}\right)^{\sin B} < e^{\frac{3\sqrt{3}}{2}}$$

Proposed by Khaled Abd Imouti-Syria

S.2668 Let $a,b,c>0$ such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that:

$$a^{2023}+b^{2023}+c^{2023}+2024(abc)^{2024} \geq 3(abc)^{2022}+2024(abc)^{2023}$$

Proposed by Nguyen Van Canh – Vietnam

S.2669 If $x,y,z>0, x+y+z=1$ and $n \in \mathbb{N}, \lambda \geq 0$ then

$$\sum \frac{x^n}{\lambda+\sqrt{yz}} \geq \frac{1}{(3\lambda+1)\cdot 3^{n-2}}$$

Proposed by Marin Chirciu – Romania

S.2670 Let $f(x)=ax^4+bx^3+cx^2+dx+e, (a,b,c,d,e \in \mathbb{R})$. Find all values of $a,b,c,d,e \in \mathbb{R}$ such that: $f(1)=f(2)=f(3)=f(4)=f(5)=f'(6)$

Proposed by Nguyen Van Canh – Vietnam

S.2671 In ΔABC holds:

$$\frac{12}{R}\le\sum\left(\frac{b}{c}+\frac{c}{b}\right)(\cos B+\cos C)\le\frac{3R}{r}$$

Proposed by Marin Chirciu – Romania

S.2672 Find all values of $m\in\mathbb{R}$ such that: $f(x)=x^3+mx^2+(1-m)x+1\le e^{\pi},\forall x\in[e,\pi]$

Proposed by Nguyen Van Canh – Vietnam

S.2673 In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{r_a\left(\frac{2(bm_c+cm_b-2F)}{a}-r_a\right)}\le\sum_{cyc}n_a$$

Proposed by Bogdan Fuștei – Romania

S.2674 Let $a,b,c>0: a^2+b^2+c^2=a+b+c$. Prove that:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+3\ge\frac{\sqrt{5a^2+4bc}-a}{\sqrt{bc}}+\frac{\sqrt{5b^2+4ca}-b}{\sqrt{ca}}+\frac{\sqrt{5c^2+4ab}-c}{\sqrt{ab}}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2675 Let n_a,n_b,n_c be Nagel's cevians of ΔABC with the area F and $m\ge 0$ then:

$$\frac{(a^{2m+2}+b^{2m+2})n_c^m}{(a+b)^m}+\frac{(b^{2m+2}+c^{2m+2})n_a^m}{(b+c)^m}+\frac{(c^{2m+2}+a^{2m+2})n_b^m}{(c+a)^m}\ge 8\sqrt{3}F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Adrian Barbu– Romania

S.2676 Let be $n\in\mathbb{N},n\ge 2$ and $x_k\in\mathbb{R}_+^*=(0,\infty),\forall k=\overline{1,n}$ and $a,b>0$, then:

$$a^2\left(\frac{1}{x_1^2}+\frac{1}{x_2^2}+\cdots+\frac{1}{x_n^2}\right)b^2n^3\ge 2abn\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Bianca Negreț– Romania

S.2677 Let be $x,y>0$ then in ΔABC with the area F the following inequality holds:

$$\frac{(a^4+b^4)h_c}{xa+yb}+\frac{(b^4+c^4)h_a}{xb+yc}+\frac{(c^4+a^4)h_b}{xc+ya}\ge\frac{16\sqrt{3}}{x+y}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Ana Jipescu– Romania

S.2678 Let be $t\ge 0,n\in\mathbb{N},n\ge 2$ and $a,b,x_k\in\mathbb{R}_+^*=(0,\infty),\forall k=\overline{1,n}$, then:

$$a^2\left(\frac{1}{x_k^{2t+2}}+\frac{1}{x_k^{2t+2}}+\cdots+\frac{1}{x_k^{2t+2}}\right)+b^2n\ge\frac{2ab}{n^t}\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)^{t+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2679 In any ΔABC with the area F the following inequality holds:

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a) \ge 48F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2680 If $t,u,x,y,z>0$ then in ΔABC with the area F the following inequality holds:

$$\frac{(x^2a^4+y^2b^4+z^2c^4)h_c}{ta+ub}+\frac{(x^2b^4+y^2c^4+z^2a^4)h_a}{tb+uc}+\frac{(x^2c^4+y^2a^4+z^2b^4)h_b}{tc+ua} \ge \frac{8\sqrt{3}(x+y+z)^2}{3(t+u)}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2681 If $a,b,c>0$, then:

$$\frac{a}{1936b+86c}+\frac{b}{1936c+86a}+\frac{c}{1936a+86b} \ge \frac{1}{674}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2682 If $m\ge 0, t,u,x,y>0$ and $a,b,c>0$ then:

$$\frac{t^{m+1}a^{2m+2}+u^{m+1}b^{2m+2}}{c^m(ax+by)^m}+\frac{t^{m+1}b^{2m+2}+u^{m+1}c^{2m+2}}{a^m(bx+cy)^m}+\frac{t^{m+1}c^{2m+2}+u^{m+1}a^{2m+2}}{b^m(cx+ay)^m} \ge \frac{(t+u)^{m+1}\sqrt{3}}{2^{m-2}(x+y)^m}F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaela Dăianu– Romania

S.2683 If $x,y>0$ then in any ΔABC with the area F the following inequality holds:

$$\frac{(x^2a^4+y^2b^4)h_c}{a+b}+\frac{(x^2b^4+y^2c^4)h_a}{b+c}+\frac{(x^2c^4+y^2a^4)h_b}{c+a} \ge 2(x+y)^2F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Anca Dumitru– Romania

S.2684 Let be $x,y>0$, then in any ΔABC with semiperimeter s the following inequality holds:

$$\frac{\sin^2 A}{x\sin B+y\sin C}+\frac{\sin^2 B}{x\sin C+y\sin A}+\frac{\sin^2 C}{x\sin A+y\sin B} \ge \frac{s}{(x+y)R}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Catană– Romania

S.2685 In a scalene triangle ΔABC, prove that:

$$\frac{b+c}{(r_a-r_b)(r_a-r_c)}+\frac{a+c}{(r_b-r_a)(r_b-r_c)}+\frac{a+b}{(r_c-r_a)(r_c-r_b)}=\frac{2}{a+b+c}$$

Proposed Ertan Yildirim – Turkey

S.2686 If $a,b,c,x,y\in\mathbb{R}_+^*=(0,\infty), t\in\mathbb{R}_+=[0,\infty)$ and $u\in[1,\infty)$ then:

$$\left(\frac{a}{bx+cy}\right)^{t+u}+\left(\frac{b}{cx+ay}\right)^{t+u}+\left(\frac{c}{ax+by}\right)^{t+u} \ge \frac{3}{(x+y)^{t+u}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2687 If $m \geq 0$ and $a, b, c > 0$, then:

$$\frac{a^{m+1}}{(1936b+86c)^{m+1}}+\frac{b^{m+1}}{(1936c+86a)^{m+1}}+\frac{c^{m+1}}{(1936a+86b)^{m+1}} \geq \frac{1}{3^m \cdot 674^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2688 If $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{e^{2x}}{(y+z+2)^2 h_a^4}+2\right)\left(\frac{e^{2y}}{(z+x+2)^2 h_b^4}+2\right)\left(\frac{e^{2z}}{(x+y+z)^2 h_c^4}+2\right) \geq \frac{9}{4F^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.2689 Let the quadrilateral $ABCD$ circumscribe a circle of radius r and let $A'B'C'D'$ be the quadrilateral whose vertices are the points of contact of the sides of the quadrilateral $ABCD$ with the circle. Prove that:

$$\sqrt[4]{r}\left(\frac{1}{\sqrt[4]{AA'}}+\frac{1}{\sqrt[4]{BB'}}+\frac{1}{\sqrt[4]{CC'}}+\frac{1}{\sqrt[4]{DD'}}\right) \geq \frac{64}{\frac{\sin\frac{\hat{A}+\hat{B}}{2}\cdot\sin\frac{\hat{B}+\hat{C}}{2}\cdot\sin\frac{\hat{C}+\hat{A}}{2}}{\sin\frac{\hat{A}}{2}\cdot\sin\frac{\hat{B}}{2}\cdot\sin\frac{\hat{C}}{2}\cdot\sin\frac{\hat{D}}{2}}+12}$$

Proposed by Radu Diaconu – Romania

S.2690 If a, b, c are sides in a triangle then:

$$18RFx^2-\sqrt{(a+b+c)^3}x+a^2+b^2+c^2 \geq 0, \forall x \in \mathbb{R}$$

Proposed by Khaled Abd Imouti -Syria

S.2691 Prove that in any ΔABC with F – area, the following relationship holds:

$$F \leq \frac{\sqrt{3}}{4}\max\left(\frac{a^{n+2}+b^{n+2}}{a^n+b^n}, \frac{b^{n+2}+c^{n+2}}{b^n+c^n}, \frac{c^{n+2}+a^{n+2}}{c^n+a^n}\right), n \in \mathbb{N}^*$$

Proposed by Marian Ursărescu – Romania

S.2692 In ΔABC the following relationship holds:

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \geq a^2+b^2+c^2+\frac{\sum(b-c)^4}{2023(a^2+b^2+c^2)+2024(a+b+c)^2}$$

$$\sum\sqrt{\frac{m_a}{n_a+g_a-m_a}}+\frac{R^2}{r^2} \geq 4+\sum\sqrt{\frac{a}{b+c-a}}$$

Proposed by Nguyen Van Canh-Vietnam

S.2693 $k^3 = h_a h_b h_c$, $t^3 = (abc)^2$, $\frac{r_a^2}{k+a^2}+\frac{r_b^2}{k+b^2}+\frac{r_c^2}{k+c^2} \geq \frac{108r^3}{t+6R^3}$

Proposed by Elsen Kerimov -Azerbaijan

S.2694 Let $n \geq 4$. In ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_a+m_c}+\frac{m_c}{m_a+m_b}+\left(\frac{R}{2r}\right)^n \geq 1+\frac{w_a}{w_b+w_c}+\frac{w_b}{w_a+w_c}+\frac{w_c}{w_a+w_b}$$

Proposed by Nguyen Van Canh –Vietnam

S.2695 If $a,b,c>0, a+b+c=1$ then:

$$\sum a^2 \sum \frac{a}{a+2b} \geq \frac{1}{3}$$

Proposed by Marin Chirciu – Romania

S.2696 In ΔABC holds:

$$\sum \frac{a}{\cos\frac{A}{2}} < 3\sqrt{3}\frac{R^2}{r}$$

Proposed by Marin Chirciu – Romania

S.2697 If $a,b,c>0, a+b+c=2$ then: $a^2b^2+b^2c^2+c^2a^2 \leq 1$

Proposed by Tran Quoc Thinh-Vietnam

S.2698 Let $a,b \geq 0$. Prove that $\sqrt{|a-b|}+\sqrt{a+b} \geq 2^{\frac{1}{2}} \cdot \max\left\{a^{\frac{1}{2}}, b^{\frac{1}{2}}\right\}$

Find all values of $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha\sqrt{|a-b|}+\beta\sqrt{a+b} \geq \gamma \max\left\{a^{\frac{1}{2}}, b^{\frac{1}{2}}\right\}$

Proposed by Nguyen Van Canh-Vietnam

S.2699 If $a,b,c>0$ and $a+b+c=3$, then prove that:

$$\frac{(a^5+2a^2b^2(a+b)+b^5)^5}{(a^4+2ab(a^2+b^2)+b^4)^3}+\frac{(b^5+2b^2c^2(b+c)+c^5)^5}{(b^4+2bc(b^2+c^2)+c^4)^3}+\frac{(c^5+2c^2a^2(c+a)+a^5)^5}{(c^4+2ca(c^2+a^2)+a^4)^3} \geq 108$$

Proposed by Zaza Mzhavanadze – Georgia

S.2700 In ΔABC, the following relationship holds:

$$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+\frac{64abc}{(a+b)+(b+c)(c+a)} \geq 17$$

Proposed by Nguyen Van Canh-Vietnam

S.2701 Let $a,b,c,d>0, \alpha \geq 64$, such that: $a^4+b^4+c^4+d^4+4abcd=8$. Prove that:

$$(a+b+c+d)^4+\alpha(a^4+b^4+c^4+d^4) \geq 256+4\alpha$$

Proposed by Nguyen Van Canh-Vietnam

S.2702 In ΔABC, the following relationship holds:

$$9\overset{(1)}{\le}\left(\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\right)\left(\frac{m_a}{r_a}+\frac{m_b}{r_b}+\frac{m_c}{r_c}\right)\overset{(2)}{\le}\left(\frac{3R}{2r}\right)^2$$

$$h_a+h_b+h_c\overset{(3)}{\le}\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\overset{(4)}{\le}p\sqrt{3}$$

$$m_am_b+m_bm_c+m_cm_a+4(R^2-4r^2)\overset{(4)}{\ge}p^2$$

Proposed by Nguyen Van Canh-Vietnam

S.2703 Let $a,b,c,d\in\mathbb{R}$ such that: $|ax^4+bx^3+cx^2+dx|\le 2,\forall|x|\le 1$

Prove that: $|4a+3b+2c+d|\le 32$

Proposed by Nguyen Van Canh-Vietnam

S.2704 Let $a_i>0, i=1,2,\dots,n,\alpha>n^n$, such that:

$$\sum_{i=1}^{n}a_i^n+\prod_{i=1}^{n}a_i=n+1$$

Prove that:

$$\left(\sum_{i=1}^{n}a_i\right)^n+\alpha\left(\sum_{i=1}^{n}a_i^n\right)\ge n^n+\alpha n.$$

Proposed by Nguyen Van Canh-Vietnam

S.2705 In $\Delta ABC, n_a$ – Nagel's cevian, V – Bevan's point, the following relationship holds:

$$n_aw_a+n_bw_b+n_bw_b+r(R-2r)\ge m_a^2+m_b^2+m_c^2$$

$$2R\le AI+AV\le 2\sqrt{2(R^2-Rr)}$$

Proposed by Nguyen Van Canh-Vietnam

S.2706 Let $a_1=\alpha>0, a_{n+1}=\frac{2}{a_n}-a_n^2$. Find all positive real numbers α such that:

$$\lim_{n\to\infty}a_n=1.$$

Proposed by Nguyen Van Canh-Vietnam

S.2707 Solve in $\mathbb{R}$: a. $\sqrt{x^2-x+1}\le 2-x$ b. $2^x+3^{-x}=5^x+7^{-x}$ c. $x^3-3x+1\le|x^2-1|$

Proposed by Nguyen Van Canh-Vietnam

S.2708 Let $\alpha\ge\beta>0$. Solve in $\mathbb{R}$:

a. $\alpha^{2021x}+\alpha^{-2021x}=\beta^{2021x}+\beta^{-2021x}$ b. $\cos x-1=\cos^2 x-\cos 2x$

c. $x^2-\alpha x+\beta=|\beta x|+|\alpha x-\beta|$

Proposed by Nguyen Van Canh-Vietnam

S.2709 In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{r_a+2m_a}+\frac{r_c+r_a}{r_b+2m_b}+\frac{r_a+r_b}{r_c+2m_c}\geq 2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2710 In ΔABC the following relationship holds:

$$\frac{n_a}{m_a}+\frac{n_b}{m_b}+\frac{n_c}{m_c}\leq\frac{R}{r}+1$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2711 If $p_a, p_b, p_c -$ Spieker's cevians in ΔABC then:

$$\frac{r_b+r_c}{r_a+p_a}+\frac{r_c+r_a}{r_b+p_b}+\frac{r_a+r_b}{r_c+p_c}\geq 3$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2712 In ΔABC the following relationship holds:

$$\frac{r_b+r_c}{2r_a+n_a}+\frac{r_c+r_a}{2r_b+n_b}+\frac{r_a+r_b}{2r_c+n_c}\geq 2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2713 In ΔABC the following relationship holds:

$$w_aw_b+w_bw_c+w_cw_a+4R^2-16r^2\geq p^2$$

$$m_am_b+m_bm_c+m_cm_a+5(R-2r)^2\geq p^2$$

$$ab+bc+ac+2r(R-2r)\leq a^2+b^2+c^2$$

Proposed by Nguyen Van Canh-Vietnam

S.2714 Let $a,b\in\mathbb{R}$ such that: $\frac{|ax+b|}{1+x^2}\leq 1, \forall x\in\mathbb{R}$. Prove that: $|a+b|\leq 2^n, \forall n\geq 1, n\in\mathbb{N}$

Proposed by Nguyen Van Canh-Vietnam

S.2715 Let $\alpha,\beta>0$. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(\alpha x)f(\beta y)=\alpha\beta f(\beta x+\alpha y)+x^\alpha y^\beta f\left(x^\beta y^\alpha\right), \forall x,y\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.2716 Let $a,b,c\in\mathbb{R}$ such that: $\max\{|ax^2+bx+c|, |cx^2+ax+b|, |bx^2+cx+a|\}$,

$\sqrt{1-x^2}\leq 1, \forall|x|<1$. Prove that: $\max\{|a|,|b|,|c|,|a+b+c|\}\leq 1$

Proposed by Nguyen Van Canh-Vietnam

S.2717 Find all polynomials $p: \mathbb{R} \to \mathbb{R}$ such that:

$$p(x^2) + p(-x^2) = x^4 + 3x^2 - 2x - 6, \quad \forall x \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

S.2718 In $\Delta ABC, n_a$ – Nagel's cevian. Prove that:

$$2\sum n_a n_b + 3\sum m_a m_b \le 5\sum h_a h_b + \frac{s^3(R^2 - 4r^2)}{r^3 + sr^2}$$

Proposed by Nguyen Van Canh-Vietnam

S.2719 In $\Delta ABC, n_a$ – Nagel's cevian, the following relationship holds:

$$m_a^2 + m_b^2 + m_c^2 + \frac{3R(R-2r)^2}{2r} \ge n_a^2 + n_b^2 + n_c^2$$

$$h_a^2 + h_b^2 + h_c^2 + 2r(R - 2r) \le w_a^2 + w_b^2 + w_c^2$$

Proposed by Nguyen Van Canh-Vietnam

S.2720 Let $\alpha < \beta < \gamma$. Prove that:

$$\frac{\alpha^2}{|\alpha-\beta||\alpha-\gamma|} + \frac{\beta^2}{|\beta-\alpha||\beta-\gamma|} + \frac{\gamma^2}{|\gamma-\alpha||\gamma-\beta|} \ge \cos(\alpha^2 + \beta^2 + \gamma^2)$$

$$|\alpha\beta(\alpha-\beta)| + |\beta\gamma(\gamma-\beta)| + |\gamma\alpha(\gamma-\alpha)| \ge |(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)|$$

Proposed by Nguyen Van Canh-Vietnam

S.2721 In ΔABC, the following relationship holds:

$$m_a w_a + m_b w_b + m_c w_c + 3(R^2 - 4r^2) \ge m_a^2 + m_b^2 + m_c^2$$

$$\max\left\{\sum_{cyc}\sqrt{\frac{m_a}{r_a}}, \sum_{cyc}\sqrt{\frac{m_a}{h_a}}\right\} \le \sqrt{\frac{4R}{r} + 1}$$

$$\min\left\{\sum_{cyc}\sqrt{r_a}, \sum_{cyc}\sqrt{m_a}, \sum_{cyc}\sqrt{w_a}\right\} \le \sqrt{3(4R + r)}$$

Proposed by Nguyen Van Canh-Vietnam

S.2722 Find all functions $\varphi: \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{cases} \varphi\left(\varphi\big(\varphi(x)\big)\right) = 2020\varphi\big(\varphi(x)\big) - 2018\varphi(x), \forall x \in \mathbb{R} \\ \varphi(1) = 0 \end{cases}$$

Proposed by Nguyen Van Canh-Vietnam

S.2723 If $t, u, x, y, z > 0$ then:

$$(x^2+t^2)(x^2+u^2)(y^2+t^2)(y^2+u^2)(z^2+t^2)(z^2+u^2) \geq \frac{9}{16}t^2u^4(x+y+z)^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2724 Let be $m \geq 0, x, y, z > 0$ and ABC a triangle with the area F, then:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}} \geq \frac{2\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1}}{s^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2725 If in triangle ABC we denote by F the area and by s the semiperimeter, then:

$$(a^4+s^2+r^2)(b^4+s^2+r^2)(c^4+s^2+r^2) \geq 144F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2726 In triangle ABC with the area F such that $a, b, c > 1$ the following inequality holds:

$$a^{\sqrt{\log_a b}+\sqrt{\log_a c}}+b^{\sqrt{\log_b a}+\sqrt{\log_b c}}+c^{\sqrt{\log_c a}+\sqrt{\log_c a}} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți-Romania

S.2727 If $m, n \geq 0, m+n, x, x, z > 0$ and ABC is a triangle with the area F then:

$$\left(\left(\frac{mx}{y+z}\cdot a^4+\frac{ny}{z+x}\cdot b^4\right)^2+2\right)\cdot\left(\left(\frac{my}{z+x}\cdot b^4+\frac{nz}{x+y}\cdot c^4\right)^2+2\right)^2\cdot$$

$$\cdot\left(\left(\frac{mz}{x+y}\cdot c^4+\frac{nx}{y+z}\cdot a^4\right)^2+2\right) \geq 192\cdot(m+n)^2\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

S.2728 If $x, y > 0$ then in triangle ABC with the area F the following inequality holds:

$$(xr_a+yr_b)^2+(xr_b+yr_c)^2+(xr_c+yr_a)^2 \geq 3(x+y)^2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Cristian Catană-Romania

S.2729 If $m, x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{mx^2\cdot a^4}{(y+z)^2}+1\right)\left(\frac{my^2\cdot b^4}{(z+x)^2}+1\right)\left(\frac{mz^2\cdot c^4}{(x+y)^2}+1\right) \geq 9m\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Răzvan Lupu-Romania

S.2730 Let be $m, n > 0$ and ABC is a triangle with the area F, then:

$$((ma^2+nb^2)^2+2)\cdot((mb^2+nc^2)^2+2)\cdot((mc^2+na^2)^2+2) \geq 144\cdot(m+n)^2\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Luiza Cremeneanu-Romania

S.2731 If $x, y, z > 0$, then in triangle ABC with the area F the following inequality holds:

$$\left(\frac{xa}{y+z}+\frac{yb}{z+x}+\frac{zc}{x+y}\right)\cdot\left(\frac{xa^3}{y+z}+\frac{yb^3}{z+x}+\frac{zc^3}{x+y}\right)\geq 12F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2732 If $t, x, y, z > 0$ then:

$$\left(\frac{x^2}{(y+z)^2}+t^2\right)\cdot\left(\frac{y^2}{(z+x)^2}+t^2\right)\cdot\left(\frac{z^2}{(x+y)^2}+t^2\right)\geq\frac{27}{16}t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2733 In convex quadrilateral $ABCD$ with semiperimeter s and sides of lengths $a = AB$, $b = BC, c = CD, d = DA$ the following inequality holds:

$$\frac{a^2+b^2+c^2}{(s-d)^2}+\frac{b^2+c^2+d^2}{(s-a)^2}+\frac{c^2+d^2+a^2}{(s-b)^2}+\frac{d^2+a^2+b^2}{(s-c)^2}\geq 12$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2734 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x^2}{(y+z)^2}\cdot a^4+1\right)\cdot\left(\frac{y^2}{(z+x)^2}\cdot b^4+1\right)\cdot\left(\frac{z^2}{(x+y)^2}\cdot c^4+1\right)\geq 9F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

S.2735 In any triangle ABC with the semiperimeter s the following inequality holds:

$$\frac{a^3+b^3}{(s-c)^3}+\frac{b^3+c^3}{(s-a)^3}+\frac{c^3+a^3}{(s-b)^3}\geq 48$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

S.2736 If $m, n, x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{m^2x^2}{(y+z)^2}+\frac{m^2y^2}{(z+x)^2}+\frac{m^2z^2}{(x+y)^2}+n^2(a^8+b^8+c^8)\geq 16mnF^2$$

Proposed by D.M. Bătinețu – Giurgiu, Laura Marin-Romania

S.2737 If $m \geq 0$, then in any ΔABC the following inequality holds:

$$\frac{a^m\cdot b}{h_a\cdot h_b^m}+\frac{b^m\cdot c}{h_b\cdot h_c^m}+\frac{c^m\cdot a}{h_c\cdot h_a^m}\geq 2^{m+1}\left(\sqrt{3}\right)^{1-m}$$

Proposed by D.M. Bătinețu – Giurgiu, Ana Dumitru-Romania

S.2738 In any triangle ABC with the semiperimeter s the following inequality holds:

$$\frac{a^3+b^3}{(s-c)^3}+\frac{b^3+c^3}{(s-a)^3}+\frac{c^3+a^3}{(s-b)^3}\geq 48$$

Proposed by D.M. Bătinețu – Giurgiu, Olivia Bercea-Romania

S.2739 In any triangle ABC with the area F the following inequality holds:

$$\frac{a^4}{h_b^2+m_c^2}+\frac{b^4}{h_c^2+m_a^2}+\frac{c^4}{h_a^2+m_b^2}\geq\frac{8\sqrt{3}}{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2740 In any triangle ABC we denote with F the area and with s the semiperimeter, then:

$$(a^2b^2+s+r)(b^2c^2+s+r)(c^2a^2+s+r)\geq 144F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2741 In any triangle ABC with the area F the following inequality holds:

$$a^8+b^8+c^8+\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.2742 Let $t>0$ and $A_1A_2\ldots A_{12}$ a convex polygon having the sides of lengths $a_k=A_kA_{k+1}$, $k=\overline{1,12}, A_{13}=A_1$ and the area F then: $((a_1^2+a_2^2+a_3^2)^2+t^2)\cdot\left(\left(a_4^2+a_5^2+a_6^2+a_7^2\right)+t^2\right)\cdot$

$$\cdot((a_8^2+a_9^2+a_{10}^2+a_{11}^2+a_{12}^2)^2+t^2)\geq 12\cdot t^4\cdot F^2\cdot\tan^2\frac{\pi}{12}$$

Proposed by D.M. Bătinețu – Giurgiu, Dan Nănuți -Romania

S.2743 Let be $x,y>0$ and ABC a triangle with the area F, then:

$$((a^2x+yr_br_c)^2+2)((b^2x+yr_cr_a)^2+2)((c^2x+yr_ar_b)^2+2)\geq 9(4x+3y)^2F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2744 Solve for real numbers: $[x]^x+x^{[x]}=2, [*]$ - GIF

Proposed by Jalil Hajimir-Canada

S.2745 Solve for real numbers: $[x]^x<x^{[x]}$, $[*]$ - GIF

Proposed by Jalil Hajimir-Canada

S.2746 Solve for real numbers: $\begin{cases} x+y+z=6 \\ \left(x+\frac{1}{[y]}\right)^2+\left(y+\frac{1}{[z]}\right)^2+\left(z+\frac{1}{[x]}\right)^2=\frac{75}{4} \end{cases}$, $[*]$ – GIF

Proposed by Jalil Hajimir-Canada

S.2747 Solve for real numbers:

$$\begin{cases} z^2-12t=15 \\ \dfrac{z^2}{8t}+\dfrac{2z}{3}=\sqrt{\dfrac{z^3}{3t}+\dfrac{z^2}{4}-\dfrac{t}{2}} \end{cases}$$

Proposed by Hikmat Mammadov-Azerbaijan

S.2748 If $a,b,c>0$ then:

$$\frac{a}{\sqrt[3]{b}}+\frac{b}{\sqrt[3]{c}}+\frac{c}{\sqrt[3]{a}}\geq\sqrt[3]{9(a^2+b^2+c^2)}$$

Proposed by Tran Quoc Thinh-Vietnam

S.2749 Find all values of $x,y\in\mathbb{Z}$ such that $x^8-y^8=2024$.

Proposed by Nguyen Van Canh-Vietnam

S.2750 Let $a,b,c\geq 0: ab+bc+ca=1$. Prove that:

$$\frac{1}{\sqrt{9a+4bc}}+\frac{1}{\sqrt{9b+4ca}}+\frac{1}{\sqrt{9c+4ab}}\geq\frac{7}{6}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2751 In ΔABC:

$$9\left(\frac{2r}{R}\right)^2\leq\sum\frac{(m_b+m_c)^2}{a^2}\leq 9\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

S.2752 Prove that in any triangle ABC with area F is true the inequality

$$\sum ab\left(1+\sin^2\frac{C}{2}\right)\geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2753 In ΔABC holds:

$$2\sum_{cyc}\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)^{-1}\geq\frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a}$$

Proposed by Bogdan Fuștei – Romania

S.27541. Compare: $(\sin x)^{\sqrt{\cos x}}$ and $(\cos x)^{\sqrt{\sin x}}, \forall x\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]$.

2. Find all values of k such that $2022\sqrt{x^2-x+1}+2023\sqrt{x^2-x+2}\geq k, \forall k\in[0,1]$

Proposed by Nguyen Van Canh-Vietnam

S.2755 If $x, y, z, t \geq 0$ then:

$$x^3 + y^3 + z^3 + t^3 \geq \frac{x+y}{2}(z^2 + t^2) + xy(z+t)$$

Proposed by Marin Chirciu – Romania

S.2756 Let $a, b, c > 0$. Prove that:

$$\sqrt{ab + ac + 1} + \sqrt{bc + ba + 1} + \sqrt{ca + cb + 1} \geq$$

$$\geq 1 + \sqrt{\frac{8abc}{a+b+c}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}} + \frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}} + \frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\right)}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2757 In ΔABC the following relationship holds:

$$72\sqrt{3}r^3 \leq \frac{ab(a^3+b^3)}{a^2+b^2} + \frac{bc(b^3+c^3)}{b^2+c^2} + \frac{ca(c^3+a^3)}{c^2+a^2} < \frac{108\sqrt{3}(81R^5 - 2560r^5)}{s^2}$$

Proposed by Zaza Mzhavanadaze – Georgia

S.2758 If ω – Brocard's angle in ΔABC then:

$$2R + 5r \geq 5r + 4r \cdot \max\left(\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right) \geq 5r + \frac{2r}{\sin\omega} \geq h_a + h_b + h_c$$

Proposed by Bogdan Fuștei – Romania

S.2759 In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}} + \frac{\sin\frac{B}{2}}{\sin\frac{A}{2}} \geq \sqrt{\frac{r_a}{r_b}} + \sqrt{\frac{r_b}{r_a}} \geq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$$

Proposed by Bogdan Fuștei – Romania

S.2760 Prove that:

$$\tan^2\left(\frac{\pi}{21}\right) + \tan^2\left(\frac{4\pi}{21}\right) + \tan^2\left(\frac{5\pi}{21}\right) = 93 - 20\sqrt{21}$$

Proposed by Vasile Mircea Popa-Romania

S.2761 Find the largest positive constant k such that the following inequality

$$(a+b+c)\left(a\sqrt{a^2+bc} + b\sqrt{b^2+ca} + c\sqrt{c^2+ab}\right) \geq$$

$$\geq k\big(ab(a+b) + bc(b+c) + ca(c+a)\big)$$

holds for all nonnegative real numbers a, b, c.

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2762 1. Compare: 2024^e and 2023^π. 2. Find all value of m, k such that:

$$k \le |1-x| + |x-x^2| + |x^2-x^3| + |x^3-x^4| \le m, \forall x \in [-1,2]$$

Proposed by Nguyen Van Canh-Vietnam

S.2763 In any ΔABC the following relationship holds:

$$\frac{36r^2}{R} \le \frac{(m_a+m_b)^2}{w_b+w_c} + \frac{(w_b+w_c)^2}{h_c+h_a} + \frac{(h_c+h_a)^2}{m_a+m_b} \le \frac{9(9R^4-80r^4)}{32r^3}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2764 Find all values of $m, n \in \mathbb{Z}$ such that $2m^4 + 1945m^2n^2 + 9n^4 = 6141186$

Proposed by Nguyen Van Canh – Vietnam

S.2765 If $a, b, c > 0, a^2+b^2+c^2 = 12$ and $n \in \mathbb{N}, n \ge 2$ then:

$$\sum \frac{a^{2n}}{\sqrt{a^3+1}} \ge 4^n$$

Proposed by Marin Chirciu – Romania

S.2766 If $n \ge 1, t \in \left[-\frac{1}{n}, \frac{1}{n}\right]$ then:

$$\cosh t + |\sinh t| \le n \sinh \frac{1}{n} + \left(\frac{1}{n} + nt^2\right) e^{\frac{1}{n}}$$

Proposed by Khaled Abd Imouti-Syria

S.2767 Compute (without soft):

$$I = \int_{-2023}^{2023} (|x^{2021}-1| + |x^{2022}-2| + |x^{2023}-3| + |x^{2024}-4|)\,dx$$

Proposed by Nguyen Van Canh – Vietnam

S.2768 In ΔABC the following relationship holds:

$$\frac{r_a^{10}}{r_a^3+r_b^3} + \frac{r_b^{10}}{r_b^3+r_c^3} + \frac{r_c^{10}}{r_c^3+r_a^3} \ge \frac{3(3r)^7}{2}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2769 If $a, b, c, \lambda > 0, ab+bc+ca = \lambda$ then:

$$\frac{\sum \sqrt{a^2+\lambda}}{\sum \sqrt{ab}} \ge 2$$

Proposed by Marin Chirciu – Romania

S.2770 In any acute or right triangle ABC holds:

$$\frac{2m_a}{r_a} \geq \frac{n_b}{h_c} + \frac{n_c}{h_b}$$

Proposed by Bogdan Fuștei – Romania

S.2771 Let $a, b, c > 0: ab + bc + ca = 3$. Prove that:

$$\frac{2(a+b+c)}{abc} + 3 \geq \sqrt{\frac{5}{a^2} + 4} + \sqrt{\frac{5}{b^2} + 4} + \sqrt{\frac{5}{c^2} + 4}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2772 If $x, y \in \mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\sin^2 A}{x\cos^2\frac{A}{2} + y\sin^2\frac{B}{2}\sin^2\frac{C}{2}} + \frac{\sin^2 B}{c\cos^2\frac{B}{2} + y\sin^2\frac{C}{2}\sin^2\frac{A}{2}} +$$

$$+\frac{\sin^2 C}{x\cos^2\frac{C}{2} + y\sin^2\frac{A}{2}\sin^2\frac{B}{2}} \geq \frac{16p^2}{8(4x-y)R^2 + 8Rrx + y(p^2+r^2)}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2773 Let $a, b, c \geq 0$ such that $a^2 + b^2 + c^2 + d^2 = 2023$. Prove that:

$$\frac{1}{a^4+2} + \frac{1}{b^4+2} + \frac{1}{c^4+2} + \frac{1}{d^4+2} \geq \frac{64}{2092561}$$

Proposed by Nguyen Van Canh – Vietnam

S.2774 If $a, b, c > 0$ and $0 \leq \lambda \leq 2$ then:

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\sum (a+b)\frac{a^2 - \lambda ab + b^2}{a^2 + \lambda ab + b^2} \geq 18\frac{2-\lambda}{2+\lambda}$$

Proposed by Marin Chirciu – Romania

S.2775 If $x_1, x_2, \dots, x_2 > 0$ such that $x_1 x_2 \dots x_n = 1$ and $\lambda \geq 0$ then:

$$\sum_{i=1}^{n}\left(\sqrt{x_i^4 - (2\lambda - 1)x_i^2 + \lambda^2} + \lambda x_i\right) \geq (\lambda + 1)n$$

Proposed by Marin Chirciu – Romania

S.2776 If $m, n \in \mathbb{R}_+^*$, then in any triangle ABC holds:

$$\left(m + n\cdot\cot\frac{A}{2}\cdot\cot\frac{B}{2}\right)^2 + \left(m + n\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}\right)^2 + \left(m + n\cdot\cot\frac{C}{2}\cdot\cot\frac{A}{2}\right)^2$$
$$\geq \frac{(3m+n)^2r^2 + 8n(3m+n)Rr + 16n^2R^2}{3r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2777 In ΔABC the following relationship holds:

$$\frac{m_a}{w_b}+\frac{w_b}{h_c}+\frac{h_c}{m_a}\le\frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3-64\right)$$

Proposed by Zaza Mzhavanadze – Georgia

S.2778 Prove that in any triangle ABC, with area S and usual notations is true the inequality:

$$\frac{a^{2(m+1)}}{(F_n m_a^2+F_{n+1}m_b^2+F_{n+2}m_c^2)^m}+\frac{b^{2(m+1)}}{(F_n m_b^2+F_{n+1}m_c^2+F_{n+2}m_a^2)}+$$
$$+\frac{c^{2(m+1)}}{(F_n m_c^2+F_{n+1}m_a^2+F_{n+2}m_b^2)}\ge\frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m}S$$

where $m>0$ and F_k is the k-th Fibonacci number.

Proposed D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

S.2779 In ΔABC holds:

$$\frac{24R}{r}\le\sum\left(\csc\frac{B}{2}\csc\frac{C}{2}\right)^2\le\frac{6R^3}{r^3}$$

Proposed by Marin Chirciu – Romania

S.2780 Let be $m,n\in\mathbb{N},m,n\ge 3$ and the convex polygons $A_1A_2\dots A_n, B_1B_2\dots B_n$ having the sides of lengths $a_k=A_kA_{k+1}, b_j=B_jB_{j+1}, \forall k=\overline{1,m}, \forall j=\overline{1,n}, A_{n+1}=A_1, B_{n+1}=B_1$ and the areas F_1 respectively F_2. Prove that:

$$\sum_{k=1}^{m}a_k^2+\sum_{j=1}^{n}b_j^2\ge 8\sqrt{F_1F_2\cdot\tan\frac{\pi}{m}\cdot\tan\frac{\pi}{n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2781 Let $A_1A_2A_3$ be a triangle with the area F_1, $B_1B_2B_3B_4$ a convex quadrilateral having the sides of lengths b_1,b_2,b_3,b_4 having the area F_2 and $C_1C_2C_3C_4C_5C_6$ a convex hexagon with the area F_3 and the sides of lengths c_1,c_2,c_3,c_4,c_5,c_6. Prove that:

$$a_1a_2+a_2a_3+a_3a_1+b_1^2+b_2^2+b_3^2+b_4^2+c_1^2+c_2^2+c_3^2+c_4^2+c_5^2+c_6^2\ge 12\sqrt[3]{F_1F_2F_3}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2782 Let be $m\ge 0$ and $ABCD$ a convex quadrilateral with the semiperimeter s and the sides with the lengths $a=AB, b=BC, c=CD, d=DA$, then:

$$\frac{a^{m+1}+b^{m+1}+c^{m+1}}{(s-d)^{m+1}}+\frac{b^{m+1}+c^{m+1}+d^{m+1}}{(s-a)^{m+1}}+\frac{c^{m+1}+d^{m+1}+a^{m+1}}{(s-b)^{m+1}}+\frac{d^{m+1}+a^{m+1}+b^{m+1}}{(s-c)^{m+1}}\ge 12$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2783 Let be $m \geq 0$ and ABC a triangle with the area F, then:

$$a^{4m+8} + b^{4m+8} + c^{4m+8} + \frac{1}{a^{4m+4}} + \frac{1}{b^{4m+4}} + \frac{1}{c^{4m+4}} \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2784 If $t, x, y, z > 0$ then:

$$\left(\frac{x^2}{(y+z)^2} + t^2\right) \cdot \left(\frac{y^2}{(z+x)^2} + t^2\right) \cdot \left(\frac{z^2}{(x+y)^2} + t^2\right) \geq \frac{27}{16}t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2785 Let $ABCD$ be a convex quadrilateral having the area F and the sides $a = AB$,

$b = BC, c = CD, d = DA$, then: $((a^2 + b^2)^2 + 2)(c^4 + 2)(d^4 + 2) \geq 48F^2$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2786 Let be $x, y > 0$ and the triangle ABC with the area F, then:

$$((xa^2 + yb^2)^2 + 2)((xb^2 + yc^2)^2 + 2)((xc^2 + ya^2)^2 + 2) \geq 144(x+y)^2F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2787 Let be $A_1A_2 \dots A_n, n \geq 3$ a convex polygon with the area F and M an interior point in the polygon and d_k the distance from point M to the side $A_kA_{k+1}, k = \overline{1,n}, A_{n+1} = A_1$. Prove that:

$$\sum_{k=1}^{n} \frac{a_k^3}{d_k} \geq 8F \cdot \tan^2\frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2788 If $m, n, p \in \mathbb{N}, m, n, p \geq 2$ and $t = m + n + p$ and $A_1A_2 \dots A_t$ is a convex polygon having the sides of lengths $a_k = A_kA_{k+1}, \forall k = \overline{1,t}, A_{t+1} = A_1$ and the area F, then:

$$\left(\left(\sum_{i=1}^{m} a_i^2\right)^2 + 2\right) \cdot \left(\left(\sum_{j=1}^{n} a_{m+j}^2\right)^2 + 2\right) \cdot \left(\left(\sum_{k=1}^{p} a_{m+n+p}^2\right)^2 + 2\right) \geq 48F^2 \cdot \tan^2\frac{\pi}{t}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2789. If $t, u, x, y, z > 0$ then:

$$\left(\left(\frac{x}{ty+uz} + \frac{y}{tz+ux}\right)^2 + 2\right) \cdot \left(\left(\frac{y}{tz+ux} + \frac{z}{tx+uy}\right)^2 + 2\right) \cdot \left(\left(\frac{z}{tx+uy} + \frac{x}{ty+uz}\right)^2 + 2\right) \geq \frac{108}{(t+u)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2790 Let be $t > 0, m, n, p \in \mathbb{N}, m, n, p \geq 2$ and $w = m + n + p$ and $A_1A_2 \dots A_w$ a convex polygon having the sides of lengths $a_k = A_kA_{k+1}, k = \overline{1,w}, A_{w+1} = A_1$ and the area F then:

$$\left(\left(\sum_{i=1}^{m} a_i^2\right)^2 + t^2\right) \cdot \left(\left(\sum_{j=1}^{n} a_{m+j}^2\right)^2 + t^2\right) \cdot \left(\left(\sum_{k=1}^{p} a_{m+n+k}^2\right)^2 + t^2\right) \geq 12t^4 \cdot F^2 \cdot \tan^2 \frac{\pi}{w}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2791 Let $x, y > 0$ and ABC be a triangle with the area F, then:

$$(xa + yb)^2 + (xb + yc)^2 + (xc + ya)^2 \geq 4(x + y)^2 F$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2792 If $x, y > 0$ and ABC is a triangle with the area F, then:

$$\left((xa^2 + ym_b^2)^2 + 2\right)((xb^2 + ym_c^2)^2 + 2)((xc^2 + ym_a^2)^2 + 2) \geq 441(x + y)^2 F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2793 If $t, u, v, x, y, z > 0$ then:

$$(x^2 + t^2)^2(y^2 + u^2)^2(z^2 + v^2)^2 \geq \frac{9}{16}(tyz + yxz + vxy)^2(xuv + ytv + ztu)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2794 If $x, y > 0$, then in triangle ABC with the area F the following inequality holds:

$$\left(\frac{xa + yb}{h_c}\right)^2 + \left(\frac{xb + y_c}{h_a}\right)^2 + \left(\frac{xc + ya}{h_b}\right)^2 \geq 4(x + y)^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2795 If $x, y > 0$ and ABC is a triangle with the area F, then:

$$((xr_a^2 + yr_br_c)^2 + 2)\left((xr_b^2 + yr_cr_a)^2 + 2\right)((xr_c^2 + yr_ar_b)^2 + 2) \geq 81(x + y)^2 F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2796 Let be $x, y > 0$ and ABC a triangle with the area F, then:

$$((xm_a^2 + yr_br_c)^2 + 2)\left((xm_b^2 + yr_cr_a)^2 + 2\right)((xm_c^2 + yr_ar_b)^2 + 2) \geq 81(x + y)^2 F^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2797 If $m, x, y, z > 0$ and $xyz = 1$ then: $(x^m + 2)(y^m + 2)(z^m + 2) \geq 27$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2798 If $x, y > 0$, then in triangle ABC with the area F the following inequality holds:

$$(a^2x + b^2y)^2 + (b^2x + c^2y)^2 + (c^2x + a^2y)^2 \geq 16(x + y)^2 F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2799 If $x, y > 0$ then in triangle ABC with the area F, the following inequality holds:

$$((x \cdot m_a^2 + ybc)^2 + 2) \cdot \left((x \cdot m_b^2 + yca)^2 + 2\right) \cdot ((x \cdot m_c^2 + yab)^2 + 2) \geq 9(3x + 4y)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2800 If $t, u, v, w, x, y, z > 0$ then:

$$\left(\left(\frac{tx}{vy + wz} + \frac{uy}{vz + wx}\right)^2 + 2\right)\left(\left(\frac{ty}{vz + wx} + \frac{uz}{vx + wy}\right)^2 + 2\right)\left(\left(\frac{tz}{vx + wy} + \frac{ux}{vy + wz}\right)^2 + 2\right) \geq 27 \cdot \left(\frac{t + u}{v + w}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2801 Let be ABC a triangle with the semiperimeter s, then:

$$\frac{a^2 + b^2}{(a + b - c)^2} + \frac{b^2 + c^2}{(b + c - a)^2} + \frac{c^2 + a^2}{(c + a - b)^2} \geq 6$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

S.2802 If $x, y, z > 0$ and $x^2 + y^2 = z^2$, then:

$$(x^4 + y^4 + z^4) \cdot \left(\frac{1}{x^4} + \frac{1}{y^4} + \frac{1}{z^4}\right) \geq \frac{100}{9}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.2803 Let $A_1A_2A_3$ be a triangle with the area F_1 and $B_1B_2B_3B_4B_5B_6$ a convex hexagon having the sides of lengths $b_1, b_2, b_3, b_4, b_5, b_6$ and the area F_2. Prove that:

$$a_1a_2 + a_2a_3 + a_3a_1 + b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 \geq 8\sqrt{F_1 \cdot F_2}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.2631 Prove the integral:

$$\int_{-\infty}^{\infty}\frac{\cosh^3\left(\frac{\pi x}{4}\right)+\cosh^2\left(\frac{\pi x}{4}\right)+\cosh\left(\frac{\pi x}{4}\right)}{2\cosh\left(\frac{3\pi x}{2}\right)+2\cosh(\pi x)-1}dx=\sqrt{\frac{3}{2}+\frac{1}{4}\left(1-\frac{5}{\sqrt{3}}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2632

$$\Psi=\int_0^1\frac{\ln(x^2)+\arctan(x)}{(1+x^2)^2}dx=\frac{\pi^2}{64}-\frac{3\pi}{16}-G-\frac{1}{8}$$

Note: $\{G-Catalan's\ constant\}$

Proposed by Shirvan Tahirov – Azerbaijan

U.2633 If $x>0$ then:

$$\pi(1+(\arctan x\cdot\arctan x)^{-1})>8+2(\ln^2(\arctan x)+\ln^2(\operatorname{arccot} x))$$

Proposed by Rovsen Pirguliyev-Azerbaijan

U.2634 In ΔABC the following relationship holds:

$$\sqrt[3]{(1+e^{\sin A})(1+e^{\sin B})(1+e^{\sin C})}\geq 1+e^{\sqrt[3]{\sin A\sin B\sin C}}$$

Proposed by Khaled Abd Imouti-Syria

U.2635 Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sum_{i=1}^{k}i\left(i+\frac{1}{2}\right)\right]^{-1}$$

Proposed by Vasile Mircea Popa – Romania

U.2636 Prove that:

$$4\sum_{n=0}^{\infty}\frac{16^n}{\binom{4n}{2n}(4n+1)(4n+2)(4n+3)^2}=4G-\operatorname{arcsinh}(1)\left(4\sqrt{2}+\log\left(3-2\sqrt{2}\right)\right)$$

where G is Catalan's constant.

Proposed by Narendra Bhandari –Nepal

U.2637 If $(a_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}^2}{a_n^2\cdot n}=a>0$$

Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^2}-\sqrt[n]{a_n^2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

U.2637 Let be $H_n=\sum_{k=1}^{n}\frac{1}{k}, \forall n\in\mathbb{N}^*$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot e^{H_n}}=a>0$$

Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

U.2638 Let $(a_n)_{n\geq 1}$ be a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{(2n-1)!!}}=a>0$$

Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

U.2639

$$\sqrt{x^2+x\sqrt{x^2+x\sqrt{x^2+\cdots}}}=5x,\quad x=?$$

Proposed by Ghulam Shah Naseri-Afghanistan

U.2640 In ΔABC the following relationship holds:

$$\tan A\cdot\tan B\cdot\tan C+3\sqrt{3}>3\pi$$

Proposed Khaled Abd Imouti-Syria

U.2641 If

$$\Omega:\int\int\int_{[0,1]^3}\ln\left(\frac{1}{x+xy+xyz}\right)dx\,dy\,dz$$

Then, show that

$$\Omega = Li_2\left(\frac{1}{3}\right) - \frac{\pi^2}{12} + \frac{\ln^2(3)}{2} - \ln(2)\ln(3) - \ln\left(\frac{27}{4}\right) + 3$$

Where, $Li_2(z)$ is a spence or dilogarithm function.

Proposed by Ankush Kumar Parcha -India

U.2642 Compute (without soft):

$$I = \int_{-\pi}^{\pi} |2022\sin^2 x - 2023\cos^2 x|\, dx$$

Proposed by Nguyen Van Canh – Vietnam

U.2643 If $n \in \mathbb{N}, n \geq 2$ then:

$$\tan\left(\frac{1}{n-1}\sum_{k=2}^{n} \arctan\frac{1}{k}\right) < \frac{2}{5} + \frac{\gamma}{n-1}$$

Proposed by Khaled Abd Imouti -Syria

U.2644 Compute (without soft):

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} |\tan x - \cot x|\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left|\tan\frac{x}{2} - \cot\frac{x}{2}\right| dx$$

Proposed by Nguyen Van Chanh – Vietnam

U.2645 In any triangle ABC we have:

$$\frac{1}{8}\prod_{cyc}\left(b^{\sin^{n+\frac{1}{n}}\hat{A}} + c^{\sin^{n+\frac{1}{n}}\hat{A}}\right) \leq 2^{-\sum_{cyc}\left(\sin^{n+\frac{1}{n}}\hat{A}\right)}\prod_{cyc}(b+c)^{\sin^{n+\frac{1}{n}}\hat{A}} \leq (2R)^{\sum_{cyc}\sin^{n+\frac{1}{n}}\hat{A}}\prod_{cyc}\left(\cos\frac{\hat{A}}{2}\right)^{\sin^{n+\frac{1}{n}}\hat{A}}$$

Proposed by Radu Diaconu – Romania

U.2646 Find:

$$\Omega = \int_1^{\infty} \frac{x\ln x}{x^3 + x\sqrt{x} + 1}\, dx$$

Proposed by Vasile Mircea Popa– Romania

U.2647 Prove:

$$\cos\frac{\pi}{13}\cos\frac{8\pi}{13} = \frac{\sqrt{13}}{6}\cos\left(\frac{1}{3}\left(\arccos\left(\frac{5}{2\sqrt{13}}\right) + 2\pi\right)\right) + \frac{1}{12}$$

Proposed by Vasile Mircea Popa – Romania

U.2648 If $n \geq 1$ then:

$$2+\log\left(\frac{6\left((n+1)!-1\right)^2}{n(2n^2+3n+1)}\right) \leq 2\sqrt{\sum_{k=1}^{n}(k!)^2}$$

Proposed by Khaled Abd Imouti-Syria

U.2649 If

$$\zeta := \int_0^1 \frac{\ln\left(\sqrt{1+x}\right)}{\sqrt{1+x}+\sqrt{1-x}}dx$$

Then, show that:

$$\zeta = -Li_2\left(\frac{1}{\sqrt{2}}\right)-\frac{\ln^2\left(1+\sqrt{2}\right)}{2}+\frac{\ln(2)\ln\left(\sqrt{2}-1\right)}{2}+\sqrt{2}\ln\left(1+\sqrt{2}\right)-\frac{\ln^2(2)}{8}+\frac{\ln(2)}{\sqrt{2}}+\frac{7}{48}\pi^2-\sqrt{2}$$

Where, $Li_2(z)$ is dilogarithm of Spence's function

Proposed by Ankush Kumar Parcha-India

U.2650 Prove the below closed form.

$$\int_0^1 \frac{\ln(1+x)}{(1+x)(2+x)(3+x)(4+x)}dx =$$

$$= \frac{1}{6}\left[Li_2\left(-\frac{1}{3}\right)-Li_2\left(-\frac{2}{3}\right)\right]-\frac{\ln(2)}{3}\tanh^{-1}\left(\frac{13}{77}\right)$$

Where, $Li_2(z)$ is a dilogarithm of Spence's function

Proposed by Ankush Kumar Parcha-India

U.2651 Prove that:

$$\Delta = \int_0^1 \frac{x\left(\ln\left(\sqrt{1+x^2}\right)+\arctan^2(x)\right)}{x^2+1}dx = \frac{1}{64}\left(16\pi G-21\zeta(3)+8\ln^2(2)-\pi^2\ln(4)\right)$$

Note: $\begin{cases} \zeta(3) \Rightarrow [Aper'y\ constant] & :) \\ G \Rightarrow [Catalan's\ constant] & :) \end{cases}$

Proposed by Shirvan Tahirov-Azerbaijan

U.2652 Prove that:

$$\sum_{k=1}^{\infty} \frac{\prod_{l=1}^{k}(6l-3k-1)}{3^k kk!}\csc\left(\frac{\pi k}{2}+\frac{\pi}{6}\right) = \log\left(756-432\sqrt{3}\right)$$

where

$$\prod_{l=1}^{k}(6l-3k-1)=(5-3k)(11-3k)\dots(3k-1)$$

and

$$\csc(z)=\frac{1}{\sin z}$$

Proposed by Abdulhafeez Ayinde Abdulsalam-Nigeria

U.2653 Compute (without soft):

$$I=\int_0^1 \frac{x\,dx}{\sqrt{x+2022}+\sqrt{x+2023}+\sqrt{x+2024}}$$

Proposed by Nguyen Van Canh-Vietnam

U.2654 Prove the integral relation

$$\int_{-\infty}^{\infty}\frac{\cosh\left(\frac{\pi}{2}(1+x)\right)}{\cosh(2\pi x)-\frac{1}{2}}dx=\sqrt{2+\sqrt{3}}\int_{-\infty}^{\infty}\frac{\cosh\left(\frac{\pi}{2}(1+x)\right)}{\cosh(2\pi x)+\frac{1}{2}}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2655 If $\Omega_1=\int_0^1 x^2\ln(1+x^2)\ln\left(x+\frac{1}{x}\right)dx,\ \Omega_2=\int_0^1\frac{\ln(1+x^2)\ln\left(x+\frac{1}{x}\right)}{x^2}dx$

$\Omega_1-\Omega_2=\frac{4}{3}\alpha+\frac{8}{3}\beta-\frac{68}{27}$. Prove that: $\alpha+\beta=G-\pi\ln(2)+\ln^2(2)+\pi$

Proposed by Abbaszade Yusif –Azerbaijan

U.2656 Find:

$$\Omega=\lim_{x\to\infty}\left[\int_{x-\frac{1}{x}}^{x+\frac{1}{x}} t^2\arctan\left(\frac{1}{t}\right)dt\right]$$

Proposed by Vasile Mircea Popa – Romania

U.2657 Let

$$P_N:=\sum_{n=1}^{2N}(-1)^n\,(2n+1)\log(2n+1)$$

Show that $\lim_{N\to\infty}[H_{2N+1}+2N\log(4N+2)-P_N]=\frac{2G}{\pi}+\gamma-\log(2)$

where G is Catalan's constant, γ is the Euler Mascheroni constant, and H_N denotes the $N-$th harmonic number.

Proposed by Vincent Nguyen-USA

U.2658 Prove the below closed form

$$\int_0^1\int_0^1 \frac{\sin^{-1}(x)+\cos^{-1}(y)}{\sqrt{xy}}\,dx\,dy=\tau$$

Where, τ is the ratio of the circumference to the radius of circle.

Proposed by Ankush Kumar Parcha-India

U.2659 If:

$$\Omega_1=\sum_{n=1}^{\infty}\tan^{-1}\left(\tanh^{-1}\left(e^{-4\sqrt{3}\pi n}\right)\right)$$

$$\Omega_2=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\sin\left(2\tan^{-1}\left(\tanh^{-1}\left(e^{-4\sqrt{3}\pi n}\right)\right)\right)}{\cos\left(2\tan^{-1}\left(\tanh^{-1}\left(e^{-4\sqrt{3}\pi n}\right)\right)\right)+\cosh(2k)}$$

$$\Omega_3=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{\sinh(k\pi^2)\sinh\left(2\pi k\cdot\tan^{-1}\left(\tanh^{-1}\left(e^{-4\sqrt{3}\pi n}\right)\right)\right)}$$

then prove that:

$$\Omega_1-\Omega_2+\pi\Omega_3=\frac{1}{64}\left(\ln\left(-\frac{746496(\sqrt{3}-2)\pi^4\Gamma^8\left(\frac{2}{3}\right)}{\Gamma^8\left(\frac{1}{6}\right)}\right)-4\ln\left(4+\sqrt{2+\sqrt{3}}\right)\right)$$

Proposed by Toubal Fethi -Algeria

U.2660 Prove that:

$$\sum_{k=0}^{\infty}\frac{\left((2k)!\right)^2}{2^{2k}\cdot(k!)^4}>16\left(\log\frac{e}{2}\right)^2\cdot\left(\sum_{k=0}^{\infty}\frac{1}{k^4}\right)^{-1}$$

Proposed by Khaled Abd Imouti -Syria

U.2661 If $H_n=\sum_{k=1}^{n}\frac{1}{k}, \forall n\in\mathbb{N}^*$, find:

$$\lim_{n\to\infty}\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot e^{H_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2662 Let be $(x_n)_{n\ge1}, x_n=\sum_{k=1}^{n}\frac{1}{k^4}$ and $(H_n)_{n\ge1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$. Find:

$$\lim_{n\to\infty}\left(\frac{\pi^4}{90}-x_n\right)e^{3H_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2663 If $(x_n)_{n\geq 1}, x_n = \sum_{k=1}^{3n} \frac{1}{n+k}$, find $\lim\limits_{n\to\infty} x_n$.

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2664 Prove that:

$$\Omega = \sum_{n=1}^{\infty} \frac{(-1)^n \overline{H}_n \psi^{(0)}(n)}{n^2} = Li_4\left(\frac{1}{2}\right) - \gamma\left(\frac{5\zeta(3)}{4} - \frac{\pi^2}{4}\ln(2)\right) + \frac{\ln^4(2)}{24} - \frac{\pi^2}{24}\ln^2(2) - \frac{\pi^4}{288}$$

Skew Harmonic series: $\overline{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$

Proposed by Shirvan Tahirov -Azerbaijan

U.2665 Compute:

$$I = \int_{1980}^{2022} \sqrt{40 + \sqrt{20x + 11}}\, dx$$

Proposed by Nguyen Van Canh-Vietnam

U.2666 Prove that:

$$\Omega = \int_0^1 x(\ln(\ln(x)) + \arctan(x-1))\, dx = 0.5(i\pi - \gamma - 1)$$

γ is the Euler – Mascheroni constant.

Proposed by Shirvan Tahirov-Azebaijan

U.2667 Show that

$$\int_0^{\frac{\pi}{2}} \frac{\tan^{-1}\left(\sqrt{\tan(x)}\right)}{\tan(x)} \log(\cot(x))\, dx = \frac{11\pi^3}{96}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2668 Prove the below closed form

$$\int_0^1 \int_0^1 \frac{x^2 \tan^{-1}(xy)\, y^2 \cot^{-1}(xy)}{\ln(x) + \ln(y)}\, dx\, dy = \frac{1+G}{3} - \frac{\pi}{12} - \frac{\pi}{6}\ln(2) - \frac{\pi^2}{48}$$

Proposed by Ankush Kumar Parcha-India

U.2669 Prove that

$$\lim_{n\to\infty} \frac{4^n}{\sqrt{n}} \int_0^1 \frac{\sqrt{x^2 + \frac{1}{e}}}{\left(x^{\frac{1}{n}} + 1\right)^{2n}}\, dx = \sqrt{\frac{\pi}{e}}$$

Proposed by Vincent Nguyen-USA

U.2670 Evaluate:

$$\lim_{n\to\infty}\sum_{k=1}^{m}\sqrt[n]{\int_0^1 (1-x^k)^n\,dx}$$

Proposed by Naren Bhandari -Nepal

U.2671 For all $n\in\mathbb{N}$, define the limit

$$F(n)=\lim_{x\to 0^+}\left(\sum_{k=1}^{n}\sqrt[x]{k}\right)^x$$

Let α be number of integral solutions for the equation $M^\beta = kF(n)\big(k+F(n)\big)$ where $M,k,\beta\in\mathbb{N}$. Then show that

$$\lim_{y\to\alpha}\sqrt[y]{y!!}=\sqrt{2}\lim_{y\to\alpha}\sqrt[y]{y!}=\frac{\sqrt{2}}{e^\gamma}$$

Note: !! is double factorial notation.

Proposed by Naren Bhandari -Nepal

U.2672 $a_{n+1}=a_n-\arcsin(r\sin a_n)$, $a_1=\frac{\pi}{2}$. Prove: for any fixed $n\in\mathbb{N}$

$a_n=\frac{\pi}{2}-(n-1)r+\frac{(-4+11n-9n^2+2n^3)}{12}r^3+o(r^3)$ when $r\to 0$

Proposed by Hikmat Mammadov-Azerbaijan

U.2673 If $S=\sin^2\frac{1}{x}+\sin^2 1+\cos^2\frac{2}{x}+\cos^2 2+\tan^2\frac{3}{x}+\tan^2 3+\cot^2\frac{4}{x}+\cot^2 4$

Find: $\Omega=\lim_{x\to\infty}\frac{1\cdot 2\cdot 3\cdot 4}{x}\cdot S$

Proposed by Hikmat Mammadov-Azerbaijan

U.2674 Prove

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k+1}\left(H_{\frac{k}{2}}-H_{\frac{k-1}{2}}\right)=\frac{21}{32}\zeta(3)+\frac{\pi^2}{8}\log(2)-\frac{G}{2}$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}}{2k+1}\left(H_k-H_{\frac{k-1}{2}}\right)=\frac{21}{64}\zeta(3)+\frac{3}{8}\zeta(2)\log(2)-\frac{G}{4}+\frac{\pi}{8}\log^2(2)$$

where G is Catalan's constant, $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2}$, $\zeta(z)$ is Riemann zeta function, $\sum_{k=1}^{\infty}\frac{1}{k^z}$ and H_n is defined by $\int_0^1\frac{1-x^n}{1-x}dx$.

Proposed by Narendra Bhandari –Nepal

U.2675 Prove

$$\sum_{n=1}^{\infty}\left(n\left(\sum_{k=1}^{\infty}\frac{1}{(k+n)^2}\right)^2-\frac{1}{n}\right)=\frac{3}{2}\big(1-\zeta(3)\big)-\frac{\pi^2}{12}$$

where $\zeta(3)=\sum_{k=1}^{\infty}\frac{1}{k^3}$

Proposed by Narendra Bhandari –Nepal

U.2676 Prove

$$\int_0^1\int_0^1\frac{x^2\log^2(x)}{(1-x)(1+x^2y)}\,dx\,dy=2G^2+\frac{35}{16}\zeta(3)\log(2)-\frac{199}{5760}\pi^4,$$

where G is Catalan's constant defined by $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ and $\zeta(3)$ is Apery's constant defined by $\sum_{n=1}^{\infty}\frac{1}{n^3}$.

Proposed by Narendra Bhandari –Nepal

U.2677 Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n(2n+1)4^n}\sum_{k=1}^{\left\lfloor n+\frac{1}{2}\right\rfloor}\frac{1}{2k-1}\binom{2n}{n}=\frac{\pi^2}{4}-\pi\log(2),$$

where $\lfloor . \rfloor$ denotes the floor function.

Proposed by Narendra Bhandari –Nepal

U.2678 Prove that:

$$\Delta=\int_0^1 x\ln(\arccos^2(1-x^2))+\frac{\ln^2(1-x)}{x}=2\zeta(3)+Ci\left(\frac{\pi}{2}\right)-\gamma$$

Proposed by Shirvan Tahirov-Azerbaijan

U.2679 If $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is n th harmonic number, $\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}$ is Riemann zeta function and for $m>1$, prove

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(2)-\sum_{k=1}^{n+m}\frac{1}{k^2}\right)=$$

$$=\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\left(\frac{1}{(k+1)^4}-(m-1)\sum_{p=1}^{\infty}\frac{1}{p(p+k+1)^4}\right)$$

Proposed by Narendra Bhandari –Nepal

U.2680 Prove that

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{n^s}=\int_0^1\left(\frac{Li_s(x)}{x(x-1)}-\frac{\zeta(s)}{\ln(x)}\right)dx$$

Proposed by Vincent Nguyen-USA

U.2681 Evaluate

$$\sum_{k=0}^{n}\frac{\pi}{4^k}\binom{2k}{k}$$

Proposed by Vincent Nguyen-USA

U.2682 Prove that:

$$\sum_{p=0}^{k}\frac{1}{ap+1}\binom{2p}{p}\binom{2k-2p}{k-p}=\prod_{p=1}^{k}\frac{4ap-2a+4}{ap+1}$$

Proposed by Fao Ler-Iraq

U.2683 Find:

$$\Omega=\int_0^1\tan^{-1}x\cdot Li_3(x)\,dx+\int_0^1\frac{x\cdot Li_3(x)}{1+x^2}dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.2684 Find:

$$I=\int_0^1\log(x)\log(x^2+x+1)\,dx$$

Proposed by Togrul Ehmedov-Azerbaijan

U.2685 Prove the integral relation

$$\int_0^{\frac{\pi}{2}}\min\left(\sqrt{\frac{\sin(2x)+1}{\sin\left(\frac{x}{2}\right)+1}},\sqrt{\frac{\cos(2x)+1}{\cos\left(\frac{x}{2}\right)+1}}\right)\tan(x)\,dx$$

$$=\frac{4}{3}\left(\sqrt{2\left(10+\sqrt{2}\right)}-4\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2686 If we have the function for $a\geq 1$

$$T(a)=\int\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^{\frac{n}{2}(n+1)}}{(an+(-1)^n)^2}\right)da$$

then show that

$$T(-a)+T(a)=\frac{\pi}{\cos\left(\frac{\pi}{2a}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2687 Solve for a, b:

$$\sqrt{\frac{a}{b}}+b=7, \qquad \sqrt{\frac{b}{a}}+a=11$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2688 Prove that there are exactly 1729 positive integer solutions to the below equation.

$$4x^4+3y^3+2z^2+w=4311$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2689 Find:

$$\Omega(x)=\lim_{n\to\infty}\left(\sum_{0\le 2k\le n}\binom{2k}{n}x^k\right)\left(\sum_{0\le 2k+1\le n}\binom{2n+1}{n}x^k\right)^{-1}, x>0$$

Proposed by Khaled Abd Imouti-Syria

U.2690 If $0<m<n<2m$ and

$$A=\frac{\sqrt{2+\sqrt{5}}}{\sqrt{2+\sqrt{5}}}, B=\frac{\sqrt[6]{-38+17\sqrt{5}-\frac{1}{m}\sqrt{\frac{2m}{n}}}}{\sqrt[6]{-38+17\sqrt{5}+\frac{1}{m}\sqrt{\frac{2m}{n}}}}, C=\sqrt{\frac{1-n\left(\frac{1}{m}\sqrt{\frac{2m}{n}}\right)}{1+n\left(\frac{1}{m}\sqrt{\frac{2m}{n}}\right)}}$$

Then prove that $A:B:C=1$

Proposed by Hikmat Mammadov-Azerbaijan

U.2691 Prove that (where $m>0$ and $n\in\mathbb{R}$)

$$\int_{-\infty}^{\infty}\frac{\cos\left(\frac{n}{2}\ln(1+m^2x^2)\right)\cosh(n\cdot\tan^{-1}mx)}{1+x^2}dx=\pi\cos(n\cdot\ln(1+m))$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2692 If: $U_1+U_2+\cdots+U_n=\frac{(n+k+1)!}{(k+2)(n-1)!}$, $(k\in\{1,2,\dots,n\})$ then:

$$\frac{1}{U_1}+\frac{1}{U_2}+\cdots+\frac{1}{U_n}=\frac{1}{k}\cdot\left(\frac{1}{k!}-\frac{(n)!}{(n+k)!}\right)$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

U.2693 Let $\alpha = 1 + \sqrt[3]{2}, \beta = 2 + \sqrt[3]{2}, \gamma = 3 + \sqrt[3]{2}$. Find all functions

$f(x) = x^3 + ax^2 + bx + c\ (a, b, c \in \mathbb{R})$ such that $f(\alpha) = f(\beta) = f(\gamma) = 0$.

Proposed by Nguyen Van Canh -Vietnam

U.2694 Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = a > 0, \lim_{n\to\infty}\frac{b_{n+1}}{b_n\left(\sqrt[n]{a_n}\right)^2} = b$$

Calculate:

$$\lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{b_{n+1}}} - \frac{n^3}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2695 If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ are sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0, \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0, \lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_a} = c$$

Prove that:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n^2}}{n^2} + t\left(\frac{\sqrt[n]{n!}}{n}\right)^2\right)\cdot\left(\frac{\sqrt[n]{b_n^2}}{n^2} + t\left(\frac{\sqrt[n]{n!}}{n}\right)^2\right)\cdot\left(\frac{\sqrt[n]{c_n^2}}{n^2} + t\left(\frac{\sqrt[n]{n!}}{n}\right)^2\right) \geq \frac{3t^2}{4e^6}(a+b+c)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2696. Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty}(a_{n+1} - a_n) = a > 0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0$$

Calculate:

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{b_n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2697 In ΔABC the following relationship holds:

a. $11.\min\left\{\sum\frac{a}{b}, \sum\frac{b}{a}\right\} + \frac{R^{20}}{r^{20}} \geq 2^{20} + 11.\max\left\{\sum\sqrt[4]{\frac{a^4+b^4}{b^4+c^4}}, \sum\sqrt[3]{\frac{a^3+b^3}{b^3+c^3}}\right\}$

b. $\sqrt[3]{(a+b)(b+c)(c+a)} + \frac{R^4}{r^3} \geq 16r + \frac{2(a+b+c)}{3}$

Proposed by Nguyen Van Canh-Vietnam

U.2698 If $a, b, c, d > 0$, then prove that:

$$\left(\frac{a}{b}+1\right)\left(\frac{b}{c}+1\right)\left(\frac{c}{d}+1\right)\left(\frac{d}{a}+1\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

U.2699 In $\Delta ABC, AD, BE, CF$ – internal bisectors, R_A, R_B, R_C – circumradii of $\Delta AFE, \Delta BDF, \Delta CDE$. Prove that:

$$R_A^2 + R_B^2 + R_C^2 \leq \frac{3R^4}{16r^2}$$

Proposed by George Apostolopoulos – Greece

U.2700 If $x, y > 0$ then:

$$\sqrt{2x^2 - 3xy + 2y^2} \geq \frac{\sqrt{x^9} + \sqrt{y^9}}{\sqrt{x^7} + \sqrt{y^7}}$$

Proposed by Rahim Shahbazov-Azerbaijan

U.2701 In ΔABC the following relationship holds:

$$\sqrt[3]{\left(\frac{h_a}{r_a}\right)^2} + \sqrt[3]{\left(\frac{h_b}{r_b}\right)^2} + \sqrt[3]{\left(\frac{h_c}{r_c}\right)^2} \geq 3$$

Proposed by Rahim Shahbazov-Azerbaijan

U.2702 In ΔABC the following relationship holds:

$$\sqrt{\sum \frac{w_a}{w_b + w_c}} + \sqrt{\sum \frac{m_a}{m_b + m_c}} + \sqrt{\sum \frac{w_b + w_c}{w_a}} + \sqrt{\sum \frac{m_a^2}{m_b^2 + m_c^2}} \leq \frac{R\sqrt{6}}{r}$$

Proposed by Nguyen Van Canh-Vietnam

U.2703 In ΔABC the following relationship holds:

$$\frac{3R}{4r\sqrt[3]{abc}} \geq \frac{a}{c(a+b)} + \frac{b}{a(b+c)} + \frac{c}{b(c+a)} \geq \frac{3}{2\sqrt[3]{abc}}$$

Proposed by Nguyen Van Canh-Vietnam

U.2704 In $\Delta ABC, p = \frac{a+b+c}{2}$, prove that:

$$9 \leq \left(\sum \sqrt[3]{\frac{r_a}{r_b^2 + r_c^2}}\right)\left(\sum \sqrt[3]{\frac{r_b^2 + r_c^2}{r_a}}\right) \leq \frac{27R^2 + 8p^2}{36r^2}$$

Proposed by Nguyen Van Canh-Vietnam

U.2705 Find all values of $k \geq 0$ such that $(k+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq k\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) + 3, \forall \Delta ABC$.

Proposed by Nguyen Van Canh-Vietnam

U.2707 In ΔABC the following relationship holds:

$$2\sum_{cyc}\frac{h_a}{s-n_a}\geq\sum_{cyc}\frac{b+c-a}{a}\left(\frac{n_a}{h_a}+\frac{m_b}{b}+\frac{m_c}{c}\right)$$

Proposed by Bogdan Fuștei – Romania

U.2708 In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sqrt{\frac{R}{r}}\cdot\sqrt{\frac{h_a}{s-n_a}}\cdot\left(\sqrt{\frac{n_a}{h_a}+\frac{m_b}{b}+\frac{m_c}{c}}\right)^{-1}\geq\frac{AI}{2r}$$

Proposed by Bogdan Fuștei – Romania

U.2709 If ω – Brocard's angle $\Delta ABC, M\in Int(\Delta ABC)$ then:

$$\frac{MA}{h_b}+\frac{MB}{h_c}+\frac{MC}{h_a}\geq\frac{1}{\sin\omega}$$

Proposed by Bogdan Fuștei – Romania

U.2710 In acute ΔABC:

$$\frac{2R^2}{r^2}+1\leq\sum\frac{(1+\sec A)^2}{\tan^2 A}\leq\left(\frac{2R}{r}-1\right)^2$$

Proposed by Marin Chirciu – Romania

U.2711 Let $f(x)=x^3+ax^2+bx+1, g(x)=x^3+bx^2+ax+1\ \ (a,b\in\mathbb{Z})$

Find all values of $a,b\in\mathbb{Z}$ such that: $\max f(x)\leq\min g(x),\ \ \forall x\in[0,1]$

Proposed by Nguyen Van Canh-Vietnam

U.2712 Prove that for: $a_1,a_2,\dots,a_n\in\mathbb{R}; n\geq 2; q\in\left(1;\frac{4}{3};\frac{6}{5}\right)$

$$q\cdot(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n)\leq a_1^2+a_2^2+\cdots+a_n^2$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2713 Let $a,b,c>0$ such that $abc=1$. Prove that:

$$15+12(a^4+b^4+c^4)(a^2+b^2+c^2)\geq 11(a^6+b^6+c^6)+30(a^3+b^3+c^3)$$

Proposed by Nguyen Van Canh -Vietnam

U.2714 In ΔABC the following relationship holds:

$$\Omega_1\geq\Omega_2$$

$$\Omega_1 = \sqrt{\left(\sum_{cyc} n_a^2 n_b^2\right)\left(\sum_{cyc} n_a\right)^{-1}\left(\prod_{cyc}(n_a+n_b-n_c)\right)^{-1}}$$

$$\Omega_2 = \frac{1}{2}\max\left(\frac{n_a}{n_b}+\frac{n_b}{n_a}, \frac{n_b}{n_c}+\frac{n_c}{n_b}, \frac{n_c}{n_a}+\frac{n_a}{n_c}\right)$$

Proposed by Bogdan Fuștei – Romania

U.2715 In ΔABC. Prove that:

$$\sqrt{3\sum\frac{a}{b}}+\frac{R^2}{r^2} \ge 4+\max\left\{\sum\frac{a}{b}, \sum\frac{b}{a}\right\}$$

Propose by Nguyen Van Canh-Vietnam

U.2716 Let $a, b, c, d > 0$ such that $\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{c}}+\frac{c}{\sqrt{d}}+\frac{d}{\sqrt{a}} = 4$. Find the minimum values of expression:

$$P = \sum\frac{a^2}{b} - a^{2022}b^{2022}c^{2022}d^{2022}$$

Proposed by Nguyen Van Canh-Vietnam

U.2717 In $\Delta ABC, I$ – incenter. Prove that:

$$16\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\sum\cos\frac{A}{2} \le \sum\sin\left(\widehat{BIC}\right) < \frac{12R}{r}$$

Proposed by Nguyen Van Canh-Vietnam

U.2718 Let $a, b, c > 0$ such that $ab+bc+ca = 3$. Prove that:

$$\sum\frac{1}{a^2+b^2+2022} \le \frac{3}{2024}$$

Proposed by Nguyen Van Canh-Vietnam

U.2719 In $\Delta ABC, I_a, I_b$ and I_c are the excenters. Show that:

$$r_a\cdot\frac{I_bI_c}{AI_a}+r_b\cdot\frac{I_aI_c}{AI_c}+r_c\cdot\frac{I_aI_b}{AI_c} = a+b+c$$

Proposed by Ertan Yildirm-Turkiye

U.2720 Let be $t \in \mathbb{N}$ fixed and $x \in \mathbb{R}_+^*$. Calculate:

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{(2n+1)!!}\right)^{t+1}}\sum_{k=1}^{n}[k^t\cdot x]$$

where $[a]$ is the integer part of $a \in \mathbb{R}$.

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2721 Calculate:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{n\cdot\sqrt[n+1]{(n+1)!}}-\frac{n}{\sqrt[n]{n!}}\right)\cdot\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2722 If $(H_n)_{n\geq1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\geq1}$ is a sequence of real strictly positive numbers such that $a_{n+1}=a_n+e^{2H_n}\cdot\sin\frac{\pi}{n^2}, \forall n\in\mathbb{N}^*$. Calculate:

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{(2n-1)!!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2723 Let be $(r_n)_{n\geq1}, r_n>0, \forall n\in\mathbb{N}^*$ such that $\lim\limits_{n\to\infty}r_n=r>0$ and $(a_n)_{n\geq1}$ a sequence of real strictly positive numbers with the property that $a_{n+1}=a_n\cdot n^{t+1}r_n, \forall n\in\mathbb{N}^*$ where

$t\geq0$. Calculate:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t}-\frac{\sqrt[n]{a_n}}{n^t}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2724 Let be $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ sequences of real strictly positive real numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$$

Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}}}-\sqrt[n]{\frac{a_n}{b_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2725 Let be the sequence $(H_n)_{n\geq1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$. Calculate:

$$\lim_{n\to\infty}\left(H_n-\sqrt[n]{n!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2726 Let be $t,s>0$ and $(a_n)_{n\geq1}, (b_n)_{n\geq1}, (c_n)_{n\geq1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{t+s+1}}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t}=b>0,\ \lim_{n\to\infty}\frac{c_{n+1}}{c_n\cdot n^s}=c>0$$

Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}\cdot c_{n+1}}}-\sqrt[n]{\frac{a_n}{b_n\cdot c_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2727 Let be $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^2\cdot a_n}=a>0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$$

Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}}}-\sqrt[n]{\frac{a_n}{b_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2728 Let $(a_n)_{n\geq 1}$ sequence of real strictly positive numbers such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a>0$$

Calculate:

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2729 Let be $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}(a_{n+1}-a_n)=a>0 \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{a_n\cdot b_n}=b>0$$

Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2730 Let be $H_n=\sum_{k=1}^{n}\frac{1}{k},\forall n\in\mathbb{N}^*$. Calculate:

$$\lim_{n\to\infty}\left(\frac{n+1}{n\sqrt[n+1]{(n+1)!}}-\frac{n}{(n+1)\sqrt[n]{n!}}\right)e^{2H_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2731 If $(a_n)_{n\geq 1},(b_n)_{n\geq 1},(c_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0,\ \lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=c>0$$

Prove that:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n^2}}{n^2}+\frac{\sqrt[n]{(2n-1)!!}}{e_n^n}\right)\cdot\left(\frac{\sqrt[n]{e_n^2}}{n^2}+\frac{\sqrt[n]{(2n+1)!!}}{e_n^n}\right)\cdot$$

$$\cdot\left(\frac{\sqrt[n]{c_n^2}}{n^2}+\frac{\sqrt[n]{(2n+1)!!}}{e_n^n}\right)\geq\frac{3(a+b+c)^2}{e^6}$$

where $e_n=\left(\frac{n+1}{n}\right)^n, \forall n\geq 1$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2732 Calculate:

$$\lim_{n\to\infty}\left(\tan\frac{\pi\cdot\sqrt[n+1]{(n+1)!!}}{4\cdot\sqrt[n]{n!}}-1\right)\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2733 Let be $(H_n)_{n\geq1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$. Calculate:

$$\lim_{n\to\infty}\left(H_n-\ln\sqrt[n]{(2n-1)!!}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2734 Calculate:

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{(n+1)\sqrt[n]{n!}}\right)\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2735 Let be $(a_n)_{n\geq1}, (b_n)_{n\geq1}, (c_n)_{n\geq1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{n^4\cdot a_n}=a>0, \lim_{n\to\infty}\frac{b_{n+1}}{n^2\cdot b_n}=b>0, \lim_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n}=c>0$$

Calculate:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}\cdot c_{n+1}}}-\sqrt[n]{\frac{a_n}{b_n\cdot c_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Gabriela Bondoc-Romania

U.2736 Let be $t\in\mathbb{N}$ fixed, $(a_n)_{n\geq1}$ a sequence of real strictly positive numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$ and $x\in\mathbb{R}_+^*$. Calculate:

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{a_n}\right)^{t+1}}\cdot\sum_{k=1}^{n}[k^t\cdot x]$$

where $[a]$ is the integer part of $a \in \mathbb{R}$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2738 If $H_n = \sum_{k=1}^{n} \frac{1}{k}, \forall n \in \mathbb{N}^*$. Calculate:

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)e^{2H_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2739 Let be $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that $\lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$. Calculate:

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{n \cdot \sqrt[n+1]{(n+1)!}} - \frac{n}{\sqrt[n]{n!}}\right) \cdot a_n$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2740 Let be $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!} \cdot e^{H_n}} = a > 0$$

Compute:

$$\lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.2741 Calculate:

$$\lim_{n\to\infty}\left(\frac{n+1}{n \cdot \sqrt[n+1]{(n+1)!}} - \frac{n}{(n+1)\sqrt[n]{n!}}\right) \cdot \left(\sqrt[n]{n!}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2742 Let be $t \geq 0$ and $(a_n)_{n\geq 1}(b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{t+1}} = a > 0, \lim_{n\to\infty} \frac{b_{n+1}}{b_n \cdot n^t} = b > 0$$

Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}}{b_{n+1}}} - \sqrt[n]{\frac{a_n}{n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2743 Let be $t \geq 0$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^{t+1}} = a > 0$$

Find:

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{a_n}}{n^t} \right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2744 If $(a_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that: $\lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$. Find:

$$\lim_{n\to\infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{(n+1)\sqrt[n]{n!}} \right) a_n$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2745. If $(a_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that:

$$\lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$$

Find:

$$\lim_{n\to\infty} \left(\frac{n+1}{n\sqrt[n+1]{(n+1)!}} - \frac{n}{(n+1)\sqrt[n]{n!}} \right) a_n^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2746 Find without any software:

$$\int_0^\infty \frac{x^2 \ln x}{(x^2+1)^2} dx$$

Proposed by Jalil Hajimir-Canada

U.2747 Solve for real numbers:

$$2\tan^{-1}(x + [x] + 1) = \tan^{-1}(2x + 1) + \tan^{-1}(2[x] + 1)$$

Proposed by Jalil Hajimir-Canada

U.2748 Prove without any software:

$$\frac{\pi}{2} < \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin^5 x + \cos^5 x} dx < 2$$

Proposed by Jalil Hajimir-Canada

U.2749 Find the general solution of:

$$\frac{dy}{dx} = 4 + 3x^2 y$$

Proposed by Jalil Hajimir-Canada

U.2750 Solve for real numbers:

$$x[\log[x]] = \frac{\sin \pi x}{\sqrt{x+1}}, \quad [*] \text{ is the greatest integer part of } *$$

Proposed by Jalil Hajimir-Canada

U.2751 Find the general solution for:

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

Proposed by Jalil Hajimir-Canada

U.2752 Find $U(x, y)$:

$$\begin{cases} \sqrt{4 + x^2}\, U'_x + U'_x = 1 \\ U(x, 0) = x \end{cases}, x, y \in \mathbb{R}$$

Proposed by Jalil Hajimir-Canada

U.2753 Solve the following integral equation:

$$y(t) = e^{-t} + 4\int_0^t \cos(2t - 3x)\, y(x) dx$$

Proposed by Jalil Hajimir-Canada

U.2754 Find $f(x)$ if: $\int_0^\infty x f(x) \cos(tx)\, dx = -\int_0^\infty f(x) \sin(tx)\, dx$ and $f(1) = 1$

Proposed by Jalil Hajimir-Canada

U.2755 Prove without any software:

$$\int_0^\pi \frac{4 \sin^3 x}{5 - \cos x} dx > 1$$

Proposed by Jalil Hajimir-Canada

U.2756 Find the general form of solutions:

$$\frac{dy}{dx} = \frac{1}{1 + 2xe^{x-y}}$$

Proposed by Jalil Hajimir-Canada

U.2757 Prove that:

$$\tan x \tan^{-1} x \geq x[x], \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Proposed by Jalil Hajimir-Canada

U.2758 Prove that:

$$\sum_{k=1}^{\infty} \left\{ \frac{k^2-1}{(k^2+1)^2} + \frac{k^2-1}{(k^2+1)^2} \cdot e^{-\pi}(-1)^{k+1} \right\} = \frac{e^{\pi}-1}{2e^{\pi}} - \frac{\pi^2}{e^{2\pi}-1}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2759 If $\Omega = \lim\limits_{m\to\infty} \left(\frac{(m+1)\cdot(m+2)\cdot\ldots\cdot(m+2m)}{m^{2m}} \right)^{\frac{1}{m}}$ then prove that $\Omega = \frac{27}{e^2}$

Proposed by Hikmat Mammadov-Azerbaijan

U.2760 Find a closed form:

$$\Omega = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!\,(n+1)!\,(2n+1)(2n+3)(2n+5)(2n+7)} \psi\left(n+\frac{1}{2}\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2761 If $\alpha;\beta > 0$ and $1-\alpha < \gamma < 1+\min(\alpha;\beta)$ find:

$$J = \int_0^{\infty} \frac{\cos(z^{\alpha}) - e^{-z^{\beta}}}{z^{1+\gamma}} dz$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2762 If $\Omega(m;n;r) = \int_0^1 \frac{mr \tan^{-1} \frac{1}{n\sqrt{1+m^2x^2}}}{(1+(1+r^2)m^2x^2)\sqrt{1+m^2x^2}} dx$ then:

$$\Omega(m;n;r) + \Omega\left(\frac{1}{n};\frac{1}{m};\frac{1}{r}\right) = \tan^{-1}\frac{\sqrt{1+r^2}}{nr} \tan^{-1} m\sqrt{1+r^2}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2763 Prove that:

$$\int_0^1 \frac{\tan^{-1} \frac{2\sqrt{1+2x^2}}{\sqrt{5}(1+x^2)}}{\sqrt{1+2x^2}(1+3x^2)} dx = \frac{\pi}{2} \tan^{-1} \sqrt{\frac{3}{5}} - \frac{\pi^2}{15}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2764 Given:

$$Q=\sum_{n=0}^{\infty}\int_0^1 \frac{2x^{n-\frac{1}{2}}\ln(x)}{(\ln^2 x+\pi^2)}dx$$

and

$$P=\pi\int_0^1 \frac{1}{x(\ln^2 x+\pi^2)}dx$$

Then prove that: $Q+P=1$

Proposed by Hikmat Mammadov-Azerbaijan

U.2765 Prove that:

$$\lim_{n\to\infty}\frac{\prod_{m=1}^{n}(n^6+m^6)}{2^n\left(2+\sqrt{3}\right)^{\sqrt{3}n}n^{6n}e^{(\pi-6)n}}=\sqrt{2}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2766 Find a closed form:

$$\Omega=\int\left(\int e^x\left(\frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}\right)dx\right)dx$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2767

$$\Omega=\int_0^{\pi}\cos\left(nx-m\arctan\left(\frac{z\sin x}{1+z\cos x}\right)\right)(1+2z\cos x+z^2)^{-\frac{m}{2}}\left(2\cos\frac{x}{2}\right)^{2n}dx$$

Prove that: $\Omega=\pi$

Proposed by Hikmat Mammadov, Nazenin Eyvazova-Azerbaijan

U.2768 Find:

$$\Omega=\int_0^1 \frac{x\ln^2 x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2769 If we have the function

$$T(x)=\left(1-x\tan^{-1}\left(\frac{1}{x}\right)\right)\tan^{-1}(x)$$

then prove that

$$\int_0^{\infty}T(x)\,T\left(\frac{1}{x}\right)\frac{dx}{x}$$

$$= 3\zeta(2)\log(2) - \frac{7\zeta(3)}{2} - \frac{21}{8}\zeta(2)\zeta(3) + \frac{93\zeta(5)}{16}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2770 Prove that:

$$\int_{-\infty}^{+\infty}\int_{1}^{\infty}\frac{\ln^3(x)}{(x^3-2x^2+x)(1+y+y^2)^2}\,dx\,dy = \frac{8}{\sqrt{3}}\pi\left(\zeta(3)-\frac{\pi^4}{90}\right)$$

Proposed by Memmedov Cosqun – Azerbaijan

U.2771 Prove the below closed form

$$\int\int\int_{[0,1]^3}\ln\left(\frac{1}{xy+yz+zx}\right)dx\,dy\,dz = Li_2\left(\frac{1}{4}\right) + 2\ln^2(2) - 3\ln 3 + \frac{8}{3}$$

Where, $Li_2(z)$ is the spence or dilogarithm function

Proposed by Ankush Kumar Parcha-India

U.2772 If $n \in \mathbb{N}, n \geq 2$ then:

$$e^{\frac{(n-2)(n+3)}{2}} \geq \left(\frac{n!}{2}\right)^e$$

Proposed by Khaled Abd Imouti-Syria

U.2773 Prove the below closed form

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} xyz\ln\left(\frac{\sin(x)}{\cos(y)}\tan(z)\right)dx\,dy\,dz = \frac{7\pi^4}{256}\zeta(3)$$

Where, $\zeta(3)$ is an Apery's constant.

Proposed by Ankush Kumar Parcha-India

U.2774 Prove that:

$$\frac{\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{5n+2}}-\frac{1}{\sqrt{5n+3}}\right)(-1)^{\frac{n}{2}(n+1)}}{\sum_{n=0}^{\infty}\left(\frac{1}{\sqrt{5n+1}}-\frac{1}{\sqrt{5n+4}}\right)(-1)^{\frac{n}{2}(n+1)}} = \frac{1}{4}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2775 Prove that:

$$\int_0^1\int_0^1\frac{\arctan(x)\ln(xy)}{(1+x)^2(1+y)}\,dx\,dy = \frac{\ln(2)}{96}\left(48G - 12\pi\ln(2) - 5\pi^2\right)$$

(G – Catalan's constant)

Proposed by Memmedov Cosqun – Azerbaijan

U.2776 If p is an odd integer, then find the possible closed form of the following:

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)16^n}\binom{2n}{n}\binom{2n+1+p}{n+\frac{1+p}{2}}$$

Proposed by Narendra Bhandari – Nepal

U.2777 Find:

$$\Omega=\int_0^{\infty}\frac{\arctan(x)}{x(x^3+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2778 Find all values of $a\in\mathbb{R}$ such that

$$\int_{2023}^{+\infty}\frac{dx}{ax^4-x^2+2023a}<+\infty$$

Proposed by Nguyen Van Canh – Vietnam

U.2779 Find:

$$\Omega=\int_0^{\infty}\frac{\arctan(x)}{x(x+1)^3}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2780 Prove that:

$$\Delta=\int_0^{\infty}\frac{x\ln^2(1+x)}{(1+x)(2+x)^3}dx=\frac{1}{6}(2\pi^2-9\zeta(3)-6\ln(4))$$

Proposed by Shirvan Tahirov-Azerbaijan

U.2781 Prove that:

$$\int_0^{\frac{\pi}{4}}\ln(\cos x)\cos^{\mu-1}2x\tan 2x\,dx=\frac{\beta(\mu)}{4(1-\mu)}\quad(\mu>0)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2782 Prove that:

$$\int_0^1\left(\frac{\tan^{-1}\frac{1}{\sqrt{2+x^3}}}{\sqrt{2+x^2}}-\frac{\tan^{-1}\frac{1}{\sqrt{3+x^3}}}{\sqrt{3+x^2}}+\frac{\tan^{-1}\frac{1}{\sqrt{3+x^3}}}{(1+x^2)(2+x^2)\sqrt{3+x^2}}\right)dx+$$

$$+\frac{1}{2}\int_0^1\frac{\tan^{-1}\frac{1}{\sqrt{3+x}}\sin^{-1}\frac{1-x}{1+x}}{(3+x)\sqrt{3+x}}dx=\frac{\pi^2}{12\sqrt{3}}+\frac{\pi}{4}\ln\frac{9}{8}-\frac{\pi}{4\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2783 Find a closed form of:

$$\Omega=\sum_{n=1}^{\infty}\left((4n^2-4n+1)\tanh^{-1}\left(\frac{1}{32n^2-32n+7}\right)-\frac{1}{8}\right)$$

Proposed by Toubal Fethi -Algeria

U.2784 Prove that:

$$\frac{1}{3}\int_0^{\infty} t\,\phi\left(-\frac{t}{\pi}\right)dt=-\zeta(2),$$

where $\phi(-z)$ is the exponential integral defined as $\phi(-z)=-\int_z^{\infty}\frac{e^{-y}}{y}dy$.

Proposed by Said Attaoui –Algeria

U.2785 $N(t)\sim$ Poisson (λt), $X_1, X_2 \dots X_{N(t)} \sim N(\mu,\sigma^2)$, $X_{(m)}$ is the m^{th} order statistic & $EX_{(m)}$ is its expectation define: when $N(t)<m, X_{(m)}=0$. Prove: $EX_{(m)}\sim\sigma\sqrt{2\ln(\lambda t)}$ as $t\to\infty$

Proposed by Hikmat Mammadov-Azerbaijan

U.2786 Prove that:

$$\int_0^{\infty}\int_0^{\infty}\frac{e^{-x}x^4\ln(x)\,(\ln(y)-\ln(x))}{(x^2+y^2)^2}dx\,dy=\frac{\pi}{4}(\gamma-1)$$

Where γ is Euler – Mascheroni constant

Proposed by Cosqun Memmedov-Azerbaijan

U.2787 Prove the following:

$$\int_0^1\frac{Li_2(x)\ln^2(x)}{x(1-x)}dx=\pi^2\zeta(3)-9\zeta(5)$$

$$\int_0^1\frac{Li_3(x)\ln^2(x)}{x(1-x)}dx=\zeta^2(3)+\frac{\pi^6}{945}$$

$$\int_0^1\frac{Li_4(x)\ln^2(x)}{x(1-x)}dx=\frac{\pi^4}{45}\zeta(3)+\frac{10\pi^2}{3}\zeta(5)-34\zeta(7)$$

Proposed by Vincent Nguyen-USA

U.2788 Prove that:

$$\Omega=\int_0^1\int_0^{\infty}\frac{\ln^2(x)\ln^2(1+y^2)}{y(1+x^2)}dx=\frac{\pi^3\zeta(3)}{64}$$

Proposed by Shirvan Tahirov-Azerbaijan

U.2789 Find:

$$I=\int_0^p\int_0^1\frac{\arcsin(xy)\arccos(xy)}{y}\,dx\,dy$$

Proposed by Togrul Ehmedov-Azerbaijan

U.2790 Find:

$$\Omega=\int_0^\infty\frac{\arctan(x)}{x(x+1)^2}\,dx$$

Proposed by Vasile Mircea Popa – Romania

U.2791 Compute:

$$I=\int_{\frac{1}{2023}}^{\frac{1}{2022}}\sqrt{2022+\sqrt{1-x^2}}\,dx$$

Proposed by Nguyen Van Canh-Vietnam

U.2792 Let:

$$\Omega_1=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}e^{-\pi(2k-1)}\cos\left(\frac{\pi}{4}(2k-1)\right)}{1-e^{-\pi(2k-1)}}$$

and

$$\Omega_2=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}e^{-\pi k}\sin^2\left(\frac{\pi k}{4}\right)}{k(1+e^{-\pi k})}$$

Prove that:

$$\ln\left(1+2\sqrt{2}\Omega_1\right)+4\Omega_2=\ln\left(\frac{2\Gamma^2\left(\frac{5}{4}\right)}{\sqrt{\pi^3}}-\frac{\sqrt{2}\Gamma\left(\frac{5}{4}\right)}{\sqrt[4]{\pi^3}}+\frac{1}{4}\right)$$

Proposed by Toubal Fethi -Algeria

U.2793 Find the value of α.

$$\int_0^\infty\int_0^\infty\int_0^\infty xy^2z^3\phi_1\left(-\frac{x}{\pi}\right)\phi_2\left(-\frac{2y}{\pi^2}\right)\phi_3\left(-\frac{3z}{\pi^3}\right)dx\,dy\,dz=$$

$$=\frac{\pi^{2\alpha^2+\alpha-1}\Gamma\left(\frac{1}{\alpha}\right)}{\alpha^5(5\alpha+1)\sqrt[\alpha]{\alpha}}\sqrt{\frac{\pi}{\alpha-1}}$$

If $\phi_1(-x):=\int_x^\infty\frac{e^{-t}}{t}dt,\phi_2(-y):=(2!)^2\int_y^\infty\frac{e^{-2t^2}}{t}dt,\phi_3(-z):=(3!)^3\int_z^\infty\frac{e^{-3t^3}}{t}dt$

Proposed by Ankush Kumar Parcha-India

U.2794 Find a closed form:

$$\Omega(x,p)=\sum_{n=0}^{\infty}\left(\frac{(n+p)(n+p-1)\cdot\ldots\cdot(p-1)}{n!}\left(1-\frac{2}{1+e^{x}}\right)^{n}\right)$$

Proposed by Khaled Abd Imouti -Syria

U.2795 Let be $t\in\mathbb{N}$ fixed, then calculate:

$$\lim_{n\to\infty}\frac{1}{\left(\sqrt[n]{n!}\right)^{t+1}}\left(\sum_{k=1}^{n}[k^{t}\cdot x]\right)$$

where $x\in\mathbb{R}_{+}^{*}$, where $[a]$ is the integer part of $a\in\mathbb{R}$.

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2796 In triangle ABC with the semiperimeter s and the area F the following inequality holds:

$$\frac{a^{2023}+b^{2023}}{(s-c)^{2023}}+\frac{b^{2023}+c^{2023}}{(s-a)^{2023}}+\frac{c^{2023}+a^{2023}}{(s-b)^{2023}}\geq 3\cdot 2^{2024}$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.2797 If $a,b,m,x,y,z>0$, then:

$$\left(\frac{mx^{2}}{(ay+bz)+1}\right)\cdot\left(\frac{my^{2}}{(az+bx)^{2}}+1\right)\cdot\left(\frac{mz^{2}}{(ax+by)^{2}}+1\right)\geq\frac{27m}{4(a+b)^{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2798 Let be $x,y>0$ then in triangle ABC with the area F the following inequality holds:

$$\frac{1}{b(ax+cy)}+\frac{1}{c(bx+ay)}+\frac{1}{a(cx+by)}+(a+b+c)\geq\frac{216}{\sqrt{x+y}}F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.2799 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$\frac{x^{2}a^{3}}{h_a}+\frac{y^{2}b^{3}}{h_b}+\frac{z^{2}c^{3}}{h_c}\geq\frac{8}{3}(xy+yz+zx)\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.2800 If ABC is a triangle in which we denote by F the area, then:

$$(a^{2}b^{2}c^{4}+a^{2}b^{2}+c^{4}+1)\cdot(b^{2}c^{2}a^{4}+b^{2}c^{2}+a^{4}+1)\cdot$$

$$\cdot(c^{2}a^{2}b^{4}+c^{2}a^{2}+b^{4}+1)\geq 1296F^{4}$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.2801 If $m \geq 0$ and ABC is a triangle, then the following inequality holds:

$$\frac{a^{m+1}+b^{m+1}}{(a+b-c)^{m+1}}+\frac{b^{m+1}+c^{m+1}}{(-a+b+c)^{m+1}}+\frac{c^{m+1}+a^{m+1}}{(a-b+c)^{m+1}} \geq 24$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.2802 If $m \geq 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{a^{m+1}\cdot b}{h_b^m}+\frac{b^{m+1}\cdot c}{h_c^m}+\frac{c^{m+1}\cdot a}{h_a^m} \geq 2^{m+2}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu,Flavius Cristian Verde-Romania

U.2803 If $x, y > 0$, then in ΔABC with the area F the following inequality holds:

$$\frac{a^4}{xh_b^2+ym_c^2}+\frac{b^4}{xh_c^2+ym_a^2}+\frac{c^4}{xh_a^2+ym_b^2} \geq \frac{16\sqrt{3}}{3(x+y)}F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.2804 If $a, b, x, y, z > 0$, then:

$$\left(\frac{x^2}{(ay+bz)^2}+1\right)\cdot\left(\frac{y^2}{(az+bx)^2}+1\right)\cdot\left(\frac{z^2}{(ax+by)^2}+1\right) \geq \frac{27}{4(a+b)^2}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.2805 Let be ABC a triangle in which we denote by F its area, then:

$$(a^4b^2c^2+2b^2c^2+a^4+2)\cdot(b^4c^2a^2+2c^2a^2+b^4+2)\cdot$$
$$\cdot(c^4a^2b^2+2a^2b^2+c^4+2) \geq 5184F^4$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SPRING 2025

PROBLEMS FOR JUNIORS

JP.526 Let $a, b, c \in (0,1)$. Prove that:

$$\frac{b+c}{1-a}+\frac{c+a}{1-b}+\frac{a+b}{1-c} \geq \frac{2\sqrt{ab}}{1-\sqrt{ab}}+\frac{2\sqrt{bc}}{1-\sqrt{bc}}+\frac{2\sqrt{ca}}{1-\sqrt{ca}}$$

Proposed by Marin Chirciu – Romania

JP.527 If $x, y, z > 0$ with $x+y+z=1$ and $\lambda \geq 2$ then:

$$\frac{1}{x^3+y^3+z^3}+\frac{\lambda}{xy+yz+zx} \geq 3(\lambda+3)$$

Proposed by Marin Chirciu – Romania

JP.528 If $a, b, c > 0$ such that $a+b+c=3$ and $0 \leq \lambda \leq \frac{1}{2}$ then

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}+\frac{1}{c^3+\lambda} \geq \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

JP.529 If $a, b, c > 0; x \in \mathbb{R}$ then:

$$\frac{a}{(b\sin^2 x+c\cos^2 x)^3}+\frac{b}{(c\sin^2 x+b\cos^2 x)^3}+\frac{c}{(a\sin^2 x+c\cos^2 x)^3} \geq \frac{27}{(a+b+c)^2}$$

Proposed by Daniel Sitaru – Romania

JP.530 In ΔABC, $O-$ circumcenter. A_1, B_1, C_1 are the intersection points of AO, BO, CO with BC, AC and AB respectively. R_1, R_2 and R_3 are circumradii of $\Delta BOC, \Delta AOC$ and ΔAOB respectively. Show that:

$$R\left(\frac{1}{OA_1}+\frac{1}{OB_1}+\frac{1}{OC_1}\right)+3=\frac{4F}{R^2}\left(\frac{R_1}{BC}+\frac{R_2}{AC}+\frac{R_3}{AB}\right)$$

Proposed by Ertan Yildirim-Turkiye

JP.531 If $a, b, c > 0$ and $abc = 1$ then:

$$\left(\frac{a}{2}+\frac{b}{c}\right)^3+\left(\frac{b}{2}+\frac{c}{a}\right)^3+\left(\frac{c}{2}+\frac{a}{b}\right)^3 \ge 3\sqrt[4]{18}$$

Proposed by Khaled Abd Imouti-Syria

JP.532 In any $\Delta ABC, I -$ incenter, $r -$ radii, $R -$ circumradius, $s -$ semiperimeter, the following relationship holds:

$$AB + BI + CI \le 2(R + r)$$

Proposed by Marian Ursărescu – Romania

JP.533 Let be the triangle ABC with $AD, BE, CF -$ altitudes and H – orthocenter. Prove that:

$$\frac{HA}{HD}+\frac{HB}{HE}+\frac{HC}{HF} \ge 2\left(\left(\frac{R}{r}\right)^2 - 1\right)$$

Proposed by Marian Ursărescu – Romania

JP.534 In $\Delta ABC, I -$ incenter and D, E, F the points of contact of the cevians AI, BI, CI with the circle, then the following relationship holds:

$$ID + IE + IF \le \frac{2(R^2 - Rr + r^2)}{r}$$

Proposed by Marian Ursărescu – Romania

JP.535 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a^4 + r_b^2 r_c^2}{r_b^2 + r_c^2} \ge s^2$$

Proposed by Marian Ursărescu – Romania

JP.536 In △ABC the following relationship holds:

$$\frac{2R}{r} \ge \frac{(4R + r)^2}{s^2} + 1$$

Proposed by Alex Szoros-Romania

JP.537 Find the angles of a triangle ABC if

$$\frac{\sin A + 2\sin B}{\sqrt{\sin^2 B + \sin^2 C + 2\cos A \sin B \sin C}} + 1 = \frac{3\sqrt{3}}{2\sin C}$$

Proposed by Cristian Miu – Romania

JP.538 In △ABC the following relationship holds:

$$\frac{3}{2R} \le \sum \frac{\cos^2 \frac{A}{2}}{h_a} \le \frac{3}{4r}$$

Proposed by Alex Szoros-Romania

JP.539 In $\Delta ABC, O \in (AB), OQ \parallel BC$, where $Q \in (AC). P \in (OC)$ such that $RP \parallel BC$, where $R \in (AC)$ and $T \in (AB)$. If the lengths of the segment RT is the geometric mean of the lengths of the segments OQ and BC, then $OP < \frac{OC}{2}$.

Proposed by Gheorghe Molea – Romania

JP.540 Let be ΔACD with $m(\widehat{CAD}) > 90°, B \in (CD)$, such that $m(\widehat{BAC}) = 90°$ and $AC > AB$. The bisector $\widehat{ACD}$ intersects AD in E. If $(BE$ is the bisector $\widehat{ABD}$, prove that:

$$\frac{1}{AD} = \frac{\sqrt{2}}{2}\left(\frac{1}{AB} - \frac{1}{AC}\right)$$

Proposed by Gheorghe Molea – Romania

PROBLEMS FOR SENIORS

SP.526 If $a, b, c, \lambda > 0, a + b + c = \lambda$ then:

$$\sum \sqrt{\frac{bc}{a} + \lambda} \ge 2(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

Proposed by Marin Chirciu – Romania

SP.527 If $x, y, z \ge 0$ with $x + y + z = 1$ and $0 \le \lambda \le \frac{9}{4}$ then

$$xy + yz + zx - \lambda xyz \le \frac{9 - \lambda}{27}$$

Proposed by Marin Chirciu – Romania

SP.528 If $a, b, c \ge 1$ then:

$$\frac{1}{9}(a + b + c) + \frac{1}{3\sqrt{2}} \ge \frac{\sqrt[3]{ab - 1}}{b + c + \sqrt{2}} + \frac{\sqrt[3]{bc - 1}}{c + a + \sqrt{2}} + \frac{\sqrt[3]{ca - 1}}{a + b + \sqrt{2}}$$

Proposed by Marin Chirciu – Romania

SP.529 Let ABC be a triangle with inradius r and circumradius R and let the interior points D, E, F be chosen on the sides BC, CA, AB respectively, so that AD, BE, CF are the bisectors of the triangle ABC.Let r_A, r_B, r_C be the inradii of the triangles AEF, BFD, CDE respectively. Prove that:

$$r_A^2 + r_B^2 + r_C^2 \le \frac{3R^4}{64r^2}$$

Proposed by George Apostolopoulos– Greece

SP.530 Let a, b, c be the lengths of the sides of a triangle with inradius circumradius R. Prove that:

$$\frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)R}} \le \frac{1}{\sqrt[n]{m\cdot a+b}} + \frac{1}{\sqrt[n]{m\cdot b+c}} + \frac{1}{\sqrt[n]{m\cdot c+a}} \le \frac{3^{1-\frac{1}{2n}}}{\sqrt[n]{(m+1)\cdot 2r}}$$

for all integers $m \ge 0$ and $n \ge 1$.

Proposed by George Apostolopoulos– Greece

SP.531 If $a, b, c \ge 1$**, then:**

$$\sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc} \ge \sqrt{\frac{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}{3}} - \frac{1}{\sqrt[3]{abc}}$$

Proposed by Vasile Mircea Popa-Romania

SP.532 Prove that in any right triangle with the cathetus b and c we have the inequality:

$r \le \frac{2-\sqrt{2}}{4}(b+c)$, where r is the inradii of the triangle.

Proposed by Laura Molea and Gheorghe Molea – Romania

SP.533 Prove that $k = \frac{4}{5}$ is the largest positive value of the constant k such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e} - 5 \ge k(a+b+c+d+e-5)$$

for any positive real numbers a, b, c, d, e satisfying $ab+bc+cd+de+ea=5$

Proposed by Vasile Cîrtoaje – Romania

SP.534 If the lengths a, b, c of the sides of a triangle are the roots of the equation

$kx^3 - lx^2 + 9kx - l = 0 \quad (k\cdot l \ne 0)$, then find the area of the triangle.

Proposed by George Apostolopoulos–Greece

SP.535 Determine all the numbers $\overline{abcd}$ such that:

$$1 + a + b + c + a \cdot b + b \cdot c + c \cdot a = a \cdot b \cdot c \cdot d$$

Proposed by Neculai Stanciu, Titu Zvonaru – Romania

SP.536 Let be the triangle ABC with AD, BE, CF – altitudes and H – orthocenter. Prove that:

$$\frac{HA}{HD} + \frac{HB}{HE} + \frac{HC}{HF} \geq 2\left(\left(\frac{R}{r}\right)^2 - 1\right)$$

Proposed by Marian Ursărescu – Romania

SP.537 Solve for real numbers:

$$3e^x + 3e^{3x} + 1 = 4e^{2x} + 3 \cdot \sqrt[3]{e^{4x}}$$

Proposed by Daniel Sitaru – Romania

SP.538 In acute ΔABC the following relationship holds:

$$36 \leq 4\left(\sum_{cyc} \tan A \tan B\right) \leq 9 + \prod_{cyc} \tan^2 A$$

Proposed by Daniel Sitaru – Romania

SP.539 If $a > 0; f: \mathbb{R} \to \mathbb{R}$ is a function such that:

$$f\left(ax - \frac{1}{a}\right) \leq ax \leq f(x) - 1; (\forall)x \in \mathbb{R} \text{ then:}$$

$$f(2) + f(4) + f(8) > \frac{12\sqrt{a}}{a}$$

Proposed by Daniel Sitaru – Romania

SP.540 If $x, y \in [3, 4]; z, t \in [1, 12]$ then:

$$(x + y + z + t)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{t}\right) \leq \frac{100}{3}$$

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

UP.526 Prove the identity:

$$\int_0^\infty \frac{|\sin(x)|}{1+x^2}dx = 1 - 2\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)e^{2n}}$$

Proposed by Vasile Mircea Popa – Romania

UP.527 Prove the closed form:

$$\int_0^\infty \frac{\ln x}{x^3 + x\sqrt{x} + 1}dx = -\frac{32\pi^2}{81}\sin\frac{\pi}{18}$$

Proposed by Vasile Mircea Popa – Romania

UP.528 If $a_n > 0; r_n > 0; a_{n+1} = a_n + n \cdot r_n; n \in \mathbb{N}^*$ **and** $\lim_{n\to\infty} r_n = r > 0$

then find: $\Omega = \lim_{n\to\infty}(2H_n - \log a_n)$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP.529 If $a_n > 0; n \in \mathbb{N}^*; \lim_{n\to\infty}(a_{n+1} - a_n) = a > 0$ **then find:**

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{n!}} - \frac{1}{\sqrt[n+1]{(n+1)!}}\right)\cdot a_n^2$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP.530 Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[n]{(2n-1)!!}} - \frac{1}{\sqrt[n+1]{(2n+1)!!}}\right)\cdot e^{2H_n}$$

Proposed by D.M. Bătinețu-Giurgiu, Daniel Sitaru – Romania

UP.531 Prove that:

$$\int_0^\infty \frac{x^2\sinh(2x)}{\cosh^2(2x)}dx = \frac{3\pi^3}{128}$$

Proposed by Said Attaoui – Algeria

UP.532 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}; f(0) = 0$ **such that:**

$$f(x) = f\left(\frac{x}{5}\right) + \frac{x}{7}; (\forall)x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

UP.533 Calculate the integral:

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{4x^4 + x^2 + 4}} dx$$

Proposed by Vasile Mircea Popa – Romania

UP.534 If $a, b > 0$ then:

$$\int_a^b \int_a^b \left(\frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2}\right) dxdy \leq \ln^2\left(\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

UP.535 If $x, y, z > 1; x \neq y \neq z \neq x$ and $\log_{\frac{y}{z}} x + \log_{\frac{z}{x}} y + \log_{\frac{x}{y}} z = 0$

then:

$$\frac{\log_2 x}{\log_2^2\left(\frac{y}{z}\right)} + \frac{\log_2 y}{\log_2^2\left(\frac{z}{x}\right)} + \frac{\log_2 z}{\log_2^2\left(\frac{x}{y}\right)} = 0$$

Proposed by Daniel Sitaru – Romania

UP.536 If $1 < a \leq b; m \geq 1$ then:

$$\frac{(b-1)^{m+1} - (a-1)^{m+1}}{m+1} + b - a \leq \frac{b^{m+1} - a^{m+1}}{m+1}$$

Proposed by Daniel Sitaru – Romania

UP.537 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n \cdot 8^n} \binom{4n}{n} \binom{4n}{2n} \binom{3n}{n}^{-2}$$

Proposed by Daniel Sitaru-Romania

UP.538 Solve for real numbers:

$$\int_4^x \frac{t^2 + 1}{t^3 + 1} dt = 2(\sqrt{x} - 2)$$

Proposed by Daniel Sitaru-Romania

UP.539 If $0 < a \leq b$ then:

$$\int_a^b e^{x^2} dx \geq (b - a) \cdot \sqrt[3]{a^2 + ab + b^2}$$

Proposed by Daniel Sitaru – Romania

UP.540 If $a, b \in \mathbb{R}, a \leq b, f: [a, b] \to (0, \infty), f -$ continuous then:

$$3\int_a^b f(x)\,dx + \left(\int_a^b \frac{1}{f(x)}\,dx\right)^3 \geq 4(b-a)$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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