ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY
Mehedinți Branch

R. M. M. - 45

ROMANIAN MATHEMATICAL MAGAZINE

SUMMER EDITION 2025

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Nr.45-SUMMER EDITION 2025

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CONTENT

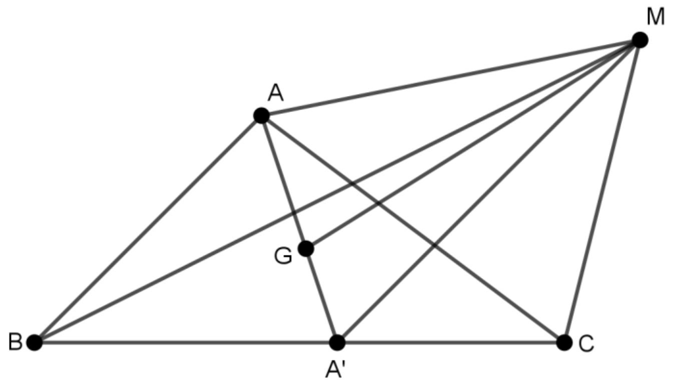
A NEW PROOF FOR IONESCU – WEITZENBOCK'S INEQUALITY USING THE TORICELLI'S POINT

By Daniel Sitaru, Claudia Nănuți -Romania

Abstract: In this paper we will find the distance between the Toricelli's point and the centroid in any triangle and we will use this to prove the Ionescu – Weitzenbock's inequality.

Lemma: Let M be an arbitrary point in the plane of ΔABC and G the centroid of ΔABC. In these conditions: $MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2$.

Proof:



$A'B = A'C; AA'$ - median in ΔABC; $AB = c; BC = a; CA = b; AA' = m_a$

$GA = \frac{2}{3}m_a; GA' = \frac{1}{3}m_a; G$ – centroid, $A'B = A'C = \frac{a}{2}$, MA' - median in ΔMBC:

$$\boldsymbol{MA'^2 = \frac{1}{2}(MB^2 + MC^2) - \frac{1}{4}BC^2} \quad \textbf{(1)}$$

We will use Stewart's theorem in $\Delta MAA'$:

$$MG^2 \cdot AA' = MA^2 \cdot GA' + MA'^2 \cdot GA - GA \cdot GA' \cdot AA'$$

$$MG^2 \cdot m_a = MA^2 \cdot \frac{1}{3}m_a + MA'^2 \cdot \frac{2}{3}m_a - \frac{2}{3}m_a \cdot \frac{1}{3}m_a \cdot m_a$$

$$MG^2 = \frac{1}{3}MA^2 + \frac{2}{3}MA'^2 - \frac{2}{9}m_a^2$$

By (1):

$$MG^2 = \frac{1}{3}MA^2 + \frac{2}{3}\left(\frac{1}{2}(MB^2 + MC^2) - \frac{1}{4}BC^2\right) - \frac{2}{9}m_a^2$$

$$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{1}{2}BC^2 - \frac{2}{3}\left(\frac{b^2 + c^2}{2} - \frac{a^2}{4}\right)$$

$$3MG^2 = MA^2 + MB^2 + MC^2 - \frac{a^2}{2} - \frac{b^2+c^2}{3} + \frac{a^2}{6}$$

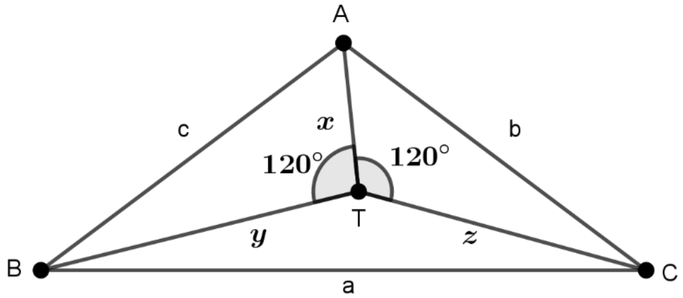
$$\boldsymbol{MA^2 + MB^2 + MC^2 = 3MG^2 + \frac{1}{3}(a^2+b^2+c^2)} \quad \textbf{(2)}$$

$$GA^2 + GB^2 + GC^2 = \frac{4}{9}\left(m_a^2 + m_b^2 + m_c^2\right) = \frac{4}{9}\cdot\frac{3}{4}(a^2+b^2+c^2) = \frac{1}{3}(a^2+b^2+c^2)$$

Replacing in (2): $MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2$

Back to the main proof: We take $M = T$ – Toricelli's point in lemma.

$$\boldsymbol{TA^2 + TB^2 + TC^2 = 3TG^2 + GA^2 + GB^2 + GC^2} \quad \textbf{(3)}$$



Denote: $TA = x; TB = y; TC = z$

Replacing in (3): $x^2 + y^2 + z^2 = 3TG^2 + GA^2 + GB^2 + GC^2$

$$3TG^2 = x^2 + y^2 + z^2 - (GA^2 + GB^2 + GC^2)$$

$$TG^2 = \frac{1}{3}(x^2+y^2+z^2) - \frac{1}{3}\cdot\frac{4}{9}\left(m_a^2 + m_b^2 + m_c^2\right)$$

$$TG^2 = \frac{1}{3}(x^2+y^2+z^2) - \frac{4}{27}\cdot\frac{3}{4}(a^2+b^2+c^2)$$

$$\boldsymbol{TG^2 = \frac{1}{3}(x^2+y^2+z^2) - \frac{1}{9}(a^2+b^2+c^2)} \quad \textbf{(4)}$$

$$[ATB] = \frac{1}{2}TA\cdot TB\cdot\sin(\measuredangle ATB)$$

$$[ATB] = \frac{1}{2}xy\sin 120^\circ = \frac{xy}{2}\sin(180^\circ - 60^\circ)$$

$$[ATB] = \frac{xy}{2}\cdot\sin 60^\circ = \frac{xy}{2}\cdot\frac{\sqrt{3}}{2} = \frac{xy\sqrt{3}}{4}$$

$$\boldsymbol{[ATB] = \frac{xy\sqrt{3}}{4}} \quad \textbf{(5)}$$

Analogous:

$$\boldsymbol{[BTC] = \frac{yz\sqrt{3}}{4}} \quad \textbf{(6)}$$

$$[\boldsymbol{CTA}]=\frac{\boldsymbol{zx}\sqrt{\mathbf{3}}}{\mathbf{4}}\quad \textbf{(7)}$$

By adding (5); (6); (7): $[ATB]+[BTC]+[CTA]=\frac{\sqrt{3}}{4}(xy+yz+zx)$

$$[ABC]=\frac{\sqrt{3}}{4}(xy+yz+zx),\qquad F=\frac{\sqrt{3}}{4}(xy+yz+zx)$$

$$\boldsymbol{xy+yz+zx=\frac{4F}{\sqrt{3}}}\quad \textbf{(8)}$$

By cosine's law in ΔATB:

$$BC^2=TA^2+TB^2-2TATB\cos(\sphericalangle ATB),\qquad a^2=x^2+y^2-2xy\cos 120^\circ$$

$$a^2=x^2+y^2-2xy\cos(180^\circ-60^\circ),\qquad a^2=x^2+y^2-2xy(-\cos 60^\circ)$$

$$\boldsymbol{a^2=x^2+y^2+xy}\quad \textbf{(9)}$$

Analogous:

$$\boldsymbol{b^2=y^2+z^2+yz}\quad \textbf{(10)}$$

$$\boldsymbol{c^2=z^2+x^2+zx}\quad \textbf{(11)}$$

By adding (9); (10); (11): $a^2+b^2+c^2=2(x^2+y^2+z^2)+(xy+yz+zx)$

By (8): $a^2+b^2+c^2=2(x^2+y^2+z^2)+\frac{4F}{\sqrt{3}}$

$$2(x^2+y^2+z^2)=a^2+b^2+c^2-\frac{4F}{\sqrt{3}}$$

$$\boldsymbol{x^2+y^2+z^2=\frac{1}{2}(a^2+b^2+c^2)-\frac{2F}{\sqrt{3}}}\quad \textbf{(12)}$$

Replacing (12) in (4):

$$TG^2=\frac{1}{3}\left(\frac{1}{2}(a^2+b^2+c^2)-\frac{2F}{\sqrt{3}}\right)-\frac{1}{9}(a^2+b^2+c^2)$$

$$TG^2=\frac{1}{6}(a^2+b^2+c^2)-\frac{2F}{3\sqrt{3}}-\frac{1}{9}(a^2+b^2+c^2)$$

$$TG^2=\frac{3(a^2+b^2+c^2)-2(a^2+b^2+c^2)}{18}-\frac{2F\sqrt{3}}{9}$$

$$TG^2=\frac{a^2+b^2+c^2-4F\sqrt{3}}{18},\qquad TG^2\ge 0\Rightarrow\frac{a^2+b^2+c^2-4F\sqrt{3}}{18}\ge 0$$

$$\Rightarrow a^2+b^2+c^2-4F\sqrt{3}\ge 0,\qquad a^2+b^2+c^2\ge 4F\sqrt{3}$$

which is **Ionescu – Weitzenbock's** inequality

Reference: www.ssmrmh.ro – Romanian Mathematical Magazine

A SIMPLE PROOF FOR SANDOR'S INEQUALITY AND APPLICATIONS

By Daniel Sitaru-Romania, Hikmat Mammadov-Azerbaijan

Abstract: In this paper we will give a simple proof for Sandor's inequality and a few applications.

SANDOR'S INEQUALITY

If $x > 0$ then:

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x} \quad (1)$$

Proof: Let be $f:[0,\infty) \to \mathbb{R}; f(x) = \sinh x - x$

$$f'(x) = \cosh x - 1$$

$$f'(x) = 0 \Rightarrow \cosh x - 1 = 0 \Rightarrow \cosh x = 1$$

$$\frac{e^x + e^{-x}}{2} - 1 = 0 \Rightarrow e^x - 2 + e^{-x} = 0$$

$$\left(\sqrt{e^x} - \sqrt{e^{-x}}\right)^2 = 0 \Rightarrow \sqrt{e^x} = \sqrt{e^{-x}}$$

$$e^x = e^{-x} \Rightarrow x = -x \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\min_{x\geq 0} f(x) = f(0) = 0 \Rightarrow f(x) \geq 0; (\forall) x \geq 0$$

$$\sinh x - x \geq 0 \Rightarrow \sinh x \geq x$$

$$\sinh^2 x \geq x^2 \Rightarrow 1 + \sinh^2 x \geq 1 + x^2$$

$$\cosh^2 x \geq 1 + x^2 \Rightarrow \sqrt[4]{\cosh^2 x} \geq \sqrt[4]{1 + x^2}$$

$$\sqrt{\cosh x} \geq \sqrt[4]{1 + x^2}$$

$$\frac{\sqrt{\cosh x}}{\sqrt[4]{1 + x^2}} \geq 1 \quad (2)$$

By Cauchy – Schwarz's inequality (integral form):

$$\int_0^x \left(\sqrt{\cosh x}\right)^2 dx \cdot \int_0^x \left(\frac{1}{\sqrt[4]{1+x^2}}\right)^2 dx \geq \left(\int_0^x \frac{\sqrt{\cosh x}}{\sqrt[4]{1+x^2}} dx\right)^2 \overset{(2)}{\geq} \left(\int_0^x dx\right)^2 = x^2$$

$$\int_0^x \cosh x \, dx \cdot \int_0^x \frac{1}{\sqrt{1+x^2}} dx \geq x^2$$

$$\sinh x \cdot \ln\left(x + \sqrt{1 + x^2}\right) \geq x^2$$

$$\sinh x \cdot \operatorname{arcsinh} x \geq x^2$$

For $x > 0$ we obtain (1):

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x}$$

Application 1: If $0 < a \le b$ then:

$$\boldsymbol{e^a + e^{-a} + 2\int_0^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le e^b + e^{-b}}$$

Solution: By (1):

$$\frac{x}{\operatorname{arcsinh} x} < \frac{\sinh x}{x} \Rightarrow \frac{x^2}{\operatorname{arcsinh} x} < \sinh x$$

$$\int_a^b \frac{x^2}{\operatorname{arcsinh} x}dx \le \int_a^b \sinh x\, dx$$

$$\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le \cosh b - \cosh a$$

$$\cosh a + \int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le \cosh b$$

$$\frac{e^a + e^{-a}}{2} + \int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le \frac{e^b + e^{-b}}{2}$$

$$e^a + e^{-a} + 2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{1+x^2}\right)}dx \le e^b + e^{-b}$$

Application 2: In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sinh A\right)\left(\sum_{cyc} \operatorname{arcsinh} A\right) > \pi^2$$

Solution: $\sinh A + \sinh B + \sinh C \overset{(1)}{\ge} \frac{A^2}{\operatorname{arcsinh} A} + \frac{B^2}{\operatorname{arcsinh} B} + \frac{C^2}{\operatorname{arcsinh} C} \ge$

$$\overset{BERGSTROM}{\ge} \frac{(A+B+C)^2}{\operatorname{arcsinh} A + \operatorname{arcsinh} B + \operatorname{arcsinh} C} = \frac{\pi^2}{\operatorname{arcsinh} A + \operatorname{arcsinh} B + \operatorname{arcsinh} C}$$

$$\left(\sum_{cyc} \sinh A\right)\left(\sum_{cyc} \operatorname{arcsinh} A\right) > \pi^2$$

Application 3: In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sinh a\right)\left(\sum_{cyc} \operatorname{arcsinh} a\right) > 108r^2$$

Solution: $\sinh a + \sinh b + \sinh C \overset{(1)}{\geq} \frac{a^2}{\operatorname{arcsinh} a} + \frac{b^2}{\operatorname{arcsinh} b} + \frac{c^2}{\operatorname{arcsinh} c} \geq$

$$\overset{BERGSTROM}{\geq} \frac{(a+b+c)^2}{\operatorname{arcsinh} a + \operatorname{arcsinh} b + \operatorname{arcsinh} c}$$

$$\left(\sum_{cyc} \sinh a\right)\left(\sum_{cyc} \operatorname{arcsinh} a\right) > (a+b+c^2) = 4s^2 \overset{MITRINOVIC}{\geq} 4 \cdot \left(3\sqrt{3}\right)^2 \cdot r^2 = 108r^2$$

Reference: www.ssmrmh.ro – Romanian Mathematical Magazine

A SIMPLE PROOF FOR DURELL'S INEQUALITY

By Daniel Sitaru-Romania

Abstract: In this paper we will give a simple proof for Durell's inequality in any triangle ABC and a few connections with Docuet's; Euler's and Mitrinovic's inequalities.

DURELL'S INEQUALITY

In any triangle ABC the following inequality holds:

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geq 1 \quad (1)$$

Proof: Lemma 1:

If $x, y, z \in \mathbb{R}$:

$$3(xy + yz + zx) \leq (x+y+z)^2 \quad (2)$$

Solution: $3xy + 3yz + 3z \leq x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$$x^2 + y^2 + z^2 \geq xy + yz + zx, \quad 2x^2 + 2y^2 + 2z^2 \geq 2xy + 2yz + 2zx$$

$$x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + z^2 - 2zx + x^2 \geq 0$$

$$(x-y)^2 + (y-z)^2 + (z-x)^2 \geq 0$$

Lemma 2: If r_a, r_b, r_c are exradies in ΔABC then:

$$r_a r_b + r_b r_c + r_c r_a = s^2 \quad (3)$$

Solution:

$$r_a r_b + r_b r_c + r_c r_a = \sum_{cyc} r_a r_b = \sum_{cyc} \frac{F}{s-a} \cdot \frac{F}{s-b} =$$

$$=\sum_{cyc}\frac{F^2}{(s-a)(s-b)}=\sum_{cyc}\frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)}=$$

$$=s\sum_{cyc}(s-c)=s\left(3s-\sum_{cyc}c\right)=s(3s-2s)=s^2$$

Lemma 3: If r_a, r_b, r_c are exradies in ΔABC then:

$$\boldsymbol{r_a+r_b+r_c=r+4R} \quad \textbf{(4)}$$

Solution:

$$r_a+r_b+r_c=\sum_{cyc}r_a=\sum_{cyc}\frac{F}{s-a}=F\sum_{cyc}\frac{1}{s-a}=\frac{F}{(s-a)(s-b)(s-c)}\sum_{cyc}(s-b)(s-c)=$$

$$=\frac{Fs}{s(s-a)(s-b)(s-c)}\sum_{cyc}(s^2-s(b+c)+bc)=\frac{Fs}{F^2}\sum_{cyc}(s^2-s(2s-a)+bc)=$$

$$=\frac{s}{F}\left(3s^2-6s^2+s\sum_{cyc}a+\sum_{cyc}bc\right)=\frac{s}{rs}(-3s^2+2s^2+s^2+r^2+4Rr)=$$

$$=\frac{1}{r}(r^2+4Rr)=r+4R$$

Lemma 4: (Doucet's inequality): In ΔABC the following relationship holds:

$$\boldsymbol{s\sqrt{3}\le r+4} \quad \textbf{(5)}$$

Solution: We replace in (2): $x=r_a; y=r_b; z=r_c$

$$3(r_ar_b+r_br_c+r_cr_a)\le(r_a+r_b+r_c)^2$$

By (3);(4):

$$3s^2\le(r+4R)^2$$

$$s\sqrt{3}\le r+4R$$

Back to the main result: Durell's inequality (1) can be written:

$$\sum_{cyc}\tan^2\frac{A}{2}\ge 1,\quad \sum_{cyc}\frac{(s-b)(s-c)}{s(s-a)}\ge 1$$

$$\sum_{cyc}(s-b)^2(s-c)^2\ge s(s-a)(s-b)(s-c)=F^2$$

By C-B-S inequality:

$$\sum_{cyc}(s-b)^2(s-c)^2 \geq \frac{1}{3}\left(\sum_{cyc}(s-b)(s-c)\right)^2 =$$

$$=\frac{1}{3}\left(\sum_{cyc}(s^2-s(b+c)+bc)\right)^2 = \frac{1}{3}\left(3s^2-s\sum_{cyc}(2s-a)+\sum_{cyc} bc\right)^2 =$$

$$=\frac{1}{3}\left(3s^2-6s^2-s\sum_{cyc}a+s^2+r^2+4Rr\right)^2 =$$

$$=\frac{1}{3}(-3s+2s^2+s^2+r^2+4Rr)^2 = \frac{1}{3}(r^2+4Rr)^2$$

Remains to prove that:

$$\frac{1}{3}(r^2+4Rr)^2 \geq F^2, \qquad r^2(r+4R)^2 \geq 3r^2s^2, \qquad 3s^2 \leq (r+4R)^2$$

$$s\sqrt{3} \leq r+4R$$

which is (5) – Doucet's inequality. Using Euler's inequality:

$$r \leq \frac{R}{2}$$

we can obtain by (5):

$$s\sqrt{3} \leq r+4R \leq \frac{R}{2}+4R = \frac{9R}{2}$$

$$\boldsymbol{s \leq \frac{9R}{2\sqrt{3}} = \frac{3\sqrt{3}R}{2}} \qquad \textbf{(6)}$$

which is Mitrinovic's inequality. Another proof for (6) is based on the concavity of the function:

$$f:(0,\pi)\to\mathbb{R}; f(x)=\sin x, \qquad f'(x)=\cos x; f''(x)=-\sin x<0$$

By Jensen's inequality:

$$f(A)+f(B)+f(C) \leq 3f\left(\frac{A+B+C}{3}\right)$$

$$\sin A+\sin B+\sin C \leq 3\sin\frac{\pi}{3}, \qquad 2R\sin A+2R\sin B+2R\sin C \leq 6R\cdot\frac{\sqrt{3}}{2}$$

$$a+b+c \leq 3\sqrt{3}R, \qquad \frac{a+b+c}{2} \leq \frac{3\sqrt{2}}{2}R, \qquad s \leq \frac{3\sqrt{3}}{2}R$$

Observation 1: In ΔABC (not right angled) the following relationship holds:

$$\boldsymbol{\cot^2 A+\cot^2 B+\cot^2 C \geq 1}$$

Proof: Let's consider the triangle with angles:

$$\pi - 2A; \pi - 2B; \pi - 2C$$

Let's observe that: $(\pi - 2A) + (\pi - 2B) + (\pi - 2C) = 3\pi - 2(A + B + C) = 3\pi - 2\pi = \pi$

By (1):

$$\tan^2\left(\frac{\pi - 2A}{2}\right) + \tan^2\left(\frac{\pi - 2B}{2}\right) + \tan^2\left(\frac{\pi - 2C}{2}\right) \geq 1$$

$$\tan^2\left(\frac{\pi}{2} - A\right) + \tan^2\left(\frac{\pi}{2} - B\right) + \tan^2\left(\frac{\pi}{2} - C\right) \geq 1$$

$$\cot^2 A + \cot^2 B + \cot^2 C \geq 1$$

Observation 2: In ΔABC the following relationship holds:

$$\boldsymbol{\tan^2\frac{5A}{2} + \tan^2\frac{5B}{2} + \tan^2\frac{5C}{2} \geq 1}$$

Proof: Let's consider the triangle with angles:

$$2\pi - 5A; 2\pi - 5B; 2\pi - 5C$$

Let's observe that:

$$(2\pi - 5A) + (2\pi - 5B) + (2\pi - 5C) = 6\pi - 5(A + B + C) = 6\pi - 5\pi = \pi$$

By (1):

$$\tan^2\left(\frac{2\pi - 5A}{2}\right) + \tan^2\left(\frac{2\pi - 5B}{2}\right) + \tan^2\left(\frac{2\pi - 5C}{2}\right) \geq 1$$

$$\tan^2\left(\pi - \frac{5A}{2}\right) + \tan^2\left(\pi - \frac{5B}{2}\right) + \tan^2\left(\pi - \frac{5C}{2}\right) \geq 1$$

$$\tan^2\frac{5A}{2} + \tan^2\frac{5B}{2} + \tan^2\frac{5C}{2} \geq 1$$

Reference: Romanian Mathematical Magazine – www.ssmrmh.ro

A NEW GENERALIZATION FOR HADWIGER – FINSLER'S INEQUALITY IN TRIANGLE

By D.M. Bătinețu – Giurgiu, Mihaly Bencze, Claudia Nănuți – Romania

Abstract: In this paper we will give a generalization for Hadwiger – Finsler's inequality.

Main result: If $m \geq 0$ then in any triangle ABC the following relationship holds:

$$\boldsymbol{\sum_{cyc} a^{2m+2} \geq 4^{m+1} \cdot \left(\sqrt{3}\right)^{1-m} \cdot F^{m+1} + \frac{1}{2}\sum_{cyc}\left(a^{m+1} - b^{m+1}\right)^2} \quad \textbf{(1)}$$

If $m=0$ then (1) becomes the classical Hadwiger – Finsler's inequality:

$$\sum_{cyc} a^2 \geq 4\sqrt{3}F + \frac{1}{2}\sum_{cyc}(a-b)^2$$

Proof 1: $\sum_{cyc}(a^{m+1}-b^{m+1})^2 = 2\sum_{cyc} a^{2m+2} - 2\sum_{cyc}(ab)^{m+1} \Rightarrow$

$$\sum_{cyc} a^{2m+2} = \sum_{cyc}(ab)^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{RADON}{\geq} \frac{(ab+bc+ca)^{m+1}}{(1+1+1)^m} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{GORDON}{\geq} \frac{\left(4\sqrt{3}F\right)^{m+1}}{3^m} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= 4^{m+1}\cdot\left(\sqrt{3}\right)^{m+1-2m}\cdot F^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= 4^m\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2$$

Proof 2:

$$\sum_{cyc} a^{2m+2} = \sum_{cyc}(ab)^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{AM-GM}{\geq} 3\cdot\left(\sqrt[3]{ab\cdot bc\cdot ca}\right)^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= \frac{3\cdot 3^{m+1}}{3^{m+1}}\cdot\left(\sqrt[3]{(abc)^2}\right)^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 \geq$$

$$\overset{CARLITZ}{\geq} \frac{3}{3^{m+1}}\left(4\sqrt{3}F\right)^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= \frac{1}{\left(\sqrt{3}\right)^{2m}}\cdot 4^{m+1}\cdot\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2 =$$

$$= 4^{m+1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1} + \frac{1}{2}\sum_{cyc}(a^{m+1}-b^{m+1})^2$$

Equality holds for $a=b=c$.

Reference: [1] – Romanian Mathematical Magazine – www.ssmrmh.ro

A NEW GENERALIZATION FOR GORDON'S INEQUALITY IN TRIANGLE

By D.M. Bătinețu – Giurgiu, Mihaly Bencze, Claudia Nănuți-Romania

Abstract: In this paper we will give a generalization for Gordon's inequality.

Main result:

If $m \geq 0, M \in Int(\Delta ABC); d_a = d(A, BC); d_b = d(B, AC); d_c = d(C, AB)$ then in ΔABC the following relationship holds:

$$\frac{a^{m+1} \cdot b}{d_b^m} + \frac{b^{m+1} \cdot c}{d_c^m} + \frac{c^{m+1} \cdot a}{d_a^m} \geq 2^{m+2} \cdot \left(\sqrt{3}\right)^{m+1} \cdot F \quad (1)$$

If $m = 0$ then (1) becomes the classical Gordon's inequality:

$$ab + bc + ca \geq 4\sqrt{3}F$$

Proof 1: Denote: $F_a = [MBC]; F_b = [MCA]; F_c = [MAB]$

$$\sum_{cyc} \frac{a^{m+1} \cdot b}{d_b^m} = \sum_{cyc} \frac{a^{m+1} \cdot b^{m+1}}{(b \cdot d_b)^m} = \sum_{cyc} \frac{(ab)^{m+1}}{(2F_b)^m} =$$

$$= \frac{1}{2^m} \sum_{cyc} \frac{(ab)^{m+1}}{F_b^m} \overset{RADON}{\geq} \frac{(ab + bc + ca)^{m+1}}{2^m (F_a + F_b + F_c)^m} =$$

$$= \frac{(s^2 + 4Rr + r^2)^{m+1}}{2^m \cdot F^m} = \frac{1}{(2F)^m} \left(s^2 + r(4R + r)\right)^{m+1} \geq$$

$$\overset{DOUCET}{\geq} \frac{1}{(2F)^m} \cdot \left(s^2 + s\sqrt{3} \cdot r\right)^{m+1} \overset{MITRINOVIC}{\geq} \frac{1}{(2F)^m} \cdot \left(s \cdot 3\sqrt{3}r + sr\sqrt{3}\right)^{m+1} =$$

$$= \frac{1}{2^m \cdot F^m} \cdot \left(3\sqrt{3}F + \sqrt{3}F\right)^{m+1} = \frac{1}{2^m \cdot F^m} \cdot 4^{m+1} \cdot \left(\sqrt{3}\right)^{m+1} \cdot F^{m+1} = 2^{m+2} \cdot \left(\sqrt{3}\right)^{m+1} \cdot F$$

Proof 2:

$$\sum_{cyc} \frac{a^{m+1} \cdot b}{d_b^m} = \sum_{cyc} \frac{a^{m+1} \cdot b^{m+1}}{(b \cdot d_b)^m} = \sum_{cyc} \frac{(ab)^{m+1}}{(2F_b)^m} =$$

$$= \frac{1}{2^m} \cdot \sum_{cyc} \frac{(ab)^{m+1}}{F_b^m} = \frac{1}{2^m \cdot F^m} \cdot (ab + bc + ca)^{m+1} \geq$$

$$\overset{AM-GM}{\geq} \frac{1}{2^m \cdot F^m} \cdot \left(3\sqrt[3]{(abc^2)}\right)^{m+1} \overset{CARLITZ}{\geq}$$

$$\geq \frac{1}{2^m \cdot F^m} \cdot \left(4\sqrt{3}F\right)^{m+1} = \frac{2^{2m+2} \cdot 3^{\frac{m+1}{2}} \cdot F^{m+1}}{2^m \cdot F^m} = 2^{m+2} \cdot \left(\sqrt{3}\right)^{m+1} \cdot F$$

Equality holds for $a = b = c$.

Reference: Romanian Mathematical Magazine – www.ssmrmh.ro

A NEW GENERALIZATION FOR TSINTSIFAS' INEQUALITY IN TRIANGLE

By D.M. Bătinețu – Giurgiu, Mihaly Bencze, Claudia Nănuți-Romania

Abstract: In this paper we will give a new generalization for Tsintsifas' inequality and its consequence: Goldner's inequality.

Main result:

If $m \geq 0$; $M \in Int\ (\Delta ABC)$; $x, y, z > 0$, $d_a = d(A, BC)$; $d_b = d(B, AC)$, $d_c = d(C, AB)$ then in ΔABC the following relationship holds:

$$\sum_{cyc} \frac{x^{m+1}}{(y+z)^{m+1}} \cdot \frac{a^{3m+4}}{d_a^m} \geq 2^{2m+3} \cdot F^{m+2} \quad (1)$$

If $m = 0$ then (1) becomes the classical Tsintsifas' inequality:

$$\frac{x}{y+z} \cdot a^4 + \frac{y}{z+x} \cdot b^4 + \frac{z}{x+y} \cdot c^4 \geq 8F^2 \quad (2)$$

Proof: Denote: $F_a = [MBC]; F_b = [MCA]; F_c = [MAB]$

$$\sum_{cyc} \frac{x^{m+1}}{(y+z)^{m+1}} \cdot \frac{a^{3m+4}}{d_a^m} = \sum_{cyc} \frac{x^{m+1}}{(y+z)^{m+1}} \cdot \frac{a^{4m+4}}{(ad_a)^m} =$$

$$= \sum_{cyc} \frac{\left(\frac{xa^4}{y+z}\right)^{m+1}}{(2F_a)^m} \overset{RADON}{\geq} \frac{\left(\sum_{cyc} \frac{xa^4}{y+z}\right)^{m+1}}{2^m(F_a + F_b + F_c)^m} = \frac{1}{2^m \cdot F^m} \cdot \left(\sum_{cyc} \frac{xa^4}{y+z}\right)^{m+1} \quad (3)$$

$$\sum_{cyc} \frac{xa^4}{y+z} = \sum_{cyc} \frac{x^2a^4}{xy+xz} \overset{BERGSTROM}{\geq} \frac{(xa^2 + yb^2 + zc^2)^2}{(xy+xz) + (yz+yx) + (zx+zy)} \overset{KLAMKIN}{\geq}$$

$$\geq \frac{16(xy+yz+zx)F^2}{2(xy+yz+zx)} = 8F^2 \quad (4)$$

By (3):

$$\sum_{cyc} \frac{x^{m+1}}{(y+z)^{m+1}} \cdot \frac{a^{3m+4}}{d_a^m} \geq \frac{1}{2^m \cdot F^m} \left(\sum_{cyc} \frac{xa^4}{y+z}\right)^{m+1} \geq$$

$$\overset{(4)}{\geq} \frac{1}{2^m \cdot F^m} \cdot (8F^2)^{m+1} = \frac{2^{3m+3} \cdot F^{2m+2}}{2^m \cdot F^m} = 2^{2m+3} \cdot F^{m+2}$$

Observation: If we take in (2): $x = y = z$ then we obtain Goldner's inequality:

$$\frac{1}{1+1} \cdot a^4 + \frac{1}{1+1} \cdot b^4 + \frac{1}{1+1} \cdot c^4 \geq 8F^2$$

$$\frac{1}{2}(a^4 + b^4 + c^4) \geq 8F^2$$

$$a^4 + b^4 + c^4 \geq 16F^2$$

Equality holds for: $a = b = c$.

Reference: Romanian Mathematical Magazine – www.ssmrmh.ro

NAGEL'S AND GERGONNE'S CEVIANS - APPLICATIONS AND RESULTS

By Bogdan Fuștei-Romania

Extended Abstract: This paper is a new study in the field of geometry of triangle involving Nagel and Gergonne cevians. We obtain new identities and inequalities involving those two types of cevians using very well-known relationships in triangle geometry involving other elements of a triangle. After obtaining new identities involving those type of cevians, we obtain new inequalities using well known inequalitys both geometric and algebraic.
Those type of cevians as we will see are very close related with very well-known elements of triangle, which helps us to manipulate the expressions we obtain for a better form.

This paper is not the first written by me involving Nagel and Gergonne's cevians, but the results are new, we also use previous results in papers written by me with this topic for keeping fresh the interest of the readers, sometimes the expressions obtain are not friendly and they must be manipulated to obtain a better form, for using in future research. This paper like others was born from passion and curiosity for obscure elements which in school of math from Europe and USA was not studied so much. This topic is very rich and as we will see, we can obtain hundreds of results, even more. I hope this topic will be researched in the future and new results will be obtained.

Keywords: Nagel's and Gergonne's cevians, geometric inequalities, identities in triangle

1. Research methodology

In this paper we will present applications and results with Nagel and Gergonne cevians. Results about those type of cevians are very rare in Europe and USA math school. We will present the new connections of those cevians with very well-known elements of a triangle. These new results will lead to obtain new geometric inequalities in triangle and inequalities related with already very well-known.

We consider triangle ABC with sides $BC = a$, $AC = b$, $AB = c$ and $\mathrm{p} = \frac{1}{2}(a + b + c)$ circumradius **R**, inradius r, l_a, l_b, l_c: the angle-bisectors; r_a, r_b, r_c the radii of excircles; m_a, m_b, m_c: the medians; h_a, h_b, h_c: the altitudes; n_a, g_a - cevians of Nagel and Gergonne from A to BC (and analogs);

We know that: $g_a^2 = (p-a)\left[p - \frac{(b-c)^2}{a}\right]$ (and analogs) [1;3]

$4r_a r = 4(p-b)(p-c) = a^2 - (b-c)^2$ (and analogs), we will obtain:

$g_a^2 - (p-a)^2 = \frac{4r_a r(p-a)}{a}$ (and analogs), but we know that $r_a = \frac{S}{p-a}$ (and analogs), $2S = ah_a = 2pr$ and after banal computations we obtain a new formula:

$$\mathbf{g_a^2 = (p-a)^2 + 2rh_a} \text{ (and analogs)} \qquad (1)$$

From $r_a = \frac{S}{p-a}$ (and analogs) $\rightarrow \frac{p-a}{r} = \frac{p}{r_a}$ (and analogs)

From S^2 =p(p-a)(p-b)(p-c) (Heron) and $ctg\frac{A}{2} = \sqrt{\frac{p(p-a)}{(p-b)(p-c)}}$ (and analogs) we will obtain $ctg\frac{A}{2} = \frac{p-a}{r} = \frac{p}{r_a}$ (and analogs); From (1) and this identity we obtain:

$$\mathbf{\frac{g_a^2}{r^2} = ctg^2\frac{A}{2} + \frac{2h_a}{r}} \text{ (and analogs)} \qquad (2)$$

Now we use $p^2 = n_a^2 + 2r_a h_a$ (and analogs) [2] and we obtain:

$$\mathbf{\left(\frac{g_a}{r}\right)^2 = \left(\frac{n_a}{r_a}\right)^2 + \frac{2h_a}{r} + \frac{2h_a}{r_a}} \text{ (and analogs)} \qquad (3)$$

From $r_a = \frac{2S}{2(p-a)} = \frac{ah_a}{b+c-a} \rightarrow \frac{r_a}{h_a} = \frac{a}{b+c-a} \rightarrow \frac{h_a}{r_a} = \frac{b+c}{a} - 1$ (and analogs)

$ah_a = 2pr = (a+b+c)r \rightarrow \frac{h_a}{r} = 1 + \frac{b+c}{a}$ (and analogs)

$\frac{2h_a}{r} + \frac{2h_a}{r_a} = 2\left(\frac{b+c}{a} - 1 + 1 + \frac{b+c}{a}\right) = 4\frac{b+c}{a}$ (and analogs)

From (3) we obtain a new result:

$$\mathbf{\left(\frac{g_a}{r}\right)^2 = \left(\frac{n_a}{r_a}\right)^2 + 4\frac{b+c}{a}} \text{ (and analogs)} \qquad (4)$$

After summation we obtain new identities:

$$\mathbf{\frac{g_a^2+g_b^2+g_c^2}{r^2} = \sum ctg^2\frac{A}{2} + \frac{2(h_a+h_b+h_c)}{r}} \qquad (5)$$

$$\mathbf{\frac{g_a^2+g_b^2+g_c^2}{r^2} = \sum\left(\frac{n_a}{r_a}\right)^2 + 4\sum\frac{b+c}{a}} \qquad (6)$$

From $ctg^2\frac{A}{2} = \left(\frac{p}{r_a}\right)^2\frac{2}{2} = \frac{n_b^2+n_c^2+2h_b r_c+2h_c r_c}{2r_a^2} \ge \frac{2(n_b n_c+h_b r_c+h_c r_c)}{2r_a^2}$ we obtain:

$$\mathbf{ctg^2\frac{A}{2} \ge \frac{n_b n_c+h_b r_c+h_c r_c}{r_a^2}} \text{ (and analogs)} \qquad (7)$$

From (2) and (7) we obtain: $\left(\frac{g_a}{r}\right)^2 \ge \frac{n_b n_c+h_b r_c+h_c r_c}{r_a^2} + \frac{2h_a}{r}$ (and analogs) (8)

From $2rh_a = g_a^2 - (p-a)^2 = (g_a+p-a)(g_a+a-p)$ we obtain:

$\frac{h_a}{g_a+a-p} = \frac{g_a+p-a}{2r}$ (and analogs), and after summation we obtain:

$$\mathbf{\sum\frac{h_a}{g_a+a-p} = \frac{g_a+g_b+g_c+p}{2r}} \qquad (9)$$

$\frac{h_a}{g_a+p-a}=\frac{g_a+a-p}{2r}$ (and analogs), after summation we obtain:

$$\sum\frac{\mathbf{h_a}}{\mathbf{g_a+p-a}}=\frac{\mathbf{g_a+g_b+g_c-p}}{\mathbf{2r}} \quad \textbf{(10)}$$

From [4] we have: $\frac{p}{r}=\sum\frac{n_a}{h_a}+2\sum\frac{r_a}{n_a+p}$ and $\frac{3p}{r}=\frac{n_a+n_b+n_c}{r}+2\sum\frac{2r_a+h_a}{n_a+p}$ and using (9) and (10) we obtain new results:

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+p-a}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}-\left(\sum\frac{\mathbf{n_a}}{\mathbf{h_a}}+\mathbf{2}\sum\frac{\mathbf{r_a}}{\mathbf{n_a+p}}\right) \quad \textbf{(11)}$$

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+p-a}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}-\left(\frac{\mathbf{n_a+n_b+n_c}}{\mathbf{3r}}+\frac{\mathbf{2}}{\mathbf{3}}\sum\frac{\mathbf{2r_a+h_a}}{\mathbf{n_a+p}}\right) \quad \textbf{(12)}$$

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+a-p}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}+\sum\frac{\mathbf{n_a}}{\mathbf{h_a}}+\mathbf{2}\sum\frac{\mathbf{r_a}}{\mathbf{n_a+p}} \quad \textbf{(13)}$$

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+a-p}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}+\frac{\mathbf{n_a+n_b+n_c}}{\mathbf{3r}}+\frac{\mathbf{2}}{\mathbf{3}}\sum\frac{\mathbf{2r_a+h_a}}{\mathbf{n_a+p}} \quad \textbf{(14)}$$

From $p^2=n_a^2+2r_ah_a$ (and analogs) we have: $2r_ah_a=(p+n_a)(p-n_a)$ and

$p-n_a=\frac{2r_ah_a}{p+n_a}\to\frac{p-n_a}{r_a}=\frac{2h_a}{p+n_a}$ (and analogs) also we use $\sum\frac{1}{r_a}=\frac{1}{r}$ we obtain:

$$\frac{\mathbf{p}}{\mathbf{r}}=\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}} \quad \textbf{(15)}$$

and using (9) and (10) we obtain:

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+p-a}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}-\left(\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}}\right) \quad \textbf{(16)}$$

$$\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{g_a+a-p}}=\frac{\mathbf{g_a+g_b+g_c}}{\mathbf{r}}+\sum\frac{\mathbf{n_a}}{\mathbf{r_a}}+\sum\frac{\mathbf{2h_a}}{\mathbf{p+n_a}} \quad \textbf{(17)}$$

From (1) we obtain: $\left(\frac{g_a}{h_a}\right)^2=\left(\frac{p-a}{h_a}\right)^2+\frac{2r}{h_a}$ (and analogs); We will use the well-known relations: $\sum\frac{1}{h_a}=\frac{1}{r}$; $r_br_c=p(p-a)$ (and analogs); $2r_br_c=h_a(r_b+r_c)$ (and analogs); $\sum r_br_c=p^2$; and we have: $\frac{p-a}{h_a}=\frac{r_b+r_c}{2p}$ (and analogs) and after simple manipulation we obtain: $\sum\left(\frac{p-a}{h_a}\right)^2=\frac{1}{2}+\frac{r_a^2+r_b^2+r_c^2}{2p^2}$ and $\sum\left(\frac{g_a}{h_a}\right)^2=2+\sum\left(\frac{p-a}{h_a}\right)^2$.

We will obtain:

$$\sum\left(\frac{\mathbf{g_a}}{\mathbf{h_a}}\right)^{\mathbf{2}}=\frac{\mathbf{5}}{\mathbf{2}}+\frac{\mathbf{r_a^2+r_b^2+r_c^2}}{\mathbf{2p^2}} \quad \textbf{(18)}$$

From (18) we can write: $\sum\left(\frac{g_a}{h_a}\right)^2=\frac{5}{2}+\frac{r_a^2+r_b^2+r_c^2}{2p^2}+\frac{1}{2}-\frac{1}{2}=3+\frac{r_a^2+r_b^2+r_c^2-p^2}{2p^2}$

Is easy to see $r_a^2+r_b^2+r_c^2-p^2=r_a^2+r_b^2+r_c^2-\sum r_br_c$ and we obtain:

$$r_a^2+r_b^2+r_c^2-\sum r_br_c=\frac{1}{2}[(r_a-r_b)^2+(r_c-r_b)^2+(r_c-r_a)^2]$$

$$\sum\left(\frac{\mathbf{g_a}}{\mathbf{h_a}}\right)^{\mathbf{2}}=\mathbf{3}+\frac{\mathbf{1}}{\mathbf{4p^2}}[(\mathbf{r_a}-\mathbf{r_b})^2+(\mathbf{r_c}-\mathbf{r_b})^2+(\mathbf{r_c}-\mathbf{r_a})^2] \quad \textbf{(19)}$$

From well-known relation: $r_a r_b r_c = Sp = p^2 r$ and $2r_b r_c = h_a(r_b + r_c)$ (and analogs) $\rightarrow \frac{r_a h_a(r_b+r_c)}{2r} = p^2$ (and analogs). From $\frac{r_a h_a(r_b+r_c)}{2r} = p^2$ and $p^2 = n_a^2 + 2r_a h_a$ (and analogs) we obtain:

$$\mathbf{n_a^2 = r_a h_a\left(\frac{r_b+r_c}{2r} - 2\right)} \text{ (and analogs)} \quad (20)$$

From $r_a + r_b + r_c = 4R + r$ and (20) we obtain:

$$\sum \frac{\mathbf{n_a^2}}{\mathbf{r_a h_a}} = \frac{4R}{r} - 5 \quad (21)$$

From $h_a = \left(1 + \frac{b+c}{a}\right)r$ (and analogs) and $\frac{h_a}{r_a} = \frac{b+c}{a} - 1$ (and analogs) we obtain

$r_a h_a = (2r_a + h_a)r$ (and analogs). From $r_a h_a = (2r_a + h_a)r$ (and analogs) and (20) we obtain:

$$\mathbf{n_a^2 = (2r_a + h_a)\left(\frac{r_b+r_c}{2} - 2r\right)} \text{ (and analogs)} \quad (22)$$

From $m_a l_a \geq p(p-a) = r_b r_c$ (Panaitopol inequality) [5] and $2r_b r_c = h_a(r_b + r_c)$ (and analogs) we obtain $\frac{m_a l_a}{h_a} \geq \frac{r_b+r_c}{2}$ (and analogs) and from (22) we obtain:

$$\mathbf{(2r_a + h_a)\left(\frac{m_a l_a}{h_a} - 2r\right) \geq n_a^2} \text{ (and analogs)} \quad (23)$$

We will use now well-known inequality $\sqrt{\frac{R}{2r}} \geq \frac{l_a}{h_a}$ (and analogs) (for proof see [6]) and using (23) we obtain:

$$\mathbf{(2r_a + h_a)\left(m_a\sqrt{\frac{R}{2r}} - 2r\right) \geq n_a^2} \text{ (and analogs)} \quad (24)$$

From (23) and (24) after summation we obtain two new inequalities:

$$\sum \mathbf{(2r_a + h_a)\left(\frac{m_a l_a}{h_a} - 2r\right) \geq \sum n_a^2} \quad (25)$$

$$\sum \mathbf{(2r_a + h_a)\left(m_a\sqrt{\frac{R}{2r}} - 2r\right) \geq \sum n_a^2} \quad (26)$$

From (1) and $abc = 4RS$; $a^2 + b^2 + c^2 = 2(p^2 - 4Rr - r^2)$;

$a^3 + b^3 + c^3 = 2p(p^2 - 6Rr - 3r^2)$ and $2S = ah_a = 2pr$ (and analogs) we have:

$$\sum \frac{\mathbf{g_a^2}}{\mathbf{h_a}} = 6r + \sum \frac{(p-a)^2}{h_a} \text{ and } \sum \frac{\mathbf{g_a^2}}{\mathbf{h_a}} = \mathbf{2R + 5r} \quad (27)$$

From (27) and Bergstrom inequality: if x_k-real numbers and $a_k > 0$,

$k \in \{1,2,\ldots,n\}$ then $\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_n^2}{a_n} \geq \frac{(x_1+x_2+\cdots+x_n)^2}{a_1+a_2+\cdots+a_n}$ with equality only if

$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$, we obtain: $2R + 5r \geq \frac{(g_a+g_b+g_c)^2}{h_a+h_b+h_c}$ wich can be written as:

$$\mathbf{\sqrt{(2R + 5r)(h_a + h_b + h_c)} \geq g_a + g_b + g_c} \quad (28)$$

From $g_a \geq h_a$(and analogs) and (28) we obtain:

$$\mathbf{2R+5r \geq g_a+g_b+g_c} \quad \textbf{(29)}$$

From (28) and (9), (10) we obtain two new inequalitys:

$$\sum \frac{\mathbf{h_a}}{\mathbf{g_a+a-p}} \leq \frac{p+\sqrt{(\mathbf{2R+5r})(\mathbf{h_a+h_b+h_c})}}{\mathbf{2}r} \quad \textbf{(30)}$$

$$\sum \frac{\mathbf{h_a}}{\mathbf{g_a+p-a}} \leq \frac{-p+\sqrt{(\mathbf{2R+5r})(\mathbf{h_a+h_b+h_c})}}{\mathbf{2}r} \quad \textbf{(31)}$$

From (4), (8) and (28) we obtain another two inequalitys:

$$\sum \sqrt{\left(\frac{\mathbf{n_a}}{\mathbf{r_a}}\right)^2+4\frac{\mathbf{b+c}}{\mathbf{a}}} \leq \frac{\sqrt{(\mathbf{2R+5r})(\mathbf{h_a+h_b+h_c})}}{r} \quad \textbf{(32)}$$

$$\sum \sqrt{\frac{\mathbf{n_b n_c+h_b r_c+h_c r_c}}{\mathbf{r_a^2}}+\frac{\mathbf{2h_a}}{\mathbf{r}}} \leq \frac{\sqrt{(\mathbf{2R+5r})(\mathbf{h_a+h_b+h_c})}}{r} \quad \textbf{(33)}$$

We use now Wolstenholme's inequality: if x, y, z are real numbers, A, B, C angles of triangle ABC then we have:

$x^2+y^2+z^2 \geq 2xy\cos C+2yz\cos A+2zx\cos B$ with equality if and only if:

$\frac{x}{\sin A}=\frac{y}{\sin B}=\frac{z}{\sin C}$ [7] and (25) and (26) and we obtain:

$$\sum(\mathbf{2r_a+h_a})\left(\frac{\mathbf{m_a l_a}}{\mathbf{h_a}}-2r\right) \geq \mathbf{2n_b n_c\cos A+2n_a n_c\cos B+2n_a n_b\cos C} \quad \textbf{(34)}$$

$$\sum(\mathbf{2r_a+h_a})\left(\mathbf{m_a}\sqrt{\frac{\mathbf{R}}{\mathbf{2r}}}-\mathbf{2r}\right) \geq \mathbf{2n_b n_c\cos A}+2n_a n_c\cos B+2n_a n_b\cos C \quad \textbf{(35)}$$

Using Wolstenholme's inequality and $\frac{3p}{r}=\frac{n_a+n_b+n_c}{r}+2\sum\frac{2r_a+h_a}{n_a+p}$ we obtain:

$$\mathbf{3p} \geq \mathbf{2}\sum\cos A\sqrt{\mathbf{n_b n_c}}+\mathbf{2r}\sum\frac{\mathbf{2r_a+h_a}}{\mathbf{n_a+p}} \quad \textbf{(36)}$$

From $\frac{p}{r}=\sum\frac{n_a}{h_a}+2\sum\frac{r_a}{n_a+p}$ and Wolstenholme's inequality we obtain:

$$\frac{\mathbf{p}}{\mathbf{r}} \geq \mathbf{2}\sum\cos A\sqrt{\frac{\mathbf{n_b n_c}}{\mathbf{h_b h_c}}}+\mathbf{2}\sum\frac{\mathbf{r_a}}{\mathbf{n_a+p}} \quad \textbf{(37)}$$

From (6) and Wolstenholme's inequality we obtain:

$$\frac{\mathbf{g_a^2+g_b^2+g_c^2}}{\mathbf{r^2}} \geq \mathbf{2}\sum\frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}}\cos A+\mathbf{4}\sum\frac{\mathbf{b+c}}{\mathbf{a}} \quad \textbf{(38)}$$

$$\sum\left(\frac{\mathbf{n_a}}{\mathbf{r_a}}\right)^2+\mathbf{4}\sum\frac{\mathbf{b+c}}{\mathbf{a}} \geq \mathbf{2}\sum\frac{\mathbf{g_b g_c}}{\mathbf{r^2}}\cos A \quad \textbf{(39)}$$

From (15) and Wolstenholme's inequality we obtain:

$$\frac{\mathbf{p}}{\mathbf{r}} \geq \mathbf{2}\sum\cos A\sqrt{\frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}}}+\mathbf{2}\sum\frac{\mathbf{h_a}}{\mathbf{n_a+p}} \quad \textbf{(40)}$$

From (27) and Wolstenholme's inequality we obtain:

$$\mathbf{2R+5r} \geq \mathbf{2}\sum\frac{\mathbf{g_b g_c}}{\sqrt{\mathbf{h_b h_c}}}\cos A \quad \textbf{(41)}$$

From (18) and Wolstenholme's inequality we obtain:

$$\frac{5}{2}+\frac{r_a^2+r_b^2+r_c^2}{2p^2}\geq 2\sum\frac{g_b g_c}{h_b h_c}\cos A \quad (42)$$

The last result presented is the following:

$$l_a\leq m_a\leq p_a\leq n_a \text{ (and analogs)} \quad (43)$$

p_a-Spieker cevian from A to BC

For the demonstration of this result, we will use this theorem: Points I, G, S_P, N_a are colinear, line that passes through these points is called Nagel line.[8]

I (incenter), G (triangle centroid), S_P (Spieker center), N_a (Nagel point)

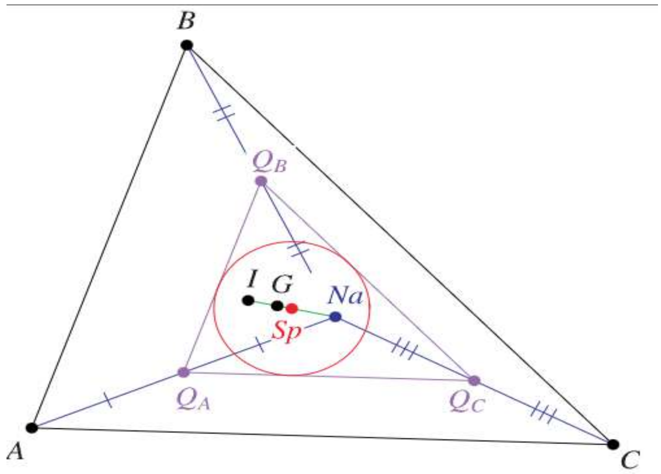


Figure 1.

2. Conclusion

The field of geometry is full of surprises and we can find new connections between well known elements of a triangle.

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RMM SOLVED PROBLEMS-IV

By Marin Chirciu – Romania

J.2563. In ΔABC the following relationship holds:

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\ge\frac{36r^2}{\sqrt[4]{3R^2}}$$

Daniel Sitaru – Romania

Solution:

$$LHS=\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\overset{AM-GM}{\ge}3\sqrt[3]{(abc)^{\frac{3}{2}}}=3(abc)^{\frac{1}{2}}\overset{Carlitz}{\ge}3\left(\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}\right)^{\frac{1}{2}}=3\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{4}}\overset{(1)}{\ge}\frac{36r^2}{\sqrt[4]{3R^2}}$$
$$=RHS$$

where (1) $\Leftrightarrow 3\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{4}}\ge\frac{36r^2}{\sqrt[4]{3R^2}}\Leftrightarrow\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{4}}\ge\frac{12r^2}{\sqrt[4]{3R^2}}\Leftrightarrow\left(\frac{4sr}{\sqrt{3}}\right)^3\ge\frac{(12r^2)^4}{3R^2}$ which follows from Mitrinovic inequality $s\ge 3\sqrt{3}r$. It remains to prove that:

$$\left(\frac{4r\cdot3\sqrt{3}r}{\sqrt{3}}\right)^3\ge\frac{(12r^2)^4}{3R^2}\Leftrightarrow(12r^2)^3\ge\frac{(12r^2)^4}{3R^2}\Leftrightarrow 3R^2\ge12r^2\Leftrightarrow R^2\ge4r^2,\text{ Euler.}$$

We've used above Carlitz's inequality: $abc\ge\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be strengthened.

In ΔABC the following relationship holds:

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\ge 6r\sqrt[4]{108r^2}$$

Marin Chirciu

Solution:

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{b^2}{\sqrt{a}}\cdot\frac{c^2}{\sqrt{b}}\cdot\frac{a^2}{\sqrt{c}}}=3\sqrt[3]{(abc)^{\frac{3}{2}}}=3(abc)^{\frac{1}{2}}\overset{Carlitz}{\ge}3\left(\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}\right)^{\frac{1}{2}}=$$

$$=3\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{4}}=3\left(\frac{4sr}{\sqrt{3}}\right)^{\frac{3}{4}}\overset{Mitrinovic}{\ge}3\left(\frac{4r\cdot3\sqrt{3}r}{\sqrt{3}}\right)^{\frac{3}{4}}=3(12r^2)^{\frac{3}{4}}=3\sqrt[4]{(12r^2)^3}=$$

$$=6r\sqrt[4]{108r^2}\overset{(1)}{\ge}\frac{36r^2}{\sqrt[4]{3R^2}}$$

where (1) follows from Euler's inequality $R\ge2r$.

We've used above Carlitz's inequality: $abc \geq \left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}$

Equality holds if and only if the triangle is equilateral.

Remark: We can write the following inequalities:

In ΔABC the following relationship holds:

$$\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}} \geq 6r\sqrt[4]{108r^2} \geq \frac{36r^2}{\sqrt[4]{3R^2}}$$

Solution:

See $\frac{b^2}{\sqrt{a}}+\frac{c^2}{\sqrt{b}}+\frac{a^2}{\sqrt{c}} \geq 6r\sqrt[4]{108r^2}$ and Euler's inequality $R \geq 2r$.

Note: The inequality strengthen Problem J.2563 from RMM – 43. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{b^n}{\sqrt{a}}+\frac{c^n}{\sqrt{b}}+\frac{a^n}{\sqrt{c}} \geq 3(12r^2)^{\frac{2n-1}{4}}, n \in \mathbb{N}^*$$

Marin Chirciu

Solution:

$$LHS = \frac{b^n}{\sqrt{a}}+\frac{c^n}{\sqrt{b}}+\frac{a^n}{\sqrt{c}} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{b^n}{\sqrt{a}}\cdot\frac{c^n}{\sqrt{b}}\cdot\frac{a^n}{\sqrt{c}}} = 3\sqrt[3]{(abc)^{\frac{2n-1}{2}}} = 3(abc)^{\frac{2n-1}{6}} \overset{Carlitz}{\geq}$$

$$\geq 3\left(\left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}\right)^{\frac{2n-1}{6}} = 3\left(\frac{4S}{\sqrt{3}}\right)^{\frac{2n-1}{4}} = 3\left(\frac{4sr}{\sqrt{3}}\right)^{\frac{2n-1}{4}} \overset{Mitrinovic}{\geq} 3\left(\frac{4r\cdot 3\sqrt{3}r}{\sqrt{3}}\right)^{\frac{2n-1}{4}} = 2(12r^2)^{\frac{2n-1}{4}}$$

We have used above Carlitz inequality: $\geq \left(\frac{4S}{\sqrt{3}}\right)^{\frac{3}{2}}$ .Equality holds if and only if the triangle is equilateral. **Note:** For $n = 2$ the inequality strengthen Problem J.2563 from RMM – 43.

J.2564. In ΔABC the following relationship holds:

$$\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3} \geq 120r^2$$

Daniel Sitaru, Claudia Nănuți – Romania

Solution: Lemma. In ΔABC the following relationship holds:

$$\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3} = \frac{5}{3}\cdot\frac{3s^4+2s^2(Rr+r^2)-r^2(8R^2+6Rr+r^2)}{s^2+2Rr+r^2}$$

Proof: We have $\sum a^5 = 2s[s^4 - 10s^2(Rr + r^2) + 5r^2(8R^2 + 6Rr + r^2)]$ and

$$\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$$

Using Lemma, the inequality can be written:

$$\frac{5}{3} \cdot \frac{3s^4 + 2s^2(Rr + r^2) - r^2(8R^2 + 6Rr + r^2)}{s^2 + 2Rr + r^2} \geq 120r^2 \Leftrightarrow$$

$$\Leftrightarrow 3s^4 + 2s^2(Rr + r^2) - r^2(8R^2 + 6Rr + r^2) \geq 72r^2(s^2 + 2Rr + r^2) \Leftrightarrow$$

$$\Leftrightarrow s^2(3s^2 + 2Rr + 2r^2) - r^2(8R^2 + 6Rr + r^2) \geq 72r^2(s^2 + 2Rr + r^2),$$

which follows from Gerrentsen's inequality $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$(16Rr - 5r^2)(3(16Rr - 5r^2) + 2Rr + 2r^2) - r^2(8R^2 + 6Rr + r^2) \geq$$

$$\geq 72r^2(4R^2 + 4Rr + 3r^2 + 2Rr + r^2)$$

$\Leftrightarrow 504R^2 - 896Rr - 224r^2 \geq 0 \Leftrightarrow 9R^2 - 16Rr - 4r^2 \geq 0 \Leftrightarrow (R - 2r)(9R + 2r)$, which follows from Euler's inequality $R \geq 2r$. Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality with opposite sense.

In ΔABC the following relationship holds:

$$\frac{32s^5 - a^5 - b^5 - c^5}{8s^3 - a^3 - b^3 - c^3} \leq 30R^2$$

Marin Chirciu-Romania

Solution: In ΔABC the following relationship holds:

$$\frac{32s^5 - a^5 - b^5 - c^5}{8s^3 - a^3 - b^3 - c^3} = \frac{5}{3} \cdot \frac{3s^4 + 2s^2(Rr + r^2) - r^2(8R^2 + 6Rr + r^2)}{s^2 + 2Rr + r^2}$$

Proof: We have $\sum a^5 = 2s[s^4 - 19s^2(Rr + r^2) + 5r^2(8R^2 + 6Rr + r^2)]$ and

$$\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$$

Let's get back to the main problem. Using the Lemma, the inequality can be written:

$$\frac{5}{3} \cdot \frac{3s^4 + 2s^2(Rr + r^2) - r^2(8R^2 + 6RR + r^2)}{s^2 + 2Rr + r^2} \leq 30R^2 \Leftrightarrow$$

$$\Leftrightarrow 3s^4 + 2s^2(Rr + r^2) - r^2(8R^2 + 6Rr + r^2) \leq 18R^2(s^2 + 2Rr + r^2) \Leftrightarrow$$

$$\Leftrightarrow s^2(18R^2 - 2Rr - 2r^2 - 3s^2) + r(36R^3 + 26R^2r + 6Rr^2 + r^3) \geq 0$$

We distinguish the cases:

Case 1. If $(18R^2 - 2Rr - 2r^2 - 3s^2) \geq 0$, the inequality is obvious.

Case 2. If $(18R^2-2Rr-2r^2-3s^2)<0$, the inequality cand be rewritten:

$$r(36R^3+26R^2r+6Rr^2+r^3)\ge s^2(3s^2+2r^2+2Rr-18R^2),$$

which follows from Gerretsen's inequality $s^2\le 4R^2+4Rr+3r^2$.

It remains to prove that:

$$r(36R^3+26R^2r+6Rr^2+r^3)\ge$$

$$\ge(4R^2+4Rr+3r^2)(3(4R^2+4Rr+3r^2)+2r^2+2Rr-18R^2)$$

$$\Leftrightarrow 244R^4+4R^3r-56R^2r^2-80Rr^3-32r^4\ge 0\Leftrightarrow$$

$$\Leftrightarrow 6R^4+R^3r-14R^2r^2-20Rr^3-8r^4\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(6R^3+13R^2r+12Rr^2+4r^3)\ge 0$, which follows from Euler's inequality

$$R\ge 2r.$$

Equality holds if and only if the triangle is equilateral. **Remark:** We can write the double inequality:

In ΔABC the following relationship holds:

$$\mathbf{120r^2\le\frac{32s^5-a^5-b^5-c^5}{8s^3-a^3-b^3-c^3}\le 30R^2.}$$

Solution: See above.

J.2423. In ΔABC the following relationship holds:

$$\sum\frac{r_a}{b+c}\sqrt{\sin A}\le\frac{9}{8}\sqrt{\frac{9R^2-24r^2}{2}}$$

Mehmet Șahin – Turkey

Remark: The problem can be developed: **In ΔABC the following relationship holds:**

$$\frac{3}{2R}\sqrt{\frac{F}{2}}\le\sum\frac{r_a}{b+c}\sqrt{\sin A}\le\left(\frac{1}{r}-\frac{1}{2R}\right)\sqrt{\frac{F}{2}}$$

Marin Chirciu-Romania

Solution: Right inequality.

$$\sum\frac{r_a}{b+c}\sqrt{\sin A}\overset{AM-GM}{\le}\sum\frac{r_a}{2\sqrt{bc}}\sqrt{\sin A}=\sum\frac{\sqrt{r_a}}{2\sqrt{bc}}\sqrt{r_a\sin A}\overset{CBS}{\le}\frac{1}{2}\sqrt{\sum\frac{r_a}{bc}\sum r_a\sin A}=$$

$$=\frac{1}{2}\sqrt{\frac{2R-r}{2Rr}\cdot\frac{s(2R-r)}{R}}=\frac{2R-r}{2R}\sqrt{\frac{s}{2r}}=\left(\frac{1}{r}-\frac{1}{2R}\right)\sqrt{\frac{F}{2}}.$$

We have used above $\sum \frac{r_a}{bc} = \frac{2R-r}{2Rr}$ and $\sum r_a \sin A = \frac{s(2R-r)}{R}$.

Left inequality:

$$\sum \frac{r_a}{b+c}\sqrt{\sin A} \overset{AM-GM}{\geq} 3\sqrt[3]{\prod \frac{r_a\sqrt{\sin A}}{b+c}} = 3\sqrt[3]{\frac{\prod r_a}{\prod(b+c)}\sqrt{\prod \sin A}} = 3\sqrt[3]{\frac{rs^2}{2s(s^2+r^2+2Rr)}\sqrt{\frac{sr}{2R^2}}}$$

$$\geq 3\sqrt[3]{\frac{rs^2}{16sR^2}\frac{1}{R}\sqrt{\frac{sr}{2}}} = \frac{3}{2R}\sqrt[3]{\frac{sr}{2}\sqrt{\frac{sr}{2}}} = \frac{3}{2R}\sqrt[3]{\left(\frac{sr}{2}\right)^{\frac{3}{2}}} = \frac{3}{2R}\sqrt{\frac{F}{2}}.$$

We have used above $\prod(b+c) = 2s(s^2+r^2+2Rr) \overset{Gerretsen\ \&\ Euler}{\leq} 16sR^2$.

Equality holds if and only if the triangle is equilateral.

J.2399. In acute ΔABC the following relationship holds:

$$\sum \frac{b^2+c^2}{a^2}\cos A \geq 24\left(\frac{r}{R}\right)^2 - 3$$

Mehmet Şahin – Turkey

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum \frac{b^2+c^2}{a^2}\cos A = \frac{s^6 - s^4(12Rr+r^2) + s^2r^2(8R^2-8Rr-r^2) + r^3(4R+r)^3}{8R^2r^2s^2}$$

Proof:

$$\sum \frac{b^2+c^2}{a^2}\cos A = \sum \frac{b^2+c^2}{a^2}\cdot\frac{b^2+c^2-a^2}{2bc} = \frac{1}{2abc}\sum \frac{(b^2+c^2)(b^2+c^2-a^2)}{a} =$$

$$= \frac{1}{2\cdot 4Rrs}\cdot\frac{s^6 - s^4(12Rr+r^2) + s^2r^2(8R^2-8Rr-r^2) + r^3(4R+r)^3}{Rrs} =$$

We have used above:

$$\sum \frac{(b^2+c^2)(b^2+c^2-a^2)}{a} = \frac{s^6 - s^4(12Rr+r^2) + s^2r^2(8R^2-8Rr-r^2) + r^3(4R+r)^3}{Rrs}$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$\sum \frac{b^2+c^2}{a^2}\cos A = \frac{s^6 - s^4(12Rr+r^2) + s^2r^2(8R^2-8Rr-r^2) + r^3(4R+r)^3}{8R^2r^2s^2} =$$

$$= \frac{1}{8R^2r^2}\left[s^2(s^2-r^2-12Rr) + r^2(8R^2-8Rr-r^2) + \frac{r^3(4R+r)^3}{s^2}\right] \overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq} \frac{1}{8R^2r^2}\left[(16Rr-5r^2)(16Rr-5r^2-r^2-12Rr) + r^2(8R^2-8Rr-r^2) + \frac{r^3(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$=\frac{1}{8R^2}\left[(16R-5r)(4R-6r)+(8R^2-8Rr-r^2)+\frac{2r(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{72R^3-108R^2r+25Rr^2-2r^3}{8R^2}\overset{Euler}{\geq}\frac{24R^3}{8R^2}=3\overset{Euler}{\geq}24\left(\frac{r}{R}\right)^2-3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\mathbf{3\leq\sum\frac{b^2+c^2}{a^2}\cos A\leq 3\left(\frac{R}{2r}\right)^3}$$

Marin Chirciu-Romania

Solution: Right inequality:

$$\sum\frac{b^2+c^2}{a^2}\cos A=\frac{s^6-s^4(12Rr+r^2)+s^2r^2(8R^2-8Rr-r^2)+r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2-r^2-12Rr)+r^2(8R^2-8Rr-r^2)+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{1}{8R^2r^2}\begin{bmatrix}(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2-r^2-12Rr)+\\+r^2(8R^2-8Rr-r^2)+\dfrac{r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\end{bmatrix}=$$

$$=\frac{1}{8R^2r^2}[(4R^2+4Rr+3r^2)(4R^2-8Rr+2r^2)+r^2(8R^2-8Rr-r^2)+r^2(4R+r)(R+r)]=$$

$$=\frac{16R^4-16R^3r-19Rr^3+6r^4}{8R^2r^2}\overset{Euler}{\leq}\frac{\frac{3R^5}{r}}{8R^2r^2}=\frac{3R^3}{8r^3}=3\left(\frac{R}{2r}\right)^3$$

Left inequality. Using the Lemma, we obtain.

$$\sum\frac{b^2+c^2}{a^2}\cos A=\frac{s^6-s^4(12Rr+r^2)+s^2r^2(8R^2-8Rr-r^2)+r^3(4R+r)^3}{8R^2r^2s^2}=$$

$$=\frac{1}{8R^2r^2}\left[s^2(s^2-r^2-12Rr)+r^2(8R^2-8Rr-r^2)+\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{1}{8R^2r^2}\begin{bmatrix}(16Rr-5r^2)(16Rr-5r^2-r^2-12Rr)+r^2(8R^2-8Rr-r^2)+\\+\dfrac{r^3(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\end{bmatrix}=$$

$$=\frac{1}{8R^2}\left[(16R-5r)(4R-6r)+(8R^2-8Rr-r^2)+\frac{2r(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{72R^3-108R^2r+25Rr^2-2r^3}{8R^2}\overset{Euler}{\geq}\frac{24R^3}{8R^2}=3\overset{Euler}{\geq}24\left(\frac{r}{R}\right)^2-3$$

Equality holds if and only if the triangle is equilateral.

Remark.

We can write the following inequalities:

$$24\left(\frac{r}{R}\right)^2-3\le 3\le\sum\frac{b^2+c^2}{a^2}\cos A\le 3\left(\frac{R}{2r}\right)^3.$$

Note: The right inequality strengthens the Problem J.2399 from RMM – 42.

J.2381 TRUE OR FALSE. If I – incenter in ΔABC then holds:

$$\frac{IA^2}{BC}+\frac{IB^2}{CA}+\frac{IC^2}{AB}\le R\sqrt{3}.$$

George Apostolopoulos – Greece

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum\frac{IA^2}{a}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}$$

Proof:

$$\sum\frac{IA^2}{a}=\sum\frac{\left(\frac{r}{\sin\frac{A}{2}}\right)^2}{a}=r^2\sum\frac{1}{a\sin^2\frac{A}{2}}=r^2\sum\frac{1}{a\cdot\frac{(s-b)(s-c)}{bc}}=$$

$$=r^2\sum\frac{bc}{a(s-b)(s-c)}=r^2\frac{\sum b^2c^2}{abc\prod(s-a)}=s^2\frac{s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]}{4Rs\cdot sr^2}$$

$$=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}$$

$$=\frac{1}{8R^3}\sum\frac{a^3(b^2+c^2-a^2)}{bc}=\frac{1}{8R^3}\frac{\sum a^4\,(b^2+c^2-a^2)}{abc}=\frac{F(s^2-3R^2-4Rr-r^2)}{R^4}$$

We have used above $\sum b^2c^2=s[s^2(s^2+2r^2-12Rr)+r^3(4R+r)]$

Let's get back to the main problem. Using the Lemma the inequality $\frac{IA^2}{BC}+\frac{IB^2}{CA}+\frac{IC^2}{AB}\le R\sqrt{3}$ can be written:

$$\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}\le R\sqrt{3}$$

With Gerretsen's inequality $s^2\le 4R^2+4Rr+3r^2$ we obtain:

$$\sum\frac{IA^2}{a}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}\overset{Gerretsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}=$$

$$=\frac{(4R^2+4Rr+3r^2)(4R^2-8Rr-5r^2)+r^3(4R+r)}{4Rrs}=\frac{16R^4-16R^3r+16r^4}{4Rrs}=$$

$$=\frac{4(R^4-R^3r+r^4)}{Rrs}\overset{Euler}{\leq}\frac{4\cdot\frac{9R^5}{32r}}{Rrs}=\frac{9R^4}{8r^2s}\overset{Mitrinovic}{\leq}\frac{9R^4}{8r^2\cdot 3\sqrt{3}r}=\frac{\sqrt{3}R^4}{8r^2\cdot r}=R\sqrt{3}\left(\frac{R}{2r}\right)^3$$

The inequality $\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}\leq R\sqrt{3}$ is false. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{4r}{Rs}(2R-r)^2\leq\frac{IA^2}{a}+\frac{IB^2}{b}+\frac{IC^2}{c}\leq R\sqrt{3}\left(\frac{R}{2r}\right)^3$$

Marin Chirciu-Romania

Solution: Using the Lemma, we obtain: Right inequality:

With Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$ we obtain:

$$\sum\frac{IA^2}{a}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}=$$

$$=\frac{4(R^4-R^3r+r^4)}{Rrs}\overset{Euler}{\leq}\frac{4\cdot\frac{9R^5}{32r}}{Rrs}=\frac{9R^4}{8r^2s}\overset{Mitrinovic}{\leq}\frac{9R^4}{8r^2\cdot 3\sqrt{3}r}=\frac{\sqrt{3}R^4}{8r^2\cdot r}=R\sqrt{3}\left(\frac{R}{2r}\right)^3.$$

Equality holds if and only if the triangle is equilateral.Left inequality:

$$\sum\frac{IA^2}{a}=\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(16Rr-5r^2+2r^2-12Rr)+r^3(4R+r)}{4Rrs}=$$

$$=\frac{(16Rr-5r^2)(4Rr-3r^2)+r^3(4R+r)}{4Rrs}=\frac{64R^2r^2-64Rr^3+16r^4}{4Rrs}=$$

$$=\frac{16r^2(4R^2-4Rr+r^2)}{4Rrs}=\frac{4r(4R^2-4Rr+r^2)}{Rs}=\frac{4r(2R-r)^2}{Rs}.$$

Equality holds if and only if the triangle is equilateral.

J.2346. If $x,y>0$ and $m\geq 0$ then:

$$(x^{2m+2}+1)(y^{2m+2}+1)\geq\frac{(x+y)^{2m+2}}{4^m}$$

D.M. Bătinețu – Giurgiu – Romania

Solution: $LHS=(x^{2m+2}+1)(y^{2m+2}+1)=(x^{2m+2}+1)(1+y^{2m+2})\overset{CBS}{\geq}$

$$\overset{CBS}{\geq}(x^{m+1}+y^{m+1})^2\overset{Holder}{\geq}\left(\frac{(x+y)^{m+1}}{2^m}\right)^2=\frac{(x+y)^{2m+2}}{4^m}=RHS.$$

Equality holds if and only if $x=y=1$. **Remark:** The problem can be developed.

If $x,y>0,m\geq 0$ and $\lambda\geq 0$ then

$$(x^{2m+2}+\lambda)(y^{2m+2}+\lambda)\geq\frac{\lambda(x+y)^{2m+2}}{4^m}$$

Marin Chirciu-Romania

Solution:

$$LHS=(x^{2m+2}+\lambda)(y^{2m+2}+\lambda)=\left((x^{m+1})^2+\left(\sqrt{\lambda}\right)^2\right)\left(\left(\sqrt{\lambda}\right)^2+(y^{m+1})^2\right)\overset{CBS}{\geq}$$

$$\overset{CBS}{\geq}\left(\sqrt{\lambda}x^{m+1}+\sqrt{\lambda}y^{m+1}\right)^2=\lambda(x^{m+1}+y^{m+1})^2\overset{Holder}{\geq}\lambda\left(\frac{(x+y)^{m+1}}{2^m}\right)^2=\lambda\frac{(x+y)^{2m+2}}{4^m}=RHS$$

Equality holds if and only if $x=y=\lambda^{\frac{1}{2m+2}}$.

Note: For $\lambda=1$ we obtain Problem J.2346 from RMM -42.

J.2484. If $m,n>0$ then in ΔABC holds:

$$\sum\frac{\tan^3\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}}\geq\frac{((4R+r)^2-2s^2)^2}{(m+n)s^4}$$

D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution:

$$LHS=\sum\frac{\tan^3\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}}=\sum\frac{\tan^4\frac{A}{2}}{m\cdot\tan\frac{A}{2}\tan\frac{B}{2}+n\cdot\tan\frac{A}{2}\tan\frac{C}{2}}\overset{Holder}{\geq}$$

$$\overset{Holder}{\geq}\frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{\sum\left(m\cdot\tan\frac{A}{2}\tan\frac{B}{2}+n\cdot\tan\frac{A}{2}\tan\frac{C}{2}\right)}=\frac{\left(\sum\tan^2\frac{A}{2}\right)^2}{(m+n)\sum\tan\frac{B}{2}\tan\frac{C}{2}}=$$

$$=\frac{\left(\frac{(4R+r)^2-2s^2}{s^2}\right)^2}{(m+n)\cdot 1}=\frac{((4R+r)^2-2s^2)^2}{(m+n)s^4}=RHS.$$

We have used above $\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2-2s^2}{s^2}$ and $\sum\tan\frac{B}{2}\tan\frac{C}{2}=1$.

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

If $m,n>0$ and $k\in\mathbb{N}$ then in ΔABC holds:

$$\sum \frac{\tan^{2k+1}\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}} \ge \frac{\left(2-\frac{2r}{R}\right)^{k+1}}{3^{k-1}(m+n)}$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{\tan^{2k+1}\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}} = \sum \frac{\tan^{2k+2}\frac{A}{2}}{m\cdot\tan\frac{A}{2}\tan\frac{B}{2}+n\cdot\tan\frac{A}{2}\tan\frac{C}{2}} \overset{Holder}{\ge}$$

$$\overset{Holder}{\ge} \frac{\left(\sum\tan^2\frac{A}{2}\right)^{k+1}}{3^{k-1}\sum\left(m\cdot\tan\frac{A}{2}\tan\frac{B}{2}+n\cdot\tan\frac{A}{2}\tan\frac{C}{2}\right)} = \frac{\left(\sum\tan^2\frac{A}{2}\right)^{k+1}}{3^{k-1}(m+n)\sum\tan\frac{B}{2}\tan\frac{C}{2}} =$$

$$= \frac{\left(2-\frac{2r}{R}\right)^{k+1}}{3^{k-1}(m+n)\cdot 1} = \frac{\left(2-\frac{2r}{R}\right)^{k+1}}{3^{k-1}(m+n)} = RHS$$

$$\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1$$

Equality holds if and only if the triangle is equilateral.

Note: For $k=1$ we obtain Problem J.2484 from RMM – 43.

J.2550. In ΔABC the following relationship holds:

$$\sum \frac{\tan\frac{A}{2}}{h_a^4} \ge \frac{1}{3}\sum \frac{\cot\frac{A}{2}}{h_a^4}$$

D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

Solution: Lemma 1: In ΔABC the following relationship holds:

$$\sum \frac{\tan\frac{A}{2}}{h_a^4} = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4}$$

Proof:

$$\sum \frac{\tan\frac{A}{2}}{h_a^4} = \sum \frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{\left(\frac{2S}{a}\right)^4} = \frac{1}{16S^4}\sum a^4 \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{1}{16S^4}\frac{1}{S}\sum a^4(s-b)(s-c) =$$

$$= \frac{1}{16s^5r^5}\cdot 4rs^2[s^2(R-2r)+r^2(5R+2r)] = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4}.$$

We have used above $\sum a^4(s-b)(s-c) = 4rs^2[s^2(R-2r)+r^2(5R+2r)]$.

Lemma 2: In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{h_a^4} = \frac{s^2(R+2r) - r(12R^2 + 11Rr + 2r^2)}{4s^3r^4}$$

Proof:

$$\sum \frac{\cot\frac{A}{2}}{h_a^4} = \sum \frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{\left(\frac{2S}{a}\right)^4} = \frac{1}{16S^4}\sum a^4 \frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{1}{16S^4}\frac{s}{S}\sum a^4(s-a) = \frac{s}{16s^5r^5}\cdot 4rs[s^2(R+2r) - r(12R^2+11Rr+2r^2)] =$$

$$= \frac{s^2(R+2r) - r(12R^2+11Rr+2r^2)}{4s^3r^4}$$

We have used above $\sum a^4(s-a) = 4rs[s^2(R+2r) - r(12R^2+11Rr+2r^2)]$

Let's get back to the main problem. Using the above Lemmas we have the sums:

$\sum \frac{\tan\frac{A}{2}}{h_a^4} = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4}$ and $\sum \frac{\cot\frac{A}{2}}{h_a^4} = \frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}$.

The inequality can be written:

$$\frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4} \ge \frac{1}{3}\cdot\frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4} \Leftrightarrow$$

$$\Leftrightarrow 3s^2(R-2r) + 3r^2(5R+2r) \ge s^2(R+2r) - r(12R^2+11Rr+2r^2) \Leftrightarrow$$

$\Leftrightarrow s^2(R-4r) + r(6R^2+13Rr+4r^2) \ge 0$. We distinguish the cases:

Case 1. If $(R-4r) \ge 0$ the inequality is obvious.

Case 2. If $(R-4r) < 0$ the inequality cand be rewritten:

$r(6R^2+13Rr+4r^2) \ge s^2(4r-R)$, which follows from Gerretsen's inequality

$s^2 \le 4R^2+4Rr+3r^2$. It remains to prove that:

$$r(6R^2+13Rr+4r^2) \ge (4R^2+4Rr+3r^2)(4r-R) \Leftrightarrow 2R^3 - 3R^2r - 4r^3 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2+Rr+2r^2) \ge 0$, obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{3}{rs^3}\left(\frac{R}{2r}\right)^2 \le \sum \frac{\tan\frac{A}{2}}{h_a^4} \le \frac{3}{rs^3}\left(\frac{R}{2r}\right)^4$$

Marin Chirciu-Romania

Solution:Lemma: In ΔABC the following relationship holds:

$$\sum \frac{\tan \frac{A}{2}}{h_a^4} = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4}.$$

Proof:

$$\sum \frac{\tan \frac{A}{2}}{h_a^4} = \sum \frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{\left(\frac{2S}{a}\right)^4} = \frac{1}{16S^4}\sum a^4 \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{1}{16S^4}\frac{1}{S}\sum a^4 (s-b)(s-c) =$$

$$= \frac{1}{16s^5r^5} \cdot 4rs^2[s^2(R-2r)+r^2(5R+2r)] = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4}.$$

We have used above $\sum a^4 (s-b)(s-c) = 4rs^2[s^2(R-2r)+r^2(5R+2r)]$.

Let's get back to the main problem.Using the Lemma we obtain:

Right inequality:

$$\sum \frac{\tan \frac{A}{2}}{h_a^4} = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4} \overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq} \frac{(4R^2+4Rr+3r^2)(R-2r)+r^2(5R+2r)}{4s^3r^4} = \frac{R^3-R^2r-r^3}{s^3r^4} \overset{Euler}{\leq} \frac{\frac{3R^4}{16r}}{s^3r^4} =$$

$$= \frac{3R^4}{16s^3r^5} = \frac{3}{rs^3}\left(\frac{R}{2r}\right)^4.$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\sum \frac{\tan \frac{A}{2}}{h_a^4} = \frac{s^2(R-2r)+r^2(5R+2r)}{4s^3r^4} \overset{Gerretsen}{\geq} \frac{(16Rr-5r^2)(R-2r)+r^2(5R+2r)}{4s^3r^4} =$$

$$= \frac{4R^2-8Rr+3r^2}{s^3r^3} \overset{Euler}{\geq} \frac{\frac{3R^2}{4}}{s^3r^3} = \frac{3R^2}{4s^3r^3} = \frac{3}{rs^3}\left(\frac{R}{2r}\right)^2.$$

Equality holds if and only if the triangle is equilateral.**Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{9}{rs^3}\left(\frac{R}{2r}\right) \leq \sum \frac{\cot \frac{A}{2}}{h_a^4} \leq \frac{9}{rs^3}\left(\frac{R}{2r}\right)^3$$

Marin Chirciu -Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum\frac{\cot\frac{A}{2}}{h_a^4}=\frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}.$$

Proof:

$$\sum\frac{\cot\frac{A}{2}}{h_a^4}=\sum\frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{\left(\frac{2S}{a}\right)^4}=\frac{1}{16S^4}\sum a^4\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{1}{16S^4}\frac{s}{S}\sum a^4(s-a)=\frac{s}{16s^5r^5}\cdot 4rs[s^2(R+2r)-r(12R^2+11Rr+2r^2)]=$$

$$=\frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}.$$

We have used above $\sum a^4(s-a)=4rs[s^2(R+2r)-r(12R^2+11Rr+2r^2)]$.

Let's get back to the main problem. Using the Lemma we obtain:

$$\sum\frac{\cot\frac{A}{2}}{h_a^4}=\frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}=$$

$$=\frac{R^3+r^3}{s^3r^4}\overset{Euler}{\leq}\frac{\frac{9R^3}{8}}{s^3r^4}=\frac{9R^3}{8s^3r^4}=\frac{9}{rs^3}\left(\frac{R}{2r}\right)^3$$

Equality holds if and only if the triangle is equilateral.

Left inequality: $\sum\frac{\cot\frac{A}{2}}{h_a^4}=\frac{s^2(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}\overset{Gerretsen}{\geq}$

$$\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(R+2r)-r(12R^2+11Rr+2r^2)}{4s^3r^4}=$$

$$=\frac{R^2+4Rr-3r^2}{s^3r^3}\overset{Euler}{\geq}\frac{\frac{9Rr}{2}}{s^3r^3}=\frac{9R}{s^3r^2}=\frac{9}{rs^3}\left(\frac{R}{2r}\right)$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{3}{rs^3}\leq\sum\frac{\tan\frac{A}{2}}{r_a^4}\leq\frac{3}{rs^3}\left(\frac{R}{2r}\right)^3$$

Marin Chirciu-Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum \frac{\tan\frac{A}{2}}{r_a^4} = \frac{s^2 - 12Rr}{s^3 r^3}$$

Proof:

$$\sum \frac{\tan\frac{A}{2}}{r_a^4} = \sum \frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{\left(\frac{S}{s-a}\right)^4} = \frac{1}{S^4}\sum (s-a)^4 \frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$= \frac{1}{S^4}\frac{\prod(s-a)}{S}\sum (s-a)^3 = \frac{r^2 s}{s^5 r^5}\cdot s(s^2-12Rr) = \frac{s^2-12Rr}{s^3 r^3}$$

We have used above $\sum (s-a)^3 = s(s^2-12Rr)$. Let's get back to the main problem.

Using the above Lemma, we obtain: Right inequality:

$$\sum \frac{\tan\frac{A}{2}}{r_a^4} = \frac{s^2-12Rr}{s^3r^3} \overset{Gerretsen}{\leq} \frac{4R^2+4Rr+3r^2-12Rr}{s^3r^3} =$$

$$= \frac{4R^2-8Rr+3r^2}{s^3r^3} \overset{Euler}{\leq} \frac{\frac{3R^3}{8r}}{s^3r^3} = \frac{3R^3}{8s^3r^4} = \frac{3}{rs^3}\left(\frac{R}{2r}\right)^3.$$

Equality holds if and only if the triangle is equilateral. Left inequality:

$$\sum \frac{\tan\frac{A}{2}}{r_a^4} = \frac{s^2-12Rr}{s^3r^3} \overset{Gerretsen}{\geq} \frac{16Rr-5r^2-12Rr}{s^3r^3} = \frac{4Rr-5r^2}{s^3r^3} \overset{Euler}{\geq} \frac{3r^2}{s^3r^3} = \frac{3}{s^3r}$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\frac{9}{rs^3}\left(\frac{R}{2r}\right)^2 \leq \sum \frac{\cot\frac{A}{2}}{r_a^4} \leq \frac{9}{rs^3}\left(\frac{R}{2r}\right)^5$$

Marin Chirciu-Romania

Solution : Lemma: In ΔABC the following relationship holds:

$$\sum \frac{\cot\frac{A}{2}}{r_a^4} = \frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}.$$

Proof:

$$\sum \frac{\cot\frac{A}{2}}{r_a^4} = \sum \frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{\left(\frac{S}{s-a}\right)^4} = \frac{1}{S^4}\sum (s-a)^4 \frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} =$$

$$=\frac{1}{S^4}\frac{s}{S}\sum(s-a)^5=\frac{s}{s^5r^5}\cdot s[s^2(s^2-20Rr)+20Rr^2(4R+r)]=$$
$$=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}$$

We have used a above $\sum(s-a)^5=s[s^2(s^2-20Rr)+20Rr^2(4R+r)]$.

Let's get back to the main problem. Using the above Lemma, we obtain:

Right inequality:

$$\sum\frac{\cot\frac{A}{2}}{r_a^4}=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}=$$

$$=\frac{16R^4-48R^3r+40R^2r^2-16Rr^3+9r^4}{s^3r^3}\overset{Euler}{\leq}\frac{\frac{9R^5}{32r}}{s^3r^3}=\frac{9R^5}{32s^3r^4}=\frac{9}{rs^3}\left(\frac{R}{2r}\right)^5.$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\sum\frac{\cot\frac{A}{2}}{r_a^4}=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(16Rr-5r^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}=$$

$$=\frac{16R^2-40Rr+25r^2}{s^3r^3}\overset{Euler}{\geq}\frac{\frac{9R^2}{4}}{s^3r^3}=\frac{9R^2}{4s^3r^3}=\frac{9}{rs^3}\left(\frac{R}{2r}\right)^2.$$

Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed. **In ΔABC the following relationship holds:**

$$\sum\frac{\tan\frac{A}{2}}{r_a^4}\geq\frac{1}{3}\left(\frac{2r}{R}\right)^5\sum\frac{\cot\frac{A}{2}}{r_a^4}$$

Marin Chirciu-Romania

Solution: Lemma 1: In ΔABC the following relationship holds:

$$\sum\frac{\tan\frac{A}{2}}{r_a^4}=\frac{s^2-12Rr}{s^3r^3}$$

Proof:

$$\sum\frac{\tan\frac{A}{2}}{r_a^4}=\sum\frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}{\left(\frac{S}{s-a}\right)^4}=\frac{1}{S^4}\sum(s-a)^4\frac{(s-b)(s-c)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{1}{S^4}\frac{\prod(s-a)}{S}\sum(s-a)^3=\frac{r^2s}{s^5r^5}\cdot s(s^2-12Rr)=\frac{s^2-12Rr}{s^3r^3}.$$

We have use above $\sum(s-a)^3=s(s^2-12Rr)$.

Lemma 2: In ΔABC the following relationship holds:

$$\sum\frac{\cot\frac{A}{2}}{r_a^4}=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}$$

Proof:

$$\sum\frac{\cot\frac{A}{2}}{r_a^4}=\sum\frac{\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}}{\left(\frac{S}{s-a}\right)^4}=\frac{1}{S^4}\sum(s-a)^4\frac{s(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}}=$$

$$=\frac{1}{S^4}\frac{s}{S}\sum(s-a)^5=\frac{s}{s^5r^5}\cdot s[s^2(s^2-20Rr)+20Rr^2(4R+r)]=$$

$$=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}$$

We have used above $\sum(s-a)^5=s[s^2(s^2-20Rr)+20Rr^2(4R+r)]$.

Let's get back to the main problem. Using the above Lemmas, we have the sums:

$$\sum\frac{\tan\frac{A}{2}}{r_a^4}=\frac{s^2-12Rr}{s^3r^3}\ ;\quad \sum\frac{\cot\frac{A}{2}}{r_a^4}=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}$$

We obtain:

$$\frac{\sum\frac{\tan\frac{A}{2}}{r_a^4}}{\sum\frac{\cot\frac{A}{2}}{r_a^4}}=\frac{\frac{s^2-12Rr}{s^3r^3}}{\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}}\overset{(1)}{\geq}\frac{\frac{3}{rs^3}}{\frac{9}{rs^3}\left(\frac{R}{2r}\right)^5}=\frac{1}{3\left(\frac{R}{2r}\right)^5}=\frac{1}{3}\left(\frac{2r}{R}\right)^5.$$

where (1) follows from: $\sum\frac{\tan\frac{A}{2}}{r_a^4}\geq\frac{s^2-12Rr}{s^3r^3}$ and $\sum\frac{\cot\frac{A}{2}}{r_a^4}\leq\frac{9}{rs^3}\left(\frac{R}{2r}\right)^5$, see:

$$\sum\frac{\tan\frac{A}{2}}{r_a^4}=\frac{s^2-12Rr}{s^3r^3}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2-12R}{s^3r^3}=\frac{4R^2-8Rr+3r^2}{s^3r^3}\overset{Euler}{\leq}$$

$$\overset{Euler}{\leq}\frac{\frac{3R^3}{8r}}{s^3r^3}=\frac{3R^3}{8s^3r^4}=\frac{3}{rs^3}\left(\frac{R}{2r}\right)^3.$$

$$\sum\frac{\cot\frac{A}{2}}{r_a^4}=\frac{s^2(s^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2-20Rr)+20Rr^2(4R+r)}{s^3r^5}=$$

$$\frac{16R^4-48R^3r+40R^2r^2-16Rr^3+9r^4}{s^3r^3}\overset{Euler}{\leq}\frac{\frac{9R^5}{32r}}{s^3r^3}=\frac{9R^5}{32s^3r^4}=\frac{9}{rs^3}\left(\frac{R}{2r}\right)^5.$$

Equality holds if and only if the triangle is equilateral.

RMM SOLVED PROBLEMS V

By Marin Chirciu – Romania

J.2585. In $\Delta ABC, I$ – incenter the following relationship holds:

$$IA^4+IB^4+IC^4\geq\frac{(a^2+b^2+c^2)^2}{27}$$

Daniel Sitaru – Romania

Solution:

$$LHS=\sum IA^4\overset{CS}{\geq}\frac{(\sum IA^2)^2}{3}\overset{(1)}{\geq}\frac{(a^2+b^2+c^2)^2}{27}=RHS$$

where (1) $\Leftrightarrow\frac{(\sum IA^2)^2}{3}\geq\frac{(a^2+b^2+c^2)^2}{27}\Leftrightarrow\sum IA^2\geq\frac{a^2+b^2+c^2}{3}\Leftrightarrow 3\sum IA^2\geq a^2+b^2+c^2$.

Lemma: In $\Delta ABC, I$ – incenter the following relationship holds:

$$3\sum IA^2\geq a^2+b^2+c^2$$

Proof: Using $\sum IA^2=s^2+r^2-8Rr$ and $\sum a^2=2(s^2-r^2-4Rr)$ the inequality can be written:

$3(s^2+r^2-8Rr)\geq 2(s^2-r^2-4Rr)\Leftrightarrow s^2\geq 16Rr-5r^2$, (Gerretsen).

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In $\Delta ABC, I$ – incenter and $n\in\mathbb{N}$ the following relationship holds:

$$IA^{2n}+IB^{2n}+IC^{2n}\geq\frac{(a^2+b^2+c^2)^n}{3^{2n-1}}$$

Marin Chirciu – Romania

Solution: For $n=0$ we obtain equality $3=3$. For $n=1$ we obtain the Lemma.

For $n\geq 2$ we use Holder's inequality.

$$LHS = \sum IA^{2n} \overset{Holder}{\geq} \frac{(\sum IA^2)^n}{3^{n-1}} \overset{(1)}{\geq} \frac{(a^2+b^2+c^2)^n}{3^{2n-1}} = RHS,$$

where (1) $\Leftrightarrow \frac{(\sum IA^2)^n}{3^{n-1}} \geq \frac{(a^2+b^2+c^2)^n}{3^{2n-1}} \Leftrightarrow \sum IA^2 \geq \frac{a^2+b^2+c^2}{3}$.

Lemma: In $\Delta ABC, I$ – incenter the following relationship holds:

$$\sum IA^2 \geq \frac{a^2+b^2+c^2}{3}.$$

Proof:

Using $\sum IA^2 = s^2 + r^2 - 8Rr$ and $\sum a^2 = 2(s^2 - r^2 - 4Rr)$ the inequality can be written:

$3(s^2 + r^2 - 8Rr) \geq 2(s^2 - r^2 - 4Rr) \Leftrightarrow s^2 \geq 16Rr - 5r^2$, (Gerretsen).

Equality holds if and only if the triangle is equilateral.

Note: For $n = 2$ we obtain Problem J.2585 from RMM – 43.

J.2555. If $x, y, z \geq 0$ then:

$$x^3 + y^3 + 2z^3 \geq z(xz + yz + 2xy).$$

Daniel Sitaru – Romania

Solution: We have $x^3 + z^3 \geq xz(x+z) \Leftrightarrow (x+z)(x-z)^2 \geq 0$, with equality for $x = z$.

Analogous $y^3 + z^3 \geq yz(y+z)$. Adding $x^3 + z^3 \geq xz(x+z)$ and $y^3 + z^3 \geq yz(y+z)$ we obtain:

$$LHS = x^3 + y^3 + 2z^3 \geq xz(x+z) + yz(y+z) \overset{(1)}{\geq} z(xz + yz + 2xy) = RHS,$$

where (1) $\Leftrightarrow xz(x+z) + yz(y+z) \geq z(xz + yz + 2xy) \Leftrightarrow z(x^2 + y^2) \geq 2xyz \Leftrightarrow$

$\Leftrightarrow z(x-y)^2 \geq 0$. Equality holds if and only if $x = y = z$. **Remark:** The problem can be developed.

If $x, y, z, t \geq 0$ then: $x^3 + y^3 + z^3 + 3t^3 \geq t(xt + yt + zt + xy + yz + zx)$

Marin Chirciu – Romania

Solution: We have $x^3 + t^3 \geq xt(x+t) \Leftrightarrow (x+z)(x-z)^2 \geq 0$, with equality for $x = z$.

Analogous $y^3 + t^3 \geq yt(y+t)$ and $z^3 + t^3 \geq zt(z+t)$

Adding $x^3 + t^3 \geq xt(x+t), y^3 + t^3 \geq yt(y+t)$ and $z^3 + t^3 \geq zt(z+t)$ we obtain:

$$LHS = x^3 + y^3 + z^3 + 3t^3 \geq xt(x+t) + yt(y+t) + zt(z+t) \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} t(xt + yt + zt + xy + yz + zx) = RHS,$$

where (1) $\Leftrightarrow xt(x+t) + yt(y+t) + zt(z+t) \geq t(xt + yt + zt + xy + yz + zx) \Leftrightarrow$

$$\Leftrightarrow t(x^2+y^2+z^2) \ge t(xy+yz+zx) \Leftrightarrow t\sum(x-y)^2 \ge 0.$$

Equality holds if and only if $x=y=z=t$.

Note: Inequality is an extension of the Problem J.2555 from RMM – 43.

J.2539. If $x,y,z>0$ then

$$\mathbf{6\sum(x+y)^4 \ge 96xyz(x+y+z)+\sum(y-x)(x+y+2z)}$$

Daniel Sitaru – Romania

Solution:

We have $\sum(y-x)(x+y+2z)=2\sum xy-2\sum x^2 \le 0$, see $\sum x^2 \ge \sum xy$, with equality for $x=y=z$. It suffices to prove that:

$$6\sum(x+y)^4 \ge 96xyz(x+y+z) \Leftrightarrow \sum(x+y)^4 \ge 16xyz(x+y+z) \Leftrightarrow$$

$$\Leftrightarrow 2\sum x^4+4\sum xy\,(x^2+y^2)+6\sum x^2y^2 \ge 16xyz(x+y+z) \Leftrightarrow$$

$\Leftrightarrow \sum x^4+2\sum xy\,(x^2+y^2)+3\sum x^2y^2 \ge 8xyz(x+y+z)$, which follows from:

$\sum x^4 \ge \sum x^2y^2$ and $\sum xy(x^2+y^2) \ge 2\sum x^2y^2$, see $x^2+y^2 \ge 2xy$.

It remains to prove that:

$\sum x^2y^2+2\sum x^2y^2+3\sum x^2y^2 \ge 8xyz\sum x \Leftrightarrow \sum x^2y^2 \ge xyz\sum x$, see $\sum a^2 \ge \sum ab$,

for $(a,b,c)=(xy,yz,zx)$.

Equality holds if and only if $x=y=z$. **Remark:** The problem can be developed.

If $x,y,z>0$ and $\lambda \ge 0$ then

$$\boldsymbol{\lambda\sum(x+y)^4 \ge 6\lambda xyz(x+y+z)+\sum(y-x)(x+y+2z)}$$

Marin Chirciu – Romania

Solution:

We have $\sum(y-x)(x+y+2z)=2\sum xy-2\sum x^2 \le 0$, see $\sum x^2 \ge \sum xy$, with equality for $x=y=z$. It suffices to prove that:

$$\lambda\sum(x+y)^4 \ge 6\lambda xyz(x+y+z) \Leftrightarrow \sum(x+y)^4 \ge 16xyz(x+y+z) \Leftrightarrow$$

$$\Leftrightarrow 2\sum x^4+4\sum xy\,(x^2+y^2)+6\sum x^2y^2 \ge 16xyz(x+y+z) \Leftrightarrow$$

$\Leftrightarrow \sum x^4+2\sum xy\,(x^2+y^2)+3\sum x^2y^2 \ge 8xyz(x+y+z)$, which follows from:

$\sum x^4 \geq \sum x^2y^2$ and $\sum xy(x^2+y^2) \geq 2\sum x^2y^2$, see $x^2+y^2 \geq 2xy$.

It remains to prove that:

$\sum x^2y^2 + 2\sum x^2y^2 + 3\sum x^2y^2 \geq 8xyz\sum x \Leftrightarrow \sum x^2y^2 \geq xyz\sum x$, see $\sum a^2 \geq \sum ab$, for $(a,b,c) = (xy,yz,zx)$. Equality holds if and only if $x=y=z$.

J.2545. If $a,b,c>0, a+b+c=3$ then

$$\sum \frac{1}{a^3+b^3} + 3\sum \frac{1}{ab(a+b)} \geq 6$$

Daniel Sitaru, Dan Nănuți – Romania

Solution: Lemma: If $a,b>0$ then

$$\frac{1}{a^3+b^3} + \frac{3}{ab(a+b)} \geq \frac{16}{(a+b)^3}.$$

Proof:

$$\frac{1}{a^3+b^3} + \frac{3}{ab(a+b)} = \frac{1}{(a+b)(a^2-ab+b^2)} + \frac{1}{ab(a+b)} + \frac{1}{ab(a+b)} + \frac{1}{ab(a+b)}$$

$$\overset{AM-GM}{\geq} 4\sqrt[4]{\frac{1}{(a+b)(a^2-ab+b^2)} \cdot \frac{1}{ab(a+b)} \cdot \frac{1}{ab(a+b)} \cdot \frac{1}{ab(a+b)}} =$$

$$= \frac{4}{a+b} \frac{1}{\sqrt[4]{(a^2-ab+b^2)\cdot ab \cdot ab \cdot ab}} \overset{AM-GM}{\geq} \frac{4}{a+b} \cdot \frac{1}{\frac{(a^2-ab+b^2)+ab+ab+ab}{4}} =$$

$= \frac{4}{a+b} \cdot \frac{4}{a^2+2ab+b^2} = \frac{4}{a+b} \cdot \frac{4}{(a+b)^2} = \frac{16}{(a+b)^3}$, with equality for $a=b$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS = \sum \frac{1}{a^3+b^3} + 3\sum \frac{1}{ab(a+b)} \overset{Lemma}{\geq} \sum \frac{16}{(a+b)^3} = 16\sum \left(\frac{1}{a+b}\right)^3 \overset{Holder}{\geq}$$

$$\overset{Holder}{\geq} 16\frac{\left(\sum \frac{1}{a+b}\right)^3}{9} \overset{CS}{\geq} 16\frac{\left(\frac{9}{\sum(a+b)}\right)^3}{9} = 16\frac{\left(\frac{9}{2\sum a}\right)^3}{9} = 16\frac{\left(\frac{9}{2\cdot 3}\right)^3}{9} =$$

$$= 16\frac{\left(\frac{3}{2}\right)^3}{9} = 6 = RHS.$$

Equality holds if and only if $a=b=c=1$. **Remark:** The problem can be developed.

If $a,b,c>0, a+b+c=3$ then:

$$\sum \frac{1}{a^2-ab+b^2} + \frac{9}{abc} \geq 12$$

Marin Chirciu – Romania

Solution: Lemma: If $a,b>0$ then

$$\frac{1}{a^2-ab+b^2}+\frac{3}{ab}\ge\frac{16}{(a+b)^2}.$$

Proof:

$$\frac{1}{a^2-ab+b^2}+\frac{3}{ab}\ge\frac{16}{(a+b)^2}\Leftrightarrow(3a^2-2ab+3b^2)(a+b)^2\ge16ab(a^2-ab+b^2)\Leftrightarrow$$

$\Leftrightarrow 3a^4-12a^3b+18a^2b^2-12ab^3+3b^4\ge0\Leftrightarrow3(a-b)^4\ge0$, with equality for $a=b$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$\sum\frac{1}{a^2-ab+b^2}+3\sum\frac{1}{ab}\overset{Lemma}{\ge}\sum\frac{16}{(a+b)^2}=16\sum\left(\frac{1}{a+b}\right)^2\overset{CS}{\ge}$$

$$\overset{CS}{\ge}16\frac{\left(\sum\frac{1}{a+b}\right)^2}{3}\overset{CS}{\ge}16\frac{\left(\frac{9}{\sum(a+b)}\right)^2}{3}=16\frac{\left(\frac{9}{2\sum a}\right)^2}{3}=16\frac{\left(\frac{9}{2\cdot3}\right)^2}{3}=16\frac{\left(\frac{3}{2}\right)^2}{3}=12.$$

From $\sum\frac{1}{a^2-ab+b^2}+3\sum\frac{1}{ab}\ge12$ and $a+b+c=3\Rightarrow\sum\frac{1}{a^2-ab+b^2}+3\sum\frac{1}{ab}\ge12\Leftrightarrow$

$$\Leftrightarrow\sum\frac{1}{a^2-ab+b^2}+3\cdot\frac{a+b+c}{abc}\ge12\Leftrightarrow\sum\frac{1}{a^2-ab+b^2}+3\cdot\frac{3}{abc}\ge12\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{a^2-ab+b^2}+\frac{9}{abc}\ge12.$$

Equality holds if and only if $a=b=c=1$. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\sum\frac{1}{\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}}\ge3(36r^2-s^2)$$

Marin Chirciu – Romania

Solution: Lemma: If $a,b,c>0, a+b+c=3$ then

$$\sum\frac{1}{a^2-ab+b^2}+\frac{9}{abc}\ge12$$

Proof:

$$\frac{1}{a^2-ab+b^2}+\frac{3}{ab}\ge\frac{16}{(a+b)^2}\Leftrightarrow(3a^2-ab+3b^2)(a+b)^2\ge16ab(a^2-ab+b^2)\Leftrightarrow$$

$\Leftrightarrow 3a^4-12a^3b+18a^2b^2-12ab^3+3b^4\ge0\Leftrightarrow3(a-b)^4\ge0$, with the equality $a=b$.

Let's get back to the main problem. It is known the identity in triangle

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\Leftrightarrow\frac{3r}{r_a}+\frac{3r}{r_b}+\frac{3r}{r_c}=3$$

Using the Lemma for $(a,b,c)=\left(\frac{3r}{r_a},\frac{3r}{r_b},\frac{3r}{r_c}\right)$ we obtain:

$$\sum\frac{1}{\left(\frac{3r}{r_a}\right)^2-\frac{3r}{r_a}\cdot\frac{3r}{r_b}+\left(\frac{3r}{r_b}\right)^2}+\frac{9}{\frac{3r}{r_a}\cdot\frac{3r}{r_b}\cdot\frac{3r}{r_c}}\geq 12\Leftrightarrow\sum\frac{1}{\frac{9r^2}{r_a^2}-\frac{9r^2}{r_ar_b}+\frac{9r^2}{r_b^2}}+\frac{9}{\frac{27r^3}{r_ar_br_c}}\geq 12\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{\frac{9r^2}{r_a^2}-\frac{9r^2}{r_ar_b}+\frac{9r^2}{r_b^2}}+\frac{1}{\frac{3r^3}{rs^2}}\geq 12\Leftrightarrow\sum\frac{1}{\frac{9r^2}{r_a^2}-\frac{9r^2}{r_ar_b}+\frac{9r^2}{r_b^2}}+\frac{1}{\frac{3r^2}{s^2}}\geq 12\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{9r^2\left(\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}\right)}+\frac{1}{\frac{3r^2}{s^2}}\Leftrightarrow\sum\frac{1}{3\left(\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}\right)}+\frac{1}{\frac{1}{s^2}}\geq 36r^2\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{3\left(\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}\right)}+s^2\geq 36r^2\Leftrightarrow\sum\frac{1}{3\left(\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}\right)}\geq 36r^2-s^2\Leftrightarrow$$

$$\Leftrightarrow\sum\frac{1}{\frac{1}{r_a^2}-\frac{1}{r_ar_b}+\frac{1}{r_b^2}}\geq 36(36r^2-s^2).$$

Equality holds if and only if the triangle is equilateral.

J.2526. In ΔABC the following relationship holds:

$$\sum\frac{r_a}{b+c}\sqrt{\sin A}\leq\frac{9}{8}\sqrt{\frac{9R^2-24r^2}{2}}$$

Mehmet Șahin – Turkey

Remark: The problem can be developed. **In ΔABC the following relationship holds:**

$$\frac{3}{2R}\sqrt{\frac{sr}{2}}\leq\sum\frac{r_a}{b+c}\sqrt{\sin A}\leq\frac{1}{2R}\sqrt{\frac{s}{2r}(8R^2-23r^2)}$$

Marin Chirciu – Romania

Solution: Right inequality.

$$\sum\frac{r_a}{b+c}\sqrt{\sin A}\overset{CBS}{\leq}\sqrt{\sum\left(\frac{r_a}{b+c}\right)^2\sum\sin A}\leq\sqrt{\frac{8R^2-23r^2}{8Rr}\cdot\frac{s}{R}}=\frac{1}{2R}\sqrt{\frac{s}{2r}(8R^2-23r^2)}.$$

We have used above:

$$\sum\left(\frac{r_a}{b+c}\right)^2=\sum\frac{r_a^2}{(b+c)^2}\overset{SOS}{\leq}\sum\frac{r_a^2}{4bc}=\frac{1}{4}\sum\frac{r_a^2}{bc}=\frac{1}{4}\cdot\frac{8R^2+2Rr-s^2}{2Rr}=$$

$$=\frac{8R^2+2Rr-s^2}{8Rr}\overset{Gerretsen}{\le}\frac{8R^2+2Rr-16Rr+5r^2}{8Rr}=\frac{8R^2-14Rr+5r^2}{8Rr}\overset{Gerrtsen}{\le}$$

$$\overset{Gerretsen}{\le}\frac{8R^2-28r^2+5r^2}{8Rr}=\frac{8R^2-23r^2}{8Rr},\qquad \sum\frac{r_a^2}{bc}=\frac{8R^2+2Rr-s^2}{2Rr}.$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\sum\frac{r_a}{b+c}\sqrt{\sin A}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{r_a}{b+c}\sqrt{\sin A}}=3\sqrt[3]{\frac{\prod r_a}{\prod(b+c)}\sqrt{\prod\sin A}}=$$

$$=3\sqrt[3]{\frac{\prod r_a}{\prod(b+c)}\sqrt{\prod\sin A}}=3\sqrt[3]{\frac{rs^2}{16sR^2}\sqrt{\frac{sr}{2R^2}}}=\frac{3}{2}\sqrt[3]{\frac{rs}{2R^2}\sqrt{\frac{sr}{2R^2}}}=\frac{3}{2}\sqrt[6]{\left(\frac{rs}{2R^2}\right)^3}=$$

$$=\frac{3}{2}\sqrt{\frac{rs}{2R^2}}=\frac{3}{2R}\sqrt{\frac{rs}{2}}.$$ We have used above:

$$\prod(b+c)=2s(s^2+r^2+2Rr)\overset{Gerretsen}{\le}2s(4R^2+4Rr+3r^2+r^2+2Rr)=$$

$$=2s(4R^2+6Rr+4r^2)=4s(2R^2+3Rr+2r^2)\overset{Euler}{\le}4s\cdot 4R^2=16sR^2.$$

Equality holds if and only if the triangle is equilateral.

J.2486 In ΔABC the following relationship holds:

$$\sum\frac{\cot^2\frac{A}{2}}{2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}\ge\frac{s}{6r^2}$$

D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution: We have $2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)>0$ and the analogs.

$$LHS=\sum\frac{\cot^2\frac{A}{2}}{2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}\overset{CS}{\ge}\frac{\left(\sum\cot\frac{A}{2}\right)^2}{\sum\left(2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)\right)}=\frac{\left(\frac{s}{r}\right)^2}{6s}=\frac{s}{6r^2}=RHS.$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\sum\frac{\cot^n\frac{A}{2}}{2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}\ge\frac{s}{2\cdot 3^{n-1}r^2},\ n\in\mathbb{N},n\ge 2$$

Marin Chirciu – Romania

Solution: We have $2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)>0$ and the analogs.

$$LHS=\sum\frac{\cot^n\frac{A}{2}}{2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)}\overset{Holder}{\geq}\frac{\left(\sum\cot\frac{A}{2}\right)^2}{3^{n-2}\sum\left(2s-\left(\cot\frac{A}{2}-\cot\frac{C}{2}\right)\right)}=\frac{\left(\frac{s}{r}\right)^2}{3^{n-2}\cdot 6s}=$$

$=\frac{s}{2\cdot 3^{n-1}r^2}=RHS$. Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\sum\frac{\tan^n\frac{A}{2}}{2s-\left(\tan\frac{A}{2}-\tan\frac{C}{2}\right)}\geq\frac{1}{2\cdot 3^{n-2}s}, n\in\mathbb{N}, n\geq 2$$

Marin Chirciu – Romania

Solution: We have $2s-\left(\tan\frac{A}{2}-\tan\frac{C}{2}\right)>0$ and the analogs.

$$LHS=\sum\frac{\tan^2\frac{A}{2}}{2s-\left(\tan\frac{A}{2}-\tan\frac{C}{2}\right)}\overset{Holder}{\geq}\frac{\left(\sum\tan\frac{A}{2}\right)^2}{3^{n-2}\sum\left(2s-\left(\tan\frac{A}{2}-\tan\frac{C}{2}\right)\right)}=\frac{\left(\frac{4R+r}{s}\right)^2}{3^{n-2}\cdot 6s}=$$

$$=\frac{\frac{(4R+r)^2}{s^2}}{3^{n-2}\cdot 6s}\overset{Gerretsen}{\geq}\frac{\frac{(4R+r)^2}{\frac{R(4R+r)^2}{2(2R-r)}}}{3^{n-2}\cdot 6s}=\frac{\frac{2(2R-r)}{R}}{3^{n-2}\cdot 6s}=\frac{\frac{2R-r}{R}}{3^{n-2}\cdot 3s}\overset{Euler}{\geq}\frac{\frac{3}{2}}{3^{n-2}\cdot 3s}=$$

$$=\frac{1}{2\cdot 3^{n-2}s}=RHS.$$

Equality holds if and only if the triangle is equilateral.

J.2480. In ΔABC the following relationship holds:

1. $\boldsymbol{m_a-w_a+\sqrt{m_a^2+w_a^2}\geq\sqrt{2(m_a^2-m_aw_a+w_a^2)}}$

2. $\boldsymbol{m_a-h_a+\sqrt{m_a^2-m_ah_a+h_a^2}\geq\sqrt{2m_a^2-3m_ah_a+2h_a^2}}$

Nguyen Van Canh – Vietnam

Solution:

1. $m_a-w_a+\sqrt{m_a^2+w_a^2}\geq\sqrt{2(m_a^2-m_aw_a+w_a^2)}$.

Squaring the inequality transform itself equivalently:

$$m_a-w_a+\sqrt{m_a^2+w_a^2}\geq\sqrt{2(m_a^2-m_aw_a+w_a^2)}\Leftrightarrow$$

$$\Leftrightarrow\left(m_a-w_a+\sqrt{m_a^2+w_a^2}\right)^2\ge 2(m_a^2-m_aw_a+w_a^2)\ \Leftrightarrow$$

$$\Leftrightarrow m_a^2-2m_aw_a+w_a^2+2(m_a-w_a)\sqrt{m_a^2+w_a^2}+m_a^2+w_a^2\ge 2(m_a^2-m_aw_a+w_a^2)\Leftrightarrow$$

$\Leftrightarrow 2(m_a-w_a)\sqrt{m_a^2+w_a^2}\ge 0$, which follows from $m_a\ge w_a$.

Equality holds if and only if the triangle is isosceles, with $b=c$.

2. $m_a-h_a+\sqrt{m_a^2-m_ah_a+h_a^2}\ge\sqrt{2m_a^2-3m_aw_a+2h_a^2}$.

Squaring the inequality transform itself equivalently:

$$m_a-h_a+\sqrt{m_a^2-m_ah_a+h_a^2}\ge\sqrt{2m_a^2-3m_aw_a+2h_a^2}\Leftrightarrow$$

$$\Leftrightarrow m_a^2-2m_ah_a+h_a^2+2(m_a-h_a)\sqrt{m_a^2-m_ah_a+h_a^2}+m_a^2-m_ah_a+h_a^2\ge$$

$$\ge 2m_a^2-3m_ah_a+2h_a^2\Leftrightarrow$$

$\Leftrightarrow 2(m_a-h_a)\sqrt{m_a^2-m_ah_a+h_a^2}\ge 0$, which follows from $m_a\ge h_a$.

Equality holds if and only if the triangle is isosceles, with $b=c$. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

3. $\boldsymbol{m_a-w_a+\sqrt{m_a^2-\lambda m_aw_a+w_a^2}\ge\sqrt{2m_a^2-(\lambda+2)m_aw_a+2w_a^2},-2\le\lambda\le 2}$

4. $\boldsymbol{m_a-h_a+\sqrt{\lambda m_a^2-m_ah_a+\lambda h_a^2}\ge\sqrt{(\lambda+1)m_a^2-3m_ah_a+(\lambda+1)h_a^2}\cdot\lambda\ge\frac{1}{2}}$

Marin Chirciu – Romania

Solution:

3. $m_a-w_a+\sqrt{m_a^2-\lambda m_aw_a+w_a^2}\ge\sqrt{2m_a^2-(\lambda+2)m_aw_a+2w_a^2}$.

We have $m_a^2-\lambda m_aw_a+w_a^2\ge 0$, see $\Delta=\lambda^2-4\le 0$, for $-2\le\lambda\le 2$.

We have $2m_a^2-(\lambda+2)m_aw_a+2w_a^2\ge 0$, see $\Delta=(\lambda+2)^2-16\le 0$, for $-6\le\lambda\le 2$.

Squaring the inequality transform itself equivalently:

$$m_a-w_a+\sqrt{m_a^2-\lambda m_aw_a+w_a^2}\ge\sqrt{2m_a^2-(\lambda+2)m_aw_a+2w_a^2}\Leftrightarrow$$

$$m_a^2-2m_aw_a+w_a^2+2(m_a-w_a)\sqrt{m_a^2-\lambda m_aw_a+w_a^2}+m_a^2-\lambda m_aw_a+w_a^2\ge$$

$$\ge 2m_a^2-(\lambda+2)m_aw_a+2w_a^2$$

$\Leftrightarrow 2(m_a - w - a)\sqrt{m_a^2 - \lambda m_a w_a + w_a^2} \geq 0$, which follows from $m_a \geq w_a$.

Note: For $\lambda = 0$ we obtain Problem J.2480, 1. from RMM – 43.

Equality holds if and only if the triangle is isosceles, with $b = c$.

4. $m_a - h_a + \sqrt{\lambda m_a^2 - m_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)m_a^2 - 3m_a w_a + (\lambda+1)h_a^2}$.

We have $\lambda m_a^2 - m_a h_a + \lambda h_a^2 \geq 0$, see $\Delta = 1 - 4\lambda^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

$(\lambda+1)m_a^2 - 3m_a w_a + (\lambda+1)h_a^2$, see $\Delta = 9 - 4(\lambda+1)^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

Squaring the inequality transforms itself equivalently:

$$m_a - h_a + \sqrt{\lambda m_a^2 - m_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)m_a^2 - 3m_a w_a + (\lambda+1)h_a^2} \Leftrightarrow$$

$$\Leftrightarrow m_a^2 - 2m_a h_a + h_a^2 + 2(m_a - h_a)\sqrt{\lambda m_a^2 - m_a h_a + \lambda h_a^2} + \lambda m_a^2 - m_a h_a + \lambda h_a^2 \geq$$

$$\geq (\lambda+1)m_a^2 - 3m_a w_a + (\lambda+1)h_a^2$$

$$\Leftrightarrow 2(m_a - h_a)\sqrt{\lambda m_a^2 - m_a h_a + \lambda h_a^2} \geq 0, \text{ which follows from } m_a \geq h_a.$$

Equality holds if and only if the triangle is isosceles, with $b = c$.

Note: For $\lambda = 1$ we obtain Problem J.2480, 2. From RMM – 43. **Remark:** In the same way.

In ΔABC the following relationship holds:

5. $\boldsymbol{w_a - h_a + \sqrt{w_a^2 - \lambda w_a h_a + h_a^2} \geq \sqrt{2w_a^2 - (\lambda+2)w_a h_a + 2h_a^2}, -2 \leq \lambda \leq 2.}$

6. $\boldsymbol{w_a - h_a + \sqrt{\lambda w_a^2 - w_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)w_a^2 - 3w_a h_a + (\lambda+1)h_a^2}. \lambda \geq \frac{1}{2}.}$

Marin Chirciu – Romania

Solution:

5. $w_a - h_a + \sqrt{w_a^2 - \lambda w_a h_a + h_a^2} \geq \sqrt{2w_a^2 - (\lambda+2)w_a h_a + 2h_a^2}$.

We have $w_a^2 - \lambda w_a h_a + h_a^2 \geq 0$, see $\Delta = \lambda^2 - 4 \leq 0$, for $-2 \leq \lambda \leq 2$.

We have $2w_a^2 - (\lambda+2)w_a h_a + 2h_a^2 \geq 0$, see $\Delta = (\lambda+2)^2 - 16 \leq 0$, for $-6 \leq \lambda \leq 2$.

Squaring the inequality transforms itself equivalently:

$$w_a - h_a + \sqrt{w_a^2 - \lambda w_a h_a + h_a^2} \geq \sqrt{2w_a^2 - (\lambda+2)w_a h_a + 2h_a^2} \Leftrightarrow$$

$$w_a^2 - 2w_a h_a + h_a^2 + 2(w_a - h_a)\sqrt{w_a^2 - \lambda w_a h_a + h_a^2} + w_a^2 - \lambda w_a h_a + h_a^2 \geq$$

$$\geq 2w_a^2 - (\lambda+2)w_a h_a + 2h_a^2 \Leftrightarrow$$

$2(w_a - h_a)\sqrt{w_a^2 - \lambda w_a h_a + h_a^2} \geq 0$, which follows from $w_a \geq w_a$.

Equality holds if and only if the triangle is isosceles, with $b = c$.

6. $w_a - h_a + \sqrt{\lambda w_a^2 - w_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)w_a^2 - 3w_a h_a + (\lambda+1)h_a^2}$.

We have $\lambda w_a^2 - w_a h_a + \lambda h_a^2 \geq 0$, see $\Delta = 1 - 4\lambda^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

$(\lambda+1)w_a^2 - 3w_a h_a + (\lambda+1)h_a^2 \geq 0$, see $\Delta = 9 - 4(\lambda+1)^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

Squaring the inequality transforms itself equivalently:

$$w_a - h_a + \sqrt{\lambda w_a^2 - w_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)w_a^2 - 3w_a h_a + (\lambda+1)h_a^2} \Leftrightarrow$$

$$\Leftrightarrow w_a^2 - 2w_a h_a + h_a^2 + 2(w_a - h_a)\sqrt{\lambda w_a^2 - w_a h_a + \lambda h_a^2} + \lambda w_a^2 - w_a h_a + \lambda h_a^2 \geq$$

$$\geq (\lambda+1)w_a^2 - 3w_a h_a + (\lambda+1)h_a^2 \Leftrightarrow$$

$$\Leftrightarrow 2(w_a - h_a)\sqrt{\lambda w_a^2 - w_a h_a + \lambda h_a^2} \geq 0, \text{ which follows from } w_a \geq h_a.$$

Equality holds if and only if the triangle is isosceles, with $b = c$. **Remark:** In the same way.

In ΔABC the following relationship holds:

7. $\boldsymbol{m_a - s_a + \sqrt{m_a^2 - \lambda m_a s_a + s_a^2} \geq \sqrt{2m_a^2 - (\lambda+2)m_a s_a + 2s_a^2}, -2 \leq \lambda \leq 2.}$

8. $\boldsymbol{m_a - s_a + \sqrt{\lambda m_a^2 - m_a s_a + \lambda s_a^2} \geq \sqrt{(\lambda+1)m_a^2 - 3m_a s_a + (\lambda+1)s_a^2}. \lambda \geq \frac{1}{2}}$

Marin Chirciu – Romania

Solution:

7. $m_a - s_a + \sqrt{m_a^2 - \lambda m_a s_a + s_a^2} \geq \sqrt{2m_a^2 - (\lambda+2)m_a s_a + 2s_a^2}$.

We have $m_a^2 - \lambda m_a s_a + s_a^2 \geq 0$, see $\Delta = \lambda^2 - 4 \leq 0$, for $-2 \leq \lambda \leq 2$.

We have $2m_a^2 - (\lambda+2)m_a s_a + 2s_a^2 \geq 0$, see $\Delta = (\lambda+2)^2 - 16 \leq 0$, for $-6 \leq \lambda \leq 2$.

Squaring the inequality transforms itself equivalently:

$$m_a - s_a + \sqrt{m_a^2 - \lambda m_a s_a + s_a^2} \geq \sqrt{2m_a^2 - (\lambda+2)m_a s_a + 2s_a^2} \Leftrightarrow$$

$$m_a^2 - 2m_a s_a + s_a^2 + 2(m_a - s_a)\sqrt{m_a^2 - \lambda m_a s_a + s_a^2} + m_a^2 - \lambda m_a s_a + s_a^2 \geq$$

$$\geq 2m_a^2 - (\lambda+2)m_a s_a + 2s_a^2 \Leftrightarrow$$

$2(m_a - s_a)\sqrt{m_a^2 - \lambda m_a s_a + s_a^2} \geq 0$, which follows from $m_a \geq s_a$.

Equality holds if and only if the triangle is isosceles, with $b = c$.

$$8.\ m_a - s_a + \sqrt{\lambda m_a^2 - m_a s_a + \lambda s_a^2} \ge \sqrt{(\lambda+1)m_a^2 - 3m_a s_a + (\lambda+1)s_a^2}.$$

We have $\lambda m_a^2 - m_a s_a + \lambda s_a^2 \ge 0$, see $\Delta = 1 - 4\lambda^2 \le 0$, for $\lambda \ge \frac{1}{2}$.

$(\lambda+1)m_a^2 - 3m_a s_a + (\lambda+1)s_a^2 \ge 0$, see $\Delta = 9 - 4(\lambda+1) \le 0$, for $\lambda \ge \frac{1}{2}$.

Squaring the inequality transforms itself equivalently:

$$m_a - s_a + \sqrt{\lambda m_a^2 - m_a s_a + \lambda s_a^2} \ge \sqrt{(\lambda+1)m_a^2 - 3m_a s_a + (\lambda+1)s_a^2} \Leftrightarrow$$

$$\Leftrightarrow m_a^2 - 2m_a s_a + s_a^2 + 2(m_a - s_a)\sqrt{\lambda m_a^2 - m_a s_a + \lambda s_a^2} + \lambda m_a^2 - m_a s_a + \lambda s_a^2 \ge$$

$$\ge (\lambda+1)m_a^2 - 3m_a s_a + (\lambda+1)s_a^2 \Leftrightarrow$$

$$\Leftrightarrow 2(m_a - s_a)\sqrt{\lambda m_a^2 - m_a s_a + \lambda s_a^2} \ge 0,\ \text{wich follows from } m_a \ge s_a.$$

Equality holds if and only if the triangle is isosceles with $b = c$. **Remark:** In the same way.

In ΔABC the following relationship holds:

9. $\boldsymbol{s_a - h_a + \sqrt{s_a^2 - \lambda s_a h_a + h_a^2} \ge \sqrt{2s_a^2 - (\lambda+2)s_a h_a + 2h_a^2}, -2 \le \lambda \le 2.}$

10. $\boldsymbol{s_a - h_a + \sqrt{\lambda s_a^2 - s_a h_a + \lambda h_a^2} \ge \sqrt{(\lambda+1)s_a^2 - 3s_a h_a + (\lambda+1)h_a^2}. \lambda \ge \frac{1}{2}}$

Marin Chirciu – Romania

Solution:

$$9.\ s_a - h_a + \sqrt{s_a^2 - \lambda s_a h_a + h_a^2} \ge \sqrt{2s_a^2 - (\lambda+2)s_a h_a + 2h_a^2}.$$

We have $s_a^2 - \lambda s_a h - a + h_a^2 \ge 0$, see $\Delta = \lambda^2 - 4 \le 0$, for $-2 \le \lambda \le 2$.

We have $2s_a^2 - (\lambda+2)s_a h_a + 2h_a^2 \ge 0$, see $\Delta = (\lambda+2)^2 - 16 \le 0$, for $-6 \le \lambda \le 2$.

Squaring the inequality transform itself equivalently:

$$s_a - h_a + \sqrt{s_a^2 - \lambda s_a h_a + h_a^2} \ge \sqrt{2s_a^2 - (\lambda+2)s_a h_a + 2h_a^2} \Leftrightarrow$$

$$s_a^2 - 2s_a h_a + h_a^2 + 2(s_a - h_a)\sqrt{s_a^2 - \lambda s_a h_a + h_a^2} + s_a^2 - \lambda s_a h_a + h_a^2 \ge$$

$$\ge 2s_a^2 - (\lambda+2)s_a h_a + 2h_a^2 \Leftrightarrow$$

$$2(s_a - h_a)\sqrt{s_a^2 - \lambda s_a h_a + h_a^2} \ge 0,\ \text{which follows from } s_a \ge h_a.$$

Equality holds if and only if the triangle is isosceles, with $b = c$.

10. $s_a - h_a + \sqrt{\lambda s_a^2 - s_a h_a + \lambda h_a^2} \ge \sqrt{(\lambda+1)s_a^2 - 3s_a h_a + (\lambda+1)h_a^2}.$

We have $\lambda s_a^2 - s_a h_a + \lambda h_a^2 \geq 0$, see $\Delta = 1 - 4\lambda^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

$(\lambda+1)s_a^2 - 3s_a h_a + (\lambda+1)h_a^2 \geq 0$, see $\Delta = 9 - 4(\lambda+1)^2 \leq 0$, for $\lambda \geq \frac{1}{2}$.

Squaring the inequality transforms itself equivalently:

$$s_a - h_a + \sqrt{\lambda s_a^2 - s_a h_a + \lambda h_a^2} \geq \sqrt{(\lambda+1)s_a^2 - 3s_a h_a + (\lambda+1)h_a^2} \Leftrightarrow$$

$$\Leftrightarrow s_a^2 - 2s_a h_a + s_a^2 + 2(s_a - h_a)\sqrt{\lambda s_a^2 - s_a h_a + \lambda h_a^2} + \lambda s_a^2 - s_a h_a + \lambda h_a^2 \geq$$

$\geq (\lambda+1)s_a^2 - 3s_a h_a + (\lambda+1)h_a^2 \Leftrightarrow 2(s_a - h_a)\sqrt{\lambda s_a^2 - s_a h_a + \lambda h_a^2} \geq 0$, which follows from $s_a \geq h_a$.

Equality holds if and only if the triangle is isosceles, with $b = c$.

J.2487. In ΔABC the following relationship holds:

$$\sum \frac{w_a^3}{(1+w_b)(1+w_c)} \geq \frac{324r^3}{9R^2 + 12R + 4}$$

Zaza Mzhavanadze – Georgia

Solution:

$$LHS = \sum \frac{w_a^3}{(1+w_b)(1+w_c)} \overset{Holder}{\geq} \frac{(\sum w_a)^3}{3\sum(1+w_b)(1+w_c)} \overset{(1)}{\geq} \frac{(9r)^3}{3 \cdot \frac{3}{4}(9R^2 + 12R + 4)} =$$

$$= \frac{324r^3}{9R^2 + 12R + 4} = RHS$$

where (1) it follows from $\sum w_a \geq 9r$ and $\sum(1+w_b)(1+w_c) \leq \frac{3}{4}(9R^2 + 12R + 4)$, see

$$\sum(1+w_b)(1+w_c) = 3 + 2\sum w_a + \sum w_b w_c \leq 3 + 2 \cdot \frac{9R}{2} + \frac{27R^2}{4} = \frac{3}{4}(9R^2 + 12R + 4).$$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\sum \frac{w_a^n}{(1+w_b)(1+w_c)} \geq \frac{12(3r)^n}{(3R+2)^2}, n \in \mathbb{N}, n \geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{w_a^n}{(1+w_b)(1+w_c)} \overset{Holder}{\geq} \frac{(\sum w_a)^n}{3^{n-2}\sum(1+w_b)(1+w_c)} \overset{(1)}{\geq}$$

$$\geq \frac{(9r)^n}{3^{n-2}\cdot\frac{3}{4}(3R+2)^2} = \frac{12(3r)^n}{(3R+2)^2} = RHS,$$

where (1) it follows from $\sum w_a \geq 9r$ and $\sum(1+w_b)(1+w_c) \leq \frac{3}{4}(3R+2)^2$, see

$$\sum(1+w_b)(1+w_c) = 3+2\sum w_a + \sum w_b w_c \leq 3+2\cdot\frac{9R}{2}+\frac{27R^2}{4} = \frac{3}{4}(9R^2+12R+4) = \frac{3}{4}(3R+2)^2$$

Equality holds if and only if the triangle equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\sum\frac{m_a^n}{(1+m_b)(1+m_c)} \geq \frac{12(3r)^n}{(3R+2)^2}, n \in \mathbb{N}, n \geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum\frac{m_a^n}{(1+m_b)(1+m_c)} \overset{Holder}{\geq} \frac{(\sum m_a)^n}{3^{n-2}\sum(1+m_b)(1+m_c)} \overset{(1)}{\geq} \frac{(9r)^n}{3^{n-2}\cdot\frac{3}{4}(3R+2)^2} =$$

$$= \frac{12(3r)^n}{(3R+2)} = RHS$$

where (1) it follows from $\sum m_a \geq 9r$ and $\sum(1+m_b)(1+m_c) \leq \frac{3}{4}(3R+2)^2$, see

$$\sum(1+m_b)(1+m_c) = 3+2\sum m_a + \sum m_b m_c \leq 3+2\cdot\frac{9R}{2}+\frac{27R^2}{4} =$$

$$= \frac{3}{4}(9R^2+12R+4) = \frac{3}{4}(3R+2)^2.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\sum\frac{s_a^n}{(1+s_b)(1+s_c)} \geq \frac{12(3r)^n}{(3R+2)^2}, n \in \mathbb{N}, n \geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum\frac{s_a^n}{(1+s_b)(1+s_c)} \overset{Holder}{\geq} \frac{(\sum s_a)^n}{3^{n-2}\sum(1+s_b)(1+s_c)} \overset{(1)}{\geq} \frac{(9r)^n}{3^{n-2}\cdot\frac{3}{4}(3R+2)^2} =$$

$$= \frac{12(3r)^n}{(3R+2)^2} = RHS,$$

where (1) it follows from $\sum s_a \geq 9r$ and $\sum(1+s_b)(1+s_c) \leq \frac{3}{4}(3R+2)^2$, see

$$\sum(1+s_b)(1+s_c) = 3+2\sum s_a + \sum s_b s_c \leq 3+2\cdot\frac{9R}{2}+\frac{27R^2}{4} = \frac{3}{4}(9R^2+12R+4)$$

$=\frac{3}{4}(3R+2)^2$. Equality holds if and only if the triangle is equilateral. **Remark:** In the same way:

In ΔABC the following relationship holds:

$$\sum \frac{h_a^n}{(1+h_b)(1+h_c)} \geq \frac{12(3r)^n}{(3R+2)^2}, n \in \mathbb{N}, n \geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{h_a^n}{(1+h_b)(1+h_c)} \overset{Holder}{\geq} \frac{(\sum h_a)^n}{3^{n-2}\sum(1+h_b)(1+h_c)} \overset{(1)}{\geq}$$

$$\overset{(1)}{\geq} \frac{(9r)^n}{3^{n-2}\cdot\frac{3}{4}(3R+2)^2} = \frac{12(3r)^n}{(3R+2)^2} = RHS,$$

where (1) it follows from $\sum h_a \geq 9r$ and $\sum(1+h_b)(1+h_c) \leq \frac{3}{4}(3R+2)^2$, see

$$\sum(1+h_b)(1+h_c) = 3+2\sum h_a + \sum h_b h_c \leq 3+2\cdot\frac{9R}{2}+\frac{27R^2}{4} =$$

$$= \frac{3}{4}(9R^2+12R+4) = \frac{3}{4}(3R+2)^2.$$

Equality holds if and only if the triangle is equilateral. **Remark:**In the same way.

In ΔABC the following relationship holds:

$$\sum \frac{r_a^n}{(1+r_b)(1+r_c)} \geq \frac{12(3r)^n}{(3R+2)^2}, n \in \mathbb{N}, n \geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{r_a^n}{(1+r_b)(1+r_c)} \overset{Holder}{\geq} \frac{(\sum r_a)^n}{3^{n-2}\sum(1+r_b)(1+r_c)} \overset{(1)}{\geq} \frac{(9r)^n}{3^{n-2}\cdot\frac{3}{4}(3R+2)^2} =$$

$$= \frac{12(3r)^n}{(3R+2)^2} = RHS,$$

where (1) it follows from $\sum r_a \geq 9r$ and $\sum(1+r_b)(1+r_c) \leq \frac{3}{4}(3R+2)^2$, see

$$\sum(1+r_b)(1+r_c) = 3+2\sum r_a + \sum r_b r_c \leq 3+2\cdot\frac{9R}{2}+\frac{27R^2}{4} = \frac{3}{4}(9R^2+12R+4) =$$

$$=\frac{3}{4}(3R+2)^2.$$

Equality holds if and only if the triangle is equilateral.

J.2527. In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a^5(r_b+r_c)} \geq \frac{32}{243R^6}$$

Zaza Mzhavanadze – Georgia

Solution:

$$LHS=\sum \frac{1}{r_a^5(r_b+r_c)}=\sum \frac{\frac{1}{r_a^5}}{r_b+r_c} \overset{Holder}{\geq} \frac{\left(\sum \frac{1}{r_a}\right)^5}{3^3\sum(r_b+r_c)}=\frac{\left(\sum \frac{1}{r_a}\right)^5}{3^3\cdot 2\sum r_a}=$$

$$=\frac{\left(\frac{1}{r}\right)^5}{3^3\cdot 2(4R+r)}=\frac{1}{3^3r^5\cdot 9R}=\frac{1}{243Rr^5} \overset{Euler}{\geq} \frac{32}{243R^6}=RHS.$$

We used above: $\sum \frac{1}{r_a}=\frac{1}{r}$ and $\sum r_a=4R+r$

Equality holds if and only if the triangle is equilateral. **Remark:** The problem can be developed.

In ΔABC the following relationship holds:

$$\sum \frac{1}{r_a^n(r_b+r_c)} \geq \frac{1}{(3r)^n R}, n\in \mathbb{N}, n\geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum \frac{1}{r_a^n(r_b+r_c)}=\sum \frac{\frac{1}{r_a^n}}{r_b+r_c} \overset{Holder}{\geq} \frac{\left(\sum \frac{1}{r_a}\right)^n}{3^{n-2}\sum(r_b+r_c)}=\frac{\left(\sum \frac{1}{r_a}\right)^n}{3^{n-2}\cdot 2\sum r_a}=$$

$$=\frac{\left(\frac{1}{r}\right)^n}{3^{n-2}\cdot 2(4R+r)} \overset{Euler}{\geq} \frac{1}{3^{n-2}r^n\cdot 9R}=\frac{1}{3^n r^n\cdot R}-\frac{1}{(3r)^n R}=RHS.$$

We have used above: $\sum \frac{1}{r_a}=\frac{1}{r}$ and $\sum r_a=4R+r$. Equality holds if and only if the triangle is equilateral.

Note: For $n=5$ we obtain a straighten problem of the Problem J.2527 from RMM – 43.

Remark: In the same way. **In ΔABC the following relationship holds:**

$$\sum \frac{1}{h_a^n(h_b+h_c)} \geq \frac{1}{(3r)^n R}, n\in \mathbb{N}, n\geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{h_a^n(h_b+h_c)}=\sum\frac{\frac{1}{h_a^n}}{h_b+h_c}\overset{Holder}{\geq}\frac{\left(\sum\frac{1}{h_a}\right)^n}{3^{n-2}\sum(h_b+h_c)}=\frac{\left(\sum\frac{1}{h_a}\right)^n}{3^{n-2}\cdot 2\sum h_a}\geq$$

$$\geq\frac{\left(\frac{1}{r}\right)^n}{3^{n-2}\cdot 2(4R+r)}\overset{Euler}{\geq}\frac{1}{3^{n-2}r^n\cdot 9R}=\frac{1}{3^n r^n\cdot R}=\frac{1}{(3r)^n R}=RHS.$$

We have used above: $\sum\frac{1}{h_a}=\frac{1}{r}$ and $\sum h_a\leq 4R+r$. Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$\sum\frac{1}{w_a^n(w_b+w_c)}\geq\frac{1}{R}\left(\frac{2}{3R}\right)^n,n\in\mathbb{N},n\geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{w_a^n(w_b+w_c)}=\sum\frac{\frac{1}{w_a^n}}{w_b+w_c}\overset{Holder}{\geq}\frac{\left(\sum\frac{1}{w_a}\right)^n}{3^{n-2}\sum(w_b+w_c)}=\frac{\left(\sum\frac{1}{w_a}\right)^n}{3^{n-2}\cdot 2\sum w_a}\geq$$

$$\geq\frac{\left(\frac{2}{R}\right)^n}{3^{n-2}\cdot 2(4R+r)}\overset{Euler}{\geq}\frac{2^n}{3^{n-2}R^n\cdot 9R}=\frac{2^n}{3^n R^n\cdot R}=\frac{2^n}{(3R)^n R}=\frac{1}{R}\left(\frac{2}{3R}\right)^n=RHS.$$

We have used above: $\sum\frac{1}{w_a}\geq\frac{2}{R}$ and $\sum w_a\leq 4R+r$, see $\sum w_a\leq\sum m_a\leq 4R+r$ (Leuenberger).

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$\sum\frac{1}{m_a^n(m_b+m_c)}\geq\frac{1}{R}\left(\frac{2}{3R}\right)^n,n\in\mathbb{N},n\geq 2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{m_a^n(m_b+m_c)}=\sum\frac{\frac{1}{m_a^n}}{m_b+m_c}\overset{Holder}{\geq}\frac{\left(\sum\frac{1}{m_a}\right)^n}{3^{n-2}\sum(m_b+m_c)}=\frac{\left(\sum\frac{1}{m_a}\right)^n}{3^{n-2}\cdot 2\sum m_a}\geq$$

$$\geq\frac{\left(\frac{2}{R}\right)^n}{3^{n-2}\cdot 2(4R+r)}\overset{Euler}{\geq}\frac{2^n}{3^{n-2}R^n\cdot 9R}=\frac{2^n}{3^n R^n\cdot R}=\frac{2^n}{(3R)^n R}=\frac{1}{R}\left(\frac{2}{3R}\right)^n=RHS.$$

We have used above: $\sum\frac{1}{m_a}\geq\frac{2}{R}$ and $\sum m_a\leq 4R+r$ (Leuenberger).

Equality holds if and only if the triangle is equilateral.**Remark:** In the same way.

In ΔABC the following relationship holds:

$$\sum \frac{1}{s_a^n(s_b+s_c)} \ge \frac{1}{R}\left(\frac{2}{3R}\right)^n, n \in \mathbb{N}, n \ge 2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{1}{s_a^n(s_b+s_c)} = \sum \frac{\frac{1}{s_a^n}}{s_b+s_c} \overset{Holder}{\ge} \frac{\left(\sum\frac{1}{s_a}\right)^n}{3^{n-2}\sum(s_b+s_c)} = \frac{\left(\sum\frac{1}{s_a}\right)^n}{3^{n-2}\cdot 2\sum s_a} \ge$$

$$\ge \frac{\left(\frac{2}{R}\right)^n}{3^{n-2}\cdot 2(4R+r)} \overset{Euler}{\ge} \frac{2^n}{3^{n-2}R^n\cdot 9R} = \frac{2^n}{3^nR^n\cdot R} = \frac{2^n}{(3R)^nR} = \frac{1}{R}\left(\frac{2}{3R}\right)^n = RHS.$$

We have used above: $\sum\frac{1}{s_a} \ge \frac{2}{R}$, see $\sum\frac{1}{s_a} \ge \sum\frac{1}{m_a} \ge \frac{2}{R}$ and $\sum s_a \le 4R+r$, see

$\sum s_a \le \sum m_a \le 4R+r$ (Leuenberger). Equality holds if and only if the triangle is equilateral.

J.2502. If $a,b,c > 0$ then:

$$\mathbf{1.}\ \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \le a+b,$$

$$\mathbf{2.}\ \sqrt{\frac{a^2+b^2+c^2}{3}} + 2\sqrt[3]{abc} \le a+b+c.$$

Lucian Tuțescu – Romania

Solution:

$$1.\ \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \le a+b.$$

$$\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \le a+b \Leftrightarrow \left(\sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab}\right)^2 \le (a+b)^2 \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2+b^2}{2} + ab + 2\sqrt{\frac{a^2+b^2}{2}}\cdot\sqrt{ab} \le a^2+2ab+b^2 \Leftrightarrow 2\sqrt{\frac{a^2+b^2}{2}}\cdot\sqrt{ab} \le \frac{a^2+ab+b^2}{2} \Leftrightarrow$$

$$\Leftrightarrow 4\sqrt{\frac{a^2+b^2}{2}}\cdot\sqrt{ab} \le a^2+ab+b^2 \Leftrightarrow 8ab(a^2+b^2) \le (a^2+ab+b^2)^2 \Leftrightarrow$$

$\Leftrightarrow a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \ge 0 \Leftrightarrow (a-b)^4 \ge 0$, with equality for $a=b$.

Equality holds if and only if $a=b$.

2. $\sqrt{\frac{a^2+b^2+c^2}{3}}+2\sqrt[3]{abc}\le a+b+c.$

$$\sqrt{\frac{a^2+b^2+c^2}{3}}+2\sqrt[3]{abc}\le a+b+c\Leftrightarrow\left(\sqrt{\frac{a^2+b^2+c^2}{3}}+2\sqrt[3]{abc}\right)^2\le(a+b+c)^2\Leftrightarrow$$

$$\Leftrightarrow\frac{a^2+b^2+c^2}{3}+4\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\sqrt[3]{abc}+4\sqrt[3]{a^2b^2c^2}\le$$

$$\le a^2+b^2+c^2+2ab+2bc+2ca\Leftrightarrow$$

$$\Leftrightarrow 12\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\sqrt[3]{abc}+12\sqrt[3]{a^2b^2c^2}\le 2a^2+2b^2+2c^2+6ab+6bc+6ca\Leftrightarrow$$

$\Leftrightarrow 6\sqrt{\frac{a^2+b^2+c^2}{3}}\cdot\sqrt[3]{abc}+6\sqrt[3]{a^2b^2c^2}\le a^2+b^2+c^2+3(ab+bc+ca)$, which follows from:

We denote $\sqrt{\frac{a^2+b^2+c^2}{3}}=t$ and $\sqrt[3]{abc}=x$. From $ab+bc+ca\ge 3\sqrt[3]{(abc)^2}=3x^2$, it suffices to prove that:

$$6t\cdot x+6x^2\le 3t^2+3\cdot 3x^2\Leftrightarrow 3t^2+3x^2-6tx\ge 0\Leftrightarrow 3(t-x)^2\ge 0,$$

with equality for $t=x\Leftrightarrow\sqrt{\frac{a^2+b^2+c^2}{3}}=\sqrt[3]{abc}\Leftrightarrow a=b=c$.

Equality holds if and only if $a=b=c$. **Remark:** The problem can be developed.

If $a,b,c,d>0$ then:

$$\mathbf{3.}\ \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}+3\sqrt[4]{abcd}\le a+b+c+d.$$

Marin Chirciu – Romania

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}+3\sqrt[4]{abcd}\le a+b+c+d\Leftrightarrow$$

$$\left(\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}+3\sqrt[4]{abcd}\right)^2\le(a+b+c+d)^2\Leftrightarrow$$

$$\Leftrightarrow\frac{a^2+b^2+c^2+d^2}{4}+6\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}\cdot\sqrt[4]{abcd}+9\sqrt[4]{(abcd)^2}\le$$

$$\le a^2+b^2+c^2+d^2+2\sum ab\Leftrightarrow$$

$$24\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}\cdot\sqrt[4]{abcd}+36\sqrt[4]{(abcd)^2}\le 3a^2+3b^2+3c^2+3d^2+8\sum ab,$$

which follows from:

We denote $\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}=t$ and $\sqrt[4]{abcd}=x$. From $\sum ab\ge 6\sqrt[6]{(abcd)^3}=6\sqrt{abcd}=6x^2$, it suffices to prove that:

$24t\cdot x+36x^2\le 3\cdot 4t^2+8\cdot 6x^2\Leftrightarrow 12t^2+12x^2-24tx\ge 0\Leftrightarrow 12(t-x)^2\ge 0$, with equality for $t=x\Leftrightarrow\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}=\sqrt[4]{abcd}\Leftrightarrow a=b=c=d$.

Equality holds if and only if $a=b=c=d$.

J.2499. Solve in $\mathbb{C}$ the equation:

$$x^2+\frac{25x^2}{(2x+5)^2}=\frac{74}{49}$$

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Solution: We transform the equation equivalent:

$$x^2+\frac{25x^2}{(2x+5)^2}=\frac{74}{49}\Leftrightarrow x^2+\left(\frac{5x}{2x+5}\right)^2=\frac{74}{49}\Leftrightarrow\left(x-\frac{5x}{2x+5}\right)^2+2x\cdot\frac{5x}{2x+5}=\frac{74}{49}\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{2x^2}{2x+5}\right)^2+\frac{10x^2}{2x+5}=\frac{74}{49}\Leftrightarrow 4\left(\frac{x^2}{2x+5}\right)^2+10\cdot\frac{x^2}{2x+5}=\frac{74}{49}\Leftrightarrow$$

$$\Leftrightarrow 2\left(\frac{x^2}{2x+5}\right)^2+5\cdot\frac{x^2}{2x+5}=\frac{37}{49}.$$

With the substitution $\frac{x^2}{2x+5}=t$ we obtain the equation $2t^2+5t=\frac{37}{49}\Leftrightarrow$

$\Leftrightarrow 98t^2+245t-37=0$, with $\Delta=273^2$

We obtain $t_1=\frac{1}{7}$ and $t_2=\frac{-37}{14}$. Returning to the substitution we obtain:

$\frac{x^2}{2x+5}=\frac{1}{7}\Leftrightarrow 7x^2-2x-5=0$, with the solutions $x_{1,2}=\frac{1\pm 6}{7}\Rightarrow x_1,x_2=\frac{-5}{7}$.

$\frac{x^2}{2x+5}=\frac{-37}{14}\Leftrightarrow 14x^2+74x+185=0$, with the solutions $x_{1,2}=\frac{-37\pm i\sqrt{1221}}{7}=\frac{-37\pm 2i\sqrt{303}}{7}$.

Remark: The problem can be developed: **Let $\lambda>0, n>0$ fixed. Solve in $\mathbb{C}$ the equation:**

$$x^2+\frac{\lambda^2x^2}{(nx+\lambda)^2}=\frac{2\lambda+2\lambda n+n^2}{(\lambda+n)^2}$$

Marin Chirciu – Romania

Solution: The left-hand side of the equation becomes equivalent to:

$$x^2+\frac{\lambda^2x^2}{(nx+\lambda)^2}\Leftrightarrow x^2+\left(\frac{\lambda x}{nx+\lambda}\right)^2\Leftrightarrow\left(x-\frac{\lambda x}{nx+\lambda}\right)^2+2x\cdot\frac{\lambda x}{nx+\lambda}\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{nx^2}{nx+\lambda}\right)^2+\frac{2\lambda x^2}{nx+\lambda}\Leftrightarrow n^2\left(\frac{x^2}{nx+\lambda}\right)^2+2\lambda\cdot\frac{x^2}{nx+\lambda}.$$

With the substitution $\frac{x^2}{nx+\lambda}=t$ we obtain the equation $n^2t^2+2\lambda t=\frac{2\lambda^2+2\lambda n+n^2}{(\lambda+n)^2}\Leftrightarrow$

$\Leftrightarrow n^2(\lambda+n)^2t^2+2\lambda(\lambda+n)^2t-2\lambda^2+2\lambda n+n^2=0$, with

$\Delta'=(\lambda+n)^2(\lambda^2+\lambda n+n^2)^2$. It follows $t_1=\frac{1}{\lambda+n}$ and $t_2=\frac{-2\lambda^2-2\lambda n-n^2}{n^2(\lambda+n)}$.

Returning to the substitution we obtain:

$\frac{x^2}{nx+\lambda}=\frac{1}{\lambda+n}\Leftrightarrow(\lambda+n)x^2-nx-\lambda=0$, with the solutions $x_{1,2}=\frac{n\pm(2\lambda+n)}{2(\lambda+n)}$,

$$x_1=1,x_2=\frac{-\lambda}{\lambda+n};\qquad\frac{x^2}{nx+\lambda}=\frac{-2\lambda^2-2\lambda n-n^2}{n^2(\lambda+n)}\Leftrightarrow$$

$$\Leftrightarrow n^2(\lambda+n)x^2+n(2\lambda^2+2\lambda n+n^2)+\lambda(2\lambda^2+2\lambda n+n^2)=0,$$

with $\Delta=n^2(n^4-4\lambda^2(n+\lambda)^2)$ and the solutions:

$$x_{3,4}=\frac{-n(n+2n\lambda+2\lambda^2)\pm n\sqrt{n^4-4\lambda^2(n+\lambda)^2}}{2n^2(n+\lambda)}=\frac{-(n+2n\lambda+2\lambda^2)\pm\sqrt{n^4-4\lambda^2(n+\lambda)^2}}{2n(n+\lambda)}.$$

Note: For $n=2,\lambda=5$ we obtain the Problem J.2499 from RMM – 43.

RMM SOLVED PROBLEMS VI

By Marin Chirciu – Romania

J.2594. Solve for real numbers:

$$\mathbf{\frac{1}{1+\tan^4 x}+\frac{1}{10}=\frac{2}{1+2\tan^2 x}}$$

Daniel Sitaru – Romania

Solution: Lemma:

Denoting $\tan^2 x=t$ we obtain: $\frac{1}{1+t^2}+\frac{1}{10}=\frac{2}{1+2t}\Leftrightarrow 3t^3-19t^2+33t-9=0\Leftrightarrow$

$\Leftrightarrow(t-3)^2(3t-1)-0$, with the solutions $t_1=3$ and $t_2=\frac{1}{3}$.

From $\tan^2 x=3\Rightarrow x=\pm\sqrt{3}\Rightarrow x=\pm\frac{\pi}{3}+k\pi,k\in\mathbb{Z}$.

From $\tan^2 x = \frac{1}{3} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}} \Rightarrow x = \pm\frac{\pi}{6} + k\pi, k \in \mathbb{Z}$.

The set of the equation's solutions is $S = \left\{\pm\frac{\pi}{3} + k\pi, k \in \mathbb{Z}\right\} \cup \left\{\pm\frac{\pi}{6} + k\pi, k \in \mathbb{Z}\right\}$

Remark: The problem can be developed: **Let $\lambda > 0$ fixed. Solve for real numbers:**

$$\frac{50\lambda}{1+\tan^4 x} + 4\lambda + 3 = \frac{3(3\lambda+1)^2}{1+\lambda\tan^2 x}$$

Marin Chirciu – Romania

Solution:

Denoting $\tan^2 x = t$ we obtain: $\frac{50\lambda}{1+t^2} + 4\lambda + 3 = \frac{3(3\lambda+1)^2}{1+\lambda t} \Leftrightarrow$

$$(4\lambda+3)t^3 - (27\lambda+14)t^2 + (54\lambda+3)t - 27\lambda + 36 = 0 \Leftrightarrow$$

$$\Leftrightarrow (t-3)^2\big((4\lambda+3)t + 4 - 3\lambda\big) = 0$$

with the solutions $t_1 = 3$ and $t_2 = \frac{3\lambda-4}{4\lambda+3}$.

From $\tan^2 x = 3 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow x = \pm\frac{\pi}{3} + k\pi, k \in \mathbb{Z}$

From $\tan^2 x = \frac{3\lambda-4}{4\lambda+3} \Rightarrow \tan x = \pm\sqrt{\frac{3\lambda-4}{4\lambda+3}}$, for $\lambda \geq \frac{4}{3} \Rightarrow x = \pm\arctan\sqrt{\frac{3\lambda-4}{4\lambda+3}} + k\pi, k \in \mathbb{Z}$

The set of equation's solutions is $S = \left\{\pm\frac{\pi}{3} + k\pi, k \in \mathbb{Z}\right\} \cup \left\{\pm\arctan\sqrt{\frac{3\lambda-4}{4\lambda+3}} + k\pi, k \in \mathbb{Z}\right\}$, for $\lambda \geq \frac{4}{3}$.

If $\lambda \in \left(0, \frac{4}{3}\right)$ the set of equation's solutions is $S = \left\{\pm\frac{\pi}{3} + k\pi, k \in \mathbb{Z}\right\}$.

Note: For $\lambda = 3$ we obtain the Problem $J.2594$ form RMM – 43

J.2592 In ΔABC the following relationship holds:

$$\sum(2b+a)(2c+a) \leq 81R^2$$

Daniel Sitaru – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum(2b+a)(2c+a) = 2(5s^2 + 3r^2 + 12Rr).$$

Proof:

$$\sum(2b+a)(2c+a) = 8\sum bc + \sum a^2 = 2(5s^2 + 3r^2 + 12Rr)$$

We have used above $\sum bc = s^2 + r^2 + 4Rr$ and $\sum a^2 = 2(s^2 - r^2 - 4Rr)$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS=\sum(2b+a)(2c+a)=2(5s^2+3r^2+12Rr)\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}2(5(4R^2+4Rr+3r^2)+3r^2+12Rr)=4(10R^2+16Rr+9r^2)\overset{Euler}{\leq}81R^2=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality with opposite sense.

In ΔABC the following relationship holds:

$$\sum(2b+a)\,(2c+a)\geq 324r^2$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum(2b+a)\,(2c+a)=2(5s^2+3r^2+12Rr)$$

Proof:

$$\sum(2b+a)(2c+a)=8\sum bc+\sum a^2=2(5s^2+3r^2+12Rr)$$

We have used above: $\sum bc=s^2+r^2+4Rr$ and $\sum a^2=2(s^2-r^2-4Rr)$

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS=\sum(2b+a)\,(2c+a)=2(5s^2+3r^2+12Rr)\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}2(5(16Rr-5r^2)+3r^2+12Rr)=4(46Rr-11r^2)\overset{Euler}{\geq}324r^2=RHS$$

Equality holds if and only if the triangle is equilateral.

Remark: We can write the double inequality: **In ΔABC the following relationship holds:**

$$324r^2\leq\sum(2b+a)(2c+a)\leq 81R^2$$

Solution: See above. Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

If $\lambda\geq 0$, in ΔABC the following relationship holds:

$$36(\lambda+1)^2r^2\leq\sum(\lambda b+a)(\lambda c+a)\leq 9(\lambda+1)^2R^2$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\sum(\boldsymbol{\lambda b}+\boldsymbol{a})(\boldsymbol{\lambda c}+\boldsymbol{a})=(\boldsymbol{\lambda^2}+\mathbf{2}\boldsymbol{\lambda}+\mathbf{2})\boldsymbol{s^2}+(\boldsymbol{\lambda^2}+\mathbf{2}\boldsymbol{\lambda}-\mathbf{2})\boldsymbol{r^2}+(\mathbf{4}\boldsymbol{\lambda^2}+\mathbf{8}\boldsymbol{\lambda}-\mathbf{8})$$

Proof:

$$\sum(\lambda \boldsymbol{b}+\boldsymbol{a})(\lambda c+a)=(\lambda^2+2\lambda)\sum bc+\sum a^2=$$

$$=(\lambda^2+2\lambda+2)s^2+(\lambda^2+2\lambda-2)r^2+(4\lambda^2+8\lambda-8)$$

We have used above $\sum bc=s^2-r^2+4Rr$ and $\sum a^2=2(s^2-r^2-4Rr)$

Let's get back to the main problem.Using the Lemma, we obtain:

Right inequality:

$$\sum(\lambda b+a)(\lambda c+a)=(\lambda^2+2\lambda+2)s^2+(\lambda^2+2\lambda-2)r^2+(4\lambda^2+8\lambda-8)\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}(\lambda^2+2\lambda+2)(4R^2+4Rr+3r^2)+(\lambda^2+2\lambda-2)r^2+(4\lambda^2+8\lambda-8)=$$

$$=4\big((2\lambda^2+2\lambda+2)R^2+(8\lambda^2+16\lambda)Rr+(4\lambda^2+8\lambda+4)r^2\big)\overset{Euler}{\leq}36(\lambda+1)^2r^2$$

Equality holds if and only if the triangle is equilateral.

Note: For $\lambda=2$ we obtain the right inequality from Problem J.2592 from RMM – 43.

J.2586 In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$\boldsymbol{a\cdot IM^2+b\cdot IN^2+c\cdot IP^2\geq\frac{abc}{4}}$$

Daniel Sitaru – Romania

Solution: Lemma: In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$\boldsymbol{a\cdot IM^2+b\cdot IN^2+c\cdot IP^2=\frac{s(s^2+5r^2-14Rr)}{2}}$$

Proof: Using median theorem in ΔIBC, we obtain:

$$IM^2=\frac{2IB^2+2IC^2-a^2}{4}\Rightarrow$$

$$\sum a\cdot IM^2=\sum a\cdot\frac{2IB^2+2IC^2-a^2}{4}=\frac{2\sum(b+c)IA^2-\sum a^3}{4}=\frac{s(s^2+5r^2-14Rr)}{2}$$

We have used above $\sum(b+c)IA^2=2s(s^2+r^2-10Rr)$ and $\sum a^3=2s(s^2-3r^2-6Rr)$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS=\sum a\cdot IM^2\overset{Lemma}{=}\frac{s(s^2+5r^2-14Rr)}{2}\overset{(1)}{\geq}\frac{abc}{4}=RHS.$$

where (1) $\Leftrightarrow \frac{s(s^2+5r^2-14Rr)}{2} \ge \frac{abc}{4} \Leftrightarrow \frac{s(s^2+5r^2-14Rr)}{2} \ge \frac{4Rrs}{4} \Leftrightarrow s^2 \ge 16Rr - 5r^2$, (Gerretsen).

Equality holds if and only if the triangle is equilateral. **Remark:** Let's find an inequality of opposite sense.

In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$a \cdot IM^2 + b \cdot IN^2 + c \cdot IP^2 \le 2s(R^2 - 3r^2)$$

Marin Chirciu – Romania

Solution: Lemma: In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$a \cdot IM^2 + b \cdot IN^2 + c \cdot IP^2 = \frac{s(s^2 + 5r^2 - 14Rr)}{2}$$

Proof: Using median theorem in ΔIBC we obtain:

$$IM^2 = \frac{2IB^2 + 2IC^2 - a^2}{4} \Rightarrow$$

$$\sum a \cdot IM^2 = \sum a \cdot \frac{2IB^2 + 2IC^2 - a^2}{4} = \frac{2\sum(b+c)IA^2 - \sum a^3}{4} = \frac{s(s^2 + 5r^2 - 14Rr)}{2}$$

We have used above $\sum(b+c)IA^2 = 2s(s^2 + r^2 - 10Rr)$ and $\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$.

Let's get back to the main problem. Using the Lemma, we obtain:

$$\sum a \cdot IM^2 \overset{Lemma}{=} \frac{s(s^2 - 5r^2 - 14Rr)}{2} \overset{Gerretsen}{\le} \frac{s(4R^2 + 4Rr + 3r^2 + 5r^2 - 14Rr)}{2} =$$

$$= \frac{s(4R^2 - 10Rr + 8r^2)}{2} = s(2R^2 - 5Rr + 4r^2) \overset{Euler}{\le} s(2R^2 - 10r^2 + 4r^2) =$$

$$= s(2R^2 - 6R^2) = 2s(R^2 - 3r^2).$$

Equality holds if and only if the triangle is equilateral.

Remark: We can write the double inequality:

In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$Rrs \le a \cdot IM^2 + b \cdot IN^2 + c \cdot IP^2 \le 2s(R^2 - 3r^2)$$

Solution: See above. Equality holds if and only if the triangle is equilateral.

Remark: The problem can be developed.

In $\Delta ABC, AM, BN, CP$ – medians, I – incenter, then:

$$36r^4 \le (a \cdot IM)^2 + (b \cdot IN)^2 + (c \cdot IP)^2 \le 2(5R^4 - 62r^4).$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC, AM, BN, CP – medians, I – incenter, then:

$$(a\cdot IM)^2+(b\cdot IN)^2+(c\cdot IP)^2=\frac{s^2(s^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}$$

Proof: Using the median theorem we obtain:

$$IM^2=\frac{2IB^2+2IC^2-a^2}{4}\Rightarrow\sum a^2\cdot IM^2=\sum a^2\cdot\frac{2IB^2+2IC^2-a^2}{4}=$$

$$=\frac{2\sum(b^2+c^2)IA^2-\sum a^4}{4}=\frac{s^2(s^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}.$$

We have used above $\sum(b^2+c^2)\,IA^2=2[s^2(s^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)]$

and $\sum a^4=2[s^4-s^2(8Rr+6r^2)+r^2(4R+r)^2]$.

Let's get back to the main problem.

Right inequality:

$$\sum(a\cdot IM)^2=\frac{s^2(s^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}\overset{Gerretsen}{\leq}$$

$$\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}$$

$$=\frac{1}{2}(16R^4-32R^3r+16R^2r^2-8Rr^3+24r^4)=4(2R^4-4R^3r+2R^2r^2-Rr^3+3r^4)$$

$$\overset{Euler}{\leq}4\left(\frac{5}{2}R^4-31r^4\right)=2(5R^4-62r^4).$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\sum(a\cdot IM)^2=\frac{s^2(s^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}\overset{Gerretsen}{\geq}$$

$$\overset{Gerretsen}{\geq}\frac{(16Rr-5r^2)(16Rr-5r^2+6r^2-16Rr)+r^2(16R^2-8Rr-3r^2)}{2}=$$

$$=\frac{r^2}{2}(16R^2-8Rr-8r^2)=4r^2(2R^2-Rr-r^2)\overset{Euler}{\geq}4r^2\cdot 9r^2=36r^4$$

Equality holds if and only if the triangle is equilateral.

J.2590 In ΔABC the following relationship holds:

$$\sqrt[3]{(a+b)(b+c)(c+a)}\geq 4\sqrt{3}r$$

Daniel Sitaru, Dan Nănuți – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$(a+b)(b+c)(c+a)=2s(s^2+r^2+2Rr)$$

The inequality can be written:

$2s(s^2+r^2+2Rr)\geq\left(4\sqrt{3}r\right)^3 \Leftrightarrow 2s(s^2+r^2+2Rr)\geq 64\cdot 3\sqrt{3}r^3$, which follows from Mitrinovic inequality $s\geq 3\sqrt{3}r$. It remains to prove that:

$$2\cdot 3\sqrt{3}r(s^2+r^2+2Rr)\geq 64\cdot 3\sqrt{3}r^3 \Leftrightarrow s^2+r^2+2Rr\geq 32r^2,$$

true from Gerretsen's inequality: $s^2\geq 16R-5r^2$.

It suffices to prove that: $16Rr-5r^2+r^2+2Rr\geq 32r^2 \Leftrightarrow 18Rr\geq 36r^2 \Leftrightarrow R\geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: Let's find an inequality of opposite sense.

In ΔABC **the following relationship holds:**

$$\sqrt[3]{(a+b)(b+c)(c+a)}\leq 2\sqrt{3}R$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(b+c)=2s(s^2+r^2+2Rr)$$

Solution: The inequality can be written:

$2s(s^2+r^2+2Rr)\geq\left(2\sqrt{3}R\right)^3 \Leftrightarrow 2s(s^2+r^2+2Rr)\geq 8\cdot 3\sqrt{3}R^3$, which follows from Mitrinovic inequality $s\leq\frac{3\sqrt{3}R}{2}$. It remains to prove that:

$$2\cdot\frac{3\sqrt{3}R}{2}(s^2+r^2+2Rr)\leq 8\cdot 3\sqrt{3}R^3 \Leftrightarrow s^2+r^2+2Rr\leq 8R^2,$$

true from Gerretsen's inequality: $s^2\leq 4R^2+4Rr+3r^2$.

It suffices to prove that:

$4R^2+4Rr+3r^2+r^2+2Rr\leq 8R^2 \Leftrightarrow 2R^2-3R-2r^2\geq 0 \Leftrightarrow (R-2r)(2R+r)\geq 0$, obvious from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark: We can write the double inequality:

In ΔABC the following relationship holds:

$$4\sqrt{3}\leq\sqrt[3]{(a+b)(b+c)(c+a)}\leq 2\sqrt{3}R$$

Solution: See above. Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$\mathbf{5\sqrt{3}r \le \sqrt[3]{(s+a)(s+b)(s+c)} \le \frac{5\sqrt{3}R}{2}}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(\boldsymbol{s+a}) = \boldsymbol{s(4s^2+r^2+8Rr)}$$

Left inequality:

$$\prod(s+a) = s(4s^2+r^2+8Rr) \overset{Gerretsen}{\le} s(s(4R^2+4Rr+3r^2)+r^2+8Rr) =$$

$$= s(16R^2+24Rr+13r^2) \overset{Euler}{\le} s\cdot\frac{125R^2}{4} \overset{Mitrinovic}{\le} \frac{3\sqrt{3}R}{2}\cdot\frac{125R^2}{4} = \left(\frac{5\sqrt{3}R}{2}\right)^3$$

Equality holds if and only if the triangle is equilateral.

Right inequality:

$$\prod(s+a) = s(4s^2+r^2+8Rr) \overset{Gerretsen}{\ge} s(4(16Rr-5r^2)+r^2+8Rr) =$$

$$= s(72Rr-19r^2) \overset{Euler}{\ge} sr\cdot 125r \overset{Mitrinovic}{\ge} 3\sqrt{3}r\cdot 125r = \left(5\sqrt{3}r\right)^3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\mathbf{8\sqrt{3}r \le \sqrt[3]{(2s+a)(2s+b)(2s+c)} \le 4\sqrt{3}R}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(\boldsymbol{2s+a}) = \boldsymbol{2s(9s^2+r^2+6Rr)}$$

Left inequality:

$$\prod(2s+a) = 2s(9s^2+r^2+6Rr) \overset{Gerretsen}{\le} 2s(9(4R^2+4rR+3R^2)+R^2+6Rr) =$$

$$= 4s(18R^2+21Rr+14r^2) \overset{Euler}{\le} 4s\cdot 32R^2 \overset{Mitrinovic}{\le} 4\cdot\frac{3\sqrt{3}R}{2}\cdot 32R^2 = \left(4\sqrt{3}R\right)^3$$

Equality holds if and only if the triangle is equilateral.

Right inequality:

$$\prod(2s+a)=2s(9s^2+r^2+6Rr)\overset{Gerretsen}{\geq}2s(6(16Rr-5r^2)+r^2+8Rr)=$$

$$=4sr(75R-22r)\overset{Euler}{\geq}4sr\cdot 128r\overset{Mitrinovic}{\geq}4\cdot 3\sqrt{3}r\cdot 128r=\left(8\sqrt{3}r\right)^3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$4r\leq\sqrt[3]{(r+r_a)(r+r_b)(r+r_c)}\leq 2R$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(r+r_a)=2r(s^2+r^2+2Rr)$$

Right inequality:

$$\prod(r+r_a)=2r(s^2+r^2+2Rr)\overset{Gerretsen}{\leq}2r(4R^2+4Rr+3r^2+r^2+2Rr)=$$

$$=4r(2R^2+3Rr+2r^2)\overset{Euler}{\leq}4r\cdot 4R^2\overset{Euler}{\leq}2R\cdot 4R^2=(2R)^3$$

Equality holds if and only if the triangle is equilateral.

Left inequality.

$$\prod(r+r_a)=2r(s^2+r^2+2Rr)\overset{Gerretsen}{\geq}2r(16Rr-5r^2+r^2+2Rr)=$$

$$=4r^2(9R-2r)\overset{Euler}{\geq}4^2\cdot 16r\overset{Mitrinovic}{\geq}64r^3=(8r)^3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$2r\leq\sqrt[3]{(r_a-r)(r_b-r)(r_c-r)}\leq R$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(r_a-r)=4Rr^2$$

Right inequality:

$$\prod(r_a-r)=4Rr^2\overset{Euler}{\leq}R^3.$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\prod(r_a - r) = 4Rr^2 \overset{Euler}{\geq} 8r^3$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\mathbf{9r\left(\frac{2r}{R}\right)^{\frac{1}{3}} \leq \sqrt[3]{(h_a + 2r_a)(h_b + 2r_b)(h_c + 2r_c)} \leq \frac{9R}{2}}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\mathbf{\prod(h_a + 2r_a) = \frac{2s^4}{R}}$$

Right inequality:

$$\prod(h_a + 2r_a) = \frac{2s^4}{R} \overset{Mitrinovic}{\leq} \frac{2}{R}\frac{27R^2}{4}\frac{27R^2}{4} = \left(\frac{9R}{2}\right)^3.$$

Equality holds if and only if the triangle is equilateral.

Left inequality.

$$\prod(h_a + 2r_a) = \frac{2s^4}{R} \overset{Mitrinovic}{\geq} \frac{2}{R} \cdot 27r^2 \cdot 27r^2 = \frac{2r}{R} \cdot (9r)^3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In ΔABC the following relationship holds:

$$\mathbf{6r \leq \sqrt[3]{(r_a + r_b)(r_b + r_c)(r_c + r_a)} \leq 3R}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\mathbf{\prod(r_b + r_c) = 4Rs^2}$$

Right inequality:

$$\prod(r_b + r_c) = 4Rs^2 \overset{Mitrinovic}{\leq} 4R \cdot \frac{27R^2}{4} = (3R)^3.$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\prod(r_b + r_c) = 4Rs^2 \overset{Euler\&Mitrinovic}{\geq} 8r \cdot 27r^2 = (6r)^3.$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

In ΔABC the following relationship holds:

$$6r\left(\frac{2r}{R}\right)^{\frac{2}{3}} \le \sqrt[3]{(h_a+h_b)(h_b+h_c)(h_c+h_a)} \le 3R$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(h_b+h_c) = \frac{r^2(s^2+r^2+2Rr)}{R^2}$$

Right inequality:

$$\prod(h_b+h_c) = \frac{s^2(s^2+r^2+2Rr)}{2R} \overset{Mitrinovic\ \&\ Gerretsen}{\le} \frac{\frac{27R^2}{4}(4R^2+4Rr+3r^2+r^2+2Rr)}{2R} =$$

$$= \frac{\frac{27R^2}{4}(4R^2+6Rr+4r^2)}{2R} = \frac{27(2R^2+3Rr+2r^2)}{4} \overset{Euler}{\le} \frac{27R\cdot 4R^2}{4} = 27R^3 = (3R)^3$$

Equality holds if and only if the triangle is equilateral.

Left inequality.

$$\prod(h_a+r_a) = \frac{s^2(s^2+r^2+2Rr)}{2R} \overset{Mitrinovic\ \&\ Gerretsen}{\ge} \frac{27r^2(16Rr-5r^2+r^2+2Rr)}{2R} =$$

$$= \frac{27r^2(18Rr-4r^2)}{2R} = \frac{27r^2\cdot 2r(9R-2r)}{2R} \overset{Euler}{\ge} \frac{27r^2\cdot 2r\cdot 16r}{2R} = \left(\frac{2r}{R}\right)(6r)^3$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$2r\left(\frac{2r}{R}\right)^{\frac{1}{3}} \le \sqrt[3]{(h_a-r)(h_b-r)(h_c-r)} \le R$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod(h_a-r) = \frac{r^2}{2R}(s^2+r^2+2Rr)$$

Right inequality:

$$\prod(h_a-r) = \frac{R^2}{2R}(s^2+r^2+2Rr) \overset{Gerretsen}{\le} \frac{r^2(4R^2+4Rr+3r^2+r^2+2Rr)}{2R} =$$

$$= \frac{r^2(4R^2+6Rr+4r^2)}{2R} = \frac{r^2(2R^2+3Rr+2r^2)}{R} \overset{Euler}{\le} \frac{\frac{R^2}{4}\cdot 4R^2}{R} = R^3$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\prod(h_a - r) = \frac{r^2}{2R}(s^2 + r^2 + 2Rr) \overset{Gerretsen}{\geq} \frac{r^2(16Rr - 5r^2 + r^2 + 2Rr)}{2R} =$$

$$= \frac{r^2(18Rr - 4r^2)}{2R} = \frac{r^2 \cdot 2r(9R - 2r)}{2R} \overset{Euler}{\geq} \frac{r^2 \cdot 2r \cdot 16r}{2R} = \left(\frac{2r}{R}\right)(2r)^3$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$\boldsymbol{r\left(\frac{2r}{R}\right)^{\frac{1}{3}} \leq \sqrt[3]{(h_a - 2r)(h_b - 2r)(h_c - 2r)} \leq \frac{R}{2}}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\boldsymbol{\prod(h_a - 2r) = \frac{2r^4}{R}}$$

Right inequality:

$$\prod(h_a - 2r) = \frac{2r^4}{R} \overset{Euler}{\leq} \frac{2\frac{r^4}{16}}{R} = \frac{R^3}{8} = \left(\frac{R}{2}\right)^3$$

Equality holds if and only if the triangle is equilateral.

Left inequality:

$$\prod(h_a - 2r) = \frac{2r^4}{R} = \frac{2r}{R} \cdot r^3 =$$

$$= \frac{r^2(18Rr - 4r^2)}{2R} = \frac{r^2 \cdot 2r(9R - 2r)}{2R} \overset{Euler}{\geq} \frac{r^2 \cdot 2r \cdot 16r}{2R} = \left(\frac{2r}{R}\right)(2r)^3$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way: **In ΔABC the following relationship holds:**

$$\boldsymbol{\sqrt[3]{(r_a - 2r)(r_b - 2r)(r_c - 2r)} \leq r}$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\boldsymbol{\prod(r_a - 2r) = r(16Rr - 4r^2 - s^2)}$$

$$\prod(r_a - 2r) = r(16Rr - 4r^2 - s^2) \overset{Gerretsen}{\leq} r(16Rr - 4r^2 - 16Rr + 5r^2) = r \cdot r^2 = r^3.$$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

In acute ΔABC the following relationship holds:

$$\sqrt[3]{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)} \le 3R^2$$

Marin Chirciu – Romania

Solution: Lemma: In ΔABC the following relationship holds:

$$\prod (b^2+c^2-a^2) = 32r^2s^2(s^2-(2R+r)^2)$$

$$\prod (b^2+c^2-a^2) = 32r^2s^2(s^2-(2R+r)^2) \overset{Mitrinovic\ \&\ Gerretsen}{\le}$$

$$= 32r^2\frac{27R^2}{4}(4R^2+4Rr+3r^2-(2R+r)^2) = 8r^2\cdot 27R^2\cdot 2r^2 =$$

$$= 16r^4\cdot 27R^2 \overset{Euler}{\le} R^4\cdot 27R^2 = 27R^6 = (3R^2)^3.$$

Equality holds if and only if the triangle is equilateral.

J.2571 If $M\in Int\ (\Delta ABC), d_a = d(M,BC), d_b = d(M,CA), d_c = d(M,AB)$ then:

$$\frac{a^2}{d_ah_a}+\frac{b^2}{d_bh_b}+\frac{c^2}{d_ch_c}\ge 12$$

D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

Solution: Lemma: If $M\in Int(\Delta ABC), d_a = d(M,BC), F_1 = Aria[MBC]$ then:

$$\frac{a^2}{d_ah_a} = \frac{a^4}{4FF_1}$$

Proof:

$$\frac{a^2}{d_ah_a} = \frac{a}{d_a}\cdot\frac{a}{h_a} = \frac{a^2}{ad_a}\cdot\frac{a^2}{ah_a} = \frac{a^2}{2F_1}\cdot\frac{a^2}{2F} = \frac{a^4}{4FF_1}$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS = \sum\frac{a^2}{d_ah_a} = \sum\frac{a^4}{4FF_1} \overset{CS}{\ge} \frac{(\sum a^2)^2}{\sum 4FF_1} \overset{I-W}{\ge} \frac{\left(4F\sqrt{3}\right)^2}{4F\sum F_1} = \frac{16F^2\cdot 3}{4F\cdot F} = 12 = RHS.$$

We have used above Ionescu – Weitzenbock inequality: $\sum a^2 \ge 4F\sqrt{3}$.

Equality holds if and only if the triangle is equilateral and $M\equiv 0$.

Remark: The problem can be developed.

If $M\in Int(\Delta ABC), d_a = d(M,BC), d_b = d(M,CA), d_c = d(M,AB)$ and $n\in\mathbb{N}$ then:

$$\frac{a^{2n}}{d_ah_a}+\frac{b^{2n}}{d_bh_b}+\frac{c^{2n}}{d_ch_c}\geq 12\left(\frac{4F}{\sqrt{3}}\right)^{n-1}$$

Solution: Lemma: If $M\in Int\ (\Delta ABC), d_a=d(M,BC), F_1=Aria\ [MBC]$ and $n\in\mathbb{N}$ then:

$$\frac{a^{2n}}{d_ah_a}=\frac{a^{2n+2}}{4FF_1}$$

Proof:

$$\frac{a^{2n}}{d_ah_a}=\frac{a}{d_a}\cdot\frac{a}{h_a}\cdot a^{2n-2}=\frac{a^2}{ad_a}\cdot\frac{a^2}{ah_a}\cdot a^{2n-2}=\frac{a^2}{2F_1}\cdot\frac{a^2}{2F}\cdot a^{2n-2}=\frac{a^{2n+2}}{4FF_1}$$

Let's get back to the main problem. Using the Lemma, we obtain:

$$LHS=\sum\frac{a^{2n}}{d_ah_a}\overset{Lemma}{=}\sum\frac{a^{2n+2}}{4FF_1}\overset{Holder}{\geq}\frac{(\sum a^2)^{n+1}}{3^{n-1}\sum 4FF_1}\overset{I-W}{\geq}\frac{\left(4F\sqrt{3}\right)^{n+1}}{3^{n-1}\cdot 4F\sum F_1}=$$

$$=\frac{16F^2\cdot 3}{3^{n-1}\cdot 4F\cdot F}=12\left(\frac{4F}{\sqrt{3}}\right)^{n-1}=RHS.$$

We have used above Ionescu – Weitzenbock inequality: $\sum a^2\geq 4F\sqrt{3}$. Equality holds if and only if the triangle is equilateral and $M\equiv 0$. **Note:** For $n=1$ we obtain the Problem J.2571 form RMM – 43.

J.2567. If $x,y,z>0$ then in ΔABC:

$$\frac{yz}{h_a^2}+\frac{yz}{h_b^2}+\frac{yz}{h_c^2}\leq\frac{R^2}{4F^2}(x+y+z)^2$$

D.M. Bătinețu – Giurgiu, Dan Nănuți – Romania

Solution:

$$LHS=\sum\frac{yz}{h_a^2}=\sum\frac{yz}{\left(\frac{2F}{a}\right)^2}=\frac{1}{4F^2}\sum yza^2\overset{Lemma}{\leq}\frac{R^2}{4F^2}(x+y+z)^2=RHS$$

Lemma: If $x,y,z>0$ then in ΔABC:

$$\sum yza^2\overset{Lemma}{\leq}R^2(x+y+z)^2$$

Proof: We use:

$(x+y+z)(xMA^2+yMB^2+zMC^2)\geq yza^2+zxb^2+xyc^2$, where $x,y,z>0$ and M an arbitrary point in the plane ΔABC. Indeed:

$$0\leq\left(x\cdot\overrightarrow{MA}+y\cdot\overrightarrow{MB}+z\cdot\overrightarrow{MC}\right)^2=$$

$$=(x+y+z)(x\cdot MA^2+y\cdot MB^2+z\cdot MC^2)-(yza^2+zxb^2+xyc^2)$$

Putting $M \equiv O$ in $(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2$ we obtain:

$$(x+y+z)(xOA^2+yOB^2+zOC^2) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow (x+y+z)(xR^2+yR^2+zR^2) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)(x+y+z) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)^2 \geq yza^2+zxb^2+xyc^2$$

Equality holds if and only if the triangle is equilateral and $x=y=z$.

Remark: The problem can be developed.

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{yz}{m_a^2}+\frac{yz}{m_b^2}+\frac{yz}{m_c^2} \leq \frac{R^2}{4F^2}(x+y+z)^2 = RHS$$

Proof: Putting $M \equiv O$ in $(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2$ we obtain:

$$(x+y+z)(xOA^2+yOB^2+zOC^2) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow (x+y+z)(xR^2+yR^2+zR^2) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)(x+y+z) \geq yza^2+zxb^2+xyc^2 \Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)^2 \geq yza^2+zxb^2+xyc^2$$

Equality holds if and only the triangle is equilateral and $x=y=z$.

Remark: The problem can be developed: **If $x,y,z>0$ then in ΔABC:**

$$\frac{yz}{w_a^2}+\frac{yz}{w_b^2}+\frac{yz}{w_c^2} \leq \frac{R^2}{4F^2}(x+y+z)^2$$

Marin Chirciu – Romania

Solution:

$$LHS = \sum \frac{yz}{w_a^2} \leq \sum \frac{yz}{h_a^2} = \sum \frac{yz}{\left(\frac{2F}{a}\right)^2} = \frac{1}{4F^2}\sum yza^2 \overset{Lemma}{\leq} \frac{R^2}{4F^2}(x+y+z)^2 = RHS$$

The problem can be developed.

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{yz}{s_a^2}+\frac{yz}{s_b^2}+\frac{yz}{s_c^2} \leq \frac{R^2}{4F^2}(x+y+z)^2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{yz}{s_a^2}\le\sum\frac{yz}{h_a^2}=\sum\frac{yz}{\left(\frac{2F}{a}\right)^2}=\frac{1}{4F^2}\sum yza^2\overset{Lemma}{\le}\frac{R^2}{4F^2}(x+y+z)^2=RHS$$

Equality holds if and only ift the triangle is equilateral and $x=y=z$.

Remark: The problem can be developed.

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{yz}{r_a^2}+\frac{yz}{r_b^2}+\frac{yz}{r_c^2}\le\frac{13R^2}{4F^2}(x+y+z)^2$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{yz}{r_a^2}=\sum\frac{yz}{\left(\frac{F}{s-a}\right)^2}=\frac{1}{F^2}\sum yz(s-a)^2\overset{Lemma}{\le}\frac{13R^2}{4F^2}(x+y+z)^2=RHS$$

Lemma: If $x,y,z>0$ then in ΔABC the following relationships holds:

$$\sum yz(s-a)^2\le\frac{13R^2}{4}(x+y+z)^2;\quad \sum yza^2\le R^2(x+y+z)^2$$

Proof: We use:

$(x+y+z)(xMA^2+yMB^2+zMC^2)\ge yza^2+zxb^2+xyc^2$, where $x,y,z>0$ and M an arbitrary point in ΔABC's plane. Indeed:

$$0\le\left(x\cdot\overrightarrow{MA}+y\cdot\overrightarrow{MB}+z\cdot\overrightarrow{MC}\right)=(x+y+z)(x\cdot MA^2+y\cdot MB^2+z\cdot MC^2)-$$

$$-(yza^2+zxb^2+xyc^2)$$

Putting $M\equiv O$ in $(x+y+z)(xMA^2+yMB^2+zMC^2)\ge yza^2+zxb^2+xyc^2$ we obtain:

$$(x+y+z)(xOA^2+yOB^2+zOC^2)\ge yza^2+zxb^2+xyc^2\Leftrightarrow$$

$$\Leftrightarrow(x+y+z)(xR^2+yR^2+zR^2)\ge yza^2+zxb^2+xyc^2\Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)(x+y+z)\ge yza^2+zxb^2+xyc^2\Leftrightarrow$$

$$\Leftrightarrow R^2(x+y+z)^2\ge yza^2+zxb^2+xyc^2$$

Equality holds if and only if the triangle is equilateral and $x=y=z$.

$$\sum yz(s-a)^2=\sum yz(s^2-2sa+a^2)=s^2\sum yz-2s\sum yz\cdot a+\sum yz\cdot a^2\le$$

$$\le s^2\cdot\frac{1}{3}(x+y+z)^2-2s\sum yz\cdot a+R^2(x+y+z)^2\le$$

$$\le\frac{27R^2}{4}\cdot\frac{1}{3}(x+y+z)^2+R^2(x+y+z)^2-2\textstyle\sum yz\cdot a=$$

$$=\frac{9R^2}{4}(x+y+z)^2+R^2(x+y+z)^2-2s\sum yz\cdot a=\frac{13R^2}{4}(x+y+z)^2-2s\sum yz\cdot a$$

$$\le\frac{13R^2}{4}(x+y+z)^2.$$

J.2601 If $x,y,z>0$, in ΔABC the following relationship holds:

$$\sum\frac{1}{w_a^2}\cdot\frac{y+z}{x}\ge\frac{18}{s^2}$$

Mehmet Șahin – Turkey

Solution:

$$LHS=\sum\frac{1}{w_a^2}\frac{y+z}{x}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{1}{w_a^2}\frac{y+z}{x}}=3\sqrt[3]{\prod\frac{y+z}{x}\prod\frac{1}{w_a^2}}\overset{Cesaro}{\ge}3\sqrt[3]{8\prod\frac{1}{w_a^2}}=$$

$$=\frac{6}{\sqrt[3]{\prod w_a^2}}\overset{(1)}{\ge}\frac{6}{\frac{s^2}{3}}=\frac{18}{s^2}=RHS,$$

Where (1) $\Leftrightarrow\sqrt[3]{\prod w_a^2}\le\frac{s^2}{3}$, see $\prod w_a^2\le\prod r_a^2=(rs^2)^2=r^2s^4\overset{Mitrinovic}{\le}\frac{s^2}{27}\cdot s^4=\frac{s^6}{27}$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way: **If $x,y,z>0$, in ΔABC the following relationship holds:**

$$\sum\frac{1}{h_a^2}\frac{y+z}{x}\ge\frac{18}{s^2}$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{h_a^2}\frac{y+z}{x}\overset{AM-GM}{\ge}3\sqrt[3]{\prod\frac{1}{h_a^2}\frac{y+z}{x}}=3\sqrt[3]{\prod\frac{y+z}{x}\prod\frac{1}{h_a^2}}\overset{Cesaro}{\ge}3\sqrt[3]{8\prod\frac{1}{h_a^2}}=$$

$$=\frac{6}{\sqrt[3]{\prod h_a^2}}\overset{(1)}{\ge}\frac{6}{\frac{s^2}{3}}=\frac{18}{s^2}=RHS,$$

where (1) $\Leftrightarrow\sqrt[3]{\prod h_a^2}\le\frac{s^2}{3}$, see $\prod h_a^2\le\prod w_a^2\le\prod r_a^2=(rs^2)^2=r^2s^4\overset{Mitrinovic}{\le}\frac{s^2}{27}\cdot s^4=\frac{s^6}{27}$

Equality holds if and only if the triangle is equilateral. **Remark:** In the same way.

If $x,y,z>0$, in ΔABC the following relationship holds:

$$\sum\frac{1}{m_a^2}\frac{y+z}{x}\ge\frac{8}{3R^2}$$

Marin Chirciu - Romania

Solution:

$$LHS=\sum\frac{1}{m_a^2}\frac{y+z}{x}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{1}{m_a^2}\frac{y+z}{x}}=3\sqrt[3]{\prod\frac{y+z}{x}\prod\frac{1}{m_a^2}}\overset{Cesaro}{\geq}3\sqrt[3]{8\prod\frac{1}{m_a^2}}=$$

$$=\frac{6}{\sqrt[3]{\prod m_a^2}}\overset{(1)}{\geq}\frac{6}{\left(\frac{3R}{2}\right)^2}=\frac{8}{3R^2}=RHS,$$

Where (1) $\Leftrightarrow \sqrt[3]{\prod m_a^2}\leq\left(\frac{3R}{2}\right)^2$, see $\prod m_a^2\leq\left(\frac{Rs^2}{2}\right)^2\overset{Mitrinovic}{\leq}\left(\frac{R\cdot\frac{27R^2}{4}}{2}\right)^2=\left(\frac{27R^3}{8}\right)^2=\left(\frac{3R}{2}\right)^6$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

If $x,y,z>0$, in ΔABC the following relationship holds:

$$\sum\frac{1}{s_a^2}\frac{y+z}{x}\geq\frac{8}{3R^2}$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{s_a^2}\frac{y+z}{x}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{1}{s_a^2}\frac{y+z}{x}}=3\sqrt[3]{\prod\frac{y+z}{x}\prod\frac{1}{s_a^2}}\overset{Cesaro}{\geq}3\sqrt[3]{8\prod\frac{1}{s_a^2}}=$$

$=\frac{6}{\sqrt[3]{\prod s_a^2}}\overset{(1)}{\geq}\frac{6}{\left(\frac{3R}{2}\right)^2}=\frac{8}{3R^2}=RHS,$ where (1) $\Leftrightarrow \sqrt[3]{s_a^2}\leq\left(\frac{3R}{2}\right)^2$, see

$$\prod s_a^2\leq\prod m_a^2\leq\left(\frac{Rs^2}{2}\right)^2\overset{Mitrinovic}{\leq}\left(\frac{R\cdot\frac{27R^2}{4}}{2}\right)^2=\left(\frac{27R^3}{8}\right)^2=\left(\frac{3R}{2}\right)^6$$

Equality holds if and only if the triangle is equilateral.

Remark: In the same way.

If $x,y,z>0$, in ΔABC the following relationship holds:

$$\sum\frac{1}{r_a^2}\frac{y+z}{x}\geq\frac{18}{s^2}$$

Marin Chirciu – Romania

Solution:

$$LHS=\sum\frac{1}{r_a^2}\frac{y+z}{x}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{1}{r_a^2}\frac{y+z}{x}}=3\sqrt[3]{\prod\frac{y+z}{x}\prod\frac{1}{r_a^2}}\overset{Cesaro}{\geq}3\sqrt[3]{8\prod\frac{1}{r_a^2}}=$$

$$=\frac{6}{\sqrt[3]{\prod r_a^2}}\overset{(1)}{\geq}\frac{6}{\frac{s^2}{3}}=\frac{18}{s^2}=RHS,$$

where (1) $\Leftrightarrow \sqrt[3]{\prod r_a^2}\leq\frac{s^2}{3}$,m see $\prod r_a^2=(rs^2)^2=r^2s^4\overset{Mitrinovic}{\leq}\frac{s^2}{27}\cdot s^4=\frac{s^6}{27}$.

Equality holds if and only if the triangle is equilateral.

ABOUT THE SPIEKER'S CEVIANS IN TRIANGLE

By Bogdan Fuștei-Romania

ABSTRACT: We consider ABC a triangle with usual notations. We will study the Spieker's cevians in a triangle ABC. Spieker center is obtain as follow: Le be M_1-middle of the side BC,M_2-middle of the side AC,M_3-middle of the side AB. Spieker center of $\triangle$ABC is the center of the circle inscribed in the medial triangle of $\triangle$ABC. ($\Delta M_1M_2M_3$).Notation for Spieker center: S_p ;

RESULTS:

Notation for Spieker center: S_p ; We consider $AA_1= p_a$ Spieker cevian from A (and analogs);

We consider: $\alpha=\measuredangle BAA_1$ and $\beta=\measuredangle CAA_1$; $\alpha+\beta=A$, M_1M_2 , M_2M_3 , M_1M_3-middle lines in triangle ABC , $M_1M_2=\frac{1}{2}c$, $M_2M_3=\frac{1}{2}a$, $M_1M_3=\frac{1}{2}b$

From direct manipulations using Heron formula we obtain: area $\Delta M_1M_2M_3=\frac{1}{4}S$

S = area of triangle ABC.

$$\frac{1}{2}(M_1M_2+M_2M_3+M_1M_3)=\frac{1}{2}\frac{1}{2}(a+b+c)=\frac{1}{4}2p=\frac{1}{2}p.$$

We consider r_1-inradius of the circle inscribed in medial triangle of $\triangle$ABC.

From $\frac{1}{4}pr=\frac{1}{2}pr_1\rightarrow r_1=\frac{1}{2}r$. We consider $\triangle AM_3S_p$, $M_3S_p=\frac{r_1}{\sin\frac{C}{2}}=\frac{r}{2\sin\frac{C}{2}}$ (and analogs)

In $\triangle AM_3S_p$ we use sinus law and obtain: $\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin AM_3S_p}$

$\measuredangle AM_3M_2=\measuredangle ABC$ because $M_2M_3\parallel BC$

$\measuredangle M_1M_3M_2=\measuredangle ACB$ because $M_1M_3M_2C$ parallelogram and two angles are opposite angles.

$$\measuredangle AM_3S_p=\measuredangle AM_3M_2+\frac{1}{2}(\measuredangle M_1M_3M_2)=\measuredangle B+\measuredangle\frac{1}{2}C$$

$$\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin AM_3S_p}\rightarrow\frac{M_3S_p}{\sin\alpha}=\frac{AS_p}{\sin\left(B+\frac{1}{2}C\right)},\ \frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(B+\frac{1}{2}C\right)}\rightarrow\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(\frac{2B+C}{2}\right)}$$

$$A+B+C=\pi\rightarrow B+C=\pi-A\rightarrow 2B+C=\pi+B-A$$

$$\frac{r}{2\sin\alpha\sin\frac{C}{2}}=\frac{AS_p}{\sin\left(\frac{\pi+B-A}{2}\right)}=\frac{AS_p}{\sin\left(\frac{\pi}{2}-\frac{A-B}{2}\right)}$$

We use the well-known formula:$\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and obtain:

$$\sin\left(\frac{\pi}{2}-\frac{A-B}{2}\right)=\cos\frac{A-B}{2},\ \cos\frac{A-B}{2}=\frac{a+b}{c}\sin\frac{C}{2}\text{ (and analogs) (Karl Mollweide formulas)}$$

We obtain:

$$\frac{\mathbf{r}}{\mathbf{2\sin\alpha\sin\frac{C}{2}}}=\frac{\mathbf{AS_p}}{\mathbf{\frac{a+b}{c}\sin\frac{C}{2}}}\rightarrow\mathbf{\sin\alpha}=\frac{\mathbf{a+b}}{\mathbf{c}}\frac{\mathbf{r}}{\mathbf{2AS_p}}\ \mathbf{(1)}$$

Using same method for $\triangle AM_2S_p$ obtain:

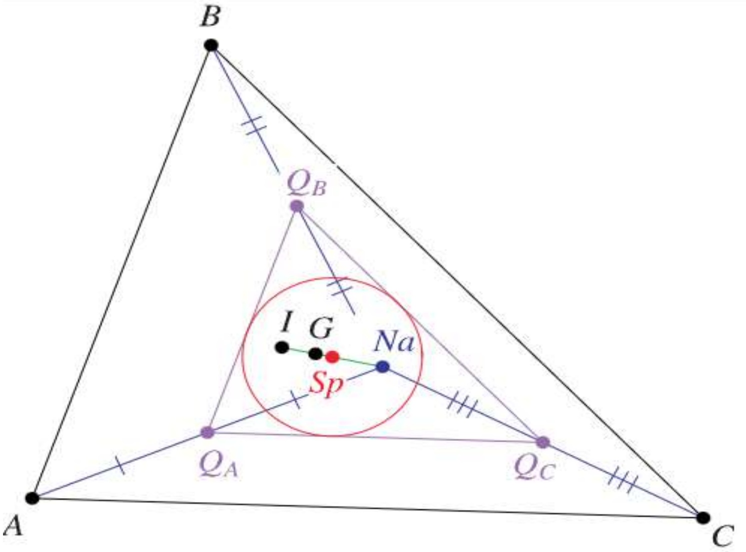
$$\mathbf{\sin\beta}=\frac{\mathbf{a+c}}{\mathbf{b}}\frac{\mathbf{r}}{\mathbf{2AS_p}}\ \mathbf{(2)}$$

$$S_{\triangle BAA_1}+S_{\triangle CAA_1}=S=p\,r,\ S_{\triangle BAA_1}=\frac{1}{2}cp_a\sin\alpha=\frac{1}{2}\frac{r(a+b)}{2AS_p}p_a\text{ and}$$

$$S_{\triangle CAA_1}=\frac{1}{2}bp_a\sin\beta=\frac{1}{2}\frac{r(a+c)}{2AS_p}p_a,\ \frac{1}{2}\frac{r}{2AS_p}p_a(a+b+a+c)=p\,r\rightarrow$$

$$\mathbf{p_a}=\frac{\mathbf{4p}}{\mathbf{2p+a}}\mathbf{AS_p}\ \textbf{(and analogs) (3)}$$

We will use this theorem: Points I, G,S_P,N_a are colinear, line that passes through these points is called Nagel line. I (incenter), G (triangle centroid), S_P (Spieker center), N_a (Nagel point).[1]



From this we obtain:

$$\mathbf{l_a\le m_a\le p_a\le n_a}\ \textbf{(and analogs) (4)}$$

From (3) and (4) →

$$\mathbf{l_a\le m_a\le}\frac{\mathbf{4p}}{\mathbf{2p+a}}\mathbf{AS_p\le n_a}\ \textbf{(and analogs) (5)}$$

We obtain

$$\mathbf{AI\le AG\le AS_p\le AN_a}\ \mathbf{(6)}$$

AI=$\frac{r}{\sin\frac{A}{2}}$=4R$\sin\frac{B}{2}\sin\frac{C}{2}$=$\sqrt{bc-4Rr}$(and analogs), AG=$\frac{2}{3}m_a$(and analogs)

$AN_a=\sqrt{4r^2+(b-c)^2}=\frac{an_a}{p}$(and analogs). From (3), (6) and

$$AN_a=\frac{an_a}{p} \to p_a \le \frac{4a}{2p+a}n_a (7)$$

From (3), (6) and AG=$\frac{2}{3}m_a$ $\to$

$$\frac{8p}{3(2p+a)}m_a \le p_a (8)$$

From (7) after some banal manipulations and summation we obtain

$$\frac{p}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{2} \le \frac{n_a}{p_a}+\frac{n_b}{p_b} (9)$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}\right) \le 4\left(\frac{n_a}{p_a}+\frac{n_b}{p_b}\right)-2 (10)$$

From (8) after some banal manipulation and summation we obtain

$$\frac{8p}{3}\left[\frac{4p+a+b}{(2p+a)(2p+b)}\right] \le \frac{p_a}{m_a}+\frac{p_b}{m_b} (11)$$

Now we will use some proprieties of Nagel line and well-known results:

1)$N_aS_P = S_PI \to AS_P$-median in $\triangle AIN_a$

2)$2N_aI=3N_aG=4S_PI=6GI=12GS_P$

3)$9GI^2=p^2+5r^2-16Rr \to N_aI^2=p^2+5r^2-16Rr$ [1]

$4AS_P{}^2=2(AI^2+AN_a{}^2)-N_aI^2=2(AI^2+AN_a{}^2)-p^2-5r^2+16Rr$(and analogs) (12)

and an equivalent form

$$4AS_P{}^2=2(b^2+c^2-bc)+8Rr+3r^2-p^2 \text{(and analogs) } (13)$$

$$4AS_P{}^2=b^2+c^2+(b-c)^2+8Rr+3r^2-p^2 \text{(and analogs) } (14)$$

From (3), (13), (14),3) we obtain:

$$p_a=\frac{2p}{2p+a}\sqrt{2(b^2+c^2-bc)+8Rr+3r^2-p^2} \text{(and analogs) } (15)$$

$$p_a=\frac{2p}{2p+a}\sqrt{2(AI^2+AN_a{}^2)-p^2-5r^2+16Rr} \text{(and analogs) } (16)$$

$$p_a=\frac{2p}{2p+a}\sqrt{b^2+c^2+(b-c)^2+8Rr+3r^2-p^2} \text{ (and analogs) } (17)$$

$$p_a=\frac{2p}{2p+a}\sqrt{2(AI^2+AN_a{}^2)-9GI^2} \text{(and analogs) } (18)$$

$4AS_P{}^2=2(a^2+b^2+c^2-bc-a^2)+8Rr+3r^2-p^2=2(a^2+b^2+c^2)-2(a^2+bc)+8Rr+3r^2-p^2$. We use $a^2+b^2+c^2=2p^2-8Rr-2r^2$

$$4AS_P{}^2=3p^2-8Rr-r^2-2(a^2+bc) \text{ (and analogs) } (19)$$

$$p_a=\frac{2p}{2p+a}\sqrt{3p^2-8Rr-r^2-2(a^2+bc)} \text{ (and analogs) } (20)$$

From (13) and $4m_a^2=2(b^2+c^2)-a^2$ (and analogs) we obtain:

$$4AS_P{}^2=4m_a^2+a^2-2bc+8Rr+3r^2-p^2 \text{(and analogs) } (21)$$

From (3) and (21) we obtain:

$$p_a = \frac{2p}{2p+a}\sqrt{4m_a^2 + a^2 - 2bc + 8Rr + 3r^2 - p^2}\text{(and analogs) (22)}$$

From (21) and $4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c$ (and analogs) [2] we obtain:

$$4AS_P{}^2 = n_a^2 + g_a^2 + 2r_b r_c + a^2 - 2bc + 8Rr + 3r^2 - p^2\text{(and analogs)}$$

We know that: bc=$rr_a + r_b r_b$(and analogs) and using (22) we obtain

$$4AS_P{}^2 = n_a^2 + g_a^2 + a^2 - 2\,rr_a + 8Rr + 3r^2 - p^2\text{(and analogs) (23)}$$

From (23) and (3) we obtain:

$$p_a = \frac{2p}{2p+a}\sqrt{n_a^2 + g_a^2 + a^2 - 2\,rr_a + 8Rr + 3r^2 - p^2}\text{(and analogs) (24)}$$

From (23) and $p^2 = n_a^2 + 2r_a h_a$ (and analogs) we obtain:

$4AS_P{}^2 = n_a^2 + g_a^2 + a^2 - 2\,rr_a + 8Rr + 3r^2 - n_a^2 - 2r_a h_a$

$$4AS_P{}^2 = g_a^2 + a^2 + 8Rr + 3r^2 - 2\,r_a(h_a + r)\text{(and analogs) (25)}$$

From (3) and (25) we obtain:

$$p_a = \frac{2p}{2p+a}\sqrt{g_a^2 + a^2 + 8Rr + 3r^2 - 2\,r_a(h_a + r)}\text{(and analogs) (26)}$$

From (19) and $p^2 = n_a^2 + 2r_a h_a$ (and analogs) [3] we obtain:

$4AS_P{}^2 = n_a^2 + n_b^2 + n_c^2 + 2r_a h_a + 2r_b h_b + 2r_c h_c - 8Rr - r^2 - 2(a^2 + bc)$

(and analogs) (27)

From (3) and (27) we obtain:

$$p_a = \frac{2p}{2p+a}\sqrt{n_a^2 + n_b^2 + n_c^2 + 2r_a h_a + 2r_b h_b + 2r_c h_c - 8Rr - r^2 - 2(a^2 + bc)}$$

(and analogs) (28)

From (19) and $n_a n_b + n_b n_c + n_a n_c \geq p^2$ [4] we obtain:

$$4AS_P{}^2 \leq 3(n_a n_b + n_b n_c + n_a n_c) - 8Rr - r^2 - 2(a^2 + bc)\text{(and analogs) (29)}$$

From (3) and (29) we obtain:

$$p_a \leq \frac{2p}{2p+a}\sqrt{3(n_a n_b + n_b n_c + n_a n_c) - 8Rr - r^2 - 2(a^2 + bc)}\text{(and analogs) (30)}$$

From (19) and $\sum\sqrt{n_a m_a l_a g_a} \geq p^2$ [5] we obtain:

$$4AS_P{}^2 \leq 3\sum\sqrt{n_a m_a l_a g_a} - 8Rr - r^2 - 2(a^2 + bc)\text{ (and analogs) (31)}$$

From (31) and (3) we obtain:

$$p_a \leq \frac{2p}{2p+a}\sqrt{3\sum\sqrt{n_a m_a l_a g_a} - 8Rr - r^2 - 2(a^2 + bc)}\text{ (and analogs) (32)}$$

From $m_a l_a \geq p(p-a)$ (and analogs) (Panaitopol) after summation we obtain:

$m_a l_a + m_b l_b + m_c l_c \geq p^2$ and from (19) we obtain:

$$4AS_P{}^2 \leq 3(m_a l_a + m_b l_b + m_c l_c) - 8Rr - r^2 - 2(a^2 + bc) \text{ (and analogs) (33)}$$

From (33) and (3) we obtain:

$$p_a \leq \frac{2p}{2p+a}\sqrt{3(m_a l_a + m_b l_b + m_c l_c) - 8Rr - r^2 - 2(a^2 + bc)} \text{ (and analogs) (34)}$$

From [6]: p_a, p_b, p_c- are sides of a triangle regardless the shape of triangle ABC.

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https://www.ssmrmh.ro/2020/02/16/about-nagels-and-gergonnes-cevians-iii/

PROPOSED PROBLEMS

PROBLEMS FOR JUNIORS

J.2806 In any triangle ABC with the area F the following inequality holds:

$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2807 Let be $t>0$, then in ΔABC the following inequality holds:

$$\left(\frac{a^4}{r_b^2r_c^2}+t^2\right)\cdot\left(\frac{b^4}{r_c^2r_a^2}+t^2\right)\cdot\left(\frac{c^4}{r_a^2r_b^2}+t^2\right)\geq 12t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2808 If $x,y>0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{r_aa^4}{xr_b+yr_c}+\frac{r_bb^4}{xr_c+yr_a}+\frac{r_cc^4}{xr_a+yr_b}\geq\frac{16}{x+y}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2809 If $m\geq 0, x,y>0$, then in triangle ABC with the area F the following inequality holds:

$$\left(\frac{a^{2m+4}}{(bx+cy)^{2m}}+t^2\right)\cdot\left(\frac{b^{2m+4}}{(cx+ay)^{2m}}+t^2\right)\cdot\left(\frac{c^{2m+4}}{(ax+by)^{2m}}+t^2\right)\geq\frac{36t^4}{(x+y)^{2m}}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2810 If $t,x,y,z>0$ then:

$$(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)\cdot\left(\frac{1}{(x+y)^4}+t^2\right)\cdot\left(\frac{1}{(y+z)^4}+t^2\right)\cdot\left(\frac{1}{(z+x)^4}+t^2\right)\geq\frac{729}{256}t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2811 If $m \geq 0, a, b, c, t > 0$ then:

$$(a^{2m+2}(a+b)^{2m+2}+t^2)((ab)^{2m+2})(b^{4m+4}+t^2) \geq \frac{t^4}{4 \cdot 3^{2m-1}}(a+b)^{4m+4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2812 In triangle ABC with the area F the following inequality holds:

$$(a^4+2)(b^4+2)(c^4+2) \geq 144F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2813 If $x, y > 0$ and $x^2+y^2=1$, then:

$$(x^4+1)(y^4+1) \geq \frac{3}{2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2814 If $m \geq 0$ and $a, b, c, x, y, z > 0$ then:

$$\left(\frac{a^{2m+2}}{x^{2m}}+2\right) \cdot \left(\frac{b^{2m+2}}{y^{2m}}+2\right) \cdot \left(\frac{x^{2m+2}}{z^{2m}}+2\right) \geq \frac{3(a+b+c)^{2m+2}}{(x+y+z)^{2m}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2815 If $x, y, z > 0$ then:

$$\left(\frac{x^2}{(y+z)^2}+2\right) \cdot \left(\frac{y^2}{(z+x)^2}+2\right) \cdot \left(\frac{z^2}{(x+y)^2}+2\right) \geq \frac{27}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2816 In triangle ABC with the area F the following inequality holds:

$$(a^8+2)(b^8+2)(c^8+2) \geq 728F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2817 If $a, b, c, x, y, z > 0$ then:

$$\left(\frac{a^4}{x^2}+2\right)\left(\frac{b^4}{y^2}+2\right)\left(\frac{c^4}{z^2}+2\right) \geq \frac{3(a+b+c)^4}{(x+y+z)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2818 If $t, u, x, y, z > 0$ then:

$$(x^2y^2+t^2)(y^2z^2+t^2)(z^2x^2+t^2) \cdot \left(\frac{1}{(x+y)^4}+u^2\right) \cdot \left(\frac{1}{(y+z)^4}+u^2\right) \cdot \left(\frac{1}{(z+x)^4}+u^2\right) \geq \frac{729}{256}t^4u^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2819 If $a,b,c,t>0$ then:

$$(a^4(a+b)^4+t^2)(a^4b^4+t^2)(b^8+t^2)\geq\frac{1}{12}t^4(a+b)^8$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2820 If $t>0$ then in triangle ABC with the area F the following inequality holds:

$$(a^2b^2+t^2)(b^2c^2+t^2)(c^2a^2+t^2)\geq 36F^2t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2821 If $x,y>0$ then in any ΔABC with the area F the following inequality holds:

$$\frac{a^3}{bx+cy}+\frac{b^3}{cx+ay}+\frac{c^3}{ax+by}\geq\frac{4\sqrt{3}}{x+y}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2822 If $t,x,y,z>0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x^2a^4}{(y+z)^2}+t^2\right)\cdot\left(\frac{y^2b^4}{(z+x)^2}+t^2\right)\cdot\left(\frac{z^2c^4}{(x+y)^2}+t^2\right)\geq 9t^4F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2823 Let M be an interior point in ΔABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA,AB. Prove that:

$$\frac{a^2b}{d_b}+\frac{b^2c}{d_c}+\frac{c^2a}{d_a}\geq 24F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2824 If $x,y,z>0$ then:

$$(x^2y^2+2)(y^2z^2+2)(z^2x^2+2)\left(\frac{1}{(x+y)^4}+\frac{1}{(y+z)^4}+\frac{1}{(z+x)^4}\right)\geq\frac{81}{16}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2825 If M is an interior point in ΔABC with the semiperimeter s and the area F and

$x=MA,y=MB,z=MC$, then:

$$\left(\frac{x^4}{a^4}+s^2\right)\left(\frac{y^4}{b^4}+s^2\right)\left(\frac{z^4}{c^4}+s^2\right)\geq\frac{81}{4}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.2826 In ΔABC the following relationship holds:

$$\sum \sqrt[3]{\cot\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}} \le 9\sqrt[3]{3}\frac{r}{s}$$

Proposed by Marin Chirciu – Romania

J.2827 In ΔABC the following relationship holds:

$$2F\left(5-\frac{r}{R}\right) \le \sum r_a h_a \cot\frac{A}{2} \le 2F\left(\frac{R}{r}+\frac{r}{R}+2\right)$$

Proposed by Marin Chirciu – Romania

J.2828 In ΔABC the following relationship holds:

$$2F\left(4+\frac{r}{R}\right) \le \sum h_b h_c \cot\frac{A}{2} \le 9F$$

Proposed by Marin Chirciu – Romania

J.2829 In ΔABC the following relationship holds:

$$3\sum r_b r_c \tan\frac{A}{2} \le \sum r_b r_c \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2830 In ΔABC the following relationship holds:

$$9\sqrt{3}r^2 \le \sum r_b r_c \tan\frac{A}{2} \le \frac{9Rr\sqrt{3}}{2}$$

Proposed by Marin Chirciu – Romania

J.2831 In ΔABC the following relationship holds:

$$\sum r_a h_a \tan\frac{A}{2} \le \frac{1}{3}\left(\frac{R}{2r}\right)^2 \sum r_a h_a \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2832 In ΔABC the following relationship holds:

$$F\left(\frac{8R}{r}-7\right) \le \sum r_b r_c \cot\frac{A}{2} \le F\left(\frac{2R}{r}-1\right)^2$$

Proposed by Marin Chirciu – Romania

J.2833 In ΔABC the following relationship holds:

$$\sum \frac{s_a}{m_a^2 + h_b h_c} \le \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

J.2834 In ΔABC the following relationship holds:

$$\sum \frac{s_a}{m_a^2 + r_b r_c} \le \frac{1}{2R}$$

Proposed by Marin Chirciu – Romania

J.2835 In ΔABC the following relationship holds:

$$4F\left(1 + \frac{r}{R}\right) \le \sum h_a\,(h_b + h_c)\tan\frac{A}{2} \le 6F$$

Proposed by Marin Chirciu – Romania

J.2836 In ΔABC the following relationship holds:

$$\sum h_a(h_b + h_c)\tan\frac{A}{2} \le \sum r_a(r_b + r_c)\tan\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2837 In ΔABC the following relationship holds:

$$12F\left(2 - \frac{r}{R}\right) \le \sum h_a(h_b + h_c)\cot\frac{A}{2} \le 4F\left(\frac{2R}{r} + \frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

J.2838 In ΔABC the following relationship holds:

$$3\sum r_a(r_b + r_c)\tan\frac{A}{2} \ge \sum r_a(r_b + r_c)\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2839 In ΔABC the following relationship holds:

$$\sum h_a(h_b + h_c)\cot\frac{A}{2} \le \sum r_a(r_b + r_c)\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2840 In ΔABC the following relationship holds:

$$3\sum h_a(h_b + h_c)\tan\frac{A}{2} \le \sum h_a(h_b + h_c)\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2841 In ΔABC the following relationship holds:

$$\frac{2}{R\sqrt{3}} \leq \sum \frac{1}{r_a} \tan \frac{A}{2} \leq \frac{1}{r\sqrt{3}}$$

Proposed by Marin Chirciu – Romania

J.2842 In ΔABC the following relationship holds:

$$\frac{9}{s} \leq \sum \frac{1}{h_a} \cot \frac{A}{2} \leq \frac{1}{s}\left(\frac{4R}{r} + 1\right)$$

Proposed by Marin Chirciu – Romania

J.2843 In ΔABC the following relationship holds:

$$\frac{1}{s}\left(\frac{8r}{r} - 7\right) \leq \sum \frac{1}{r_a} \cot \frac{A}{2} \leq \frac{1}{s}\left(\frac{2R}{r} - 1\right)^2$$

Proposed by Marin Chirciu – Romania

J.2844 In ΔABC the following relationship holds:

$$\sum \frac{1}{h_a} \cot \frac{A}{2} \leq \sum \frac{1}{r_a} \cot \frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2845 Let be $n \geq 2, n \in \mathbb{N}$. Prove that if one of the numbers $4^n - 1$ or $4^n + 1$ is a prime number, the other one is a compose number.

Proposed by Olivia Bercea– Romania

J.2846 Solve in $\mathbb{R}$ the system:

$$\begin{cases} xy = 2x - y + 2 \\ yz = 2y - z + 2 \\ zx = 2z - x + 2 \end{cases}$$

Proposed by Lavinia Trincu –Romania

J.2847 Prove that there it doesn't exists a number of four digits $\overline{abcd}$ such that

$$\overline{abcd} = 5 \cdot a \cdot b \cdot c \cdot d$$

Proposed by Cornelia Neacșu, Laura Zaharia –Romania

J.2848 Let be p a natural number such that p and $16p - 1$ to be prime numbers. Prove that

$16p + 1$ is a compose number. ***Proposed by Gigi Zaharia, Monica Matei –Romania***

J.2849 a. Prove that there it exists an infinity of natural numbers n such that $n + 2024$ divides $n!$.

b. Prove that there it exists an infinity of natural numbers n such that $n+2024$ does not divides $n!$
$n!=1\cdot 2\cdot \ldots \cdot n, 0!=1.$

Proposed by Mihaela Dăianu, Simona Chiriță –Romania

J.2850 In any ΔABC with the area F the following inequality holds:

$$\frac{a^4}{r_b r_c}+\frac{b^4}{r_c r_a}+\frac{c^4}{r_a r_b}\geq \frac{16\sqrt{3}}{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

J.2851 If $a,b,c,d,x,y>0$, then:

$$\frac{a^2+d^2}{bx+cy}+\frac{b^2+d^2}{cx+ay}+\frac{c^2+d^2}{ax+by}\geq \frac{6d}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

J.2852 If $a,b,c,x,y>0$, then:

$$\frac{ax+by}{c^2}+\frac{bx+cy}{a^2}+\frac{cx+ay}{b^2}\geq \frac{9(x+y)}{a+b+c}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

J.2853 In any ΔABC with the area F the following inequality holds:

$$r_a^4+r_b^4+r_c^4\geq 9F^2+\frac{1}{2}\left(\left(r_a^2-r_b^2\right)^2+\left(r_b^2-r_c^2\right)^2+(r_c^2-r_a^2)^2\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

J.2854 Let M be an interior point in ΔABC, then:

$$\left(\frac{AM^2}{h_a^2}+2\right)\cdot\left(\frac{BM^2}{h_b^2}+2\right)\cdot\left(\frac{CM^2}{h_c^2}+2\right)\geq 12$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2855 In any ΔABC with the area F the following inequality holds:

$$a^4+b^4+c^4\geq 16\cdot F^2+\frac{1}{2}\cdot((a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2856 If $m\geq 0$ and g_a,g_b,g_c are Gergone's cevians of ΔABC with the area F, then:

$$\frac{g_a}{a^m}+\frac{g_b}{b^m}+\frac{g_c}{c^m}\geq \frac{2F\cdot\left(\sqrt{3}\right)^{1-m}}{R^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

J.2857 If $m \geq 0$ and $a, b, c > 0$ then:

$$(a^{m+2}+b^{m+2}+c^{m+2})\left(\frac{1}{a^{m+1}}+\frac{1}{b^{m+1}}+\frac{1}{c^{m+1}}\right) \geq 3(a+b+c)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

J.2858 In any ΔABC with the area F the following inequality holds:

$$\left(m_a^2 \cdot \cos^2\frac{A}{2}+2\right)\cdot\left(m_b^2 \cdot \cos^2\frac{B}{2}+2\right)\cdot\left(m_c^2 \cdot \cos^2\frac{C}{2}+2\right) \geq \frac{81\sqrt{3}}{4}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

J.2859 If $ABC, A_1B_1C_1$ are triangles with the area F respectively F_1 and M and interior point in ΔABC and d_a, d_b, d_c are the distances of point M to the sides BC, CA, AB then:

$$\frac{a^3 \cdot b_1^2}{d_a}+\frac{b^3 \cdot c_1^2}{d_b}+\frac{c^3 \cdot a_1^2}{d_c} \geq 32\sqrt{3}\cdot F \cdot F_1$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

J.2860 If $m \geq 0$, then in any ΔABC with the area F the following inequality holds:

$$\frac{a^{2m+2}}{r_b^m \cdot r_c^m}+\frac{b^{2m+2}}{r_c^m \cdot r_a^m}+\frac{c^{2m+2}}{r_a^m \cdot r_b^m} \geq \frac{4^{m+1}\sqrt{3}}{3^m}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2861. In any ΔABC with the semiperimeter s the following inequality holds:

$$\left(\frac{1}{r_a^2 \cdot r_b^2}+2\right)\cdot\left(\frac{1}{r_b^2 \cdot r_c^2}+2\right)\cdot\left(\frac{1}{r_c^2 \cdot r_a^2}+2\right) \geq \frac{243}{s^4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2862 If $x, y, x > 0$, then in ΔABC with the area F the following inequality holds:

$$\frac{a^x \cdot b^y}{h_c^z}+\frac{b^x \cdot c^y}{h_a^z}+\frac{c^x \cdot a^y}{h_b^z} \geq 2^{x+y}\cdot\left(\sqrt[4]{3}\right)^{4-x-y-z}\cdot F^{\frac{x+y-z}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2863 In ΔABC let be I_a, I_b, I_c excenters and $m \geq 0$ then:

$$I_aI_b^{2m+2}+I_bI_c^{2m+2}+I_cI_a^{2m+2} \geq 2^{4m+4}\cdot 3^{m+2}\cdot r^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2864 If $m \geq 0, x, y > 0$ and $ABC, A_1B_1C_1$ are triangles with the areas F respectively F_1, then:

$$x^2\cdot(a^{2m+2}+b^{2m+2}+c^{2m+2})+y^2(a_1^{2m+2}+b_1^{2m+2}+c_1^{2m+2}) \geq$$

$$\geq 2^{2m+3}\cdot\left(\sqrt{3}\right)^{1-m}\left(\sqrt{F\cdot F_1}\right)^{m+1}+\sum_{cyc}\left(xa^{m+1}-ya_a^{m+1}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2865 Triangle $T_0=A_0B_0C_0=ABC$ with the area F is associated with the triangle

$T_k=A_kB_kC_k$ with the sides $a_k=\sqrt[2^k]{a}, b_k=\sqrt[2^k]{b}, c_k=\sqrt[2^k]{c}$ and the area F_k. Prove that:

$$a_ka_{k+1}+b_kb_{k+1}+c_kc_{k+1}\geq\sqrt{8}\cdot\sqrt[8]{243}\cdot\sqrt[4]{F_k^3}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2866 In any ΔABC with the area F the following inequality holds:

$$\left(\frac{1}{r_a^2}+2\right)\cdot\left(\frac{1}{r_b^2}+2\right)\cdot\left(\frac{1}{r_c^2}+2\right)\geq\frac{9\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

J.2867 Triangle ABC with the area F is associated with triangles MNP, UVW with the sides $m=\sqrt{a}, n=\sqrt{b}, p=\sqrt{c}, u=\sqrt[4]{a}, v=\sqrt[4]{b}, w=\sqrt[4]{c}$ with the areas F_1 respectively F_2. Prove that:

$$mu+nv+pw\geq 2^{\frac{3}{4}}\cdot 3^{\frac{21}{32}}\cdot F^{\frac{3}{8}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2868 If $a,b,c>0$ and $m\geq 0$, then:

$$\left(\frac{a}{b}\right)^{m+2}+\left(\frac{b}{c}\right)^{m+1}+\left(\frac{c}{a}\right)^{m+1}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2869 In any ΔABC with the area F the following inequality holds:

$$m_a^2+m_b^2+m_c^2\geq 3\sqrt{3}\cdot F+\frac{3}{8}\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

J.2870 In ΔABC the following inequality holds:

$$3\sum\frac{1}{h_a}\tan\frac{A}{2}\geq\sum\frac{1}{h_a}\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2871 In ΔABC the following inequality holds:

$$\frac{3r}{R^2} \le \sum \frac{h_b + h_c}{b^2 + c^2} \le \frac{1}{2}\left(\frac{1}{R} + \frac{1}{r}\right) \le \frac{3}{4r}$$

Proposed by Marin Chirciu – Romania

J.2872 In ΔABC the following inequality holds:

$$3\sum \frac{1}{r_A} \tan\frac{A}{2} \le \sum \frac{1}{r_a} \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2873 In ΔABC the following inequality holds:

$$\frac{72r^4}{R} \le \sum h_a \cdot IA^2 \le \frac{9R^4}{4r}$$

Proposed by Marin Chirciu – Romania

J.2874 In ΔABC the following inequality holds:

$$\sum h_a \cdot IA^2 \ge \sum r_a \cdot IA^2$$

Proposed by Marin Chirciu – Romania

J.2875 In ΔABC the following inequality holds:

$$\sum \frac{h_b + h_c}{bc} \le \sum \frac{r_b + r_c}{bc}$$

Proposed by Marin Chirciu – Romania

J.2876 In ΔABC the following inequality holds:

$$4r^3\left(4 + \frac{r}{R}\right)^2 \le \sum h_a w_a r_a \le \frac{3Rr(4R + r)^2}{2(2R - r)}$$

Proposed by Marin Chirciu – Romania

J.2877 In ΔABC the following inequality holds:

$$4\left(\frac{2R}{r} - 1\right) \le \sum \csc^2\frac{A}{2} \le 4\left(\frac{R^2}{r^2} - \frac{R}{r} + 1\right)$$

Proposed by Marin Chirciu – Romania

J.2878 In ΔABC the following inequality holds:

$$3\sum \sec^2\frac{A}{2} \le \sum \csc^2\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2879 In ΔABC the following inequality holds:

$$\frac{3}{32R}\left(28-\frac{27r}{R}+\frac{6r^2}{R^2}\right)\le\sum\frac{m_a}{a^2}\le\frac{3R}{8r^2}$$

Proposed by Marin Chirciu – Romania

J.2880 In ΔABC the following inequality holds:

$$3\sqrt{6r}\le\sum\sqrt{r_b+r_c}\ \le 3\sqrt{3}R$$

Proposed by Marin Chirciu – Romania

J.2881 Let M be an arbitrary point in ΔABC's plane. Find:

$$\Omega=\min\left(\frac{a\cdot MA^2+b\cdot MB^2+c\cdot MC^2}{abc}\right)$$

Proposed by Marin Chirciu – Romania

J.2882 In ΔABC the following inequality holds:

$$\left(2-\frac{r}{R}\right)^2\le\sum\frac{m_a w_a}{a^2}\le\frac{9}{4}\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu – Romania

J.2883 In ΔABC the following inequality holds:

$$\sum h_a+\frac{\lambda r}{R}(R-2r)\le\sum r_a,$$

where $\lambda\le 5$.

Proposed by Marin Chirciu – Romania

J.2884 In ΔABC the following inequality holds: $\lambda\sum\csc A-\sum\cot A\le\sqrt{3}(2\lambda-1)\left(\frac{R}{2r}\right)^2$,

where $\lambda\ge 1$.

Proposed by Marin Chirciu – Romania

J.2885 In ΔABC the following inequality holds:

$$x\sum\csc A-y\sum\cot A\le\sqrt{3}(2x-y)\left(\frac{R}{2r}\right)^2$$

where $x\ge y\ge 0$.

Proposed by Marin Chirciu – Romania

J.2886 If $x, y > 0$ then prove that:

$$\left(\frac{1}{2}+\frac{x}{2y}\right)^2\left(\frac{1}{2}+\frac{y}{2x}\right)^2 \geq \frac{1}{16}\left(1+\frac{2x+y}{\sqrt[3]{x^2y}}\right)\left(1+\frac{2y+x}{\sqrt[3]{xy^2}}\right) \geq 1$$

Proposed by Neculai Stanciu – Romania

J.2887 Prove that:

$$\frac{\sum_{k=1}^{n} a_k}{n^2} \cdot \sum_{cyc}\left(\frac{a_2+a_3}{a_1^2}+\frac{a_1+a_3}{a_2^2}+\frac{a_1+a_2}{a_3^2}\right) \geq 6, \forall a_k > 0$$

Proposed by Neculai Stanciu – Romania

J.2888 If $x, y, z, > 0, x + y + z = 1$, then prove that:

$$\left(\sum x^3+\sum x^2-\sum xy\right)\cdot\frac{1}{xyz} \geq 3$$

Proposed by Neculai Stanciu – Romania

J.2889 Prove that in all triangles ABC with usual notations holds:

$$\frac{9}{\sum m_a} \leq \sum\frac{2}{m_a+m_b} < \frac{10}{\sum m_a}$$

Proposed by Neculai Stanciu – Romania

J.2890 If $a, b, c > 0$ then prove that:

$$\prod(a^2+ab+b^2) \leq \left(\frac{1}{2}(a-b)(b-c)(c-a)\right)^2+\sum(3ab)^2\left(\frac{a+b}{2}\right)^2$$

Proposed by Neculai Stanciu – Romania

J.2891 If $a_k > 0$ $(k = 1,2, \dots, n)$ and $p \in \mathbb{N}$, then prove that:

$$\sum_{cyc}\left(\frac{a_1^{2p+2}}{a_2}+\frac{a_2^{2p+2}}{a_1}\right) \geq \sum_{cyc}\left(a_1a_2^{2p}+a_1^{2p}a_2\right)$$

Proposed by Neculai Stanciu – Romania

J.2892 Prove that in all triangles ABC holds:

$$\sum\frac{a^2}{4s(s-a)} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.2893 Which of the following is true and which is false?

(i) If ABC is a triangle, then $\sum a(\sum a^2 + \sum ab) \geq 3\sum a^2(b+c)$;

(ii) If ABC is a triangle, then $3s^2 \leq 5(r^2 + 4Rr)$.

Proposed by Neculai Stanciu – Romania

J.2894 If $a_i > 0$ $(i = 1,2, \dots, n), k \in \{1,2, \dots, n\}$ such that $\sum_{cyc} a_1 a_2 \dots a_k = 1$, then prove that:

$$\sum_{cyc} \frac{(n+1)a_1 a_2 \dots a_k}{\left((1+a_1^k)(1+a_2^k)\dots(1+a_k^k)\right)^{\frac{1}{k}} n} \leq 1$$

Proposed by Neculai Stanciu – Romania

J.2895 If $a_i > 0$ $(i = 1,2, \dots, n), k \in \{1,2, \dots, n\}$ such that $\sum_{cyc} a_1 a_2 \dots a_k = 1$, then prove that:

$$\sum_{cyc} \frac{(n+1)a_1 a_2 \dots a_k}{\left((1+a_1^k)(1+a_2^k)\dots(1+a_k^k)\right)^{\frac{1}{k}} n} \leq 1$$

Proposed by Neculai Stanciu – Romania

J.2896 Prove that in all triangles ABC with usual notations holds:

$$\frac{2}{3}\sum m_a \leq \sqrt{\frac{R(s^2 + r^2 + Rr)}{2r}}$$

Proposed by Neculai Stanciu – Romania

J.2897 If $a, b, c > 0$ then prove that:

$$\sum \frac{4a}{3(2a+b+c)} \leq 1 \Leftrightarrow \sum \frac{2a}{3(b+c)} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.2898 If $x, y, z > 0, n \in \mathbb{N}$ and $(xy)^{n-1} + (yz)^{n-1} + (zx)^{n-1} = 1$, then prove that:

$$\frac{x^{2n}}{yz} + \frac{y^{2n}}{zx} + \frac{z^{2n}}{xy} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.2899 If $x, y, z > 0$, then prove that:

$$\sum \frac{1}{2x+y+z} + \frac{16xyz}{(\sum x)(\prod(2x+y+z))} \leq \frac{5}{2\sum x}$$

Proposed by Neculai Stanciu – Romania

J.2900 Prove or disprove the following statement: In all triangles ABC holds: $5s^2 < 3r^2 + 2Rr$

Proposed by Neculai Stanciu – Romania

J.2901 If I is the incenter of ΔABC and R_a, R_b, R_c are circumradii of triangles BIC, CIA respectively AIB, then:

$$\left(\frac{R_a^4}{r_a^2(\sin B+\sin C)^2}+2\right)\cdot\left(\frac{R_b^4}{r_b^2(\sin C+\sin A)^2}+2\right)\cdot\left(\frac{R_c^4}{r_c^2(\sin A+\sin B)^2}+2\right)\geq\frac{16}{27}\cdot s^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania

J.2902 In any ΔABC the following inequality holds:

$$\left(\frac{a^4}{(h_b+h_c)^2}+2\right)\cdot\left(\frac{b^4}{(h_c+h_a)^2}+2\right)\cdot\left(\frac{c^4}{(h_a+h_b)^2}+2\right)\geq 27\cdot R^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

J.2903 In any ΔABC with the area F the following inequality holds:

$$(ab+2)\cdot(bc+2)\cdot(ca+2)\geq 36\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania

J.2904 If $x, y, z > 0$ and $xyz \geq 1$, then in any ΔABC with the area F the following inequality holds:

$$(a^2+x^2+y^2)\cdot(b^2+y^2+z^2)\cdot(c^2+z^2+x^2)\geq 36\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.2905 If $t > 0$, then in any triangle ABC with the area F the following inequality holds:

$$(a^2b^2+t^2)\cdot(b^2c^2+t^2)\cdot(c^2a^2+t^2)\geq 36\cdot t^4\cdot F^2$$

Proposed by *D.M. Bătinețu – Giurgiu, Claudia Nănuți*– ***Romania***

J.2906 If $u, b \geq 0, u+v, x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{ux}{y+z}a^2+\frac{vy}{z+x}b^2\right)^2+\left(\frac{uy}{z+x}b^2+\frac{vz}{x+y}\cdot c^2\right)^2+\left(\frac{uz}{x+y}c^2+\frac{vx}{y+x}a^2\right)^2\geq 4(u+v)^2\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.2907 If $x, y, z > 0$ and $x^3+y^3+z^3=6$, then:

$$\frac{x+2}{2x^2+1}+\frac{y+2}{2y^2+1}+\frac{z+2}{2z^2+1}\geq\frac{9}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2908 If $t, u, x, y, z > 0$ and $x+y+z=1$ then:

$$\frac{x}{t+uyz}+\frac{y}{t+uzx}+\frac{z}{t+uxy}\geq\frac{9}{9t+u}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2909 If $a,b,c,x,y,z>0$ then:

$$\left(\frac{(ax+by)^2}{z^2}+c^2\right)\cdot\left(\frac{(by+cz)^2}{x^2}+a^2\right)\cdot\left(\frac{(cz+ax)^2}{y^2}+b^2\right)\geq\frac{243}{8}\cdot a^2b^2c^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2910 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x^3a^3}{(y+z)^2h_a}+F\right)\cdot\left(\frac{y^2b^2}{(z+x)^2h_b}+F\right)\cdot\left(\frac{z^2c^3}{(x+y)^2h_c}+F\right)\geq\frac{9}{2}F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2911 If $m,n,t,x,y,z>0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{a^4}{(my+nz)^2}+t^2\right)\cdot\left(\frac{b^4}{(mz+nz)^2}+t^2\right)\cdot\left(\frac{c^4}{(mx+ny)^2}+t^2\right)\geq\frac{324\cdot t^4\cdot F^2}{(m+n)^2\cdot(x+y+z)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2912 In any ΔABC with the area F the following inequality holds:

$$\frac{ab^2}{c}+\frac{bc^2}{a}+\frac{ca^2}{b}\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2913 If $m,n,x,y,z>0$ and $xyz\geq 1$ then in any ΔABC with the area F the following inequality holds:

$$(a^2+x^m+y^n)\cdot(b^2+y^m+z^n)\cdot(c^2+z^m+x^n)\geq 36\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2914 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$(x^2\cdot a^4+y^2+z^2)\cdot(y^2\cdot b^4+z^2+x^2)\cdot(z^2\cdot c^4+x^2+y^2)\geq 144\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2915 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$((x+y)^2a^4+2z^2)\cdot((y+z)^2b^4+2x^2)\cdot((z+x)^2c^4+2y^2)\geq 576\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2916 If $u,v>0$, then in ΔABC with the area F the following inequality holds:

$$u^4(a^8+v^2)\cdot(b^8+v^2)\cdot(c^8+v^2)+v^4(a^4b^4+u^2)\cdot(b^4c^4+u^2)\cdot(c^4a^4+u^2)\geq 384\cdot u^4v^4\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2917 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$(x^2\cdot a^8+2(y+z)^2)\cdot(y^2\cdot b^8+2(z+x)^2)\cdot(z^2\cdot c^8+2(x+y)^2)\geq 1536x^2y^2z^2\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2918 If $t>0$, then in any ΔABC with the area F the following inequality holds:

$$(a^8+t^2)\cdot(b^8+t^2)\cdot(c^8+t^2)\geq 192\cdot t^4\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2919 Let be M an interior point in ΔABC and $x=MA, y=MB, z=MC$, and t,u,v the distances of point M to the sides BC, CA, respectively AB. Let $m,n,p>0$ then:

$$\left(\left(m\cdot\frac{x}{u+v}+n\cdot\frac{y}{\sqrt{vt}}+p\cdot\frac{z^2}{vu}\right)^2+2\right)\cdot\left(\left(m\cdot\frac{y}{v+t}+n\cdot\frac{z}{\sqrt{tu}}+p\cdot\frac{x^2}{tv}\right)^2+2\right)\cdot$$

$$\cdot\left(\left(m\cdot\frac{z}{t+u}+n\cdot\frac{x}{\sqrt{uv}}+p\cdot\frac{y^2}{ut}\right)^2+2\right)\geq 27\cdot(m+2n+4p)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2920 Let be $x,y\geq 0, x+y>0$ and $A_1B_1C_1, A_2B_2C_2$ triangles of areas F_1 respectively F_2,then:

$$(xa_1a_2+yb_1b_2)^{m+1}+(xb_1b_2+yc_1c_2)^{m+1}+(xc_1c_2+yb_1b_2)^{m+1}\geq$$

$$\geq 4^{m+1}\left(\sqrt{3}\right)^{1-m}\cdot(x+y)^{m+1}\cdot\left(\sqrt{F_1F_2}\right)^{m+1}, \forall m\in\mathbb{R}_+=[0,\infty)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2921 If $t,v,x,y,z>0$ and $x+y+z=s$, then:

$$\frac{x}{ts+uyz}+\frac{y}{ts+uzx}+\frac{z}{ts+uxy}\geq\frac{9}{9t+us}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

J.2922 If $x,y,z>0$ and $x+y+z=s$ then:

$$\left(\frac{1}{sx+yz}+2\right)\cdot\left(\frac{1}{sy+zx}+2\right)\cdot\left(\frac{1}{sz+xy}+2\right)\geq\frac{243}{4s^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

J.2923 In any triangle ABC the following inequality holds:

$$\left(\tan^2\frac{A}{2}\cdot r_b^{2n}+2\right)\cdot\left(\tan^2\frac{B}{2}\cdot r_c^{2n}+2\right)\cdot\left(\tan^2\frac{C}{2}\cdot r_a^{2n}+2\right)\geq\frac{r^2\cdot s^{2(n-1)}}{3^{n-5}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

J.2924 If $a,b,c,x,y,z>0$, then:

$$\left(\frac{a^2}{(bx+cy)^4}+2\right)\cdot\left(\frac{b^2}{(cx+ay)^4}+2\right)\cdot\left(\frac{c^2}{(ax+by)^4}+2\right)\geq\frac{243}{(x+y)^4\cdot(a+b+c)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.2925 If $u,x,y,z>0$ and $u+x+y+z=1$, then:

$$\frac{u}{1+uyz}+\frac{x}{1+uyz}+\frac{y}{1+uzx}+\frac{z}{1+uxy}\geq\frac{64}{65}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.2926 If $x,y,z>0$ then in any ΔABC with the area F the following relationship holds:

$$\left(\frac{x^2\cdot a^6}{(y+z)^2\cdot h_a^2}+F\right)\cdot\left(\frac{y^2\cdot b^6}{(z+x)^2\cdot h_b^2}+F\right)\cdot\left(\frac{z^2\cdot c^6}{(x+y)^2\cdot h_c^2}+F\right)\geq 12\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.2927 If $x,y>0$ then in ΔABC the following inequality holds:

$$\left(\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\right)\cdot\left(\frac{r_a^2}{xr_b+yr_c}+\frac{r_b^2}{xr_c+yr_a}+\frac{r_c^2}{xr_a+yr_b}\right)\geq\frac{3\sqrt{3}\cdot s^2}{2(x+y)(2R-r)}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2928 If $t\geq 0,x,y,z,u,v,w>0$ and $x+u=y+v=z+w=1$ then in any ΔABC with the area F and the semiperimeter s the following inequality holds:

$$\left(a^2+2^{t+1}\cdot s\cdot(x^{t+1}+u^{t+1})\right)\cdot\left(b^2+2^{t+1}\cdot s\cdot(y^{t+1}+v^{t+1})\right)\cdot\left(c^2+2^{t+1}\cdot s\cdot(z^{t+1}+w^{t+1})\right)\geq 324F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2929 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x}{y+z}\cdot a+\frac{y}{z+x}\cdot b+\frac{z}{x+y}\cdot c\right)\cdot\left(\frac{y+z}{x}\cdot a^3+\frac{z+x}{y}\cdot b^3+\frac{x+y}{z}\cdot c^3\right)\geq 48F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2930 Let $ABC, A_1B_1C_1$ triangles of area F respectively F_1, then:

$$a_1^2\cdot a^4+b_1^2\cdot b^4+c_1^2\cdot c^4\geq\frac{64\sqrt{3}}{3}\cdot F_1\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2931 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$(xa^2+yb^2+zc^2)\cdot\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq 36\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2932 If $x, y, z > 0$, then: $(x^2+y^2+z^2)\cdot\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2933 Let be $x, y, z > 0$ and ABC a triangle with the area F, then:

$$\frac{x^2\cdot a^3}{h_a}+\frac{y^2\cdot b^3}{h_b}+\frac{z^2\cdot c^3}{h_c}\geq\frac{8}{3}(xy+yz+zx)\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2934 If $a, b, c, A_1B_1C_1$ are two triangles with the area F respectively F_1 and $h_{a_1}, h_{b_1}, h_{c_1}$ are the heights of $A_1B_1C_1$ then:

$$\frac{1}{h_{a_1}\cdot h_b}+\frac{1}{h_{b_1}\cdot h_c}+\frac{1}{h_{c_1}\cdot h_a}\geq\frac{\sqrt{3}}{\sqrt{F\cdot F_1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2935 Let be $x, y, z > 0, M$ an interior point in ΔABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA respectively AB, then:

$$\frac{x^3\cdot a^4}{d_a^2}+\frac{y^3\cdot b^4}{d_b^2}+\frac{z^3\cdot c^4}{d_c^2}\geq 16(xy+yz+zx)^{\frac{3}{2}}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2936 In any ΔABC with the area F the following inequality holds:

$$((a+b)^2+2)\cdot(c^2+2)\geq 24\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2937 In any ΔABC with the area F the following inequality holds:

$$\left((m_a^2+m_b^2)+2\right)\cdot\left((m_b^2+m_c^2)^2+2\right)\cdot((m_c^2+m_a^2)^2+2)\geq 324\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2938 If $a, b, c, x, y > 0$ then:

$$\left(\frac{a^2}{(xb+yc)^2}+2\right)\cdot\left(\frac{b^2}{(xc+ya)^2}+2\right)\cdot\left(\frac{c^2}{(xa+yb)^2}+2\right)\geq\frac{27}{(x+y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2939 If $a, b, c, m, n > 0$, then:

$$\left(\left(\frac{a}{b+c}+\frac{b}{mc+na}\right)^2+2\right)\cdot\left(\left(\frac{b}{c+a}+\frac{c}{ma+nb}\right)^2+2\right)\cdot\left(\left(\frac{c}{a+b}+\frac{a}{mb+nc}\right)^2+2\right)\geq\frac{27}{4}\cdot\left(\frac{m+n+2}{m+n}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2940 If $a, b, c > 0$ and $a + b + c = 1$ then:

$$\left(\frac{a^2}{(1+bc)^2}+2\right)\cdot\left(\frac{b^2}{(1+ca)^2}+2\right)\cdot\left(\frac{c^2}{(1+ab)^2}+2\right)\geq\frac{243}{100}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2941 In ΔABC the following relationship holds:

$$\frac{4r(4R+r)^2}{R+r}\leq\sum r_a(b+c)\cot\frac{A}{2}\leq\frac{2R(4R+r)^2}{2R-r}$$

Proposed by Marin Chirciu – Romania

J.2942 In ΔABC the following relationship holds:

$$\frac{4r(4R+r)^2}{R+r}\leq\sum h_a(b+c)\cot\frac{A}{2}\leq\frac{R^3(4R+r)^2}{2r^2(2R-r)}$$

Proposed by Marin Chirciu – Romania

J.2943 In ΔABC the following relationship holds:

$\frac{s}{r}-\frac{\lambda}{s}\sum(r_a+h_a)\geq(3-2\lambda)\sum\cot A$, where $\lambda\geq\frac{2}{3}$.

Proposed by Marin Chirciu – Romania

J.2944 In ΔABC the following relationship holds:

$r_a^n+r_b^n+r_c^n\geq 3^{1+\frac{n}{4}}F^{\frac{n}{2}}$, where $n\in\mathbb{N}^*$

Proposed by Marin Chirciu – Romania

J.2945 In ΔABC the following relationship holds:

$$\sum a\cos\left(A-\frac{\pi}{6}\right)\leq 0$$

Proposed by Marin Chirciu – Romania

J.2946 In ΔABC the following relationship holds:

$$\frac{8}{F}(2R-r)^2\leq\sum\frac{\csc^2\frac{A}{2}}{\sin A}\leq\frac{8}{F}\cdot\frac{R^4-R^3r+r^4}{r^2}$$

Proposed by Marin Chirciu – Romania

J.2947 In ΔABC the following relationship holds:

$$\sum\frac{\csc^2\frac{A}{2}}{\sin A}\geq 3\sum\frac{\sec^2\frac{A}{2}}{\sin A}$$

Proposed by Marin Chirciu – Romania

J.2948 In ΔABC the following relationship holds: $\frac{3R}{s} \le \sum \frac{\tan^2 \frac{A}{2}}{\sin A} \le \frac{3R^3}{4r^2 s}$

Proposed by Marin Chirciu – Romania

J.2949 In ΔABC the following relationship holds:

$$\sum h_a(b+c)\cot\frac{A}{2} \ge \sum r_a(b+c)\cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.2950 In ΔABC the following relationship holds:

$$\frac{2s}{r} \le \sum \frac{\cot^2 \frac{A}{2}}{\sin A} \le \frac{2s(R-r)^2}{r^3}$$

Proposed by Marin Chirciu – Romania

J.2951 If $t, x, y, z > 0$, then:

$$\left(\frac{x^2}{(y+z)^2} + t^2\right) \cdot \left(\frac{y^2}{(z+x)^2} + t^2\right) \cdot \left(\frac{z^2}{(x+y)^2} + t^2\right) \ge \frac{27}{16} \cdot t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2952 Let be $x, y, z, u > 0, v \in \mathbb{R}$ then in any ΔABC with the area F the following inequality holds:

$$\left(\left(\frac{ux}{y+z} \cdot a^4 \cdot \sin^2 v + \frac{uy}{z+x} \cdot b^4 \cdot \cos^2 v\right)^2 + 2\right) \cdot \left(\left(\frac{uy}{z+x} \cdot b^4 \cdot \sin^2 v + \frac{uz}{x+y} \cdot c^4 \cdot \cos^2 v\right)^2 + 2\right) \cdot$$

$$\cdot \left(\left(\frac{uz}{x+y} \cdot c^4 \cdot \sin^2 v + \frac{ux}{y+z} \cdot a^4 \cdot \cos^2 v\right)^2 + 2\right) \ge 192 \cdot u^2 \cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2953 Let M be an interior point in $\Delta ABC, x = MA, y = MB, z = MC$ and u, v, w the distances from point M to the sides BC, CA, AB, then:

$$\left(\left(\frac{x^2}{vw} + \frac{y}{\sqrt{wu}}\right)^2 + 2\right) \cdot \left(\left(\frac{y^2}{wu} + \frac{z}{\sqrt{uv}}\right)^2 + 2\right) \cdot \left(\left(\frac{z^2}{uv} + \frac{x}{\sqrt{vw}}\right)^2 + 2\right) \ge 972$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2954 In any ΔABC with the area F, the following inequality holds:

$$\left(\frac{1}{h_a^2} + 2\right) \cdot \left(\frac{1}{h_b^2} + 2\right) \cdot \left(\frac{1}{h_c^2} + 2\right) \ge \frac{9\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2955 If $t, x, y, z > 0$ and ABC is a triangle with the area F then:

$$\left(\frac{x^4\cdot a^{12}}{(y+z)^4}+\frac{x^2\cdot a^4}{(y+z)^2}\cdot(1+a^4)t^2+t^4\right)\cdot\left(\frac{y^4\cdot b^{12}}{(z+x)^4}+\frac{y^2\cdot b^4}{(z+x)^2}\cdot(1+b^4)t^2\right)\cdot$$
$$\cdot\left(\frac{z^4\cdot c^{12}}{(x+y)^4}+\frac{x^2\cdot c^4}{(x+y)^2}\cdot(1+c^4)t^2+t^4\right)\geq 432\cdot t^8\cdot F^6$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2956 If $x,y,z>0$, then in any ΔABC with the area F the following inequality holds:

$$(x^2a^4+y^2b^4+z^2c^4)\cdot\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{3}{4}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2957 If $t\geq 0$ then in ΔABC with the area F the following inequality holds:

$$\left(\frac{n_a+g_a}{\mu(A)}\right)^{t+1}+\left(\frac{n_b+g_b}{\mu(B)}\right)^{t+1}+\left(\frac{n_c+g_c}{\mu(C)}\right)^{t+1}\geq\frac{2^{2t+2}\cdot\left(\sqrt{3}\right)^{t+3}}{\pi^{t+1}\cdot R^{t+1}}\cdot F^{t+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.2958 If $t\in\mathbb{R}$, then in any ΔABC with area F the following inequality holds:

$$((a^2\cdot\cos^2 t+b^2\cdot\sin^2 t)^2+2)\cdot((b^2\cdot\cos^2 t+c^2\cdot\sin^2 t)^2+2)\cdot\cdot((c^2\cdot\cos^2 t+a^2\cdot\sin^2 t)^2+2)\geq 144\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2959 If $x,y,z>0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\left(\frac{x}{y+z}\cdot a^2+\frac{y}{z+x}\cdot b^2\right)^2+2\right)\cdot\left(\left(\frac{y}{z+x}\cdot b^2+\frac{z}{x+y}\cdot c^2\right)^2+2\right)\cdot\left(\left(\frac{z}{x+y}\cdot c^2+\frac{x}{y+z}\cdot a^2\right)^2+2\right)$$
$$\geq 144\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2960 If $t>0$, then in any ΔABC with the area F the following inequality holds:

$$(a^4r_a^2+t^2)\cdot\left(b^4r_b^2+t^2\right)\cdot(c^4r_c^2+t^2)\geq 36\cdot\sqrt{3}t^4\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.2961 In ΔABC the following relationship holds:

$$m_a^n m_b+m_b^n m_c+m_c^n m_a\geq 3(3r)^{n+1}, n\in\mathbb{N}, n\geq 2$$

Proposed by Marin Chirciu – Romania

J.2962 In ΔABC the following relationship holds:

$$r_a^n r_b+r_b^n r_c+r_c^n r_a\geq 81r^3\left(\frac{4R+r}{3}\right)^{n-2} n\in\mathbb{N}, n\geq 2.$$

Proposed by Marin Chirciu – Romania

J.2963 Let $x, y \in \mathbb{R}$. Find the maximum and minimum value of:

$$A = \sin^2 x + \sin^2 y + \sin^2(x+y)$$

Proposed by Nguyen Hung Cuong – Vietnam

J.2964 If $a+b+c=3$ then: $a^a + b^b + c^c \geq 3$

Proposed by Nguyen Hung Cuong – Vietnam

J.2965 In any ΔABC the following relationship holds:

$$\frac{r_a^5 + r_b^5}{r_a^3 r_b^3 (r_a^2 + r_b^2)} + \frac{r_b^5 + r_c^5}{r_b^3 r_c^3 (r_b^2 + r_c^2)} + \frac{r_c^5 + r_a^5}{r_c^3 r_a^3 (r_c^2 + r_a^2)} \geq \frac{16r}{9R^4}$$

Proposed by Zaza Mzhavanadze – Georgia

J.2966 Let $a, b, c \in \mathbb{R}$. Prove that:

$$3^{|a|} + 3^{|b|} + 3^{|c|} \geq 3 + \sqrt{a^2 + b^2 + c^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

J.2967 In any ΔABC the following relationship holds:

$$\frac{m_a^5 + m_b^5}{m_a^4 m_b^4 (m_a + m_b)} + \frac{m_b^5 + m_c^5}{m_b^4 m_c^4 (m_b + m_c)} + \frac{m_c^5 + m_a^5}{m_c^4 m_a^4 (m_c + m_a)} \geq \frac{16}{27R^4}$$

Proposed by Zaza Mzhavanadze – Georgia

J.2968 In ΔABC holds:

$$F\left(4 - \frac{2r}{R}\right)^2 \leq \sum h_a^2 \cot\frac{A}{2} \leq 9F\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu – Romania

J.2969 In ΔABC holds: $9F \leq \sum r_a^2 \cot\frac{A}{2} \leq 9F\left(\frac{R}{2r}\right)$

Proposed by Marin Chirciu – Romania

J.2970 In ΔABC the following relationship holds:

$$\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} + \frac{R^2}{4r^2} \geq 1 + \frac{r_a}{m_a} + \frac{r_b}{m_b} + \frac{r_c}{m_c}$$

Proposed by Nguyen Van Canh – Vietnam

J.2971 In ΔABC holds:

$$4r\left(4 + \frac{r}{R}\right)(2R - r)^2 \leq \sum r_b r_c (r_b + r_c) \leq R(4R + r)^2$$

Proposed by Marin Chirciu – Romania

J.2972 If $a_i \in (0,\infty), i \in \overline{1,n}$ then:

$$1+\frac{2}{n}\log\left(\prod_{i=1}^{n} a_i\right) \le \frac{1}{n}\sum_{i=1}^{n} a_i^2$$

Proposed by Khaled Abd Imouti-Syria

J.2973 If $x,y,z > 0$ then:

$$\sum_{cyc} x^8z^4 \cdot \sum_{cyc}\frac{1}{(xy^2+yz^2)^4} \ge \frac{9}{16}$$

Proposed by Khaled Abd Imouti-Syria

J.2974 If $a,b > 0$ then:

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{2}{a^2+b^2} \ge 3$$

Proposed by Nguyen Hung Cuong – Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.2804 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{3R}{r}\left(\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\right)^2} \le \frac{4(R-r)}{r}$$

Proposed by Bogdan Fuștei – Romania

S.2805 In any acute triangle ABC holds:

$$\frac{a}{h_a}\sqrt{\sin A}+\frac{b}{h_b}\sqrt{\sin B}+\frac{c}{h_c}\sqrt{\sin C}>\frac{4\sqrt{3}}{3}$$

Proposed by Vasile Mircea Popa – Romania

S.2806 In ΔABC the following relationship holds:

$$\frac{n_a n_b n_c}{r_a r_b r_c}\geq\frac{\sqrt{3}}{R}(\max(a,b,c)-\min(a,b,c))$$

Proposed by Bogdan Fuștei – Romania

S.2807 In any acute triangle ABC holds:

$$\frac{a}{b+c}\sqrt{\sin A}+\frac{b}{c+a}\sqrt{\sin B}+\frac{c}{a+b}\sqrt{\sin C}>1$$

Proposed by Vasile Mircea Popa, Mihai Neghină – Romania

S.2808 In ΔABC the following relationship holds:

$$\sin^8 A\cdot\cos A+\sin^8 B\cdot\cos B+\sin^8 C\cdot\cos C\leq\frac{243}{512}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.2809 If $a=\min(a,b,c)\,,I-$ incenter in acute ΔABC then:

$$\frac{1}{r}\sum_{cyc}AI\geq\sqrt{2\left(\frac{n_b}{h_c}+\frac{n_c}{h_b}\right)}+\sqrt{\frac{2(n_b+h_b)}{r_b}}+\sqrt{\frac{2(n_c+h_c)}{r_c}}$$

Proposed by Bogdan Fuștei – Romania

S.2810 If $x,y,z\in\mathbb{R}^+,\sum_{cyc}\left(\frac{y}{x}+\frac{y^2}{xz}\right)=6$ then: $\sum_{cyc}\left(\frac{x}{y}+\frac{x^2}{yz}\right)\geq\sum_{cyc}\left(\frac{x}{yz}+\frac{y}{xz}\right)$

Proposed by Kerimov Elsen-Azerbaijan

S.2811 1. Compare: e^{2023} and π^{2020}. 2. Find all values of k,m such that:

$$m\leq\sqrt{x^4-x^2+2022}+2023|x^2-x|\leq k,\forall x\in[-1,1]$$

Proposed by Nguyen Van Canh-Vietnam

S.2812 Let $\lambda\geq 0$ fixed. Solve in $\mathbb{R}$:

$$\frac{(x-\lambda)^4}{(x^2-2\lambda-1)^2}+(x^2-2\lambda-1)^4+\frac{1}{(x-\lambda)^2}=3x^2-2\lambda x+\lambda^2-4\lambda-2$$

Proposed by Marin Chirciu – Romania

S.2813 If $a, b, c > 0$ and $\lambda \geq \frac{1}{9}$ then:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\lambda\frac{(a+b+c)^3}{abc} \geq 27\lambda+1$$

Proposed by Marin Chirciu – Romania

S.2814 In ΔABC the following relationship holds:

$$\prod \frac{\cos A}{\tan^2 \frac{A}{2}} \leq \left(\frac{3}{2}\right)^3$$

Proposed by Marin Chirciu – Romania

S.2815 Let $a, b, c \geq 0$: $\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{ab+bc+ca} > 0$. Find Min value of P:

$$P=\frac{a+b+c}{\sqrt{abc}+\sqrt{ab+bc+ca}}+\sqrt{ab+bc+ca}$$

Proposed by Phan Ngoc Chau-Vietnam

S.2816 If $a, b, c > 0, a+b+c=1$ and $n, k > 0$ then:

$$\sum \frac{a^2+ab+b^2}{na+kb} \geq \frac{3}{n+k}$$

Proposed by Marin Chirciu – Romania

S.2815 If $x, y, z \in \mathbb{R}_+^*$, in any acute triangle ABC holds:

$$\frac{1}{x+y\sin A+z\cos B}+\frac{1}{x+y\sin B+z\cos C}+\frac{1}{x+y\sin C+z\cos A} \geq \frac{9R}{(3x+z)R+yp+zr}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2816 Determine all pairs (p, q) of prime numbers such that

$$q|p^2+p+1, \frac{q-1}{p-1}=\frac{p^2+p+1}{q}$$

Proposed by Neculai Stanciu – Romania

S.2817 If $m, n \in \mathbb{R}_+^*$, then in any triangle ABC holds:

$$\left(m+n\cdot\cot\frac{A}{2}\cdot\cot\frac{B}{2}\right)^2+\left(m+n\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}\right)^2+$$

$$+\left(m+n\cdot\cot\frac{C}{2}\cdot\cot\frac{A}{2}\right)^2 \geq \frac{(3m+n)^2r^2+8m(3m+n)Rr+16n^2R^2}{3r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2818 If $a,b,c>0$, then prove that:

$$\sum\frac{ac(b+c)}{(a+b+c)\sqrt{(a^2+b^2)(a^2+c^2)}}\le 1$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2819 If $x,y\in\mathbb{R}_+$, then in any triangle ABC holds:

$$\frac{\sin^2 A}{x\cos^2\frac{A}{2}+y\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+\frac{\sin^2 B}{x\cos^2\frac{B}{2}+y\sin^2\frac{C}{2}\sin^2\frac{A}{2}}+$$

$$+\frac{\sin^2 C}{x\cos^2\frac{C}{2}+y\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\ge\frac{16s^2}{8(4x-y)R^2+8Rrx+y(s^2+r^2)}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2820 If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\tan^3\frac{A}{2}}{m\cdot\cot\frac{B}{2}+n\cdot\cot\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\cot\frac{C}{2}+n\cdot\cot\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\cot\frac{A}{2}+n\cdot\cot\frac{B}{2}}\ge\frac{(4R+r)r}{(m+n)p^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2821 If $x_k>0$ $(k=1,2,\dots,n)$ then prove that: $\sum_{cyclic}\frac{1}{x_1x_2^2}\left(\frac{x_1^2+x_1x_2+x_2^2}{2x_1+x_2}\right)^3\ge n$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2822 Prove that in any triangle ABC holds:

$$\frac{\sin^2 A}{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}+\frac{\sin^2 B}{\cos^2\frac{C}{2}\cos^2\frac{A}{2}}+\frac{\sin^2 C}{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}\ge\frac{16p^2}{p^2+(4R+r)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2823 If $x,y\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\sin^2 A}{x\sin^2\frac{A}{2}+y\cos^2\frac{B}{2}\cos^2\frac{C}{2}}+\frac{\sin^2 B}{x\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\cos^2\frac{A}{2}}+\frac{\sin^2 C}{x\sin^2\frac{C}{2}+y\cos^2\frac{A}{2}\cos^2\frac{B}{2}}\ge$$

$$\ge\frac{4p^2}{16(x+y)R^2+8(2y-x)Rr+yp^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2824 If $x,y\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\sin^4 A}{x\sin^2\frac{A}{2}\sin^2\frac{B}{2}+y\cos^2\frac{C}{2}\cos^2\frac{A}{2}}+\frac{\sin^4 B}{x\sin^2\frac{B}{2}\sin^2\frac{C}{2}+y\cos^2\frac{A}{2}\cos^2\frac{B}{2}}+$$

$$+\frac{\sin^4 C}{x\sin^2\frac{C}{2}\sin^2\frac{A}{2}+y\cos^2\frac{B}{2}\cos^2\frac{C}{2}}\geq\frac{4(s^2-4Rr-r^2)^2}{R^2\big((x+y)p^2+(x+y)r^2+8(2y-x)R^2+8Rry\big)}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2825 If $x,y\in\mathbb{R}_+^*, m\in\mathbb{R}_+$, then prove that in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{\left(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}\right)^m}+\frac{\tan\frac{B}{2}}{\left(x+y\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}\right)^m}+\frac{\tan\frac{C}{2}}{\left(x+y\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}\right)^m}\geq\frac{(4R+r)^{m+1}}{p(x(4R+r)+3ry)^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2826 In any triangle ABC holds:

$$\frac{\sin^2 A}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}+\frac{\sin^2 B}{\sin^2\frac{C}{2}\sin^2\frac{A}{2}}+\frac{\sin^2 C}{\sin^2\frac{A}{2}\sin^2\frac{B}{2}}\geq\frac{16p^2}{p^2+r^2-8R^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2827 If $x_k>0$ $(k=1,2,\dots,n)$ then prove that

$$\sum_{cyclic}\frac{(x_1^2+x_1x_2+x_2^2)^2}{(2x_1+x_2)(x_1+2x_2)}\geq\sum_{cyclic}x_1x_2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2828 Prove that in any in all nonisosceles triangles ABC with usual notations holds the following identities:

i. $\left(\sum\frac{h_a-h_b}{h_c}\right)\left(\sum\frac{h_c}{h_a-h_b}\right)=\frac{48R^3+16s^2Rr(s^2+r^2+4Rr)-(s^2+r^2+4Rr)^3-48s^2R^2r^2}{16s^2R^2r^2}$

ii. $\left(\sum\frac{r_a-r_b}{r_c}\right)\left(\sum\frac{r_c}{r_a-r_b}\right)=\frac{(4R+r)((4R+r)^2-s^2)}{s^2r}-6$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2829 If $x,y,z\in\mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{\cot^{2m+1}\frac{A}{2}}{\left(x\cot\frac{A}{2}+y\cot\frac{B}{2}+z\tan\frac{B}{2}\tan\frac{C}{2}\right)^m}+\frac{\cot^{2m+1}\frac{B}{2}}{\left(x\cot\frac{B}{2}+y\tan\frac{C}{2}+z\tan\frac{C}{2}\tan\frac{A}{2}\right)}+$$

$$+\frac{\cot^{2m+1}\frac{C}{2}}{\left(x\cot\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}\right)^m}\geq\frac{p^{2m+1}}{\big((3x+y)p+3zr\big)^m r^{m+1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2830 If $m,n\in\mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\geq\frac{(4R+r)^2}{s(m(4R+r)+3mr)}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2831 If $a,b,c,d>0$, then prove that:

$$\left(\frac{a}{b}+1\right)\left(\frac{b}{c}+1\right)\left(\frac{c}{d}+1\right)\left(\frac{d}{a}+1\right)\geq(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq 16$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2832 Solve for real numbers: $x^3-7x+7=0$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2833 If $x,y>0$, then what can you say about the following double inequality

$x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}\geq\big((n-1)x+y\big)\sqrt[n]{xy^{n-1}}\geq nxy$?

Proposed by Neculai Stanciu – Romania

S.2834 In all nonisosceles triangle holds

$$\left(\sum\frac{\sin\frac{A-B}{2}\sin^2\frac{C}{2}}{\cos^2\frac{C}{2}}\right)\left(\sum\frac{\cos^2\frac{C}{2}}{\sin\frac{A-B}{2}\sin^2\frac{C}{2}}\right)=5-\frac{16R}{r}+\left(\frac{s}{r}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2835 In all nonisosceles triangle holds

$$\left(\sum\frac{\sin\frac{C-B}{2}\tan\frac{A}{2}}{\sin\frac{C-B}{2}-\sin\frac{A}{2}}\right)\left(\sum\frac{\cos\frac{C-B}{2}-\sin\frac{A}{2}}{\sin\frac{C-B}{2}\tan\frac{A}{2}}\right)=5-\frac{16R}{r}+\left(\frac{s}{r}\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2836 Prove that in all triangle ABC holds:

$$\sum(a+b)^4+4abc\sum a\geq 4\sum ab(a+b)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2837 In all nonisosceles triangle holds:

$$\left(\sum\frac{\sin(A-B)\sin^2C}{\cos C}\right)\left(\sum\frac{\cos C}{\sin(A-B)\sin^2C}\right)=6-\frac{(s^2-4Rr-r^2)(4s^2r^2-(s^2-r^2-4Rr)+48s^2R^2r^2}{4sr(s^2-(2R+r)^2)}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2838 If $x, y, z \in \mathbb{N}$ such that $x^2 + y^2 + z^2 = 2002$, then prove that $x + y + z \leq 70$.

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2839 If $x, y, z > 0$ and $x^2 + y^2 + z^2 \leq 1$, then prove that:

$$\sum \frac{1}{\sqrt{1+x^2}} \geq \frac{9}{4}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2840 If $m, n \in \mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{\tan^3 \frac{A}{2}}{m \cdot \tan\frac{B}{2} + n \cdot \tan\frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cdot \tan\frac{C}{2} + n \cdot \tan\frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cdot \tan\frac{A}{2} + n \cdot \tan\frac{B}{2}} \geq \frac{((4R+r)^2 - 2s^2)^2}{(m+n)p^4}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2841 Prove that in any triangle ABC holds:

$$\frac{\cot^2 \frac{A}{2}}{2s - \left(\cot\frac{A}{2} - \cot\frac{C}{2}\right)} + \frac{\cot^2 \frac{B}{2}}{2s - \left(\cot\frac{B}{2} - \cot\frac{A}{2}\right)} + \frac{\cot^2 \frac{C}{2}}{2s - \left(\cot\frac{C}{2} - \cot\frac{B}{2}\right)} \geq \frac{s}{6r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2842 If $m, n \in \mathbb{R}_+^*$, then prove that in any triangle ABC holds:

$$\frac{\cot^3 \frac{A}{2}}{m \cdot \tan\frac{B}{2} + n \cdot \tan\frac{C}{2}} + \frac{\cot^3 \frac{B}{2}}{m \cdot \tan\frac{C}{2} + n \cdot \tan\frac{A}{2}} + \frac{\cot^3 \frac{C}{2}}{m \cdot \tan\frac{A}{2} + n \cdot \tan\frac{B}{2}} \geq \frac{s^2}{(m+n)r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2843 If $x, y, z \in \mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\cot^3 \frac{A}{2}}{x + y\tan\frac{B}{2} + z\tan\frac{B}{2}\tan\frac{C}{2}} + \frac{\cot^3 \frac{B}{2}}{x + y\tan\frac{C}{2} + z\tan\frac{C}{2}\tan\frac{A}{2}} +$$

$$+ \frac{\cot^3 \frac{C}{2}}{x + y\tan\frac{A}{2} + z\tan\frac{A}{2}\tan\frac{B}{2}} \geq \frac{s^3}{\big((4R+r)x + sy + 3zr\big)r}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2844 If $x, y, z \in \mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\cot^3 \frac{A}{2}}{x\tan\frac{A}{2} + y\tan\frac{B}{2}} + \frac{\cot^3 \frac{B}{2}}{x\tan\frac{B}{2} + y\tan\frac{C}{2} + z\tan\frac{C}{2}\tan\frac{A}{2}} +$$

$$+\frac{\cot^3\frac{C}{2}}{x\tan\frac{C}{2}+y\tan\frac{A}{2}+z\tan\frac{A}{2}\tan\frac{B}{2}}\geq\frac{s^4}{((4R+r)^2x+(y-2x)s^2+3zsr)r^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2845 If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\tan\frac{A}{2}}{m+n\cdot\cot^2\frac{A}{2}}+\frac{\tan\frac{B}{2}}{m+n\cdot\cot^2\frac{B}{2}}+\frac{\tan\frac{C}{2}}{m+n\cdot\cot^2\frac{C}{2}}\geq\frac{(4R+r)^2r}{\big(ns^2+mr(4R+r)\big)r}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2846 If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\cot\frac{A}{2}}{m+n\cdot\tan^2\frac{A}{2}}+\frac{\cot\frac{B}{2}}{m+n\cdot\tan^2\frac{B}{2}}+\frac{\cot\frac{C}{2}}{m+n\cdot\tan^2\frac{C}{2}}\geq\frac{s^3}{\big(ms^2+n(4Rr+r^2)\big)r}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2847 If $m,n\in\mathbb{R}_+^*$, then in any triangle ABC holds:

$$\frac{\cot\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\cot\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\cot\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\geq\frac{9s}{4mR+(m+3n)r}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

S.2848 Prove that in all triangles ABC holds the following inequalities:

a. $\sum\sqrt{\frac{h_ah_b}{(r-h_a)(r-h_b)}}\geq\frac{9}{2}$ b. $\sum\sqrt{\frac{r_ar_b}{(r-r_a)(r-r_b)}}\geq\frac{9}{2}$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2849 If $a,b,c>0$, then prove that: $4\leq\left(\frac{a+b}{2a}+\frac{2a}{a+b}\right)\left(\frac{a+b}{2b}+\frac{2b}{a+b}\right)\leq\left(\frac{a}{b}+\frac{b}{a}\right)^2$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2850 If $a,b,c>0$, then:

$$\sum\frac{a(2a+3b+3c)}{3(b+c)(2a+b+c)}\geq 1$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2851 If $a,b,c>0$ then:

$$\left(\frac{(2b+5c)^2}{a^2+3bc}+2\right)\cdot\left(\frac{(2c+5a)^2}{b^2+3ca}+2\right)\cdot\left(\frac{(2a^2+5b)^2}{c^2+3ab}+2\right)\geq\frac{1323}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2852 If $x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$(x^2a^4 + 2)(y^2b^4 + 2)(z^2c^4 + 2) \geq 144\sqrt[3]{x^2y^2z^2}F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2853 If $t > 0$, then in ΔABC with the area F the following inequality holds:

$$(a^4b^4 + t^2)(b^4c^4 + t^2)(c^4a^4 + t^2) \geq 192t^4F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2854 If $m \geq 0, t, u, x, y, z > 0$ then:

$$\frac{tx + uy}{z^m} + \frac{ty + yz}{x^m} + \frac{tz + ux}{y^m} \geq (t + u)\left(\sqrt{3}\right)^{m+1}(xy + yz + zx)^{\frac{1-m}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2855 If $x, y, z > 0$ and $xy + yz + zx = 1$ then:

$$\left(\frac{3xy + 1}{(xy + z)^2} + 2\right) \cdot \left(\frac{3yz + 1}{(yz + x)^2} + 2\right) \cdot \left(\frac{3zx + 1}{zx + y} + 2\right) \geq 48$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2856 If $x, y, z > 0$ then:

$$\left(\frac{x^2}{(x + y)^2} + 2\right)\left(\frac{y^2}{(z + x)^2} + 2\right)\left(\frac{z^2}{(x + y)^2} + 2\right) \geq \frac{27}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2857 If $x, y, z > 0$ then in triangle ABC with the area F the following inequality holds:

$$\left(\frac{x^2}{(y + z)^2}a^8 + 2\right) \cdot \left(\frac{y^2}{(z + x)^2}b^8 + 2\right) \cdot \left(\frac{z^2}{(x + y)^2}c^8 + 2\right) \geq \frac{64}{3} \cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2858. Let be I_a, I_b, I_c excenters of ΔABC and ρ_a, ρ_b, ρ_c exradii of the triangles ACI_a, CAI_b, ABI_c, then:

$$\left(\frac{1}{\rho_a^2 w_2^2} + 2\right) \cdot \left(\frac{1}{\rho_b^2 w_b^2} + 2\right) \cdot \left(\frac{1}{\rho_c^2 w_c^2} + 2\right) \geq \frac{12}{R^4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2859 In ΔABC the following inequality holds:

$$\left(\frac{a^2}{w_a^2} + 2\right) \cdot \left(\frac{b^2}{w_b^2} + 2\right) \cdot \left(\frac{c^2}{w_c^2} + 2\right) \geq \frac{4(a^2 + b^2 + c^2)(a + b + c)}{abc} \geq 36$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2860 If $a,b,c,t,u>0$ and $a+b+c=1$ then:

$$\left(\frac{a^2}{1-2a^2+a^4}+t^2\right)\left(\left(a+\frac{1}{a}\right)^4+u^2\right)\left(\frac{b^2}{1-2b^2+b^4}+t^2\right)\left(\left(b+\frac{1}{b}\right)^4+u^2\right)\cdot$$

$$\cdot\left(\frac{c^2}{1-2c^2+c^4}+t^2\right)\left(\left(c+\frac{1}{c}\right)^4+u^2\right)\geq 67500t^4u^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2861 If $x,y,z>0$ and $xy+yz+zx=3$, then:

$$\left(\frac{\left(\sqrt{4-xy}+\sqrt{xy}\right)^2}{\left(2z+\sqrt{xy}\right)^2}+2\right)\cdot\left(\frac{\left(\sqrt{4-yz}\right)^2+\sqrt{yz}}{\left(2x+\sqrt{yz}\right)^2}+2\right)\cdot\left(\frac{\left(\sqrt{4-zx}\right)^2+\sqrt{zx}}{\left(2y+\sqrt{zx}\right)^2}+2\right)\geq 6\left(2+\sqrt{3}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2862 If $a,b,c,t>0$ then:

$$(a^2(a+b)^2+t^2)(a^2b^2+t^2)(b^4+t^2)\geq\frac{3}{4}(a+b)^4t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2863. If $t>0$ then in any ΔABC with the area F the following inequality holds:

$$(a^4b^4+t^2)(b^4c^4+t^2)(c^4a^4+t^2)\geq 192t^4F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2864. If $x,y>0$ then in any ΔABC with the area F the following inequality holds:

$$(a^4+x^2)(b^4+x^2)(c^4+x^2)\geq 36x^2F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2865 In any triangle ABC with the area F the following inequality holds:

$$\left(r_a^2r_b^2+2\right)\left(r_b^2r_c^2+2\right)(r_c^2r_a^2+2)\geq 81F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2866 If $t\geq 0$ and $m,n,x,y,z>0$ then:

$$\frac{x}{(my+nz)^{t+1}}+\frac{y}{(mz+nx)^{t+1}}+\frac{z}{(mx+ny)^{t+1}}\geq\frac{3^{t+1}}{(m+n)^{t+1}(x+y+z)^t}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2867 In triangle ABC with the area F the following inequality holds:

$$(r_a^2+2)(r_b^2+2)(r_c^2+2) \geq 27\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2868 Let M be an interior point in ΔABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA, respectively AB, then:

$$\left(\frac{a^6}{d_a^2}+2\right)\cdot\left(\frac{b^6}{d_b^2}+2\right)\cdot\left(\frac{c^6}{d_c^2}+2\right) \geq 36F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2869 If $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$((x+y)^2a^8+2z^2)((y+z)b^8+2x^2)((z+x)c^8+2y^2) \geq 3072F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2870 Let $t > 0$, then in any ΔABC with the area F the following inequality holds:

$$(a^4+t^2)(b^4+t^2)(c^4+t^2) \geq 36t^4F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2871 If $k(a^2+b^2+c^2)=1$ and $a+b+c=\sqrt{\frac{3k+1}{k(k+1)}}$, then find:

$$\lim_{n\to\infty}\sum_{k=1}^{n}((a^2-bc)^2+(b^2-ca)^2+(c^2-ab)^2)$$

Proposed by Neculai Stanciu – Romania

S.2872 Prove that:

$$\frac{1}{6}\sum_{cyc}\frac{(5a_1+a_2)(5a_1+4a_2)}{8a_1+a_2} \geq \sum_{k=1}^{n} a_k\,, \forall a_k > 0\ \ (k=1,2,\dots,n)$$

Proposed by Neculai Stanciu – Romania

S.2873 Prove the following inequalities:

(i) If $x, y, z > 0$, then $\sum\frac{1}{x+y} \leq \sum\frac{4x}{3y^2+2yx+3z^2} \leq \sum\frac{x^2}{2xyz}$;

(ii) In all triangles ABC with usual notations holds:

$$\frac{5s^2+r^2+4Rr}{8s(s^2+r^2+2Rr)} \leq \sum\frac{a}{3b^2+2bc+3c^2} \leq \frac{s^2-r^2-4Rr}{16sRr}$$

Proposed by Neculai Stanciu – Romania

S.2874 If $a_i > 0$ $(i = 1,2, \dots, n), k \in \{1,2, \dots, n\}$ such that $\sum_{cyc} a_1 a_2 \dots a_k = 1$, then prove that:

$$\sum_{cyc} \frac{(n+1)a_1 a_2 \dots a_k}{\left((1+a_1^k)(1+a_2^k)\dots(1+a_k^k)\right)^{\frac{1}{k}} n} \le 1$$

Proposed by Neculai Stanciu – Romania

S.2875 In ΔABC the following relationship holds:

$$\sum h_a + \frac{5r}{R}(R-2r) \le \sum r_a$$

Proposed by Marin Chirciu – Romania

S.2876 Let be a, b, c real positive numbers and $k \ge 2$. Prove that:

$$\frac{a}{\sqrt{ka+b}} + \frac{b}{\sqrt{kb+c}} + \frac{c}{\sqrt{kc+a}} \le \sqrt{\frac{3(a+b+c)}{k+1}}$$

Proposed by Marin Chirciu – Romania

S.2877 Let $0 < a < b$ fixed. Solve in $\mathbb{R}$:

$$x = \left[\sqrt{x + a^2 + (a-b)^2} - b\right]\left(\sqrt{x+a^2} + a\right)$$

Proposed by Marin Chirciu – Romania

S.2878 In ΔABC the following relationship holds:

$$36r \le \sum (b+c)\cot\frac{A}{2} \le \frac{9R^2}{r}$$

Proposed by Marin Chirciu – Romania

S.2879 In ΔABC the following relationship holds:

$$\frac{1}{4}\left(7\frac{R}{r} - 6\right) \le \sum \frac{\cos A}{\sin^2 A} \le \frac{R}{r}\left(\frac{R}{r} - 1\right)$$

Proposed by Marin Chirciu – Romania

S.2880 Solve in $\mathbb{R}$ the ecuation:

$$\log_a(a^2 - b^2 + 2bx - x^2) = n^{x-b} + n^{b-x},$$

where $a > 1, b > 0, n > 0, n \ne 1$, fixed.

Proposed by Marin Chirciu – Romania

S.2881 Let $a>1$ fixed. Solve in real numbers:

$$x^{\log_a \sqrt{a+1}}=\sqrt{x}+1$$

Proposed by Marin Chirciu – Romania

S.2882 Let be $\mathbb{R}\setminus\left\{\frac{1}{a}\right\}, a>1$ and $f:A\to \Upsilon, f(x)=\frac{ax^2-3ax+a+1}{(ax-1)^2}$. Find $\min\limits_{x\in A} f(x)$

Proposed by Marin Chirciu – Romania

S.2883 In acute ΔABC the following relationship holds:

$$\frac{s}{r}\le\sum\frac{\sin^2 B+\sin^2 C}{\sin 2A}\le\frac{s}{9r}\cdot\frac{3R^2-8Rr-5r^2}{R^2-2Rr-r^2}$$

Proposed by Marin Chirciu – Romania

S.2884 In ΔABC the following relationship holds:

$$6F\le\sum a^2\cos\frac{A}{2}\le\frac{9\sqrt{3}}{2}R^2.$$

Proposed by Marin Chirciu – Romania

S.2885 In ΔABC the following relationship holds:

$$\frac{1}{32R^2r(2R-3r)}\le\sum\frac{\sin^2 A}{a^4+b^4}\le\frac{1}{32R^2r^2}$$

Proposed by Marin Chirciu – Romania

S.2886 In ΔABC the following relationship holds:

$$\sum\frac{h_bh_c}{a^2}\le\sum\frac{r_br_c}{a^2}$$

Proposed by Marin Chirciu – Romania

S.2887 In ΔABC the following relationship holds:

$$\frac{4s}{\sqrt{3}}\le\sum\frac{a}{\cos\frac{A}{2}}\le 3R\sqrt{2+\frac{R}{r}}$$

Proposed by Marin Chirciu – Romania

S.2888 In ΔABC the following relationship holds:

$$\frac{4}{3R}\le\sum\frac{\sec^2\frac{A}{2}}{r_b+r_c}\le\frac{R}{3r^2}$$

Proposed by Marin Chirciu – Romania

S.2889 In ΔABC the following relationship holds:

$$\frac{3}{R} \le \sum \frac{r_b + r_c}{a^2} \le \frac{3R^2}{8r^3}$$

Proposed by Marin Chirciu – Romania

S.2890 In ΔABC the following relationship holds:

$$\frac{3}{R} \le \sum \frac{r_b + r_c}{bc} \le \frac{3}{2r}$$

Proposed by Marin Chirciu – Romania

S.2891 In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{a^2} \le \sum \frac{r_b + r_c}{a^2}$$

Proposed by Marin Chirciu – Romania

S.2892 In ΔABC the following relationship holds:

$$\frac{3}{R} \le \sum \frac{h_b + h_c}{bc} \le \frac{3}{2r}$$

Proposed by Marin Chirciu – Romania

S.2893 In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{bc} \le \sum \frac{r_b + r_c}{bc}$$

Proposed by Marin Chirciu – Romania

S.2894 Let be n_a, n_b, n_c Nagel's cevians of ΔABC with the area F, then:

$$(a^4 + 2x \cdot r_b) \cdot (b^4 + 2y \cdot r_c) \cdot (c^4 + 2z \cdot r_a) \ge 144\sqrt{3} \cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2895 Let be $m \ge 0; t, u > 0$, then in any ΔABC with the semiperimeter s the following inequality holds:

$$\frac{a}{(ts + ubc)^m} + \frac{b}{(ts + uca)^m} + \frac{c}{(ts + uab)^m} \ge \frac{2 \cdot 9^m \cdot s^{1-m}}{(5us + 9t)^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2896 In any ΔABC the following relationship holds:

$$\left(\frac{r_a^2}{b^2c^2} \cdot \cos^4 \frac{A}{2} + 2\right) \cdot \left(\frac{r_b^2}{c^2a^2} \cdot \cos^4 \frac{B}{2} + 2\right) \cdot \left(\frac{r_c^2}{a^2b^2} \cdot \cos^4 \frac{C}{2} + 2\right) \ge \frac{243}{16} \cdot \frac{r^2}{R^4}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2897 In any ΔABC with the area F the following inequality holds:

$$\left(\frac{a^4}{(r_b+r_c)^2}+2\right)\cdot\left(\frac{b^4}{(r_c+r_a)^2}+2\right)\cdot\left(\frac{c^4}{(r_a+r_b)^2}+2\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2898 If s_a, s_b, s_c are the lengths of the symedians of ΔABC with the area F, then:

$$(a^4+2s_b)(b^4+2s_c)(c^4+2s_a)\geq 144\sqrt{3}\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2899 In any ΔABC the following inequality holds:

$$\left(r_a^2\cdot r_b^2+2\right)\left(r_b^2\cdot r_c^6+2\right)\cdot(r_c^2\cdot r_a^6+2)\geq\frac{1}{3}\cdot r^2\cdot(4R+r)^6$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2900 In any ΔABC the following inequality holds:

$$\left(h_a^2\cdot r_b^6+2\right)\cdot(h_c^2\cdot r_a^6+2)\cdot\left(h_a^2\cdot r_b^6+2\right)\geq\frac{r^2(4R+r)^6}{3}\geq 9r^2s^6$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2901 In any ΔABC with the area F the following inequality holds:

$$\left(h_a^2\cdot r_b^4+2\right)\left(h_b^2r_c^4+2\right)(h_c^2r_a^4+2)\geq 27F^2\cdot s^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.2902 In ΔABC the following relationship holds:

$$h_a^3+h_b^3+h_c^3\leq r_a^3+r_b^3+r_c^3$$

Proposed by Marin Chirciu – Romania

S.2903 In ΔABC the following relationship holds:

$$\sum\frac{1}{2\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\leq\frac{3\sqrt{3}}{16}\left(\frac{R}{r}+2\right)$$

Proposed by Marin Chirciu – Romania

S.2904 In ΔABC the following relationship holds:

$$\frac{8}{3R}\leq\sum\frac{\sec^2\frac{A}{2}}{r_a}\leq\frac{4}{3r}$$

Proposed by Marin Chirciu – Romania

S.2905 In ΔABC the following relationship holds:

$$\sum \frac{\sec^2 \frac{A}{2}}{h_a} \geq \sum \frac{\sec^2 \frac{A}{2}}{r_a}$$

Proposed by Marin Chirciu – Romania

S.2906 In ΔABC the following relationship holds:

$$\frac{9}{4} \leq \sum \left(\frac{r_a}{a}\right)^2 \leq \frac{9R}{8r}$$

Proposed by Marin Chirciu – Romania

S.2907 In ΔABC the following relationship holds:

$$\sum \frac{h_a}{r_a} \geq \sum \frac{r_a}{h_a}$$

Proposed by Marin Chirciu – Romania

S.2908 In ΔABC the following relationship holds:

$$\frac{2r}{R} \leq \prod \frac{m_a}{r_a} \leq \frac{R^2}{4r^2}$$

Proposed by Marin Chirciu – Romania

S.2909 If $x, y, z > 0$ and $\lambda \geq \frac{2}{3}$ then:

$$\frac{3\lambda}{xy + yz + zx} - \frac{3}{(x + y + z)^2} \geq \frac{3\lambda - 1}{x^2 + y^2 + z^2}$$

Proposed by Marin Chirciu – Romania

S.2910 Let be x are real positive number and a a real number, such that:

$$x^2 - a(a^2 + 2)\sqrt{x} = a^2 + 1$$

Proposed by Marin Chirciu – Romania

S.2911 In ΔABC the following relationship holds:

$$1 \leq \prod \frac{m_a}{h_a} \leq \frac{R^3}{8r^3}$$

Proposed by Marin Chirciu – Romania

S.2912 In ΔABC the following relationship holds:

$$\frac{12r^2}{R} \leq \frac{a^2}{m_b + m_c} + \frac{b^2}{m_c + m_a} + \frac{c^2}{m_a + m_b} \leq \frac{3R^2}{2r}$$

Proposed by Marin Chirciu – Romania

S.2913 In ΔABC the following relationship holds:

$$(a+b)(b+c)(c+a) \geq 8abc + \lambda r^2(R-2r)$$

where $\lambda \leq 12\sqrt{3}$.

Proposed by Marin Chirciu – Romania

S.2914 In ΔABC the following relationship holds:

$$(a+b)(b+c)(c+a) \geq 8abc + \lambda r^3 \cdot \frac{R-2r}{R+2r}$$

where $\lambda \leq 48\sqrt{3}$.

Proposed by Marin Chirciu – Romania

S.2915 In ΔABC the following relationship holds:

$$\sqrt{\frac{b+c}{s-a}} + \sqrt{\frac{c+a}{s-b}} + \sqrt{\frac{a+b}{s-c}} \geq 6$$

Proposed by Marin Chirciu – Romania

S.2916 In ΔABC the following relationship holds:

$$\sum m_a\sqrt{h_a h_b h_c} \geq \frac{2r}{R}\sum h_a\sqrt{r_a r_b r_c}$$

Proposed by Marin Chirciu – Romania

S.2917 In ΔABC the following relationship holds:

$$\sum a^4 \geq \frac{16}{3}F^2\sum\left(\frac{m_a}{m_b}\right)^2$$

Proposed by Marin Chirciu – Romania

S.2918 If $x, y, z > 0$ then:

$$\frac{2}{xy+yz+zx} - \frac{3}{(x+y+z)^2} \geq \frac{1}{x^2+y^2+z^2}$$

Proposed by Marin Chirciu – Romania

S.2919 If $a, b, c > 0$ such that $ab+bc+ca=1$ and $\lambda \geq \frac{2}{3}$ then:

$$\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} \leq \frac{9}{3\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.2920 In ΔABC the following relationship holds:

$$\sum \frac{a^3}{\lambda + \lambda c - a} \geq \frac{4\sqrt{3}F}{2\lambda - 1}$$

where $\lambda \geq \frac{1}{2}$

Proposed by Marin Chirciu – Romania

S.2921 If $x, y, z > 0$, then prove that:

$$3\prod(x^2 + 3y^2 + z^2 + 3xy + 3yz + zx) \geq 4\left(\sum x\right)^2\left(\sum x^2 + 3\sum xy\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2922 Solve for real numbers the following system:

$$\sqrt{x_1 - 3} + \sqrt{x_2^2 - 4x_2 + 3} - \sqrt{(x_3 - 2)^3} = 0$$

$$\sqrt{x_2 - 3} + \sqrt{x_3^2 - 4x_3 + 3} - \sqrt{(x_4 - 2)^3} - 0$$

$$-\ -$$

$$\sqrt{x_n - 3} + \sqrt{x_1^2 - 4x_1 + 3} - \sqrt{(x_2 - 2)^3} = 0$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2923 If $a, b, c > 0$, then prove that:

$$\sum \frac{(a+b)(a^2+b^2)}{4c} \geq \sum a^2$$

Proposed by Neculai Stanciu – Romania

S.2924 If $a, b, c > 0$, then:

$$\left(\sum a\right)\left(\sum a^2\right)\left(\sum a^3\right) \leq 9\sum a^6$$

Proposed by Neculai Stanciu – Romania

S.2925 If $x_k > 0$ $(k = 1,2, \dots, n)$, then prove that:

$$\sum_{cyc} \frac{(x_1 + x_2 + x_3)^5 - x_1^5 - x_2^5 - x_3^5}{(x_1 + x_2 + x_3)^3 - x_1^3 - x_2^3 - x_3^3} < 10\sum_{k=1}^{n} x_k^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2926 If $a_i > 0$ $(i = 1,2,\dots,n)$, then prove that:

$$\left(\sum \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}}\right)\left(\sum \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}\right)\cdots\left(\sum \frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\right) < \left(\sum_{i=1}^{n} a_i\right)^{k-1}, k \in \{1,2,\dots,n-1\}$$

Proposed by Neculai Stanciu – Romania

S.2927 Solve in $\mathbb{R}_+$ the system:

$$\begin{cases} x^2+xy+y^2 = (2x+y)\sqrt[3]{xz^2} \\ y^2+yz+z^2 = (2y+z)\sqrt[3]{yx^2} \\ z^2+zx+z^2 = (2z+x)\sqrt[3]{zy^2} \end{cases}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2928 If $a,b,c,\lambda > 0$ and $a+b+c=1$, then prove that:

$$\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}} \geq \frac{1}{\sqrt{\lambda \sum ab + 3(1-\lambda)abc}}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2929 If $a_k > 0$ $(k = 1,2,\dots,n), \lambda \geq 2n+1$ and $\sum_{k=1}^{n} a_k = n$, then prove that:

$\sum_{k=1}^{n} \frac{1}{\lambda + a_k^2} \leq \frac{n}{1+\lambda}.$ What happens if λ does not verify the hypothesis?

Proposed by Neculai Stanciu – Romania

S.2930 Prove that in all triangles ABC with usual notations the following inequality is true:

$$2Rr \leq \sum \left(\frac{m_a}{3\cos\frac{A}{2}}\right)^2$$

Proposed by Neculai Stanciu – Romania

S.2931 Prove that:

$$\sum_{cyc} \frac{(x_1+x_2)(x_2+x_3)(x_3+x_4)(x_4+x_1)}{4(x_1x_2x_3+x_2x_3x_1+x_3x_4x_1+x_4x_1x_2)} - \sum_{k=1}^{n} x_k \geq 0, \quad \forall x_k > 0, (k = 1,2,\dots,n)$$

Proposed by Neculai Stanciu – Romania

S.2932 If $a,b,c > 0$ then prove that:

$$\sum \sqrt{a^2+b^2} \geq \sqrt{2}\sum a$$

Proposed by Neculai Stanciu – Romania

S.2933 If ABC is a triangle, then prove that:

$$\sum \frac{2}{\sqrt{3}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)} \leq 1$$

Proposed by Neculai Stanciu – Romania

S.2934 If ABC is a triangle, then prove that:

$$\left(\frac{4R+r}{s}\right)-\frac{s^2}{8r^2} \leq 1$$

Proposed by Neculai Stanciu – Romania

S.2935 If p and $p+2$ are both prime numbers, then prove that: $p^3-p^2-4p-4>0$

Proposed by Neculai Stanciu – Romania

S.2936 If $a,b,c>0$, then prove that:

$$\sum_{cyc}\left(\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{c+a}{b^2}\right) \geq \frac{54}{a+b+c}$$

Proposed by Neculai Stanciu – Romania

S.2937 Prove that 10^n+10^{n+1} can be written as a sum of four perfect squares for any natural number n.

Proposed by Neculai Stanciu – Romania

S.2938 If ABC is a triangle such that $\sum \tan^2\frac{A}{2} \geq 1$, then prove that $\sum \tan\frac{A}{2} \geq \sqrt{3}$.

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2939 Prove or disprove that:

$$\{(x,y)\in\mathbb{Z}\times\mathbb{Z}|2xy+3y+y+2=0\}=\{(x,y)\in\mathbb{Z}\times\mathbb{Z}|2xy+x+3y+8=0\}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.2940 If n_a,n_b,n_c are Nagel's cevians of ΔABC with the area F and semiperimeter s, then:

$$(n_a b^2+n_b c^2+n_c a^2)^2 \geq 32\cdot\frac{s}{R}\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

S.2941 In any ΔABC with the area F the following inequality holds:

$$\frac{a(a^3+b^3)}{a^2+ab+b^2}+\frac{b(b^3+a^3)}{b^2+bc+c^2}+\frac{c(c^3+a^3)}{c^2+ca+a^2} \geq \frac{8\sqrt{3}}{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

S.2942 If $a, b, c > 0$ and $a + b + c = 1$, then:

$$\left(\frac{a^2}{81b^2c^2 + 18bc + 1} + 2\right) \cdot \left(\frac{b^2}{81c^2a^2 + 17ca + 1} + 2\right) \cdot \left(\frac{c^2}{81a^2b^2 + 18ab + 1} + 2\right) \geq \frac{3}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

S.2943 If $a, b, c, x, y > 0$ then:

$$\frac{a}{(bx + cy)^3} + \frac{b}{(cx + ay)^3} + \frac{c}{(ax + by)^3} \geq \frac{9}{(x + y)^2(a + b + c)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2944 If $a, b, c > 0$ and $a + b + c = 1$ then:

$$\left(\frac{a^2}{(b + c)^2} + 2\right) \cdot \left(\frac{b^2}{(c + a)^2} + 2\right) \cdot \left(\frac{c^2}{(a + b)^2} + 2\right) \geq \frac{243}{100}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2945 If $a, b, x, y, z > 0$ then:

$$x \cdot \sqrt{\frac{x}{ay + bz}} + y \cdot \sqrt{\frac{y}{az + bx}} + z \cdot \sqrt{\frac{z}{ax + by}} \geq \frac{2(x + y + z)}{1 + a + b}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2946 If $m, n \geq 0$ and $m + n, t, x, y, z > 0$ then:

$$\left(\frac{x^2}{(my + nz)^2} + t^2\right) \cdot \left(\frac{y^2}{(mz + nz)^2} + t^2\right) \cdot \left(\frac{z^2}{(mx + ny)^2} + 2\right) \geq \frac{27}{4(m + n)^2} \cdot t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2947 If $u, v \geq 0, u + v, x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\left(\frac{ux}{y + z} \cdot a^2 + \frac{vy}{z + x} \cdot b^2\right)^2 + 2\right) \cdot \left(\left(\frac{vy}{z + x} \cdot b^2 + \frac{vz}{x + y} \cdot c^2\right) + 2\right) \cdot \left(\left(\frac{uz}{x + y} \cdot c^2 + \frac{vx}{y + z} \cdot a^2\right)^2 + 2\right) \geq 36(u + v)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2948 If $m, u, v \geq 0, u + v, x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{ux}{y + z} \cdot a^4 + \frac{vy}{z + x} \cdot b^4\right)^{m+1} + \left(\frac{uy}{z + x} \cdot b^4 + \frac{vz}{x + y} \cdot c^4\right)^{m+1} + \left(\frac{uz}{x + y} \cdot c^4 + \frac{vx}{y + z} \cdot a^4\right)^{m+1} \geq \frac{8^{m+1}(u + v)^{m+1}}{3^m} \cdot F^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2949 In any ΔABC with the area F the following inequaity holds:

$$\left(h_a^2 \cdot r_b^{2n} + 2\right) \cdot \left(h_b^2 \cdot r_c^{2n} + 2\right) \cdot \left(h_c^2 r_a^{2n} + 2\right) \geq \frac{r^2 \cdot s^n}{3^{n-5}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2950 In ΔABC the following inequality holds: $\left(\frac{s_a}{a}\right)^2 + \left(\frac{s_b}{b}\right)^2 + \left(\frac{s_c}{c}\right)^2 \leq \frac{9}{32}\left(\frac{R}{r}\right)^3$

Proposed by Marin Chirciu – Romania

S.2951. In any ΔABC with the area F the following inequality holds:

$$\left(r_a^2 \cdot r_b^4 + 2\right) \cdot \left(r_b^2 \cdot r_c^4 + 2\right) \cdot \left(r_c^2 \cdot r_a^4 + 2\right) \geq 108 \cdot \sqrt{3} \cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

S.2952 If $x, y, z > 0$ and $xyz = 1$ then in any ΔABC with the area F the following inequality holds:

$$(a^3 + x \cdot m_b + y) \cdot (b^3 + y \cdot m_c + z) \cdot (c^3 + z \cdot m_a + x) \geq 72 \cdot \sqrt{3} \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

S.2954 If $m \geq 0$ and $x, y, z > 0$ then:

$$(x^2 + xy + z^2)^{m+1} + (y^2 + yz + z^2)^{m+1} + (z^2 + zx + x^2)^{m+1} \geq 3 \cdot (xy + yz + zx)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2955 If $a, b, c, d > 0$ and $a + b + c + d = s$, then:

$$\frac{a}{s + bcd} + \frac{b}{s + cda} + \frac{c}{s + dab} + \frac{d}{s + abc} \geq \frac{64}{s^2 + 64}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2956 If $u, v \geq 0, u + v, x, y, z > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{ux}{y+z} \cdot a^4 + \frac{vy}{z+x} \cdot b^4\right)^2 + \left(\frac{uy}{z+x} \cdot b^4 + \frac{vz}{x+y} \cdot c^4\right)^2 + \left(\frac{uz}{x+y} \cdot c^4 + \frac{vx}{y+z} \cdot a^4\right)^2 \geq \frac{64}{3}(u+v)^2 \cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2957 In any ΔABC with the area F the following inequality holds:

$$(a^4 + 1) \cdot (b^4 + 1) \cdot (c^4 + 1) \geq 36 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.2958 In any ΔABC with the area F the following inequality holds:

$$(a + 2 \cdot h_b) \cdot (b + 2 \cdot m_c) \cdot (c + 2 \cdot w_a) \geq 144 \cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2959 Let s be the semiperimeter of ΔABC, then:

$$\frac{1}{\sqrt{2sa+bc}}+\frac{1}{\sqrt{2sb+ca}}+\frac{1}{\sqrt{2sc+ab}}\geq\frac{9}{4s}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.2960 Let be M an interior point in ΔABC and $x=MA, y=MB, z=MC$ and u,v,w, are the distances from M to the sides BC, CA respectively AB, then:

$$\left(\left(\frac{x^2}{vw}+\frac{y}{w+u}\right)^2+2\right)\cdot\left(\left(\frac{y^2}{wu}+\frac{z}{u+c}\right)^2+2\right)\cdot\left(\left(\frac{z^2}{uv}+\frac{x}{v+w}\right)^2+2\right)\geq 675$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.2961 Let $x,y\in\mathbb{R}$ such that $x^2+y^2=x^3+y^3$. Find the maximum and the minimum value of:

$A=x+y$.

Proposed by Nguyen Hung Cuong – Vietnam

S.2962 If $m,n,p\in\mathbb{N}, p-$ odd, then: $4(p-3)\left(2^{3^n}+4^m\right)+(p-1)p^2\equiv 0(mod\ 3)$

Proposed by Khaled Abd Imouti -Syria

S.2963 a. Find all numbers $a,b\in\mathbb{Z}$ such that $\frac{a-b}{a^2-ab+b^2}\in\mathbb{Z}$

b. Find all numbers $a,b\in\mathbb{Z}$ such that $\frac{a^2-ab+b^2}{a+b}\in\mathbb{Z}$

Proposed by Nguyen Van Canh – Vietnam

S.2964 Find n,m,k natural numbers such that: $n!+2^m=3^k$.

Proposed by Elsen Kerimov-Azerbaijan

S.2965 If $a,b,c\geq 0$ with $ab+bc+ca\neq 0$ then prove that:

$$\sum_{cyc}\frac{a^3}{b^2-bc+c^2}\geq a+b+c+AB\sum_{cyc}(a-b)^2(a+b-c)^2((a-b)^2+ab)$$

$$A=\frac{2(a^3+b^3+c^3)}{2(a^2+b^2+c^2)+(a-b)^2+(b-c)^2+(c-a)^2}$$

$$B=\frac{(a+b)(b+c)(c+a)}{(a^3+b^3)(b^3+c^3)(c^3+a^3)}$$

When does equality holds?

Proposed by Sidi Abdullah Lemrabott-Mauritania

S.2966 If $a,b,c>0, ab+bc+ca=3$ then:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{2}{a+b+c+1}+\frac{3}{a+b+c}$$

Proposed by Marin Chirciu – Romania

S.2967 If $n\in\mathbb{N},n\geq 1$ then:

$$\frac{1}{2(2n+1)^2}+\sum_{k=0}^{n}\frac{1}{(2k+1)^2}<\frac{\pi^2}{8}$$

Proposed by Khaled Abd Imouti-Syria

S.2968 In ΔABC holds:

$$\sum\frac{h_a^2}{\sin^2\frac{A}{2}}\geq\frac{2r}{R}\sum\frac{r_a^2}{\sin^2\frac{A}{2}}$$

Proposed by Marin Chirciu – Romania

S.2969 In ΔABC the following relationship holds:

$$n_a^2+2r_ah_a\geq r_br_c+m_bh_b+m_ch_c$$

Proposed by Bogdan Fuștei – Romania

S.2970 Let $\lambda,n\geq 0$ fixed. If $a,b,c>0,a^2+b^2+c^2=1$ then find min of

$$S=\lambda(a^3+b^3+c^3)-n\,(a+b+c)$$

Proposed by Marin Chirciu – Romania

S.2971 In any ΔABC the following relationship holds:

$$\frac{\left(h_a^2+w_b^2+m_c^2\right)^5}{\left(r_a^5+r_b^5\right)^2}+\frac{\left(h_b^2+w_c^2+m_a^2\right)^5}{\left(r_b^5+r_c^5\right)^2}+\frac{\left(h_c^2+w_a^2+m_b^2\right)^5}{\left(r_c^5+r_a^5\right)^2}\geq\frac{4\cdot 6^6\cdot r^{10}}{(81R^5-2560r^5)^2}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2972 If $a,b,c>0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\leq 3$ then:

$$\frac{1}{\sqrt{7a^2-2ab+4b^2}}+\frac{1}{\sqrt{7b^2-2bc+4c^2}}+\frac{1}{\sqrt{7c^2-2ca+4a^2}}\leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

S.2973 If $a_i,b_i,c_i\geq 0,1\leq i\leq 3$ then:

$$M=A+B-6a_1b_1c_1a_2b_2c_2a_3b_3c_3\geq 0$$

Such that:

$$A=a_1^3(b_2^3c_3^3+c_2^3b_3^3)+b_1^3(c_2^3a_3^3+a_2^3c_3^3)+c_1^3(a_2^3b_3^3+b_2^3a_3^3)$$

$$B=\sum_{cyc}^{\substack{a_1,\ a_2,\ a_3\\ b_1,\ b_2,\ b_3\\ c_1,\ c_2,\ c_3}}\left(\left(\sum_{cyc}^{a_i,b_i,c_i}(a_1b_1)^3\,(c_2^3+c_3^3)-3(a_1b_1c_1)^2(a_2b_2c_2+a_3b_3c_3)\right)\right).$$

And prove that:

$$\left(\sum_{i=1}^{3}a_i^3\right)\left(\sum_{i=1}^{3}b_i^3\right)\left(\sum_{i=1}^{3}c_i^3\right)=\left(\sum_{i=1}^{3}a_ib_ic_i\right)^3+M$$

Proposed by Sidi Abdullah Lemrabott-Mauritania

S.2974 Solve the equation:

$$\sqrt{4-x^2}=3x-x^3$$

Proposed by Ibrahim Abdullayev Masalli – Azerbaijan

S.2975 In ΔABC the following relationship holds:

$$\sum\frac{m_a^3}{\tan\frac{B}{2}\tan\frac{C}{2}}+\frac{3R^2}{4R^2}\geq\sum\frac{a}{b}+9g_ag_bg_c$$

Proposed by Nguyen Van Canh-Vietnam

S.2976 If $a,b,c>0$ and $a^5+b^5+c^5=3$ than prove that:

$$\frac{a\sqrt[3]{a(a^5+a^3b^2+b^5)^5}}{\sqrt[3]{(a^5+a^2b^3+b^5)^2}}+\frac{b\sqrt[3]{b(b^5+b^3c^2+c^5)^5}}{\sqrt[3]{(b^5+b^2c^3+c^5)^2}}+\frac{c\sqrt[3]{c(c^5+c^3a^2+a^5)^5}}{\sqrt[3]{(c^5+c^2a^3+a^5)^2}}\geq$$

$$\geq\sqrt[3]{3(a^2+b^2+c^2)^5}$$

Proposed by Zaza Mzhavanadze – Georgia

S.2977 In the bicentric quadrilateral $ABCD$ with sides a,b,c,d in which $a\leq b\leq c\leq d$, we have the inequality:

$$\sqrt[4]{\frac{a}{r}}+\sqrt[4]{\frac{b}{r}}+\sqrt[4]{\frac{c}{r}}\leq3\sqrt{\frac{R}{r}}$$

Proposed by Emil C. Popa – Romania

S.2978 Let $ABCD$ be a bicentric quadrilateral with sides of lengths a,b,c,d. Then:

$$3R\sqrt{2}\cdot\min\left(\frac{1}{a^{-3}+b^{-3}+c^{-3}},\frac{1}{a^{-3}+b^{-3}+d^{-3}},\frac{1}{a^{-3}+c^{-3}+d^{-3}},\frac{1}{b^{-3}+c^{-3}+d^{-3}}\right)\leq$$

$$\leq F^2\leq\frac{R\sqrt{2}}{3}\cdot\max(a^3+b^3+c^3,a^3+b^3+d^3,a^3+c^3+d^3,b^3+c^3+d^3)$$

Proposed by Emil C. Popa – Romania

S.2979 If $a, b, c > 0$ and $a + b + c = 3$ then:

$$a^{b+c}b^{c+a}c^{a+b} \leq 1$$

Proposed by Nguyen Hung Cuong – Vietnam

S.2980 If $a, b > 0$ then:

$$a^b b^a \leq \left(\frac{a+b}{2}\right)^{a+b}$$

Proposed by Nguyen Hung Cuong – Vietnam

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.2806 Find:

$$\Omega = \int_0^1 \frac{x^2 \ln x}{x^2 - x + 1} dx$$

Proposed by Vasile Mircea Popa – Romania

U.2807 Show that:

$$\int_0^{\frac{\pi}{4}} \tan(x) \ln^2(\sin(2x))\, dx = \frac{\zeta(3)}{4} + \frac{\ln^3(2)}{6} - \frac{\pi^2}{24}\ln(2)$$

Proposed by Vincent Nguyen-USA

U.2808 Prove that:

$$\lim_{s \to 1} \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{\Lambda(\mathrm{n})}{n^s} = 1$$

where $\Lambda(n)$ is Von Mangoldt function.

Proposed by Fao Ler-Iraq

U.2809 Find all functions $f(x)=x^4+ax^3+x^2+ax-2 \ \ (a\in\mathbb{R})$ such that:

$$\min f'(x)+\max f''(x)=\max f'''(x), \forall x\in[1966,2022]$$

Proposed by Nguyen Van Canh – Vietnam

U.2810 Prove that:

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\cos^{\mu-1}2x\tan 2x\,dx=\frac{\beta(\mu)}{4(1-\mu)} \qquad (\mu>0)$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2811 Find:

$$\Omega=\int_0^\infty \frac{\ln(1+x)}{x(x+1)^3}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2812 Show that:

$$\int_0^1\int_0^1\int_0^1 \frac{dx\,dy\,dz}{1-xyz+\sqrt{1-xyz}}=2\zeta(3)+\frac{\ln^3(4)}{6}-\frac{\pi^2}{6}\ln(4)$$

Proposed by Vincent Nguyen-USA

U.2813 If $n\in\mathbb{N}-\{0\}$ then:

$$\sum_{m=1}^{n}\left(\sum_{k=0}^{m}\frac{1}{k!\,(n-k)!}\right)\le 2\left(2^{n^2}-1\right)$$

Proposed by Khaled Abd Imouti-Sirya

U.2814 Prove that:

$$\Omega=\int_0^\infty\int_0^1 \frac{\ln(1+y)\ln^2(1+x^2)}{x(1+y^2)}dx\,dy=\frac{G\zeta(3)}{8}+\frac{\pi\ln(2)}{32}$$

Proposed by Shirvan Tahirov-Azerbaijan

U.2815 Prove the integral

$$\int_{-\infty}^{\infty}\left(\frac{\cos\left(\frac{x}{2}\right)+\cos(2x)}{1+x^2}\right)^2\cos(x)\,dx=\frac{\pi\left(6+9e^{\frac{3}{2}}+4e^2+7e^{\frac{5}{2}}+3e^3+5e^{\frac{7}{2}}+8e^4+3e^{\frac{9}{2}}+e^5\right)}{8e^5}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2816 Prove that:

$$\Delta=\int_1^{\infty}\frac{\ln^2(x)}{(x+1)(x+2)}dx=\frac{\zeta(3)}{4}+\frac{2\ln^3(2)}{3}+\frac{\pi^2\ln(2)}{6}-\frac{Li_3\left(\frac{1}{4}\right)}{2}$$

Proposed by Shirvan Tahirov – Azerbaijan

U.2817 Prove that:

$$2ae^x-(a^2+x^2)e^a\le e^a-e^{-a}, x\in[-a,a]$$

Proposed by Khaled Abd Imouti-Sirya

U.2818 Let $f(x)=\frac{ax^2+bx+c}{dx+e}$ such that $f(1)=\frac{2}{3}, f(2)=\frac{11}{13}, f(-1)=2, f(-2)=-1, f(-3)=-\frac{6}{7}$. Compute: $f(2023)+f(-2023)=?$

Proposed by Nguyen Van Canh-Vietnam

U.2819 If $x,y,z>1$ then find the minimum value of

$$P=\frac{x^3}{(y-1)^2}+\frac{y^3}{(z-1)^2}+\frac{z^3}{(x-1)^2}$$

Proposed by Marin Chirciu – Romania

U.2820 If $x,y,z>1$ and $n\in\mathbb{N}, n\ge 3$ then find the minimum value of

$$P=\frac{x^n}{(y-1)^2}+\frac{y^n}{(z-1)^2}+\frac{z^n}{(x-1)^2}$$

Proposed by Marin Chirciu – Romania

U.2821 If $a,b,c,m,n>0, mna+mb+nc=abc$ then find:

$$\Omega=\max\left(\frac{1}{\sqrt{1+a^2}}+\frac{n}{\sqrt{n^2+b^2}}+\frac{m}{\sqrt{m^2+c^2}}\right)$$

Proposed by Khaled Abd Imouti-Sirya

U.2822 Find $\lim_{n\to\infty}\frac{\sqrt[2n]{n!\cdot L_n^2}}{n^2\sqrt{a_n}}$, where $(a_n)_{n\ge1}, a_1=1, a_{n+1}=(n+1)!\cdot a_n, \forall n\in\mathbb{N}$ and L_k is k-th Lucas number.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

U.2823 If f, f', f'', f''' are continuous functions $[0,\lambda]$ with $f(\lambda)=a, f'(\lambda)=b, f'''(\lambda)=c$ and $\int_0^{\lambda}f(x)\,dx=d$ then compute

$$\int_0^{\lambda}x^3f'''(x)\,dx$$

Proposed by Marin Chirciu – Romania

U.2824 $a,b,c\in\mathbb{N}, 7a^2+3a^2c-b^2c=72,\ \frac{ab(a^2+b^2)+2ac(a^2+c^2)+4bc(b^2+c^2)}{8a+4b+2c}=abc$

$$a,b,c=?$$

Proposed by Samed Ahmedov – Azerbaijan

U.2825 In ΔABC the following relationship holds:

$$\frac{6ar_a^2}{6+\sqrt{3}a(a+b+c)}+\frac{6br_b^2}{6+\sqrt{3}b(a+b+c)}+\frac{6cr_c^2}{6+\sqrt{3}(a+b+c)}\geq\frac{324Sr^2}{R(8+20S)-4Sr}$$

Proposed by Elsen Kerimov-Azerbaijan

U.2826 In ΔABC the following relationship holds:

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{A}{2}}\geq\frac{r_a}{r_b}+\frac{r_b}{r_a}$$

Proposed by Bogdan Fuștei – Romania

U.2827 In ΔABC the following relationship holds:

$$(4R+r_a)(r_b+r_c-h_a)\geq n_a^2+g_a^2+m_bh_b+m_ch_c-r_a(h_a-2r)$$

Proposed by Bogdan Fuștei – Romania

U.2828 In any ΔABC and the following relationship holds:

$$\left(\frac{m_a}{w_b}+\frac{w_c}{h_a}\right)^3+\left(\frac{m_b}{w_c}+\frac{w_a}{h_b}\right)^3+\left(\frac{m_c}{w_a}+\frac{w_b}{h_c}\right)^3\geq\frac{3\cdot 2^9\cdot r^6}{3\cdot(9R^3-64r^3)^2-128r^6}$$

Proposed by Zaza Mzhavanadze – Georgia

U.2829 In any ΔABC and the following relationship holds:

$$\frac{m_aw_b}{w_ch_a}+\frac{w_bh_c}{h_am_b}+\frac{h_cm_a}{m_bw_c}\leq\frac{3}{8}\cdot\left(9\left(\frac{R}{r}\right)^3-64\right)$$

Proposed by Zaza Mzhavanadze – Georgia

U.2830 In ΔABC the following relationship holds: $\frac{2m_aw_a}{h_a}\geq r_b+r_c\geq\frac{m_bh_b+m_ch_c}{r_a}$

Proposed by Bogdan Fuștei – Romania

U.2831 In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}+\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}}\geq\sqrt{\frac{r_a}{r_b}}+\sqrt{\frac{r_b}{r_a}}\geq\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}$$

Proposed by Bogdan Fuștei – Romania

U.2832 In ΔABC show that:

$$\sum \frac{1+\cos(A-B)\cdot\cos C}{h_c\cdot\sec C}=\frac{3}{2R}$$

Proposed by Ertan Yildirim-Turkiye

U.2833 Let $a,b,c,d\in\left[\frac{1}{2};1\right]$ and $a\geq b\geq c\geq d$. Prove that:

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\geq\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}+4$$

Proposed by Minh Vu – Vietnam

U.2834 If $a,b,c>0, a+b+c=1$ and $n,k>0$ then

$$\sum\frac{a^2+ab+b^2}{na+kb}\geq\frac{3}{n+k}$$

Proposed by Marin Chirciu – Romania

U.2835 Let $a,b,c>0$ such that $a+b>c, b+c>a, a+c>b$. Find all values of $k\geq 0$ such that

$$\left(k+\frac{a}{b+c-a}\right)\left(k+\frac{b}{a+c-b}\right)\left(k+\frac{c}{a+b-c}\right)\geq(k+1)^3$$

Proposed by Nguyen Van Canh – Vietnam

U.2836 Compute:

$$K=\int_{1980}^{2022}\frac{17x+2023}{\sqrt{40+\sqrt{20x+11}}}dx$$

Proposed by Nguyen Van Canh –Vietnam

U.2837 $n\in\mathbb{N},\lambda\in\mathbb{R},\lambda-$ fixed. Find $\alpha\in\mathbb{R}$ such that:

$$x^{n-\lambda+1}\cdot\left(x^{\lambda}\cdot\ln x\right)^{(n+1)}-(\lambda-n)\cdot x^{n-\lambda}=a,\forall x>0$$

Proposed by Khaled Abd Imouti -Sirya

U.2838 Find all values of $x,y\in\mathbb{Z}$ such that: $x^6-y^6=32$.

Proposed by Nguyen Van Canh-Vietnam

U.2839 Let $a,b,c>0: ab+bc+ca=a+b+c$. Prove that:

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}+33\geq 4\left(\sqrt{8a+1}+\sqrt{8b+1}+\sqrt{8c+1}\right)$$

Proposed by Phan Ngoc Chau-Vietnam

U.2840 Let $a,b,c>0: abc=1$. Prove that:

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+a)}\ge\frac{a+b+c}{ab+bc+ca-1}$$

Proposed by Phan Ngoc Chau-Vietnam

U.2841 Prove that for any acute triangle ABC the following inequality holds:

$$\frac{\sec A+\sec B+\sec C}{\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}}\ge 2\sqrt{2}$$

Proposed by Vasile Mircea Popa – Romania

U.2842 Prove that:

$$\Psi=\int_0^1\int_0^1\frac{\ln(1+x^2)\ln(xy)}{x(1+y^2)^2}dxdy=\frac{1}{384}(-8\pi^2G-9\pi\zeta(3)-18\zeta(3)-2\pi^3)$$

Note: G – Catalan's constant, $\zeta(3)$ – Apery's constant

Proposed by Shirvan Tahirov– Azerbaijan

U.2843 In ΔABC the following relationship holds:

$$\frac{m_a^4}{w_b^3+2w_bw_c+w_c^3}+\frac{m_b^4}{w_c^3+2w_cw_a+w_a^3}+\frac{m_c^4}{w_a^3+2w_aw_b+w_b^3}\ge\frac{108\cdot r^4}{27R^3+2R^2-192r^3}$$

Proposed by Zaza Mzhavanadze – Georgia

U.2844 Find all functions $f(x)=x^4+ax^3+bx^2+cx+d$ such that:

$$f(20)=f(11)=f(1980)=f(2022)=0$$

Proposed by Nguyen Van Canh-Vietnam

U.2845 If $a,b,c>0, a+b+c=\frac{3}{2}$ then

$$\sum\frac{a+b}{(b+c)^2}\ge\frac{9}{4}+a^2+b^2+c^2$$

Proposed by Marin Chirciu – Romania

U.2846 Prove that in all triangle ABC with usual notations holds:

$$\frac{9}{\sum m_a}\le\sum\frac{2}{m_a+m_b}<\frac{10}{\sum m_a}$$

Proposed by Neculai Stanciu – Romania

U.2847 Let $P(x)=x^2-x+1$ and $Q(x)=x^2+ax+b, (a,b\in\mathbb{R})$. Find all values of a,b such that:

$$P\big(Q(x+2)\big)=Q\big(P(x-2)\big), \forall x\in\mathbb{R}$$

Proposed by Nguyen Van Canh-Vietnam

U.2848 In ΔABC

$$4(R-r) \le \sum \frac{IA^2}{r_a} \le \frac{4(R-r)^2}{r}$$

Proposed by Marin Chirciu – Romania

U.2849 If

$$\Omega := \int\int_{[0,1]^2} \ln\left(\frac{y^2-x^2}{y^2+x^2}\right) \frac{dx\,dy}{x\sqrt{1-x^4}}$$

then show that: $\Im\{\Omega\} = -\frac{\pi}{2}\varpi$, where, $\Im(z)$ is the imaginary part, and ϖ is lemniscate constant

Proposed by Ankush Kumar Parcha-India

U.2850 Prove that:

$$2\pi \int_{-1}^{1} \left(\int_{-\frac{1}{2}x^2+\frac{1}{2}}^{1-|x|} \frac{1-x-y}{\sqrt{2}} dy \right) dx = \frac{2\sqrt{2}}{15}\pi$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2851 Prove that:

$$\sum_{k=1}^{n-1} \frac{H_k^2}{\frac{1}{k} - \log\left(1+\frac{1}{k}\right)} \ge \frac{\left((n-1)\gamma + H_{n-1} + H_{n-2}\right)^2}{H_n - \frac{1}{n} - \log n}, n \in \mathbb{N}^*$$

Proposed by Khaled Abd Imouti-Sirya

U.2852 Prove the below closed form

$$\int\int_{[0,1]^2} \ln^{-1}(xy)^{-1} \beta\left(1-\frac{xy}{2}, 1+\frac{xy}{2}\right) dx\,dy = \frac{4G}{\pi}$$

Where, $\beta(x,y)$ is the Euler integral of the first kind and G is the Catalan's constant

Proposed by Ankush Kumar Parcha-India

U.2853 Prove that:

$$\Omega = \int_0^1 \frac{x\ln^2(x)}{x^2+x+1} dx = \frac{8\zeta(3)}{9} - \frac{4\pi^3}{81\sqrt{3}}$$

Note: $\zeta(3)$ – Apery's constant

Proposed by Shirvan Tahirov-Azerbaijan

U.2854 Prove that:

$$\Delta = \int_0^1 \frac{x\ln(x)}{(x+1)(x^2+1)} dx = \frac{1}{32}(\pi^2 - 16G)$$

Note: G – Catalan's constant

Proposed by Shirvan Tahirov-Azerbaijan

U.2855 If in ΔABC $\alpha:\beta:\gamma = 1:3:6$ then find: $\Omega = \frac{s}{r}$.

Proposed by Samir Cabiyev-Azerbaijan

U.2856 a. Find all values of $x, y \in \mathbb{Z}$ such that $x^2 + xy + y^2 = 2023$

b. Find all value of $m \in \mathbb{Z}$ such that $\frac{m^5+2m+2024}{m^2-m+1} \in \mathbb{Z}$

Proposed by Nguyen Van Canh-Vietnam

U.2857 Prove that in all triangle ABC with usual notations holds:

$$\frac{2}{3}\sum m_a \leq \sqrt{\frac{R(s^2+r^2+Rr)}{2r}}$$

Proposed by Neculai Stanciu – Romania

U.2858 In any ΔABC the following relationship holds:

1. $\frac{h_a w_a}{h_a+w_b+2m_c} + \frac{w_b m_c}{w_b+m_c+2h_a} + \frac{m_c h_a}{m_c+h_a+2w_b} \leq \frac{9R}{8}$

2. $\frac{27r^3}{R} \leq \frac{h_a w_b^2}{h_a+w_b} + \frac{w_b m_c^2}{w_b+m_c} + \frac{m_c h_a^2}{m_c+h_a} \leq \frac{27R^2}{8}$

Proposed by Zaza Mzhavanadze – Georgia

U.2859 Let $a, b, c \geq 0: ab + bc + ca = 3$. Prove that:

$$\sqrt{a+10b+1} + \sqrt{b+10c+1} + \sqrt{c+10a+1} \geq 6\sqrt{3}$$

Proposed by Phan Ngoc Chau-Vietnam

U.2860 Prove that:

$$\int_0^1 \frac{K(-x)-E(-x)}{x\sqrt{x+1}} \ln\left(\frac{1-x}{1+x}\right) dx = \frac{\pi - 4\ln 2}{4\sqrt{\pi}} \Gamma^2\left(\frac{1}{4}\right)$$

where:

$$E(x) = \int_0^1 \frac{\sqrt{1-t^2x}}{\sqrt{1-t^2}} dt, K(x) = \int_0^1 \frac{1}{\sqrt{1-t^2}\sqrt{1-t^2x}} dt$$

Proposed by Fao Ler-Iraq

U.2861 Let $a, b, c \geq 0: a + b + c = 2$ and $ab + bc + ca > 0$. Prove that

$$\sqrt{\frac{ab+1}{ab+c}}+\sqrt{\frac{bc+1}{bc+a}}+\sqrt{\frac{ca+1}{ca+b}}\geq 2+\sqrt{2}$$

Proposed by Phan Ngoc Chau-Vietnam

U.2862 In ΔABC the following relationship holds:

$$\frac{a}{b}\sqrt{\cot\frac{A}{2}}+\frac{b}{c}\sqrt{\cot\frac{B}{2}}+\frac{c}{a}\sqrt{\cot\frac{C}{2}}\leq 3\sqrt[4]{3}\cdot\frac{R}{2r}$$

Proposed by George Apostolopoulos-Greece

U.2863 If $x,y,z>0$, then prove that:

$$\sum\frac{1}{2x+y+z}+\frac{16xyz}{(\sum x)(\prod(2x+y+z))}\leq\frac{5}{2\sum x}$$

Proposed by Neculai Stanciu – Romania

U.2864 In ΔABC

$$\sum\frac{\cot^{2n+1}\frac{A}{2}}{\cot^{2n-1}\frac{B}{2}}\geq 2\left(\frac{4R}{r}-3\right), n\in\mathbb{N}^*$$

Proposed by Marin Chirciu – Romania

U.2865 Find the general term of the sequence: 0,2,4,1,1,0,3, …

Proposed by Khaled Abd Imouti -Sirya

U.2866 Let $a,b,c>0$ and $a+b+c\leq 3$. Find all values of $k\in\mathbb{R}$ such that:

$$\frac{1}{a^k+a+1}+\frac{1}{b^k+b+1}+\frac{1}{c^k+c+1}\geq 1$$

Proposed by Nguyen Van Canh –Vietnam

U.2867 In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{m_a}{h_a+h_b}}+\frac{R^4}{r^2}\geq 16+\sum\sqrt{\frac{h_a}{m_b+m_c}}$$

Proposed by Nguyen Van Canh – Vietnam

U.2868 In ΔABC the following relationship holds:

$$3+\sum_{cyc}\frac{b+c}{a}\leq\frac{1}{r}\sum_{cyc}m_a$$

Proposed by Hasan Mammadov-Azerbaijan

U.2869 In ΔABC the following relationship holds:

$$\sum(b^2+c^2)\cos\frac{A}{2}\geq 12F$$

Proposed by Marin Chirciu – Romania

U.2870 Let $a,b,c>0$ such that $abc=1$. Prove that:

$$\frac{1}{a^3+a^2+a}+\frac{1}{b^3+b^2+b}+\frac{1}{c^3+c^2+c}+\frac{8}{a+b+c}\geq\frac{11}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.2871 In ΔABC prove that:

$$\sum_{cyc}\frac{m_a^{1000}}{b^{500}+c^{500}}\gtreqless\frac{1}{2\cdot 4^{374}\cdot 3^{499}}\left(\sum_{cyc}\sqrt[4]{\left(2b^2-\frac{1}{2}a^2\right)\left(2c^2-\frac{1}{2}a^2\right)}\right)^{500}\cdot\frac{\left(\sum_{cyc}a^2\right)^{249}\cdot\left(\sum_{cyc}m_a\cdot\cos A\right)^2}{\sum_{cyc}a^{500}\cdot\sum_{cyc}\cos^2 A}$$

Proposed by Elsen Kerimov–Azerbaijan

U.2872 Prove the below closed form.

$\int_0^1\frac{\ln(x)}{x}\ln\left(\frac{x^2+x+1}{x^2-x+1}\right)dx=-\frac{14}{9}\zeta(3)$, where $\zeta(3)$ is an Apery's constant.

Proposed by Ankush Kumar Parcha-India

U.2873 Prove the below closed form:

$$\int\int_{[0,1]^2}\left(\frac{x+y}{x-y}\right)\tan^{-1}\left(\frac{x-y}{x+y}\right)dx\,dy=2G-\frac{\pi}{4}\ln 2-\frac{\ln 2}{2}$$

where G is a Catalan's constant

Proposed by Ankush Kumar Parcha-India

U.2874 Let $f(x)=x^3+x^2+x+m,(m\in\mathbb{R})$. Find all values of m such that: $f\left(x^3-f(x)\right)\leq 0,\forall x\in\mathbb{R}$

Proposed by Nguyen Van Canh-Vietnam

U.2875 If $x\in\left(0,\frac{\pi}{4}\right)$ then:

$$2\sin x+\tan x<3x+\ln x\ln(1-x)$$

Proposed by Khaled Abd Imouti -Sirya

U.2876 a. Find all values of $a\in\mathbb{Z}$ such that $P=\frac{a^2-a+2023}{a^3+a^2+2023}+\frac{a^3+a^2+2023}{a^2-a+2023}\in\mathbb{Z}$

b. Find all values of $a,b\in\mathbb{Z}$ such that $2023a^2-ab+2024b^2=0$

Proposed by Nguyen Van Canh – Vietnam

U.2877 If $x, y > 0$ then:

$$\frac{1}{x}+\frac{1}{y}+\frac{2}{x+y}\geq\frac{3}{\sqrt{xy}}$$

Proposed by Nguyen Hung Cuong – Vietnam

U.2878 Prove that:

$$\sum_{0<n}\frac{(-1)^{n-1}(4n-1)}{(2n)^5\beta_n^5}\sum_{m=0}^{n-1}(-1)^m(4m+1)\beta_m^5\sum_{k=1}^{m}\frac{(-1)^{k-1}(4k-1)}{(2k)^3\beta_k^3}\sum_{l=0}^{k-1}(-1)^l(4l+1)\beta_i^3=\frac{\pi^6}{480\Gamma\left(\frac{3}{4}\right)^8}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.2879 If

$$\sum_{n=1}^{\infty}\frac{\pi^{-\frac{n}{2}}\xi(1-n)}{\Gamma\left(\frac{n}{2}\right)}=\frac{\pi^2+2\pi\psi^{(a)}\left(1-\frac{1}{\pi}\right)-\psi^{(b)}\left(1-\frac{1}{\pi}\right)}{2c}\sum_{n=1}^{\infty}\frac{H_n^{(2)}+H_n^2}{6n\times 2^{n-2}}=d\zeta(3)$$

Prove that:

$$\int_0^{(a+b)\sqrt{c}}\int_0^d\frac{x^2\ln\left(\frac{1}{1-x}\right)\ln^2(1+x)}{y+y\sin x}dy\,dx=\frac{3\pi^2}{8}\zeta(4)+\frac{3\pi}{4}\zeta(4)\ln(2)-3G\zeta(4)$$

Note: $\xi(x)$ is Riemann Xi function

Proposed by Abbaszade Yusif –Azerbaijan

U.2880 If $0 < a \leq b$ then:

$$b-a+\ln\left(\frac{a+1}{b+1}\right)\leq\int_a^b\ln\left(\frac{e^x}{x+1}\right)^2dx\leq\frac{1}{3}(b^3-a^3)$$

Proposed by Khaled Abd Imouti-Sirya

U.2881 In any ΔABC and $n\in\mathbb{N}: n\geq 2$; the following relationship holds:

$$\frac{h_a^n}{r_b^2r_c(r_b^3+r_c^3)}+\frac{w_b^n}{r_c^2r_a(r_c^3+r_a^3)}+\frac{m_c^n}{r_a^2r_b(r_a^3+r_b^3)}\geq\frac{32\cdot 3^{n-5}\cdot r^n}{3(9R^3-64r^3)^2-128r^6}$$

Proposed by Zaza Mzhavanadze – Georgia

U.2882 Let $a, b\in\mathbb{R}$. Find the maximum value of:

$$A=\left(\frac{1}{5}\right)^{a^2+b^2}-\left(\frac{1}{2}\right)^{(a+b)^2}$$

Proposed by Nguyen Hung Cuong – Vietnam

U.2883 Let: $n \geq 1$ be positive integer. Prove that:

$$\frac{16^n lcm(1,3,5,\dots,2n+1)}{\binom{2n}{n}(2n+1)}$$

is an integer.

Proposed by Toubal Fethi -Algeria

U.2884 Let be $a, b \in \mathbb{R}$. Find the minimum value of:

$$A = 2^{\sqrt{a^2+b^2}} - |a+b|$$

Proposed by Nguyen Hung Cuong - Vietnam

U.2885 If $a, b, c, \lambda > 0$ and $a+b+c=1$, then prove that:

$$\sum \frac{a}{\sqrt{\lambda(b^2+c^2)+bc}} \geq \frac{1}{\sqrt{\lambda \sum ab + 3(1-\lambda)abc}}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

U.2886 In any ΔABC and $n \in \mathbb{N}: n \geq 2$; the following relationship holds:

$$\frac{h_a^n}{r_b^3 r_c + w_b^3 m_c} + \frac{w_b^n}{r_c^3 r_a + m_c^3 h_a} + \frac{m_c^n}{r_a^3 r_b + h_a^3 w_b} \geq \frac{16 \cdot 3^{n-3} \cdot r^n}{3 \cdot (3R^2 - 8r^2)^2 + 3R^4 - 64r^4}$$

Proposed by Zaza Mzhavanadze – Georgia

U.2887 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that: $f(x^{2023}) + f(-x^{2023}) = x^{2024}, x \in \mathbb{R}$

Proposed by Nguyen Van Canh – Vietnam

U.2888 If $a, b \in [0; 1]$ then:

$$0 \leq \frac{a}{3+b^2} + \frac{b}{3+a^2} \leq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

U.2889 If $x, y, z > 0$ and $x^3 + y^3 + z^3 \leq 3$, then:

$$\left(\left(\frac{x+2}{2x^2+1}\right)^2 + 2\right) \cdot \left(\left(\frac{y+2}{2y^2+1}\right)^2 + 2\right) \cdot \left(\frac{z+2}{2z^2+1} + 1\right) \geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.2890. If $a, b, c, t > 0$ then:

$$\left(\frac{a^2}{(b+c)^2} + 2t^2\right) \cdot \left(\frac{b^2}{(c+a)^2} + 2t^2\right) \cdot \left(\frac{c^2}{(a+b)^2} + 2t^2\right) \geq \frac{27}{4} t^4$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.2891 If $m \geq n > 0$ and $a, b, c > 0$, then:

$$\left(\frac{a^{2m}}{(b^n + c^n)^2} + 2\right) \cdot \left(\frac{b^{2m}}{(c^n + a^n)^2} + 2\right) \cdot \left(\frac{c^{2m}}{(a^n + b^n)^2} + 2\right) \geq 4^{m-n-1}\left(\sqrt{3}\right)^{-m+n} F^{m-n}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.2892 If $a, b, c > 0$ then:

$$\left((1+a)^{2a^2-2bc} + 2\right)\left((1+b)^{2b^2-2ca} + 2\right)\left((1+c)^{2c^2-2ab} + 2\right) \geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2893 If $t, u > 0$ then in any ΔABC with the area F the following inequality holds:

$$t^4(a^4 + u^2)(b^4 + u^2)(c^4 + u^2) + u^4(a^2b^2 + t^2)(b^2c^2 + t^2)(c^2a^2 + t^2) \geq 72 \cdot t^4 \cdot u^4 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2894 Let be $m, n, x, y, z > 0$ then:

$$\frac{x^2}{(my + nz)^3} + \frac{y^2}{(mz + nx)^3} + \frac{z^2}{(mx + ny)^3} \geq \frac{9}{(m+n)^3(x+y+z)}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2895 If $a, b, c, x, y, m, n > 0$ and $m + n \geq 1$ then:

$$a^{2m} + b^{2m} + c^{2m} + \frac{a^{2n}}{(bx + cy)^{2(m+n)}} + \frac{b^{2n}}{(cx + ay)^{2(m+n)}} + \frac{c^{2n}}{(ax + by)^{2(m+n)}} \geq \frac{6}{(x+y)^{m+n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2896 Let be $a, b, c, m, n > 0$ such that $a + b + c = 1$. Prove that:

$$\left(\frac{a^2}{(m + nbc)} + 2\right) \cdot \left(\frac{b^2}{(m + nca)^2} + 2\right) \cdot \left(\frac{c^2}{(m + nab)^2} + 2\right) \geq \frac{243}{(9m + n)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2897 If $a, b, c > 0$ then:

$$\left(\frac{(a^5 + a^3)^2}{b^8} + 2\right) \cdot \left(\frac{(b^5 + b^3)^2}{c^8} + 2\right) \cdot \left(\frac{(c^5 + c^3)^2}{a^8} + 2\right) \geq 108$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2898 In any ΔABC the following inequality holds:

$$\left(\frac{1}{\left(\sin\frac{A}{2} + \sin\frac{B}{2}\right)^4} + 2\right) \cdot \left(\frac{1}{\left(\sin\frac{B}{2} + \sin\frac{C}{2}\right)^4} + 2\right) \cdot \left(\frac{1}{\left(\sin\frac{C}{2} + \sin\frac{A}{2}\right)^4} + 2\right) \geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2899 Let g_a, g_b, g_c Gergone's cevians of ΔABC with the area F, then:

$$(a^4 + x \cdot g_b + y \cdot g_c)(b^4 + yg_c + zg_a)(c^4 + zg_a + xg_b) \geq 144\sqrt{3}F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2900 Let be $x, y, z \geq 0, x + y + z > 0$ and the triangles ABC, MNP, UVW with the areas F, F_1 respectively F_2, then:

$$a^x \cdot a^y \cdot w^z + b^x \cdot p^y \cdot u^z + c^x \cdot m^y \cdot v^z \geq 2^{x+y+z}\left(\sqrt[4]{3}\right)^{4-(x+y+z)} F^{\frac{x}{2}} F_1^{\frac{y}{2}} F_2^{\frac{z}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2901 If ABC is a triangle with the area F, then:

$$((a^2 + b^2)^2 + 2) \cdot (c^4 + 2) \geq 96F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

U.2902 In any ΔABC with the area F the following inequality holds:

$$((a^2 + bc)^2 + 2) \cdot ((b^2 + ca)^2 + 2) \cdot ((c^2 + ab)^2 + 2) \geq 576 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

U.2903 If $x, y, z > 0$, then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{x}{y+z}a^2 + \frac{y}{z+x}b^2 + \frac{z}{x+y}c^2\right) \cdot \left(\frac{y+z}{x}b^2 + \frac{z+x}{y}c^2 + \frac{x+y}{z}a^2\right) \geq 48 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

U.2904 If $x, y \geq 0, x + y > 0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\left(x \cdot m_a^2 + y \cdot m_b^2\right)^2 + 2\right) \cdot \left(\left(x \cdot m_b^2 + y \cdot m_c^2\right)^2 + 2\right) \cdot$$
$$\cdot ((x \cdot m_c^2 + y \cdot m_a^2)^2 + 2) \geq 81(x + y)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2905 If $x, y \geq 0, x + y > 0$ then in any ΔABC with the area F the following inequality holds:

$$((xa^2 + yb^2) + 2 \cdot ((xb^2 + yc^2)^2 + 2) \cdot ((xc^2 + ya^2)^2 + 2) \geq 144 \cdot (x + y)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2906 If M is an interior point in $\Delta ABC, x = MA, y = MB, z = MC$ and u, v, w are the distances of point M to the sides BC, CA, AB then:

$$\left(\frac{x^2}{(v+w)^2} + 2\right) \cdot \left(\frac{y^2}{(w+u)^2} + 2\right) \cdot \left(\frac{z^2}{(u+v)^2} + 2\right) \geq 27$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.2907 In any ΔABC with the area F the following inequality holds:

$$(w_a^4+2)\cdot(w_b^4+2)\cdot(w_c^4+2)\geq 36\cdot\sqrt{3}\cdot\frac{F^3}{R^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2908 If ABC is a triangle with the area F and semiperimeter s and $x,y>0, x+y=s$ then:

$$\left(a^2+2\cdot(x^2+y^2)\right)\left(b^2+2(x^2+y^2)\right)\cdot\left(c^2+2(x^2+y^2)\right)\geq 243\cdot\sqrt{3}\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2909 If $x,y,z>0$ then in ΔABC with the area F the following inequality holds:

$$\left(\left(\frac{x}{y+z}a^2+\frac{y+z}{x}b^2\right)+2\right)\cdot\left(\left(\frac{y}{z+x}b^2+\frac{z+x}{y}c^2\right)^2+2\right)\cdot\left(\left(\frac{z}{x+y}c^2+\frac{x+y}{z}a^2\right)^2+2\right)\geq 900\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2910 If $x,y,z>0$ and ABC is a triangle with the area F then:

$$\frac{x^2\cdot a^2}{h_a^2}+\frac{y^2\cdot b^2}{h_b^2}+\frac{z^2\cdot c^2}{h_c^2}\geq\frac{4}{3}\cdot(xy+yz+zx)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2911 If $x,y,z>0$ then:

$$((x+y)^2+2z^2)\cdot((y+z)^2+2x^2)\cdot((z+x)^2+2y^2)\geq 108\cdot x^2y^2z^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2912 In any ΔABC the following inequality holds:

$$\left(\frac{1}{h_b}+\frac{1}{h_c}\right)\cdot\sin^2\frac{A}{2}+\left(\frac{1}{h_c}+\frac{1}{h_a}\right)\cdot\sin^2\frac{B}{2}+\left(\frac{1}{h_a}+\frac{1}{h_b}\right)\cdot\sin^2\frac{C}{2}\geq\frac{1}{2r}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2913 Let be $x,y,z>0$ then in ΔABC with the area F the following inequality holds:

$$\frac{x}{y+z}\cdot\frac{a^3}{h_a}+\frac{y}{z+x}\cdot\frac{b^3}{h_b}+\frac{z}{x+y}\cdot\frac{c^3}{h_c}\geq 4F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2914 In any ΔABC with the area F, the following inequality holds:

$$(a+b)(b+c)+(b+c)(c+a)+(c+a)(a+b)\geq 16\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2915 In any ΔABC with the area F the following inequality holds:

$$(a^2+2b^2)(b^2+2c^2)(c^2+2a^2) \geq 192 \cdot \sqrt{3} \cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2916 In any ΔABC with the area F the following inequality holds:

$$\frac{a}{h_b}+\frac{b}{h_c}+\frac{c}{h_a} \geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2917 Let be $m \geq 0$, then in any ΔABC the following inequality holds:

$$\frac{a}{(bx+cy)^{m+1}}+\frac{b}{(cx+ay)^{m+1}}+\frac{c}{(ax+by)^{m+1}} \geq \frac{\left(\sqrt{3}\right)^{2-m}}{(x+y)^{m+1} \cdot R^m}; \forall x,y>0$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2918 Let be $x,y,z>0$ and ABC a triangle, then:

$$\frac{x}{y+z}\sqrt{\frac{a}{h_a}}+\frac{y}{z+x}\sqrt{\frac{b}{h_b}}+\frac{z}{x+y}\sqrt{\frac{c}{h_c}} \geq \frac{\sqrt[4]{27}}{\sqrt{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2919 In any ΔABC with the area F the following inequality holds:

$$(a+b+1)(b+c+1)(c+a+1) \geq 36\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.2920 In any ΔABC the following inequality holds:

$$(a+b+c)^2 \cdot \sum_{cyc} \frac{1}{(r_a+r_b)^2} \geq 9$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.2931 If $a,b,c,d>0$ then:

$$\frac{a+b}{d^2}+\frac{b+c}{a^2}+\frac{c+d}{b^2}+\frac{d+a}{c^2}-\frac{16}{a+b+c+d} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

Proposed by Marin Chirciu – Romania

U.2932 In ΔABC the following relationship holds:

$$\sum \cot^{3n-1}\frac{A}{2}\cot\frac{B}{2} \geq 9\left(\frac{s}{3r}\right)^n\left(\frac{4R}{r}-5\right)^{n-1}$$

where $n \in \mathbb{N}^*$

Proposed by Marin Chirciu – Romania

U.2933 In ΔABC the following relationship holds:

$$\sum\left(\frac{\tan\frac{A}{2}}{\cos\frac{A}{2}}\right)^n \geq 3\left(\frac{2}{3}\right)^n$$

where $n \in \mathbb{N}$

Proposed by Marin Chirciu – Romania

U.2934 In ΔABC the following relationship holds:

$$\sum\frac{a^2}{b^2+c^2} \leq 6\left(1-\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

U.2935 If $x, y, z > 0$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$ then find the maximum of

$$A=\frac{1}{(x+y+4)^3}+\frac{1}{(y+z+4)^3}+\frac{1}{(z+x+4)^3}$$

Proposed by Marin Chirciu – Romania

U.2936 In ΔABC the following relationship holds:

$$\sum a^2 \geq 4\sqrt{3}S+\lambda r(R-2r),\ \text{where } \lambda \leq 6$$

Proposed by Marin Chirciu – Romania

U.2937 If $x, y, z > 0$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$ and $n \in \mathbb{N}$ then find the maximum of

$$A=\frac{1}{(x+y+2n)^{n+1}}+\frac{1}{(y+z+2n)^{n+1}}+\frac{1}{(z+x+2n)^{n+1}}$$

Proposed by Marin Chirciu – Romania

U.2938 In ΔABC the following relationship holds:

$$\sum\frac{a^2}{b^2+c^2} \geq \frac{3}{2}+\frac{\lambda}{2R}(R-2r)$$

where $\lambda \leq 1$

Proposed by Marin Chirciu – Romania

U.2939 In ΔABC the following relationship holds:

$$\prod\left(r_a^2+r_ar_b+r_b^2\right) \geq 27(3r)^6$$

Proposed by Marin Chirciu – Romania

U.2940 If $a, b > 0$ with $na + b \leq n + 1, n \in \mathbb{N}^*$ and $\lambda \geq 0$ then:

$$\frac{n}{\sqrt{a+\lambda}} + \frac{1}{\sqrt{b+\lambda}} \geq \frac{n+1}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu – Romania

U.2941 If $a, b, c > 0$ with $a^2 + b^2 + c^2 = 3$ and $n \geq 0$ then:

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{3n}{n+1} + \frac{1}{ab+n} + \frac{1}{bc+n} + \frac{1}{ca+n}$$

Proposed by Marin Chirciu – Romania

U.2942 In ΔABC the following relationship holds:

$$3\left(\frac{2r}{R}\right)^{\frac{4}{3}} \leq \sum \left(\frac{w_a}{r_a}\right)^2 \leq \frac{s^2 - 12Rr}{r^2}$$

Proposed by Marin Chirciu – Romania

U.2943 In ΔABC the following relationship holds:

$$\prod \left(h_a^2 + h_a h_b + h_b^2\right) \geq 27(3r)^6$$

Proposed by Marin Chirciu – Romania

U.2944 In ΔABC the following relationship holds:

$$1 \leq 3\sqrt[3]{\left(\frac{4R^2}{2R^2 + 3Rr + 2r^2}\right)^2} \leq \sum \left(\frac{w_a}{h_a}\right)^2 \leq 1 + \frac{R}{r}$$

Proposed by Marin Chirciu – Romania

U.2945 In ΔABC the following relationship holds:

$$1 \leq \sum \frac{m_a^2}{w_b w_c} \leq 3\left(\frac{R}{2r}\right)^5$$

Proposed by Marin Chirciu – Romania

U.2946 In ΔABC the following relationship holds:

$$\frac{2}{R} \leq \sum \frac{m_a}{w_b w_c} \leq \frac{1}{r}\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

U.2947 In ΔABC the following relationship holds:

$$\sum \frac{1}{w_a^{2n}} \geq \sum \frac{1}{r_b^n r_c^n}$$

where $n \in \mathbb{N}$.

Proposed by Marin Chirciu – Romania

U.2948 In ΔABC the following relationship holds:

$$\frac{S}{2R^2}\left(5-\frac{2r}{R}\right) \le \sum \sin^3 A \le \frac{s}{2R}\left(2-\frac{r}{R}\right)$$

Proposed by Marin Chirciu – Romania

U.2949 Let f be a function that is continuous on the interval $\left(0,\frac{\pi}{2}\right)$ such that $f(x)+f\left(\frac{\pi}{2}-x\right)=1$.

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \frac{f(x)}{\tan x+\cot x}\,dx$$

Proposed by Marin Chirciu – Romania

U.2950 If $m \ge 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a\cdot b^{m+1}}{c^m}+\frac{b\cdot c^{m+1}}{a^m}+\frac{c\cdot a^{m+1}}{b} \ge 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze

U.2951 In ΔABC the following relationship holds:

$$w_a^n w_b + w_b^n w_c + w_c^n w_a \ge 3(3r)^{n+1}, n \in \mathbb{N}, n \ge 2$$

Proposed by Marin Chirciu – Romania

U.2952 In ΔABC the following relationship holds:

$$r_a^2+h_a^2+w_a^2+m_a^2 \ge 4S\sqrt{\frac{a+b+c}{a}}$$

Proposed by Marin Chirciu – Romania

U.2953 Let x,y,z be positive real numbers such that $x+y+z=3$. Prove that:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{9xyz}{xy+yz+zx} \ge 6$$

Proposed by Marin Chirciu – Romania

U.2954 Prove the below closed form:

$$\int\int\int_{[0,1]^3} \frac{xyz}{(x+y)(y+z)(z+x)}\,dx\,dy\,dz = -\frac{\pi^2}{12}-\ln^2(2)+2\ln 2$$

Proposed by Ankush Kumar Parcha-India

U.2955 Prove the below closed form:

$$\int\int\int_{[0,1]^3}\sum_{x,y,z}\sqrt{\frac{\sqrt{x}}{\sqrt{y}+\sqrt{z}}}dx\,dy\,dz=\frac{128}{35}\left(2-\sqrt{2}\right)$$

Proposed by Ankush Kumar Parcha-India

U.2956 In ΔABC the following relationship holds:

$$81r^3\le h_a^3+h_b^3+h_c^3\le\frac{81}{8}R^3$$

Proposed by Marin Chirciu – Romania

U.2957 In ΔABC the following relationship holds:

$$\left(\frac{s_a}{b+c}\right)^2+\left(\frac{s_b}{c+a}\right)^2+\left(\frac{s_c}{a+b}\right)^2\le\frac{9}{16}$$

Proposed by Marin Chirciu – Romania

U.2958 In ΔABC the following relationship holds:

$$\frac{R}{2}(4R+r)^2\le r_a^3+r_b^3+r_c^3\le\frac{1}{3R}(4R+r)^2(4R^2-3r^2)$$

Proposed by Marin Chirciu – Romania

U.2959 Prove the integral relation:

$$\int_0^\infty e^{-\frac{\pi x}{3}}\sqrt[3]{\coth(3\pi x)+1}dx$$

$$=\frac{\Gamma\left(\frac{1}{18}\right)\Gamma\left(\frac{19}{18}\right)}{\Gamma\left(\frac{7}{18}\right)\Gamma\left(\frac{13}{18}\right)}\int_0^\infty e^{\frac{-\pi x}{3}}\sqrt[3]{\coth(3\pi x)-1}dx$$

Proposed by Srinivasa Raghava-AIRMC-India

U.2960 Prove the below closed form:

$$\int\int\int_{[0,1]^3}\frac{dx\,dy\,dz}{\sqrt{x+2y+3z}}=\frac{4}{45}\left(-31+4\sqrt{2}-25\sqrt{5}+36\sqrt{6}\right)$$

Proposed by Ankush Kumar Parcha-India

U.2961 Show that:

$$\int_0^1\frac{x\log(x^3+1)}{x^4+1}dx=\frac{\pi}{16}\log\left(3\sqrt{2}-4\right)-\frac{\pi}{6}\log\left(2-\sqrt{3}\right)-\frac{G}{3}$$

where G is Catalan's constant

Proposed by Vincent Nguyen-USA

U.2962 Find:

$$\Omega = \int_0^1 \frac{1}{\sqrt[4]{(1-x)(1+x)^3}}dx$$

Proposed by Vasile Mircea Popa – Romania

U.2963 Prove the below closed form:

$$\int\int_{[0,1]^2} \ln\left(\frac{x\sqrt{x}+y\sqrt{y}}{x\sqrt{x+y}+y\sqrt{x+y}}\right)dx\,dy = \frac{\pi}{\sqrt{3}} - 3\ln 2$$

Proposed by Ankush Kumar Parcha-India

U.2964 Prove the integral:

$$\int_0^\infty {}_3\tilde{F}_2\left(\frac{1}{2},\frac{3}{2},\frac{5}{2};2{,}2;e^{-x}\right)e^{-x}dx = \frac{8}{3}\left(1-\frac{4\Gamma\left(\frac{3}{4}\right)^4}{\pi^3}\right)$$

${}_3\tilde{F}_2(a, bc; x, y; q)$ is Regularized Hypergeometric function

Proposed by Srinivasa Raghava-AIRMC-India

U.2965 Find all values of $a \geq 1$ such that $\int_0^a |x^3 - x^2 + x - 1|\,dx \leq 2023$.

Proposed by Nguyen Van Canh – Vietnam

U.2966 Find:

$$\Omega(n) = \int_0^{\frac{\pi}{4}} \frac{\sin(nx)\cos(nx)}{\sin x}\,dx,\quad n \in \mathbb{N}^*$$

Proposed by Khaled Abd Imouti-Syria

U.2967 Prove without using Dilogarithm identities:

$$\Im\left\{Li_2\left(\frac{1-i\sqrt{3}}{4}\right) - Li_2\left(-\frac{i}{\sqrt{3}}\right)\right\} = \frac{\pi}{12}\ln\left(\frac{4}{3}\right)$$

$\Im$ denotes the imaginary part

Proposed by Rana Ranino-Algeria

U.2968 If,

$$\Omega := \int\int\int_{[0,1]^3} \sum_{x,y,z} \frac{\sqrt{x}}{\sqrt{x+y+z}}dx\,dy\,dz$$

Then, show that:

$$\Omega = \frac{4 - 25\sqrt{2}}{3} + 8\sqrt{3} + 2\ln 2 + \ln\left(1 + \sqrt{2}\right) - 4\ln\left(1 + \sqrt{3}\right)$$

Proposed by Ankush Kumar Parcha-India

U.2969 If $\Omega := \int_0^\infty \int_0^\infty \frac{x^3}{1+x^4} e^{ixy} dx\, dy$ then, show that:

$$\Im\{\Omega\} = \frac{\pi}{2\sqrt{2}}$$

where, $\Im(z)$ is the imaginary part

Proposed by Ankush Kumar Parcha-India

U.2970 If $n \in \mathbb{N}, n \geq 2$ then:

$$\sqrt[3]{\sum_{k=2}^{n} H_k^{-3} \cdot \left(\sum_{k=2}^{n} H_k^3\right)^2} > \frac{1}{2} + n(H_n - \gamma - 1)$$

Proposed by Khaled Abd Imouti-Syria

U.2971 Compute (without any software):

$$I = \int_0^\pi \left|\frac{\sin x - \cos x}{\sin x + \cos x + 1}\right| dx$$

Proposed by Nguyen Van Canh – Vietnam

U.2972 Prove that:

$$\int_0^1 e^{x^x} dx = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \cdot n^k}{n! \cdot (k+1)^{k+1}};$$

$$\int_0^1 x^{e^x} dx = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n \cdot n^k}{k! \cdot (k+1)^{n+1}};\ \int_0^1 x^{x^e} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n \cdot (ne+1)^{n+1}}$$

Proposed by Emil C. Popa – Romania

U.2973 Find all functions $F \in C^1(\mathbb{R})$ such that:

$$f(5^x) = 5^x \cdot f'(5^x) + 5^{2x},\ \ \forall x \in \mathbb{R}$$

Proposed by Khaled Abd Imouti-Syria

U.2974 Find all functions $f: \mathbb{R} \to \mathbb{R}$ $s$uch that:

$$\sum_{k=1}^{n} sign\left(f(k)\right) = \begin{cases} 0 & If: n = 2k, k \in \mathbb{N} \\ 1 & If: n = 2k+1, k \in \mathbb{N} \end{cases}$$

Proposed by Toubal Fethi -Algeria

U.2975 Prove the below closed form:

$$\int_0^1\int_0^1\int_0^1 \frac{xyz}{x+y+z}dx\,dy\,dz = -\frac{9}{10}\ln 3+\frac{4}{5}\ln 2+\frac{1}{2}$$

Proposed by Ankush Kumar Parcha-India

U.2976 Prove the below closed form:

$$\int\int\int_{[0,\infty)^3}\frac{dx\,dy\,dz}{(1+x^2y^2)(1+y^2z^2)(1+z^2x^2)}=\frac{\pi^3}{4\sqrt{2}}$$

Proposed by Ankush Kumar Parcha-India

U.2977 Prove the integral relation

$$\int_0^1 \mathcal{L}_x\left[\frac{\cos\left(\frac{\pi}{5}-x\right)\cos\left(x+\frac{\pi}{5}\right)}{\sqrt{\pi x}}\right](y)dy=\sqrt{\frac{1}{2}\left(1+\sqrt{5}\right)+\frac{1}{4}\left(\sqrt{5}-5\right)}$$

where $\mathcal{L}_x[\dots](y)$ is Laplace transform

Proposed by Srinivasa Raghava-AIRMC-India

U.2978 Find all values $a\in\mathbb{R}$ such that:

$$\int_0^{2\pi}\left|\sin\frac{ax}{2}\right|dx+\int_0^{\pi}\left|\cos\frac{ax}{3}\right|dx\le 2022$$

Proposed by Nguyen Van Canh – Vietnam

U.2979 Find:

$$\Omega(x)=\frac{\sum_{k=1}^{\infty}\left(\frac{\sin k}{k}\right)x^k}{\prod_{k=1}^{\infty}\frac{(2k-1)(2k+1)}{4k^2}},\quad x\in[-1,1]$$

Proposed by Khaled Abd Imouti-Syria

U.2980 Prove that:

$$\frac{1}{\sin^2\frac{7\pi}{18}}-\frac{1}{\sin^2\frac{\pi}{18}}=8\sqrt{3}\left(-3\sin\frac{4\pi}{9}+\sin\frac{2\pi}{9}\right)$$

Proposed by Vasile Mircea Popa – Romania

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ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SUMMER 2025

PROBLEMS FOR JUNIORS

JP.541 If a, b, c sides in acute ΔABC with s – semiperimeter; r – inradii and $x, y, z \in \left(0, \frac{\pi}{2}\right)$ are such that:

$$\cos x = \frac{a}{b+c}; \cos y = \frac{b}{c+a}; \cos z = \frac{c}{a+b}$$

then:

$$\left(\tan^2\frac{x}{2} + \tan^2\frac{y}{2} + \tan^2\frac{z}{2}\right)\tan^2\frac{x}{2}\tan^2\frac{y}{2}\tan^2\frac{z}{2} = \frac{r^2}{s^2}$$

Proposed by Daniel Sitaru – Romania

JP.542 Solve for real numbers:

$$\begin{cases} \dfrac{6x+6y}{9+4xy} = z \\ \dfrac{6y+6z}{9+4yz} = x \\ \dfrac{6z+6x}{9+4zx} = y \end{cases}$$

Proposed by Daniel Sitaru – Romania

JP.543 Find $x, y, z > 0$ such that $x + y + z = 1$ and

$$\left(\frac{x^5}{y+1} + \frac{y^5}{zx+1} + \frac{z^5}{xy+1}\right)\left(\frac{x^7}{yz+1} + \frac{y^7}{zx+1} + \frac{z^7}{xy+1}\right) = \frac{1}{72900}$$

Proposed by Daniel Sitaru – Romania

JP.544 If $x, y \in \mathbb{R}$ then:

$$\log(1 + 3\sin^2 x) \cdot \log(1 + 3\cos^2 x \sin^2 y) \cdot \log(1 + 3\cos^2 x \cos^2 y) \le \log^3 2$$

Proposed by Daniel Sitaru – Romania

JP.545 If $x, y, z > 0$ then:

$$\frac{x}{7x+5y+5z}+\frac{y}{5x+7y+5z}+\frac{z}{5x+5y+7z}\le\frac{3}{17}$$

Proposed by Daniel Sitaru – Romania

JP.546 In ΔABC the following relationship holds:

$$\sum\frac{\sin^4\frac{A}{2}+\sin^4\frac{B}{2}}{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}}\ge\frac{3}{4}$$

Proposed by Marin Chirciu – Romania

JP.547 In ΔABC the following relationship holds:

$$\sum\frac{\cos^4\frac{A}{2}+\cos^4\frac{B}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}}\ge\frac{9r}{2R}$$

Proposed by Marin Chirciu – Romania

JP.548 If $x, y, z \in [0, \infty)$ solve the system:

$$\begin{cases} x^2 = y(y+z) \\ y^2 = z(z+x) \end{cases}$$

Proposed by Cristian Miu – Romania

JP.549 Let ABC be a triangle with inradius r and circumradius R. Prove that:

$$\sum\frac{\sin^3 A+\sin^3 B}{\sin^5 A+\sin^5 B}\le\left(\frac{R}{r}\right)^2$$

The sum is over all cyclic permutations of (A, B, C).

Proposed by George Apostolopoulos –Greece

JP.550 The non-coplanar points are given: A, B,C and D. If K is the middle of the segment $[BD]$, $(KM =$ bisector $\widehat{AKB}, M \in (AB)$, $(KP =$ bisector $\widehat{AKD}, P \in (AD)$, and $N \in (AC)$, such that $\frac{AC}{AN}-\frac{BD}{2AK}=1$. Prove that:

$$AN\cdot NC+AM\cdot MB\ge 2PD\cdot(AN+AM-AP)$$

Proposed by Gheorghe Molea – Romania

JP.551 Find the real numbers x, y, z knowing that they meet the conditions:

$$x+y+z=1;\, xy+(x+y)(z+1)=\frac{4}{3}$$

Proposed by Gheorghe Molea – Romania

JP.552 Justify if exists non-zero natural numbers a, b, c, d, different in pairs, such that we have:

$$a(b+c-a)=b(a+c-b)=c(a+b-c)=\frac{a+b+c}{d}$$

Proposed by Gheorghe Molea – Romania

JP.553 Prove that for any $a, b, c \in \mathbb{R}$, we have the inequality:

$$\frac{a^3+a}{a^4+a^2+1}+\frac{b^3+b}{b^4+b^2+1}+\frac{c^3+c}{c^4+c^2+1} \leq 2$$

Proposed by Laura Molea and Gheorghe Molea – Romania

JP.554 Solve for integers: $x(x-1)^2+y(y-1)^2=x(3x+7y)$

Proposed by Laura Molea and Gheorghe Molea – Romania

JP.555 In ΔABC we know: $m(\widehat{A})=90°, AD \perp BC, D \in (BC), DE \perp AB, E \in (AB),$

$DF \perp AC, F \in (AC), BE=a, CF=b, BC=c, a, b, c > 0$. Prove that $c \leq 2\sqrt{a^2+b^2}$.

Proposed by Gheorghe Molea – Romania

PROBLEMS FOR SENIORS

SP.541 Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\left(\frac{x}{3}\right)-3f(x)=15x; (\forall)x \in \mathbb{R}$$

Proposed by Daniel Sitaru – Romania

SP.542 If $z_1, z_2, z_3 \in \mathbb{C}; |z_1|=|z_2|=|z_3|=1; z_1+z_2+z_3=1$ then find:

$$\Omega=\left(\sum_{k=1}^{n} z_i^3\right)\left(\sum_{i=1}^{n} z_i^5\right)\left(\sum_{i=1}^{n} z_i^7\right)$$

Proposed by Daniel Sitaru – Romania

SP.543 Let be $f: [0,1] \to [0,20]; f(x)=2 \cdot 3^x+4 \cdot 5^x-6$. Find:

$$\Omega=\int_0^{20} f^{-1}(x)dx$$

Proposed by Daniel Sitaru – Romania

SP.544 Find the maximum value of $n \in \mathbb{N}^*$ such that:

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k+1} + k\sqrt{k}} < \frac{31}{32}$$

Proposed by Daniel Sitaru – Romania

SP.545 Let a, b, c be positive real numbers such that $a = \max\{a, b, c\}$ and $a^2b^5c^5 \geq 1$, and let

$$F(a,b,c) = \sqrt{\frac{ab+bc+ca}{3}} - \sqrt[3]{abc}$$

Prove that

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$$

Proposed by Vasile Cîrtoaje and Vasile Mircea Popa – Romania

SP.546 If $a, b, c \in (0, 1)$ then:

$$\sqrt{3a - a^2} + \sqrt{5b - b^2} + \sqrt{7c - c^2} \leq \sqrt{15(a+b+c) - (a+b+c)^2}$$

Proposed by Daniel Sitaru – Romania

SP.547 Prove that if $x, y > 1$ then:

$$\ln x \cdot \ln y \left(\sqrt[3]{\log_x y} + \sqrt[3]{\log_y x}\right)^3 \leq 2\ln^2(xy)$$

Proposed by Daniel Sitaru – Romania

SP.548 If $a, b, c > 0$ then:

$$\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 + 2\left(\frac{ab}{a+b} + \frac{bc}{c+a} + \frac{ca}{a+b}\right) \geq 4\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)$$

Proposed by Daniel Sitaru – Romania

SP.549 Let be ΔABC with sides a, b, c. Let be $x, y, z \in \mathbb{R}$ such that:

$$\cos x = \frac{a}{b+c}; \cos y = \frac{b}{c+a}; \cos z = \frac{c}{a+b}$$

Prove that:

$$\tan\frac{x}{2}\tan\frac{y}{2}\tan\frac{z}{2} \leq \frac{\sqrt{3}}{9}$$

Proposed by Daniel Sitaru – Romania

SP.550 Prove that $(\forall)x \in (0;1)$ and $n \in \mathbb{N}^*$, we have the inequality:

$$x(1+x)(1-x^n) < \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n+1}$$

Proposed by Gheorghe Molea – Romania

SP.551 Find:

$$\Omega = \lim_{n\to\infty} n\int_0^1 \frac{x^3(1+x^{4n-8})}{(1+x^4)^n}dx$$

Proposed by Daniel Sitaru – Romania

SP.552 If $a > 0$ then find:

$$\Omega = \int_{-a}^{a} \log_a\left(\sqrt{a^2x^2+1}-ax\right)dx$$

Proposed by Daniel Sitaru – Romania

SP.553 If $0 \le a \le b < 1$ then:

$$6\int_a^b \log\left(\frac{1+x}{1-x}\right)dx \ge (b^2-a^2)(b^2+a^2+6)$$

Proposed by Daniel Sitaru – Romania

SP.554 Find:

$$\Omega = \int \frac{8x-1}{e^{8x}+7x}dx; x \in (0,\infty)$$

Proposed by Daniel Sitaru – Romania

SP.555 Find:

$$\Omega = \lim_{x\to 0}\frac{1}{x}\left(\frac{1}{x}\int_0^x \frac{dt}{t+e^t}-1\right)$$

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

UP.541 Find:

$$\Omega = \int_1^{\sqrt{3}} \frac{x-\tan^{-1}x}{(1+x^2)^2(\tan^{-1}x)^3}dx$$

Proposed by Daniel Sitaru – Romania

UP.542 Find:

$$\Omega=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{49}{49+(7n+1)(7n+8)}\right)$$

Proposed by Daniel Sitaru – Romania

UP.543 Prove:

$$\int_0^1\left(\frac{x^2\log(x)}{1+x^2}\right)^2dx=G+2-\frac{3\pi^3}{32}$$

G represents the Catalan's constant.

Proposed by Said Attaoui –Algeria

UP.544 For $x,y,z>0$ let us denote:

$$F(x,y,z)=\frac{xyz(xy+yz+zx)}{x^3y^3+y^3z^3+z^3x^3}[x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2]$$

If $u,v,w\geq 1$, prove that:

$$F(u,v,w)\geq F\left(\frac{1}{u},\frac{1}{v},\frac{1}{w}\right)$$

Proposed by Vasile Mircea Popa – Romania

UP.545 Find:

$$\Omega=\int_0^{\frac{1}{2}}\frac{x^5-3x^3}{3x^6-x^4-3x^2+1}dx$$

Proposed by Daniel Sitaru – Romania

UP.546 Prove without any software:

$$\int_0^2\sqrt[3]{\frac{x}{2}+\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx+\int_0^2\sqrt[3]{\frac{x}{2}-\sqrt{\frac{1}{27}+\frac{x^2}{4}}}dx>\frac{5}{4}$$

Proposed by Daniel Sitaru – Romania

UP.547 Find without any software:

$$\Omega=\int_1^2\frac{4x^4-6x-9}{x^4}e^{2x+\frac{3}{x}}dx$$

Proposed by Daniel Sitaru – Romania

UP.548 Find:

$$\Omega = \int_1^e \frac{1-\ln x}{x^2+\ln^2 x}\,dx$$

Proposed by Daniel Sitaru – Romania

UP.549 If $0 < a \le b$ then find:

$$\Omega(a,b) = \int_a^b \frac{\ln x}{x^2+(a+b)x+ab}\,dx$$

Proposed by Daniel Sitaru – Romania

UP.550 If $f:[0,1]\to\mathbb{R}$; f continuous and

$$\int_0^1 xf(x)\,dx = a;\ \int_0^1 f(x)\,dx = b;\ a,b\in\mathbb{R}$$

then:

$$\int_0^1 f^2(x)\,dx \ge 3(a-b)^2$$

Proposed by Daniel Sitaru – Romania

UP.551 Calculate the integral:

$$\int_0^\infty \frac{arctanx}{\sqrt{3x^4+x^2+3}}\,dx$$

Proposed by Vasile Mircea Popa – Romania

UP.552 Calculate the integral:

$$\int_1^\infty \frac{\sqrt{x}\ln x}{(x+1)(x^2+1)}\,dx$$

Proposed by Vasile Mircea Popa – Romania

UP.553 Prove the equality:

$$\int_0^\infty \frac{|\cos(x)|}{1+x^2}\,dx = 1-2\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)e^{2n}}$$

Proposed by Vasile Mircea Popa – Romania

UP.554 We consider the equation: $(1+iz)^{2n} = i\cdot(1+z^2)^n$, where $n\ge 1$ natural number and $i^2=-1$.

a. Prove that the complex number i is a solution of the equation for any $n \ge 1$.

b. Solve the equation in the case $n = 1$ and in one of the cases $n = 2$ or $n = 3$.

c. Find the solution of the equation in the general case $n \in \mathbb{N}^*$

Proposed by Adalbert Kovacs – Romania

UP.555 Find the solutions of the system:

$$\sqrt{8x+5}+\sqrt{9y+6}=\sqrt{8x+9y+29} \qquad \sqrt{12x+19}-\sqrt{3y+15}=\sqrt{12x+3y-6}$$

Proposed by Bela Kovacs – Romania

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NOTĂ: Pentru a publica probleme propuse, articole și note matematice în RMM puteți trimite materialele pe mailul: dansitaru63@yahoo.com