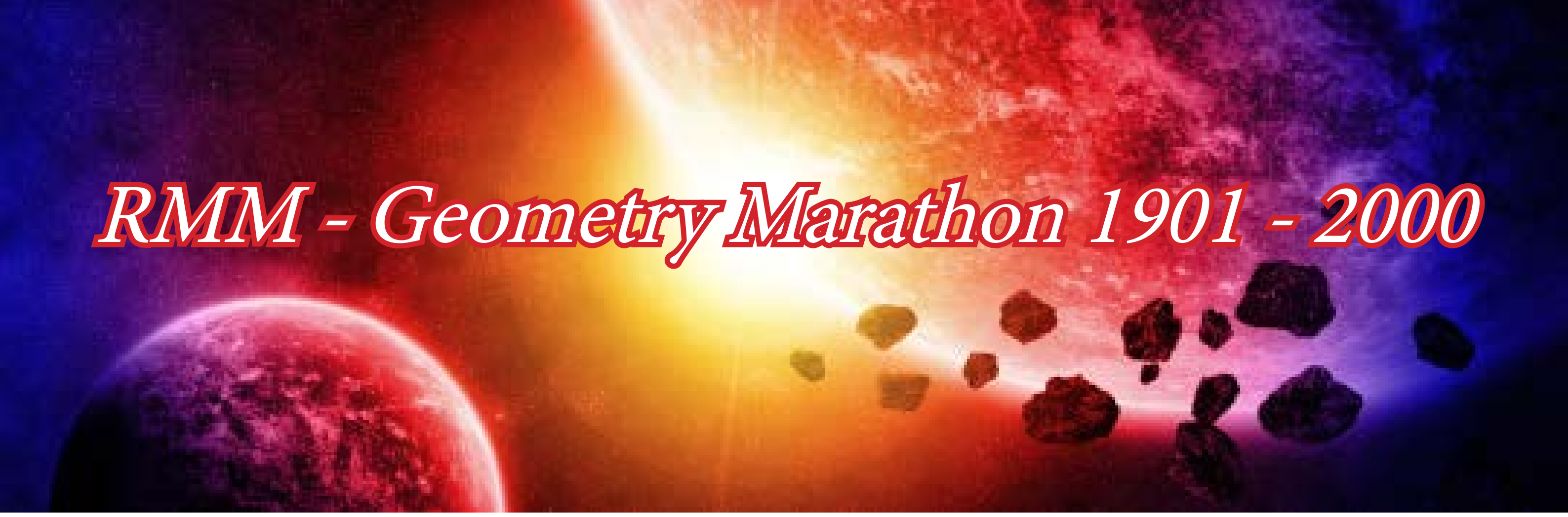


RMM - Geometry Marathon 1901 - 2000

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Proposed by

Tapas Das-India, Mohamed Amine Ben Ajiba-Morroco

Marin Chirciu-Romania, Zaza Mzhavanadze-Georgia

Vasile Mircea Popa-Romania, Ertan Yildirim-Turkiye

Samir Cabiyev-Azerbaijan, Thanasis Gakopoulos-Greece

Bogdan Fuștei-Romania, George Apostolopoulos-Greece

Mehmet Șahin-Turkiye, Nguyen Hung Cuong-Vietnam

Nguyen Van Canh-Vietnam, Dang Ngoc Minh-Vietnam

R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

Solutions by

Daniel Sitaru – Romania, Soumava Chakraborty-India

Mohamed Amine Ben Ajiba-Morroco, Tapas Das-India

Mirsadix Muzefferov-Azerbaijan, Samir Cabiyev-Azerbaijan

1901. In ΔABC the following relationship holds:

$$\pi\left(\frac{3}{2}+\frac{r}{R}\right)<\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2\leq\frac{\pi}{4r^2}(5R^2-Rr+6r^2)$$

Proposed by Radu Diaconu-Romania

Solution by Tapas Das-India

$$We\ know\ that\ \forall x,y,z>0,(x+y+z)^2>x^2+y^2+z^2$$

$$WLOG\ a\geq b\geq c\ \ then\ A\geq B\geq C\ and\ \frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a}$$

$$\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2>$$

$$>A\cdot\frac{h_b+h_c}{h_a}+B\cdot\frac{h_c+h_a}{h_b}+C\cdot\frac{h_a+h_b}{h_c}=$$

$$=A\left(\frac{a}{b}+\frac{a}{c}\right)+B\left(\frac{b}{c}+\frac{b}{a}\right)+C\left(\frac{c}{a}+\frac{c}{b}\right)\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}(A+B+C)\left(\left(\frac{a}{b}+\frac{a}{c}\right)+\left(\frac{b}{c}+\frac{b}{a}\right)+\left(\frac{c}{a}+\frac{c}{b}\right)\right)\overset{AM-GM}{>}$$

$$>\frac{1}{3}\pi 6\sqrt{\left(\frac{a}{b}.\frac{a}{c}\right).\left(\frac{b}{c}.\frac{b}{a}\right).\left(\frac{c}{a}.\frac{c}{b}\right)}=2\pi=\pi\left(\frac{3}{2}+\frac{1}{2}\right)\overset{Euler}{>}\pi\left(\frac{3}{2}+\frac{r}{R}\right)$$

$$\left(\sqrt{A.\frac{h_b+h_c}{h_a}}+\sqrt{B.\frac{h_c+h_a}{h_b}}+\sqrt{C.\frac{h_a+h_b}{h_c}}\right)^2\overset{C-S}{\leq}(A+B+C)\sum\frac{h_b+h_c}{h_a}=$$

$$=\pi\sum\left(\frac{a}{b}+\frac{a}{c}\right)=\pi\sum\left(\frac{a}{b}+\frac{b}{a}\right)\overset{Bandila}{\leq}\frac{3\pi R}{r}$$

$$We\ \ need\ to\ show\ \frac{3\pi R}{r}\leq\frac{\pi}{4r^2}(5R^2-Rr+6r^2)\ or$$

$$5R^2-13Rr+6r^2\geq 0\ or\ (R-2r)(5R-3r)\geq 0\ True\ (Euler)$$

Equality holds for $a=b=c$.

1902.

Let ABC be a triangle with the measures of all its angle smaller than $\frac{2\pi}{3}$ and T its Torricelli's point. Prove that:

$$m_a^2.TA+m_b^2.TB+m_c^2.TC\ge\sqrt[4]{432F^6}$$

Proposed by Tapas Das-India

Solution 1 by Mohamed Amine Ben Ajiba-Morocco

Since m_a, m_b, m_c can be the sides of a triangle Δ_m with area $F_m=\frac{3}{4}F$, then by using Oppenheim's inequality in triangle Δ_m, we have, for all $x,y,z>0$,

$$m_a^2.x+m_b^2.y+m_c^2.z\ge 4F_m\sqrt{xy+yz+zx}. \quad (1)$$

Let $x=TA, y=TB, z=TC$. Since we have

$$TA.TB+TB.TC+TC.TA$$
$$=\frac{4}{\sqrt{3}}.\left(\frac{1}{2}TA.TB.\sin\frac{2\pi}{3}+\frac{1}{2}TB.TC.\sin\frac{2\pi}{3}+\frac{1}{2}TC.TA.\sin\frac{2\pi}{3}\right)$$
$$=\frac{4\sqrt{3}}{3}([TAB]+[TBC]+[TCA])=\frac{4\sqrt{3}}{3}F,$$

then the inequality (1) becomes $m_a^2.TA+m_b^2.TB+m_c^2.TC\ge 4.\frac{3}{4}F.\sqrt{\frac{4\sqrt{3}}{3}F}$
$=\sqrt[4]{432F^6}$.

Equality holds iff ΔABC is equilateral.

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

Let $x=TA, y=TB, z=TC$. We have

$$F=[TAB]+[TBC]+[TCA]=\frac{1}{2}TA.TB.\sin\frac{2\pi}{3}+\frac{1}{2}TB.TC.\sin\frac{2\pi}{3}+\frac{1}{2}TC.TA.\sin\frac{2\pi}{3}$$
$$=\frac{\sqrt{3}}{4}(TA.TB+TB.TC+TC.TA)=\frac{\sqrt{3}}{4}(xy+yz+zx),$$

$$a^2=TB^2+TC^2-2TB.TC.\cos\frac{2\pi}{3}=y^2+z^2+yz \text{ (and analogs).}$$

$$4{m_a}^2=2(b^2+c^2)-a^2=4x^2+y^2+z^2+2xy+2xz-yz \text{ (and analogs).}$$

$$4({m_a}^2.TA+{m_b}^2.TB+{m_c}^2.TC)=\sum_{cyc}4{m_a}^2.TA$$

$$=\sum_{cyc}(4x^2+y^2+z^2+2xy+2xz-yz)x$$

$$=4\sum_{cyc}x^3+3\sum_{cyc}x^2(y+z)-3xyz\overset{AM-GM}{\geq}3\sum_{cyc}x^3+3\sum_{cyc}x^2(y+z)$$

$$=3\sum_{cyc}x^2.\sum_{cyc}x\geq 3\sum_{cyc}yz.\sqrt{3\sum_{cyc}yz}=\sqrt{3(xy+yz+zx)}^3=4\sqrt[4]{432F^6}.$$

So the proof is complete. Equality holds iff ΔABC is equilateral.

1903. If in Δ ABC holds : $\cos A+2\cos B+\cos C=2$, then :

$$9\leq(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq 6+\frac{3R}{2r}$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$\cos A+2\cos B+\cos C=2\Rightarrow 2\cos B+\left(1+\frac{r}{R}-\cos B\right)=2$$
$$\Rightarrow \cos B=1-\frac{r}{R}\rightarrow(1)$$

$$\text{Again, }\cos A+2\cos B+\cos C=2\Rightarrow\cos A+\cos C=2(1-\cos B)$$

$$\Rightarrow 2\sin\frac{B}{2}\cos\frac{C-A}{2}=4\sin^2\frac{B}{2}\Rightarrow\frac{c+a}{b}\sin\frac{B}{2}=2\sin\frac{B}{2}\Rightarrow c+a+b=3b$$

$$\Rightarrow 2s=6R\sin B\Rightarrow s^2=9R^2(1-\cos^2B)\overset{via\,(1)}{=}9R^2\left(1-\left(1-\frac{r}{R}\right)^2\right)$$
$$=9R^2\left(\frac{2r}{R}-\frac{r^2}{R^2}\right)=9R^2\left(\frac{2Rr-r^2}{R^2}\right)\Rightarrow s^2=18Rr-9r^2\rightarrow(2)$$

$$\text{Now, } (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le 6+\frac{3R}{2r} \Leftrightarrow \frac{2s(s^2+4Rr+r^2)}{4Rrs} \le \frac{3R+12r}{2r}$$

$$\Leftrightarrow s^2 \le 3R^2+8Rr-r^2 \overset{\text{via }(2)}{\Leftrightarrow} 18Rr-9r^2 \le 3R^2+8Rr-r^2$$

$$\Leftrightarrow 3R^2-10Rr+8r^2 \ge 0 \Leftrightarrow (R-2r)(3(R-2r)+2r) \ge 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le 6+\frac{3R}{2r} \text{ and finally, } (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \overset{AM-HM}{\ge} 9$$

$$\therefore 9 \le (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le 6+\frac{3R}{2r}\ \forall \Delta ABC \text{ with } \cos A+2\cos B+\cos C=2$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1904. If in ΔABC holds : $\cos A + 2\cos B + \cos C = 2$, then :

$$9 \le (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le 6+\frac{3R}{2r}$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$\cos A+2\cos B+\cos C=2 \Rightarrow 2\cos B+\left(1+\frac{r}{R}-\cos B\right)=2$$
$$\Rightarrow \cos B = 1-\frac{r}{R} \to (1)$$

$$\text{Again, } \cos A+2\cos B+\cos C=2 \Rightarrow \cos A+\cos C = 2(1-\cos B)$$

$$\Rightarrow 2\sin\frac{B}{2}\cos\frac{C-A}{2} = 4\sin^2\frac{B}{2} \Rightarrow \frac{c+a}{b}\sin\frac{B}{2} = 2\sin\frac{B}{2} \Rightarrow c+a+b=3b$$

$$\Rightarrow 2s = 6R\sin B \Rightarrow s^2 = 9R^2(1-\cos^2 B) \overset{\text{via }(1)}{=} 9R^2\left(1-\left(1-\frac{r}{R}\right)^2\right)$$
$$= 9R^2\left(\frac{2r}{R}-\frac{r^2}{R^2}\right) = 9R^2\left(\frac{2Rr-r^2}{R^2}\right) \Rightarrow s^2 = 18Rr-9r^2 \to (2)$$

$$\text{Now, } (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \le 6+\frac{3R}{2r} \Leftrightarrow \frac{2s(s^2+4Rr+r^2)}{4Rrs} \le \frac{3R+12r}{2r}$$

$$\Leftrightarrow s^2 \le 3R^2 + 8Rr - r^2 \overset{\text{via }(2)}{\Leftrightarrow} 18Rr - 9r^2 \le 3R^2 + 8Rr - r^2$$

$$\Leftrightarrow 3R^2 - 10Rr + 8r^2 \ge 0 \Leftrightarrow (R - 2r)(3(R - 2r) + 2r) \ge 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \le 6 + \frac{3R}{2r} \text{ and finally}, (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \overset{AM-HM}{\ge} 9$$

$$\therefore 9 \le (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \le 6 + \frac{3R}{2r}\ \forall \Delta\, ABC \text{ with } \cos A + 2\cos B + \cos C = 2$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1905. In any acute Δ ABC, the following relationship holds :

$$\sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2} \le \frac{3}{2}\max(m_a, m_b, m_c)$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$m_a \overset{?}{\le} \sqrt{\frac{b^2 + c^2}{2}}.\cos\frac{A}{2} \Leftrightarrow 8m_a^2 \overset{?}{\le} (b^2 + c^2).\left(4\cos^2\frac{A}{2}\right)$$

$$\Leftrightarrow 2(2b^2 + 2c^2 - a^2)bc \overset{?}{\le} (b^2 + c^2)((b + c)^2 - a^2)$$

$$\Leftrightarrow 4bc(b^2 + c^2) - 2a^2bc \overset{?}{\le} (b^2 + c^2)^2 + 2bc(b^2 + c^2) - a^2(b^2 + c^2)$$

$$\Leftrightarrow \left((b^2 + c^2)^2 - a^2(b^2 + c^2)\right) - \left(2bc(b^2 + c^2) - 2a^2bc\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow (b^2 + c^2)(b^2 + c^2 - a^2) - 2bc(b^2 + c^2 - a^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow (b^2 + c^2 - a^2)(b^2 + c^2 - 2bc) \overset{?}{\ge} 0 \Leftrightarrow (b^2 + c^2 - a^2)(b - c)^2 \overset{?}{\ge} 0 \to \text{true}$$

$$\because \Delta\, ABC \text{ being acute} \Rightarrow b^2 + c^2 - a^2 > 0 \therefore m_a \le \sqrt{\frac{b^2 + c^2}{2}}.\cos\frac{A}{2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} m_a \sin\frac{A}{2} \overset{CBS}{\le} \sqrt{3}.\sqrt{\sum_{cyc} m_a^2 \sin^2\frac{A}{2}} \le \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{b^2 + c^2}{2}.\cos^2\frac{A}{2}\sin^2\frac{A}{2}\right)}$$

$$= \sqrt{3}.\sqrt{\sum_{cyc}\left(\frac{b^2 + c^2}{2}.\frac{a^2}{16R^2}\right)} = \frac{\sqrt{3}}{2}.\sqrt{\sum_{cyc}\frac{b^2c^2}{4R^2}} = \frac{\sqrt{3}}{2}.\sqrt{\sum_{cyc} h_a^2}$$

$$\therefore \sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2} \text{ and again, } \frac{3}{2}max(m_a, m_b, m_c) \ge \frac{1}{2}\sum_{cyc} m_a \overset{Tereshin}{\ge}$$

$$\frac{1}{2}\sum_{cyc}\frac{b^2+c^2}{4R} = \frac{1}{4R}\cdot\sum_{cyc} a^2 \ge \frac{1}{4R}\cdot\sqrt{3\sum_{cyc} b^2c^2} = \frac{1}{2}\sqrt{3\sum_{cyc}\frac{b^2c^2}{4R^2}} = \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2}$$

$$\therefore \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2} \le \frac{3}{2}max(m_a, m_b, m_c) \text{ and so, } \sum_{cyc} m_a \sin\frac{A}{2} \le \frac{1}{2}\cdot\sqrt{3\sum_{cyc} h_a^2}$$

$$\le \frac{3}{2}max(m_a, m_b, m_c)\ \forall \text{ acute } \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

1906. In any Δ ABC, the following relationship holds :

$$\frac{6\sum_{cyc} m_a^2}{\sqrt{a^2+b^2+c^2}} \ge \sum_{cyc}\left((s_b+s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b+m_c)\sin^2 A\right) \ge \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

Proposed by Tapas Das-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}(s_b+s_c)\sin^2 A + \sum_{cyc}(m_b+m_c)\sin^2 A = \sum_{cyc}(\sin^2 B + \sin^2 C)s_a + \sum_{cyc}(\sin^2 B + \sin^2 C)m_a =$$

$$= \sum_{cyc}\frac{b^2+c^2}{4R^2}\cdot\frac{2bc}{b^2+c^2}m_a + \sum_{cyc}\frac{b^2+c^2}{4R^2}.m_a = \frac{1}{4R^2}\sum_{cyc}(b+c)^2 m_a \le \frac{1}{4R^2}\sum_{cyc} 2(b^2+c^2)m_a$$

$$= \frac{2}{R}\sum_{cyc}\frac{b^2+c^2}{4R}.m_a \overset{Tereshin}{\le} \frac{2}{R}\sum_{cyc} {m_a}^2 \overset{Leibniz}{\le} \frac{6({m_a}^2+{m_b}^2+{m_c}^2)}{\sqrt{a^2+b^2+c^2}}$$

Again, we have

$$\sum_{cyc}(s_b+s_c)\sin^2 A + \sum_{cyc}(m_b+m_c)\sin^2 A = \frac{1}{4R^2}\sum_{cyc}(b+c)^2 m_a \ge$$

$$\ge \frac{1}{4R^2}\sum_{cyc}(a+2(s-a))(b+c)w_a \overset{AM-GM}{\ge} \frac{1}{4R^2}\sum_{cyc} 2\sqrt{2a(s-a)}.2\sqrt{bcs(s-a)} =$$

$$= \frac{\sqrt{2sabc}}{R^2}\sum_{cyc}(s-a) = \frac{2s\sqrt{2s^2Rr}}{R^2} \overset{Gerretsen}{\ge} \frac{2s\sqrt{2(16Rr-5r^2)Rr}}{R^2} = \frac{2F}{R}\sqrt{2\left(16-\frac{5r}{R}\right)} =$$

$$=\frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2+\frac{4}{9}\left(1-\frac{2r}{R}\right)\left(8-\frac{9r}{R}\right)}\overset{Euler}{\geq}\frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_b\overset{?}{\leq}\frac{2c^2+ab}{4}\Leftrightarrow\left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right)\overset{?}{\leq}\frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2\overset{?}{\geq}0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\geq}0\Leftrightarrow(a-b)^2(a+b+c)(a+b-c)\overset{?}{\geq}0$$

$$\to \text{true}\Rightarrow m_am_b\leq\frac{2c^2+ab}{4}\text{ and analogs}\to(1)$$

$$\text{Now, }\sum_{cyc}\left((s_b+s_c)\sin^2A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2A\right)\leq$$

$$\sum_{cyc}\left((m_b+m_c)\sin^2A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2A\right)\left(\because s_a=\frac{2bc}{b^2+c^2}.m_a\overset{A-G}{\leq}m_a\text{ and analogs}\right)$$

$$=\frac{2}{4R^2}\sum_{cyc}a^2(m_b+m_c)\overset{CBS}{\leq}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b+m_c)^2}$$

$$\overset{via\,(1)}{\leq}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b^2+m_c^2)+\sum_{cyc}\left(2a^2.\frac{2a^2+bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}\frac{a^2(b^2+c^2+4a^2)}{4}+\sum_{cyc}\left(\frac{4a^4+2a^2bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{1}{2}\left(4\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+\sum_{cyc}a^2b^2+8Rrs^2\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{9((s^2+4Rr+r^2)^2-16Rrs^2)-64r^2s^2+8Rrs^2}{2}}\overset{?}{\leq}\frac{6\sum_{cyc}m_a^2}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{9\sum_{cyc}a^2}{2\sqrt{\sum_{cyc}a^2}}\Leftrightarrow 9\left((s^2+4Rr+r^2)^2-16Rrs^2\right)-64r^2s^2+8Rrs^2\overset{?}{\leq}162R^4$$

$$\Leftrightarrow 9s^4-(64Rr+46r^2)s^2+9r^2(4R+r)^2-162R^4\boxed{\overset{?}{\underset{(*)}{\leq}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq0$ and $s^2-(m+n)\leq0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq0\Rightarrow s^4-s^2(2m)+m^2-n^2\leq0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3$

$$\leq 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$
$$9s^4-(64Rr+46r^2)s^2+9r^2(4R+r)^2-162R^4$$
$$\leq 9s^4-9s^2(4R^2+20Rr-2r^2)+9r(4R+r)^3 \Leftrightarrow$$
$$(18R^2+58Rr-32r^2)s^2 \boxed{\overset{(**)}{\leq}} 81R^4+288R^3r+144R^2r^2+18Rr^3 \text{ and}$$
$$\text{again, LHS of } (**) \overset{\text{Gerretsen}}{\leq} (18R^2+58Rr-32r^2)(4R^2+4Rr+3r^2) \overset{?}{\leq}$$
$$\text{RHS of } (**) \Leftrightarrow 9t^4-16t^3-14t^2-28t+96 \geq 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(9t^2+20t+30)+12\big) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right) \leq \frac{6\sum_{cyc} m_a^2}{\sqrt{a^2+b^2+c^2}}$$
$$\text{Again,} \sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right) \geq 2\sum_{cyc}\left((h_b+h_c)\sin^2 A\right)$$
$$=\frac{1}{4R^3}\cdot\sum_{cyc} a^2(ca+ab)=\frac{1}{4R^3}\cdot\sum_{cyc}\left(a^2\left(\sum_{cyc} ab-bc\right)\right)$$
$$=\frac{(s^2+4Rr+r^2)(s^2-4Rr-r^2)-4Rrs^2}{2R^3} \overset{?}{\geq} \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right)-8\left(\frac{r}{R}\right)^2}$$
$$=\frac{6rs}{R^2}\cdot\sqrt{10Rr-8r^2}$$
$$\Leftrightarrow \frac{\left((s^2+4Rr+r^2)(s^2-4Rr-r^2)-4Rrs^2\right)^2}{4R^6} \overset{?}{\geq} \frac{36r^2s^2(10Rr-8r^2)}{R^4}$$
$$\Leftrightarrow s^8-8Rrs^6-r^2(16R^2+16Rr+2r^2)s^4-Rr^3(1312R^2-1216Rr-8r^2)s^2$$
$$+r^4(4R+r)^4 \boxed{\underset{(***)}{\overset{?}{\geq}}} 0 \text{ and } \because (s^2-16Rr+5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (***), \text{it suffices to prove : LHS of } (***) \geq (s^2-16Rr+5r^2)^4$$
$$\Leftrightarrow (14R-5r)s^6-r(388R^2-236Rr+38r^2)s^4$$
$$+r^2(3768R^3-3536R^2r+1202Rr^2-125r^3)$$
$$-r^3(16320R^4-20544R^3r+9576R^2r^2-2004Rr^3+156r^4) \boxed{\overset{(****)}{\geq}} 0 \text{ and}$$
$$\because (14R-5r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (****), \text{it suffices}$$
$$\text{to prove : LHS of } (****) \geq (14R-5r)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (284R^2-214Rr+37r^2)s^4-r(6984R^3-7024R^2r+2248Rr^2-250r^3)$$
$$+r^2(41024R^4-53696R^3r+26424R^2r^2-5746Rr^3+469r^4) \boxed{\overset{(*****)}{\geq}} 0 \text{ and}$$
$$\because (284R^2-214Rr+37r^2)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (****), \text{it suffices to prove : LHS of } (*****) \geq$$
$$(284R^2-214Rr+37r^2)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (526R^3 - 666R^2r + 269Rr^2 - 30r^3)s^2 \overset{(******)}{\boxed{\geq}} r(7920R^4 - 11632R^3r + 6097R^2r^2 - 1381Rr^3 + 114r^4)$$

$$\text{Finally, } (526R^3 - 666R^2r + 269Rr^2 - 30r^3)s^2 \overset{\text{Gerretsen}}{\geq} (526R^3 - 666R^2r + 269Rr^2 - 30r^3)(16Rr - 5r^2) \overset{?}{\geq} r(7920R^4 - 11632R^3r + 6097R^2r^2 - 1381Rr^3 + 114r^4)$$

$$\Leftrightarrow 496t^4 - 1654t^3 + 1537t^2 - 444t + 36 \geq 0$$

$$\Leftrightarrow (t-2)\left((t-2)(496t^2 + 330t + 873) + 1728\right) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (******) \Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \text{ is true}$$

$$\therefore \sum_{cyc}\left((s_b + s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b + m_c)\sin^2 A\right) \geq \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

$$\text{and so, } \frac{6\sum_{cyc} m_a^2}{\sqrt{a^2 + b^2 + c^2}} \geq \sum_{cyc}\left((s_b + s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b + m_c)\sin^2 A\right) \geq \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1907. In any ΔABC, the following relationship holds :

$$\frac{6\sum_{cyc} m_a^2}{\sqrt{a^2 + b^2 + c^2}} \geq \sum_{cyc}\left((s_b + s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b + m_c)\sin^2 A\right) \geq \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right) - 8\left(\frac{r}{R}\right)^2}$$

Proposed by Tapas Das-India

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}(s_b + s_c)\sin^2 A + \sum_{cyc}(m_b + m_c)\sin^2 A = \sum_{cyc}(\sin^2 B + \sin^2 C)s_a + \sum_{cyc}(\sin^2 B + \sin^2 C)m_a =$$

$$= \sum_{cyc}\frac{b^2 + c^2}{4R^2}\cdot\frac{2bc}{b^2 + c^2}m_a + \sum_{cyc}\frac{b^2 + c^2}{4R^2}\cdot m_a = \frac{1}{4R^2}\sum_{cyc}(b + c)^2 m_a \leq \frac{1}{4R^2}\sum_{cyc}2(b^2 + c^2)m_a$$

$$= \frac{2}{R}\sum_{cyc}\frac{b^2 + c^2}{4R}\cdot m_a \overset{\text{Tereshin}}{\leq} \frac{2}{R}\sum_{cyc}{m_a}^2 \overset{\text{Leibniz}}{\leq} \frac{6({m_a}^2 + {m_b}^2 + {m_c}^2)}{\sqrt{a^2 + b^2 + c^2}}$$

Again, we have

$$\sum_{cyc}(s_b + s_c)\sin^2 A + \sum_{cyc}(m_b + m_c)\sin^2 A = \frac{1}{4R^2}\sum_{cyc}(b + c)^2 m_a \geq$$

$$\geq \frac{1}{4R^2}\sum_{cyc}(a+2(s-a))(b+c)w_a \overset{AM-GM}{\geq} \frac{1}{4R^2}\sum_{cyc}2\sqrt{2a(s-a)}.2\sqrt{bcs(s-a)}=$$

$$=\frac{\sqrt{2sabc}}{R^2}\sum_{cyc}(s-a)=\frac{2s\sqrt{2s^2Rr}}{R^2}\overset{Gerresten}{\geq}\frac{2s\sqrt{2(16Rr-5r^2)Rr}}{R^2}=\frac{2F}{R}\sqrt{2\left(16-\frac{5r}{R}\right)}=$$

$$=\frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2+\frac{4}{9}\left(1-\frac{2r}{R}\right)\left(8-\frac{9r}{R}\right)}\overset{Euler}{\geq}\frac{6F}{R}\sqrt{10\frac{r}{R}-8\left(\frac{r}{R}\right)^2}.$$

which completes the proof. Equality holds iff ΔABC is equilateral.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_b \overset{?}{\leq} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right)\overset{?}{\leq}\frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2\overset{?}{\geq}0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\geq}0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \Rightarrow m_am_b \leq \frac{2c^2+ab}{4} \text{ and analogs} \rightarrow (1)$$

$$\text{Now, }\sum_{cyc}\left((s_b+s_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\leq$$

$$\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2 A\right)\left(\because s_a=\frac{2bc}{b^2+c^2}.m_a\overset{A-G}{\leq}m_a \text{ and analogs}\right)$$

$$=\frac{2}{4R^2}\sum_{cyc}a^2(m_b+m_c)\overset{CBS}{\leq}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b+m_c)^2}$$

$$\overset{via\ (1)}{\leq}\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}a^2(m_b^2+m_c^2)+\sum_{cyc}\left(2a^2.\frac{2a^2+bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\sum_{cyc}\frac{a^2(b^2+c^2+4a^2)}{4}+\sum_{cyc}\left(\frac{4a^4+2a^2bc}{4}\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{1}{2}\left(4\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+\sum_{cyc}a^2b^2+8Rrs^2\right)}$$

$$=\frac{1}{2R^2}.\sqrt{\sum_{cyc}a^2}.\sqrt{\frac{9((s^2+4Rr+r^2)^2-16Rrs^2)-64r^2s^2+8Rrs^2}{2}}\overset{?}{\leq}\frac{6\sum_{cyc}m_a^2}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{9\sum_{cyc}a^2}{2\sqrt{\sum_{cyc}a^2}}\Leftrightarrow 9\left(\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)-64r^2s^2+8Rrs^2\overset{?}{\leq}162R^4$$

$$\Leftrightarrow 9s^4-\left(64Rr+46r^2\right)s^2+9r^2(4R+r)^2-162R^4\boxed{\overset{?}{\underset{(*)}{\leq}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\leq 0\therefore$ in order to prove $(*)$, it suffices to prove:

$$9s^4-\left(64Rr+46r^2\right)s^2+9r^2(4R+r)^2-162R^4\leq 9s^4-9s^2\left(4R^2+20Rr-2r^2\right)+9r(4R+r)^3\Leftrightarrow$$

$$\left(18R^2+58Rr-32r^2\right)s^2\boxed{\overset{(**)}{\leq}}81R^4+288R^3r+144R^2r^2+18Rr^3 \text{ and}$$

again, LHS of $(**)\overset{\text{Gerretsen}}{\leq}\left(18R^2+58Rr-32r^2\right)\left(4R^2+4Rr+3r^2\right)\overset{?}{\leq}$ RHS of $(**)\Leftrightarrow 9t^4-16t^3-14t^2-28t+96\geq 0\ \left(t=\frac{R}{r}\right)$

$$\Leftrightarrow (t-2)\left((t-2)\left(9t^2+20t+30\right)+12\right)\geq 0\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (**)\Rightarrow (*) \text{ is true}$$

$$\therefore\sum_{cyc}\left((s_b+s_c)\sin^2A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2A\right)\leq\frac{6\sum_{cyc}m_a^2}{\sqrt{a^2+b^2+c^2}}$$

Again, $\sum_{cyc}\left((s_b+s_c)\sin^2A\right)+\sum_{cyc}\left((m_b+m_c)\sin^2A\right)\geq 2\sum_{cyc}\left((h_b+h_c)\sin^2A\right)$

$$=\frac{1}{4R^3}\cdot\sum_{cyc}a^2(ca+ab)=\frac{1}{4R^3}\cdot\sum_{cyc}\left(a^2\left(\sum_{cyc}ab-bc\right)\right)$$

$$=\frac{\left(s^2+4Rr+r^2\right)\left(s^2-4Rr-r^2\right)-4Rrs^2}{2R^3}\overset{?}{\geq}\frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right)-8\left(\frac{r}{R}\right)^2}$$

$$=\frac{6rs}{R^2}\cdot\sqrt{10Rr-8r^2}$$

$$\Leftrightarrow\frac{\left(\left(s^2+4Rr+r^2\right)\left(s^2-4Rr-r^2\right)-4Rrs^2\right)^2}{4R^6}\overset{?}{\geq}\frac{36r^2s^2\left(10Rr-8r^2\right)}{R^4}$$

$$\Leftrightarrow s^8-8Rrs^6-r^2\left(16R^2+16Rr+2r^2\right)s^4-Rr^3\left(1312R^2-1216Rr-8r^2\right)s^2$$

$+r^4(4R+r)^4\boxed{\overset{?}{\underset{(***)}{\geq}}}0$ *and* $\because\left(s^2-16Rr+5r^2\right)^4\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(***)$, it suffices to prove: LHS of $(***)\geq\left(s^2-16Rr+5r^2\right)^4$

$$\Leftrightarrow(14R-5r)s^6-r\left(388R^2-236Rr+38r^2\right)s^4$$
$$+r^2\left(3768R^3-3536R^2r+1202Rr^2-125r^3\right)$$
$$-r^3\left(16320R^4-20544R^3r+9576R^2r^2-2004Rr^3+156r^4\right)\boxed{\overset{(****)}{\geq}}0 \text{ and}$$

$$\because (14R-5r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (****), \text{it suffices}$$
$$\text{to prove : LHS of } (****) \geq (14R-5r)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (284R^2-214Rr+37r^2)s^4 - r(6984R^3-7024R^2r+2248Rr^2-250r^3)$$
$$+r^2(41024R^4-53696R^3r+26424R^2r^2-5746Rr^3+469r^4) \overset{(*****)}{\geq} 0 \text{ and}$$
$$\because (284R^2-214Rr+37r^2)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$
$$\text{to prove } (****), \text{it suffices to prove : LHS of } (*****) \geq$$
$$(284R^2-214Rr+37r^2)(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow (526R^3-666R^2r+269Rr^2-30r^3)s^2 \overset{(******)}{\geq}$$
$$r(7920R^4-11632R^3r+6097R^2r^2-1381Rr^3+114r^4)$$
$$\text{Finally}, (526R^3-666R^2r+269Rr^2-30r^3)s^2 \overset{\text{Gerretsen}}{\geq}$$
$$(526R^3-666R^2r+269Rr^2-30r^3)(16Rr-5r^2) \overset{?}{\geq}$$
$$r(7920R^4-11632R^3r+6097R^2r^2-1381Rr^3+114r^4)$$
$$\Leftrightarrow 496t^4-1654t^3+1537t^2-444t+36 \geq 0$$
$$\Leftrightarrow (t-2)\left((t-2)(496t^2+330t+873)+1728\right) \geq 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (******) \Rightarrow (*****) \Rightarrow (****) \Rightarrow (***) \text{ is true}$$
$$\therefore \sum_{cyc}\left((s_b+s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b+m_c)\sin^2 A\right) \geq \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right)-8\left(\frac{r}{R}\right)^2}$$
$$\text{and so}, \frac{6\sum_{cyc} m_a^2}{\sqrt{a^2+b^2+c^2}} \geq \sum_{cyc}\left((s_b+s_c)\sin^2 A\right) + \sum_{cyc}\left((m_b+m_c)\sin^2 A\right)$$
$$\geq \frac{6F}{R}\cdot\sqrt{10\left(\frac{r}{R}\right)-8\left(\frac{r}{R}\right)^2}\ \forall \Delta ABC, "=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

1908. In ΔABC the following relationship holds:

$$\frac{(a^2+b^2)^2}{\sin^2 A} + \frac{(b^2+c^2)^2}{\sin^2 B} + \frac{(c^2+a^2)^2}{\sin^2 C} \geq 9\cdot(4r)^4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{(a^2+b^2)^2}{\sin^2 A} + \frac{(b^2+c^2)^2}{\sin^2 B} + \frac{(c^2+a^2)^2}{\sin^2 C} = \sum \frac{(a^2+b^2)^2}{\sin^2 A} =$$
$$= 4R^2 \sum \frac{(a^2+b^2)^2}{a^2} \overset{Bergstrom}{\geq} \frac{4R^2(2a^2+2b^2+2c^2)^2}{a^2+b^2+c^2} =$$

$$= 16R^2(a^2+b^2+c^2) \overset{Neuberg}{\geq} 16R^2.36r^2 \overset{Euler}{\geq} 16(2r)^2 36r^2 = 9(4r)^4$$

$$Equality\ holds\ for\ a=b=c$$

1909. **If in $\Delta ABC, abc = 1$ then:**

$$\sum \frac{b^2\cot\frac{C}{2}+c^2\cot\frac{B}{2}}{b+c} \geq 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a\geq b\geq c\ then\ \cot\frac{A}{2}\leq\cot\frac{B}{2}\leq\cot\frac{C}{2}$$

$$\frac{b^2\cot\frac{C}{2}+c^2\cot\frac{B}{2}}{b+c} \overset{Chebyshev}{\geq} \frac{\frac{1}{2}(b^2+c^2)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{b+c} \overset{CBS}{\geq}$$

$$\geq \frac{\frac{1}{2}\frac{(b+c)^2}{2}\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{b+c} = \frac{\frac{(b+c)^2}{4}\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{b+c} =$$

$$= \frac{b+c}{4}\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right) \overset{AM-GM}{\geq} \sqrt{bc}\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\ \ (1)$$

$$\sum \frac{b^2\cot\frac{C}{2}+c^2\cot\frac{B}{2}}{b+c} \overset{(1)}{\geq} \sum\sqrt{bc}\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}} \overset{AM-GM}{\geq}$$

$$\geq 3\left(abc\prod\cos\frac{A}{2}\right)^{\frac{1}{3}} \overset{abc=1}{=} 3\left(\frac{s}{r}\right)^{\frac{1}{3}} \overset{Mitrinovic}{\geq} 3\sqrt{3}$$

Equality holds for $a=b=c=1$.

1910. **In ΔABC the following relationship holds:**

$$\sum \frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)} \geq 3\sqrt{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ then\ \cot\frac{A}{2} \le \cot\frac{B}{2} \le \cot\frac{C}{2}$$

$$\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)} \overset{Chebyshev}{\ge} \frac{\frac{1}{2}(b^3+c^3)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{\frac{a}{2R}\cdot 2R(b^2+c^2)} \overset{Chebyshev}{\ge}$$

$$\ge \frac{\frac{1}{4}(b+c)(b^2+c^2)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)}{\frac{a}{2R}\cdot 2R(b^2+c^2)} = \frac{\frac{1}{4}\left((b+c)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)\right)}{a} \overset{AM-GM}{\ge}$$

$$\ge \frac{\sqrt{bc}}{a}\sqrt{\cot\frac{C}{2}\cot\frac{B}{2}}\ \ (1)$$

$$\sum\frac{b^3\cot\frac{C}{2}+c^3\cot\frac{B}{2}}{\sin A\left(b^2\frac{c}{\sin C}+c^2\frac{b}{\sin B}\right)} \overset{(1)}{\ge} \sum\frac{\sqrt{bc}}{a}\sqrt{\cot\frac{C}{2}\cot\frac{B}{2}} \overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\prod\cot\frac{A}{2}} = 3\sqrt[3]{\frac{S}{r}} \overset{Mitrinovic}{\ge} 3\left(3\sqrt{3}\right)^{\frac{1}{3}} = 3\sqrt{3}$$

$$Equality\ holds\ for\ \ a=b=c$$

1911. In ΔABC the following relationship holds:

$$\sum\frac{(a+b)^4}{\sin\frac{A}{2}\left(1+\sin\frac{B}{2}\right)} \ge 36(4r)^4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum\sin\frac{A}{2} \overset{Jensen}{\le} 3\sin\frac{\pi}{6} = \frac{3}{2}\ (1)$$

$$\left(\sum\frac{(a+b)^4}{\sin\frac{A}{2}\left(1+\sin\frac{B}{2}\right)}\right)\left(\sum\sin\frac{A}{2}\right)\left(\sum 1+\sin\frac{B}{2}\right)(1+1+1)\overset{Holder}{\geq}(2a+2b+2c)^4$$

$$\sum\frac{(a+b)^4}{\sin\frac{A}{2}\left(1+\sin\frac{B}{2}\right)}\geq\frac{16(2s)^4}{\left(\sum\sin\frac{A}{2}\right)\left(\sum 1+\sin\frac{B}{2}\right)(1+1+1)}\overset{(1)\&Mitrinovic}{\geq}$$

$$\geq\frac{256\left(3\sqrt{3}r\right)^4}{\frac{3}{2}\left(3+\frac{3}{2}\right).3}=\frac{(4.9)(81)(4r)^4}{81}=36(4r)^4$$

Equality holds iff ΔABC is equilateral

1912. In ΔABC the following relationship holds:

$$\frac{a(b+c)}{9R^2-a^2}+\frac{b(c+a)}{9R^2-b^2}+\frac{c(a+b)}{9R^2-c^2}<4$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

We know that in ΔABC:

$$b+c>a \text{ or } 2(b+c)>a+b+c \text{ or } 2(b+c)>2s \text{ or } b+c>s$$

$$\text{similarly}, (a+b)>s \text{ and } (c+a)>s$$

$$9R^2-a^2\overset{Leibniz}{\geq}a^2+b^2+c^2-a^2=b^2+c^2\overset{CBS}{\geq}\frac{(b+c)^2}{2}\quad(1)$$

$$\text{Similarly}:\ 9R^2-b^2\geq\frac{(a+c)^2}{2}\quad(2),\ 9R^2-c^2\geq\frac{(a+b)^2}{2}\quad(3)$$

$$\frac{a(b+c)}{9R^2-a^2}+\frac{b(c+a)}{9R^2-b^2}+\frac{c(a+b)}{9R^2-c^2}\overset{(1),(2)\&(3)}{\leq}\frac{a(b+c)}{\frac{(b+c)^2}{2}}+\frac{b(c+a)}{\frac{(a+c)^2}{2}}+\frac{c(a+b)}{\frac{(a+b)^2}{2}}=$$

$$=\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}<\frac{2a}{s}+\frac{2b}{s}+\frac{2c}{s}=\frac{2(a+b+c)}{s}=2\cdot\frac{2s}{s}=4$$

Equality holds for: $a = b = c$.

1913. In ΔABC the following relationship holds:

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B} = \frac{4}{3} \textit{ then } GI^2 \geq 2r^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sin A = \frac{a}{2R}, \sin B = \frac{b}{2R}, \sin C = \frac{c}{2R}$$

$$\textit{we know that } GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$$

$$(\textit{Reference}: \textit{Useful Identities and inequalities in Geometry},$$
$$\textit{contributor}: \textit{Samer Seraj, Andrew Krik, Reda Afara, Luis Gonzales})$$

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B} = \frac{\sum a^2}{\sum ab} = \frac{2(s^2 - r^2 - 4Rr)}{s^2 + r^2 + 4Rr}$$

$$\frac{\sum \sin^2 A}{\sum \sin A \sin B} = \frac{4}{3} \textit{ or } \frac{2(s^2 - r^2 - 4Rr)}{s^2 + r^2 + 4Rr} = \frac{4}{3} \textit{ or}$$
$$6(s^2 - r^2 - 4Rr) = 4(s^2 + r^2 + 4Rr) \textit{ or}$$
$$2s^2 - 10r^2 - 40Rr = 0 \textit{ or}$$
$$s^2 = 5r^2 + 20Rr \quad (1)$$

$$\textit{We need to show } GI^2 \geq 2r^2 \textit{ or } \frac{1}{9}(s^2 + 5r^2 - 16Rr) \geq 2r^2 \textit{ or}$$

$$(s^2 + 5r^2 - 16Rr) \geq 18r^2 \textit{ or } (5r^2 + 20Rr + 5r^2 - 16Rr) \overset{(1)}{\geq} 18r^2 \textit{ or}$$

$$4Rr \geq 8r^2 \textit{ or } R \geq 2r \textit{ True(Euler)}$$

1914. In any Δ ABC, the following relationship holds:

$$\sum_{cyc} \frac{\tan^2 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \geq \frac{4R + r}{p}\left(\frac{1}{2}\left(\frac{4R + r}{p}\right)^2 - 1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\sum_{cyc}\frac{\tan^4\frac{A}{2}}{\tan^2\frac{A}{2}\tan\frac{B}{2}+\tan^2\frac{A}{2}\tan\frac{C}{2}}\overset{\text{Bergstrom}}{\geq}$$

$$\geq\frac{\left(\sum_{cyc}\tan^2\frac{A}{2}\right)^2}{\sum_{cyc}\tan^2\frac{A}{2}\tan\frac{B}{2}+\sum_{cyc}\tan^2\frac{A}{2}\tan\frac{C}{2}}=$$

$$=\frac{\frac{1}{p^4}\left((4R+r)^2-2p^2\right)^2}{\left(\sum_{cyc}\tan\frac{A}{2}\right)\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)-3\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}$$

$$=\frac{\left((4R+r)^2-2p^2\right)^2}{p^4\left(\frac{4R+r}{p}-\frac{3r}{p}\right)}\overset{?}{\geq}\frac{4R+r}{p}\left(\frac{(4R+r)^2-2p^2}{2p^2}\right)$$

$$\Leftrightarrow 2(4R+r)^2-4p^2\overset{?}{\geq}(4R-2r)(4R+r)\Leftrightarrow p^2\overset{?}{\leq}4R^2+5Rr+r^2$$

$$\Leftrightarrow\left(p^2-(4R^2+4Rr+3r^2)\right)-r(R-2r)\overset{?}{\leq}0\rightarrow\text{true}$$

$$\because p^2-(4R^2+4Rr+3r^2)\overset{\text{Gerretsen}}{\leq}0\ \textit{and}\ -r(R-2r)\overset{\text{Euler}}{\leq}0$$

$$\therefore\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}\geq\frac{4R+r}{p}\left(\frac{1}{2}\left(\frac{4R+r}{p}\right)^2-1\right)$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1915. In any acute triangle ABC, the following relationship holds :

$$m_a\sqrt{\cot A}+m_b\sqrt{\cot B}+m_c\sqrt{\cot C}>6r$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{\cot A}=x,\sqrt{\cot B}=y,\sqrt{\cot C}=z\ \therefore\ \sum_{cyc}\sqrt{\cot A}\overset{?}{>}2\Leftrightarrow\sum_{cyc}x\overset{?}{>}2$

$$\Leftrightarrow\left(\sum_{cyc} x\right)^2 \overset{?}{>} 4 = 4.\sqrt{\sum_{cyc} x^2y^2}\left(\because \sum_{cyc}\cot A\cot B = \sum_{cyc} x^2y^2 = 1\right)$$

$$\Leftrightarrow\left(\sum_{cyc} x\right)^4 \boxed{\overset{?}{>}} 16\sum_{cyc} x^2y^2 \to (2)$$

Assigning $y+z = X, z+x = Y, x+y = Z \Rightarrow X+Y-Z = 2z > 0, Y+Z-X = 2x > 0$ and $Z+X-Y = 2y > 0 \Rightarrow X+Y > Z, Y+Z > X, Z+X > Y \Rightarrow X, Y, Z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (i) \Rightarrow x = s - X, y = s - Y, z = s - Z$ and such substitutions $\Rightarrow \sum_{cyc} xy = \sum_{cyc}(s-X)(s-Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (ii)$

$$\text{and } \sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (i) and (ii)}}{=}$$

$$(4Rr+r^2)^2 - 2\left(\prod_{cyc}(s-X)\right).s = (4Rr+r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} x^2y^2 = r^2((4R+r)^2 - 2s^2) \to (iii) \therefore \text{via (i) and (iii), (2)} \Leftrightarrow$$

$$s^4 \overset{?}{>} 16r^2((4R+r)^2 - 2s^2) \Leftrightarrow s^4 + 32r^2s^2 \boxed{\underset{(*)}{\underbrace{\overset{?}{>}}}} 16r^2(4R+r)^2$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\geq} (16Rr + 27r^2)s^2 \overset{\text{Gerretsen}}{\geq} (16Rr+27r^2)(16Rr - 5r^2) \overset{?}{>} 16r^2(4R+r)^2 \Leftrightarrow 76r(R-2r) + 148Rr + r^2 \overset{?}{>} 0 \to$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (2) \Rightarrow (*)$ is true $\therefore \sum_{cyc}\sqrt{\cot A} > 2$ and WLOG assuming $a \geq b \geq c \Rightarrow m_a \leq m_b \leq m_c$

and $\sqrt{\cot A} \leq \sqrt{\cot B} \leq \sqrt{\cot C} \therefore m_a\sqrt{\cot A} + m_b\sqrt{\cot B} + m_c\sqrt{\cot C} \overset{\text{Chebyshev}}{\geq}$

$$\frac{1}{3}\left(\sum_{cyc} m_a\right)\left(\sum_{cyc}\sqrt{\cot A}\right) \overset{\text{Chebyshev}}{\geq} \frac{1}{3}\left(\sum_{cyc} h_a\right)\left(\sum_{cyc}\sqrt{\cot A}\right)$$

$$= \frac{1}{3}\left(2rs\sum_{cyc}\frac{1}{a}\right)\left(\sum_{cyc}\sqrt{\cot A}\right) \overset{\text{Bergstrom}}{\geq} \frac{1}{3}\left(2rs.\frac{9}{2s}\right)\left(\sum_{cyc}\sqrt{\cot A}\right) \overset{\sum_{cyc}\sqrt{\cot A} > 2}{>} 6r$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)

1916. In ΔABC the following relationship holds:

$$\frac{\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)^3}{\left(\sin^2 A+\cos^2\frac{A}{2}\right)^2}+\frac{\left(\sec^2\frac{B}{2}+\csc^2\frac{B}{2}\right)^3}{\left(\sin^2 B+\cos^2\frac{B}{2}\right)^2}+\frac{\left(\sec^2\frac{C}{2}+\csc^2\frac{C}{2}\right)^3}{\left(\sin^2 C+\cos^2\frac{C}{2}\right)^2}\ge\frac{16384}{81}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)=\frac{1}{\cos^2\frac{A}{2}}+\frac{1}{\sin^2\frac{A}{2}}=\frac{\left(\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right)}{\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=$$

$$=\frac{1}{\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=\frac{4}{4\cos^2\frac{A}{2}\sin^2\frac{A}{2}}=\frac{4}{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)^2}=\frac{4}{\sin^2 A}\ (1)$$

$$\sum\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)\overset{(1)}{=}4\sum\frac{1}{\sin^2 A}\overset{Bergstom}{\ge}4\frac{(1+1+1)^2}{\sum\sin^2 A}=$$

$$=\frac{36}{\sum\frac{a^2}{4R^2}}=\frac{144R^2}{\sum a^2}\overset{Leibniz}{\ge}\frac{144R^2}{9R^2}=16\ (2)$$

$$\sum\cos^2\frac{A}{2}=2+\frac{r}{2R}\overset{Euler}{\le}2+\frac{1}{4}=\frac{9}{4}\ (3)\ and$$

$$\sum\sin^2 A=\frac{a^2+b^2+c^2}{4R^2}\overset{Leibniz}{\le}\frac{9R^2}{4R^2}=\frac{9}{4}\ (4)$$

$$\frac{\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)^3}{\left(\sin^2 A+\cos^2\frac{A}{2}\right)^2}+\frac{\left(\sec^2\frac{B}{2}+\csc^2\frac{B}{2}\right)^3}{\left(\sin^2 B+\cos^2\frac{B}{2}\right)^2}+\frac{\left(\sec^2\frac{C}{2}+\csc^2\frac{C}{2}\right)^3}{\left(\sin^2 C+\cos^2\frac{C}{2}\right)^2}\overset{Radon}{\ge}$$

$$\ge\frac{\left(\sum\left(\sec^2\frac{A}{2}+\csc^2\frac{A}{2}\right)\right)^3}{\left(\sum\sin^2 A+\sum\cos^2\frac{A}{2}\right)^2}\overset{(2),(3),(4)}{\ge}\frac{(16)^3}{\left(\frac{9}{4}+\frac{9}{4}\right)^2}=\frac{256\times16\times4}{81}=\frac{16384}{81}$$

Equality holds iff ΔABC is equilateral.

1917. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2-c^2}{sin^2(A)+sin^2(B)}+\frac{m_a^2+m_c^2-b^2}{sin^2(A)+sin^2(C)}+\frac{m_b^2+m_c^2-a^2}{sin^2(B)+sin^2(C)}=3R^2$$

Proposed by Ertan Yildirim-Turkiye

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sin(A)=\frac{a}{2R}\ ;\ \sin(B)=\frac{b}{2R}\ ;\ \sin(C)=\frac{c}{2R}\quad (1)$$

$$Also\rightarrow\begin{cases}4m_a^2=2(b^2+c^2)-a^2\\4m_b^2=2(a^2+c^2)-b^2\quad (2)\\4m_c^2=2(b^2+a^2)-c^2\end{cases}$$

$$Let's\ write\ the\ expression\ \ (1)\ \ instead$$

$$R^2\left(\frac{4(m_a^2+m_b^2-c^2)}{b^2+a^2}+\frac{4(m_a^2+m_c^2-b^2)}{c^2+a^2}+\frac{4(m_c^2+m_b^2-a^2)}{b^2+c^2}\right)=$$

$$R^2\left(\frac{4m_a^2+4m_b^2-4c^2}{b^2+a^2}+\frac{4m_a^2+4m_c^2-4b^2}{c^2+a^2}+\frac{4m_c^2+4m_b^2-4a^2}{b^2+c^2}\right)\overset{(2)}{=}$$

$$R^2\Big(\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2-4c^2}{b^2+a^2}$$

$$+\frac{2(b^2+c^2)-a^2+2(a^2+b^2)-c^2-4b^2}{c^2+a^2}+$$

$$+\frac{2(a^2+c^2)-b^2+2(a^2+b^2)-c^2-4a^2}{b^2+c^2}\Big)=$$

$$R^2\left(\frac{b^2+a^2}{b^2+a^2}+\frac{c^2+a^2}{c^2+a^2}+\frac{b^2+c^2}{b^2+c^2}\right)=3R^2\ (proved)$$

1918. If in $\Delta ABC,\ A:B:C=1:3:6$ then find : $\frac{s}{r}$

Proposed by Samir Cabiyev -Azerbaijan

Solution by proposer

$$A:B:C=1:3:6\Longrightarrow B=3A,C=6A$$

$$A+B+C=180^\circ\Longrightarrow A+3A+6A=180^\circ\Longrightarrow 10A=180^\circ,$$

$$A=18^\circ,B=54^\circ,C=108^\circ$$

$$F=\frac{1}{2}bcsinA\Longrightarrow rs=\frac{1}{2}bcsinA\Longrightarrow r=\frac{bcsinA}{2s}$$

$$\frac{r}{s}=\frac{bcsinA}{2s^2}=\frac{2RsinB\cdot 2RsinC\cdot sinA}{2\cdot\left(\frac{a+b+c}{2}\right)^2}=$$

$$=\frac{2R^2sinAsinBsinC}{(RsinA+RsinB+RsinC)^2}=\frac{2sinAsinBsinC}{(sinA+sinB+sinC)^2}$$

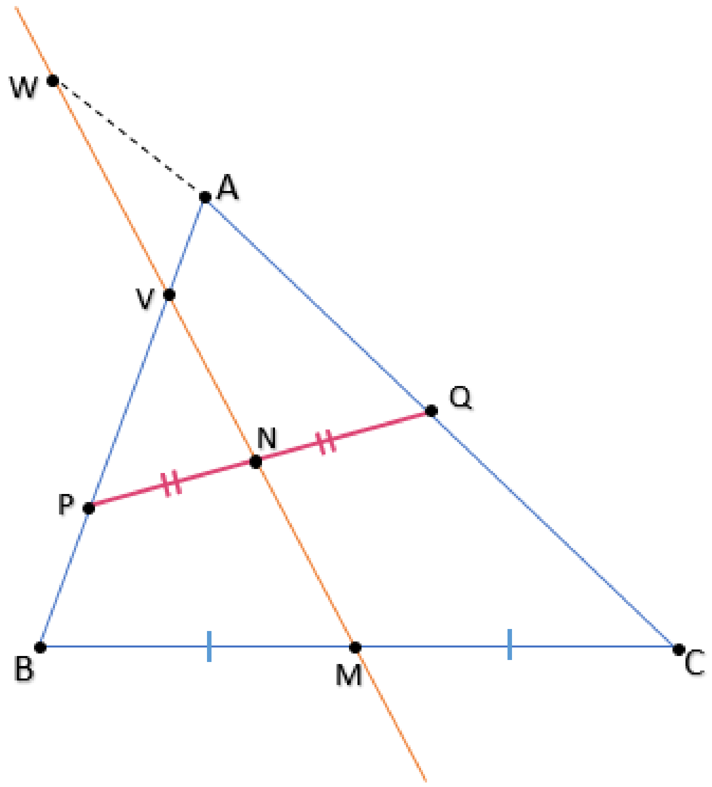
$$\frac{s}{r}=\frac{(sinA+sinB+sinC)^2}{2sinAsinBsinC}=\frac{(sin18^\circ+sin54^\circ+sin108^\circ)^2}{2sin18^\circ sin54^\circ sin108^\circ}$$

$$sin18^\circ=\frac{\sqrt{5}-1}{4},\quad sin54^\circ=\frac{\sqrt{5}+1}{4},\quad sin108^\circ=\frac{\sqrt{5+\sqrt{5}}}{\sqrt{8}}$$

Then:

$$\frac{s}{r}=\frac{15+\sqrt{5}+2\sqrt{50+10\sqrt{5}}}{\sqrt{10+2\sqrt{5}}}$$

1919.



Prove that:

$$\frac{VP-AV}{AQ}=\frac{PB}{QC}\left(=\frac{AV}{AW}\text{ by Dao Thanh Oai}\right)$$

Proposed by Thanas Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \Delta APQ\ by\ theorem\ Menelaus:$$

$$\frac{WA}{WQ}\cdot\frac{QN}{NP}\cdot\frac{PV}{VA}=1\Rightarrow\frac{PV}{AV}=\frac{WA}{WQ}\Rightarrow\frac{PV}{AV}-1=\frac{WQ}{WA}-1\Rightarrow\frac{PV-AV}{AV}=\frac{WQ-WA}{WA}\Rightarrow$$

$$\Rightarrow\frac{PV-AV}{AV}=\frac{AQ}{WA}\Rightarrow\frac{PV-AV}{AQ}=\frac{AV}{AW}$$

$$Now\ let\ us\ prove\ that:$$

$$\frac{PB}{QC}=\frac{AV}{AW}$$

$$For\ this\ also\ by\ Menelaus\ theorem\ in\ the\ triangle\ ABC$$

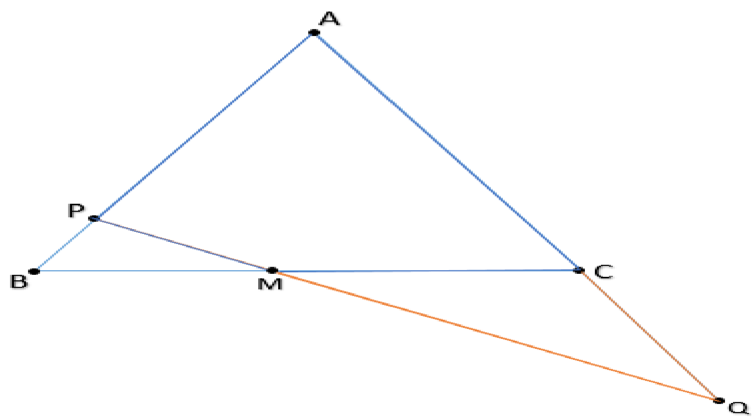
$$\frac{WA}{WC}\cdot\frac{CM}{MB}\cdot\frac{BV}{VA}=1\Rightarrow\frac{WC}{WA}=\frac{BV}{AV}\Rightarrow\frac{AV}{AW}=\frac{BV}{WC}=\frac{BP+PV}{WA+AC}\Rightarrow\frac{AV}{WA}=\frac{BP+PV}{WA+AC}\Rightarrow$$

$$\Rightarrow\frac{PB+PV}{AV}=\frac{WA+AC}{WA}\Rightarrow\frac{BP}{VA}+\frac{PV}{VA}=\frac{AC}{WA}+1\Rightarrow\frac{BP}{VA}=\frac{AC}{WA}+1-\frac{WQ}{WA}\Rightarrow$$

$$\Rightarrow\frac{BP}{VA}=\frac{AC-WQ}{WA}+1\Rightarrow\frac{AC+WA-WQ}{WA}$$

$$\frac{BP}{VA}=\frac{AQ+QC+WA-WA-AQ}{WA}=\frac{QC}{WA}\Rightarrow\frac{PB}{QC}=\frac{AV}{AW}\ (Qed)$$

1920.



$AB=BC=CA,BM=MC,[APMC]=S_1,[CMQ]=S_2,\frac{AP}{PB}=x>1,$ $\frac{S_1}{S_2}=y.\ Calculate\ y=f(x).\ If\ [APMC]=[CQM]\ then\ find\ \frac{AP}{PB}.$

Proposed by Thanasis Gakopoulos-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{S_1+S_2}{S_2}=\frac{\frac{1}{2}AQ\cdot PQ\cdot\sin Q}{\frac{1}{2}CQ\cdot QM\cdot\sin Q}=\frac{AQ\cdot PQ}{CQ\cdot QM}\quad(*)$$

$$So\ let's\ find\ \frac{AQ}{CQ}=?\ \frac{PQ}{QM}=?.\ Let's\ use\ Menelaus\ theorem\ for\ this:$$

$$\frac{QC\cdot AP\cdot BM}{QA\cdot BP\cdot MC}=1\Rightarrow\frac{QC}{QA}=\frac{1}{x}\ (1)\Rightarrow\frac{AC}{CQ}=x-1,\qquad\frac{AP}{BP}=x\Rightarrow\frac{BP}{BA}=\frac{1}{x+1}$$

Also, Menelaus theorem:

$$\frac{BP\cdot AC\cdot MQ}{BA\cdot CQ\cdot PM}=1\Rightarrow\frac{MQ}{PM}=\frac{x+1}{x-1}\Rightarrow\frac{PQ}{MQ}=\frac{2x}{x+1}\quad(2)$$

Let's use (1) and (2) in (*)

$$Then\ \frac{S_1+S_2}{S_2}=\frac{2x^2}{x+1}\Rightarrow y=\frac{S_1}{S_2}=\frac{2x^2-x-1}{x+1};\ if\ S_1=S_2\Rightarrow y=1$$

$$2x^2-x-1=x+1\Rightarrow x^2-x-1=0$$

$$x=\frac{\sqrt{5}+1}{2}=\varphi$$

1921. In any Δ ABC, the following relationship holds:

$$\sum_{cyc}\frac{r_a}{s+n_a}\le 3+\frac{s-h_a-h_b-h_c}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4sr^2s}{s-a}=as^2-2a.\frac{2F}{a}.\frac{F}{s-a}=as^2-2ah_ar_a\Rightarrow n_a^2=s^2-2h_ar_a$$

$$\text{and analogs}\Rightarrow\sum_{cyc}\frac{r_a}{s+n_a}=\sum_{cyc}\frac{r_a(s-n_a)}{s^2-n_a^2}=\sum_{cyc}\frac{r_a(s-n_a)}{2h_ar_a}=\sum_{cyc}\frac{s}{2h_a}-\sum_{cyc}\frac{n_a}{2h_a}$$

$$\Rightarrow\sum_{cyc}\frac{r_a}{s+n_a}=\frac{s}{2r}-\sum_{cyc}\frac{n_a}{2h_a}\to(1)$$

$$\text{Now, }n_a\overset{?}{\ge}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\ge}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\ge}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$

$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\overset{via\,(1)}{\Leftrightarrow}$$

$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}.\frac{b^2c^2}{4R^2}$$

$$\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{4R^2}\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\ge}\frac{b^2+c^2}{4R^2}\ (\because (b-c)^2\ge 0)$$

$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2 + c^2a^2 \rightarrow \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq} \sum_{\text{cyc}} a^2b^2$$

$$> a^2b^2 + c^2a^2 \therefore n_a \geq \frac{b^2 - bc + c^2}{2R} \text{ and analogs} \Rightarrow \frac{s}{2r} - \sum_{\text{cyc}} \frac{n_a}{2h_a} - 3$$

$$\leq \frac{s}{2r} - \sum_{\text{cyc}} \left(\frac{b^2 - bc + c^2}{2bc} + 1\right) = \frac{s}{2r} - \sum_{\text{cyc}} \frac{a(b^2 + bc + c^2)}{2abc}$$

$$= \frac{s}{2r} - \frac{(\sum_{\text{cyc}} a)(\sum_{\text{cyc}} ab) - 3abc + 3abc}{2s.4Rr} = \frac{s}{2r} - \frac{1}{2r}\sum_{\text{cyc}} \frac{bc}{2R} = \frac{s - h_a - h_b - h_c}{2r}$$

$$\overset{\text{via (1)}}{\Rightarrow} \sum_{\text{cyc}} \frac{r_a}{s + n_a} - 3 \leq \frac{s - h_a - h_b - h_c}{2r} \therefore \sum_{\text{cyc}} \frac{r_a}{s + n_a} \leq 3 + \frac{s - h_a - h_b - h_c}{2r}$$

$$\forall \Delta \text{ ABC, with equality iff } \Delta \text{ ABC is equilateral (QED)}$$

1922. **In ΔABC the following relationship holds:**

$$\cos(B - C)\cos(C - A)\cos(A - B) \geq 8\cos A \cos B \cos C$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\frac{b\cos B + c\cos C}{a} = \frac{R(\sin 2B + \sin 2C)}{2R\sin A} =$$

$$= R\frac{2\sin(B + C)\cos(B - C)}{2R\sin A} \overset{A+B+C=\pi}{=} \cos(B - C) \quad (1)$$

$$\cos(B - C)\cos(C - A)\cos(A - B) = \prod \cos(B - C) \overset{(1)}{=}$$

$$= \prod \frac{b\cos B + c\cos C}{a} \overset{AM-GM}{\geq} \prod \frac{2\sqrt{bc\cos A\cos C}}{a} = 8\cos A\cos B\cos C$$

$$\textit{Equality holds for } A = B = C = \frac{\pi}{3}$$

1923. **In ΔABC the following relationship holds:**

$$\frac{9}{4}\left(\frac{R}{2r}\right)^{-2} - \frac{15}{4} \leq \cos 2A + \cos 2B + \cos 2C \leq 5 - \frac{13}{2}\left(\frac{R}{2r}\right)^{-2}$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$\cos 2A+\cos 2B+\cos 2C=2\cos(A+B)\cos(A-B)+\cos 2C$$
$$=2\cos(\pi-C)\cos(A-B)+2\cos^2 C-1$$
$$=-2\cos C\,(\cos(A-B)-\cos C)-1$$
$$=-2\cos C\,(\cos(A-B)+\cos(A+B))-1$$
$$=-4\cos A\cos B\cos C-1\quad(1)$$

$$From\ (1):\cos 2A+\cos 2B+\cos 2C=-4\cos A\cos B\cos C-1=$$
$$=4\frac{(2R+r)^2-s^2}{4R^2}-1\overset{Gerretsen}{\le}\frac{4R^2+4Rr+r^2-16Rr+5r^2}{R^2}-1=$$
$$=\frac{3R^2-12Rr+6r^2}{R^2}=3-12\frac{r}{R}+6\left(\frac{r}{R}\right)^2=5-2-12\frac{r}{R}+6\left(\frac{r}{R}\right)^2=$$
$$=5-8.\left(\frac{1}{2}\right)^2-24\left(\frac{1}{2}\right).\left(\frac{r}{R}\right)+6\left(\frac{r}{R}\right)^2\overset{Euler}{\le}$$
$$\le\ 5-8\left(\frac{r}{R}\right)^2-24\left(\frac{r}{R}\right)^2+6\left(\frac{r}{R}\right)^2=5-26\left(\frac{r}{R}\right)^2=5-\frac{13}{2}.4\left(\frac{r}{R}\right)^2==5-\frac{13}{2}\left(\frac{R}{2r}\right)^{-2}$$

$$From\ (1):cos\,2A+cos\,2B+cos\,2C=-4cos\,A\,cos\,B\,cos\,C-1=$$
$$=4\frac{(2R+r)^2-s^2}{4R^2}-1\overset{Gerretsen}{\ge}\frac{(2R+r)^2-4R^2-4Rr-3r^2}{R^2}-1=$$
$$=\frac{-2r^2-R^2}{R^2}=-2\left(\frac{r}{R}\right)^2-1=-2\left(\frac{r}{R}\right)^2-\left(\frac{15}{4}-\frac{11}{4}\right)=$$
$$=\frac{11}{4}-2\left(\frac{r}{R}\right)^2-\frac{15}{4}=11.\left(\frac{1}{2}\right)^2-2\left(\frac{r}{R}\right)^2-\frac{15}{4}\overset{Euler}{\ge}$$
$$\ge 11\cdot\left(\frac{r}{R}\right)^2-2\left(\frac{r}{R}\right)^2-\frac{15}{4}=9\left(\frac{r}{R}\right)^2-\frac{15}{4}=\frac{9}{4}\left(\frac{2r}{R}\right)^2-\frac{15}{4}=\frac{9}{4}\left(\frac{R}{2r}\right)^{-2}-\frac{15}{4}$$

Equality holds for: $a=b=c$.

1924. In ΔABC the following relationship holds:

$$1-2\left(\frac{r}{R}\right)^2\le\sum\cos^2A-\left(\sum\cos A-1\right)^2\le 3-10\left(\frac{r}{R}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\cos^2A-\left(\sum\cos A-1\right)^2=\sum\cos^2A-\left(\sum\cos A\right)^2+2\sum\cos A-1=$$

$$=2\sum\cos A-2\sum\cos A\cos B-1=2\left(1+\frac{r}{R}\right)-2\left(\frac{s^2+r^2}{4R^2}-1\right)-1=$$

$$=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\ (1)$$

$$From\ (1):\sum\cos^2A-\left(\sum\cos A-1\right)^2=$$

$$=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\overset{Gerretsen}{\le}\frac{4Rr-16Rr+4r^2+6R^2}{2R^2}\overset{Euler}{\le}$$

$$\le\frac{(6R^2-12(2r).r+4r^2)}{2R^2}=\frac{6R^2-20Rr}{2R^2}=3-10\left(\frac{r}{R}\right)^2$$

$$From\ (1):\sum\cos^2A-\left(\sum\cos A-1\right)^2=\frac{4Rr-s^2-r^2+6R^2}{2R^2}\overset{Gerretsen}{\ge}$$

$$\frac{4Rr-4R^2-4Rr-3r^2-r^2+6R^2}{2R^2}=\frac{2R^2-4r^2}{2R^2}=1-2\left(\frac{r}{R}\right)^2$$

Equality holds iff ΔABC is equilateral

1925. In ΔABC the following relationship holds:

$$\sum\frac{(a+b)^{2n}}{w_a+r_b}\ge\frac{\left(4\sqrt{3}r\right)^{2n}}{R},n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{(a+b)^{2n}}{w_a+r_b}\overset{Holder}{\geq}\frac{(a+b+b+c+c+a)^{2n}}{\left((\sum w_a)+(\sum r_a)\right)\cdot 3^{2n-2}}\overset{w_a\leq m_a}{\geq}$$

$$\geq\frac{(4s)^{2n}}{3^{2n-2}\cdot\left((\sum m_a)+(\sum r_a)\right)}\overset{Leuenberger}{\geq}\frac{(4s)^{2n}}{3^{2n-2}\left(2(4R+r)\right)}\overset{Euler\ \&\ Mitrinovic}{\geq}$$

$$\geq\frac{\left(4.3\sqrt{3}r\right)^{2n}}{3^{2n-2}.2.\frac{9R}{2}}=\frac{\left(4\sqrt{3}r\right)^{2n}}{R}$$

Equality holds for $a=b=c$

1926. In ΔABC **the following relationship holds:**

$$\sum\frac{(a+b)^4}{w_a+r_b}\geq\frac{\left(4\sqrt{3}r\right)^4}{R}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{(a+b)^4}{w_a+r_b}\overset{Holder}{\geq}\frac{(a+b+b+c+c+a)^4}{(\sum w_a)+(\sum r_a)9}\overset{w_a\leq m_a}{\geq}\frac{256s^4}{9(\sum m_a)+(\sum r_a)}\overset{Leuenberger}{\geq}$$

$$\geq\frac{(4s)^4}{9\left(2(4R+r)\right)}\overset{Euler\ \&\ Mitrinovic}{\geq}\frac{\left(4.3\sqrt{3}r\right)^4}{9.2.\frac{9R}{2}}=\frac{81\left(4\sqrt{3}r\right)^4}{81R}=\frac{\left(4\sqrt{3}r\right)^4}{R}$$

Equality holds for $a=b=c$

1927. In ΔABC **the following relationship holds:**

$$\sum\frac{r_a^2}{r_a^2+3rr_a+9r^2}\geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{r_a^2}{r_a^2+3rr_a+9r^2}=1-\frac{3rr_a+9r^2}{r_a^2+3rr_a+9r^2}=1-\frac{3rr_a+9r^2}{(r_a^2+9r^2)+3rr_a}\overset{AM-GM}{\geq}$$

$$\geq 1-\frac{3r(r_a+3r)}{6rr_a+3rr_a}=1-\frac{r_a+3r}{3r_a}=1-\frac{1}{3}-\frac{r}{r_a}=\frac{2}{3}-\frac{r}{r_a}\ (1)$$

$$\sum\frac{1}{r_a}=\frac{1}{F}\sum s-a\ =\frac{s}{F}=\frac{1}{r}\ (2)$$

$$\sum\frac{r_a^2}{r_a^2+3rr_a+9r^2}\overset{(1)}{\geq}\sum\left(\frac{2}{3}-\frac{r}{r_a}\right)=\frac{6}{3}-r\sum\frac{1}{r_a}\overset{(2)}{=}2-1=1$$

Equality holds if ΔABC is equilateral.

1928. In ΔABC the following relationship holds:

$$\sum\frac{h_a^2}{h_a^2+3rh_a+9r^2}\geq 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{h_a^2}{h_a^2+3rh_a+9r^2}=1-\frac{3rh_a+9r^2}{h_a^2+3rh_a+9r^2}=1-\frac{3rh_a\ +9r^2}{(h_a^2+9r^2)+3rh_a}\overset{AM-GM}{\geq}$$

$$\geq 1-\frac{3r(h_a+3r)}{6rh_a+3rh_a}=1-\frac{h_a+3r}{3h_a}=1-\frac{1}{3}-\frac{r}{h_a}=\frac{2}{3}-\frac{r}{h_a}\ (1)$$

$$\sum\frac{1}{h_a}=\frac{1}{2F}\sum a\ =\frac{2s}{2F}=\frac{1}{r}\ (2)$$

$$\sum\sum\frac{h_a^2}{h_a^2+3rh_a+9r^2}\overset{(1)}{\geq}\sum\left(\frac{2}{3}-\frac{r}{h_a}\right)=\frac{6}{3}-r\sum\frac{1}{h_a}\overset{(2)}{=}2-1=1$$

Equality holds if ΔABC is equilateral.

1929. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2}+2\tan\frac{B}{2}}\geq\frac{1}{\sqrt{3}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\sqrt{3}\tan\frac{A}{2}, y=\sqrt{3}\tan\frac{B}{2}, y=\sqrt{3}\tan\frac{C}{2} \text{ and then :}$$

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2}+2\tan\frac{B}{2}}\ge\frac{1}{\sqrt{3}}\Leftrightarrow\sum_{cyc}\frac{\sqrt{3}.\frac{x^2}{3}}{\sqrt{3}.\frac{x^2}{3}+\frac{2y}{\sqrt{3}}}\ge 1$$

$$\Leftrightarrow\sum_{cyc}x^2(y^2+2z)(z^2+2x)\ge\prod_{cyc}(x^2+2y)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} x^2y^2z^2-4xyz+\sum_{cyc}x^3y^2\ge 0$$

$$\Leftrightarrow 27\prod_{cyc}\tan^2\frac{A}{2}-12\sqrt{3}.\prod_{cyc}\tan\frac{A}{2}+9\sqrt{3}.\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\ge 0$$

$$\Leftrightarrow 3\sqrt{3}.\frac{r^2}{s^2}+3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\boxed{\overset{(*)}{\ge}}\frac{4r}{s}$$

$$\text{Now, }3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}=3\sum_{cyc}\frac{\tan^3\frac{A}{2}\tan^3\frac{B}{2}}{\tan\frac{B}{2}}\overset{\text{Holder}}{\ge}\frac{3\left(\sum_{cyc}\tan\frac{A}{2}\tan\frac{B}{2}\right)^3}{3\sum_{cyc}\tan\frac{A}{2}}=\frac{s}{4R+r}$$

$$\Rightarrow\left(3\sqrt{3}.\frac{r^2}{s^2}+3\sum_{cyc}\tan^3\frac{A}{2}\tan^2\frac{B}{2}\right)^2\ge\left(3\sqrt{3}.\frac{r^2}{s^2}+\frac{s}{4R+r}\right)^2$$

$$=\frac{27r^4}{s^4}+\frac{s^2}{(4R+r)^2}+\frac{6\sqrt{3}r^2}{s(4R+r)}\overset{\text{Trucht or Doucet}}{\ge}\frac{27r^4}{s^4}+\frac{s^2}{(4R+r)^2}+\frac{18r^2}{(4R+r)^2}$$

$$\overset{?}{\ge}\frac{16r^2}{s^2}\Leftrightarrow s^4(s^2+18r^2)+27r^4(4R+r)^2-16r^2s^2(4R+r)^2\overset{?}{\ge}0$$

$$\Leftrightarrow s^6+18r^2s^4-r^2s^2(256R^2+128Rr+16r^2)+27r^4(4R+r)^2\boxed{\underset{(**)}{\overset{?}{\ge}}}0$$

$$\text{Now, LHS of }(**)\overset{\text{Gerretsen}}{\ge}(16Rr+13r^2)s^4-r^2s^2(256R^2+128Rr+16r^2)$$

$$+27r^4(4R+r)^2\overset{\text{Gerretsen}}{\ge}\begin{pmatrix}(16Rr+13r^2)(16Rr-5r^2)\\-r^2(256R^2+128Rr+16r^2)\end{pmatrix}s^2+27r^4(4R+r)^2$$

$$= -81r^4s^2 + 27r^4(4R+r)^2 = 27r^4\left((4R+r)^2 - 3s^2\right) \overset{\text{Trucht or Doucet}}{\geq} 0$$

$$\Rightarrow (**) \text{ is true} \therefore \left(3\sqrt{3}.\frac{r^2}{s^2} + 3\sum_{cyc} \tan^3\frac{A}{2}\tan^2\frac{B}{2}\right)^2 \geq \frac{16r^2}{s^2} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{\tan^2\frac{A}{2}}{\sqrt{3}\tan^2\frac{A}{2} + 2\tan\frac{B}{2}} \geq \frac{1}{\sqrt{3}}\ \forall\ \Delta\ ABC, \text{with equality iff } \Delta\ ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \sqrt{3}\tan\frac{A}{2}, y := \sqrt{3}\tan\frac{B}{2}, z := \sqrt{3}\tan\frac{C}{2}$. We have $xy + yz + zx = 3$.

The desired inequality is equivalent to

$$\sum_{cyc} \frac{x^2}{x^2 + 2y} \geq 1.$$

By CBS inequality, we have

$$\sum_{cyc} \frac{x^2}{x^2+2y} \geq \frac{\left(\sqrt{x^3} + \sqrt{y^3} + \sqrt{z^3}\right)^2}{\sum_{cyc} x(x^2+2y)} = \frac{\sum_{cyc} x^3 + 2\sum_{cyc}(yz)^{\frac{3}{2}}}{\sum_{cyc} x^3 + 2\sum_{cyc} yz} \geq$$

$$\overset{\text{Power Mean}}{\geq} \frac{\sum_{cyc} x^3 + 2.3\left(\frac{\sum_{cyc} yz}{3}\right)^{\frac{3}{2}}}{\sum_{cyc} x^3 + 6} = 1.$$

which complete the proof. Equality holds iff ΔABC is equilateral.

1930. If $x, y, z > 0$ then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\cdot\frac{a}{b+c} + \frac{y+z}{x}\cdot\frac{b}{c+a} + \frac{z+x}{y}\cdot\frac{c}{a+b} \geq \frac{6}{\sqrt[3]{4 + \frac{2R}{r}}}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\frac{abc}{(a+b)(b+c)(c+a)}=\frac{4Rrs}{(a+b+c)(ab+bc+ca)-abc}=$$
$$=\frac{4Rrs}{2s(s^2+r^2+4Rr)-4Rrs}=\frac{2Rr}{s^2+r^2+4Rr-2Rr}=$$
$$=\frac{2Rr}{s^2+r^2+2Rr}\overset{Gerretsen}{\geq}\frac{2Rr}{4R^2+4Rr+3r^2+r^2+2Rr}=$$
$$=\frac{2Rr}{4R^2+6Rr+4r^2}=\frac{1}{\frac{2R}{r}+3+\frac{2r}{R}}\overset{Euler}{\geq}\frac{1}{\frac{2R}{r}+3+1}=\frac{1}{4+\frac{2R}{r}}\ (1)$$
$$\frac{x+y}{z}\frac{a}{b+c}+\frac{y+z}{x}\frac{b}{c+a}+\frac{z+x}{y}\frac{c}{a+b}=$$
$$=\sum\frac{x+y}{z}\frac{a}{b+c}\overset{AM-GM}{\geq}2\sum\frac{\sqrt{xy}}{z}\frac{a}{b+c}\overset{AM-GM}{\geq}$$
$$\geq 6\left(\frac{abc}{(a+b)(b+c)(c+a)}\right)^{\frac{1}{3}}\overset{(1)}{\geq}\frac{6}{\sqrt[3]{4+\frac{2R}{r}}}$$

Equality holds for $x=y=z, a=b=c$.

1931. **If $\lambda,\mu>0$ then in ΔABC the following relationship holds:**

$$4\sqrt{3}(\lambda+\mu)\cdot\frac{r}{R}\leq\sum_{cyc}\frac{\lambda a+\mu b}{r_c}\leq\frac{3R(\lambda+\mu)}{F}\cdot\sqrt{9R^2-s^2}$$

Proposed by Mehmet Şahin-Turkiye

Solution by Tapas Das-India

$$\sum_{cyc}\frac{\lambda a+\mu b}{r_c}=\lambda\sum\frac{a}{r_c}+\mu\sum\frac{b}{r_c}=\frac{\lambda}{F}\sum a(s-c)+\frac{\mu}{F}\sum b(s-c)=$$
$$=\frac{\lambda}{F}\left(2s^2-\sum ac\right)+\frac{\mu}{F}\left(2s^2-\sum bc\right)=$$
$$=\frac{\lambda+\mu}{F}\left(2s^2-\sum bc\right)=\frac{\lambda+\mu}{F}(2s^2-s^2-r^2-4Rr)=$$

$$=\frac{\lambda+\mu}{F}(s^2-r^2-4Rr)=\frac{\lambda+\mu}{2F}2(s^2-r^2-4Rr)=$$

$$=\frac{\lambda+\mu}{2F}\left(\sum a^2\right)\overset{Leibniz}{\leq}\frac{\lambda+\mu}{2F}(9R^2)$$

$$We\ need\ to\ show\ \frac{\lambda+\mu}{2F}(9R^2)\leq\frac{3R(\lambda+\mu)}{F}.\sqrt{9R^2-s^2}\ or$$

$$\frac{3R}{2}\leq\sqrt{9R^2-s^2}\ or,\frac{9R^2}{4}\leq 9R^2-s^2$$

$$or\ s^2\leq\frac{27R^2}{4}\ true\ (Mitrinovic)$$

$$Again\ from\ the\ previous\ result:$$

$$\sum_{cyc}\frac{\lambda a+\mu b}{r_c}=\frac{\lambda+\mu}{2F}\left(\sum a^2\right)\overset{Ionescu-Weitzenbock}{\geq}$$

$$\geq\frac{\lambda+\mu}{2F}4\sqrt{3}\,F=4\sqrt{3}(\lambda+\mu).\frac{1}{2}\overset{Euler}{\geq}4\sqrt{3}(\lambda+\mu)\frac{r}{R}$$

1932. In ΔABC the following relationship holds:

$$\sum\frac{h_a}{b+c}\sin A\leq\frac{9}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{1}{a}=\frac{\sum ab}{abc}\leq\frac{(a+b+c)^2}{3abc}=\frac{4s^2}{12Rrs}=\frac{s}{3Rr}\overset{Doucet}{\leq}\frac{4R+r}{3\sqrt{3}Rr}\overset{Euler}{\leq}\frac{\frac{9R}{2}}{3\sqrt{3}Rr}=\frac{\sqrt{3}}{2r}\quad(1)$$

$$\sum\frac{h_a}{b+c}\sin A=\sum\frac{\frac{2F}{a}}{b+c}\left(\frac{a}{2R}\right)\leq\sum\frac{F}{R}\cdot\frac{1}{b+c}\overset{AM-HM}{\leq}$$

$$\leq\frac{F}{4R}\sum\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{F}{2R}\sum\frac{1}{a}\overset{(1)}{\leq}\frac{F}{2R}.\frac{\sqrt{3}}{2r}=\frac{1}{4}\cdot\frac{s\sqrt{3}}{R}\overset{Mitrinovic}{\leq}\frac{1}{4}\cdot\frac{3\sqrt{3}\frac{R}{2}}{R}.\sqrt{3}=\frac{9}{8}$$

$$Equality\ holds\ for\ a=b=c$$

1933. In ΔABC the following relationship holds:

$$\sum\frac{\tan^2 A+\tan^2 B}{\tan^4 A+\tan^4 B}\le 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$A+B+C=\pi\ or\ A+B=\pi-A\ or\ \tan(A+B)=-\tan C$$
$$or\ \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$
$$or\ \sum\tan A=\prod\tan A\ \ (1)\ \ and$$

$$\frac{\tan^2 A+\tan^2 B}{\tan^4 A+\tan^4 B}\overset{CBS}{\le}\frac{\tan^2 A+\tan^2 B}{\frac{(\tan^2 A+\tan^2 B)^2}{2}}=\frac{2}{\tan^2 A+\tan^2 B}\overset{AM-GM}{\le}$$
$$\le\frac{2}{2\tan A\tan B}=\frac{1}{\tan A\tan B}(1)$$

$$\sum\frac{\tan^2 A+\tan^2 B}{\tan^4 A+\tan^4 B}\overset{(1)}{\le}\sum\frac{1}{\tan A\tan B}=\frac{\sum\tan A}{\prod\tan A}\overset{(1)}{=}1$$

$$Equality\ for\ A=B=C$$

1934. In acute ΔABC holds:

$$\sum\sin A+\sum\tan A\ge\frac{9\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ be\ f(x)=\sin x+\tan x, x\in\left(0,\frac{\pi}{2}\right)$$

$$f''(x)=\sin x\,(2\sec^3 x-1)>0\ as\ \sec x>1, f\ is\ convex.$$

$$f(A)+f(B)+f(C)\overset{Jensen}{\geq}3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)=$$

$$=3\left(\sin\left(\frac{\pi}{3}\right)+\tan\left(\frac{\pi}{3}\right)\right)=3\left(\frac{\sqrt{3}}{2}+\sqrt{3}\right)=\frac{9\sqrt{3}}{2}$$

$$\sum\sin A+\sum\tan A\geq\frac{9\sqrt{3}}{2}$$

$$Equality\ for\ A=B=C=\frac{\pi}{3}$$

1935. In ΔABC the following relationship holds:

$$\sum\frac{a}{r_a}\geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{r_a}=\frac{1}{F}\sum a(s-a)=\frac{1}{F}\left(2s^2-\sum a^2\right)=$$

$$=\frac{1}{F}(2s^2-2s^2+2r^2+8Rr)=\frac{2r(4R+r)}{rs}.=\frac{2(4R+r)}{s}\overset{Doucet}{\geq}2\sqrt{3}$$

$$Equality\ for\ a=b=c$$

1936. In ΔABC the following relationship holds:

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}=\sum_{cyc}\frac{a}{h_b+h_c}=\sum_{cyc}\frac{a}{\frac{2F}{b}+\frac{2F}{c}}=$$

$$=\frac{1}{2F}\sum_{cyc}\frac{a}{\frac{1}{b}+\frac{1}{c}}=\frac{1}{2F}\sum_{cyc}\frac{abc}{b+c}=\frac{abc}{2F}\sum_{cyc}\frac{1}{b+c}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{4RF}{2F}\cdot\frac{(1+1+1)^2}{b+c+c+a+a+b}=2R\cdot\frac{9}{2(a+b+c)}=$$

$$=\frac{9R}{2s}\overset{MITRINOVIC}{\geq}\frac{9R}{2\cdot\frac{3\sqrt{3}}{2}\cdot R}=\frac{3}{\sqrt{3}}=\sqrt{3}$$

Equality holds for $a=b=c$.

1937. In ΔABC the following relationship holds:

$$\frac{a}{r_b+r_c}+\frac{b}{r_c+r_a}+\frac{c}{r_a+r_b}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a}{r_b+r_c}+\frac{b}{r_c+r_a}+\frac{c}{r_a+r_b}=\sum_{cyc}\frac{a}{r_b+r_c}=\sum_{cyc}\frac{a}{\frac{F}{s-b}+\frac{F}{s-c}}=$$

$$=\frac{1}{F}\sum_{cyc}\frac{a}{\frac{1}{s-b}+\frac{1}{s-c}}=\frac{1}{F}\sum_{cyc}\frac{a(s-b)(s-c)}{s-c+s-b}=\frac{1}{rs}\sum_{cyc}\frac{a(s-b)(s-c)}{a}=$$

$$=\frac{1}{rs}\sum_{cyc}(s-b)(s-c)=\frac{1}{rs}\cdot(4R+r)r=\frac{4R+r}{s}\overset{DOUCET}{\geq}\frac{s\sqrt{3}}{s}=\sqrt{3}$$

Equality holds for: $a=b=c$.

1938. In ΔABC the following relationship holds:

$$\frac{a^2}{m_b^2+m_c^2}+\frac{b^2}{m_c^2+m_a^2}+\frac{c^2}{m_a^2+m_b^2}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ then\ m_a \leq m_b \leq m_c\ and$$

$$m_a^2 + m_b^2 \leq m_c^2 + m_a^2 \leq m_b^2 + m_c^2$$

$$\frac{1}{m_a^2 + m_b^2} \geq \frac{1}{m_c^2 + m_a^2} \geq \frac{1}{m_b^2 + m_c^2}$$

$$\frac{a^2}{m_b^2 + m_c^2} + \frac{b^2}{m_c^2 + m_a^2} + \frac{c^2}{m_a^2 + m_b^2} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^2\right)\left(\sum \frac{1}{m_b^2 + m_c^2}\right) \overset{Bergstrom}{\geq}$$

$$\geq \frac{1}{3} \cdot \frac{(\sum a^2)(1+1+1)^2}{2(\sum m_a^2)} = \frac{1}{3} \cdot \frac{(\sum a^2)(3)^2}{2\left(\frac{3}{4}\sum a^2\right)} = 2$$

$$Equality\ for\ a = b = c$$

1939. In ΔABC the following relationship holds:

$$\sum \frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2} + \sin^{2022}\frac{C}{2}} \geq \frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ \ then\ \ \sin\frac{A}{2} \geq \sin\frac{B}{2} \geq \sin\frac{C}{2}\ and$$

$$\sin\frac{A}{2} + \sin\frac{B}{2} \geq \sin\frac{A}{2} + \sin\frac{C}{2} \geq \sin\frac{C}{2} + \sin\frac{B}{2}$$

$$\frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2} + \sin^{2022}\frac{C}{2}} = \sin^2\frac{A}{2} \cdot \frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2} + \sin^{2022}\frac{C}{2}}\ (1)$$

$$\sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R} \overset{Euler}{\geq} 1 - \frac{1}{4} = \frac{3}{4}\ (2)$$

$$\sum\frac{\sin^{2024}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\overset{1}{=}\sum\left(\sin^2\frac{A}{2}\cdot\frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\right)\overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}\left(\sum\sin^2\frac{A}{2}\cdot\right)\left(\sum\frac{\sin^{2022}\frac{A}{2}}{\sin^{2022}\frac{B}{2}+\sin^{2022}\frac{C}{2}}\right)\overset{Nesbitt\,\&\,(2)}{\geq}\frac{1}{3}\cdot\frac{3}{4}\cdot\frac{3}{2}=\frac{3}{8}$$

$$Equality\ holds\ for\ A=B=C$$

1940. In $\Delta ABC, I$ –incenter, the following relationship holds:

$$\sqrt{abc-aIA^2}+\sqrt{abc-bIB^2}+\sqrt{abc-cIC^2}\leq\sqrt{6abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum aIA^2=\sum\left(\frac{abc(s-a)}{s}\right)=abc\sum\frac{s-a}{s}=abc\ (1)$$

$$\sqrt{abc-aIA^2}+\sqrt{abc-bIB^2}+\sqrt{abc-cIC^2}\overset{CBS}{\leq}$$

$$\leq\sqrt{3\left(3abc-\sum aIA^2\right)}\overset{(1)}{=}\sqrt{(3(3abc-abc))}=\sqrt{6abc}$$

$$Equality\ holds\ for\ a=b=c$$

1941. In ΔABC the following relationship holds:

$$\frac{a}{b+c}sec\frac{A}{2}+\frac{b}{c+a}sec\frac{B}{2}+\frac{c}{a+b}sec\frac{C}{2}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be $f:(0,\pi)\to\mathbb{R}, f(x)=sec\frac{x}{2}$

$$f'(x)=\frac{1}{2}\cdot\frac{\sin\frac{x}{2}}{\cos^2\frac{x}{2}}\quad f''(x)=\frac{1}{2}\cdot\frac{\frac{1}{2}\cdot\cos^3\frac{x}{2}+\sin\frac{x}{2}\cdot 2\cdot\frac{1}{2}\cdot\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^4\frac{x}{2}}$$

$$f''(x)=\frac{1}{4}\cdot\frac{\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}}{\cos^3\frac{x}{2}}=\frac{1}{4}\cdot\frac{1+\sin^2\frac{x}{2}}{\cos^3\frac{x}{2}}$$

$x\in(0,\pi)\to\frac{x}{2}\in\left(0,\frac{\pi}{2}\right)\to\sin\frac{x}{2}>0,\cos\frac{x}{2}>0\to f''(x)>0\to f$ convex

By Jensen's inequality:

$$f(A)+f(B)+f(C)\ge 3f\left(\frac{A+B+C}{3}\right)$$

$$\sum_{cyc}\sec\frac{A}{2}\ge 3\sec\frac{\pi}{6}=\frac{3}{\frac{\sqrt{3}}{2}}=2\sqrt{3}\quad (1)$$

WLOG: $a\ge b\ge c\to$

$$\to b+c\le a+c\le a+b\to\frac{1}{b+c}\ge\frac{1}{a+c}\ge\frac{1}{a+b}\to$$

$$\frac{a}{b+c}\ge\frac{b}{a+c}\ge\frac{c}{a+b}\quad (2)$$

$$a\ge b\ge c\to\frac{A}{2}\ge\frac{B}{2}\ge\frac{C}{2}\to\cos\frac{A}{2}\le\cos\frac{B}{2}\le\cos\frac{C}{2}\to$$

$$\to\sec\frac{A}{2}\ge\sec\frac{B}{2}\ge\sec\frac{C}{2}\qquad (3)$$

By $(2),(3)$:

$$\sum_{cyc}\frac{a}{b+c}\cdot\sec\frac{A}{2}\overset{CEBYSHEV}{\ge}\frac{1}{3}\cdot\sum_{cyc}\frac{a}{b+c}\cdot\sum_{cyc}\sec\frac{A}{2}\overset{NESBITT}{\ge}$$

$$\ge\frac{1}{3}\cdot\frac{3}{2}\cdot\sum_{cyc}\sec\frac{A}{2}\overset{(1)}{\ge}\frac{1}{2}\cdot 2\sqrt{3}=\sqrt{3}$$

Equality holds for $a=b=c$.

1942. In any Δ ABC, the following relationship holds :

$$\frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)\ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with $area$ F (say) and $16F^2 =$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F = 2\sqrt{\sum_{cyc}AB} \rightarrow (1)$$

$$\text{Now}, \forall\, x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4} \Leftrightarrow \left(\sum_{cyc}xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc}x \rightarrow \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \rightarrow (2)$$

$$\text{We have}: \frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)$$

$$+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right) = \frac{x}{y+z}(B'+C')+\frac{y}{z+x}(C'+A')+\frac{z}{x+y}(A'+B')$$

$$\left(x=r_a, y=r_b, z=r_c, A'=\sec\frac{A}{2}, B'=\sec\frac{B}{2}, C'=\sec\frac{C}{2}\right)$$

$$= \frac{x}{y+z}.\sqrt{B'+C'}^2+\frac{y}{z+x}.\sqrt{C'+A'}^2+\frac{z}{x+y}.\sqrt{A'+B'}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F'.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc}A'B'}.\frac{\sqrt{3}}{2} = \sqrt{3\sum_{cyc}\left(\sec\frac{A}{2}\sec\frac{B}{2}\right)}$$

$$= \sqrt{3\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}.\sum_{cyc}\cos\frac{A}{2}}$$

$$= \sqrt{\frac{6R}{s}.\sum_{cyc}\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}} \overset{0<\cos\frac{B-C}{2}\leq 1 \text{ and analogs}}{\geq}$$

$$\sqrt{\frac{6R}{s}.\sum_{cyc}(\sin B+\sin C)} = \sqrt{\frac{12R}{s}.\frac{s}{R}} = 2\sqrt{3}$$

$$\therefore \frac{r_a}{r_b+r_c}\left(\sec\frac{B}{2}+\sec\frac{C}{2}\right)+\frac{r_b}{r_c+r_a}\left(\sec\frac{C}{2}+\sec\frac{A}{2}\right)+\frac{r_c}{r_a+r_b}\left(\sec\frac{A}{2}+\sec\frac{B}{2}\right)$$

$\geq 2\sqrt{3}\ \forall\ \Delta\ ABC$, " = " iff Δ ABC is equi$lateral$ (QED)

1943. In any Δ ABC, the following relationship holds :

$$\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\le 2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)=2\sum_{cyc}\frac{s-b}{s-a}=2\sum_{cyc}\frac{(s-b)^2}{(s-a)(s-b)}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{2\left(\sum_{cyc}(s-a)\right)^2}{\sum_{cyc}\left((s-a)(s-b)\right)}=\frac{2s^2}{4Rr+r^2}\overset{?}{\ge}\sum_{cyc}\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\sum_{cyc}\frac{\frac{(b+c)}{a}\sin\frac{A}{2}}{\sin\frac{A}{2}}$$

$$=\frac{1}{4Rrs}\cdot\sum_{cyc}\left(bc(2s-a)\right)=\frac{2s}{4Rrs}\cdot(s^2+4Rr+r^2-6Rr)$$

$$\Leftrightarrow\frac{2s^2}{4R+r}\overset{?}{\ge}\frac{s^2-2Rr+r^2}{2R}\Leftrightarrow s^2\overset{?}{\le}8R^2-2Rr-r^2$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-(4R^2-6Rr-4r^2)\overset{?}{\le}0$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(2R+r)(R-2r)\overset{?}{\le}0\to\text{true}$$

$$\because s^2-(4R^2+4Rr+3r^2)\overset{\text{Gerretsen}}{\le}0\text{ and }-2(2R+r)(R-2r)\overset{\text{Euler}}{\le}0$$

$$\therefore\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C-A}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}\le 2\left(\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\right)$$

$\forall\,\Delta$ ABC," = " iff Δ ABC is equilateral (QED)

1944.

Prove that in any acute ΔABC:

$$\cos^2A+\cos^2B+\cos^2C+6\cos A\cos B\cos C\le\cos A+\cos B+\cos C$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality can be rewritten as

$$\sum_{cyc}(1-\cos A-2\cos B\cos C)\cos A\geq 0,$$

which is true because $\cos A>0$ **(and analogs), and**

$$1-\cos A-2\cos B\cos C=$$

$$=1-\cos A-\cos(B+C)-\cos(B-C)\overset{B+C=\pi-A}{\cong}1-\cos(B-C)\geq 0\text{ (and analogs).}$$

So the proof is complete. Equality holds if ΔABC **is equilateral.**

1945. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}}\geq 2^{2026}+\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\overset{\text{Reverse Bergstrom}}{\leq}\frac{1}{4}\sum_{cyc}\left(\frac{a^{2025}}{b^{2025}}+\frac{a^{2025}}{c^{2025}}\right)$$

$$\therefore\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\leq\frac{1}{4}\sum_{cyc}\left(\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\right)\to(1)$$

$$\left(x+\frac{1}{x}\right)^{2025}=\binom{2025}{0}x^{2025}+\binom{2025}{1}x^{2024}.\frac{1}{x}+\binom{2025}{2}x^{2023}.\frac{1}{x^2}+\cdots$$

$$+\binom{2025}{2012}x^{1013}.\frac{1}{x^{1012}}+\binom{2025}{2013}x^{1012}.\frac{1}{x^{1013}}+\cdots$$

$$+\binom{2025}{2023}x^2.\frac{1}{x^{2023}}+\binom{2025}{2024}x.\frac{1}{x^{2024}}+\binom{2025}{2025}\frac{1}{x^{2025}}$$

$$=x^{2025}+\frac{1}{x^{2025}}+\binom{2025}{1}.\left(x^{2023}+\frac{1}{x^{2023}}\right)+\binom{2025}{2}.\left(x^{2021}+\frac{1}{x^{2021}}\right)+\cdots$$

$$+\binom{2025}{2012}.\left(x+\frac{1}{x}\right)\left(\because\binom{n}{r}=\binom{n}{n-r}\right)$$

$$\overset{A-G}{\geq} x^{2025}+\frac{1}{x^{2025}}+2\left(\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2012}\right)=x^{2025}+\frac{1}{x^{2025}}+$$

$$\left(\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2012}+\binom{2025}{2013}+\cdots+\binom{2025}{2023}+\binom{2025}{2024}\right)$$

$$\left(\because 2\binom{n}{r}=\binom{n}{r}+\binom{n}{n-r}\right)=x^{2025}+\frac{1}{x^{2025}}$$

$$+\left(\begin{array}{c}\binom{2025}{0}+\binom{2025}{1}+\binom{2025}{2}+\cdots+\binom{2025}{2023}+\binom{2025}{2024}+\binom{2025}{2025}\\ -\left(\binom{2025}{0}+\binom{2025}{2025}\right)\end{array}\right)$$

$$=x^{2025}+\frac{1}{x^{2025}}+\left(2^{2025}-2\right)\left(\because \binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}=2^n\right)\text{ and putting } x=\frac{b}{c},$$

$$\text{we get}:\left(\frac{b}{c}+\frac{c}{b}\right)^{2025}\geq\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}+2^{2025}-2\Rightarrow\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\leq$$

$$\left(\frac{b}{c}+\frac{c}{b}\right)^{2025}-2^{2025}+2\overset{Bandila}{\leq}\left(\frac{R}{r}\right)^{2025}-2^{2025}+2\text{ and analogs}$$

$$\Rightarrow\frac{1}{4}\sum_{cyc}\left(\frac{b^{2025}}{c^{2025}}+\frac{c^{2025}}{b^{2025}}\right)\leq\frac{3}{4}\left(\frac{R}{r}\right)^{2025}-\frac{3}{4}.2^{2025}+\frac{3}{2}\to(2)\therefore(1),(2)\Rightarrow$$

$$\boxed{\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\leq\frac{3}{4}\left(\frac{R}{r}\right)^{2025}-\frac{3}{4}.2^{2025}+\frac{3}{2}\to(i)}\text{ and }\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}}$$

$$-2^{2026}\overset{Nesbitt}{\geq}\frac{3}{2}+\left(\frac{R}{r}\right)^{2026}-2^{2026}\overset{Euler}{\geq}\frac{3}{2}+2\left(\left(\frac{R}{r}\right)^{2025}-2^{2025}\right)$$

$$\geq\frac{3}{2}+\frac{3}{4}\left(\left(\frac{R}{r}\right)^{2025}-2^{2025}\right)\left(\because\frac{R}{r}\overset{Euler}{\geq}2\Rightarrow\left(\frac{R}{r}\right)^{2025}-2^{2025}\geq 0\right)\overset{via\ (i)}{\geq}$$

$$\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}\therefore\sum_{cyc}\frac{a^{2024}}{b^{2024}+c^{2024}}+\frac{R^{2026}}{r^{2026}}\geq 2^{2026}+\sum_{cyc}\frac{a^{2025}}{b^{2025}+c^{2025}}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1946. In any Δ ABC, the following relationships hold :

$$① \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} + \left(\frac{R}{2r}\right)^{2025} \ge 1 + \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2} + \cos\frac{B}{2}}{\cos\frac{C}{2}}} \text{ and}$$

$$② \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2} + \cos\frac{B}{2}}{\cos\frac{C}{2}}} + \left(\frac{R}{2r}\right)^{2025} \ge 1 + \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakaborty-Kolkata-India

$$\prod_{cyc} \frac{\cos\frac{A}{2} + \cos\frac{B}{2}}{\cos\frac{C}{2}} = \frac{\left(\sum_{cyc}\cos\frac{A}{2}\right)\left(\sum_{cyc}\cos\frac{A}{2}\cos\frac{B}{2}\right) - \prod_{cyc}\cos\frac{A}{2}}{\prod_{cyc}\cos\frac{A}{2}}$$

$$\overset{\text{Jensen}}{\le} \frac{\frac{3\sqrt{3}}{2}.\left(\sum_{cyc}\cos^2\frac{A}{2}\right)}{\frac{s}{4R}} - 1 = \frac{\frac{3\sqrt{3}}{2}.\left(\frac{4R+r}{2R}\right)}{\frac{s}{4R}} - 1 \overset{\text{Mitrinovic}}{\le} \frac{3\sqrt{3}(4R+r)}{3\sqrt{3}r} - 1$$

$$\Rightarrow \prod_{cyc} \sqrt[2025]{\frac{\cos\frac{A}{2} + \cos\frac{B}{2}}{\cos\frac{C}{2}}} \le \sqrt[2025]{\frac{4R}{r}} \to (1)$$

$$\text{Again, } \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} = \prod_{cyc} \sqrt[2025]{\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}}} \overset{0<\cos\frac{B-C}{2}\le 1 \text{ and analogs}}{\le}$$

$$\sqrt[2025]{\frac{1}{\prod_{cyc}\sin\frac{A}{2}}} \Rightarrow \prod_{cyc} \sqrt[2025]{\frac{b+c}{a}} \le \sqrt[2025]{\frac{4R}{r}} \to (2)$$

$$\text{Now, } \sqrt[2025]{8} + \left(\frac{R}{2r}\right)^{2025} \overset{?}{\ge} 1 + \sqrt[2025]{\frac{4R}{r}} \Leftrightarrow \sqrt[2025]{8} + \left(\frac{R}{2r}\right)^{2025} \overset{?}{\ge} 1 + \sqrt[2025]{8}.\sqrt[2025]{\frac{R}{2r}}$$

$$\Leftrightarrow t^{4100625} - 1 \underset{(*)}{\overset{?}{\ge}} \sqrt[2025]{8}.(t-1) \left(t = \sqrt[2025]{\frac{R}{2r}}\right)$$

$$\text{We have : } t^{4100625} - 1 = (t-1)(t^{4100624} + t^{4100623} + \cdots + 1) \overset{\text{Euler}}{\ge}$$

$$(t-1)(4100624) \geq \sqrt[2025]{8}.(t-1)\left(\because t-1=\sqrt[2025]{\frac{R}{2r}}-1 \overset{Euler}{\geq} 0\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt[2025]{8}+\left(\frac{R}{2r}\right)^{2025} \geq 1+\sqrt[2025]{\frac{4R}{r}} \rightarrow (3) \text{ and also},$$

$$\prod_{cyc}\sqrt[2025]{\frac{b+c}{a}}, \prod_{cyc}\sqrt[2025]{\frac{\cos\frac{A}{2}+\cos\frac{B}{2}}{\cos\frac{C}{2}}} \overset{Cesaro}{\geq} \sqrt[2025]{8} \rightarrow (4) \therefore (1),(2) \text{ and } (4)$$

$\Rightarrow$ in order to prove ① *and* ②, it suffices to prove :

$$\sqrt[2025]{8}+\left(\frac{R}{2r}\right)^{2025} \geq 1+\sqrt[2025]{\frac{4R}{r}} \rightarrow \text{true via } (3)$$

$\therefore$ ① *and* ② *are* both true $\forall \Delta ABC$, " = " iff ΔABC is equil*ateral* (QED)

1947. In ΔABC the following relationship holds:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{2}{3}(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

We will show that:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2} \text{ or}$$

$$\sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right) \geq \frac{3}{2}$$

$$Proof: \sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right)=\sum_{cyc}\frac{ac}{b(b+c)}=\sum\frac{\left(\sqrt{\frac{ac}{b}}\right)^2}{b+c} \overset{Bergstrom}{\geq}$$

$$\geq \frac{\left(\sqrt{\frac{ac}{b}}+\sqrt{\frac{ab}{c}}+\sqrt{\frac{bc}{a}}\right)^2}{2(a+b+c)} \overset{\forall x,y,z>0\ (\sum x)^2 \geq 3\sum xy}{\geq} \frac{3\sum\sqrt{\frac{ac}{b}}\sqrt{\frac{ab}{c}}}{2(a+b+c)}=$$

$$=\frac{3\sum\sqrt{a^2}}{2(a+b+c)}=\frac{3(a+b+c)}{2(a+b+c)}=\frac{3}{2}$$

so the proof complete.

$$Now\ \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\geq$$

$$=\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{3}{2}\overset{Nesbitt}{\geq}$$

$$\geq\frac{1}{3}\cdot\frac{3}{2}+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\frac{3}{2}=$$

$$=\frac{1}{2}+\frac{3}{2}+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=$$

$$=2+\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=\frac{2}{3}\left(3+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)=$$

$$=\frac{2}{3}\sum\left(1+\frac{a}{b+c}\right)=\frac{2}{3}\sum\left(\frac{a+b+c}{b+c}\right)=$$

$$=\frac{2}{3}(a+b+c)\sum\frac{1}{b+c}=\frac{2}{3}(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$Equality\ if\ \ a=b=c$$

1948. Let a,b,c be sides in the right triangle $ABC, A=90^\circ$. Prove that:

$$c^{c^2}.\,b^{b^2}\geq\left(\frac{b+c}{2}\right)^{a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\Delta ABC\ right\ triangle\ at\ A,\ \ b^2+c^2=a^2\ and\ \frac{b^2+c^2}{2}\overset{CBS}{\geq}\frac{(b+c)^2}{4}\ \ or$$

$$(b^2+c^2)\geq\frac{(b+c)^2}{2}\quad(1)$$

Let us consider c with associated weight c^2 and b with associated weight b^2

$$G.M\geq\ H.M\ or\ \left(c^{c^2}.\,b^{b^2}\right)^{\frac{1}{c^2+b^2}}\geq\frac{c^2+b^2}{\frac{c^2}{c}+\frac{b^2}{b}}=\frac{c^2+b^2}{c+b}\overset{(1)}{\geq}\frac{(c+b)^2}{2(c+b)}=\frac{c+b}{2}$$

$$or\ c^{c^2}.\,b^{b^2}\geq\left(\frac{c+b}{2}\right)^{c^2+b^2}=\left(\frac{b+c}{2}\right)^{a^2}\quad(as\ b^2+c^2=a^2)$$

$$Equality\ holds\ for:\ A=\frac{\pi}{2},B=C=\frac{\pi}{4}$$

1949. If ΔABC is right in A then:

$$\sqrt[4]{\frac{(-a^6+b^6+c^6)(-a^{14}+b^{14}+c^{14})}{21(a^4-b^2c^2)^2}}=4Rr(2R+r)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\Delta ABC \text{ is right at } A, \text{ so } a^2=b^2+c^2$$

$$2R=a \text{ and for right angle triangle inradius } r=\frac{1}{2}(b+c-a) \text{ or}$$

$$b+c=2r+a=2r+2R \text{ and}$$

$$2s=a+b+c=2R+2r+2R=4R+2r \text{ or } s=2R+r\ (1)$$

$$\text{Let } a^2=x, b^2=y, c^2=z \text{ then } (-a^6+b^6+c^6)=(-x^3+y^3+z^3)$$

$$=-\left(x^3-(y^3+z^3)\right)\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}-((y+z)^3-y^3-z^3)$$

$$=-(y^3+z^3+3yz(y+z)-y^3-z^3)=-3yz(y+z)\ (2)$$

$$-a^{14}+b^{14}+c^{14}=-(x^7-y^7-z^7)\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}-((y+z)^7-y^7-z^7)=$$

$$=-\left(y^7+7y^6z+21y^5z^2+35y^4z^3+35y^3z^4+21y^2z^5+7yz^6+z^7-y^7-z^7\right)=$$

$$=-7yz(y+z)(y^4+2y^3z+3y^2z^2.+2yz^3+z^4\ (3)$$

$$(a^4-b^2c^2)^2=(x^2-yz)^2\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}((y+z)^2-yz)^2=(y^2+z^2+yz)^2$$

$$=(y^4+2y^3z+3y^2z^2.+2yz^3+z^4\ )_{\text{using formula } (p+q+r)^2=\sum p^2+2\sum pq}\ (4)$$

$$\sqrt[4]{\frac{(-a^6+b^6+c^6)(-a^{14}+b^{14}+c^{14})}{21(a^4-b^2c^2)^2}}\overset{(2),(3),(4)}{=}$$

$$=\sqrt[4]{\left(\frac{(-3yz(y+z)\ )(-7yz(y+z)(y^4+2y^3z+3y^2z^2.+2yz^3+z^4))}{(y^4+2y^3z+3y^2z^2.+2yz^3+z^4\ )21}\right)}$$

$$=\sqrt[4]{(y+z)^2y^2z^2}\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}\sqrt[4]{(x)^2y^2z^2}\overset{a^2=x,b^2=y,c^2=z}{=}$$

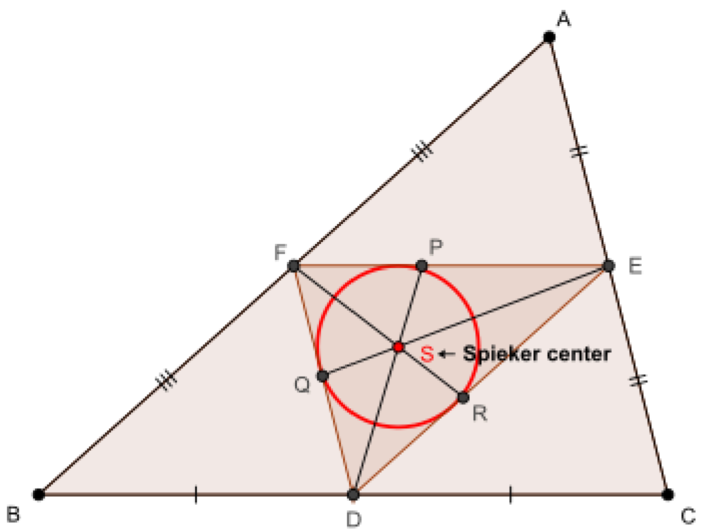
$$=\sqrt[4]{a^4b^4c^4}=abc=4Rrs\overset{(1)}{=}4Rr(2R+r)$$

1950. **In any Δ ABC, the following relationship holds :**

$$\frac{m_a m_b m_c}{h_a h_b h_c} \ge \frac{R}{2r} \cdot \sqrt[4]{\frac{p_a p_b p_c}{w_a w_b w_c}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3 + c^3 - abc + a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3 + c^3 - abc + a(4m_a^2)\right)$$

$$\Rightarrow p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3 + c^3 - abc + a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3 + c^3 - abc) - \left(1 - \frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3 + c^3 - abc) - (2b^2 + 2c^2 - a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2 + 4a(b+c)(b-c)^2 + 2(b^2 - c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2 + 2a(b+c) + (b+c)^2) + ((b+c)^2 + 2a(b+c) + a^2) - a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2 - a^2) = \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\bullet)}{=} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}$$

$$\text{Now, } m_a^4 w_a \overset{\text{Lascu + A-G}}{\geq} m_a^3.\, s(s-a) \overset{?}{\geq} s^2(s-a)^2.\, p_a \Leftrightarrow m_a^6 \overset{?}{\geq} s^2(s-a)^2.\, p_a^2$$

$$\Leftrightarrow \frac{m_a^4}{s^2(s-a)^2} - 1 \overset{?}{\geq} \frac{p_a^2}{m_a^2} - 1$$

$$\overset{\text{via } (\bullet)}{\Leftrightarrow} \frac{\left(s(s-a) + \frac{(b-c)^2}{4}\right)^2 - s^2(s-a)^2}{s^2(s-a)^2} \overset{?}{\geq} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2.\, m_a^2}$$

$$\Leftrightarrow m_a^2.\frac{\frac{(b-c)^4}{16} + s(s-a).\frac{(b-c)^2}{2}}{s^2(s-a)^2} \overset{?}{\geq} \frac{(b-c)^2(8s^2 - a^2)}{4(2s+a)^2}$$

$$\Leftrightarrow (b-c)^2\left(m_a^2.\frac{\frac{(b-c)^2}{16} + \frac{s(s-a)}{2}}{s^2(s-a)^2} - \frac{8s^2 - a^2}{4(2s+a)^2}\right) \underset{(\blacksquare)}{\overset{?}{\geq}} 0$$

$\because (b-c)^2 \geq 0$ and $m_a^2 \geq s(s-a) \therefore$ in order to prove $(\blacksquare)$, it suffices to prove :

$$\frac{s(s-a).\frac{s(s-a)}{2}}{s^2(s-a)^2} - \frac{8s^2-a^2}{4(2s+a)^2} > 0 \Leftrightarrow 2(2s+a)^2 > 8s^2-a^2 \Leftrightarrow 8sa+3a^2 > 0$$

$$\rightarrow \text{true} \Rightarrow (\blacksquare) \text{ is true} \therefore m_a^4 w_a \geq s^2(s-a)^2.p_a \Rightarrow \boxed{\frac{m_a^4}{s^2(s-a)^2} \geq \frac{p_a}{w_a}}$$

$$\Rightarrow \frac{m_a}{\sqrt{r_b r_c}} \geq \sqrt[4]{\frac{p_a}{w_a}} \Rightarrow \frac{m_a}{h_a}.\frac{h_a}{\sqrt{r_b r_c}} \geq \sqrt[4]{\frac{p_a}{w_a}} \text{ and analogs}$$

$$\Rightarrow \frac{m_a m_b m_c}{h_a h_b h_c}.\frac{h_a h_b h_c}{r_a r_b r_c} \geq \sqrt[4]{\frac{p_a p_b p_c}{w_a w_b w_c}} \Rightarrow \frac{m_a m_b m_c}{h_a h_b h_c}.\frac{\frac{2r^2s^2}{R}}{rs^2} \geq \sqrt[4]{\frac{p_a p_b p_c}{w_a w_b w_c}}$$

$$\Rightarrow \frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r}.\sqrt[4]{\frac{p_a p_b p_c}{w_a w_b w_c}} \ \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have the following known formula (see [1, pp. 1]),

$$p_a{}^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

And by the formulas for median and angle bisector of triangle ABC, $m_a{}^2 = \frac{1}{4}(2b^2+2c^2-a^2)$

and $w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}$, we can easily get

$$m_a{}^2 = s(s-a) + \frac{(b-c)^2}{4} \text{ and } w_a{}^2 = s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

Using these identities, we have

$$p_a w_a \leq \frac{p_a{}^2 + w_a{}^2}{2} = s(s-a) + \frac{1}{2}\left(\frac{s(3s+a)}{(2s+a)^2} - \frac{s(s-a)}{(2s-a)^2}\right)(b-c)^2$$

$$= s(s-a) + \frac{s(4s^3-4s^2a+sa^2+a^3)(b-c)^2}{(4s^2-a^2)^2}$$

$$= s(s-a) + \left(\frac{1}{4} - \frac{4sa(s-a)(4s+a)+a^4}{4(4s^2-a^2)^2}\right)(b-c)^2 \leq m_a{}^2.$$

Then

$$\frac{R}{2r}\cdot\sqrt[4]{\frac{p_ap_bp_c}{w_aw_bw_c}}=\frac{R}{2r}\cdot\sqrt[4]{\prod_{cyc}\frac{p_aw_a}{{w_a}^2}}\le\frac{R}{2r}\cdot\sqrt[4]{\prod_{cyc}\frac{{m_a}^2}{{w_a}^2}}$$

$$=\frac{Rm_am_bm_c}{2r\sqrt{\prod_{cyc}m_aw_a}}\overset{Panaitopol}{\le}\frac{Rm_am_bm_c}{2r\sqrt{\prod_{cyc}s(s-a)}}$$

$$=\frac{Rm_am_bm_c}{2s^2r^2}=\frac{abc.m_am_bm_c}{(2F)^3}=\frac{m_am_bm_c}{h_ah_bh_c}.$$

Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuștei, Mohamed Amine Ben Ajiba,
SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE.

1951. In ΔABC the following relationship holds:

$$\sqrt[r_a]{a}\cdot\sqrt[r_b]{b}\cdot\sqrt[r_c]{c}\le\sqrt[r]{\frac{a+b+c}{3}}\le\sqrt[h_a]{a}\cdot\sqrt[h_b]{b}\cdot\sqrt[h_c]{c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$WLOG\ a\ge b\ge c\ then\ r_a\ge r_b\ge r_c.$$

$$Now\ \frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\overset{Chebyshev}{\le}\frac{1}{3}(a+b+c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\ (1)$$

$$We\ know\ that\ \sum\frac{1}{r_a}=\sum\frac{1}{h_a}=\frac{1}{r}\ (2)$$

$$ah_a=bh_b=ch_c=a\frac{2F}{a}=b\frac{2F}{b}=c\frac{2F}{c}=2F\ (3)$$

$$Let\ us\ consider\ a\ with\ associated\ weight\ \frac{1}{r_a},b\ with\ associated\ weight\ \frac{1}{r_b},$$
$$c\ with\ associated\ weight\ \frac{1}{r_c}$$

$$GM\le AM\ or\ \left(\sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\right)^{\frac{1}{\sum\frac{1}{r_a}}}\le\frac{\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}}{\sum\frac{1}{r_a}}\overset{(1)}{\le}\frac{\frac{1}{3}(a+b+c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)}{\sum\frac{1}{r_a}}=$$

$$=\frac{a+b+c}{3}\ or\ \sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\le\left(\frac{a+b+c}{3}\right)^{\sum\frac{1}{r_a}}\overset{(2)}{=}$$

$$=\left(\frac{a+b+c}{3}\right)^{\frac{1}{r}}=\sqrt[r]{\frac{a+b+c}{3}}\ (A)$$

Let us consider a with associated weight $\frac{1}{h_a}$, b *with associated weight* $\frac{1}{h_b}$,

c *with associated weight* $\frac{1}{h_c}$

$$GM\geq HM \text{ or } \left(\sqrt[h_a]{a}\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\sum\frac{1}{h_a}}}\geq\frac{\sum\frac{1}{h_a}}{\sum\frac{1}{ah_a}}\overset{(3)}{\geq}\frac{\sum\frac{1}{h_a}}{\frac{3}{2F}}\overset{(2)}{=}\frac{2F}{3r}=2r.\frac{s}{3r}=\frac{2s}{3}=\frac{a+b+c}{3}$$

$$\text{or } \left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\sum\frac{1}{h_a}}}=\left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{\frac{1}{\frac{1}{r}}}=$$

$$=\left(\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}\right)^{r}\geq\frac{a+b+c}{3}\ \text{ or}$$

$$\sqrt[h_a]{a}\sqrt[h_b]{b}\sqrt[h_c]{c}\geq\sqrt[r]{\frac{a+b+c}{3}}\,(B)$$

$$\text{From } (A) \text{ and } (B) \text{ we have } \sqrt[r_a]{a}.\sqrt[r_b]{b}.\sqrt[r_c]{c}\leq\sqrt[r]{\frac{a+b+c}{3}}\leq\sqrt[h_a]{a}.\sqrt[h_b]{b}\sqrt[h_c]{c}$$

Equality holds for the equilateral triangle.

1952. In ΔABC the following relationship holds:

$$(h_a-2r)^{\frac{1}{h_a-2r}}.(h_b-2r)^{\frac{1}{h_b-2r}}.(h_c-2r)^{\frac{1}{h_c-2r}}\leq\left(\frac{3R^2}{2R-r}\right)^{\frac{2R-r}{r^2}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{s-a}=\frac{\sum a(s-b)(s-c)}{(s-a)(s-b)(s-c)}=\frac{\sum a(s^2-s(b+c)+bc)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{s^2(a+b+c)-2s(ab+bc+ca)+3abc}{(s-a)(s-b)(s-c)}=\frac{2s^3-2s(s^2+r^2+4Rr)+12Rrs}{sr^2}=$$

$$=\frac{2s^2-2s^2-2r^2-8Rr+12Rr}{r^2}=\frac{4R-2r}{r}\ (1)$$

$$\sum\frac{1}{(h_a-2r)}=\sum\frac{1}{\left(\frac{2rs}{a}-2r\right)}=\frac{1}{2r}\sum\frac{a}{s-a}\overset{(1)}{=}\frac{1}{2r}\cdot\frac{4R-2r}{r}=\frac{2R-r}{r^2}\quad(2)$$

Let us consider (h_a-2r) *with associated weight* $\frac{1}{(h_a-2r)}$,

(h_b-2r) *with associated weight* $\frac{1}{(h_b-2r)}$ *and* (h_c-2r) *with* $\frac{1}{(h_c-2r)}$.

By $G.M\le A.M$ *or* $\left((h_a-2r)^{\frac{1}{h_a-2r}}.(h_b-2r)^{\frac{1}{h_b-2r}}.(h_c-2r)^{\frac{1}{h_c-2r}}\right)^{\frac{1}{\sum\frac{1}{(h_a-2r)}}}\le$

$$\le\frac{(h_a-2r).\frac{1}{(h_a-2r)}+(h_b-2r).\frac{1}{(h_b-2r)}+(h_c-2r).\frac{1}{(h_c-2r)}}{\sum\frac{1}{(h_a-2r)}}\overset{(2)}{=}\frac{3}{\frac{2R-r}{r^2}}$$

$$=\frac{3r^2}{2R-r}$$

$$\left((h_a-2r)^{\frac{1}{h_a-2r}}.(h_b-2r)^{\frac{1}{h_b-2r}}.(h_c-2r)^{\frac{1}{h_c-2r}}\right)^{\frac{1}{\sum\frac{1}{(h_a-2r)}}}\le\frac{3r^2}{2R-r}$$

$$(h_a-2r)^{\frac{1}{h_a-2r}}.(h_b-2r)^{\frac{1}{h_b-2r}}.(h_c-2r)^{\frac{1}{h_c-2r}}\le\left(\frac{3r^2}{2R-r}\right)^{\sum\frac{1}{(h_a-2r)}}\overset{(2)}{=}\left(\frac{3R^2}{2R-r}\right)^{\frac{2R-r}{r^2}}$$

Equality $a=b=c$

1953. In any Δ ABC, holds :

$$r_a^3+r_b^3+r_c^3-3r_ar_br_c\ge\frac{\sqrt{3}}{2}(a^3+b^3+c^3-3abc)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$r_a^3+r_b^3+r_c^3-3r_ar_br_c\ge\frac{\sqrt{3}}{2}(a^3+b^3+c^3-3abc)$$

$$\Leftrightarrow\left(\sum_{cyc}r_a\right)^3-3\left(\left(\sum_{cyc}r_a\right)\left(\sum_{cyc}r_ar_b\right)-r_ar_br_c\right)-3r_ar_br_c\ge$$

$$\frac{\sqrt{3}}{2}.2s(s^2-6Rr-3r^2-6Rr)$$

$$\Leftrightarrow (4R+r)^3-3s^2(4R+r)\geq\sqrt{3}s(s^2-12Rr-3r^2)$$
$$\Leftrightarrow \frac{(4R+r)}{\sqrt{3}s}((4R+r)^2-3s^2)\overset{(*)}{\geq} s^2-12Rr-3r^2$$
Now, via Doucet (or Trucht), LHS of $(*)\geq 16R^2+8Rr+r^2-3s^2$
$$\overset{?}{\geq} s^2-12Rr-3r^2\Leftrightarrow s^2\overset{?}{\leq}4R^2+5Rr+r^2$$
$$\Leftrightarrow (s^2-4R^2-4Rr-3r^2)-r(R-2r)\overset{?}{\leq}0$$
$$\rightarrow \text{true} \because s^2\overset{\text{Gerretsen}}{\leq}4R^2+4Rr+3r^2 \text{ and } -r(R-2r)\overset{\text{Euler}}{\leq}0\Rightarrow(*)\text{ is true}$$
$$\therefore r_a^3+r_b^3+r_c^3-3r_ar_br_c\geq\frac{\sqrt{3}}{2}(a^3+b^3+c^3-3abc)$$
$\forall\Delta ABC$, " = " iff ΔABC is equilateral (QED)

1954. In any ΔABC, the following relationship holds:

$$\sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\leq\sqrt{3\left(1-\frac{r}{2R}\right)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\right)^{\sin\frac{B}{2}}=\left(1+\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}-1\right)\right)^{\sin\frac{B}{2}}\overset{\text{Bernoulli}}{\leq}1+\left(\sin\frac{B}{2}\right)\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}-1\right)$$

$$=1+\frac{\sin^2\frac{A}{2}}{\sin\frac{B}{2}}-\sin\frac{B}{2}\Rightarrow\left(\sin\frac{B}{2}\right)\left(\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\right)^{\sin\frac{B}{2}}\leq\sin\frac{B}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}$$

$$\Rightarrow\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\leq\sin\frac{B}{2}+\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\text{ and analogs}$$

$$\therefore\sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\leq\sum_{cyc}\sin\frac{B}{2}+\sum_{cyc}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)=$$

$$=\sum_{cyc}\sin\frac{A}{2}\overset{\text{Jensen}}{\leq}\frac{3}{2}=\sqrt{\frac{9}{4}}=\sqrt{3-\frac{3r}{4r}}\overset{\text{Euler}}{\leq}\sqrt{3-\frac{3r}{2R}}$$

$$\therefore \sum_{cyc}\left(\left(\sin\frac{A}{2}\right)^{2\sin\frac{B}{2}}.\left(\sin\frac{B}{2}\right)^{1-2\sin\frac{B}{2}}\right)\le\sqrt{3\left(1-\frac{r}{2R}\right)}$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

1955. In any $\Delta\,ABC$, the following relationship holds :

$$\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$
$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}\le 1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Weighted GM}\ge\text{Weighted HM}\Rightarrow\sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}}}$$

$$\ge\frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{A}{2}}+\frac{r}{r_b.\tan\frac{B}{2}}+\frac{r}{r_c.\tan\frac{C}{2}}}\ \text{and}\ \because\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$

$$\therefore\left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}}\ge\frac{1}{rs.\sum_{cyc}\frac{1}{r_a^2}}=\frac{1}{rs.\left(\left(\sum_{cyc}\frac{1}{r_a}\right)^2-2\sum_{cyc}\frac{1}{r_ar_b}\right)}$$

$$=\frac{1}{rs.\left(\frac{1}{r^2}-\frac{2(4R+r)}{rs^2}\right)}=\frac{1}{rs.\left(\frac{s^2-8Rr-2r^2}{r^2s^2}\right)}=\frac{rs}{s^2-8Rr-2r^2}$$

$$\therefore\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}=\frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{A}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_c}}}$$

$$\boxed{\le}\frac{\left(\frac{r}{s}\right)}{\left(\frac{rs}{s^2-8Rr-2r^2}\right)}=\frac{s^2-8Rr-2r^2}{s^2}\to(1)$$

Again, Weighted GM ≥ Weighted HM ⇒ $\sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}}}$

$$\geq \frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{B}{2}}+\frac{r}{r_b.\tan\frac{C}{2}}+\frac{r}{r_c.\tan\frac{A}{2}}} \text{ and } \because \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$

$$\therefore \left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}} \geq \frac{1}{rs.\sum_{cyc}\frac{1}{r_ar_b}} = \frac{1}{\frac{rs}{rs^2}(4R+r)} = \frac{s}{4R+r}$$

$$\therefore \left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}} = \frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{B}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_c}}}$$

$$\boxed{\leq} \frac{\left(\frac{r}{s}\right)}{\left(\frac{s}{4R+r}\right)} = \frac{4Rr+r^2}{s^2} \to (2)$$

Also, Weighted GM ≥ Weighted HM ⇒ $\sqrt[\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)]{\left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}}}$

$$\geq \frac{\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}}{\frac{r}{r_a.\tan\frac{C}{2}}+\frac{r}{r_b.\tan\frac{A}{2}}+\frac{r}{r_c.\tan\frac{B}{2}}} \text{ and } \because \frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1,$$

$$\therefore \left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}} \geq \frac{1}{rs.\sum_{cyc}\frac{1}{r_ar_c}} = \frac{1}{\frac{rs}{rs^2}(4R+r)} = \frac{s}{4R+r}$$

$$\therefore \left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}} = \frac{\left(\tan\frac{A}{2}\right)\left(\tan\frac{B}{2}\right)\left(\tan\frac{C}{2}\right)}{\left(\tan\frac{C}{2}\right)^{\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{\frac{r}{r_c}}}$$

$$\boxed{\leq} \frac{\left(\frac{r}{s}\right)}{\left(\frac{s}{4R+r}\right)} = \frac{4Rr+r^2}{s^2} \to (3) \therefore (1)+(2)+(3) \Rightarrow$$

$$\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$

$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}} \leq \frac{s^2-8Rr-2r^2}{s^2}+2.\frac{4Rr+r^2}{s^2}$$

$$\therefore \left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_c}}+\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_a}}.\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_b}}.\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_c}}$$

$$+\left(\tan\frac{C}{2}\right)^{1-\frac{r}{r_a}}\cdot\left(\tan\frac{A}{2}\right)^{1-\frac{r}{r_b}}\cdot\left(\tan\frac{B}{2}\right)^{1-\frac{r}{r_c}}\le 1$$

$\forall\,\Delta\,ABC,'' ='' \text{ iff } \Delta\,ABC$ is equilateral (QED)

1956. In ΔABC the following relationship holds:

$$\sum\frac{\cos B\cos C}{\cos(B-C)}\le\cos^2 A+\cos^2 B+\cos^2 C$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{b\cos B+c\cos C}{a}=\frac{R(\sin 2B+\sin 2C)}{2R\sin A}=$$

$$=\frac{2R\sin(B+C)\cos(B-C)}{2R\sin A}\overset{A+B+C=\pi}{=}\frac{2R\sin(A)\cos(B-C)}{2R\sin A}=$$

$$=\cos(B-C)\quad and\ \cos(B-C)=\frac{b\cos B+c\cos C}{a}=$$

$$=\frac{R(\sin 2B+\sin 2C)}{2R\sin A}=\frac{(\sin 2B+\sin 2C)}{2\sin A}\quad(1)$$

$$sin2A+\sin 2B+\sin 2C=2\sin(A+B)\cos(A-B)+2\sin C\cos C$$
$$=2\sin(\pi-C)\cos(A-B)+2\sin C\cos C=$$
$$=2\sin C\cos(A-B)+2\sin C\cos C=2\sin C\,(\cos(A-B)+\cos C)=$$
$$=2\sin C\left(\cos(A-B)+\cos(\pi-(A+B))\right)=$$
$$=2\sin C\,(\cos(A-B)-\cos(A+B))=4\sin A\sin B\sin C\quad(2)$$

$$\cos(B-C)=\cos B\cos C+\sin B\sin C\quad(3)$$

$$\sum\frac{\cos B\cos C}{\cos(B-C)}\overset{(3)}{=}\sum\left(1-\frac{\sin B\sin C}{\cos(B-C)}\right)\overset{(1)}{=}$$

$$=3-\sum\left(\frac{\sin B\sin C}{\cos(B-C)}\right)\overset{(1)}{=}3-\sum\left(\frac{2\sin A\sin B\sin C}{\sin 2B+\sin 2C}\right)=$$

$$=3-2\sin A\sin B\sin C\sum\left(\frac{1}{\sin 2B+\sin 2C}\right)\overset{Bergstrom}{\le}$$

$$\le 3-2\sin A\sin B\sin C\frac{(1+1+1)^2}{2(\sin 2A+\sin 2B+\sin 2C)}\overset{(2)}{=}$$

$$= \ \ 3-2\sin A\sin B\sin C\frac{9}{8\sin A\sin B\sin C}=3-\frac{9}{4}=\frac{3}{4}\ (4)$$

$$\cos^2 A+\cos^2 B+\cos^2 C=3-\sum\sin^2 A=3-\frac{a^2+b^2+c^2}{4R^2}\overset{Leibniz}{\geq}$$

$$\geq\ 3-\frac{9R^2}{4R^2}=3-\frac{9}{4}=\frac{3}{4}\ (5)$$

$$From\ (4) and (5) we\ \ get\sum\frac{\cos B\cos C}{\cos(B-C)}\leq\cos^2 A+\cos^2 B+\cos^2 C$$

Equality holds for $A=B=C$.

1957. In ΔABC the following relationship holds:

$$\sum\sqrt[3]{\frac{3r}{r_a}}\geq\frac{16r}{R}-5$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sqrt[3]{\frac{3r}{r_a}}=\sqrt[3]{\frac{3r}{r_a}1.1}\overset{GM\geq HM}{\geq}\frac{3}{\frac{1}{\frac{3r}{r_a}}+\frac{1}{1}+\frac{1}{1}}=\frac{3}{\frac{r_a}{3r}+2}\ \ (1)$$

$$\sum\sqrt[3]{\frac{3r}{r_a}}\overset{(1)}{\geq}\sum\frac{3}{\frac{r_a}{3r}+2}=3\sum\frac{1^2}{\frac{r_a}{3r}+2}\overset{Bergstrom}{\geq}\frac{3(1+1+1)^2}{\frac{1}{3r}(r_a+r_b+r_c)+6}=$$

$$=\frac{27}{\frac{4R+r}{3r}+6}=\frac{81r}{4R+r+18r}=\frac{81r}{4R+19r}\overset{EULER}{\geq}\frac{81r}{4R+\frac{19R}{2}}=$$

$$=81r\cdot\frac{2}{27R}=\frac{6r}{R}=\frac{16r}{R}-\frac{10r}{R}\overset{EULER}{\geq}\frac{16r}{R}-10\cdot\frac{1}{2}=\frac{16r}{R}-5$$

Equality holds for the equilateral triangle.

1958. In ΔABC the following relationship holds:

$$\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}=\sum_{cyc}\frac{r_a}{h_a}=\sum_{cyc}\frac{\frac{F}{s-a}}{\frac{2F}{a}}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}=$$

$$=\frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc}a(s-b)(s-c)=$$

$$=\frac{s}{2s(s-a)(s-b)(s-c)}\cdot 2rs(2R-r)=\frac{s^2r(2R-r)}{F^2}=$$

$$=\frac{s^2r(2R-r)}{s^2r^2}=\frac{2R-r}{r}\overset{EULER}{\geq}\frac{4r-r}{r}=3$$

Equality holds for: $a=b=c$.

1959. **In ΔABC the following relationship holds:**

$$\sum\cos A+2\sum\tan^2\frac{A}{2}\geq\frac{7}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos A+2\sum\tan^2\frac{A}{2}=\left(1+\frac{r}{R}\right)+2\left(\left(\frac{4R+r}{s}\right)^2-2\right)$$

$$We\ need\ to\ show:\ \left(1+\frac{r}{R}\right)+2\left(\left(\frac{4R+r}{s}\right)^2-2\right)\geq\frac{7}{2}\ or$$

$$2\left(\frac{4R+r}{s}\right)^2+\frac{r}{R}-3\geq\frac{7}{2}\ or$$

$$2\left(\frac{4R+r}{s}\right)^2+\frac{r}{R}\geq\frac{7}{2}+3\ or$$

$$2\left(\frac{4R+r}{s}\right)^2+\frac{r}{R}\geq\frac{13}{2}\ or$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2}+\frac{r}{R}\geq\frac{13}{2}(Gerretsen)\ or$$

$$\frac{2R(16R^2+8Rr+r^2)+r(4R^2+4Rr+3r^2)}{(4R^2+4Rr+3r^2)R}\geq\frac{13}{2}or$$

$$(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3)\geq$$

$$13(4R^2+4Rr+3r^2)R\ or\ (64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3)\geq$$
$$\geq 52R^3+52R^2r+39Rr^3\ or\ (64R^3+40R^2r+12Rr^3+6r^3)-(52R^3+52R^2r+39Rr^3)$$

$$12R^3-12R^2r-27Rr^2+6r^3\geq 0\ or$$

$$(R-2r)(12R^2+12Rr-3r^2)\geq 0\ true\ (Euler)$$

$$Equality\ holds\ for\ A=B=C$$

1960. In ΔABC the following relationship holds:

$$\sum\cos A+2\sum\sec^2\frac{A}{2}\geq\frac{19}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos A+2\sum\sec^2\frac{A}{2}=\left(1+\frac{r}{R}\right)+2\left(3+\sum\tan^2\frac{A}{2}\right)=$$
$$=\left(1+\frac{r}{R}\right)+6+2\left(\left(\frac{4R+r}{s}\right)^2-2\right)=3+2\left(\frac{4R+r}{s}\right)^2+\frac{r}{R}\overset{Gerretsen}{\geq}$$
$$\geq\ 3+\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2}+\frac{r}{R}$$

$$We\ need\ to\ show: 3+\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2}+\frac{r}{R}\geq\frac{19}{2}$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2}+\frac{r}{R}\geq\frac{19}{2}-3\ \ or$$

$$\frac{2(16R^2+8Rr+r^2)}{4R^2+4Rr+3r^2}+\frac{r}{R}\geq\frac{13}{2}$$

$$\frac{2R(16R^2+8Rr+r^2)+r(4R^2+4Rr+3r^2)}{(4R^2+4Rr+3r^2)R}\geq\frac{13}{2}$$

$$(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3)\geq 13(4R^2+4Rr+3r^2)R\ or$$

$$(64R^3+32R^2r+4Rr^2+8R^2r+8Rr^2+6r^3)\geq$$
$$\geq 52R^3+52R^2r+39Rr^3\ or(64R^3+40R^2r+12Rr^3+6r^3)-(52R^3+52R^2r+39Rr^3)$$

$$12R^3-12R^2r-27Rr^2+6r^3\geq 0$$
$$(R-2r)(12R^2+12Rr-3r^2)\geq 0\ true\ (Euler)$$

$$Equality\ holds\ for\ A=B=C$$

1961. In ΔABC the following relationship holds:

$$\sum sin\frac{A}{2}+\sum\cot^2\frac{A}{2}\geq\frac{21}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum sin\frac{A}{2}\overset{Jensen}{\leq}3\sin\frac{\pi}{6}=\frac{3}{2}\ (f(x)=\sin x\ is\ concave\ in\ \left(0,\frac{\pi}{2}\right)\ (1)$$
$$We\ need\ to\ show\sum sin\frac{A}{2}+\sum\cot^2\frac{A}{2}\geq\frac{21}{2}\ \ or$$
$$\sum sin\frac{A}{2}+\sum cosec^2\frac{A}{2}-3\geq\frac{21}{2}\ or$$
$$\sum sin\frac{A}{2}+\sum cosec^2\frac{A}{2}\geq\frac{21}{2}+3=\frac{27}{2}\ or$$
$$\sum sin\frac{A}{2}+\sum\frac{1}{\sin^2\frac{A}{2}}\geq\frac{27}{2}\ (2)$$
$$let\sin\frac{A}{2}=x,\sin\frac{B}{2}=y,\sin\frac{C}{2}=z$$
$$Using\ (1)and\ \ (2)\ we\ need\ to\ show\sum x+\sum\frac{1}{x^2}\geq\frac{27}{2}$$
$$where\ x+y+z\leq\frac{3}{2}\ and\ x,y,z\in(0,1)$$
$$we\ will\ show\ x+\frac{1}{x^2}\geq 12-15x\ (3)$$
$$proof: x+\frac{1}{x^2}\geq 12-15x\ or\ x^3+1\geq 12x^2-15x^3\ or$$
$$16x^3-12x^2+1\geq 0\ or\ (2x-1)^2(4x+1)\geq 0\ true\ as\ x\in(0,1)$$

$$\sum\left(x+\frac{1}{x^2}\right)\overset{(3)}{\geq}\sum(12-15x)=3\times 12-15(x+y+z)=$$
$$=36-15(x+y+z)\geq 36-15\times\frac{3}{2}=\frac{27}{2}\left(as\ \ x+y+z\leq\frac{3}{2}\right)\ or$$

$$\sum x+\sum\frac{1}{x^2}\geq\frac{27}{2}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{2}\ or\ \sin\frac{A}{2}=\sin\frac{B}{2}=\sin\frac{C}{2}=\frac{1}{2}\ or\ A=B=C=\frac{\pi}{3}$$

1962. If in $\Delta ABC, a\neq b\neq c\neq a$ then:

$$\frac{1}{m_a^2-m_b^2}\left(\frac{1}{h_a^2}-\frac{1}{h_b^2}\right)^2+\frac{1}{m_b^2-m_c^2}\left(\frac{1}{h_b^2}-\frac{1}{h_c^2}\right)^2+\frac{1}{m_c^2-m_a^2}\left(\frac{1}{h_c^2}-\frac{1}{h_a^2}\right)^2=0$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$m_a^2-m_b^2=\frac{2b^2+2c^2-a^2}{4}-\frac{2a^2+2c^2-b^2}{4}=\frac{3}{4}(b^2-a^2)$$
$$\left(\frac{1}{h_a^2}-\frac{1}{h_b^2}\right)^2=\left(\frac{a^2}{4F^2}-\frac{b^2}{4F^2}\right)^2=\frac{1}{16F^2}(a^2-b^2)^2$$
$$\frac{1}{m_a^2-m_b^2}\left(\frac{1}{h_a^2}-\frac{1}{h_b^2}\right)^2=\frac{1}{16F^2}(a^2-b^2)^2\cdot\frac{4}{3}\frac{1}{(b^2-a^2)}=$$

$$=-\frac{1}{12F^2}(a^2-b^2)=\frac{1}{12F^2}(b^2-a^2)$$

$$\frac{1}{m_a^2-m_b^2}\left(\frac{1}{h_a^2}-\frac{1}{h_b^2}\right)^2+\frac{1}{m_b^2-m_c^2}\left(\frac{1}{h_b^2}-\frac{1}{h_c^2}\right)^2+\frac{1}{m_c^2-m_a^2}\left(\frac{1}{h_c^2}-\frac{1}{h_a^2}\right)^2=$$

$$=\sum\frac{1}{m_a^2-m_b^2}\left(\frac{1}{h_a^2}-\frac{1}{h_b^2}\right)^2=\sum\frac{1}{12F^2}(b^2-a^2)=$$

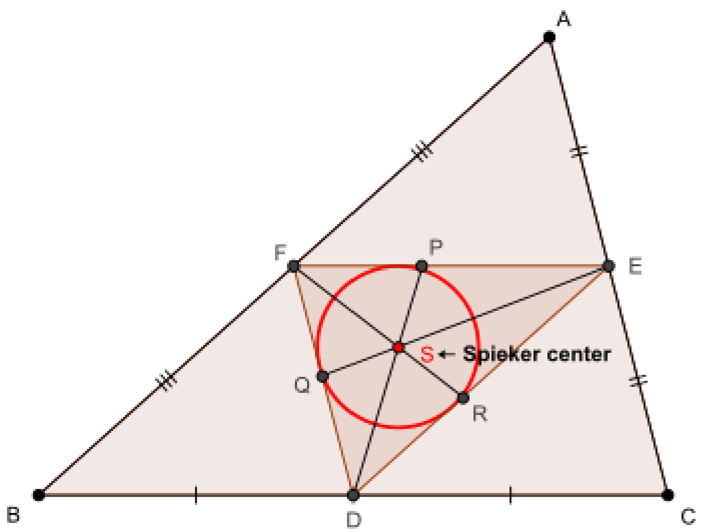
$$=\frac{1}{12F^2}(b^2-a^2+c^2-b^2+a^2-c^2)=0$$

1963. In any Δ ABC, the following relationship holds :

$$\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \cdot \sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a + w_a}{w_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

Via sine law on Δ AFS, $\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$

$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}$ and via sine law on Δ AES, $b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$

Now, $[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via }(ii)}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

Now, $2m_a^2w_a\overset{\text{Lascu}+A-G}{\geq}2m_a.s(s-a)\overset{?}{\geq}s(s-a)(p_a+w_a)$

$\Leftrightarrow m_a-w_a\overset{?}{\geq}p_a-m_a\Leftrightarrow\frac{m_a^2-w_a^2}{m_a+w_a}\overset{?}{\geq}\frac{p_a^2-m_a^2}{p_a+m_a}$ and in order

to prove this, it suffices to prove : $m_a^2-w_a^2\overset{?}{\geq}p_a^2-m_a^2$

$$\left(\because p_a \ge m_a \ge w_a \Rightarrow \frac{1}{m_a + w_a} \ge \frac{1}{p_a + m_a}\right) \overset{\text{via } (\bullet)}{\Leftrightarrow}$$

$$s(s-a) + \frac{(b-c)^2}{4} - s(s-a) + \frac{s(s-a)(b-c)^2}{(b+c)^2} \overset{?}{\ge} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{and in order to prove this, it suffices to prove : } \frac{1}{4} + \frac{s(s-a)}{(2s-a)^2} \overset{?}{>} \frac{8s^2-a^2}{4(2s+a)^2}$$

$$(\because (b-c)^2 \ge 0) \Leftrightarrow \frac{4s(s-a)+(2s-a)^2}{4(2s-a)^2} \overset{?}{>} \frac{8s^2-a^2}{4(2s+a)^2}$$

$$\Leftrightarrow 16s^3 - 12s^2a + 4sa^2 + a^3 \overset{?}{>} 0 \Leftrightarrow (s-a)(16s^2+4sa) + a^3 \overset{?}{>} 0 \to \text{true}$$

$$\because s - a > 0 \therefore 2m_a^2 w_a \ge s(s-a)(p_a + w_a)$$

$$\Rightarrow m_a \ge \sqrt{s(s-a)}.\sqrt{\frac{p_a + w_a}{2w_a}} \text{ and analogs}$$

$$\Rightarrow \frac{m_a m_b m_c}{h_a h_b h_c} \ge \frac{\prod_{cyc}\sqrt{r_b r_c}}{h_a h_b h_c}.\sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a + w_a}{w_a}} = \frac{r_a r_b r_c}{h_a h_b h_c}.\sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a + w_a}{w_a}}$$

$$= \frac{rs^2}{\left(\frac{2r^2s^2}{R}\right)}.\sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a + w_a}{w_a}} \therefore \frac{m_a m_b m_c}{h_a h_b h_c} \ge \frac{R}{2r}.\sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a + w_a}{w_a}}$$

$$\forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morooco

We have the following known formula (see [1, pp. 1]),

$$p_a^2 = s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

And by the formulas for median and angle bisector of triangle ABC,

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) \text{ and } w_a = \frac{2\sqrt{bcs(s-a)}}{b+c}, \text{we can easily get}$$

$$m_a^2 = s(s-a) + \frac{(b-c)^2}{4} \text{ and } w_a^2 = s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

Using these identities, we have

$$\left(\frac{p_a + w_a}{2}\right)^2 \le \frac{p_a^2 + w_a^2}{2} = s(s-a) + \frac{1}{2}\left(\frac{s(3s+a)}{(2s+a)^2} - \frac{s(s-a)}{(2s-a)^2}\right)(b-c)^2 =$$

$$= s(s-a) + \frac{s(4s^3 - 4s^2a + sa^2 + a^3)(b-c)^2}{(4s^2-a^2)^2} =$$

$$= s(s-a) + \left(\frac{1}{4} - \frac{4sa(s-a)(4s+a)+a^4}{4(4s^2-a^2)^2}\right)(b-c)^2 \le {m_a}^2 \Rightarrow \frac{p_a+w_a}{2} \le m_a$$

Then

$$\frac{R}{2r}\cdot\sqrt{\frac{1}{8}\prod_{cyc}\frac{p_a+w_a}{w_a}} \le \frac{R}{2r}\cdot\sqrt{\frac{1}{8}\prod_{cyc}\frac{2m_a}{w_a}} = \frac{Rm_am_bm_c}{2r\sqrt{\prod_{cyc}m_aw_a}} \overset{Panaitopol}{\le}$$

$$\le \frac{Rm_am_bm_c}{2r\sqrt{\prod_{cyc}s(s-a)}} = \frac{Rm_am_bm_c}{2s^2r^2} = \frac{abc.m_am_bm_c}{(2F)^3} = \frac{m_am_bm_c}{h_ah_bh_c}$$

Equality holds iff ΔABC is equilateral.

[1] **Bogdan Fuștei, Mohamed Amine Ben Ajiba,**
SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE.

1964. In ΔABC the following relationship holds:

$$a^{r_a}b^{r_b}c^{r_c} \ge \left(2\sqrt{3}r\right)^{9r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ then\ r_a \ge r_b \ge r_c$$

$$\sum\frac{r_a}{a} \overset{Chebyshev}{\le} \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(r_a+r_b+r_c) \overset{Leunberger\ II}{\le} \frac{1}{3}(4R+r)\frac{\sqrt{3}}{2r} = \frac{4R+r}{2\sqrt{3}r} \quad (1)$$

$$We\ know\ that\ \sum\frac{1}{r_a} = \frac{1}{r},\qquad \frac{\sum r_a}{3} \overset{AM-HM}{\ge} \frac{3}{\sum\frac{1}{r_a}} = 3r\ or$$

$$\sum r_a \ge 9r \quad (2)$$

Let us consider a with associated. weight r_a, b with r_b and c with r_c

$$GM \ge HM\ or\left(a^{r_a}b^{r_b}c^{r_c}\right)^{\frac{1}{(r_a+r_b+r_c)}} \ge \frac{(r_a+r_b+r_c)}{\sum\frac{r_a}{a}} \overset{(1)}{\ge} \frac{4R+r}{\frac{4R+r}{2\sqrt{3}r}} = \left(2\sqrt{3}r\right)$$

$$a^{r_a}b^{r_b}c^{r_c} \ge \left(2\sqrt{3}r\right)^{(r_a+r_b+r_c)} \overset{(2)}{\ge} \left(2\sqrt{3}r\right)^{9r}$$

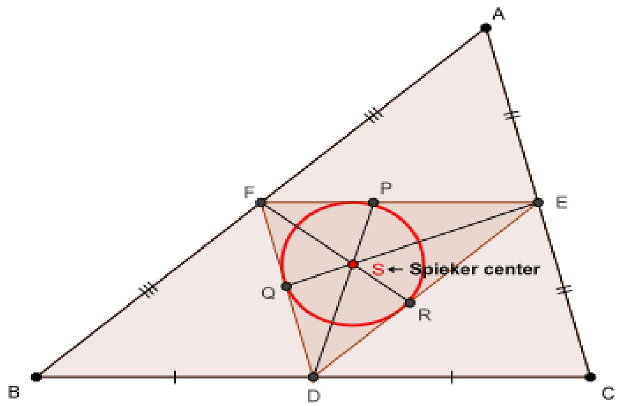
$$Equality\ holds\ for\ a = b = c$$

1965. In any Δ ABC, the following relationship holds :

$$\frac{m_a m_b m_c}{r_a r_b r_c} \geq \sqrt[4]{\frac{p_a p_b p_c}{w_a w_b w_c}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$\Rightarrow p_a^2-m_a^2=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$=\frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$=\frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2)=\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2\overset{(\bullet)}{=}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\text{Now, }m_a^4w_a\overset{\text{Lascu + A-G}}{\geq}m_a^3.s(s-a)\overset{?}{\geq}s^2(s-a)^2.p_a\Leftrightarrow m_a^6\overset{?}{\geq}s^2(s-a)^2.p_a^2$$

$$\Leftrightarrow\frac{m_a^4}{s^2(s-a)^2}-1\overset{?}{\geq}\frac{p_a^2}{m_a^2}-1$$

$$\overset{\text{via }(\bullet)}{\Leftrightarrow}\frac{\left(s(s-a)+\frac{(b-c)^2}{4}\right)^2-s^2(s-a)^2}{s^2(s-a)^2}\overset{?}{\geq}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2.m_a^2}$$

$$\Leftrightarrow m_a^2.\frac{\frac{(b-c)^4}{16}+s(s-a).\frac{(b-c)^2}{2}}{s^2(s-a)^2}\overset{?}{\geq}\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\Leftrightarrow (b-c)^2\left(m_a^2.\frac{\frac{(b-c)^2}{16}+\frac{s(s-a)}{2}}{s^2(s-a)^2}-\frac{8s^2-a^2}{4(2s+a)^2}\right)\underset{(\blacksquare)}{\overset{?}{\geq}}0$$

$\because (b-c)^2\geq 0$ *and* $m_a^2\geq s(s-a)\therefore$ **in order to prove** $(\blacksquare)$**, it suffices to prove :**

$$\frac{s(s-a).\frac{s(s-a)}{2}}{s^2(s-a)^2}-\frac{8s^2-a^2}{4(2s+a)^2}>0\Leftrightarrow 2(2s+a)^2>8s^2-a^2\Leftrightarrow 8sa+3a^2>0$$

$$\rightarrow \text{true}\Rightarrow(\blacksquare)\text{ is true}\therefore m_a^4w_a\geq s^2(s-a)^2.p_a\Rightarrow\boxed{\frac{m_a^4}{s^2(s-a)^2}\geq\frac{p_a}{w_a}}$$

$$\Rightarrow\frac{m_a}{\sqrt{r_br_c}}\geq\sqrt[4]{\frac{p_a}{w_a}}\text{ and analogs}\Rightarrow\frac{m_am_bm_c}{r_ar_br_c}\geq\sqrt[4]{\frac{p_ap_bp_c}{w_aw_bw_c}}$$

$\forall\,\Delta\,ABC,''=''$ **iff** Δ **ABC is equilateral (QED)**

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have the following known formula (see [1, pp. 1]),

$$p_a{}^2=s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}.$$

And by the formulas for median and angle bisector of triangle ABC**,**

$$m_a{}^2=\frac{1}{4}(2b^2+2c^2-a^2)\text{ and }w_a=\frac{2\sqrt{bcs(s-a)}}{b+c},\text{ we can easily get}$$

$$m_a{}^2=s(s-a)+\frac{(b-c)^2}{4}\quad\text{and}\quad w_a{}^2=s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}.$$

Using these identities, we have

$$p_aw_a\leq\frac{p_a{}^2+w_a{}^2}{2}=s(s-a)+\frac{1}{2}\left(\frac{s(3s+a)}{(2s+a)^2}-\frac{s(s-a)}{(2s-a)^2}\right)(b-c)^2=$$

$$=s(s-a)+\frac{s(4s^3-4s^2a+sa^2+a^3)(b-c)^2}{(4s^2-a^2)^2}=$$

$$=s(s-a)+\left(\frac{1}{4}-\frac{4sa(s-a)(4s+a)+a^4}{4(4s^2-a^2)^2}\right)(b-c)^2\leq m_a{}^2.$$

Then

$$\sqrt[4]{\frac{p_ap_bp_c}{w_aw_bw_c}}=\sqrt[4]{\prod_{cyc}\frac{p_aw_a}{{w_a}^2}}\le\sqrt[4]{\prod_{cyc}\frac{{m_a}^2}{{w_a}^2}}=\frac{m_am_bm_c}{\sqrt{\prod_{cyc}m_aw_a}}\overset{Panaitopol}{\le}$$
$$\le\frac{m_am_bm_c}{\sqrt{\prod_{cyc}r_br_c}}=\frac{m_am_bm_c}{r_ar_br_c}.$$

Equality holds iff ΔABC is equilateral.

[1] Bogdan Fuștei, Mohamed Amine Ben Ajiba,
SPIEKER'S CEVIANS IN THE GEOMETRY OF TRIANGLE.

1966. In ΔABC the following relationship holds:

$$\sum\frac{\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)^3}{\sin\frac{A}{2}+\sin\frac{B}{2}}\ge\frac{64\sqrt{3}}{3}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\sum\tan\frac{A}{2}+\sum\cot\frac{A}{2}=\frac{4R+r}{s}+\frac{s}{r}\overset{Doucet\ \&\ Mitrinovic}{\ge}\sqrt{3}+3\sqrt{3}=4\sqrt{3}\ (1)$$

$$\sum\sin\frac{A}{2}\overset{Jensen}{\le}3\sin\frac{\pi}{6}=\frac{3}{2}\quad(2)$$
$$\mathrm{f(x)}=\sin\frac{x}{2}\ is\ concave\in(0,\pi)$$

$$\sum\frac{\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)^3}{\sin\frac{A}{2}+\sin\frac{B}{2}}\overset{Holder}{\ge}\frac{1}{3}\frac{\left(\sum\tan\frac{A}{2}+\sum\cot\frac{A}{2}\right)^3}{2\sum\sin\frac{A}{2}}\overset{(1)\&(2)}{\ge}\frac{(4\sqrt{3})^3}{3.2.\frac{3}{2}}=\frac{192\sqrt{3}}{9}=\frac{64\sqrt{3}}{3}$$

$$Equality\ holds\ for\ A=B=C=\frac{\pi}{3}$$

1967. In ΔABC the following relationship holds:

$$\sum\frac{1}{\sqrt{4m_b^2-h_a^2}+\sqrt{c^2-h_a^2}}=\sum\frac{a}{a^2+2r_ar_c-2rr_b}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1}{\sqrt{4m_b^2-h_a^2}+\sqrt{c^2-h_a^2}}=\frac{\sqrt{4m_b^2-h_a^2}-\sqrt{c^2-h_a^2}}{\left(\sqrt{4m_b^2-h_a^2}+\sqrt{c^2-h_a^2}\right)\cdot\left(\sqrt{4m_b^2-h_a^2}-\sqrt{c^2-h_a^2}\right)}=$$

$$=\frac{\sqrt{4m_b^2-h_a^2}-\sqrt{c^2-h_a^2}}{4m_b^2-c^2}=\frac{\sqrt{4m_b^2-h_a^2}-\sqrt{c^2-h_a^2}}{a^2+(a^2+c^2-b^2)}$$

$$\boxed{\begin{array}{c}Lemma\ \ 1:\ ac=r_ar_c+rr_b\ \ (true)\\ Lemma\ \ 2:\ b^2=(a-c)^2+4rr_b\ (true)\end{array}}$$

From Lemma 1 and Lemma 2, we have :

$$b^2=a^2-2ac+c^2+4rr_b=a^2+c^2-2(r_ar_c+rr_b)+4rr_b$$
$$=a^2+c^2-2r_ar_c+2rr_b$$
$$a^2+c^2-b^2=2r_ar_c-2rr_b$$

Now let prove that : $\sqrt{4m_b^2-h_a^2}-\sqrt{c^2-h_a^2}=a$ (2)

For this : $\sqrt{4m_b^2-h_a^2}=a+\sqrt{c^2-h_a^2}$

$$4m_b^2-h_a^2=a^2+c^2-h_a^2+2a\sqrt{c-h_a^2}$$
$$4m_b^2-a^2-c^2=2a\sqrt{c-h_a^2}$$
$$a^2+c^2-b^2=2a\sqrt{c-h_a^2}$$
$$(a^2+c^2-b^2)^2=4a^2c^2-4a^2h_a^2\ (\text{Because }\Delta ABC\text{ acute})$$
$$4a^2h_a^2=4a^2c^2-(a^2+c^2-b^2)^2=(2ac-a^2-c^2+b^2)(2ac+a^2+c^2-b^2)=$$
$$[b^2-(a-c)^2][(a+c)^2-b^2]=(b-a+c)(b+a-c)(a+c-b)(a+c+b)=$$
$$(2p-2a)(2p-2c)(2p-2b)2p$$
$$4a^2h_a^2=16p(p-a)(p-b)(p-c)$$
$$2ah_a=4\sqrt{p(p-a)(p-b)(p-c)}$$
$$S=\sqrt{p(p-a)(p-b)(p-c)}\ (true)$$

So (2) true

Analogously, it is true in the other three.

1968. In any Δ ABC, the following relationship holds :

$$\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}\le\frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}=\sum_{cyc}\frac{2Rs\tan\frac{A}{2}}{4R\tan\frac{A}{2}\cos^2\frac{A}{2}.a}=\frac{s}{2}\sum_{cyc}\frac{\sec^2\frac{A}{2}}{a}\overset{\text{Chebyshev}}{\le}$$

$$\leq \frac{s}{6}\cdot\sum_{cyc}\sec^2\frac{A}{2}\cdot\sum_{cyc}\frac{1}{a}=\frac{s}{6}\cdot\frac{s^2+(4R+r)^2}{s^2}\cdot\frac{s^2+4Rr+r^2}{4Rrs}\overset{\substack{\text{Doucet or Trucht}\\ and\\ \text{Gerretsen + Euler}}}{\leq}$$

$$\leq \frac{s}{6}\cdot\frac{\frac{(4R+r)^2}{3}+(4R+r)^2}{s^2}\cdot\frac{s^2+\frac{s^2}{3}}{4Rrs}=\frac{1}{6}\cdot\frac{16}{9}\cdot\frac{(4R+r)^2}{4Rr}\overset{\text{Euler}}{\leq}\frac{1}{6}\cdot\frac{16}{9}\cdot\frac{81R^2}{16Rr}$$

$$\therefore \frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}\leq\frac{3R}{2r}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

1969. If $\lambda\geq 0$ then in ΔABC the following relationship holds:

$$\frac{1}{\lambda+\tan A\tan B}+\frac{1}{\lambda+\tan B\tan C}+\frac{1}{\lambda+\tan C\tan A}\leq\frac{3}{\lambda+3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{We know that in } \Delta ABC,\ \tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$\textit{Let } \tan A=\frac{1}{x},\tan B=\frac{1}{y},\tan C=\frac{1}{z}\ \textit{then}$$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C\ \ \textit{can be writen as } xy+yz+zx=1\ (1)$$

$$\frac{1}{\lambda+\tan A\tan B}+\frac{1}{\lambda+\tan B\tan C}+\frac{1}{\lambda+\tan C\tan A}=$$

$$=\sum\frac{1}{\lambda+\tan A\tan B}=\sum\frac{1}{\lambda+\frac{1}{x}\cdot\frac{1}{y}}=\sum\frac{xy}{\lambda xy+1}=$$

$$=\frac{1}{\lambda}\sum\left(1-\frac{1}{\lambda xy+1}\right)=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\left(\frac{1^2}{\lambda xy+1}\right)\overset{Bergstrom}{\leq}$$

$$\leq\frac{3}{\lambda}-\frac{1}{\lambda}\cdot\frac{(1+1+1)^2}{\lambda(xy+yz+zx)+3}\overset{(1)}{=}\frac{3}{\lambda}-\frac{1}{\lambda}\cdot\frac{9}{\lambda+3}=$$

$$=\frac{1}{\lambda}\left(3-\frac{9}{\lambda+3}\right)=\frac{1}{\lambda}\cdot\frac{3\lambda}{\lambda+3}=\frac{3}{\lambda+3}$$

$$\textit{Equality holds } A=B=C=\frac{\pi}{3}$$

1970. In ΔABC the following relationship holds:

$$2sr\left(7-\frac{2r}{R}\right)\leq\sum a(h_b+h_c)\leq 6Rs$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a(h_b+h_c)=\sum a\left(\frac{ac}{2R}+\frac{ab}{2R}\right)=\frac{1}{2R}\sum a^2(b+c)\overset{Chebyshev}{\le}$$
$$\le\frac{1}{3}\cdot\frac{1}{2R}\left(\sum a^2\right)\left(\sum b+c\right)\overset{Leibniz}{\le}\frac{9R^2}{2R}.2(a+b+c)\frac{1}{3}=3R\cdot 2s=6Rs$$

$$\sum a(h_b+h_c)=\sum a(h_b+h_a+h_c)-\sum ah_a=\left(\sum h_a\right)\left(\sum a\right)-\sum a\cdot\frac{2F}{a}=$$
$$=2s\left(\frac{bc}{2R}+\frac{ca}{2R}+\frac{ab}{2R}\right)-6F=2s\cdot\frac{\sum bc}{2R}-6F=$$
$$=\frac{s}{R}(s^2+r^2+4Rr)-6rs\overset{Gerretsen}{\ge}\frac{s}{R}(16Rr-5r^2+r^2+4Rr)-6rs=$$
$$=\frac{2rs(10R-2r-3R)}{R}=\frac{2rs(7R-2r)}{R}=2sr\left(7-\frac{2r}{R}\right)$$
$$Equality\ holds\ for\ a=b=c$$

1971. In ΔABC the following relationship holds:

$$\sum\frac{\csc^2\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}=\frac{\cos\frac{B}{2}\sin\frac{C}{2}+\cos\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}=$$
$$=\frac{\sin\frac{B+C}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\overset{A+B+C=\pi}{=}\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}\ (1)$$

$$\sum\frac{\csc^2\frac{A}{2}}{\cot\frac{B}{2}+\cot\frac{C}{2}}\overset{(1)}{=}\sum\frac{\csc^2\frac{A}{2}}{\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}}=\sum\frac{\csc^2\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}\overset{AM-GM}{\ge}$$

$$\geq 3\sqrt[3]{\sec\frac{A}{2}} = 3\sqrt[3]{\frac{4R}{s}} \overset{Mitrinovic}{\geq} 3\sqrt[3]{\frac{4R}{\frac{3\sqrt{3}R}{2}}} = 3.\frac{2}{\sqrt{3}} = 2\sqrt{3}$$

$(Equality\ \ holds\ for\ A = B = C)$

1972. In ΔABC the following relationship holds:

$$\sum\frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\tan\frac{B}{2}+\tan\frac{C}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}} = \frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} =$$

$$= \frac{\sin\frac{B+C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{\sin\frac{\pi-A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\ (1)$$

$$\frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \overset{(1)}{=} \frac{1}{\cos^2\frac{A}{2}\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}} = \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^3\frac{A}{2}}\ (2)$$

$$\sum\frac{\sec^2\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \overset{(2)}{=} \sum\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos^3\frac{A}{2}} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{1}{\prod\cos\frac{A}{2}}} =$$

$$= 3\sqrt[3]{\frac{4R}{s}} \overset{Mitrinovic}{\geq} 3\sqrt[3]{\frac{4R}{\frac{3\sqrt{3}R}{2}}} = 3.\sqrt[3]{\frac{8}{3\sqrt{3}}} = \frac{3.2}{\sqrt{3}} = 2\sqrt{3}$$

$Equality\ holds\ for\ A = B = C = \frac{\pi}{3}$

1973. In ΔABC the following relationship holds:

$$\sum\frac{\cot\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}} \geq \frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\ or \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)\ or,$$

$$\frac{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{C}{2}\ or\sum\tan\frac{A}{2}\tan\frac{B}{2}=1\ (1)$$

$$\sum\frac{\cot\frac{A}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}=\sum\frac{1^2}{\tan\frac{B}{2}\tan\frac{A}{2}+\tan\frac{A}{2}\tan\frac{C}{2}}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{2\sum\tan\frac{A}{2}\tan\frac{B}{2}}\overset{(1)}{=}\frac{9}{2}$$

$$Equality\ holds\ A=B=C=\frac{\pi}{3}$$

1974. In ΔABC the following relationship holds:

$$\frac{a}{l_a}+\frac{b}{l_b}+\frac{c}{l_c}\geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\boxed{l_a=\frac{2bc}{b+c}.\cos\frac{A}{2}}$$

$$\frac{a}{l_a}+\frac{b}{l_b}+\frac{c}{l_c}=\frac{a(b+c)}{2bc.\cos\frac{A}{2}}+\frac{b(a+c)}{2ac.\cos\frac{B}{2}}+\frac{c(a+b)}{2ab.\cos\frac{C}{2}}\overset{A-G}{\geq}$$

$$\geq\frac{a\sqrt{bc}}{bc.\cos\frac{A}{2}}+\frac{b\sqrt{ac}}{ac.\cos\frac{B}{2}}+\frac{c\sqrt{ab}}{ab.\cos\frac{C}{2}}\overset{A-G}{\geq}$$

$$\geq 3\left(\frac{a^2b^2c^2}{a^2b^2c^2.\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}\right)^{\frac{1}{3}}=3\left(\frac{4R}{P}\right)^{\frac{1}{3}}\geq 3\left(4\,.\,\frac{2}{3\sqrt{3}}\right)^{\frac{1}{3}}=2\sqrt{3}$$

Equality holds if the triangle is an equilateral one.

1975. In ΔABC the following relationship holds:

$$m_a^{r_a}m_b^{r_b}m_c^{r_c}\leq\left(\frac{3R}{2}\right)^{\frac{9R}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ then\ m_a \le m_b \le m_c\ and\ r_a \ge r_b \ge r_c$$

$$m_a r_a + m_b r_b + m_c r_c \overset{Chebyshev}{\le} \frac{1}{3}\left(\sum m_a\right)\left(\sum r_a\right)\ (1)$$

$$and\ \left(\sum r_a\right) = 4R + r \overset{Euler}{\le} \frac{9R}{2}(2), \left(\sum m_a\right) \overset{Leunberger}{\le} 4R + r \overset{Euler}{\le} \frac{9R}{2}(3)$$

$$\left(m_a^{r_a} m_b^{r_b} m_c^{r_c}\right)^{\frac{1}{(\sum r_a)}} \overset{AM-Gm}{\le} \frac{m_a r_a + m_b r_b + m_c r_c}{(\sum r_a)} \overset{(1)}{\le} \frac{\frac{1}{3}(\sum m_a)(\sum r_a)}{(\sum r_a)} =$$

$$= \frac{1}{3}\left(\sum m_a\right) \overset{(3)}{\le} \frac{1}{3}\frac{9R}{2} = \frac{3R}{2}$$

$$or\ m_a^{r_a} m_b^{r_b} m_c^{r_c} \le \left(\frac{3R}{2}\right)^{(\sum r_a)} \overset{(2)}{\le} \left(\frac{3R}{2}\right)^{\frac{9R}{2}}$$

$$Equality\ holds\ for\ a = b = c$$

1976. In any scalene Δ ABC, the following relationship holds :

$$\sum_{cyc} \sqrt[4]{\frac{m_a^2 - h_a^2}{w_a^2 - h_a^2}} = \frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}} + \sqrt{\frac{r}{r_c}}\right) = \sum_{cyc}\frac{\sqrt{bc}}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt[4]{\frac{m_a^2 - h_a^2}{w_a^2 - h_a^2}} = \sqrt[4]{\frac{s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{a^2}\right)}{s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{a^2}\right)}}$$

$$= \sqrt[4]{\frac{\frac{1}{4} + \frac{s(s-a)}{a^2}}{\frac{s(s-a)}{a^2} - \frac{s(s-a)}{(b+c)^2}}} = \sqrt[4]{\frac{\frac{4s^2 - 4sa + a^2}{4a^2}}{\frac{s(s-a)}{a^2(b+c)^2}.\left((b+c)^2 - a^2\right)}} = \sqrt[4]{\frac{\frac{(2s-a)^2}{4a^2}}{\frac{s(s-a)}{a^2(b+c)^2}.\left(4s(s-a)\right)}}$$

$$= \sqrt[4]{\frac{(b+c)^4}{16s^2(s-a)^2}} \Rightarrow \sqrt[4]{\frac{m_a^2 - h_a^2}{w_a^2 - h_a^2}} = \frac{1}{2}.\frac{b+c}{\sqrt{s(s-a)}}\ and\ analogs$$

$$\therefore \sum_{cyc}\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}\cdot\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\rightarrow(1)$$

$$\text{Again},\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\sum_{cyc}\left(\sqrt{\frac{r_a}{r}}+\sqrt{\frac{r}{r_a}}\right)=\sum_{cyc}\frac{r_a+r}{\sqrt{rr_a}}=\sum_{cyc}\frac{rs\left(\frac{1}{s-a}+\frac{1}{s}\right)}{\sqrt{(s-b)(s-c)}}$$

$$=\sum_{cyc}\frac{\sqrt{s(s-a)(s-b)(s-c)}.\frac{2s-a}{s(s-a)}}{\sqrt{(s-b)(s-c)}}$$

$$=\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\therefore\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\frac{1}{2}\cdot\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\rightarrow(2)\ \text{and}$$

$$\text{finally},\sum_{cyc}\frac{\sqrt{bc}}{w_a}=\sum_{cyc}\frac{\sqrt{bc}}{\frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}}\therefore\sum_{cyc}\frac{\sqrt{bc}}{w_a}=\frac{1}{2}\cdot\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\rightarrow(3)$$

$$\therefore(1),(2),(3)\Rightarrow\sum_{cyc}\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\sum_{cyc}\frac{\sqrt{bc}}{w_a}\ \text{(QED)}$$

1977. In any Δ ABC, the following relationship holds :

$$\frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}}\le\frac{1}{8}\left(1+\frac{5R}{2r}\right)$$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } s-a=x, s-b=y \text{ and } s-c=z\therefore s=x+y+z$$
$$\Rightarrow a=y+z, b=z+x \text{ and } c=x+y$$

$$\text{Now},\frac{s^2}{r^2}=\frac{s^4}{F^2}=\frac{s^4}{s(s-a)(s-b)(s-c)}\Rightarrow\frac{s^2}{r^2}\overset{(*)}{=}\frac{\left(\sum_{cyc}x\right)^3}{xyz}\ \text{and}\ 1+\frac{4R}{r}$$

$$=1+\frac{4sabc}{4s(s-a)(s-b)(s-c)}=1+\frac{\prod_{cyc}(y+z)}{xyz}$$

$$\Rightarrow 1+\frac{4R}{r}\overset{(**)}{=}\frac{xyz+\prod_{cyc}(y+z)}{xyz}$$

$$\text{Also},\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{y+z}{z+x}\Rightarrow\sum\frac{a}{b}\overset{(***)}{=}\frac{\sum_{cyc}((x+y)(y+z)^2)}{\prod_{cyc}(y+z)}\therefore(*),(**),(***)\Rightarrow$$

$$\frac{s^2}{r^2} \ge \left(\sum_{cyc} \frac{a}{b}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{xyz} \ge$$
$$\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}((x+y)(y+z)^2)}{\prod_{cyc}(y+z)}\right)$$
$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc} x\right)^3 \ge \left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}((x+y)(y+z)^2)\right)$$
$$\Leftrightarrow \sum_{cyc} x^2y^4 + \sum_{cyc} x^3y^3 \overset{(\bullet)}{\ge} xyz\sum_{cyc} x^2y + 3x^2y^2z^2$$

Now, if $u, v, w > 0$, *then* : $v^3+v^3+u^3 \overset{A-G}{\ge} 3v^2u$,
$w^3+w^3+v^3 \overset{A-G}{\ge} 3w^2v$ *and* $u^3+u^3+w^3 \overset{A-G}{\ge} 3u^2w$ *and adding these three* :
$\sum_{cyc} u^3 \ge \sum_{cyc} uv^2$ *and* choosing $u = xy, v = yz$ and $w = zx$, we get :

$$\sum_{cyc} x^3y^3 \overset{(\bullet\bullet)}{\ge} xyz\left(\sum_{cyc} x^2y\right) \text{ and } \underbrace{\sum_{cyc} x^2y^4 \overset{A-G}{\ge} 3x^2y^2z^2}_{(\bullet\bullet\bullet)} \therefore (\bullet\bullet)+(\bullet\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \boxed{\frac{s^2}{r^2} \ge \left(\sum_{cyc} \frac{a}{b}\right)\left(1+\frac{4R}{r}\right)} \to (1)$$

Moreover, $\sum_{cyc} \frac{b}{a} = \sum_{cyc} \frac{z+x}{y+z} \Rightarrow \sum_{cyc} \frac{b}{a} \overset{(****)}{=} \frac{\sum_{cyc}((x+y)^2(y+z))}{\prod_{cyc}(y+z)} \therefore (*), (**), (****)$

$$\Rightarrow \frac{s^2}{r^2} \ge \left(\sum_{cyc} \frac{b}{a}\right)\left(1+\frac{4R}{r}\right) \Leftrightarrow \frac{\left(\sum_{cyc} x\right)^3}{xyz} \ge$$
$$\left(\frac{xyz+\prod_{cyc}(y+z)}{xyz}\right)\left(\frac{\sum_{cyc}((x+y)^2(y+z))}{\prod_{cyc}(y+z)}\right)$$
$$\Leftrightarrow \left(\prod_{cyc}(y+z)\right)\left(\sum_{cyc} x\right)^3 \ge \left(xyz+\prod_{cyc}(y+z)\right)\left(\sum_{cyc}((x+y)^2(y+z))\right)$$
$$\Leftrightarrow \sum_{cyc} x^4y^2 + \sum_{cyc} x^3y^3 \overset{(\bullet\bullet\bullet\bullet)}{\ge} xyz\sum_{cyc} xy^2 + 3x^2y^2z^2$$

Now, if $u, v, w > 0$, *then* : $u^3+u^3+v^3 \overset{A-G}{\ge} 3u^2v$,
$v^3+v^3+w^3 \overset{A-G}{\ge} 3v^2w$ *and* $w^3+w^3+u^3 \overset{A-G}{\ge} 3w^2u$ *and adding these three* :

$$\sum_{cyc} u^3 \ge \sum_{cyc} u^2 v \text{ and choosing } u = xy, v = yz \text{ and } w = zx, \text{ we get :}$$

$$\sum_{cyc} x^3y^3 \overset{(\bullet\bullet\bullet\bullet\bullet)}{\ge} xyz\left(\sum_{cyc} xy^2\right) \text{ and } \sum_{cyc} x^4y^2 \underset{(\bullet\bullet\bullet\bullet\bullet\bullet)}{\underbrace{\overset{A-G}{\ge}}} 3x^2y^2z^2 \therefore (\bullet\bullet\bullet\bullet\bullet) + (\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true } \therefore \boxed{\frac{s^2}{r^2} \ge \left(\sum_{cyc} \frac{b}{a}\right)\left(1 + \frac{4R}{r}\right)} \to (2)$$

$$\text{Now, } m_a m_b \overset{?}{\le} \frac{2c^2 + ab}{4} \Leftrightarrow \left(\frac{2b^2 + 2c^2 - a^2}{4}\right)\left(\frac{2c^2 + 2a^2 - b^2}{4}\right) \overset{?}{\le} \frac{(2c^2 + ab)^2}{16}$$

$$\Leftrightarrow a^4 + b^4 - 2a^2b^2 - a^2c^2 + 2abc^2 - b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2 - c^2(a-b)^2 \overset{?}{\ge} 0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\ge} 0$$

$$\to \text{true} \Rightarrow \boxed{m_a m_b \le \frac{2c^2 + ab}{4}} \text{ and analogs} \to (3)$$

$$\text{Now, } \frac{1}{2}\left(1 + \frac{5m_a m_b m_c}{9F^2}\left(\sum_{cyc} m_a\right)\right) \overset{\substack{\text{Lascu} \\ \text{and} \\ \text{Tereshin}}}{\ge}$$

$$\frac{1}{2}\left(1 + \frac{5}{9r^2s^2}\left(\prod_{cyc}\left(\frac{b+c}{2}\cos\frac{A}{2}\right)\right)\left(\sum_{cyc} \frac{b^2+c^2}{4R}\right)\right)$$

$$= \frac{1}{2}\left(1 + \frac{5}{9r^2s^2}\left(\frac{2s(s^2 + 2Rr + r^2)}{8} \cdot \frac{s}{4R}\right)\left(\frac{s^2 - 4Rr - r^2}{R}\right)\right)$$

$$\overset{?}{\ge} \frac{s^2}{r(4R+r)} \Leftrightarrow \frac{5(s^2 + 2Rr + r^2)(s^2 - 4Rr - r^2)}{144R^2r^2} \overset{?}{\ge} \frac{2s^2 - 4Rr - r^2}{r(4R+r)}$$

$$\Leftrightarrow (20R + 5r)s^4 - rs^2(328R^2 + 10Rr) + r^2(416R^3 - 16R^2r - 50Rr^2 - 5r^3) \boxed{\underset{(\blacksquare)}{\underbrace{\overset{?}{\ge}}}} 0$$

$$\text{and } \because (20R + 5r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (\blacksquare),$$

$$\text{it suffices to prove : LHS of } (\blacksquare) \ge (20R + 5r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (156R^2 - 25Rr - 25r^2)s^2 \boxed{\overset{(\blacksquare\blacksquare)}{\ge}} r(2352R^3 - 952R^2r - 125Rr^2 + 65r^3)$$

$$\text{Finally, } (156R^2 - 25Rr - 25r^2)s^2 \overset{\text{Gerretsen}}{\ge} (156R^2 - 25Rr - 25r^2)(16Rr - 5r^2)$$

$$\overset{?}{\ge} r(2352R^3 - 952R^2r - 125Rr^2 + 65r^3) \Leftrightarrow 24t^3 - 38t^2 - 25t - 10 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(24t^2 + 10t - 5) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore \frac{1}{2}\left(1+\frac{5m_a m_b m_c}{9F^2}\left(\sum_{cyc} m_a\right)\right) \ge \frac{s^2}{r(4R+r)} \ge \sqrt{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{b}{a}\right)}$$

$$\left(\because (1) \text{ and } (2) \Rightarrow \frac{s^2}{r(4R+r)} \ge \left(\sum_{cyc}\frac{a}{b}\right), \left(\sum_{cyc}\frac{b}{a}\right) \Rightarrow \sqrt{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{b}{a}\right)} \le \frac{s^2}{r(4R+r)}\right)$$

$$\therefore \boxed{\frac{1}{2}\left(1+\frac{5m_a m_b m_c}{9F^2}\left(\sum_{cyc} m_a\right)\right) \ge \sqrt{\left(\sum_{cyc}\frac{a}{b}\right)\left(\sum_{cyc}\frac{b}{a}\right)}} \to (4)$$

Now, implementing (4) on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians and area as a consequence of trivial calculations are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ and $\frac{F}{3}$ respectively, we get :

$$\frac{1}{2}\left(1+\frac{5.\frac{abc}{8}}{9.\frac{F^2}{9}}\left(\frac{1}{2}\sum_{cyc} a\right)\right) \ge \sqrt{\left(\sum_{cyc}\frac{\frac{2m_a}{3}}{\frac{2m_b}{3}}\right)\left(\sum_{cyc}\frac{\frac{2m_b}{3}}{\frac{2m_a}{3}}\right)}$$

$$\Rightarrow \sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)} \le \frac{1}{2}\left(1+\frac{5.\frac{4Rrs}{8}}{9.\frac{r^2s^2}{9}}.s\right)$$

$$\therefore \boxed{\sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)} \le \frac{1}{2}\left(1+\frac{5R}{2r}\right)} \to (5)$$

Now, $\frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}} \overset{A-G}{\le} \sum_{cyc}\frac{m_a}{\sqrt{4bc+8a^2}}$

$$\overset{via\ (3)}{\le} \sum_{cyc}\frac{m_a}{\sqrt{16m_b m_c}} = \frac{1}{4}\sum_{cyc}\left(\sqrt{\frac{m_a}{m_b}}.\sqrt{\frac{m_a}{m_c}}\right) \overset{A-G}{\le} \frac{1}{4}.\sqrt{\left(\sum_{cyc}\frac{m_a}{m_b}\right)\left(\sum_{cyc}\frac{m_b}{m_a}\right)}$$

$$\overset{via\ (5)}{\le} \frac{1}{8}\left(1+\frac{5R}{2r}\right) \therefore \frac{m_a}{\sqrt{(b+c)^2+8a^2}}+\frac{m_b}{\sqrt{(c+a)^2+8b^2}}+\frac{m_c}{\sqrt{(a+b)^2+8c^2}}$$

$$\le \frac{1}{8}\left(1+\frac{5R}{2r}\right)\ \forall\,\Delta\,ABC, \text{"="} \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(b+c)^2+8a^2=(b+c)^2+4a^2+4a^2\geq 4a(b+c)+4a^2=$$

$$=4a\cdot 2s=8sa\quad \text{(and analogs). Then:}$$

$$\sum_{cyc}\frac{m_a}{\sqrt{(b+c)^2+8a^2}}\leq\sum_{cyc}\frac{m_a}{\sqrt{8sa}}\overset{CBS}{\leq}\sqrt{\frac{3}{8s}\sum_{cyc}\frac{{m_a}^2}{a}}=\sqrt{\frac{3}{8s}\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4a}}$$

$$=\sqrt{\frac{3}{32sabc}\left(2\sum_{cyc}bc(b^2+c^2)-abc\sum_{cyc}a\right)}$$

$$=\sqrt{\frac{3}{128s^2Rr}\left[2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}bc\right)-3abc\sum_{cyc}a\right]}$$

$$=\sqrt{\frac{3}{128s^2Rr}[4(s^2-r^2-4Rr)(s^2+r^2+4Rr)-3.4Rsr2s]}$$

$$=\sqrt{\frac{3[s^4-r^2(4R+r)^2-6s^2Rr]}{32s^2Rr}}$$

$$=\sqrt{\frac{3}{32Rr}\left[s^2-6Rr-\frac{r^2(4R+r)^2}{s^2}\right]}\overset{\substack{Gerretsen\\Doucet}}{\leq}\sqrt{\frac{3}{32Rr}\left[4R^2-2Rr+3r^2-\frac{r^2.3s^2}{s^2}\right]}$$

$$=\sqrt{\frac{3(2R-r)}{16r}}\overset{AM-GM}{\leq}\frac{1}{2}\left(\frac{3}{4}+\frac{2R-r}{4r}\right)=\frac{1}{4}\left(1+\frac{R}{r}\right)\overset{Euler}{\leq}\frac{1}{8}\left(1+\frac{5R}{2r}\right).$$

Equality holds iff ΔABC is equilateral.

1978. **In any Δ ABC, the following relationship holds :**

$$\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}}+\left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}}+\left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}}\geq\left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)-\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right)}$$

Proposed by Tapas Das-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1}{am_a}\sum_{cyc}a^2 \geq 2\sqrt{3} \Leftrightarrow \frac{1}{a^2m_a^2} \geq \frac{12}{\left(\sum_{cyc}a^2\right)^2} \Leftrightarrow$$

$$\left(\sum_{cyc}a^2\right)^2 - 3a^2(2b^2+2c^2-a^2) \geq 0 \Leftrightarrow \left(\sum_{cyc}a^2\right)^2 - 3a^2\left(2\sum_{cyc}a^2 - 3a^2\right) \geq 0$$

$$\Leftrightarrow \left(\sum_{cyc}a^2\right)^2 - 6a^2\sum_{cyc}a^2 + 9a^4 \geq 0 \Leftrightarrow \left(\sum_{cyc}a^2 - 3a^2\right)^2 \geq 0$$

$$\Leftrightarrow (b^2+c^2-2a^2)^2 \geq 0 \to \text{true} \Rightarrow am_a \leq \frac{\sum_{cyc}a^2}{2\sqrt{3}} \text{ and analogs} \to (1)$$

$$\text{Now, } \left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}} = \left(1+\left(\frac{n_a^2}{h_a^2}-1\right)\right)^{\frac{m_a}{h_a}} \overset{\text{Bernoulli}}{\geq} 1+\left(\frac{n_a^2}{h_a^2}-1\right).\frac{m_a}{h_a} \geq 1 \Rightarrow \left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}} \geq 1$$

$$\text{and analogs } \therefore \left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}} + \left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}} + \left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}} \geq 3 \to (2)$$

$$\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right) = \sum_{cyc}\frac{m_a^2}{h_a^2} + 3\sum_{cyc}\frac{m_bm_c}{h_bh_c} \leq$$

$$\sum_{cyc}\frac{m_a^2}{h_a^2} + 2\sum_{cyc}\frac{m_bm_c}{h_bh_c} + \frac{1}{3}.\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2 = \frac{4}{3}.\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2$$

$$\Rightarrow \left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)} \leq \frac{2s}{\sqrt{3}r}.\sum_{cyc}\frac{am_a}{2rs} \overset{\text{via }(1)}{\leq}$$

$$\frac{1}{\sqrt{3}r^2}.\frac{3}{2\sqrt{3}}\sum_{cyc}a^2 \therefore \left(\sum_{cyc}\cot\frac{A}{2}\right).\sqrt{\sum_{cyc}\left(\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)\right)} \leq \frac{s^2-4Rr-r^2}{r^2}$$

$$\to (3) \therefore (2) \text{ and } (3) \Rightarrow \text{it suffices to prove} : 3+\left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right) \geq$$

$$\frac{s^2-4Rr-r^2}{r^2} \Leftrightarrow \frac{4R^2+Rr}{r^2} \geq \frac{s^2-4Rr-r^2}{r^2} \Leftrightarrow s^2 \leq 4R^2+5Rr+r^2$$

$$\Leftrightarrow (s^2-4R^2-4Rr-3r^2) - r(R-2r) \leq 0 \to \text{true via Gerretsen and Euler}$$

$$\Rightarrow \text{main inequality is true}, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $n_a \geq h_a$ (and analogs), then

$$\left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}} + \left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}} + \left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}} \geq 1+1+1=3. \quad (1)$$

By Panaitopol's inequality, we have $\frac{m_a}{h_a} \leq \frac{R}{2r}$ (and analogs), then

$$\sqrt{\sum_{cyc}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)} \leq \sqrt{\sum_{cyc}\left(\frac{R}{2r}+\frac{R}{2r}\right)\left(\frac{R}{2r}+\frac{R}{2r}\right)} = \frac{\sqrt{3}R}{r}. \quad (2)$$

By Doucet's inequality, we have

$$\sum_{cyc}\cot\frac{A}{2} = \frac{s}{r} \leq \frac{4R+r}{\sqrt{3}r}. \quad (3)$$

From (1), (2) and (3), we get

$$\sum_{cyc}\cot\frac{A}{2}.\sqrt{\sum_{cyc}\left(\frac{m_a}{h_a}+\frac{m_b}{h_b}\right)\left(\frac{m_b}{h_b}+\frac{m_c}{h_c}\right)} - \left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right)$$

$$\leq \frac{4R+r}{\sqrt{3}r}.\frac{\sqrt{3}R}{r} - \left(\frac{R}{r}+1\right)\left(\frac{4R}{r}-3\right) = 3$$

$$\leq \left(\frac{n_a}{h_a}\right)^{\frac{2m_a}{h_a}} + \left(\frac{n_b}{h_b}\right)^{\frac{2m_b}{h_b}} + \left(\frac{n_c}{h_c}\right)^{\frac{2m_c}{h_c}}.$$

Equality holds iff ΔABC is equilateral.

1979. In ΔABC the following relationship holds:

$$\sum \frac{1}{h_b+h_c} \geq \sum \frac{1}{r_b+r_c}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{1}{h_b+h_c} = \sum \frac{1}{\frac{ac}{2R}+\frac{ab}{2R}} = 2R\sum \frac{1^2}{ab+ac} \overset{Bergstrom}{\geq}$$

$$\geq 2R\frac{(1+1+1)^2}{2\sum ab} \overset{(\sum x)^2 \geq 3\sum xy}{\geq} \frac{9R.3}{(\sum a)^2} = \frac{27R}{(2s)^2} = \frac{27R}{4s^2} \ (1)$$

$$\sum \frac{1}{r_b+r_c} = \frac{\sum(r_a+r_c)(r_a+r_b)}{\prod(r_b+r_c)} = \frac{\sum(r_a^2+r_ar_b+r_br_c+r_ar_c)}{\prod(r_b+r_c)} =$$

$$= \frac{\sum r_a^2 + 3\sum r_ar_b}{\prod(r_b+r_c)} = \frac{(\sum r_a)^2 + \sum r_ar_b}{(\sum r_a)(\sum r_ar_b) - \prod r_a} = \frac{(4R+r)^2+s^2}{(4R+r)s^2 - s^2r} =$$

$$=\frac{(4R+r)^2+s^2}{4Rs^2}\overset{Gerretsen}{\le}\frac{16R^2+8Rr+r^2+4R^2+4Rr+3r^2}{4Rs^2}=$$

$$=\frac{20R^2+12Rr+4r^2}{4Rs^2}\ (2)$$

$$We\ need\ to\ show\ \sum\frac{1}{h_b+h_c}\ge\sum\frac{1}{r_b+r_c}\ or$$

$$\frac{27R}{4s^2}\overset{using\ 2\ \&3}{\ge}\frac{20R^2+12Rr+4r^2}{4Rs^2}\ or,$$

$$27R^2\ge 20R^2+12Rr+4r^2\ or,$$

$$7R^2-12Rr-4r^2\ge 0\ or\ (R-2r)(7R+2r)\ge 0\ true\ (Euler)$$

$$Equality\ holds\ for\ an\ equilateral\ triangle$$

1980. In ΔABC the following relationship holds:

$$\frac{1}{R}\le\sum\frac{1}{h_b+h_c}\le\frac{R}{4r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{1}{h_b+h_c}=\sum\frac{1^2}{h_b+h_c}\overset{Bergstrom}{\ge}\frac{(1+1+1)^2}{2\sum h_a}\overset{m_a\ge h_a}{\ge}$$

$$\ge\frac{9}{2\sum m_a}\overset{Leuenberger}{\ge}\frac{9}{2(4R+r)}\overset{Euler}{\ge}\frac{9}{2.\left(4R+\frac{R}{2}\right)}=\frac{1}{R}$$

$$\sum\frac{1}{h_b+h_c}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{2}\sum\left(\frac{1}{h_b}\right)=\frac{1}{2r}=\frac{R}{2Rr}\overset{Euler}{\le}\frac{R}{2.r.2r}=\frac{R}{4r^2}$$

$$Equality\ holds\ for\ a=b=c$$

1981. In ΔABC the following relationship holds:

$$\sum \frac{\sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \geq 4$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\tan \frac{B}{2} + \tan \frac{C}{2} = \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} = \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} =$$

$$\frac{\sin \frac{B+C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\sin \frac{\pi - A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} = \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \quad (1)$$

$$\frac{\sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \overset{(1)}{=} \frac{1}{\cos^3 \frac{A}{2} \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}} = \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos^4 \frac{A}{2}} \quad (2)$$

$$\sum \frac{\sec^3 \frac{A}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}} \overset{(2)}{=} \sum \frac{\cos \frac{B}{2} \cos \frac{C}{2}}{\cos^4 \frac{A}{2}} \overset{AM-GM}{\geq} 3 \sqrt[3]{\frac{1}{\prod \cos^2 \frac{A}{2}}} =$$

$$= 3 \sqrt[3]{\left(\frac{4R}{s}\right)^2} \overset{Mitrinovic}{\geq} 3 \sqrt[3]{\left(\frac{4R}{\frac{3\sqrt{3}R}{2}}\right)^2} = 3. \sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2} = 3. \frac{4}{3} = 4$$

$$Equality\ holds\ for\ A = B = C = \frac{\pi}{3}$$

1982. In ΔABC the following relationship holds:

$$12rs \leq \sum a(r_b + r_c) \leq 6Rs$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ \ then\ r_a \geq r_b \geq r_c$$

$$\sum a(r_b+r_c)\overset{Chebyshev}{\le}\frac{1}{3}\left(\sum a\right)\left(\sum(r_b+r_c)\right)=\frac{1}{3}2s.2\sum r_a=$$

$$=\frac{4}{3}s(4R+r)\overset{Euler}{\le}\frac{4}{3}s.\left(4R+\frac{R}{2}\right)=\frac{4}{3}\cdot\frac{9Rs}{2}=6Rs$$

$$\sum a(r_b+r_c)\overset{AM-GM}{\ge}2\sum a\sqrt{r_br_c}\overset{Am-Gm}{\ge}6\sqrt[3]{abcr_ar_br_c}=$$

$$=6\sqrt[3]{4Rrs.s^2r}\overset{Euler}{\ge}6\sqrt[3]{4.2r.rs.s^2r}=12rs$$

$$Equality\ holds\ for\ \ a=b=c$$

1983. In ΔABC the following relationship holds:

$$\sum a(h_b+h_c)\le\sum a(r_b+r_c)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum ar_a=F\sum\frac{a}{s-a}=\frac{F}{(s-a)(s-b)(s-c)}\left(\sum a(s-b)(s-c)\right)=$$
$$=\frac{F}{sr^2}(s^2(a+b+c)-2s(ab+bc+ca)+3abc)$$
$$=\frac{F}{sr^2}(2s^3-2s(s^2+r^2+4Rr)+12Rrs)=\frac{F}{sr^2}(2Rr-r^2)2s=\frac{2s(2Rr-r^2)}{r}\quad(1)$$

$$\sum h_a=\sum\frac{bc}{2R}=\frac{\sum bc}{2R}=\frac{s^2+r^2+4Rr}{2R}\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2+r^2+4Rr}{2R}=\frac{4(R+r)^2}{2R}=\frac{2(R+r)^2}{R}\quad(2)$$

$$\sum a(h_b+h_c)=\sum a(h_a+h_b+h_c)-\sum a\,h_a=$$
$$=\left(\sum a\right)\left(\sum h_a\right)-\sum a\frac{2F}{a}\overset{(2)}{\le}2s.\frac{2(R+r)^2}{R}-6F=\frac{2s}{R}(2(R+r)^2-3Rr)\ (3)$$

$$\sum a(r_b+r_c)=\sum a(r_b+r_a+r_c)-\sum a(r_a)\overset{(1)}{=}$$

$$=\left(\sum a\right)\left(\sum r_a\right)-\frac{2s(2Rr-r^2)}{r}=2s.(4R+r)-2s(2R-r)=2s(2R+r)\ (4)$$

$$We\ need\ to\ show\ \sum a(h_b+h_c)\le\sum a(r_b+r_c)\ or$$

$$2s(2R+r)\overset{(3)\&(4)}{\ge}=\frac{2s}{R}(2(R+r)^2-3Rr)\ or$$

$$2R^2+2Rr\ge 2R^2+4Rr+2r^2-3Rr\ or\ Rr\ge 2r^2\ or$$

$$R\ge 2r\ True\ Euler$$

$$Equality\ holds\ for\ a=b=c$$

1984. In any Δ ABC, the following relationship holds :

$$\sum_{cyc}\left(m_a.\sqrt{\frac{r_a}{r}-1}\right)\ge\sqrt{\left(5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}\right)\left(\sum_{cyc}h_a\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\text{and } b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)\text{ and via summation, we get :}$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)=$$
$$2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)=2(n_a^2+g_a^2)+$$
$$a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 4m_a^2+(b-c)^2=$$
$$2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2\overset{(**)}{=}s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\text{Via } (*) \text{ and } (**), g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right)$$

$$= (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a} = (s-a)^2 + \frac{4r^2 s}{a} = (s-a)^2 + 2rh_a$$

$$\Rightarrow \frac{g_a^2}{h_a} = \frac{a(s-a)^2}{2rs} + 2r \text{ and analogs} \Rightarrow \sum_{cyc} \frac{g_a^2}{h_a} = \frac{1}{2rs}.\sum_{cyc} a(s^2 - 2sa + a^2) + 6r$$

$$= \frac{1}{2rs}.\left(s^2(2s) - 4s(s^2 - 4Rr - r^2) + 2s(s^2 - 6Rr - 3r^2)\right) + 6r$$

$$= \frac{4Rrs - 2r^2 s}{2rs} + 6r \therefore \boxed{\sum_{cyc} \frac{g_a^2}{h_a} = 2R + 5r} \rightarrow (1)$$

$$\text{Again}, an_a^2 = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} = as^2 - \frac{4r^2 s}{s-a}$$

$$= as^2 - 2a.\frac{rs}{s-a}.\frac{2rs}{a} = as^2 - 2ar_a h_a \Rightarrow \frac{n_a^2}{h_a} = \frac{s^2}{h_a} - 2r_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} \frac{n_a^2}{h_a} = \sum_{cyc} \frac{s^2}{h_a} - 2\sum_{cyc} r_a = \frac{s^2}{r} - 2(4R + r) \therefore \boxed{\sum_{cyc} \frac{n_a^2}{h_a} = \frac{s^2 - 8Rr - 2r^2}{r}} \rightarrow (2)$$

$$\therefore \sqrt{\left(5R - r + \sum_{cyc} \frac{n_a g_a}{h_a}\right)\left(\sum_{cyc} h_a\right)} \overset{CBS}{\leq} \sqrt{\left(5R - r + \sqrt{\sum_{cyc} \frac{n_a^2}{h_a}}.\sqrt{\sum_{cyc} \frac{g_a^2}{h_a}}\right)\left(\sum_{cyc} h_a\right)}$$

$$\overset{A-G}{\leq} \sqrt{\left(5R - r + \frac{1}{2}\left(\sum_{cyc} \frac{n_a^2}{h_a} + \sum_{cyc} \frac{g_a^2}{h_a}\right)\right)\left(\sum_{cyc} h_a\right)} \overset{\text{via (1) and (2)}}{=}$$

$$\sqrt{\left(5R - r + \frac{1}{2}\left(\frac{s^2 - 8Rr - 2r^2}{r} + 2R + 5r\right)\right)\left(\sum_{cyc} h_a\right)}$$

$$= \sqrt{\frac{s^2 + 4Rr + r^2}{2r}}.\sqrt{\frac{s^2 + 4Rr + r^2}{2R}}$$

$$\therefore \boxed{\sqrt{\left(5R - r + \sum_{cyc} \frac{n_a g_a}{h_a}\right)\left(\sum_{cyc} h_a\right)} \leq \frac{s^2 + 4Rr + r^2}{\sqrt{4Rr}}} \rightarrow (3) \text{ and also,}$$

$$\sum_{cyc}\left(m_a\cdot\sqrt{\frac{r_a}{r}-1}\right)\overset{\text{Lascu}}{\geq}\sum_{cyc}\left(\frac{b+c}{2}.\cos\frac{A}{2}.\sqrt{\frac{s}{s-a}-1}\right)$$

$$=\sum_{cyc}\left(\frac{b+c}{2}.\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{a^2}{a(s-a)}}\right)$$

$$=\frac{1}{\sqrt{4Rr}}.\sum_{cyc}\frac{a(b+c)}{2}=\frac{1}{\sqrt{4Rr}}.\sum_{cyc}ab\therefore\boxed{\sum_{cyc}\left(m_a\cdot\sqrt{\frac{r_a}{r}-1}\right)\geq\frac{s^2+4Rr+r^2}{\sqrt{4Rr}}}$$

$$\rightarrow(4)\therefore(3)\text{ and }(4)\Rightarrow\sum_{cyc}\left(m_a\cdot\sqrt{\frac{r_a}{r}-1}\right)\geq\sqrt{\left(5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}\right)\left(\sum_{cyc}h_a\right)}$$

$$\forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}$ **(and analogs), then**

$$\sum_{cyc}m_a\sqrt{\frac{r_a}{r}-1}\geq\sum_{cyc}\frac{b+c}{2}\sqrt{\frac{s(s-a)}{bc}}.\sqrt{\frac{s}{s-a}-1}=\sum_{cyc}\frac{b+c}{2}\sqrt{\frac{sa}{bc}}$$

$$=\sum_{cyc}\frac{(b+c)a}{2}\sqrt{\frac{s}{abc}}$$

$$=\frac{ab+bc+ca}{2\sqrt{Rr}}=\frac{2R(h_a+h_b+h_c)}{2\sqrt{Rr}}=\sqrt{\frac{R}{r}}.\sum_{cyc}h_a.$$

By AM − GM inequality, we have

$$\sum_{cyc}\frac{n_a g_a}{h_a}\leq\sum_{cyc}\frac{n_a{}^2+g_a{}^2}{2h_a}=\sum_{cyc}\frac{s\left(s-a+\frac{(b-c)^2}{a}\right)+(s-a)\left(s-\frac{(b-c)^2}{a}\right)}{2h_a}=$$

$$=\sum_{cyc}\frac{a[2s(s-a)+(b-c)^2]}{4F}=\frac{1}{4F}\left(2s^2\sum_{cyc}a-2s\sum_{cyc}a^2+\sum_{cyc}a.\sum_{cyc}bc-9abc\right)=$$

$$=\frac{1}{4F}\left(2s^2.2s-2s.2(s^2-r^2-4Rr)+2s.\sum_{cyc}2Rh_a-9.4Rsr\right)=r-5R+\frac{R}{r}.\sum_{cyc}h_a.$$

$$\sqrt{\left(5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}\right)\sum_{cyc}h_a} \le \sqrt{\frac{R}{r}}\cdot\sum_{cyc}h_a \le \sum_{cyc}m_a\sqrt{\frac{r_a}{r}-1}.$$

Equality holds iff ΔABC is equilateral.

1985. In ΔABC the following relationship holds:

$$(2b+2c-a)a \ge 4\sqrt{3}F$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$(2b+2c-a)a = 2ab+2ac-a^2 = 2(ab+bc+ca)-a^2-2bc \overset{AM-GM}{\ge}$$

$$\ge 2(ab+bc+ca)-a^2-(b^2+c^2) = 2(ab+bc+ca)-(a^2+b^2+c^2) =$$

$$= 2(s^2+r^2+4Rr)-2(s^2-r^2-4Rr) = 4r(4R+r) \overset{Doucet}{\ge} 4r\sqrt{3}s = 4\sqrt{3}F$$

$$Equality\ holds\ \ for\ a=b=c$$

1986. In ΔABC the following relationship holds:

$$7+\cot^2 A+\cot^2 B+\cot^2 C \le 8\left(\frac{R}{2r}\right)^4$$

Proposed by George Apostolopoulos-Greece

Solution by Tapas Das-India

$$7+\cot^2 A+\cot^2 B+\cot^2 C = 7+\sum\csc^2 A-3 = 4+\sum\frac{4R^2}{a^2} =$$

$$= 4+4R^2\sum\frac{1}{a^2} \overset{Steining}{\le} 4+\frac{4R^2}{4r^2} = (2)^2+\left(\frac{R}{r}\right)^2 \overset{Euler}{\le} \left(\frac{R}{r}\right)^2+\left(\frac{R}{r}\right)^2 = 2\left(\frac{R}{r}\right)^2 =$$

$$= 2\cdot\frac{R^2R^2}{r^2R^2} \overset{Euler}{\le} \frac{2(R)^4}{r^2\cdot 4r^2} = \frac{8(R^4)}{16r^4} = 8\left(\frac{R}{2r}\right)^4$$

$$Equality\ holds\ for\ A=B=C$$

1987. **In ΔABC the following inequality holds:**

$$\prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\geq 2\prod\tan\frac{A}{2}\left(\sum\tan^2\frac{A}{2}+3\right)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ know\ that\ \tan\frac{A}{2}=\frac{r_a}{s},\tan\frac{B}{2}=\frac{r_b}{s},\tan\frac{C}{2}=\frac{r_c}{s},and$$

$$\left(\sum\tan^2\frac{A}{2}\right)=\left(\frac{4R+r}{s}\right)^2-2,\prod\tan\frac{A}{2}=\frac{r}{s},\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)=1$$

$$\prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=\left(\sum\tan\frac{A}{2}\right)\left(\sum\tan\frac{A}{2}\tan\frac{B}{2}\right)-\prod\tan\frac{A}{2}=\frac{4R+r}{s}-\frac{r}{s}=\frac{4R}{s}$$

$$2\prod\tan\frac{A}{2}\left(\sum\tan^2\frac{A}{2}+3\right)=2.\frac{r}{s}\left(\left(\frac{4R+r}{s}\right)^2-2+3\right)=\frac{2r}{s}\left(\left(\frac{4R+r}{s}\right)^2+1\right)$$

$$We\ need\ to\ show\ \frac{2r}{s}\left(\left(\frac{4R+r}{s}\right)^2+1\right)\leq\frac{4R}{s}\ or$$

$$2Rs^2\geq r(4R+r)^2+rs^2$$

$$or\ using\ Gerretsen$$

$$2R(16Rr-5r^2)\geq r(16R^2+8Rr+r^2)+r(4R^2+4Rr+3r^2)$$

$$or\ 12R^2r-22Rr^2-4r^3\geq 0\ or\ 2r(6R+r)(R-2r)\geq 0\ true\ (Euler)$$

$$Equality\ holds\ for\ A=B=C$$

1988. **In ΔABC the following relationship holds:**

$$\sum h_a\sum\frac{1}{r_a}\leq\sum r_a\sum\frac{1}{h_a}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ know\ that\ \sum\frac{1}{h_a}=\sum\frac{1}{r_a}=\frac{1}{r}\ (1)$$

$$We\ \ need\ to\ \ show$$

$$\sum h_a\sum\frac{1}{r_a}\le\sum r_a\sum\frac{1}{h_a}\,or,\sum h_a\overset{(1)}{\le}\sum r_a$$

$$or\ \sum m_a\overset{m_a\ge h_a}{\le}.4R+r$$

$$(which\ is\ true\ by\ \ Leuenberger\ inquality)$$

$$Equality\ holds\ for\ equilateral\ triangle$$

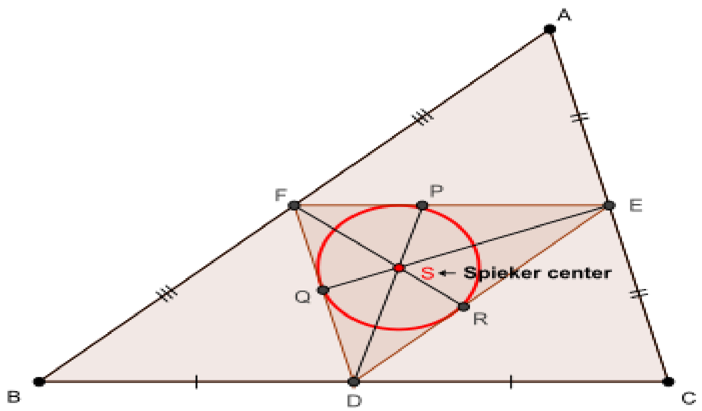
1989.

In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, $n_a, n_b, n_c \to$ Nagel cevians, the following relationship holds :

$$n_a+n_b+n_c\ge p_a+p_b+p_c+\frac{2}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$Let\ AS\ produced\ meet\ BC\ at\ X\ and\ m(\measuredangle BAX)=\alpha\ and\ m(\measuredangle CAX)=\beta\ (say)$$

$$and\ inradius\ of\ \Delta\ DEF=r'(say)$$

$$Now, 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(\text{i}),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\big(b^2+c^2+ab+ca\big)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)}{4s}\Rightarrow 2AS^2\overset{(\text{ii})}{=}\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{4s}$$

$$\text{Via sine law on }\Delta\,\text{AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,\text{AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\big(4m_a^2\big)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\Big(b^3+c^3-abc+a\big(4m_a^2\big)\Big)}$$

$$\text{Now, }b^3+c^3-abc+a\big(4m_a^2\big)=b^3+c^3-abc+a\big(2b^2+2c^2-a^2\big)$$

$$=(b+c)\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2\big)+a\big(b^2+c^2-a^2\big)$$

$$=2s\big(b^2-bc+c^2\big)+a\big(b^2-bc+c^2+bc-a^2\big)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\big(b^2-bc+c^2\big)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$ $\Rightarrow s(b^2+c^2)-bc(2s-a) = an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc = an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2-as^2 \Rightarrow an_a^2 = as^2+$ $s(2bc\cos A-2bc) = as^2-4sbc\sin^2\frac{A}{2} = as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$$= as^2-\frac{as(c+a-b)(a+b-c)}{a} = as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s-\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s(s-a)+\frac{s}{a}.(b-c)^2$$

$$\text{Now, } n_a-p_a \overset{?}{\geq} \frac{(b-c)^2}{3a} \Leftrightarrow n_a \overset{?}{\geq} p_a+\frac{(b-c)^2}{3a}$$

$$\Leftrightarrow n_a^2 \overset{?}{\geq} p_a^2+\frac{(b-c)^4}{9a^2}+\frac{2p_a}{3a}.(b-c)^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)\text{ and }(\bullet\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s}{a}.(b-c)^2 \overset{?}{\geq} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{9a^2}$$

$+\frac{2p_a}{3a}.(b-c)^2$ and $\because (b-c)^2 \geq 0 \therefore$ in order to prove this, it suffices to prove : $\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2} \overset{?}{>} \frac{(b-c)^2}{9a^2}+\frac{2p_a}{3a}$ and $\because (b-c)^2 < a^2 \therefore$ in order to prove this,

it suffices to prove : $\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}-\frac{1}{9} \overset{?}{>} \frac{2p_a}{3a} \Leftrightarrow \frac{36s^3+5s^2a-4sa^2-a^3}{9a(2s+a)^2} \overset{?}{>} \frac{2p_a}{3a}$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4} \overset{?}{>} \frac{4}{9a^2}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\left(\because 36s^3+5s^2a-4sa^2-a^3 = 36s^3+a(s-a)(5s+a) \overset{s>a}{>} 36s^3 > 0\right)$$

$$\text{and} \because (b-c)^2 < a^2 \therefore \text{in order to prove this, it suffices to prove :}$$

$$\frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4} \overset{?}{>} \frac{4\left(s(s-a)(2s+a)^2+sa^2(3s+a)\right)}{9a^2(2s+a)^2}$$

$$\Leftrightarrow 720t^6-216t^5-407t^4-112t^3+6t^2+8t+1 \overset{?}{>} 0 \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\begin{pmatrix}720t^5+504t^4+72t^3+15t^2(t-1)+9t(t+1)(t-1)\\ +(t-1)(t^2+t+1)\end{pmatrix} \overset{?}{>} 0$$

$$\to \text{true} \because t=\frac{s}{a}>1 \therefore n_a-p_a \ge \frac{(b-c)^2}{3a} \overset{s>a}{\ge} \frac{(b-c)^2}{3s} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc} n_a - \sum_{cyc} p_a \ge \frac{2}{3s}\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)$$

$$\therefore n_a+n_b+n_c \ge p_a+p_b+p_c+\frac{2}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

$$\forall \Delta ABC, \text{with equality iff } \Delta ABC \text{ is equilateral (QED)}$$

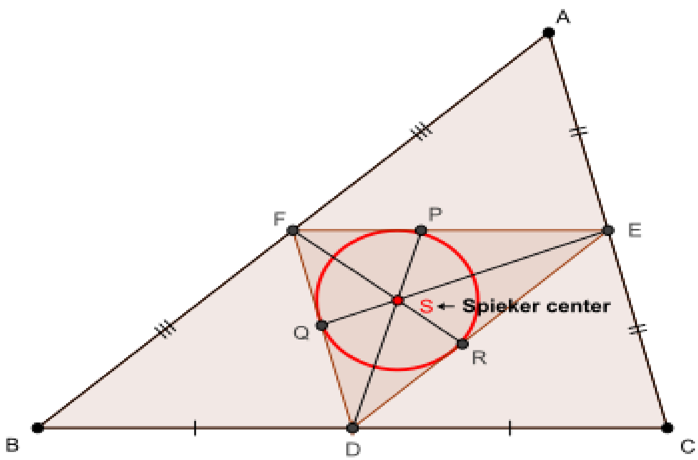
1990.

In any ΔABC with $p_a, p_b, p_c \to$ Spieker cevians, $n_a, n_b, n_c \to$ Nagel cevians, the following relationship holds :

$$\frac{n_a-p_a}{h_a}+\frac{n_b-p_b}{h_b}+\frac{n_c-p_c}{h_c} \ge \frac{a^2+b^2+c^2-ab-bc-ca}{3F}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let AS produced meet BC at X and } m(\measuredangle BAX)=\alpha \text{ and } m(\measuredangle CAX)=\beta \text{ (say)}$$
$$\text{and inradius of } \Delta DEF = r' \text{ (say)}$$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$\because$ Spieker center is incenter of Δ DEF, $\therefore$ $m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$
$=\frac{\pi}{2}-\frac{A-B}{2}$ and $m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$
$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$
$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$
$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$
$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\big(ca(s-b)+ab(s-c)\big)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\big(b^3+c^3-abc+a(4m_a^2)\big)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$= an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2\overset{(\bullet\bullet\bullet\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$

Now, $n_a-p_a\overset{?}{\geq}\frac{(b-c)^2}{3a}\Leftrightarrow n_a\overset{?}{\geq}p_a+\frac{(b-c)^2}{3a}$

$$\Leftrightarrow n_a^2\overset{?}{\geq}p_a^2+\frac{(b-c)^4}{9a^2}+\frac{2p_a}{3a}.(b-c)^2$$

$$\overset{\text{via } (\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s}{a}.(b-c)^2\overset{?}{\geq}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{9a^2}$$

$+\frac{2p_a}{3a}.(b-c)^2$ and $\because (b-c)^2\geq 0\ \therefore$ in order to prove this, it suffices to prove :

$\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}\overset{?}{>}\frac{(b-c)^2}{9a^2}+\frac{2p_a}{3a}$ and $\because (b-c)^2<a^2\ \therefore$ in order to prove this,

it suffices to prove : $\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}-\frac{1}{9}\overset{?}{>}\frac{2p_a}{3a}\Leftrightarrow\frac{36s^3+5s^2a-4sa^2-a^3}{9a(2s+a)^2}\overset{?}{>}\frac{2p_a}{3a}$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow}\frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4}\overset{?}{>}\frac{4}{9a^2}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\left(\because 36s^3+5s^2a-4sa^2-a^3=36s^3+a(s-a)(5s+a)\overset{s>a}{>}36s^3>0\right)$$

$$\text{and} \because (b-c)^2 < a^2 \therefore \text{in order to prove this, it suffices to prove :}$$

$$\frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4} \overset{?}{>} \frac{4\left(s(s-a)(2s+a)^2+sa^2(3s+a)\right)}{9a^2(2s+a)^2}$$

$$\Leftrightarrow 720t^6-216t^5-407t^4-112t^3+6t^2+8t+1 \overset{?}{>} 0 \ \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\begin{pmatrix}720t^5+504t^4+72t^3+15t^2(t-1)+9t(t+1)(t-1)\\ +(t-1)(t^2+t+1)\end{pmatrix} \overset{?}{>} 0$$

$$\rightarrow \text{true} \because t=\frac{s}{a}>1 \therefore n_a-p_a \ge \frac{(b-c)^2}{3a} \Rightarrow \frac{n_a-p_a}{h_a} \ge \frac{(b-c)^2}{3a.\frac{2rs}{a}} = \frac{(b-c)^2}{6F}$$

$$\text{and analogs} \Rightarrow \sum_{cyc}\frac{n_a-p_a}{h_a} \ge \sum_{cyc}\frac{(b-c)^2}{6F} = \frac{2}{6F}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\therefore \frac{n_a-p_a}{h_a}+\frac{n_b-p_b}{h_b}+\frac{n_c-p_c}{h_c} \ge \frac{a^2+b^2+c^2-ab-bc-ca}{3F}$$

$$\forall\,\Delta\,ABC, \text{with equality iff } \Delta\,ABC \text{ is equilateral (QED)}$$

1991. In ΔABC the following relationship holds:

$$\frac{3}{2} < \sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}} < \frac{4\pi}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{We know that in } \Delta ABC,\ b+c>a \text{ or } 2(b+c)>a+b+c \text{ or}$$

$$b+c>\frac{1}{2}(a+b+c)$$

$$\text{similarly}, (a+b)>\frac{1}{2}(a+b+c) \text{ and } c+a>\frac{1}{2}(a+b+c)$$

$$\sum\frac{a}{b+c} < \sum\frac{a}{\frac{1}{2}(a+b+c)} = \sum\frac{2a}{a+b+c} = 2 \quad (1)$$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}} \overset{CBS}{<} \sqrt{3\sum\frac{a}{b+c}} \overset{(1)}{<} \sqrt{3.2}=\sqrt{6}$$

$$\text{We need to show}$$

$$\sqrt{6}<\frac{4\pi}{5} \text{ or } 5\sqrt{6}<4\times\frac{22}{7} \text{ or}, 35\sqrt{6}<88 \text{ or},$$

$$1225\times 6 \overset{squaring}{<} 88\times 88 \text{ or}, 7350<7744 \ (True)$$

$$\text{Now } a<b+c, \text{so } \frac{a}{b+c}<1, \text{we can say } \sqrt{\frac{a}{b+c}}>\frac{a}{b+c}$$

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}>\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\overset{Nesbitt}{>}\frac{3}{2}$$

1992. In ΔABC the following relationship holds:

$$(a+b+c)\left(\frac{1}{r_a+r_b}+\frac{1}{r_b+r_c}+\frac{1}{r_c+r_a}\right)\geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$According\ to\ the\ known\ formula:\ \boxed{r_a=r\cdot\tan\frac{A}{2}}$$

$$r_a+r_b=r\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=r\cdot\frac{\sin\frac{A+B}{2}}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}}=r\cdot\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}}$$

$$=\frac{r\cdot\cos^2\frac{C}{2}}{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}}$$

$$Then\ \sum_{cyc}\frac{1}{r_a+r_b}$$

$$=\frac{1}{r}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\left(\sum_{cyc}\frac{1}{\cos^2\frac{A}{2}}\right)$$

$$=\frac{1}{r}\cdot\left(\frac{P}{4R}\right)\cdot\left(\sum_{cyc}\left(1+\tan^2\frac{A}{2}\right)\right)=$$

$$\frac{1}{r}\cdot\left(\frac{P}{4R}\right)\cdot\left(3+\sum_{cyc}\tan^2\frac{A}{2}\right)\overset{\boxed{\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}=\frac{P}{4R}}(true)\ ,\ \boxed{\sum_{cyc}\tan^2\frac{A}{2}\geq 1}(true)}{\geq}\frac{1}{r}\cdot\frac{P}{4R}\cdot(3+1)$$

$$=\frac{P}{Rr}$$

$$(a+b+c)\cdot\sum_{cyc}\frac{1}{r_a+r_b}\geq 2p\cdot\frac{P}{Rr}=2p^2\cdot\frac{2(a+b+c)}{abc}\overset{\boxed{Rr=\frac{abc}{2(a+b+c)}}}{\cong}\frac{8p^3}{abc}=$$

$$=\frac{(2p)^3}{abc}\geq\frac{\left(3\sqrt[3]{abc}\right)^3}{abc}=27$$

$$Equality\ holds\ iff\ a=b=c$$

1993. In ΔABC the following relationship holds:

$$\frac{a}{(sinB+sinC)^2}+\frac{b}{(sinC+sinA)^2}+\frac{c}{(sinA+sinB)^2}\ge\sqrt{3}R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{(sinB+sinC)^2}=\frac{2RsinA}{4sin^2\frac{B+C}{2}\,.\,cos^2\frac{B-C}{2}}=\frac{4Rsin\frac{A}{2}\,.\,cos\frac{A}{2}}{4cos^2\frac{A}{2}\,.cos^2\frac{B-C}{2}}=$$

$$\frac{R\,.\,tan\frac{A}{2}}{cos^2\frac{B-C}{2}}\ge R\,.\,tan\frac{A}{2}\ \left(Because\ cos^2\frac{B-C}{2}\le 1\right)$$

$$So\ \frac{a}{(sinB+sinC)^2}\ge R\,.\,tan\frac{A}{2}$$

Analogously:

$$\frac{b}{(sinC+sinA)^2}\ge R\,.\,tan\frac{B}{2}\ ;\ \frac{c}{(sinA+sinB)^2}\ge R\,.tan\frac{C}{2}$$

Let's summarize the results :

$$\frac{a}{(sinB+sinC)^2}+\frac{b}{(sinC+sinA)^2}+\frac{c}{(sinA+sinB)^2}\ge R\,.\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)\ge\sqrt{3}R$$

$$But\ \boxed{\left(tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}\right)\ge\sqrt{3}}\ (true)$$

Equality holds iff $a=b=c$

1994. If $n\in N$ then in ΔABC the following relationship holds:

$$\frac{a^n}{sin^2A}+\frac{b^n}{sin^2B}+\frac{c^n}{sin^2C}\ge 2^{n+2}\cdot 3^{\frac{n}{2}}\cdot r^n$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a^n}{sin^2A}+\frac{b^n}{sin^2B}+\frac{c^n}{sin^2C}\overset{A-G}{\ge}3\sqrt[3]{\frac{(abc)^n}{(sinA\,.sinB\,.\,sinC)^2}}\ge$$

$$abc=4RS\ \overset{Euler\ Mitrinovic}{\ge}\ 2^3\,.3^{\frac{3}{2}}\,.r^3$$

$$\boxed{(abc)^n\ge 2^{3n}\,.3^{\frac{3n}{2}}\,.r^{3n}}\ (1)$$

$$\overset{(1)}{\ge}3\,.2^n\,.3^{\frac{n}{2}}\,.r^n\,.\sqrt[3]{\frac{1}{(sinA\,.\,sinB\,.\,sinC)^2}}\overset{(4)}{\ge}$$

$$\boxed{sinA\,.\,sinB\,.\,sinC=\frac{S}{2R^2}}\ (2)\ \boxed{S\le\frac{3\sqrt{3}R^2}{4}}\ (3)$$

From (2) *and* (3) *we have* $\boxed{\sin A\,.\,\sin B\,.\,\sin C \le \frac{3\sqrt{3}}{8}}$ (4)

$$\overset{4}{\ge} 3\,.\,2^n\,.\,3^{\frac{n}{2}}\,.\,r^n\,.\,\sqrt[3]{\left(\frac{8}{3\sqrt{3}}\right)^2} = 2^{n+2}\,.\,3^{\frac{n}{2}}\,.\,r^n$$

1995. *In $\Delta\,ABC$, prove that theperimeter can be represented by 3 following sums:*

$$\sum\sqrt{m_a^2 + rr_a^2} = \frac{1}{2}\sum(r_c + r)\sqrt{\frac{r_a + r_b}{r_c - r}} = \sum(a+b)\cos C$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum(a+b)\cos C = (a+b)\cos C + (b+c)\cos A + (c+a)\cos B =$$

$$= \sum(b\cos C + c\cos B) = \sum a = 2s\ (A)$$

$$r_c + r = \frac{rs}{s-c} + r = \frac{r(s+s-c)}{s-c} = \frac{r(2s-c)}{s-c}$$

$$r_a + r_b = \frac{F}{s-a} + \frac{F}{s-b} = \frac{F(2s-a-b)}{(s-a)(s-b)} = \frac{cF}{(s-a)(s-b)}$$

$$r_c - r = \frac{F}{s-c} - r = \frac{r(s-s+c)}{s-c} = \frac{cr}{s-c},\ \frac{r_a + r_b}{r_c - r} = \frac{cF}{(s-a)(s-b)}\cdot\frac{s-c}{cr} = \frac{(s-c)s}{(s-a)(s-b)}$$

$$= \frac{(s-c)^2 s}{(s-a)(s-c)(s-b)} = \frac{(s-c)^2 s}{sr^2} = \frac{(s-c)^2}{r^2},\ \sqrt{\frac{r_a + r_b}{r_c - r}} = \frac{s-c}{r}$$

$$(r_c + r)\sqrt{\frac{r_a + r_b}{r_c - r}} = \frac{r(2s-c)}{s-c}\cdot\frac{s-c}{r} = 2s - c \text{ and}$$

$$\frac{1}{2}\sum(r_c + r)\sqrt{\frac{r_a + r_b}{r_c - r}} = \frac{1}{2}\sum 2s - c = \frac{1}{2}.(4s) = 2s\ \ (B)$$

(*Reference: Bogdan Fustei, Mohamed Amine Ben Ajiba about NAGEL cevians*)

$$m_a^2 + rr_a = s(s-a) + \frac{(b-c)^2}{4} + \frac{sr^2}{s-a} = s(s-a) + \frac{(b-c)^2}{4} + \frac{sr^2(s-b)(s-c)}{(s-a)(s-b)(s-c)} =$$

$$= s(s-a) + \frac{(b-c)^2}{4} + \frac{sr^2(s-b)(s-c)}{sr^2} = s(s-a) + \frac{(b-c)^2}{4} + (s-b)(s-c)$$

$$= s(s-a)+(s-b)(s-c)+\frac{(b-c)^2}{4} = 2s^2 - s(a+b+c)+bc+\frac{(b-c)^2}{4} =$$

$$= 2s^2 - 2s^2 + \frac{(b-c)^2+4bc}{4} = \frac{(b+c)^2}{4} \ and$$

$$\sum\sqrt{m_a^2+rr_a^2} = \sum\frac{b+c}{2} = a+b+c = 2s \quad (C)$$

$$from\ (A),(B),(C)\ we\ get\ \sum\sqrt{m_a^2+rr_a^2} = \frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}} = \sum(a+b)\cos C$$

1996. In ΔABC the following relationships holds:

$$\sum\frac{asinA}{2(r_b+r_c)} = \sum\frac{r_a^2}{r_a^2+s^2} = \sum\frac{a(cosB+cosC)}{2(b+c)} = \frac{1}{2}\sum sinA\sqrt{\frac{rr_a}{r_br_c}} = \sum sin^2\frac{A}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{asinA}{2(r_b+r_c)} = \frac{asinA}{2S\left(tan\frac{B}{2}+tan\frac{C}{2}\right)} = \frac{a}{2S}\cdot\frac{sinA}{\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{cos\frac{B}{2}\cdot cos\frac{C}{2}}} = \frac{a}{2S}\cdot\frac{sinA\cdot cos\frac{B}{2}\cdot cos\frac{C}{2}}{cos\frac{A}{2}} =$$

$$\frac{a}{2S}\cdot\frac{2sin\frac{A}{2}\cdot cos\frac{A}{2}.cos\frac{B}{2}\cdot cos\frac{C}{2}}{cos\frac{A}{2}} = \frac{a}{2S}.2tan\frac{A}{2}.\frac{S}{4R} = \frac{2RsinA}{4R}.tan\frac{A}{2}$$

$$= \frac{2sin\frac{A}{2}\cdot cos\frac{A}{2}}{2}.tan\frac{A}{2} = sin^2\frac{A}{2}$$

$$Therefore\ :\ \sum_{cyc}\frac{asinA}{2(r_b+r_c)} = \sum_{cyc} sin^2\frac{A}{2} \quad (1)$$

$$\frac{r_a^2}{r_a^2+s^2} = \frac{s^2tan^2\frac{A}{2}}{s^2tan^2\frac{A}{2}+s^2} = \frac{tan^2\frac{A}{2}}{tan^2\frac{A}{2}+1} = \frac{tan^2\frac{A}{2}}{\frac{1}{cos^2\frac{A}{2}}} = sin^2\frac{A}{2}$$

$$So\ :\ \sum_{cyc}\frac{r_a^2}{r_a^2+s^2} = \sum_{cyc} sin^2\frac{A}{2} \quad (2)$$

$$\frac{a(cosB+cosC)}{2(b+c)}=\frac{2RsinA(cosB+cosC)}{2\,.\,2R(sinB+sinC)}=$$

$$=\frac{sinA\,.\,2cos\frac{B+C}{2}\,.\,cos\frac{B-C}{2}}{2\,.\,2sin\frac{B+C}{2}\,.\,cos\frac{B-C}{2}}=\frac{1}{2}sinA.ctg\frac{B+C}{2}=$$

$$=\frac{1}{2}sinA\,.\,tan\frac{A}{2}=\frac{1}{2}\,.2sin\frac{A}{2}.cos\frac{A}{2}.tan\frac{A}{2}=sin^2\frac{A}{2}$$

$$So\ :\ \sum_{cyc}\frac{a(cosB+cosC)}{2(b+c)}=\sum_{cyc}sin^2\frac{A}{2}\quad(3)$$

$$\frac{1}{2}sinA\sqrt{\frac{rr_a}{r_br_c}}=\frac{1}{2}sinA\left(\frac{r\,.\,S\,.\,tan\frac{A}{2}}{S\,.\,tan\frac{B}{2}.S\,.\,tan\frac{C}{2}}\right)^{\frac{1}{2}}=\frac{1}{2}sinA(\frac{r}{S}\,.\,\frac{tan^2\frac{A}{2}}{tan\frac{A}{2}.tan\frac{B}{2}.tan\frac{C}{2}})^{\frac{1}{2}}=$$

$$\frac{1}{2}sinA\left(\frac{r}{S}.\frac{tan^2\frac{A}{2}}{\frac{r}{S}}\right)^{\frac{1}{2}}=\frac{1}{2}sinA\,.tan\frac{A}{2}=sin^2\frac{A}{2}$$

$$Also\ :\ \frac{1}{2}\sum_{cyc}sinA\sqrt{\frac{rr_a}{r_br_c}}=\sum_{cyc}sin^2\frac{A}{2}\quad(4)\quad(Proved)$$

1997. For non-right triangle ABC the following relationship holds:

$$\prod_{cyc}\sqrt[6]{\frac{\sin^2 2A}{c^2-h_a^2}}=\sum_{cyc}\frac{\cos A+\cos B}{2r_c}=\left(\sum_{cyc}\sqrt{\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}}\right)^2=\frac{1}{R}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sin^2 2A=4\sin^2 A\,(1-\sin^2 A)=\frac{4a^2}{4R^2}\left(1-\frac{a^2}{4R^2}\right)=\frac{a^2}{4R^4}(4R^2-a^2)\ and$$

$$c^2-h_a^2=c^2-\frac{b^2c^2}{4R^2}=\frac{c^2}{4R^2}(4R^2-b^2)$$

$$\prod_{cyc}\sqrt[6]{\frac{\sin^2 2A}{c^2-h_a^2}}=\prod_{cyc}\sqrt[6]{\frac{\frac{a^2}{4R^4}(4R^2-a^2)}{\frac{c^2}{4R^2}(4R^2-b^2)}}=\prod_{cyc}\sqrt[6]{\frac{a^2(4R^2-a^2)}{R^2c^2(4R^2-b^2)}}=\prod\sqrt[6]{\frac{a^2b^2c^2}{R^6a^2b^2c^2}}=\frac{1}{R}$$

$$\sum_{cyc}\frac{\cos A+\cos B}{2r_c}=\sum\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=\frac{1}{2s}\sum\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=$$

$$=\frac{1}{2s}\sum\frac{2\sin\frac{C}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=\sum\frac{2\cos\frac{C}{2}\cos\frac{A-B}{2}}{2s}=$$

$$=\frac{1}{2s}\sum 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=\frac{1}{2s}\sum(\sin A+\sin B)=\frac{2}{2s}\cdot\frac{s}{R}=\frac{1}{R}$$

$$r_a-r=r\left(\frac{s}{s-a}-1\right)=\frac{ar}{s-a}$$

$$r_a+r_b=F\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=\frac{F(2s-a-b)}{(s-a)(s-b)}=\frac{Fc}{(s-a)(s-b)}\ \text{and}$$

$$r_a+r_c=\frac{bF}{(s-a)(s-c)}\text{, using above result we get}$$

$$\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}=\frac{ar}{bcs}=\frac{a^2r}{abcs}=\frac{a^2}{4Rs^2}$$

$$\left(\sum_{cyc}\sqrt{\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}}\right)^2=\left(\sum\frac{a}{2s\sqrt{R}}\right)^2=\frac{\frac{1}{R}(a+b+c)^2}{4s^2}=\frac{1}{R}$$

1998. In ΔABC the following relationship holds:

$$\sqrt[4]{\sum\frac{Ra^8}{r_b+r_c}}\geq\frac{4F}{\sqrt{3}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum(r_b+r_c)=2\sum r_a=2(4R+r)\overset{Euler}{\leq}2\left(4R+\frac{R}{2}\right)=9R\ (1)$$

$$\left(\sum\frac{a^8}{r_b+r_c}\right)\left(\sum(r_b+r_c)\right)(1+1+1)^6\overset{Holder}{\geq}(a+b+c)^8=(2s)^8=2^8s^8$$

$$\left(\sum\frac{a^8}{r_b+r_c}\right)\geq\frac{2^8s^8}{(\sum(r_b+r_c))3^6}\overset{(1)}{\geq}\frac{2^8s^8}{3^6.9R}=\left(\frac{2}{3}\right)^8\frac{s^8}{R}\ (2)$$

$$\sqrt[4]{\sum\frac{Ra^8}{r_b+r_c}}\overset{(2)}{\geq}\sqrt[4]{R.\left(\frac{2}{3}\right)^8\frac{s^8}{R}}=\frac{2^2s^2}{3^2}=\frac{4s}{9}.s\overset{Mitrinovic}{\geq}\frac{4s}{9}.3\sqrt{3}r=\frac{4rs}{\sqrt{3}}=\frac{4F}{\sqrt{3}}$$

Equality holds for $a=b=c$

1999. In any Δ ABC, the following relationship holds :

$$2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}\le\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}-\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)=$$

$$=2+\sum_{cyc}\frac{s(s-c)}{\frac{rs}{s-c}.4R.\frac{s(s-c)}{ab}}-\frac{7}{16}.\prod_{cyc}\frac{4R\cos^2\frac{C}{2}}{r_b}=2+\sum_{cyc}\frac{s-c}{\left(\frac{abc}{ab}\right)}-\frac{7}{16}.\frac{64R^3.\frac{s^2}{16R^2}}{rs^2}=$$

$$=2+\frac{s(s^2+4Rr+r^2)}{4Rrs}-3-\frac{7R}{4r}=\frac{s^2+r^2}{4Rr}-\frac{7R}{4r}=\frac{s^2+r^2-7R^2}{4Rr}\overset{\text{Gerretsen}}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2+r^2-7R^2}{4Rr}=-\frac{3R^2-4Rr-4r^2}{4Rr}=-\frac{(R-2r)(3R+2r)}{4Rr}\overset{\text{Euler}}{\le}0$$

$$\therefore 2+\sum_{cyc}\frac{r_a r_b}{r_c(r_a+r_b)}\le\frac{7}{16}\prod_{cyc}\left(1+\frac{r_a}{r_b}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)

2000. In ΔABC the following relationship holds:

$$\sum\left(a^{8n}+\frac{1}{a^{4n}}\right)\ge 6\left(\frac{4F}{\sqrt{3}}\right)^n, n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\left(a^{8n}+\frac{1}{a^{4n}}\right)\overset{AM-GM}{\ge}\sum 2\sqrt{a^{8n}.\frac{1}{a^{4n}}}=2\sum a^{2n}\overset{AM-GM}{\ge}$$

$$\ge 6\sqrt[3]{((abc)^2)^n}\overset{Carlitz}{\ge}6\left(\left(\frac{4F}{\sqrt{3}}\right)^3\right)^{\frac{n}{3}}=6\left(\frac{4F}{\sqrt{3}}\right)^n$$

$\textit{Equality holds for } a = b = c$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru