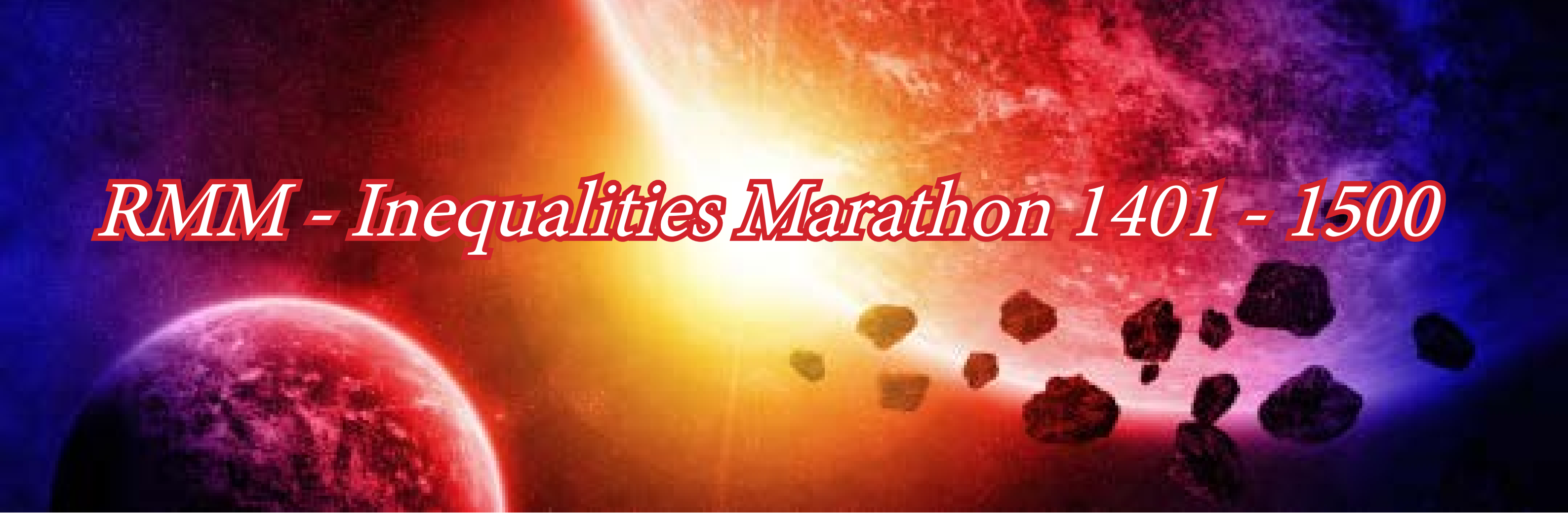
RMM - Inequalities Marathon 1401 - 1500



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Proposed by

Nguyen Hung Cuong-Vietnam

Shirvan Tahirov-Azerbaijan, Zaza Mzhavanadze-Georgia

Marin Chirciu-Romania, Phan Ngoc Chau-Vietnam

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Solutions by

Soumava Chakraborty-India

Tapas Das-India, Hikmat Mammadov-Azerbaijan

Pham Duc Nam-Vietnam, Le Thu-Vietnam

Mohamed Amine Ben Ajiba-Morocco

1401. If $a, b \in \mathbb{R}$, $ab(a^6+64)(b^6+64) \geq 4400194256896$ then:

$$a^2+b^2 \geq 128$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let us assume}: a^2+b^2 < 128 \text{ and then}:$$
$$128 > a^2+b^2 \geq 2ab \Rightarrow ab < 64 \rightarrow (1)$$

$$\text{We have}:$$
$$4400194256896 \leq ab(a^6+64)(b^6+64) = ab(a^6b^6+64(a^6+b^6)+64^2)$$

$$= ab\left(a^6b^6+64\left((a^2+b^2)^3-3a^2b^2(a^2+b^2)\right)+64^2\right) \overset{\substack{\text{assumption}\\ \text{and}\\ \because -3a^2b^2(a^2+b^2)\leq -6a^3b^3}}{<}$$

$$ab(a^6b^6+64(128^3-6a^3b^3)+64^2)\left(\because ab \geq \frac{4400194256896}{(a^6+64)(b^6+64)} > 0\right)$$

$$\therefore t^7+64.128^3t-384t^4+64^2t > 4400194256896 \ (t=ab)$$

$$\Rightarrow t^7-384t^4+134221824t-4400194256896 > 0$$

$$\Rightarrow (t-64)\begin{pmatrix} t^6+64t^5+4096t^4+261760t^3+16752640t^2 \\ +1072168960t+68753035264 \end{pmatrix} > 0 \Rightarrow t > 64$$

$$(\because t = ab > 0) \text{ which is a contradiction to } (1) \therefore \text{ our assumption is incorrect}$$

$$\text{and hence we conclude that}: a^2+b^2 \geq 128, '' ='' \text{ iff } a=b=8 \ (\text{QED})$$

1402. If $a, b > 0$ then:

$$\frac{a^2+b^2}{ab}+\frac{8\sqrt{ab}}{a+b} \geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a+b \overset{AM-GM}{\geq} 2\sqrt{ab} \text{ or } m \overset{m=a+b, ab=u^2}{\geq} 2u \ (1)$$

$$\frac{a^2+b^2}{ab}+\frac{8\sqrt{ab}}{a+b}\ge 6 \text{ or}$$

$$\frac{(a+b)^2-2ab}{ab}+\frac{8\sqrt{ab}}{a+b}\ge 6 \text{ or}$$

$$\frac{(a+b)^2}{ab}-2+\frac{8\sqrt{ab}}{a+b}\ge 6$$

$$\text{or}\frac{(a+b)^2}{ab}+\frac{8\sqrt{ab}}{a+b}\ge 8 \text{ or}$$

$$\frac{m^2}{u^2}+\frac{8m}{m}\ge 8\ (\ m=a+b, ab=u^2\ )$$

$$\text{or } m^3-8mu^2+8u^3\ge 0 \text{ or}$$

$$(m-2u)(m^2-4u^2+2mu)\ge 0 \text{ true } (\text{as } m\ge 2u)$$

$$\text{Equality for } a=b=1$$

1403. **If $a,b,c>0$ then:**

$$(a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge\frac{3}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=$$

$$=2\left(\left(\frac{x}{y}+\frac{y}{x}\right)^3-3\,\frac{x}{y}\cdot\frac{y}{x}\left(\frac{x}{y}+\frac{y}{x}\right)\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=$$

$$=2\left(\frac{x}{y}+\frac{y}{x}\right)^3-6\left(\frac{x}{y}+\frac{y}{x}\right)-3\left(\frac{x}{y}+\frac{y}{x}\right)+2=2\left(\frac{x}{y}+\frac{y}{x}\right)^3+9\left(\frac{x}{y}+\frac{y}{x}\right)+2\ \ (1)$$

$$\text{we have to show } (a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge\frac{3}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)$$

$$\text{or }\left(3+\frac{a^3}{b^3}+\frac{a^3}{c^3}+\frac{b^3}{a^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{c^3}{b^3}\right)\ge\frac{3}{2}\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)$$

$$\text{or}2\left(3+\frac{a^3}{b^3}+\frac{a^3}{c^3}+\frac{b^3}{a^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}+\frac{c^3}{b^3}\right)-3\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)\ge 0$$

$$or\left(6+2\frac{a^3}{b^3}+2\frac{a^3}{c^3}+2\frac{b^3}{a^3}+2\frac{b^3}{c^3}+2\frac{c^3}{a^3}+2\frac{c^3}{b^3}\right)-3\left(\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}\right)\geq 0$$

$$or\left(2\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)-3\left(\frac{a}{b}+\frac{b}{a}\right)+2\right)+\left(2\left(\frac{a^3}{c^3}+\frac{c^3}{a^3}\right)-3\left(\frac{a}{c}+\frac{c}{a}\right)+2\right)$$
$$+\left(2\left(\frac{b^3}{c^3}+\frac{c^3}{b^3}\right)-3\left(\frac{b}{c}+\frac{c}{b}\right)+2\right)\overset{(1)}{\geq}0$$

$$or\left(2\left(\frac{a}{b}+\frac{b}{a}\right)^3-9\left(\frac{a}{b}+\frac{b}{a}\right)+2\right)+\left(2\left(\frac{a}{c}+\frac{c}{a}\right)^3-9\left(\frac{a}{c}+\frac{c}{a}\right)+2\right)$$
$$+\left(2\left(\frac{b}{c}+\frac{c}{a}\right)^3-9\left(\frac{b}{c}+\frac{c}{b}\right)+2\right)\geq 0\ (2)$$

$$Let\ p=\left(\frac{a}{b}+\frac{b}{a}\right)\overset{AM-GM}{\geq}2,q=\left(\frac{a}{c}+\frac{c}{a}\right)\overset{AM-GM}{\geq}2,r=\left(\frac{b}{c}+\frac{c}{b}\right)\overset{AM-GM}{\geq}2$$

$$From\ (2)we\ get\ (2p^3-9p+2)+(2q^3-9q+2)+(2r^3-9r+2)\geq 0$$
$$or$$
$$(p-2)(2p^2+4p-1)+(q-2)(2q^2+4q-1)+(r-2)(2r^2+4r-1)\geq 0\ true\ (as\ p,q,r\geq 2)$$

$$Equality\ holds\ for\ a=b=c$$

1404. **If $a,b,c\geq 0$ and $a+b+c=3$, then prove that :**

$$\frac{a}{b^4+16}+\frac{b}{c^4+16}+\frac{c}{a^4+16}\geq\frac{5}{32}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b^4+16}+\frac{b}{c^4+16}+\frac{c}{a^4+16}\geq\frac{5}{32}\Leftrightarrow\frac{1}{16}.\sum_{cyc}\frac{a(16+b^4-b^4)}{b^4+16}\geq\frac{5}{32}$$

$$\overset{a+b+c=3}{\Leftrightarrow}\frac{6}{32}-\frac{5}{32}\geq\frac{1}{16}.\sum_{cyc}\frac{ab^4}{b^4+16}\Leftrightarrow\sum_{cyc}\frac{ab^4}{b^4+16}\leq\frac{1}{2}\to(1)$$

WLOG we may assume $b\geq 2$ and then : $0\leq c+a\leq 1$ ($\because a+b+c=3$)

$$\Rightarrow 0\leq c,a\leq 1\to(i)$$

$$\text{Now, }\sum_{cyc}\frac{ab^4}{b^4+16}\leq\frac{1}{2}\Leftrightarrow\frac{(3-b-c)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+\frac{ca^4}{a^4+16}\leq\frac{1}{2}$$

$$\Leftrightarrow\frac{(3-b)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+c\left(\frac{a^4}{a^4+16}-\frac{b^4}{b^4+16}\right)\leq\frac{1}{2}$$

$$\Leftrightarrow\frac{(3-b)b^4}{b^4+16}+\frac{bc^4}{c^4+16}+\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)}-\frac{15b^4c}{(a^4+16)(b^4+16)}\leq\frac{1}{2}$$

$$\Leftrightarrow \frac{(3-b)b^4}{b^4+16}+\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)}+bc\left(\frac{c^3}{c^4+16}-\frac{15b^3}{(a^4+16)(b^4+16)}\right)\boxed{\overset{(\blacksquare)}{\leq}}\frac{1}{2}$$

$$\text{Let } F(b)=\frac{(3-b)b^4}{b^4+16}\ \forall\, b\in[2,3] \text{ and then}: F'(b)=\frac{-b^3(b^5+80b-192)}{(b^4+16)^2}$$

$$=\frac{-b^3(b^4+2b^3+4b^2+8b+96)}{(b^4+16)^2}=0 \text{ iff } b=2 \text{ and } F''(b)=-\frac{5}{4}<0$$

$$\therefore F(b) \text{ attains a maxima at } b=2 \text{ and}$$

$$\because F(b) \text{ never attains a minima in } [2,3] \therefore F(b)\leq F(2)=\frac{1}{2}\Rightarrow \boxed{\frac{(3-b)b^4}{b^4+16}\overset{(\bullet)}{\leq}\frac{1}{2}}$$

$$\text{Now, via (i)}, a\leq 1\leq \frac{b}{2}\Rightarrow 16a^4-b^4\leq 0 \text{ and } \because c\geq 0$$

$$\therefore \boxed{\frac{c(16a^4-b^4)}{(a^4+16)(b^4+16)}\overset{(\bullet\bullet)}{\leq} 0}$$

$$\text{Again, let } f(t)=\frac{t^3}{t^4+16}\ \forall\, b\in[0,3] \text{ and then}: f'(t)=\frac{t^2(48-t^4)}{(t^4+16)^2}=0 \text{ iff}$$

$$t=2.\sqrt[4]{3} \therefore f'(t)\geq 0\ \forall\, t\in\left[0,2.\sqrt[4]{3}\right] \text{ and } f'(t)\leq 0\ \forall\, t\in\left[2.\sqrt[4]{3},3\right]$$

$$\Rightarrow f(t) \text{ is } \uparrow \text{ on } \left[0,2.\sqrt[4]{3}\right] \text{ and is } \downarrow \text{ on } \left[2.\sqrt[4]{3},3\right] \therefore \forall\, t\in[0,1], f(t)\leq f(1) \text{ and}$$

$$\because 0\leq c\leq 1 \therefore \frac{c^3}{c^4+16}\boxed{\overset{(*)}{\leq}}\frac{1^3}{1^4+16}=\frac{1}{17}$$

$$\text{Also}, f(b) \text{ is } \uparrow \text{ on } \left[2,2.\sqrt[4]{3}\right] \text{ and is } \downarrow \text{ on } \left[2.\sqrt[4]{3},3\right] \text{ and } \left.\frac{b^3}{b^4+16}\right|_{b=2}=\frac{1}{4} \text{ and}$$

$$\left.\frac{b^3}{b^4+16}\right|_{b=3}=\frac{27}{97}>\frac{1}{4} \text{ and hence, we conclude that } \forall\, b\in[2,3], \frac{b^3}{b^4+16}\boxed{\overset{(**)}{\geq}}\frac{1}{4}$$

$$\therefore (*),(**) \text{ and } 0\leq a\leq 1\Rightarrow \frac{c^3}{c^4+16}-\frac{15b^3}{(a^4+16)(b^4+16)}\leq\frac{1}{17}-\frac{15}{17}.\frac{1}{4}=-\frac{11}{68}$$

$$<0 \text{ and } \because bc\geq 0 \therefore \boxed{bc\left(\frac{c^3}{c^4+16}-\frac{15b^3}{(a^4+16)(b^4+16)}\right)\overset{(\bullet\bullet\bullet)}{\leq} 0}$$

$$\therefore (\bullet)+(\bullet\bullet)+(\bullet\bullet\bullet)\Rightarrow(\blacksquare)\Rightarrow(1) \text{ is true} \therefore \frac{a}{b^4+16}+\frac{b}{c^4+16}+\frac{c}{a^4+16}\geq\frac{5}{32}$$

$$\forall\, a,b,c\geq 0 \,\middle|\, a+b+c=3, ''='' \text{ iff } (b=2,c=0,a=1) \text{ or } (c=2,a=0,b=1)$$

$$\text{or } (a=2,b=0,c=1) \text{ (QED)}$$

1405.

Let $\{x,y,z\}$ be positive real numbers such that : $x^2+y^2+z^2=3$.

$$\text{Prove that}: \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z}\leq 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$4-x,4-y,4-z\geq 4-\sqrt{3}>0\Rightarrow(4-x)(4-y)(4-z)>0$$

$$\therefore\frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z}\leq 1$$

$$\Leftrightarrow 48-8\sum_{cyc}x+\sum_{cyc}xy\leq 64-16\sum_{cyc}x+4\sum_{cyc}xy-xyz$$

$$\Leftrightarrow 3\sum_{cyc}xy+16\geq 8\sum_{cyc}x+xyz\overset{x^2+y^2+z^2=3}{\Leftrightarrow}3\sum_{cyc}xy+\frac{16}{3}.\left(\sum_{cyc}x^2\right)\geq$$

$$\frac{8}{\sqrt{3}}\left(\sum_{cyc}x\right).\sqrt{\sum_{cyc}x^2}+\frac{xyz.\sqrt{3}}{\sqrt{\sum_{cyc}x^2}}=\frac{8(\sum_{cyc}x)(\sum_{cyc}x^2)+3xyz}{\sqrt{3\sum_{cyc}x^2}}$$

$$\Leftrightarrow\frac{1}{9}\left(9\sum_{cyc}xy+16\sum_{cyc}x^2\right)^2\boxed{\overset{(*)}{\geq}}\frac{1}{3\sum_{cyc}x^2}.\left(8\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)+3xyz\right)^2$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(1)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$\Rightarrow xyz\overset{(2)}{=}r^2s$ and via such substitutions, $\sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2$

$$\Rightarrow\sum_{cyc}xy\overset{(3)}{=}4Rr+r^2\Rightarrow\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy=s^2-2(4Rr+r^2)$$

$$\Rightarrow\sum_{cyc}x^2\overset{(4)}{=}s^2-8Rr-2r^2\therefore(1),(2),(3),(4)\Rightarrow$$

$$(*)\Leftrightarrow(s^2-8Rr-2r^2)\left(16(s^2-8Rr-2r^2)+9(4Rr+r^2)\right)^2$$
$$\geq 3(8s(s^2-8Rr-2r^2)+3r^2s)^2$$
$$\Leftrightarrow 32s^6-(960Rr+312r^2)s^4+r^2s^2(9864R^2+5508Rr+747r^2)$$
$$-r^3(33856R^3+25392R^2r+6348Rr^2+529r^3)\boxed{\overset{(**)}{\geq}}0$$

and $\because(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\geq}0\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\geq 32(s^2-16Rr+5r^2)^3$

$$\Leftrightarrow(576R-792r)s^4-r(14712R^2-20868Rr+1653r^2)s^2+$$
$$r^2(97216R^3-148272R^2r+32052Rr^2-4529r^3)\boxed{\overset{(***)}{\geq}}0$$

$$and \because (576R-792r)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (***),$$
$$\text{it suffices to prove : LHS of } (***) \geq (576R-792r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (3720R^2-10236Rr+6267r^2)s^2 \boxed{\overset{(****)}{\geq}}$$
$$r(50240R^3-146640R^2r+109068Rr^2-15271r^3)$$
$$\text{Now, } 3720R^2-10236Rr+6267r^2 = (R-2r)(3720R-2796r)+675r^2 \overset{Euler}{\geq}$$
$$675r^2 > 0 \therefore \text{LHS of } (****) \overset{Gerretsen}{\geq} (3720R^2-10236Rr+6267r^2)(16Rr-5r^2)$$
$$\overset{?}{\geq} r(50240R^3-146640R^2r+109068Rr^2-15271r^3)$$
$$\Leftrightarrow 1160t^3-4467t^2+5298t-2008 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\big((t-2)(1160t+173)+1350\big) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$
$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z} \leq 1$$
$$\forall x,y,z>0 \,|\, x^2+y^2+z^2=3, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

1406. If $x,y,z>0, x^2+y^2+z^2=3$ then:

$$\frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z} \leq 1$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Hikmat Mammadov-Azerbaijan

$$x^2+y^2+z^2=3$$
$$\rightarrow We\ set: \begin{cases} x=\sqrt{3}sin\theta cos\alpha \\ y=\sqrt{3}sin\theta cos\alpha \\ z=\sqrt{3}cos\theta \end{cases} \rightarrow \theta,$$
$$\alpha \in R \rightarrow Then: \ S=\frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z}=$$
$$=\frac{1}{4-\sqrt{3}sin\theta cos\alpha}+\frac{1}{4-\sqrt{3}sin\theta cos\alpha}+\frac{1}{4-\sqrt{3}cos\theta}$$

Note : θ has nothing to do with α.

Hence we adjust for Local Maximum value as treating θ,α constant.

$$\rightarrow S(\alpha)=\frac{A}{B-cos\alpha}+\frac{A}{B-sin\alpha}+C \rightarrow S'(\alpha)=A.\left[\frac{sin\alpha}{(B-cos\alpha)^2}-\frac{cos\alpha}{(B-sin\alpha)^2}\right]$$
$$Let: \ S'(\alpha)=0, clearly\ we\ have\ sin\alpha=cos\alpha, we\ take\ \alpha=45^\circ=\frac{\pi}{4}$$
$$\rightarrow Hence: S(\alpha) \leq S\left(\frac{\pi}{4}\right)=\frac{1}{4-\frac{\sqrt{6}}{2}sin\theta}+\frac{1}{4-\frac{\sqrt{6}}{2}sin\theta}+\frac{1}{4-\sqrt{3}cos\theta}=$$

$$=\frac{4}{8-\sqrt{3}sin\theta}+\frac{1}{4-\sqrt{3}cos\theta}$$

$$Use\ the\ same\ trick\ to\ \theta\rightarrow S'(\theta)=\frac{\sqrt{3}sin\theta}{(4-\sqrt{3}cos\theta)^2}-\frac{4\sqrt{6}cos\theta}{(8-\sqrt{6}sin\theta)^2}$$

$$Let:\ S'(\theta)=0\ :\ sin\theta=\frac{2}{\sqrt{6}}\ and\ cos\theta=\frac{1}{\sqrt{3}}$$

$$\rightarrow\ Hence: S=S(\alpha)\le S(\theta)\le\frac{1}{4-\frac{\sqrt{6}}{2}.\frac{2}{\sqrt{6}}}+2.\frac{1}{4-\sqrt{3}.\frac{1}{\sqrt{3}}}=1$$

$$Therefore\ :\ \frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z}\le 1\rightarrow proved.$$

Solution 2 by Pham Duc Nam-Vietnam

$$p=x+y+z,\quad q=xy+yz+zx,\quad r=xyz$$

$$x^2+y^2+z^2=3\rightarrow p^2-2q=3\rightarrow q=\frac{p^2-3}{2}$$

$$And\ we\ also\ have:\ (x+y+z)^2\le 3(x^2+y^2+z^2)=3.3=9\ p\le 3$$

$$\frac{1}{4-x}+\frac{1}{4-y}+\frac{1}{4-z}\le 1\rightarrow$$

$$(4-y)(4-z)+(4-x)(4-z)+(4-x)(4-y)\le(4-x)(4-y)(4-z)\rightarrow$$

$$8(x+y+z)-3(xy+yz+zx)+xyz-16\le 0\rightarrow 8p-3q+r-16\le 0$$

$$By\ pqr\ transformation\ we\ have\ :r\le\frac{1}{27}p^3\rightarrow$$

$$8p-3q+r-16\le 8p-3q+\frac{1}{27}p^3-16$$

$$Now, replace\ q=\frac{p^2-3}{2}\rightarrow 8p-3q+r-16\le 8p-3.\frac{p^2-3}{2}+\frac{1}{27}p^3-16=$$

$$\frac{1}{54}(p-3)^2(2p-69)\le 0$$

$$This\ is\ true\ since\ (p-3)^2\ge 0\ and\ p\le 3\rightarrow 2p-69<0$$

$$Equality\ holds\ iff\ x=y=z=1.$$

Solution 3 by Le Thu-Vietnam

$$By\ tangent\ line\ method,\quad we\ claim\ that:$$

$$f(x)=\frac{18}{x^2+5}\overset{?}{\le}f(1)+f'(1)(x-1)=4-x\rightarrow\frac{1}{4-x}\overset{?}{\le}\frac{18}{x^2+5}<->(2-x)(x-1)^2\ge 0$$

$$Which\ is\ true\ \forall x\in(0,\sqrt{3}<2).$$

$$\therefore x,y\ and\ z>0:\ \sum\frac{1}{4-x}\le\sum\left(\frac{x^2+5}{8}\right)\overset{\sum x^2=3}{=}1$$

$$Equality\ holds\ iff\ x=y=z=1.$$

1407. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

Let $a=x^2, b=y^2, c=z^2$ $(x,y,z>0)$ and then :

$$\frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}=\frac{z^2(y^2+z^2)}{\frac{z}{y}+\frac{y}{x}}+\frac{x^2(z^2+x^2)}{\frac{x}{z}+\frac{z}{y}}+\frac{y^2(x^2+y^2)}{\frac{y}{x}+\frac{x}{z}}$$

$$\overset{xyz=1}{=}\frac{z(y^2+z^2)}{zx+y^2}+\frac{x(z^2+x^2)}{xy+z^2}+\frac{y(x^2+y^2)}{yz+x^2}\overset{CBS}{\ge}$$

$$\frac{z(y^2+z^2)}{\sqrt{y^2+z^2}.\sqrt{x^2+y^2}}+\frac{x(z^2+x^2)}{\sqrt{z^2+x^2}.\sqrt{y^2+z^2}}+\frac{y(x^2+y^2)}{\sqrt{x^2+y^2}.\sqrt{z^2+x^2}}\overset{A-G}{\ge}3\sqrt[3]{xyz}=3$$

$$\therefore\frac{c(b+c)}{\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}}+\frac{a(c+a)}{\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{b}}}+\frac{b(a+b)}{\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{c}}}\ge 3\ \forall\ a,b,c>0\ |\ abc=1,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1408. If $x+y\ge 0, x-y\ge 0$ and $\sqrt{\left(\frac{x+y}{2}\right)^3}+\sqrt{\left(\frac{x-y}{2}\right)^3}=27$,

then prove that : $x\ge 9$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{x+y}{2}}^3+\sqrt{\frac{x-y}{2}}^3=27$$

$$\Rightarrow\left(\sqrt{\frac{x+y}{2}}+\sqrt{\frac{x-y}{2}}\right)\left(\frac{x+y}{2}+\frac{x-y}{2}-\sqrt{\frac{x^2-y^2}{4}}\right)=27$$

$$\Rightarrow \sqrt{\frac{x+y}{2}+\frac{x-y}{2}+2\sqrt{\frac{x^2-y^2}{4}}.\left(x-\sqrt{\frac{x^2-y^2}{4}}\right)}=27$$

$$\Rightarrow \sqrt{x+2m}.(x-m)=27 \to (1)\ \left(m=\sqrt{\frac{x^2-y^2}{4}}\right)$$

$$\text{Now}, 27=\sqrt{\left(\frac{x+y}{2}\right)^3}+\sqrt{\left(\frac{x-y}{2}\right)^3}\overset{A-G}{\geq} 2.\sqrt{\sqrt{\frac{x^2-y^2}{4}}^3}=2\sqrt{m^3}\Rightarrow \frac{729}{4}\geq m^3$$

$$\Rightarrow m\leq \frac{9}{\sqrt[3]{4}}<9\Rightarrow 9-m>0\to(2)$$

$$\text{Now, we assume } x<9 \text{ and then}: \sqrt{x+2m}.(x-m)<\sqrt{x+2m}.(9-m)$$

$$<\sqrt{9+2m}.(9-m)\ (\because 9-m>0 \text{ via } (2)) \overset{\text{via }(1)}{\Rightarrow} \boxed{27\overset{(*)}{<}\sqrt{9+2m}.(9-m)}$$

$$\text{Let } f(m)=\sqrt{9+2m}.(9-m)\ \forall m\in\left[0,\frac{9}{\sqrt[3]{4}}\right]$$

$$\left(\because x+y\geq 0, x-y\geq 0\Rightarrow m=\sqrt{\frac{x^2-y^2}{4}}\geq 0\right) \text{ and then}: f'(m)=\frac{-3m}{\sqrt{9+2m}}$$

$$\leq 0\ \forall m\in\left[0,\frac{9}{\sqrt[3]{4}}\right]\Rightarrow f(m) \text{ is} \downarrow \text{on} \left[0,\frac{9}{\sqrt[3]{4}}\right] \therefore f(m)\leq f(0)=27$$

$$\Rightarrow \boxed{\sqrt{9+2m}.(9-m)\leq 27} \text{ which is a contradiction to } (*)$$

$$\Rightarrow \text{our assumption is incorrect} \therefore x\geq 9 \text{ (QED)}$$

1409.

If $x, y>0$ and $x+y\leq 1$, then prove that:

$$\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\geq 2\sqrt{5}-\frac{4}{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{4x^2+\frac{1}{x^2}}=\sqrt{4x^2+\frac{1}{4x^2}+\frac{1}{4x^2}+\frac{1}{4x^2}+\frac{1}{4x^2}}\overset{A-G}{\geq}$$

$$\sqrt{5.\sqrt[5]{4x^2.\frac{1}{4x^2}.\frac{1}{4x^2}.\frac{1}{4x^2}.\frac{1}{4x^2}}}\Rightarrow \sqrt{4x^2+\frac{1}{x^2}}\geq \frac{\sqrt{5}}{2^{\frac{3}{5}}}.\left(\frac{1}{x^{\frac{3}{5}}}\right) \text{ and similarly,}$$

$$\sqrt{4y^2+\frac{1}{y^2}}\ge\frac{\sqrt{5}}{2^{\frac{3}{5}}}\cdot\left(\frac{1}{y^{\frac{3}{5}}}\right)\therefore\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}\ge\frac{\sqrt{5}}{2^{\frac{3}{5}}}\cdot\left(\frac{1}{x^{\frac{3}{5}}}+\frac{1}{y^{\frac{3}{5}}}\right)$$

$$\overset{\text{Jensen}}{\ge}\frac{2\sqrt{5}}{2^{\frac{3}{5}}}\cdot\left(\frac{1}{\left(\frac{x+y}{2}\right)^{\frac{3}{5}}}\right)\left(\because f(t)=\frac{1}{t^{\frac{3}{5}}}\ \forall\, t\in(0,1)\text{ is convex as }f''(t)=\frac{24}{25t^{\frac{13}{5}}}>0\right)$$

$$\overset{\text{via }(1)}{\ge}\frac{2\sqrt{5}}{2^{\frac{3}{5}}}\cdot\left(\frac{1}{\left(\frac{1}{2}\right)^{\frac{3}{5}}}\right)=2\sqrt{5}\therefore\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}\ge2\sqrt{5}\to(1)$$

$$\text{Again},\frac{x}{x^2+1}=\frac{x}{x^2+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}\overset{A-G}{\le}\frac{x}{5.\sqrt[5]{x^2.\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}}}\Rightarrow\frac{x}{x^2+1}\le\frac{2^{\frac{8}{5}}}{5}\cdot\left(x^{\frac{3}{5}}\right)$$

$$\text{and similarly},\frac{y}{y^2+1}\le\frac{2^{\frac{8}{5}}}{5}\cdot\left(y^{\frac{3}{5}}\right)\therefore\frac{x}{x^2+1}+\frac{y}{y^2+1}\le\frac{2^{\frac{8}{5}}}{5}\cdot\left(x^{\frac{3}{5}}+y^{\frac{3}{5}}\right)\overset{\text{Jensen}}{\le}$$

$$2.\frac{2^{\frac{8}{5}}}{5}\cdot\left(\frac{x+y}{2}\right)^{\frac{3}{5}}\left(\because F(t)=t^{\frac{3}{5}}\ \forall\, t\in(0,1)\text{ is concave as }F''(t)=-\frac{6}{25t^{\frac{7}{5}}}<0\right)$$

$$=2.\frac{2^{\frac{8}{5}}}{5}\cdot\left(\frac{1}{2}\right)^{\frac{3}{5}}=\frac{4}{5}\Rightarrow-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\ge-\frac{4}{5}\to(2)$$

$$\therefore(1)+(2)\Rightarrow\sqrt{4x^2+\frac{1}{x^2}}+\sqrt{4y^2+\frac{1}{y^2}}-\left(\frac{x}{x^2+1}+\frac{y}{y^2+1}\right)\ge2\sqrt{5}-\frac{4}{5},$$

$$''='' \text{ iff } x=y=\frac{1}{2}\ \text{(QED)}$$

1410. If $0<x\le y\le z\le 4$ and $xyz=1$, then prove that :

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}\le\sqrt{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}=$$

$$=\frac{1}{2\sqrt{1+x^2}}+\frac{1}{2\sqrt{1+x^2}}+\frac{1}{2\sqrt{1+y^2}}+\frac{1}{2\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}$$

$$\overset{CBS}{\leq}\sqrt{5}.\sqrt{\frac{1}{4(1+x^2)}+\frac{1}{4(1+x^2)}+\frac{1}{4(1+y^2)}+\frac{1}{4(1+y^2)}+\frac{1}{1+z}}\overset{?}{\leq}\sqrt{5}$$

$$\Leftrightarrow\frac{(1+z)(1+y^2)+(1+z)(1+x^2)+2(1+x^2)(1+y^2)}{2(1+x^2)(1+y^2)(1+z)}\overset{?}{\leq}1\Leftrightarrow$$

$$2x^2y^2z+z(x^2+y^2)-x^2-y^2-2\overset{?}{\geq}0\overset{z=\frac{1}{xy}}{\Leftrightarrow}2xy+\frac{x^2+y^2}{xy}-(x^2+y^2)-2\overset{?}{\geq}0$$

$$\Leftrightarrow 2x^2y^2+x^2+y^2-xy(x^2+y^2)-2xy\overset{?}{\geq}0$$

$$\Leftrightarrow(x^2+y^2-2xy)-xy(x^2+y^2-2xy)\overset{?}{\geq}0$$

$$\Leftrightarrow(x-y)^2(1-xy)\overset{?}{\geq}0\overset{xy=\frac{1}{z}}{\Leftrightarrow}(x-y)^2\left(1-\frac{1}{z}\right)\overset{?}{\geq}0\Leftrightarrow(z-1)(x-y)^2\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because z\geq x,y\Rightarrow z^2\geq xy\Rightarrow z^3\geq xyz=1\Rightarrow z-1\geq 0$$

$$\therefore\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z}}\leq\sqrt{5}\text{ for }0<x\leq y\leq z\leq 4\wedge xyz=1,$$

$$\text{with equality iff } x=y \text{ and } \frac{1}{2\sqrt{1+x^2}}=\frac{1}{\sqrt{1+z}}=\frac{1}{\sqrt{1+\frac{1}{x^2}}}\left(\because z=\frac{1}{xy}=\frac{1}{x^2}\right)$$

$$\Rightarrow x=y=\frac{1}{2}\Rightarrow z=\frac{1}{\left(\frac{1}{2}\right)^2}=4\therefore ''='' \text{ iff }\left(x=y=\frac{1}{2},z=4\right)\text{ (QED)}$$

1411. If $a,b>0, ab=1$ and $\lambda\geq 0$, then :

$$(a^2+b^2)(a+b+\lambda)+\frac{4\lambda}{a^2+b^2}\geq 4(\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } Q=\sqrt{\frac{a^2+b^2}{2}} \text{ and } A=\frac{a+b}{2} \text{ and then :}$$

$$(a^2+b^2)(a+b+\lambda)+\frac{4\lambda}{a^2+b^2}\geq 4(\lambda+1)\Leftrightarrow 2Q^2(2A+\lambda)+\frac{2\lambda}{Q^2}\geq 4(\lambda+1)$$

$$\Leftrightarrow(2A+\lambda)Q^4-2(\lambda+1)Q^2+\lambda\overset{(*)}{\geq}0$$

Now, LHS of $(*)$ is a quadratic polynomial in Q^2 with discriminant $\Delta=$
$4(\lambda+1)^2-4\lambda(2A+\lambda)=4(\lambda^2+2\lambda+1-2A\lambda-\lambda^2)\therefore\Delta=4\big(1-2\lambda(A-1)\big)$
Now, if $2\lambda(A-1)\geq 1$, then : $\Delta\leq 0$ and then : LHS of $(*)\geq 0\Rightarrow(*)$ is true
and we now focus on : $2\lambda(A-1)<1$ and we have : the bigger zero of LHS of $(*)$

$$=\frac{2(\lambda+1)+2\sqrt{1-2\lambda(A-1)}}{2(2A+\lambda)}=\frac{\lambda+1+\sqrt{1-2\lambda(A-1)}}{2A+\lambda}$$

$$\le \frac{\lambda+1+\sqrt{1-2\lambda(1-1)}}{2+\lambda}\left(\because A=\frac{a+b}{2}\overset{A-G}{\ge}\sqrt{ab}=1\Rightarrow A\ge 1 \text{ and } \lambda\ge 0\right)=1$$

$\therefore$ the bigger zero of LHS of $(*)\le 1\to(1)$ and $Q^2\overset{Q-A}{\ge}A^2\ge 1\overset{via\,(1)}{\ge}$ the bigger zero of LHS of $(*)\Rightarrow(*)$ is true $\therefore (a^2+b^2)(a+b+\lambda)+\frac{4\lambda}{a^2+b^2}\ge 4(\lambda+1)$

$\forall a,b>0\,|\,ab=1\wedge\lambda\ge 0,''=''$ iff $a=b=1$ (QED)

1412. If $x,y,z>0$ with $x+y+z=xyz$, then :

$$8\prod_{cyc}\sqrt{x^2+1}\le\left(1+\frac{1}{3}\sum_{cyc}yz\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$8\prod_{cyc}\sqrt{x^2+1}=8\prod_{cyc}\sqrt{x^2+\frac{xyz}{x+y+z}}=$$

$$=8\prod_{cyc}\sqrt{\frac{x}{\sum_{cyc}x}.(x^2+xy+xz+yz)}=8\prod_{cyc}\sqrt{\frac{x}{\sum_{cyc}x}.(x+y)(x+z)}=$$

$$=8\sqrt{\frac{xyz}{(\sum_{cyc}x)^3}}.\prod_{cyc}(x+y)\overset{x+y+z=xyz}{=}8\sqrt{\frac{\sum_{cyc}x}{(\sum_{cyc}x)^3}}.\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-xyz\right)=$$

$$\overset{x+y+z=xyz}{=}\frac{8}{\sum_{cyc}x}.\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\sum_{cyc}x\right)=8\left(\sum_{cyc}xy-1\right)$$

$$\Rightarrow 8\prod_{cyc}\sqrt{x^2+1}=8\left(\sum_{cyc}xy-1\right)\to(1)$$

$$\text{Now, } xyz=\sum_{cyc}x\overset{A-G}{\ge}3\sqrt[3]{xyz}\Rightarrow\sqrt[3]{x^2y^2z^2}\ge 3\therefore\sum_{cyc}xy\overset{A-G}{\ge}3\sqrt[3]{x^2y^2z^2}\ge 3.3$$

$$\Rightarrow t=\sum_{cyc}xy\ge 9\to(2)$$

$$\text{Now, }(1)\Rightarrow 8\prod_{cyc}\sqrt{x^2+1}\le\left(1+\frac{1}{3}\sum_{cyc}yz\right)^3\Leftrightarrow\left(1+\frac{t}{3}\right)^3\ge 8(t-1)$$

$$\Leftrightarrow (t+3)^3-216(t-1)\ge 0 \Leftrightarrow t^3+9t^2-189t+243\ge 0$$

$$\Leftrightarrow (t-9)\left(t^2+15t+3(t-9)\right)\ge 0 \to \text{true} \because t \overset{\text{via }(2)}{\ge} 9$$

$$\therefore 8\prod_{cyc}\sqrt{x^2+1}\le\left(1+\frac{1}{3}\sum_{cyc}yz\right)^3\ \forall\, x,y,z>0\,|\,x+y+z=xyz,$$

$$''=''\ \text{iff } x=y=z=\sqrt{3}\ (QED)$$

1413. If $x,y,z>0, x^2+y^2+z^2=3$ then:

$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}\ge 4+\frac{6}{x+y+z}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3}\overset{AM-HM}{\ge}\frac{3}{x+y+z}\ or\ \frac{1}{x+y+z}\le\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ (1)$$

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\ge\frac{3(x^2+y^2+z^2)}{x+y+z}\ (2)$$

$$Proof: (x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\ge 3(x^2+y^2+z^2)\, or$$

$$\sum\frac{x^3}{y}+\sum\frac{xy^2}{z}\ge 2\sum x^2\ or$$

$$\sum\frac{x^4}{yx}+\sum\frac{x^2y^2}{zx}\ge 2\sum x^2\ or$$

$$\frac{(\sum x^2)^2}{\sum xy}+\frac{(\sum xy)^2}{\sum xy}\ge 2\sum x^2\ (Bergstrom)$$

$$or\ \left(\sum x^2\right)^2+\left(\sum xy\right)^2-2\left(\sum xy\right)\left(\sum x^2\right)\ge 0\ or$$

$$\left(\sum x^2-\sum xy\right)^2\ge 0\ , proof\ complete$$

$$(x+y+z)^2\overset{CBS}{\le}3(x^2+y^2+z^2)\ or\ (x+y+z)\le\sqrt{3(x^2+y^2+z^2)}\ (3)$$

$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}-\frac{6}{x+y+z}=\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\left(\frac{6}{x+y+z}\right)$$

$$\overset{(1)\&(2)}{\ge}\frac{3(x^2+y^2+z^2)}{x+y+z}+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\frac{6}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=$$

$$=\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{3}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\overset{Bergstrom}{\geq}\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{1}{3}\frac{(1+1+1)^2}{x+y+z}\overset{(3)}{\geq}\frac{3(x^2+y^2+z^2)}{\sqrt{3(x^2+y^2+z^2)}}+\frac{1}{3}\frac{9}{\sqrt{3(x^2+y^2+z^2)}}$$

$$=\frac{3.3}{\sqrt{3.3}}+\frac{3}{\sqrt{3.3}}(since\ x^2+y^2+z^2=3)=3+\frac{3}{3}=4$$

$$or\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}\geq 4+\frac{6}{x+y+z}$$

$$Equality\ holds\ for\ x=y=z=1$$

1414. If $x,y>0$, then prove that :

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^4+y^4}{(x+y)^4}-\frac{1}{8}=\frac{7x^4+7y^4-4x^3y-4xy^3-6x^2y^2}{8(x+y)^4}$$

$$=\frac{3(x^2-y^2)^2+4\left(x^4+y^4-xy(x^2+y^2)\right)}{8(x+y)^4}$$

$$\geq\frac{3(x^2-y^2)^2+4\left(\frac{(x^2+y^2)^2}{2}-xy(x^2+y^2)\right)}{8(x+y)^4}$$

$$=\frac{3(x^2-y^2)^2+2(x^2+y^2)(x-y)^2}{8(x+y)^4}\geq\frac{3(x-y)^2(x+y)^2+(x+y)^2(x-y)^2}{8(x+y)^4}$$

$$=\frac{(x-y)^2}{2(x+y)^2}\Rightarrow\frac{x^4+y^4}{(x+y)^4}-\frac{1}{8}+\frac{\sqrt{xy}}{x+y}\overset{G-H}{\geq}\frac{(x-y)^2}{2(x+y)^2}+\frac{4xy}{2(x+y)^2}=\frac{(x+y)^2}{2(x+y)^2}=\frac{1}{2}$$

$$\Rightarrow\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{1}{8}+\frac{1}{2}=\frac{5}{8}\ \forall x,y>0,''=''\ \text{iff } x=y\ (\text{QED})$$

1415. If $a,b,c>0$ and $5a^2+4b^2+3c^2+2abc=60$, then prove that :

$$a+b+c\leq\sqrt{60}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let us assume } a+b+c>\sqrt{60} \text{ and } 60=5a^2+4b^2+3c^2+2abc$$

$$=(3a^2+3b^2+3c^2)+2a^2+b^2+2abc\geq(a+b+c)^2+2a^2+b^2+2abc$$

$$\overset{\text{assumption}}{>} 60+2a^2+b^2+2abc\Rightarrow 2a^2+b^2+2abc<0\rightarrow \textit{impossible}$$

$$\because a,b,c>0 \therefore \text{our assumption is incorrect and hence we conclude that :}$$

$$a+b+c\leq\sqrt{60}\ (\text{QED})$$

1416. If $a,b,c>0, abc=1$ then:

$$\sum\frac{a^3+b^3}{a^2+\lambda ab+b^2}\geq\frac{6}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a^2+\lambda ab+b^2\overset{AM-GM}{\leq} a^2+b^2+\frac{\lambda(a^2+b^2)}{2}=\frac{(a^2+b^2)(\lambda+2)}{2}\quad(1)$$

$$a^3+b^3=(a+b)(a^2+b^2-ab)$$

$$Now\frac{a^3+b^3}{a^2+\lambda ab+b^2}\overset{(1)}{\geq}\frac{(a+b)(a^2+b^2-ab)}{\frac{(a^2+b^2)(\lambda+2)}{2}}=$$

$$=\frac{2}{\lambda+2}(a+b)\left(\frac{a^2+b^2-ab}{a^2+b^2}\right)=\frac{2}{\lambda+2}(a+b)\left(1-\frac{ab}{a^2+b^2}\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{2}{\lambda+2}(a+b)\left(1-\frac{ab}{2ab}\right)=\frac{2}{\lambda+2}(a+b)\left(1-\frac{1}{2}\right)=\frac{a+b}{\lambda+2}\quad(2)$$

$$\sum\frac{a^3+b^3}{a^2+\lambda ab+b^2}\overset{(2)}{\geq}\sum\frac{a+b}{\lambda+2}\overset{AM-GM}{\geq}\frac{6}{\lambda+2}(a^2b^2c^2)^{\frac{1}{6}}=\frac{6}{\lambda+2}$$

$$\textit{Equality for } a=b=c=1$$

1417. If $a,b,c>0$ and $abc=1$, $n\in N$ **then:**

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{b^{n+2}+c^{n+2}}{a^3(b^n+c^n)}\geq 6$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{(b^{n+2}+c^{n+2})}{2}\overset{CBS}{\geq}\frac{b^n+c^n}{2}\cdot\frac{b^2+c^2}{2}\ or$$

$$(b^{n+2}+c^{n+2})\geq\frac{1}{2}(b^2+c^2)(b^n+c^n)\overset{AM-GM}{\geq}$$

$$\geq\frac{1}{2}2bc(b^n+c^n)=bc(b^n+c^n)\ (1)$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{b^{n+2}+c^{n+2}}{a^3(b^n+c^n)}\overset{(1)}{\geq}\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{bc(b^n+c^n)}{a^3(b^n+c^n)}=$$

$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\sum\frac{bc}{a^3}=$$

$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{bc}{a^3}+\frac{ca}{b^3}+\frac{ab}{c^3}\overset{AM-GM}{\geq}$$

$$\geq 6\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\frac{bc}{a^3}\cdot\frac{ca}{b^3}\cdot\frac{ab}{c^3}\right)^{\frac{1}{3}}=6\left(\frac{1}{a^2b^2c^2}\right)^{\frac{1}{3}}=6(as\ abc=1)$$

$$Equality\ a=b=c$$

1418. If $x,y>0$ and $\lambda\leq\frac{5}{4}$, **then :**

$$\frac{x^4+y^4}{(x+y)^4}+\lambda.\frac{\sqrt{xy}}{x+y}\geq\frac{4\lambda+1}{8}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^4+y^4}{(x+y)^4}+\lambda.\frac{\sqrt{xy}}{x+y}\overset{G-H}{\geq}\frac{x^4+y^4}{(x+y)^4}+\lambda.\frac{2xy}{(x+y)^2}\overset{?}{\geq}\frac{4\lambda+1}{8}$$

$$\Leftrightarrow \frac{x^4+y^4}{(x+y)^4}-\frac{1}{8}\overset{?}{\geq}\lambda.\left(\frac{1}{2}-\frac{2xy}{(x+y)^2}\right)$$

$$\Leftrightarrow \frac{7x^4+7y^4-4x^3y-4xy^3-6x^2y^2}{8(x+y)^4}\overset{?}{\geq}\lambda.\frac{(x-y)^2}{2(x+y)^2}$$

$$\Leftrightarrow \frac{(7x^4+7y^4-14x^2y^2)-(4x^3y+4xy^3-8x^2y^2)}{8(x+y)^4}\overset{?}{\geq}\lambda.\frac{(x-y)^2}{2(x+y)^2}$$

$$\Leftrightarrow \frac{7(x-y)^2(x+y)^2-4xy(x-y)^2}{4(x+y)^4}\underbrace{\overset{?}{\geq}}_{(*)}\lambda.\frac{(x-y)^2}{(x+y)^2}\text{ and } \because \lambda\leq\frac{5}{4}\therefore \text{ in order}$$

to prove $(*)$, it sufficesto prove : $\frac{7(x-y)^2(x+y)^2-4xy(x-y)^2}{4(x+y)^4}\geq\frac{5}{4}.\frac{(x-y)^2}{(x+y)^2}$

$$\Leftrightarrow \frac{(x-y)^2}{(x+y)^2}.\left(\frac{7(x+y)^2-4xy}{(x+y)^2}-5\right)\geq 0\Leftrightarrow\frac{(x-y)^2}{(x+y)^2}.\left(\frac{2(x+y)^2-4xy}{(x+y)^2}\right)\geq 0$$

$$\Leftrightarrow \frac{(x-y)^2}{(x+y)^2}.\frac{x^2+y^2}{(x+y)^2}\geq 0\to \text{true}$$

$$\therefore \frac{x^4+y^4}{(x+y)^4}+\lambda.\frac{\sqrt{xy}}{x+y}\geq\frac{4\lambda+1}{8}\ \forall x,y>0 \text{ and } \lambda\leq\frac{5}{4}, ''='' \text{ iff } x=y \text{ (QED)}$$

1419. If $x,y,z>0, xyz=1$ **then:**

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}}\geq\frac{3}{2}$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution by Tapas Das-India

$$\left(\frac{y+7}{8}\right)^{\frac{1}{3}}=\left(1+\frac{y-1}{8}\right)^{\frac{1}{3}}\overset{Bernoulli}{\leq}1+\frac{y-1}{24}=\frac{y+23}{24}$$

$$\left(\frac{8}{y+7}\right)^{\frac{1}{3}}\geq\frac{24}{y+23}\text{ or }\frac{1}{\sqrt[3]{y+7}}\geq\frac{12}{y+23}\ (1)$$

$$\forall x,y,z>0\ \left(\sum x^2\right)^2\geq 3\sum x^3y\ (Vasc\ inequality)$$

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}}\overset{(1)}{\geq}\sum\frac{12\sqrt[3]{x}}{y+23}\ (2)$$

$$let\ x^{\frac{1}{3}}=\frac{a}{b}, y^{\frac{1}{3}}=\frac{c}{a}, z^{\frac{1}{3}}=\frac{b}{c}\ then\ from\ (2):$$

$$\sqrt[3]{\frac{x}{y+7}}+\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{x+7}} \overset{(1)}{\geq} \sum \frac{12\sqrt[3]{x}}{y+23} = 12\sum \frac{\left(\frac{a}{b}\right)}{\frac{c^3}{a^3}+23} =$$

$$= 12\sum \frac{a^4}{b(c^3+23a^3)} \overset{Bergstrom}{\geq} 12\frac{(a^2+b^2+c^2)^2}{(\sum bc^3)+23(\sum a^3b)} \geq$$

$$\overset{(2)}{\geq} 12\frac{(a^2+b^2+c^2)^2}{\frac{(\sum a^2)^2}{3}+\frac{23(\sum a^2)^2}{3}} = \frac{3}{2}$$

$$(Equality\ holds\ for\ a=b=c=1)$$

1420. **If** $0<x<y<4$**, *then prove that*:**

$$\ln\frac{x(4-y)}{y(4-x)} < x-y$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\ln\frac{x(4-y)}{y(4-x)} < x-y \Leftrightarrow \ln x+\ln(4-y)-\ln y-\ln(4-x) < x-y$$

$$\Leftrightarrow \ln y - y - \ln(4-y) > \ln x - x - \ln(4-x) \Leftrightarrow f(y)-f(x)>0$$

$$\left(f(t)=\ln t - t - \ln(4-t)\ \forall t\in(0,4)\right) \overset{\text{via MVT}}{\Leftrightarrow} (y-x).f'(c)>0$$

$$(0<x<c<y<4) \Leftrightarrow (y-x)\left(\frac{1}{c}+\frac{1}{4-c}-1\right)>0 \to \text{true} \because (y-x)>0$$

$$and\ \frac{1}{c}+\frac{1}{4-c} \overset{\text{Bergstrom}}{\geq} \frac{4}{c+4-c}=1 \Rightarrow \frac{1}{c}+\frac{1}{4-c}-1>0$$

$$\therefore \ln\frac{x(4-y)}{y(4-x)} < x-y \text{ for } 0<x<y<4\ (QED)$$

1421. If $a, b, c > 0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$, then :

$$\sum_{cyc}\left(\sqrt{3b-1}\cdot\sqrt{3c-1}\right)\geq 6$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\sqrt{3b-1}.\sqrt{3c-1}\right)\overset{3=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{=}$$

$$\sum_{cyc}\left(\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)b-1}.\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)c-1}\right)=\sum_{cyc}\sqrt{\frac{b(c+a)}{ca}.\frac{c(a+b)}{ab}}$$

$$\overset{A-G}{\geq} 3.\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{Cesaro}{\geq} 3.\sqrt[3]{8}$$

$$\therefore\sum_{cyc}\left(\sqrt{3b-1}.\sqrt{3c-1}\right)\geq 6\ \forall\, a,b,c>0,''=''\ \text{ iff } a=b=c=1\ (QED)$$

1422. If $a, b \in \mathbb{R}$ with $ab \geq \frac{1}{3}$ and $\lambda \geq \frac{1}{3}$, then :

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\leq\frac{6}{3\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}\leq\frac{6}{3\lambda+1}$$

$$\Leftrightarrow (3\lambda+1)(a^2+b^2+2\lambda)\leq 6(a^2b^2+\lambda a^2+\lambda b^2+\lambda^2)$$

$$\Leftrightarrow 3\lambda(a^2+b^2)+a^2+b^2+2\lambda\leq 6a^2b^2+6\lambda(a^2+b^2)$$

$$\Leftrightarrow a^2+b^2+2\lambda-6a^2b^2-3\lambda(a^2+b^2)\overset{(*)}{\leq} 0$$

$$\text{Now}, \because ab\geq\frac{1}{3}\therefore \text{LHS of } (*)\leq a^2+b^2+6\lambda ab-2ab-3\lambda(a^2+b^2)$$

$$= (a-b)^2 - 3\lambda(a-b)^2 = (1-3\lambda)(a-b)^2 \overset{\lambda \geq \frac{1}{3}}{\leq} 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} \leq \frac{6}{3\lambda+1} \forall\, a,b \in \mathbb{R} \text{ with } ab \geq \frac{1}{3} \text{ and } \lambda \geq \frac{1}{3} \text{ (QED)}$$

1423. If $a,b,c > 0$ with $a+b+c = 3abc$ and $\lambda \geq 3$, then :

$$\sum_{cyc} \frac{bc}{\lambda a + bc + abc} \geq \frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{bc}{\lambda a + bc + abc} = \sum_{cyc} \frac{b^2c^2}{\lambda abc + b^2c^2 + abc.bc} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc} ab\right)^2}{3\lambda abc + \sum_{cyc} a^2b^2 + abc\sum_{cyc} ab} \overset{\sum_{cyc} a = 3abc}{=} \frac{\left(\sum_{cyc} ab\right)^2}{3\lambda abc.\sqrt{\frac{3abc}{\sum_{cyc} a}} + \sum_{cyc} a^2b^2 + abc\left(\sum_{cyc} ab\right).\sqrt{\frac{\sum_{cyc} a}{3abc}}}$$

$$= \frac{\left(\sum_{cyc} ab\right)^2.\sqrt{3abc\sum_{cyc} a}}{3\lambda abc.3abc + \left(\sum_{cyc} a^2b^2\right).\sqrt{3abc\sum_{cyc} a} + abc\left(\sum_{cyc} ab\right).\left(\sum_{cyc} a\right)} \overset{?}{\geq} \frac{3}{\lambda+2}$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} ab\right)^2.\sqrt{3abc\sum_{cyc} a} - 27a^2b^2c^2\right) + 2\left(\sum_{cyc} ab\right)^2.\sqrt{3abc\sum_{cyc} a}$$

$$\overset{?}{\geq} 3\left(\sum_{cyc} a^2b^2\right).\sqrt{3abc\sum_{cyc} a} + 3abc\left(\sum_{cyc} ab\right).\left(\sum_{cyc} a\right)$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} ab\right)^2 - \frac{27a^2b^2c^2}{\sqrt{3abc\sum_{cyc} a}}\right) + 2\left(\sum_{cyc} ab\right)^2 \boxed{\overset{?}{\underset{(*)}{\geq}}}$$

$$3\left(\sum_{cyc} a^2b^2\right) + \left(\sum_{cyc} ab\right).\sqrt{3abc\sum_{cyc} a}$$

$$\text{Now, } \left(\sum_{cyc} ab\right)^2.\sqrt{3abc\sum_{cyc} a} \overset{?}{\geq} 27a^2b^2c^2 \Leftrightarrow \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)^4 \overset{?}{\geq} 243a^3b^3c^3$$

$$\to \text{true} \because \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \overset{A-G}{\geq} 9abc \text{ and } \left(\sum_{cyc} ab\right)^3 \overset{A-G}{\geq} 27a^2b^2c^2$$

$$\therefore \left(\sum_{cyc} ab\right)^2 . \sqrt{3abc\sum_{cyc} a} - 27a^2b^2c^2 \geq 0 \Rightarrow \left(\sum_{cyc} ab\right)^2 - \frac{27a^2b^2c^2}{\sqrt{3abc\sum_{cyc} a}} \geq 0 \text{ and}$$

$$\because \lambda \geq 3 \therefore \text{LHS of } (*) - \text{RHS of } (*) \geq 3\left(\left(\sum_{cyc} ab\right)^2 - \frac{27a^2b^2c^2}{\sqrt{3abc\sum_{cyc} a}}\right)$$

$$+2\left(\sum_{cyc} ab\right)^2 - 3\left(\sum_{cyc} a^2b^2\right) - \left(\sum_{cyc} ab\right) . \sqrt{3abc\sum_{cyc} a}$$

$$= 2\sum_{cyc} a^2b^2 + 10abc\sum_{cyc} a - \frac{81a^2b^2c^2}{\sqrt{3abc\sum_{cyc} a}} - \left(\sum_{cyc} ab\right) . \sqrt{3abc\sum_{cyc} a} \overset{?}{\geq} 0$$

$$\Leftrightarrow 2\sum_{cyc} a^2b^2 + 10abc\sum_{cyc} a \overset{?}{\geq} \frac{81a^2b^2c^2 + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)}{\sqrt{3abc\sum_{cyc} a}}$$

$$\Leftrightarrow 3abc\left(\sum_{cyc} a\right)\left(2\sum_{cyc} a^2b^2 + 10abc\sum_{cyc} a\right)^2 \boxed{\overset{?}{\underset{(**)}{\geq}}}$$

$$\left(81a^2b^2c^2 + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)\right)^2$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - x)(s - y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \text{and } \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via } (1),(2) \text{ and } (3)}{=} (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2) \to (4)$$

and via $(1), (2), (3)$ and $(4), (**)$

$$\Leftrightarrow 3r^2s^2(2r^2((4R+r)^2-2s^2)+10r^2s^2)^2 \ge \left(81r^4s^2+3r^2s^2(4Rr+r^2)\right)^2$$

$$\Leftrightarrow (4R+r)^4+(84R^2-120Rr-582r^2)s^2+9s^4 \boxed{\overset{(***)}{\ge}} 0$$

$$\text{Now, LHS of } (***) \overset{\text{Gerretsen}}{\ge} (4R+r)^4+(84R^2-120Rr-582r^2)s^2$$

$$+9(16Rr-5r^2)s^2 \overset{?}{\ge} 0 \Leftrightarrow (84R^2+24Rr-627r^2)s^2+(4R+r)^4 \boxed{\overset{(****)}{\ge}} 0$$

$\boxed{\text{Case 1}}$ $84R^2+24Rr-627r^2 \ge 0$ and then : LHS of $(****) \ge (4R+r)^4 > 0$

$\Rightarrow (****)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $84R^2+24Rr-627r^2 < 0$ and then : LHS of $(****) \overset{\text{Gerretsen}}{\ge}$

$$(84R^2+24Rr-627r^2)(4R^2+4Rr+3r^2)+(4R+r)^4 \overset{?}{\ge} 0$$

$$\Leftrightarrow 148t^4+172t^3-516t^2-605t-470 \overset{?}{\ge} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(148t^3+468t^2+420t+235) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow (****) \text{ is true}$$

$\therefore$ combining both cases, $(****) \Rightarrow (***)$ is true for *all* triangles $\therefore (**) \Rightarrow (*)$

is true $\Rightarrow \sum_{cyc} \frac{bc}{\lambda a+bc+abc} \ge \frac{3}{\lambda+2}\ \forall a,b,c>0 \,|\, a+b+c=3abc$ and

$\lambda \ge 3$, " = " iff $a=b=c=1$ (QED)

1424. If $a,b,c>0.\ n\in N,\ n>2$ then:

$$\sum \frac{b+c}{a+\sqrt[n]{2^{n-1}(b^n+c^n)}} \le 2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{b^n+c^n}{2} \overset{CBS}{\ge} \frac{(c+b)^n}{2^n}$$

$$2^{n-1}(b^n+c^n) \ge (c+b)^n$$

$$\sqrt[n]{2^{n-1}(b^n+c^n)} \ge c+b\ (1)$$

$$\sum \frac{b+c}{a+\sqrt[n]{2^{n-1}(b^n+c^n)}} \overset{(1)}{\le} \sum \frac{b+c}{a+b+c} = \frac{2(a+b+c)}{a+b+c} = 2$$

Equality holds for $a=b=c$

1425. If $a, b, c > 0$ with $abc = 1$ and $n \in \mathbb{N}$, then :

$$\sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} \leq 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{1 + b^3 + c^3} \leq \sum_{cyc} \frac{1}{1 + bc(b + c)} \overset{abc=1}{=} \sum_{cyc} \frac{1}{1 + \frac{b + c}{a}} = \sum_{cyc} \frac{a}{a + b + c}$$

$$\therefore \sum_{cyc} \frac{1}{1 + b^3 + c^3} \leq 1 \to (1)$$

$\boxed{\text{Case 1}}$ $n = 0$ and then : $\displaystyle\sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} = \sum_{cyc} \frac{1}{1 + b^3 + c^3} \overset{\text{via }(1)}{\leq} 1$

$\boxed{\text{Case 2}}$ $n \in \mathbb{N}^*$ and then : $f(x) = x^n \Rightarrow f''(x) = n(n-1)x^{n-2} \geq 0$

$$\Rightarrow f(x) \text{ is convex} \Rightarrow \frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3} = \frac{1.a^n + b^3.b^n + c^3.c^n}{1 + b^3 + c^3} \overset{\text{Weighted Jensen}}{\geq}$$

$$\left(\frac{1.a + b^3.b + c^3.c}{1 + b^3 + c^3}\right)^n \Rightarrow \sqrt[n]{\frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3}} \geq \frac{a + b^4 + c^4}{1 + b^3 + c^3} \overset{abc=1}{=} \frac{\frac{1}{bc} + \frac{b^6}{b^2} + \frac{c^6}{c^2}}{1 + b^3 + c^3}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{(1 + b^3 + c^3)^2}{(b^2 + c^2 + bc)(1 + b^3 + c^3)} \overset{A-G}{\geq} \frac{1 + b^3 + c^3}{b^2 + c^2 + \frac{b^2 + c^2}{2}} \overset{\text{Power-Mean Inequality}}{\geq} \frac{1 + 2\left(\frac{b^2 + c^2}{2}\right)^{\frac{3}{2}}}{\frac{3}{2}(b^2 + c^2)}$$

$$= 1 + \frac{1 + 2t^3 - 3t^2}{3t^2} \left(t = \sqrt{\frac{b^2 + c^2}{2}}\right) = 1 + \frac{(2t + 1)(t - 1)^2}{3t^2} \geq 1$$

$$\therefore \sqrt[n]{\frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3}} \geq 1 \Rightarrow \frac{a^n + b^{n+3} + c^{n+3}}{1 + b^3 + c^3} \geq 1 \ (\because n \in \mathbb{N}^* \Rightarrow n > 0)$$

$$\Rightarrow \frac{1}{a^n + b^{n+3} + c^{n+3}} \leq \frac{1}{1 + b^3 + c^3} \text{ and analogs} \Rightarrow \sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}}$$

$$\leq \sum_{cyc} \frac{1}{1 + b^3 + c^3} \overset{\text{via }(1)}{\leq} 1 \therefore \text{combining both cases,} \sum_{cyc} \frac{1}{a^n + b^{n+3} + c^{n+3}} \leq 1$$

$$\forall a, b, c > 0 \mid abc = 1 \wedge n \in \mathbb{N}, " = " \text{ iff } a = b = c = 1 \text{ (QED)}$$

1426. If $a,b,c>0$, $a^2+b^2+c^2=3$, $\lambda\geq 0$ then:

$$\sum\frac{1}{a^2+\lambda a+\lambda}\geq\frac{3}{2\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$a^2+b^2+c^2=3\ or.\frac{(a+b+c)^2}{3}\leq 3(CBS)\ or\ (a+b+c)\leq 3\ \ (1)$$

$$\sum\frac{1}{a^2+\lambda a+\lambda}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{(\sum a^2)+\lambda(a+b+c)+3\lambda}\overset{(1)\&\,a^2+b^2+c^2=3}{\geq}$$

$$\geq\frac{9}{3+3\lambda+3\lambda}=\frac{3}{2\lambda+1}$$

Equality holds for $a=b=c=1$

1427. If $a,b,c>0$ and $abc=1$ then:

$$\sum\sqrt{1+8a^2}\geq\frac{7}{3}(a+b+c)+2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$1+8a^2=1+a^2+a^2+a^2+a^2+a^2+a^2+a^2+a^2\overset{CBS}{\geq}\frac{(1+8a)^2}{9}\ \ (1)$$

$$\sum\sqrt{1+8a^2}\overset{(1)}{\geq}\sum\frac{1+8a}{3}=\frac{8}{3}(a+b+c)+1=$$

$$=\frac{7}{3}(a+b+c)+\frac{1}{3}(a+b+c)+1\geq$$

$$\overset{AM-GM}{\geq}\frac{7}{3}(a+b+c)+\sqrt[3]{abc}+1=\frac{7}{3}(a+b+c)+2\ (as\ \ abc=1)$$

Equality for $a=b=c=1$

1428. If $a,b,c>0,\ abc=1,n\in N$ then:

$$\sum\frac{a^{n+2}+b^{n+2}}{a^{n+1}b^{n+1}(a^n+b^n)}\geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{a^{n+2}+b^{n+2}}{a^{n+1}b^{n+1}(a^n+b^n)}\overset{CBS}{\geq}\frac{\frac{(a^n+b^n)(a^2+b^2)}{2}}{a^{n+1}b^{n+1}(a^n+b^n)}=\frac{a^2+b^2}{2}=$$

$$=\frac{\frac{a^2+b^2}{2}}{a^{n+1}b^{n+1}}\overset{AM-GM}{\geq}\frac{ab}{a^{n+1}b^{n+1}}=\frac{1}{a^nb^n}\ (1)$$

$$\sum\frac{a^{n+2}+b^{n+2}}{a^{n+1}b^{n+1}(a^n+b^n)}\overset{(1)}{\geq}\sum\frac{1}{a^nb^n}\overset{AM-GM}{\geq}3\left(\frac{1}{abc}\right)^{\frac{2n}{3}}=3\ (abc=1)$$

Equality holds for $a=b=c=1$

1429. If $a,b,c>0,a^2+b^2+c^2=1,\ \lambda\geq 0$ then:

$$\sum\frac{a^2}{1+\lambda bc}\geq\frac{3}{\lambda+3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a^2=1\ or\ \frac{(\sum a)^2}{3}\leq 1(CBS)or\left(\sum a\right)\leq\sqrt{3}\ (1)$$

$$abc\overset{AM-GM}{\leq}.\frac{(a+b+c)^3}{27}\overset{(1)}{\leq}\frac{3\sqrt{3}}{27}=\frac{\sqrt{3}}{9}\quad(2)$$

$$\sum\frac{a^2}{1+\lambda bc}=\sum\frac{a^4}{a^2+a^2\lambda bc}\overset{Bergstrom}{\geq}$$

$$\geq \frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\lambda abc(a+b+c)} \overset{(1),(2)\&\sum a^2=1}{\geq} \cdot \frac{1}{1+\lambda\frac{\sqrt{3}}{9}\sqrt{3}} = \frac{3}{\lambda+3}$$

$$Equality\ for\ a=b=c=\frac{1}{\sqrt{3}}$$

1430. **If $a, b, c > 0$, then prove that :**

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq \frac{\sqrt{2}}{4}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\sum_{cyc}\frac{b^2}{b+c}=\sum_{cyc}\frac{b^2-c^2+c^2}{b+c}$$

$$=\sum_{cyc}(b-c)+\sum_{cyc}\frac{c^2}{b+c}=\sum_{cyc}\frac{c^2}{b+c} \Rightarrow \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}$$

$$=\frac{1}{2}\sum_{cyc}\frac{b^2}{b+c}+\frac{1}{2}\sum_{cyc}\frac{c^2}{b+c}=\sum_{cyc}\frac{b^2+c^2}{2(b+c)}=\sum_{cyc}\frac{\sqrt{b^2+c^2}.\sqrt{b^2+c^2}}{2(b+c)}$$

$$\geq \sum_{cyc}\frac{\sqrt{b^2+c^2}.\sqrt{\frac{(b+c)^2}{2}}}{2(b+c)}=\frac{\sqrt{2}}{4}.\sum_{cyc}\sqrt{b^2+c^2} \therefore \frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \geq$$

$$\frac{\sqrt{2}}{4}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)\ \forall a,b,c>0, ''='' \text{ iff } a=b=c \text{ (QED)}$$

1431. **If $x, y > 0$ and $xy \geq 1$, then prove that :**

$$\frac{x}{y+1}+\frac{y}{x+1}+\frac{1}{xy+1} \geq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Clearing denominators and simplifying, } \frac{x}{y+1}+\frac{y}{x+1}+\frac{1}{xy+1} \geq \frac{3}{2}$$

$$\Leftrightarrow 2xy(x^2+y^2)+2(x^2+y^2)+x+y \geq 3x^2y^2+xy(x+y)+4xy+1 \rightarrow (1)$$

$$\boxed{\text{Case 1}}\ xy+1>x+y \text{ and we have : LHS of (1)} - \text{RHS of (1)}$$

$$=(2xy(x^2+y^2)-4x^2y^2)+x^2y^2+(2(x^2+y^2)-4xy)+x+y-xy(x+y)-1$$

$$= 2xy(x-y)^2 + 2(x-y)^2 + (x^2y^2 - 1) - xy(x+y) + x + y$$

$$\geq (xy+1)(xy-1) - (x+y)(xy-1) = (xy-1)\big(xy+1-(x+y)\big) \geq 0$$

$$\because xy \geq 1 \text{ and } xy+1 > x+y \Rightarrow (1) \text{ is true}$$

$\boxed{\text{Case 2}}$ $x + y \geq xy + 1$ and we have : LHS of (1) – RHS of (1) =

$$2xy(x^2+y^2) + 2(x^2+y^2) - 3x^2y^2 - xy(x+y) - 3xy + (x+y-xy-1)$$

$$\geq 2xy(x^2+y^2) + 2(x^2+y^2) - 3x^2y^2 - xy(x+y) - 3xy$$

$$\overset{A-G}{\geq} 2xy(x^2+y^2) + xy - 3x^2y^2 - xy(x+y) \overset{A-G}{\geq}$$

$$xy\left(2(x^2+y^2) + 1 - \frac{3}{4}(x+y)^2 - (x+y)\right)$$

$$\geq xy\left((x+y)^2 + 1 - \frac{3}{4}(x+y)^2 - (x+y)\right) = \frac{xy}{4}.\left((x+y)^2 - 4(x+y) + 4\right)$$

$$= \frac{xy}{4}.(x+y-2)^2 \geq 0 \Rightarrow (1) \text{ is true} \therefore \text{combining both cases,}$$

$$(1) \text{ is true } \forall\, x,y > 0 \,|\, xy \geq 1 \therefore \frac{x}{y+1} + \frac{y}{x+1} + \frac{1}{xy+1} \geq \frac{3}{2}$$

$$\forall\, x,y > 0 \,|\, xy \geq 1, \text{''}=\text{''} \text{ iff } x = y = 1 \text{ (QED)}$$

1432. **If $a,b,c > 0$ and $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2$, then prove that :**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4(a+b+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+a} \overset{a>0}{<} 1 \therefore \text{we can set} : \frac{1}{1+a} = 1 - x \;(x > 0 \text{ and } x < 1)$$

$$\therefore a + 1 = \frac{1}{1-x} \Rightarrow a = \frac{1}{1-x} - 1 = \frac{x}{1-x} \to (1)$$

$$\text{Similarly, we set} : \frac{1}{1+b} = 1 - y \text{ and } \frac{1}{1+c} = 1 - z \therefore 2 = \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}$$

$$= 1 - x + 1 - y + 1 - z \Rightarrow x + y + z = 1 \to (\text{i}) \therefore (1) \text{ and (i)} \Rightarrow a = \frac{x}{y+z} \text{ and}$$

$$\text{analogously, } b = \frac{y}{z+x} \text{ and } c = \frac{z}{x+y} \text{ and hence} : \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 4(a+b+c)$$

$$\text{transforms into} : \sum_{\text{cyc}} \frac{y+z}{x} \geq 4\sum_{\text{cyc}} \frac{x}{y+z} = 4\sum_{\text{cyc}} \frac{x+y+z-(y+z)}{y+z}$$

$$= 4\left(\sum_{\text{cyc}} x\right)\sum_{\text{cyc}} \frac{1}{y+z} - 12 \Leftrightarrow \sum_{\text{cyc}}\left(\frac{y+z}{x} + 1\right) + 9 \geq 4\left(\sum_{\text{cyc}} x\right)\sum_{\text{cyc}} \frac{1}{y+z}$$

$$\Leftrightarrow \frac{1}{xyz}.\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + 9 \ge 4\left(\sum_{cyc} x\right).\frac{\sum_{cyc} x^2 + 3\sum_{cyc} xy}{\prod_{cyc}(y+z)}$$

$$\Leftrightarrow \left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + 9xyz\right).\prod_{cyc}(y+z) \ge 4xyz\left(\sum_{cyc} x\right)\left(\left(\sum_{cyc} x\right)^2 + \sum_{cyc} xy\right)$$

$$\overset{expanding\ and\ simplifying}{\Leftrightarrow} \sum_{cyc} x^4y^2 + \sum_{cyc} x^2y^4 + 2\sum_{cyc} x^3y^3 \ge 2xyz\sum_{cyc} x^3 + 6x^2y^2z^2$$

$$\to \text{true} \because \sum_{cyc} x^4y^2 + \sum_{cyc} x^2y^4 = \sum_{cyc} x^4y^2 + \sum_{cyc} x^4z^2 \overset{A-G}{\ge} \sum_{cyc} 2x^4yz$$

$$= 2xyz\sum_{cyc} x^3 \text{ and } 2\sum_{cyc} x^3y^3 \overset{A-G}{\ge} 6x^2y^2z^2 \therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 4(a+b+c)$$

$$\forall a,b,c > 0 \,\Big|\, \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2, \text{"="} \text{ iff } a = b = c = \frac{1}{2} \text{ (QED)}$$

1433. **If $a, b > 0$ and $a^3 + b^3 + 6ab \le 8$, then prove that:**

$$\frac{1}{a^2+b^2} + \frac{1}{ab} \ge \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$8 \ge a^3 + b^3 + 6ab \overset{A-G}{\ge} 2ab.\sqrt{ab} + 6ab \Rightarrow x^3 + 4x^2 - 4 \le 0 \ (x = \sqrt{ab})$$

$$\Rightarrow (x-1)(x+1)^2 \le 0 \Rightarrow x = \sqrt{ab} \le 1 \therefore ab \le 1 \to (1)$$

$$\text{Now, via Power} - \text{Mean Inequality, } a^3 + b^3 \ge 2\left(\frac{a^2+b^2}{2}\right)^{\frac{3}{2}}$$

$$\therefore 8 \ge 2\left(\frac{a^2+b^2}{2}\right)^{\frac{3}{2}} + 6ab \Rightarrow (4-ab)^{\frac{2}{3}} \overset{(\bullet)}{\ge} \frac{a^2+b^2}{2}$$

$$\text{Now, } \frac{1}{a^2+b^2} + \frac{1}{ab} \ge \frac{3}{2} \Leftrightarrow a^2 + b^2 + ab \ge \frac{3}{2}ab(a^2+b^2)$$

$$\Leftrightarrow ab \overset{(*)}{\ge} (3ab-2)\left(\frac{a^2+b^2}{2}\right) \text{ and if } ab \le \frac{2}{3}, \text{ then : RHS of } (*) \le 0 < ab$$

$$= \text{LHS of } (*) \Rightarrow (*) \text{ is true (strict inequality)}$$

$$\text{We now focus on : } ab > \frac{2}{3} \text{ and } (\bullet) \Rightarrow \text{in order to prove } (*), \text{it suffices to prove :}$$

$$ab \ge (3ab-2)(4-ab)^{\frac{2}{3}} \Leftrightarrow t^3 \ge (4-t)^2(3t-2)^3 \ (t = ab)$$

$$\Leftrightarrow 243t^5 - 1134t^4 + 2051t^3 - 1800t^2 + 768t - 128 \le 0$$

$$\Leftrightarrow (t-1)^2(243t^3 - 648t^2 + 512t - 128) \le 0$$

$$\Leftrightarrow 125(243t^3 - 648t^2 + 512t - 128) \le 0$$

$$\Leftrightarrow 30375t^3 - 81000t^2 + 64000t - 16000 \le 0$$

$$\Leftrightarrow 1215(t-1)(5t-3)^2 - 567(5t-3)^2 - 395\left(t-\frac{2}{3}\right) - \frac{676}{3} \le 0$$

$$\to \text{true (strict inequality)} \because t = ab \overset{\text{via }(1)}{\le} 1 \text{ and } t > \frac{2}{3} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{a^2+b^2} + \frac{1}{ab} \ge \frac{3}{2}\ \forall a,b > 0 \,\middle|\, a^3+b^3+6ab \le 8, ''='' \text{ iff } a = b = 1 \text{ (QED)}$$

1434. If $a,b,c > 0$ and $ab+bc+ca+abc = 4$, then prove that :

$$0 \le \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} \le \frac{2}{\sqrt{abc}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\big((2+b)(2+c)\big) - (2+a)(2+b)(2+c)$$

$$= 4 - (ab+bc+ca+abc) = 0 \therefore \sum_{cyc}\frac{1}{2+a} = 1 \to (\text{m})$$

$$\text{Now, } \frac{1}{2+a} \overset{a>0}{<} \frac{1}{2} \therefore \text{we can set : } \frac{1}{2+a} = \frac{1}{2} - x \left(x > 0 \text{ and } x < \frac{1}{2}\right)$$

$$\therefore a+2 = \frac{2}{1-2x} \Rightarrow a = \frac{2}{1-2x} - 2 = \frac{2x}{\frac{1}{2}-x} \to (1)$$

$$\text{Similarly, we set : } \frac{1}{2+b} = \frac{1}{2} - y \text{ and } \frac{1}{2+c} = \frac{1}{2} - z$$

$$\therefore 1 \overset{\text{via (m)}}{=} \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} = \frac{1}{2} - x + \frac{1}{2} - y + \frac{1}{2} - z \Rightarrow x+y+z = \frac{1}{2} \to (\text{i})$$

$$\therefore (1) \text{ and (i)} \Rightarrow a = \frac{2x}{y+z} \text{ and analogously, } b = \frac{2y}{z+x} \text{ and } c = \frac{2z}{x+y}$$

$$\text{and hence : } \sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc} \le \frac{2}{\sqrt{abc}} \text{ transforms into :}$$

$$\sum_{cyc}\sqrt{\frac{2x}{y+z}} \le \sqrt{\frac{8xyz}{\prod_{cyc}(y+z)}} + \frac{2.\sqrt{\prod_{cyc}(y+z)}}{\sqrt{8xyz}}$$

$$= \frac{8xyz + 2\prod_{cyc}(y+z)}{\sqrt{8xyz\prod_{cyc}(y+z)}} \Leftrightarrow \sum_{cyc}\sqrt{\frac{2x}{y+z}} \boxed{\overset{(*)}{\le}} \frac{4xyz + \prod_{cyc}(y+z)}{\sqrt{2xyz\prod_{cyc}(y+z)}}$$

$$\text{Now, } \sum_{cyc}\sqrt{\frac{2x}{y+z}} \overset{\text{CBS}}{\le} \sqrt{2\sum_{cyc}x}\,.\sqrt{\sum_{cyc}\frac{1}{y+z}} = \sqrt{2\sum_{cyc}x}\,.\sqrt{\frac{\left(\sum_{cyc}x\right)^2 + \sum_{cyc}xy}{\prod_{cyc}(y+z)}}$$

$$\overset{?}{\le}\frac{4xyz+\prod_{cyc}(y+z)}{\sqrt{2xyz\prod_{cyc}(y+z)}}$$

$$\Leftrightarrow 2.\sqrt{xyz\sum_{cyc}x}.\sqrt{\left(\sum_{cyc}x\right)^2+\sum_{cyc}xy}\ \boxed{\underset{(**)}{\overset{?}{\le}}}\ 4xyz+\prod_{cyc}(y+z)$$

Assigning $y+z=A, z+x=B, x+y=C \Rightarrow A+B-C=2z>0, B+C-A=2x>0$ and $C+A-B=2y>0 \Rightarrow A+B>C, B+C>A, C+A>B \Rightarrow A,B,C$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

$$\text{yielding } 2\sum_{cyc}x=\sum_{cyc}A=2s\Rightarrow\sum_{cyc}x\overset{(2)}{=}s\Rightarrow x=s-A, y=s-B, z=s-C$$

$$\Rightarrow xyz\overset{(3)}{=}r^2s \text{ and via such substitutions, } \sum_{cyc}xy=\sum_{cyc}(s-A)(s-B)=4Rr+r^2$$

$$\Rightarrow\sum_{cyc}xy\overset{(4)}{=}4Rr+r^2\ \therefore (2),(3),(4)\Rightarrow(**)$$

$$\Leftrightarrow 4r^2s^2(s^2+4Rr+r^2)\le(4r^2s+4Rrs)^2$$

$$\Leftrightarrow s^2+4Rr+r^2\le 4(R+r)^2=4R^2+8Rr+4r^2\Leftrightarrow s^2\le 4R^2+4Rr+3r^2$$

$$\to \text{true via Gerretsen}\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc}\le\frac{2}{\sqrt{abc}}$$

$$\text{Again, } 0\le\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc}\Leftrightarrow\sqrt{abc}.\sum_{cyc}\frac{1}{\sqrt{bc}}\ge\sqrt{abc}\Leftrightarrow\sum_{cyc}\frac{1}{\sqrt{bc}}\overset{(***)}{\ge}1 \text{ and}$$

$$\sum_{cyc}\frac{1}{\sqrt{bc}}=\sum_{cyc}\frac{1^{\frac{3}{2}}}{\sqrt{bc}}\overset{\text{Radon}}{\ge}\frac{3^{\frac{3}{2}}}{\sqrt{\sum_{cyc}ab}}\overset{ab+bc+ca+abc=4}{=}\frac{3^{\frac{3}{2}}}{\sqrt{4-abc}}\overset{-abc<0}{>}\frac{\sqrt{27}}{2}>1$$

$$\Rightarrow(***)\text{ is true (strict inequality)}\therefore\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc}>0\text{ and so,}$$

$$0<\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{abc}\le\frac{2}{\sqrt{abc}}\ \forall\, a,b,c>0\,|\,ab+bc+ca+abc=4,$$

$$''=''\text{ iff } a=b=c=1\text{ (QED)}$$

1435. If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\le\sqrt{2-\sqrt{ab}}+\sqrt{2-\sqrt{bc}}+\sqrt{2-\sqrt{ca}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\big((2+b)(2+c)\big)-(2+a)(2+b)(2+c)$$

$$= 4 - (ab + bc + ca + abc) = 0 \therefore \sum_{cyc} \frac{1}{2+a} = 1 \to (m)$$

$$\text{Now, } \frac{1}{2+a} \overset{a>0}{<} \frac{1}{2} \therefore \text{we can set}: \frac{1}{2+a} = \frac{1}{2} - x \left(x > 0 \text{ and } x < \frac{1}{2}\right)$$

$$\therefore a + 2 = \frac{2}{1-2x} \Rightarrow a = \frac{2}{1-2x} - 2 = \frac{2x}{\frac{1}{2} - x} \to (1)$$

$$\text{Similarly, we set}: \frac{1}{2+b} = \frac{1}{2} - y \text{ and } \frac{1}{2+c} = \frac{1}{2} - z$$

$$\therefore 1 \overset{\text{via (m)}}{=} \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} = \frac{1}{2} - x + \frac{1}{2} - y + \frac{1}{2} - z \Rightarrow x + y + z = \frac{1}{2} \to (i)$$

$$\therefore (1) \text{ and (i)} \Rightarrow a = \frac{2x}{y+z} \text{ and analogously, } b = \frac{2y}{z+x} \text{ and } c = \frac{2z}{x+y}$$

$$\text{and hence}: \sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \sqrt{2 - \sqrt{ab}} + \sqrt{2 - \sqrt{bc}} + \sqrt{2 - \sqrt{ca}}$$

$$\text{transforms into}: \sum_{cyc} \sqrt{\frac{2x}{y+z} \cdot \frac{2y}{z+x}} \overset{(*)}{\le} \sum_{cyc} \sqrt{2 - \sqrt{\frac{2x}{y+z} \cdot \frac{2y}{z+x}}}$$

Assigning $y + z = A, z + x = B, x + y = C \Rightarrow A + B - C = 2z > 0, B + C - A = 2x > 0$ and $C + A - B = 2y > 0 \Rightarrow A + B > C, B + C > A, C + A > B \Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} A = 2s \Rightarrow \sum_{cyc} x = s \Rightarrow x = s - A, y = s - B, z = s - C$$

Via such substitutions, $(*)$ becomes :

$$2\sum_{cyc} \sqrt{\frac{(s-A)(s-B)}{AB}} \le \sum_{cyc} \sqrt{2 - 2\sqrt{\frac{(s-A)(s-B)}{AB}}}$$

$$\Leftrightarrow 2\sum_{cyc} \sin\frac{\alpha}{2} \overset{(**)}{\le} \sum_{cyc} \sqrt{2 - 2\sin\frac{\alpha}{2}} \ (\alpha, \beta, \gamma \to \text{angles of triangle with sides } A, B, C)$$

$$\text{Let } f(x) = \sqrt{2 - 2\sin\frac{x}{2}} - 2\sin\frac{x}{2} + \left(x - \frac{\pi}{3}\right) \cdot \frac{3\sqrt{3}}{4} \ \forall x \in (0, \pi) \text{ and then}:$$

$$f'(x) = \frac{\sqrt{27}}{4} - \left(\cos\frac{x}{2}\right)\left(1 + \frac{1}{2\sqrt{2 - 2\sin\frac{x}{2}}}\right)$$

$$= \frac{\sqrt{27}}{4} - \left(\sqrt{1 - \sin^2\frac{x}{2}}\right)\left(1 + \frac{1}{2\sqrt{2 - 2\sin\frac{x}{2}}}\right) \to (2)$$

$$\text{Now, } \frac{\sqrt{27}}{4} \overset{?}{\leq} \left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)$$

$$\Leftrightarrow \frac{\sqrt{27}}{4} \overset{?}{\leq} \left(\sqrt{1-t^2}\right)\left(1+\frac{1}{2\sqrt{2-2t}}\right)\ \left(t=\sin\frac{x}{2}\right)$$

$$\Leftrightarrow \frac{27}{16(1-t^2)} \overset{?}{\leq} 1+\frac{1}{4(2-2t)}+\frac{1}{\sqrt{2-2t}}$$

$$\Leftrightarrow \frac{27}{16(1-t^2)}-1-\frac{1}{4(2-2t)} \overset{?}{\leq} \frac{1}{\sqrt{2-2t}} \Leftrightarrow \frac{9-2t+16t^2}{16(1-t^2)} \overset{?}{\leq} \frac{1}{\sqrt{2-2t}}$$

$$\Leftrightarrow \frac{(9-2t+16t^2)^2}{256(1-t^2)^2} \overset{?}{\leq} \frac{1}{2-2t}\ (\because 9>2>2t \Rightarrow 9-2t+16t^2>0)$$

$$\Leftrightarrow 128(1-t)(1+t)^2 \overset{?}{\geq} (9-2t+16t^2)^2$$

$$\Leftrightarrow 256t^4+64t^3+420t^2-164t-47 \overset{?}{\leq} 0$$

$$\Leftrightarrow (2t-1)(128t^3+96t^2+258t+47) \overset{?}{\leq} 0 \Leftrightarrow t \overset{?}{\leq} \frac{1}{2}$$

$$\therefore \boxed{\frac{\sqrt{27}}{4} \leq \text{ or } \geq \left(\sqrt{1-t^2}\right)\left(1+\frac{1}{2\sqrt{2-2t}}\right) \text{ according as } t \leq \frac{1}{2} \text{ or } t \geq \frac{1}{2}} \text{ and } t \geq \frac{1}{2}$$

$$\Rightarrow 1 > \sin\frac{x}{2} \geq \frac{1}{2} \Rightarrow \frac{\pi}{2} > \frac{x}{2} \geq \frac{\pi}{6} \Rightarrow \frac{\pi}{3} \leq x < \pi \text{ and } t=\sin\frac{x}{2} \leq \frac{1}{2} \Rightarrow 0 < x \leq \frac{\pi}{3}$$

$$\therefore \frac{\sqrt{27}}{4} \leq \left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)\ \forall x \in \left(0,\frac{\pi}{3}\right] \text{ and}$$

$$\frac{\sqrt{27}}{4} \geq \left(\sqrt{1-\sin^2\frac{x}{2}}\right)\left(1+\frac{1}{2\sqrt{2-2\sin\frac{x}{2}}}\right)\ \forall x \in \left[\frac{\pi}{3},\pi\right)$$

$$\overset{\text{via }(2)}{\Rightarrow} f'(x) \leq 0\ \forall x \in \left(0,\frac{\pi}{3}\right] \text{ and } f'(x) \geq 0\ \forall x \in \left[\frac{\pi}{3},\pi\right) \Rightarrow f(x) \text{ is } \downarrow \text{ on } \left(0,\frac{\pi}{3}\right]$$

$$\text{and } f(x) \text{ is } \uparrow \text{ on } \left[\frac{\pi}{3},\pi\right) \Rightarrow f(x) \geq f\left(\frac{\pi}{3}\right) = 0\ \forall x \in (0,\pi)$$

$$\therefore \boxed{\sqrt{2-2\sin\frac{x}{2}}-2\sin\frac{x}{2} \geq \left(\frac{\pi}{3}-x\right).\frac{3\sqrt{3}}{4}\ \forall x \in (0,\pi)}$$

$$\Rightarrow \sum_{\text{cyc}}\sqrt{2-2\sin\frac{\alpha}{2}}-2\sum_{\text{cyc}}\sin\frac{\alpha}{2} \geq \left(3.\frac{\pi}{3}-(\alpha+\beta+\gamma)\right).\frac{3\sqrt{3}}{4} = (\pi-\pi).\frac{3\sqrt{3}}{4} = 0$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \leq \sqrt{2-\sqrt{ab}}+\sqrt{2-\sqrt{bc}}+\sqrt{2-\sqrt{ca}}$$

$$\forall a,b,c>0 \mid ab+bc+ca+abc=4, ''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1436. If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1)\overset{a=\frac{1}{bc}}{\Leftrightarrow}\frac{1}{b^2c}+\frac{b}{c}+bc^2\ge\frac{3}{2}\left(\frac{1}{bc}+b+c-1\right)$$

$$\Leftrightarrow\frac{1+b^3+b^3c^3}{b^2c}\ge\frac{3}{2}\left(\frac{1+b^2c+bc^2-bc}{bc}\right)$$

$$\Leftrightarrow\boxed{2b^3c^3+2+2b^3+3b^2c\overset{(*)}{\ge}3b+3b^3c+3b^2c^2}$$

$\boxed{\boxed{\text{Case 1}}}$ $bc\ge1$ and $\boxed{\text{Case 1}a}$ $b\ge1$ and $\because b^3c^3+b^3+b^3\overset{A-G}{\ge}3b^3c$

$\therefore$ it remains to prove : $b^3c^3+2+3b^2c\ge3b+3b^2c^2$

$\Leftrightarrow 3b(bc-1)+(bc-1)(b^2c^2-2bc-2)\ge0$

$\Leftrightarrow (bc-1)(b^2c^2-2bc-2+3b)\ge0$; but $\because bc-1\ge0$ and $3b\ge3$

$\therefore (bc-1)(b^2c^2-2bc-2+3)=(bc-1)^3\ge0\Rightarrow(*)$ is true

$\boxed{\text{Case 1b}}$ $b\le1$ and $\because 2b^3c^3-3b^2c^2+1=(bc-1)^2(2bc+1)\ge0$ $\therefore$ it remains to prove : $2b^3-3b+1+3b^2c(1-b)\ge0$; but $\because bc\ge1$ and $1-b\ge0$

$\therefore 2b^3-3b+1+3b^2c(1-b)\ge2b^3-3b+1+3b(1-b)=2b^3-3b^2+1$

$=(b-1)^2(2b+1)\ge0\Rightarrow(*)$ is true

$\boxed{\boxed{\text{Case 2}}}$ $bc\le1$ and $\boxed{\text{Case 2}a}$ $b\le1\wedge c\ge1$ and $\because 2b^3c^3-3b^2c^2+1$ $=(bc-1)^2(2bc+1)\ge0$ $\therefore$ it remains to prove : $2b^3-3b+1+3b^2c(1-b)$ ≥0; but $\because c\ge1$ and $1-b\ge0$ $\therefore 2b^3-3b+1+3b^2c(1-b)$

$$\ge2b^3-3b+1+3b^2(1-b)=(1-b)^3\overset{b\le1}{\ge}0\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2b}}$ $b\ge1\wedge c\le1$ and $\because 2b^3c^3-3b^2c^2+1=(bc-1)^2(2bc+1)\ge0$

$\therefore$ it remains to prove : $2b^3-3b+1\ge3b^2c(b-1)$; but $\because bc\le1$ and $b-1\ge0$

$$\therefore 3b^2c(b-1)\le3b(b-1)\overset{?}{\le}2b^3-3b+1\Leftrightarrow2b^3-3b^2+1\overset{?}{\ge}0$$

$$\Leftrightarrow(b-1)^2(2b+1)\overset{?}{\ge}0\to\text{true}\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2c}}$ $b,c\le1$ and we have : $\frac{1}{b}-1,\frac{1}{c}-1\ge0$ and we let : $x=\frac{1}{b}-1$ and $y=\frac{1}{c}-1$ $(x,y\ge0)$ $\therefore b=\frac{1}{x+1}$ and $c=\frac{1}{y+1}$ $\therefore(*)$ transforms into :

$$\frac{2}{(x+1)^3(y+1)^3}+2+\frac{2}{(x+1)^3}+\frac{3}{(x+1)^2(y+1)}$$

$$-\frac{3}{x+1}-\frac{3}{(x+1)^3(y+1)}-\frac{3}{(x+1)^2(y+1)^2}\ge0\overset{\text{simplifying}}{\Leftrightarrow}$$

$2x^3y^3+6x^3y^2+3x^2y^3+6x^3y+9x^2y^2+2x^3+9x^2y+3xy^2+y^3+3x^2+3xy$

$+3y^2 \geq 0 \to \text{true} \because x, y \geq 0 \Rightarrow (*)$ is true $\therefore$ combining *all* cases, $(*)\ \forall\ b, c > 0$

$$\therefore \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{3}{2}(a + b + c - 1)\ \forall\ a, b, c > 0 \,\middle|\, abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1437. If $0 < a \leq 1, 0 < b \leq 1, 0 < c \leq 1$, *then prove that* :

$$\left(1 + \frac{1}{abc}\right)(a + b + c) \geq 3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$0 < a \leq 1 \Rightarrow \frac{1}{a} \geq 1 \text{ and we assign : } \frac{1}{a} - 1 = x\ (x \geq 0) \therefore \frac{1}{a} = x + 1$$

$$\Rightarrow a = \frac{1}{x + 1} \text{ and similarly, we assign : } \frac{1}{b} - 1 = y\ (y \geq 0) \text{ and } \frac{1}{c} - 1 = z\ (z \geq 0)$$

$$\Rightarrow b = \frac{1}{y + 1} \text{ and } c = \frac{1}{z + 1} \text{ and via such transformation, } \left(1 + \frac{1}{abc}\right)(a + b + c) \geq$$

$$3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Leftrightarrow \left(1 + \prod_{\text{cyc}}(x + 1)\right)\left(\sum_{\text{cyc}} \frac{1}{x + 1}\right) \geq 3 + \sum_{\text{cyc}} x + 3$$

$$\Leftrightarrow \sum_{\text{cyc}} \frac{1}{x + 1} + \sum_{\text{cyc}}\big((y + 1)(z + 1)\big) \geq 6 + \sum_{\text{cyc}} x$$

$$\Leftrightarrow \sum_{\text{cyc}} \frac{1}{x + 1} + 2\sum_{\text{cyc}} x + 3 + \sum_{\text{cyc}} xy \geq 6 + \sum_{\text{cyc}} x \Leftrightarrow \sum_{\text{cyc}} \frac{1}{x + 1} + \sum_{\text{cyc}} x + \sum_{\text{cyc}} xy \overset{(*)}{\geq} 3$$

$$\text{Now, } \sum_{\text{cyc}} \frac{1}{x + 1} + \sum_{\text{cyc}} x + \sum_{\text{cyc}} xy \geq \sum_{\text{cyc}} x + \sum_{\text{cyc}} \frac{1}{x + 1} \left(\because x, y, z \geq 0 \Rightarrow \sum_{\text{cyc}} xy \geq 0\right)$$

$$\overset{\text{Bergstrom}}{\geq} \sum_{\text{cyc}} x + \frac{9}{\sum_{\text{cyc}} x + 3} = \frac{\left(\sum_{\text{cyc}} x\right)^2 + 3\sum_{\text{cyc}} x + 9}{\sum_{\text{cyc}} x + 3} = \frac{\left(\sum_{\text{cyc}} x\right)^2}{\sum_{\text{cyc}} x + 3} + 3 \geq 3$$

$$\Rightarrow (*) \text{ is true} \therefore \left(1 + \frac{1}{abc}\right)(a + b + c) \geq 3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\ \forall\ a, b, c \in (0, 1],$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1438. If $a, b \in \mathbb{R}$ and $a^7b^7(a^6+b^6) \geq 2$, then prove that :

$$a^2+b^2 \geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+b^2)^{10} \overset{?}{\geq} 512a^7b^7(a^6+b^6)$$

$$\Leftrightarrow (a^2+b^2)^9 \boxed{\underbrace{\overset{?}{\geq}}_{(*)}} 512a^7b^7(a^4+b^4-a^2b^2)$$

$$\text{Now}, \forall\, a, b \in \mathbb{R}, (a-b)^2 \geq 0 \Rightarrow ab \leq \frac{a^2+b^2}{2}$$

$$\Rightarrow 512a^6b^6(a^4+b^4-a^2b^2).ab \leq 512a^6b^6(a^4+b^4-a^2b^2)\left(\frac{a^2+b^2}{2}\right)$$

$$(\because 512a^6b^6(a^4+b^4-a^2b^2) \geq 0) \overset{?}{\leq} (a^2+b^2)^9$$

$$\Leftrightarrow (a^2+b^2)^8 \overset{?}{\geq} 256a^6b^6(a^4+b^4-a^2b^2)\ (\because a^2+b^2 \geq 0)$$

$$\Leftrightarrow (x+y)^8 \boxed{\underbrace{\overset{?}{\geq}}_{(**)}} 256x^3y^3(x^2+y^2-xy)\ (x=a^2, y=b^2)$$

$$\text{Now}, (x+y)^4 - 8xy(x^2+y^2) = (x^2+y^2+2xy)^2 - 8xy(x^2+y^2)$$

$$= (x^2+y^2)^2 - 4xy(x^2+y^2) + 4x^2y^2 = (x^2+y^2-2xy)^2 = (x-y)^4 \geq 0$$

$$\therefore (x+y)^4 \geq 8xy(x^2+y^2) \Rightarrow (x+y)^8 \geq 64x^2y^2(x^2+y^2)^2$$

$$(\because x=a^2, y=b^2 \Rightarrow x, y \geq 0 \Rightarrow xy \geq 0) \overset{?}{\geq} 256x^3y^3(x^2+y^2-xy)$$

$$\Leftrightarrow (x^2+y^2)^2 \overset{?}{\geq} 4xy(x^2+y^2-xy)$$

$$\Leftrightarrow (x^2+y^2)^2 - 4xy(x^2+y^2) + 4x^2y^2 \overset{?}{\geq} 0 \Leftrightarrow (x^2+y^2-2xy)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-y)^4 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore (a^2+b^2)^{10} \geq 512a^7b^7(a^6+b^6)$$

$$\geq 1024 \Rightarrow a^2+b^2 \geq 2\ \forall\, a, b \in \mathbb{R} \,\middle|\, a^7b^7(a^6+b^6) \geq 2,$$

$$'' ='' \text{ iff } (a=b=1) \text{ or } (a=b=-1)\ (\text{QED})$$

1439. If $x, y, z \in \mathbb{R}$ and $xyz = 8$, then prove that :

$$\frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{2a}{b}, y=\frac{2b}{c}, z=\frac{2c}{a} \text{ and } \because x\neq 0, y\neq 0, z\neq 0 \therefore a\neq 0, b\neq 0, c\neq 0$$

$$\text{and via such transformation, } \frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4}\geq 1$$

$$\Leftrightarrow \frac{\frac{4a^2}{b^2}}{\frac{4a^2}{b^2}+\frac{4a}{b}+4}+\frac{\frac{4b^2}{c^2}}{\frac{4b^2}{c^2}+\frac{4b}{c}+4}+\frac{\frac{4c^2}{a^2}}{\frac{4c^2}{a^2}+\frac{4c}{a}+4}\geq 1 \Leftrightarrow \sum_{cyc}\frac{a^2}{a^2+ab+b^2}\geq 1 \to (1)$$

$$\text{Now, } a^2+ab+b^2=\frac{3}{4}(a+b)^2+\frac{1}{4}(a-b)^2 \text{ and so, if } (a^2+ab+b^2)=0, \text{ then :}$$

$$a=b \text{ and } a=-b \Rightarrow a=0, \text{ but } a\neq 0 \therefore a^2+ab+b^2\neq 0 \Rightarrow a^2+ab+b^2>0$$

$$\text{and analogously, } (b^2+bc+c^2),(c^2+ca+a^2)>0 \therefore \sum_{cyc}\frac{a^2}{a^2+ab+b^2}\geq 1$$

$$\Leftrightarrow \sum_{cyc}\left(a^2(b^2+bc+c^2)(c^2+ca+a^2)\right)\geq \prod_{cyc}(a^2+ab+b^2)$$

$$\Leftrightarrow a^4b^2+b^4c^2+c^4a^2\geq abc\sum_{cyc}ab^2 \to \text{true} \because \forall\, a,b,c\in\mathbb{R}, a^4b^2+b^4c^2+c^4a^2$$

$$\geq a^2b.b^2c+b^2c.c^2a+c^2a.a^2b=abc\sum_{cyc}ab^2 \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{x^2}{x^2+2x+4}+\frac{y^2}{y^2+2y+4}+\frac{z^2}{z^2+2z+4}\geq 1$$

$$\forall\, x,y,z\in\mathbb{R}\,\big|\, xyz=8, \text{''}=\text{''} \text{ iff } x=y=z=2 \text{ (QED)}$$

1440. If $a, b, c > 0, a+b+c=3$ then:

$$\frac{a^3}{9-a^2}+\frac{b^3}{9-b^2}+\frac{c^3}{9-c^2}\geq \frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

Lemma : For $x > 0$ ***we have:***

$$\frac{x^3}{9-x^2} \ge \frac{13x-9}{32}$$

Proof:

$$\frac{x^3}{9-x^2} \ge \frac{13x-9}{32} \text{ or}$$

$$45x^3 - 9x^2 - 117x + 81 \ge 0 \text{ or}$$

$$(x-1)^2(45x+81) \ge 0 \ (\text{true as } x > 0)$$

$$\sum \frac{a^3}{9-a^2} \ge \sum \frac{(13a-9)}{32} = \frac{(13\sum a - 27)}{32} = \frac{3}{8} \ (\text{as } a+b+c=3)$$

1441. If $a, b, c > 0, abc = 1$ then:

$$\sum \frac{1}{b+c} + \frac{1}{2} \ge \frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(a+b)(b+c)(c+a) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc \ (1)$$

$$a^2(b+c) + a(b+c) = (b+c)(a^2+a) \overset{AM-GM}{\ge} 2\sqrt{bc}\, 2\sqrt{a^3} = 4a\sqrt{abc} \overset{abc=1}{=} 4a$$

$$\sum \left(a^2(b+c) + a(b+c)\right) = \sum (a^2(b+c) + ab + ac) \ge 4\sum a \ (2)$$

$$\text{We need to show } \sum \frac{1}{b+c} + \frac{1}{2} \ge \frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)} \text{ or}$$

$$\frac{(a+b+c+1)^2}{(a+b)(b+c)(c+a)} - \sum \frac{1}{b+c} \le \frac{1}{2}$$

$$\frac{(a+b+c)^2 + 2(a+b+c) + 1 - (\sum (a+b)(a+c))}{\prod (a+b)} \le \frac{1}{2}$$

$$2\left[\left(\sum a^2\right) + 2\left(\sum ab\right) + 2(a+b+c) + 1 - \left(\sum (a^2+ab+bc+ca)\right)\right] \overset{(1)}{\le}$$

$$\leq a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc$$

$$\text{or } 4\left(\sum a\right)-2\left(\sum ab\right)+2\overset{abc=1}{\leq}\sum a^2(b+c)+2$$

$$\text{or } 4\left(\sum a\right)\leq\sum a^2(b+c)+2\left(\sum ab\right)\ \text{ or}$$

$$4\left(\sum a\right)\leq\sum\left(a^2(b+c)+ab+ac\right)\ \textit{True from } (2)$$

$$\textit{Equality holds for } a=b=c=1$$

1442. If $a,b,c>0,\ \lambda\geq 9\ ,a+b+c=3$ then:

$$\sum\frac{a^3}{\lambda-a^2}\geq\frac{3}{\lambda-1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a^3}{\lambda-a^2}\overset{Holder}{\geq}\frac{(a+b+c)^3}{\left(3\lambda-(a^2+b^2+c^2)\right)\cdot 3}=$$

$$=\frac{(a+b+c)^3}{\left(3\cdot\left(3\lambda-(a+b+c)^2+2(ab+bc+ca)\right)\right)}\geq$$

$$\geq\frac{(a+b+c)^3}{\left(3\cdot\left(3\lambda-(a+b+c)^2+\frac{2(a+b+c)^2}{3}\right)\right)}\overset{(a+b+c=3)}{=}\frac{(3)^3}{3(3\lambda-9+6)}=\frac{3}{\lambda-1}$$

$$\textit{Equality for } a=b=c=1$$

1443. If $a,b,c>0$ *with* $a+b+c=ab+bc+ca$ and $\lambda\geq 2$, then :

$$\sum_{cyc}\frac{a}{\lambda a+bc}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{\lambda a+bc}\leq\frac{3}{\lambda+1}\Leftrightarrow\frac{1}{\lambda}\cdot\sum_{cyc}\frac{\lambda a+bc-bc}{\lambda a+bc}\leq\frac{3}{\lambda+1}$$

$$\Leftrightarrow \frac{1}{\lambda}.\sum_{cyc}\frac{bc}{\lambda a+bc} \geq \frac{3}{\lambda}-\frac{3}{\lambda+1} \Leftrightarrow \sum_{cyc}\frac{bc}{\lambda a+bc} \overset{(*)}{\geq} \frac{3}{\lambda+1}$$

Now, $$\sum_{cyc}\frac{bc}{\lambda a+bc} = \sum_{cyc}\frac{b^2c^2}{\lambda abc+b^2c^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} ab\right)^2}{3\lambda abc+\sum_{cyc} a^2b^2} \overset{?}{\geq} \frac{3}{\lambda+1}$$

$$\Leftrightarrow (\lambda+1)\left(\sum_{cyc} a^2b^2\right) + 2(\lambda+1)abc\left(\sum_{cyc} a\right) \overset{?}{\geq} 9\lambda abc + 3\sum_{cyc} a^2b^2$$

$$\Leftrightarrow (\lambda-2)\left(\sum_{cyc} a^2b^2\right).\frac{\sum_{cyc} a}{\sum_{cyc} ab} + 2(\lambda+1)abc\left(\sum_{cyc} a\right).\frac{\sum_{cyc} a}{\sum_{cyc} ab}$$

$$\overset{?}{\geq} 9\lambda abc\ (\because a+b+c = ab+bc+ca)$$

$$\Leftrightarrow (\lambda-2)\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right) + 2\lambda abc\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right) + 2abc\left(\sum_{cyc} a\right)^2$$

$$\overset{?}{\geq} 3\lambda abc\left(\sum_{cyc} ab\right)$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2\right) - 3abc\left(\sum_{cyc} ab\right)\right) + 2\lambda abc\left(\left(\sum_{cyc} a\right)^2 - 3\sum_{cyc} ab\right)$$

$$\underset{(**)}{\overset{?}{\geq}} 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2b^2 - abc\left(\sum_{cyc} a\right)\right)$$

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ **and** $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ **form sides of a triangle with semiperimeter, circumradius and inradius** $= s, R, r$ (*say*);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = r^2s \rightarrow (2)$ **and such substitutions** $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \text{and} \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr+r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \rightarrow (4)$$

and via (1), (2), (3) **and** (4), **and** $\because \lambda \geq 2 \therefore$ **LHS of** $(**)$ **– RHS of** $(**) \geq$

$$2\left(sr^2((4R+r)^2-2s^2)-3r^2s(4Rr+r^2)\right)+4r^2s\left(s^2-3(4Rr+r^2)\right)$$

$$-2s(r^2((4R+r)^2-2s^2)-r^2s^2)\overset{?}{\geq}0 \overset{\text{expanding and simplifying}}{\Leftrightarrow}$$

$$6r^2s(s^2-12Rr-3r^2)\overset{?}{\geq}0 \to \text{true} \because s^2-12Rr-3r^2$$

$$=s^2-16Rr+5r^2+4r(R-2r)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\geq}0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{a}{\lambda a+bc}\leq\frac{3}{\lambda+1}\ \forall a,b,c>0\,\middle|\,a+b+c=ab+bc+ca \text{ and } \lambda\geq 2,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1444. If $x,y>0, \frac{1}{x}+\frac{1}{y}\leq 2$ then:

$$\sqrt{x^3+8y^2}+\sqrt{y^3+8x^2}\geq 6$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{x}+\frac{1}{y}\leq 2 \text{ or}, (x+y)\leq 2xy \text{ or}$$

$$2\sqrt{xy}\leq x+y\leq 2xy \ (AM-GM) \text{or } \sqrt{xy}\geq 1 \text{ or } xy\geq 1 \ (1)$$

$$\sqrt{x^3+8y^2}+\sqrt{y^3+8x^2}=\sqrt{\left(x^{\frac{3}{2}}\right)^2+\left(2\sqrt{2}y\right)^2}+\sqrt{\left(y^{\frac{3}{2}}\right)^2+\left(2\sqrt{2}x\right)^2}\overset{Minkowski}{\geq}$$

$$\geq\sqrt{\left(x^{\frac{3}{2}}+y^{\frac{3}{2}}\right)^2+\left(2\sqrt{2x}+2\sqrt{2y}\right)^2}\overset{AM-GM}{\geq}\sqrt{4(xy)^{\frac{3}{2}}+32(xy)}\overset{(1)}{\geq}\sqrt{4+32}=6$$

Equality for $a=b=1$

1445. If $a,b>0$ with $ab=1$ and $\lambda\geq 0$, then :

$$\frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}\geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{a(a^2+\lambda b^2-\lambda b^2)+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b(b^2+\lambda a^2-\lambda a^2)+\lambda a^3}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{1}{a}+\frac{\lambda b^3-\lambda ab^2}{a^2(a^2+\lambda b^2)}+\frac{1}{b}+\frac{\lambda a^3-\lambda a^2 b}{b^2(b^2+\lambda a^2)}=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda\left(\frac{a^2(a-b)}{b^2(b^2+\lambda a^2)}-\frac{b^2(a-b)}{a^2(a^2+\lambda b^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{a^4(a^2+\lambda b^2)-b^4(b^2+\lambda a^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{a^6-b^6+\lambda a^2b^2(a^2-b^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a-b)\left(\frac{(a^2-b^2)(a^4+b^4+a^2b^2+\lambda a^2b^2)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)=$$

$$=\frac{1}{a}+\frac{1}{b}+\lambda(a+b)(a-b)^2\left(\frac{a^4+b^4+a^2b^2(1+\lambda)}{a^2b^2(a^2+\lambda b^2)(b^2+\lambda a^2)}\right)\geq\frac{1}{a}+\frac{1}{b}$$

$$(\because a,b>0 \text{ and } \lambda\geq 0)\overset{A-G}{\geq}2.\sqrt{\frac{1}{ab}}\overset{ab=1}{=}2\ \therefore\ \frac{a^3+\lambda b^3}{a^2(a^2+\lambda b^2)}+\frac{b^3+\lambda a^3}{b^2(b^2+\lambda a^2)}\geq 2$$

$$\forall\, a,b>0\,\big|\,ab=1 \text{ and } \lambda\geq 0,''='' \text{ iff } a=b=c=1\ (\text{QED})$$

1446. If $a,b>0$ and $ab=1$, then prove that :

$$\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}=$$

$$=\frac{(a^3+2b^3)(a+2b)}{a^2(a^2+2b^2)(a+2b)}+\frac{(b^3+2a^3)(b+2a)}{b^2(b^2+2a^2)(b+2a)}\overset{\text{Reverse CBS}}{\geq}$$

$$\geq\frac{(a^2+2b^2)^2}{a^2(a^2+2b^2)(a+2b)}+\frac{(b^2+2a^2)^2}{b^2(b^2+2a^2)(b+2a)}=$$

$$=\frac{a^2+2b^2}{a^2(a+2b)}+\frac{b^2+2a^2}{b^2(b+2a)}\overset{ab=1}{=}\frac{1}{a+2b}+\frac{1}{b+2a}+\frac{2b^4}{a+2b}+\frac{2a^4}{b+2a}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{4}{3(a+b)}+\frac{2(a^2+b^2)^2}{3(a+b)}\geq\frac{4}{3(a+b)}+\frac{2.\frac{1}{4}.(a+b)^4}{3(a+b)}=$$

$$=\frac{t^3}{6}+\frac{4}{3t}\ (t=a+b)=\frac{t^4+8}{6t}\overset{?}{\geq}2$$

$$\Leftrightarrow t^4-12t+8\overset{?}{\geq}0\Leftrightarrow(t-2)(t^3+2t^2+4(t-2)+4)\overset{?}{\geq}0\to\text{true}\because t=a+b$$

$$\overset{A-G}{\geq}2\sqrt{ab}\overset{ab=1}{=}2\therefore\frac{a^3+2b^3}{a^2(a^2+2b^2)}+\frac{b^3+2a^3}{b^2(b^2+2a^2)}\geq 2,''='' \text{ iff } a=b=1\ (QED)$$

1447. If $a,b,c>0$, then prove that :

$$\frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}=\sum_{cyc}\frac{a^3}{a+3b}-\frac{1}{27}.\sum_{cyc}\frac{(27b^3+a^3)-a^3}{a+3b}=$$

$$=\frac{28}{27}.\sum_{cyc}\frac{a^4}{a^2+3ab}-\frac{1}{27}.\sum_{cyc}(a^2+9b^2-3ab)\overset{\text{Bergstrom}}{\geq}$$

$$\geq\frac{28}{27}.\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2+3\sum_{cyc}ab}-\frac{1}{27}.\left(10\sum_{cyc}a^2-3\sum_{cyc}ab\right)=$$

$$=\frac{28x^2-(x+3y)(10x-3y)}{27(x+3y)}\left(x=\sum_{cyc}a^2,y=\sum_{cyc}ab\right)=\frac{18x^2-27xy+9y^2}{27(x+3y)}$$

$$=\frac{9(2x^2-3xy+y^2)}{27(x+3y)}=\frac{9(x-y)(2(x-y)+y)}{27(x+3y)}\geq 0 \because x\geq y>0$$

$$as\sum_{cyc}a^2\geq\sum_{cyc}ab \therefore \frac{a^3-b^3}{a+3b}+\frac{b^3-c^3}{b+3c}+\frac{c^3-a^3}{c+3a}\geq 0,''='' \text{ iff } a=b=c \text{ (QED)}$$

1448. If $a,b,c>0$, then prove that :

$$\frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}\geq 3(a^2+b^2+c^2)-(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)=$$

$$=\frac{6b^3+14c^3-(6b^3+6bc^2-4b^2c+9b^2c+9c^3-6bc^2)}{2(2b+3c)}$$

$$\Rightarrow\frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)=\frac{5c(c^2-b^2)}{2(2b+3c)} \text{ and analogs}$$

$$\therefore \frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}\geq 3(a^2+b^2+c^2)-(ab+bc+ca)$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{3b^3+7c^3}{2b+3c}-\left(\frac{3(b^2+c^2)}{2}-bc\right)\right)\geq 0$$

$$\Leftrightarrow\frac{c(c^2-b^2)}{2b+3c}+\frac{a(a^2-c^2)}{2c+3a}+\frac{b(b^2-a^2)}{2a+3b}\geq 0$$

$$\overset{\text{clearing denominators and simplifying}}{\Leftrightarrow} 2\sum_{cyc}a^4b+3\sum_{cyc}ab^4\overset{(*)}{\geq}5abc\sum_{cyc}ab$$

$$\text{Now, }2\sum_{cyc}a^4b+3\sum_{cyc}ab^4=2abc\sum_{cyc}\frac{a^3}{c}+3abc\sum_{cyc}\frac{b^3}{c}\overset{\text{Holder}}{\geq}$$

$$2abc.\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}a}+3abc.\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}a}=\frac{5abc}{3}.\left(\sum_{cyc}a\right)^2\geq\frac{5abc}{3}.3\sum_{cyc}ab$$

$$=5abc\sum_{cyc}ab\Rightarrow(*)\text{ is true}\therefore\frac{3a^3+7b^3}{2a+3b}+\frac{3b^3+7c^3}{2b+3c}+\frac{3c^3+7a^3}{2c+3a}$$

$$\geq 3(a^2+b^2+c^2)-(ab+bc+ca)\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c \text{ (QED)}$$

1449. If $a,b,c>0, abc=1$ then:

$$\sum\frac{a(b^3+c^3)}{(a+b)(a+c)}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(1^3+1^3)(1^3+1^3)(b^3+c^3)\overset{HUYGENS}{\geq}(b+c)^3$$

$$(b^3+c^3)\geq\frac{(b+c)^3}{4}\ (1)$$

$$\prod(a+b)\overset{Cesaro}{\geq}8abc\ (2)$$

$$\sum\frac{a(b^3+c^3)}{(a+b)(a+c)}\overset{(1)}{\geq}\sum\frac{a(b+c)^3}{4(a+b)(a+c)}\overset{AM-GM}{\geq}$$

$$\geq\frac{3}{4}\sqrt[3]{abc(a+b)(b+c)(c+a)}\overset{(2)}{\geq}\frac{3}{4}\sqrt[3]{8a^2b^2c^2}=\frac{3}{2}\ (as\ abc=1)$$

$$Equality\ holds\ for\ a=b=c=1$$

1450.

$$\text{If } x,y,z>0 \text{ and } xyz=1, \text{then prove that}:$$

$$\frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}\leq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } xyz=1, \text{we can assign } x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a} \text{ and then}:$$

$$\frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}$$

$$=\frac{b}{\sqrt{4a^2+ab+4b^2}}+\frac{c}{\sqrt{4b^2+bc+4c^2}}+\frac{a}{\sqrt{4c^2+ca+4a^2}}$$

$$=\frac{1}{\sqrt{\prod_{cyc}(4a^2+ab+4b^2)}}.\sum_{cyc}\left(b.\sqrt{(4b^2+bc+4c^2)(4c^2+ca+4a^2)}\right)$$

$$\overset{CBS}{\leq}\frac{1}{\sqrt{\prod_{cyc}(4a^2+ab+4b^2)}}.\sqrt{\sum_{cyc}(4b^2c^2+cab^2+4a^2b^2)}.\sqrt{\sum_{cyc}(4b^2+bc+4c^2)}\overset{?}{\leq}1$$

$$\Leftrightarrow(4a^2+ab+4b^2)(4b^2+bc+4c^2)(4c^2+ca+4a^2)\overset{?}{\geq}$$

$$\left(8\sum_{cyc}a^2+\sum_{cyc}ab\right).\left(8\sum_{cyc}a^2b^2+abc\sum_{cyc}a\right)$$

$$\Leftrightarrow 64\sum_{cyc}(a^4b^2+a^2b^4)+16abc\sum_{cyc}a^3+16\sum_{cyc}a^3b^3+129a^2b^2c^2$$

$$+20abc\left(\sum_{cyc}a^2b+\sum_{cyc}ab^2\right)\overset{?}{\geq}$$

$$64\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)+8\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)+8abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)$$

$$+abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow 64\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-192a^2b^2c^2+16abc\sum_{cyc}a^3+16\sum_{cyc}a^3b^3$$

$$+129a^2b^2c^2+20abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-60a^2b^2c^2\overset{?}{\geq}$$

$$64\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)+8\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)+8abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)$$

$$+abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)$$

$$\Leftrightarrow abc\left(16\sum_{cyc}a^3+19\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-8\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)\right)-123a^2b^2c^2$$

$$+16\sum_{cyc}a^3b^3-8\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2b^2\right)\overset{?}{\underset{(*)}{\geq}}0$$

Assigning $b+c=X, c+a=Y, a+b=Z\Rightarrow X+Y-Z=2c>0, Y+Z-X=2a>0$ and $Z+X-Y=2b>0\Rightarrow X+Y>Z, Y+Z>X, Z+x>Y\Rightarrow X,Y,Z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

so $2\sum_{cyc}a=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-X, b=s-Y, c=s-Z$

$\therefore abc=r^2s\to(2)$ and such substitutions $\Rightarrow\sum_{cyc}ab=\sum_{cyc}(s-x)(s-y)$

$\Rightarrow\sum_{cyc}ab=4Rr+r^2\to(3)$, and $\sum_{cyc}a^2=\left(\sum_{cyc}a\right)^2-2\sum_{cyc}ab\overset{\text{via (1) and (3)}}{=}$

$s^2-2(4Rr+r^2)\Rightarrow\sum_{cyc}a^2=s^2-8Rr-2r^2\to(4)$ and also,

$\sum_{cyc}a^2b^2=\left(\sum_{cyc}ab\right)^2-2abc\left(\sum_{cyc}a\right)\overset{\text{via (1),(2) and (3)}}{=}(4Rr+r^2)^2-2r^2s.s$

$\Rightarrow\sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2)\to(5)$ and moreover, $\sum_{cyc}a^3=$

$\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)\overset{\text{via (1)}}{=}s^3-3.4Rrs\Rightarrow\sum_{cyc}a^3=s^3-12Rrs$

$\to(6)$ and finally, $\sum_{cyc}a^3b^3=\left(\sum_{cyc}ab\right)^3-3(ab+bc)(bc+ca)(ca+ab)$

$\overset{\text{via (2) and (3)}}{=}(4Rr+r^2)^3-3r^2s.4Rrs\Rightarrow\sum_{cyc}a^3b^3=(4Rr+r^2)^3-12Rr^3s^2\to(7)$

and via (1), (2), (3), (4), (5), (6) and (7), $(*)$ transforms into :

$$r^2s\left(16(s^3-12Rrs)+19s(4Rr+r^2)-8s(s^2-8Rr-2r^2)\right)-123r^4s^2$$
$$+16((4Rr+r^2)^3-12Rr^3s^2)-8(4Rr+r^2).r^2((4R+r)^2-2s^2)\geq 0$$

$\Leftrightarrow 2s^4-(45Rr+18r^2)s^2+2r(4R+r)^3\ \boxed{\geq}\ 0\to$ true $\because 2s^4-(45Rr+18r^2)s^2+2r(4R+r)^3$ is a quadratic polynomial in s^2 with discriminant $=(45Rr+18r^2)^2-16r(4R+r)^3=-r(1024R^3-1257R^2r-1428Rr^2-308r^3)$

$=-r(R-2r)(1024R^2+791Rr+154r^2)\overset{\text{Euler}}{\leq}0\Rightarrow(*)$ is true

$$\therefore\frac{1}{\sqrt{4x^2+x+4}}+\frac{1}{\sqrt{4y^2+y+4}}+\frac{1}{\sqrt{4z^2+z+4}}\leq 1$$

$\forall\, x, y, z > 0 \,\big|\, xyz = 1, " = " \text{ iff } x = y = z = 1$ (QED)

1451. *If* $x, y, z > 0$ *and* $xyz = 27$*, then prove that:*

$$\frac{1}{\sqrt{x^2+21x+9}}+\frac{1}{\sqrt{y^2+21y+9}}+\frac{1}{\sqrt{z^2+21z+9}} \geq \frac{1}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Morocco

Firstly, for all $x > 0$, we have

$$\sqrt{x^2+21x+9} \leq x+\sqrt{3x}+3 \overset{Squaring}{\Leftrightarrow} 0 \leq 2\sqrt{3x^3}-12x+6\sqrt{3x}$$
$$= 2\sqrt{3x}.\left(\sqrt{x}-\sqrt{3}\right)^2,$$

which is true.

Now, since $xyz = 27$, then there exist $a, b, c > 0$ such that $x = \frac{3bc}{a^2}, y = \frac{3ca}{b^2}, z = \frac{3ab}{c^2}$, and

$$\sum_{cyc} \frac{1}{\sqrt{x^2+21x+9}} \geq \sum_{cyc} \frac{1}{x+\sqrt{3x}+3}$$
$$= \sum_{cyc} \frac{a^2}{3\left(bc+a\sqrt{bc}+a^2\right)} \overset{AM-GM}{\geq} \frac{1}{3}\sum_{cyc} \frac{a^2}{bc+\frac{ab+ac}{2}+a^2}$$
$$\overset{CBS}{\geq} \frac{(a+b+c)^2}{3\sum_{cyc}\left(bc+\frac{ab+ac}{2}+a^2\right)} = \frac{1}{3}.$$

Equality holds iff $x = y = z = 3$.

1452. If $a, b, c > 0$ then:

$$\sum \frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2} \geq 3\sqrt{2}abc$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$WLOG\ a \geq b \geq c\ then\ a^2+b^2 \geq a^2+c^2 \geq b^2+c^2$$

$$and\ \sqrt{a^2+b^2} \geq \sqrt{a^2+c^2} \geq \sqrt{b^2+c^2}$$

$$\frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2} \overset{Chebyshev}{\geq} \frac{\frac{1}{2}(b^4+a^4)\left(\sqrt{b^2+c^2}+\sqrt{a^2+c^2}\right)}{a^2+b^2} \overset{CBS}{\geq}$$

$$\geq \frac{\frac{1}{2}\frac{(a^2+b^2)^2}{2}\left(\sqrt{\frac{(b+c)^2}{2}}+\sqrt{\frac{(a+c)^2}{2}}\right)}{a^2+b^2} =$$

$$= \frac{1}{4\sqrt{2}}(a^2+b^2)(b+c+c+a) \overset{AM-GM}{\geq} \frac{1}{4\sqrt{2}}(2ab)\left(4(abc^2)^{\frac{1}{4}}\right) = \sqrt{2}(a^5b^5c^2)^{\frac{1}{4}}\ (1)$$

$$\sum \frac{b^4\sqrt{b^2+c^2}+a^4\sqrt{c^2+a^2}}{a^2+b^2} \overset{(1)}{\geq} \sqrt{2}\sum (a^5b^5c^2)^{\frac{1}{4}} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt{2}\,(a^{12}b^{12}c^{12})^{\frac{1}{12}} = 3\sqrt{2}abc$$

$$Equality\ holds\ for\ a=b=c$$

1453. $If\ x,y,z>0\ and\ x^3+y^2+z=2\sqrt{3}+1\ then\ prove\ that:$

$$\frac{1}{x}+\frac{1}{y^2}+\frac{1}{z^3} \geq \frac{4\sqrt{3}+9}{9}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Morocco

By AM – GM inequality, we have :

$$\frac{1}{x}+\frac{1}{x}+\frac{1}{x}+x^3 \geq 4 \Rightarrow \frac{1}{x} \geq \frac{4}{3}-\frac{x^3}{3}\ (1)$$

$$\frac{1}{y^2}+\frac{y^2}{3} \geq \frac{2}{\sqrt{3}} \Rightarrow \frac{1}{y^2} \geq \frac{2\sqrt{3}}{3}-\frac{y^2}{3}\ (2)$$

$$\frac{1}{z^3}+\frac{z}{9}+\frac{z}{9}+\frac{z}{9}\geq\frac{4}{\sqrt[4]{9^3}}\ \Rightarrow\ \frac{1}{z^3}\geq\frac{4\sqrt{3}}{9}-\frac{z}{3}\ (3)$$

Adding (1), (2) and (3), we get

$$\frac{1}{x}+\frac{1}{y^2}+\frac{1}{z^3}\geq\frac{10\sqrt{3}+12}{9}-\frac{1}{3}(x^3+y^2+z)=\frac{10\sqrt{3}+12}{9}-\frac{2\sqrt{3}+1}{3}=\frac{4\sqrt{3}+9}{9}.$$

Equality holds iff $x=1, y=\sqrt[4]{3}, z=\sqrt{3}$.

1454. If $0<x<y<z$, then prove that :

$$\frac{x^3z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2z}\geq 12$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x^3z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2z}\geq 12$$

$$\Leftrightarrow\frac{\left(\frac{x}{z}\right)^3}{\left(\frac{y}{z}\right)^2\left(\frac{x}{z}+\left(\frac{y}{z}\right)^2\right)}+\frac{\left(\frac{y}{z}\right)^4}{\frac{x}{z}+\left(\frac{y}{z}\right)^2}+\frac{1+15\left(\frac{x}{z}\right)^3}{\left(\frac{x}{z}\right)^2}\geq 12$$

$$\Leftrightarrow\frac{a^3}{b^2(a+b^2)}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12\ \left(\frac{x}{z}=a,\frac{y}{z}=b\right)$$

$$\Leftrightarrow\frac{(a^3+b^6)-b^6}{b^2(a+b^2)}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12\Leftrightarrow$$

$$\frac{a^2+b^4-ab^2}{b^2}-\frac{b^4}{a+b^2}+\frac{b^4}{a+b^2}+\frac{1}{a^2}+15a\geq 12\Leftrightarrow\frac{a^2}{b^2}+b^2+\frac{1}{a^2}+14a\overset{(*)}{\geq}12$$

$$\text{Now, }\frac{a^2}{b^2}+b^2+\frac{1}{a^2}+14a\overset{A-G}{\geq}2.\sqrt{\frac{a^2}{b^2}.b^2}+\frac{1}{a^2}+14a=\frac{1}{a^2}+16a\overset{?}{\geq}12$$

$$\Leftrightarrow 16a^3-12a^2+1\overset{?}{\geq}0\Leftrightarrow(4a+1)(2a-1)^2\overset{?}{\geq}0\to\text{true}\because a=\frac{x}{z}>0$$

$$\Rightarrow(*)\text{ is true}\therefore\frac{x^3z}{y^2(xz+y^2)}+\frac{y^4}{z^2(xz+y^2)}+\frac{z^3+15x^3}{x^2z}\geq 12\text{ for }0<x<y<z,$$

$$''='' \text{ iff } a=b^2\wedge a=\frac{1}{2}\therefore ''='' \text{ iff } a=\frac{1}{2}, b=\frac{1}{\sqrt{2}}$$

$$\text{and so, }''='' \text{ iff } z=k, x=\frac{k}{2}, y=\frac{k}{\sqrt{2}}\ \forall\, k>0\ (QED)$$

1455. If $a,b,c>0,\ \lambda\geq 0$ then:

$$\sum\frac{((a+\lambda b)(c+\lambda a))}{b+\lambda c}\ge(\lambda+1)(a+b+c)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\text{Let } (a+\lambda b)=x,(b+\lambda c)=y,(c+\lambda a)=z \text{ then:}$$

$$\sum\frac{((a+\lambda b)(c+\lambda a))}{b+\lambda c}=\sum\frac{xz}{y}=\frac{x^2z^2+x^2y^2+y^2z^2}{xyz}\ge$$

$$\ge\frac{xy\cdot yz+yz\cdot zx+zx\cdot xy}{xyz}=\frac{xyz(x+y+z)}{xyz}=x+y+z=$$

$$=(a+\lambda b)+(b+\lambda c)+(c+\lambda a)=(\lambda+1)(a+b+c)$$

$$\text{Equality holds for } a=b=c=1$$

1456. If $a,b,c>0$ with $a^3+b^3+c^3=3$ and $\lambda,n\in\mathbb{N}$ with $\lambda\le 3n$, then :

$$\lambda(a+b+c)+n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge 3(\lambda+n)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\lambda(a+b+c)+n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\overset{\text{Radon}}{\ge}\lambda\left(\sum_{\text{cyc}}a\right)+n.\frac{81}{\left(\sum_{\text{cyc}}a\right)^3}\overset{a^3+b^3+c^3=3}{=}$$

$$\lambda\left(\sum_{\text{cyc}}a\right)+9n.\frac{\sum_{\text{cyc}}a^3}{\left(\sum_{\text{cyc}}a\right)^3}.\sqrt[3]{9\sum_{\text{cyc}}a^3}\overset{?}{\ge}3(\lambda+n)\overset{a^3+b^3+c^3=3}{=}3(\lambda+n).\sqrt[3]{\frac{\sum_{\text{cyc}}a^3}{3}}$$

$$\Leftrightarrow 3n\left(\frac{3\sum_{\text{cyc}}a^3}{\left(\sum_{\text{cyc}}a\right)^3}.\sqrt[3]{9\sum_{\text{cyc}}a^3}-\frac{1}{3}.\sqrt[3]{9\sum_{\text{cyc}}a^3}\right)\overset{?}{\ge}\lambda\left(3.\sqrt[3]{\frac{\sum_{\text{cyc}}a^3}{3}}-\sum_{\text{cyc}}a\right)$$

$$\Leftrightarrow 3n.\left(9\sum_{\text{cyc}}a^3-\left(\sum_{\text{cyc}}a\right)^3\right).\sqrt[3]{9\sum_{\text{cyc}}a^3}\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 3\lambda\left(\sum_{\text{cyc}}a\right)^3.\left(\sqrt[3]{9\sum_{\text{cyc}}a^3}-\sum_{\text{cyc}}a\right)$$

$$\text{Now}, \because 9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3 \overset{\text{Holder}}{\geq} 0 \text{ and } 3n \geq \lambda$$

$$\therefore 3n.\left(9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3\right).\sqrt[3]{9\sum_{cyc} a^3} \geq \lambda.\left(9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3\right).\sqrt[3]{9\sum_{cyc} a^3}$$

$$\overset{?}{\geq} 3\lambda\left(\sum_{cyc} a\right)^3.\left(\sqrt[3]{9\sum_{cyc} a^3} - \sum_{cyc} a\right)$$

$$\Leftrightarrow \left(9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3\right).\sqrt[3]{9\sum_{cyc} a^3} \boxed{\overset{?}{\geq}} 3\left(\sum_{cyc} a\right)^3.\left(\sqrt[3]{9\sum_{cyc} a^3} - \sum_{cyc} a\right)$$

$$(\because \lambda \geq 0 \; as \; \lambda, n \in \mathbb{N})$$

$$= 3\left(\sum_{cyc} a\right)^3.\frac{9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3}{\sqrt[3]{81\left(\sum_{cyc} a^3\right)^2} + \left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} a\right).\sqrt[3]{9\sum_{cyc} a^3}}$$

$$\Leftrightarrow \sqrt[3]{9\sum_{cyc} a^3}.\left(\sqrt[3]{81\left(\sum_{cyc} a^3\right)^2} + \left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} a\right).\sqrt[3]{9\sum_{cyc} a^3}\right)$$

$$\overset{?}{\geq} 3\left(\sum_{cyc} a\right)^3 \left(\because 9\sum_{cyc} a^3 - \left(\sum_{cyc} a\right)^3 \geq 0\right)$$

$$\Leftrightarrow 9\sum_{cyc} a^3 + \sqrt[3]{9\sum_{cyc} a^3}.\left(\sum_{cyc} a\right)^2 + \sqrt[3]{81\left(\sum_{cyc} a^3\right)^2}.\left(\sum_{cyc} a\right) \boxed{\underset{(**)}{\overset{?}{\geq}}} 3\left(\sum_{cyc} a\right)^3$$

$$\because 9\sum_{cyc} a^3 \geq \left(\sum_{cyc} a\right)^3 \therefore \text{LHS of } (**) \geq$$

$$\left(\sum_{cyc} a\right)^3 + \left(\sum_{cyc} a\right).\left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} a\right)^2.\left(\sum_{cyc} a\right) = 3\left(\sum_{cyc} a\right)^3$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \lambda(a+b+c) + n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) \geq 3(\lambda+n)$$

$$\forall\, a,b,c > 0 \,\middle|\, a^3+b^3+c^3 = 3 \text{ with } \lambda, n \in \mathbb{N} \,\middle|\, \lambda \leq 3n, \text{"} = \text{"} \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality can be rewritten as

$$n\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}-3\right)\geq \lambda[3-(a+b+c)] \quad (1)$$

Since $a+b+c\leq \sqrt[3]{3^2(a^3+b^3+c^3)}=3$ **and** $\lambda\leq 3n$, **then**

$$RHS_{(1)}\leq 3n[3-(a+b+c)]\overset{?}{\leq} LHS_{(1)}\Leftrightarrow n\left(3(a+b+c)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}-12\right)\geq 0,$$

which is true by AM − GM inequality,

$$3(a+b+c)+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq 12\sqrt[12]{a^3.b^3.c^3.\frac{1}{a^3}.\frac{1}{b^3}.\frac{1}{c^3}}=12.$$

Equality holds iff $a=b=c=1$.

1457. If $a,b,c>0$ with $a+b+c=1$ and $\lambda>0$, then :

$$\sqrt{\lambda a+1}+\sqrt{\lambda b+1}+\sqrt{\lambda c+1}\geq 2+\sqrt{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\lambda a+1}+\sqrt{\lambda b+1}+\sqrt{\lambda c+1}\geq 2+\sqrt{\lambda+1}$$

$$\Leftrightarrow \sum_{cyc}(\lambda a+1)+2\sum_{cyc}\sqrt{(\lambda a+1)(\lambda b+1)}\geq 4+\lambda+1+4.\sqrt{\lambda+1}$$

$$\overset{a+b+c=1}{\Leftrightarrow} \lambda+3+2\sum_{cyc}\sqrt{(\lambda a+1)(\lambda b+1)}\geq 5+\lambda+4.\sqrt{\lambda+1}$$

$$\Leftrightarrow \sum_{cyc}\sqrt{(\lambda a+1)(\lambda b+1)}\overset{(*)}{\geq} 1+2$$

$\sqrt{\lambda+1}$**Now,** $\sum_{cyc}\sqrt{(\lambda a+1)(\lambda b+1)}\overset{a+b+c=1}{=}\sum_{cyc}\sqrt{(\lambda a+a+b+c)(\lambda b+a+b+c)}$

$$=\sum_{cyc}\sqrt{((\lambda+1)a+b+c)(a+(\lambda+1)b+c)}\overset{\text{Reverse CBS}}{\geq}\sum_{cyc}\sqrt{\left(\sqrt{\lambda+1}.a+\sqrt{\lambda+1}.b+c\right)^2}$$

$$=\sqrt{\lambda+1}.\left(\sum_{cyc}a\right)+\sqrt{\lambda+1}.\left(\sum_{cyc}a\right)+\sum_{cyc}a\overset{a+b+c=1}{=}2\sqrt{\lambda+1}+1\Rightarrow(*)\text{ is true}$$

$$\therefore\sqrt{\lambda a+1}+\sqrt{\lambda b+1}+\sqrt{\lambda c+1}\geq 2+\sqrt{\lambda+1}\ \forall\, a,b,c>0\,\big|\,a+b+c=1$$

$$\text{and }\lambda>0,''='' \text{ iff } (a=b=0,c=1)\text{ or }(b=c=0,a=1)$$

$$\text{or }(c=a=0,b=1)\ (QED)$$

1458. If $a,b,c>0,\lambda\geq 0,n>0$ then:

$$\sum\frac{a^3}{b^2(\lambda a+nb)}\geq\frac{3}{\lambda+n}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Vasc\ inequality:$$

$$\forall\, x,y,z>0,\frac{(x^2+y^2+z^2)^2}{3}\geq(x^3y+y^3z+z^3x)\ \ (1)\ and$$

$$3\sum a^2b^2\leq\left(\sum a^2\right)^2\ \ (2)$$

$$\sum\frac{a^3}{b^2(\lambda a+nb)}=\sum\frac{a^4}{b^2a(\lambda a+nb)}=\sum\frac{(a^2)^2}{\lambda a^2b^2+nab^3}\overset{Bergstrom}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\lambda(a^2b^2+b^2c^2+c^2a^2)+n(ab^3+bc^3+ca^3)}\overset{(1)\&(2)}{\geq}$$

$$\geq\frac{(a^2+b^2+c^2)^2}{\frac{\lambda(c^2+b^2+a^2)^2}{3}+\frac{n(a^2+b^2+c^2)^2}{3}}=\frac{3}{\lambda+n}$$

$$Equality\ holds\ for\ a=b=c=1$$

1459. If $a,b>0\ and\ a+b=2$ then:

$$2\left(\sqrt{a+3}+\sqrt{b+3}\right)+3\left(\sqrt{a+8}+\sqrt{b+8}\right)\leq 26$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$2\left(\sqrt{a+3}+\sqrt{b+3}\right)+3\left(\sqrt{a+8}+\sqrt{b+8}\right)\overset{CBS}{\leq}$$

$$\leq 2\sqrt{2(a+3+b+3)}+3\sqrt{2(a+8+b+8)}=$$

$$=2\sqrt{2(a+b)+12}+3\sqrt{2(a+b)+32}\overset{a+b=2}{=}2\sqrt{16}+3\sqrt{36}=26$$

$$Equality\ holds\ for\ a=b=1$$

1460. If $a,b,c>0, abc=1$ **then:**

$$\sum\sqrt{1+15a^2}\geq\frac{7}{2}(a+b+c)+\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$1+15a^2=1+(\underbrace{a^2+a^2+\cdots+a^2}_{15\ times})\geq$$

$$\overset{CBS}{\geq}\frac{\left(\left(1+(\underbrace{a+a+\cdots+a}_{15\ times})\right)\right)^2}{16}=\frac{(1+15a)^2}{16}$$

$$Now\ \sqrt{1+15a^2}\geq\frac{1+15a}{4}\ (1)$$

$$\sum\sqrt{1+15a^2}\overset{(1)}{\geq}\sum\frac{1+15a}{4}=\frac{3}{4}+\frac{15}{4}(a+b+c)=$$

$$=\frac{3}{4}+\frac{1}{4}(a+b+c)+\frac{7}{2}(a+b+c)\overset{AM-GM}{\geq}$$

$$\geq\frac{3}{4}+\frac{1}{4}3\sqrt[3]{abc}+\frac{7}{2}(a+b+c)\overset{abc=1}{=}\frac{7}{2}(a+b+c)+\frac{3}{2}$$

$$Equality\ holds\ for\ a=b=c=1$$

1461. If $a,b,c>0,\ n\in N$ **then:**

$$\sum\frac{b+c}{a^n}\geq\frac{6(a+b+c)}{a^n+c^n+b^n}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c\ \ then:\ \ (a+b) \ge (a+c) \ge (b+c),\qquad \frac{1}{a^n} \le \frac{1}{b^n} \le \frac{1}{c^n}$$

$$\sum \frac{b+c}{a^n} \overset{Chebyshev}{\ge} \frac{1}{3}\left(\sum b+c\right)\left(\sum \frac{1}{a^n}\right) \overset{Bergstrom}{\ge}$$

$$\ge \frac{2}{3}(a+b+c)\frac{(1+1+1)^2}{a^n+b^n+c^n} = \frac{6(a+b+c)}{a^n+c^n+b^n}$$

Equality holds for $a=b=c=1$

1462. If $a,b,c>0$ and

$$\prod_{cyc}(2a+b+c) = 64 \text{ and } n \in \mathbb{N} \text{ then:}$$

$$\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}} \le 3$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $a^{2n}-1$ and $a-1$ have the same sign, then

$$(a^{2n}-1)(a-1) \ge 0 \text{ or } a^{2n+1}-a+1 \ge a^{2n}$$

$$\Rightarrow \sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}} \le \sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n}+b^{2n}}} \overset{AM-GM}{\le} \sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{2a^nb^n}} = \sum_{cyc} ab.$$

Now, we have

$$64 = \prod_{cyc}(2a+b+c) \overset{AM-GM}{\ge} \prod_{cyc} 2\sqrt{(a+b)(a+c)}$$

$$= 8\prod_{cyc}(b+c) \overset{AM-GM}{\ge} 8.\frac{8}{9}\sum_{cyc} a.\sum_{cyc} bc \ge$$

$$\ge \frac{64}{9}.\sqrt{3(ab+bc+ca)^3} \ \Rightarrow\ ab+bc+ca \le 3.$$

Therefore

$$\sum_{cyc}\sqrt{\frac{2a^{n+2}b^{n+2}}{a^{2n+1}-a+b^{2n}+1}}\le 3.$$

Equality holds iff $a=b=c=1$.

1463. If $a,b,c>0$ with $ab+bc+ca=3$ and $\lambda\ge 2$, then :

$$\sum_{cyc}\frac{a^2}{a^2+\lambda b}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{a^2+\lambda b}=\sum_{cyc}\frac{\left(\sqrt{a^3}\right)^2}{a^3+\lambda ab}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}\sqrt{a^3}\right)^2}{\sum_{cyc}a^3+\lambda\sum_{cyc}ab}\overset{ab+bc+ca=3}{=}$$

$$\frac{\sum_{cyc}a^3+2\sum_{cyc}\sqrt{a^3b^3}}{\sum_{cyc}a^3+3\lambda}\ge\frac{\sum_{cyc}a^3+6\left(\frac{\sum_{cyc}ab}{3}\right)^{\frac{2}{3}}}{\sum_{cyc}a^3+3\lambda}$$

$$\left(\because\left(\frac{\sum_{cyc}(ab)^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}}\overset{\text{Power Mean Inequality}}{\ge}\frac{\sum_{cyc}ab}{3}\Rightarrow\sum_{cyc}\sqrt{a^3b^3}\ge 3\left(\frac{\sum_{cyc}ab}{3}\right)^{\frac{2}{3}}\right)$$

$$\overset{ab+bc+ca=3}{=}\frac{\sum_{cyc}a^3+6}{\sum_{cyc}a^3+3\lambda}\overset{?}{\ge}\frac{3}{\lambda+1}\Leftrightarrow(\lambda-2)\left(\sum_{cyc}a^3-3\right)\overset{?}{\ge}0\to\text{true}$$

$$\because\sum_{cyc}a^3\overset{\text{Chebyshev}}{\ge}\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)\ge\frac{1}{3}\left(\sum_{cyc}ab\right)\left(\sqrt{3\sum_{cyc}ab}\right)\overset{ab+bc+ca=3}{=}3$$

$$\Rightarrow\sum_{cyc}a^3-3\ge 0\text{ and }\lambda-2\ge 0\therefore\sum_{cyc}\frac{a^2}{a^2+\lambda b}\ge\frac{3}{\lambda+1}$$

$$\forall\, a,b,c>0\,\middle|\,ab+bc+ca=3\text{ and }\lambda\ge 2,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1464. If $x,y,z\in\mathbb{R}$ such that : $x^2+y^2+z^2=8$ and $xy+yz+zx=-4$,

then prove that : $-\frac{\sqrt{3}}{6} \le \frac{x^3+y^3+z^3}{x^4+y^4+z^4} \le \frac{\sqrt{3}}{6}$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x^2+y^2+z^2+2(xy+yz+zx) = 8-8 = 0 \Rightarrow (x+y+z)^2 = 0$$

$$\Rightarrow x+y+z = 0 \rightarrow (1)$$

$$xy+yz+zx = -4 \Rightarrow xy+z(x+y) = -4 \overset{\text{via }(1)}{\Rightarrow} xy+z(-z) = -4 \Rightarrow z^2-4 \overset{(2)}{=} xy$$

$$\le \frac{(x+y)^2}{4}\ (\because (x-y)^2 \ge 0\ \forall\, x,y \in \mathbb{R}) \overset{\text{via }(1)}{=} \frac{z^2}{4} \Rightarrow \frac{3z^2}{4} \le 4 \Rightarrow 3z^2-16 \le 0 \rightarrow (3)$$

$$\text{Now,} \frac{x^3+y^3+z^3}{x^4+y^4+z^4} = \frac{\left(\sum_{cyc} x\right)^3 - 3\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - xyz\right)}{\left(\sum_{cyc} x^2\right)^2 - 2\left(\left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right)\right)} \overset{\text{via }(1)}{=} \frac{3xyz}{64-32}$$

$$\left(\because \sum_{cyc} x^2 = 8 \text{ and } \sum_{cyc} xy - 4\right) \Rightarrow \left(\frac{x^3+y^3+z^3}{x^4+y^4+z^4}\right)^2 = \frac{9x^2y^2z^2}{1024} \overset{\text{via }(2)}{=} \frac{9z^2(z^2-4)^2}{1024}$$

$$\overset{?}{\le} \frac{1}{12} \Leftrightarrow 27z^6 - 216z^4 + 432z^2 - 256 \overset{?}{\le} 0 \Leftrightarrow (3z^2-16)(3z^2-4)^2 \overset{?}{\le} 0$$

$$\rightarrow \text{true} \because 3z^2 - 16 \overset{\text{via }(3)}{\le} 0 \therefore \left(\frac{x^3+y^3+z^3}{x^4+y^4+z^4}\right)^2 \le \frac{1}{12}$$

$$\Rightarrow -\frac{1}{\sqrt{12}} = -\frac{\sqrt{3}}{6} \le \frac{x^3+y^3+z^3}{x^4+y^4+z^4} \le \frac{1}{\sqrt{12}} = \frac{\sqrt{3}}{6},$$

$$'' ='' \text{ iff } \left(x = \frac{2}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = -\frac{4}{\sqrt{3}}\right) \text{ or } \left(x = -\frac{2}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = \frac{4}{\sqrt{3}}\right)$$

$$\text{or } \left(x = -\frac{4}{\sqrt{3}}, y = \frac{2}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}\right) \text{ or } \left(x = \frac{2}{\sqrt{3}}, y = -\frac{4}{\sqrt{3}}, z = \frac{2}{\sqrt{3}}\right)$$

$$\text{or } \left(x = \frac{4}{\sqrt{3}}, y = -\frac{2}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}\right) \text{ or } \left(x = -\frac{2}{\sqrt{3}}, y = \frac{4}{\sqrt{3}}, z = -\frac{2}{\sqrt{3}}\right) \text{ (QED)}$$

1465. **If** $a, b, c > 0$ **and** $ab+bc+ca+abc = 4$**, then prove that :**

$$\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \le 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left((2+b)(2+c)\right)-(2+a)(2+b)(2+c)=4-(ab+bc+ca+abc)$$

$$=0\therefore\sum_{cyc}\frac{1}{2+a}=1\to(m)$$

$$\text{Now},\frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\therefore\text{we can set}:\frac{1}{2+a}=\frac{1}{2}-x\left(x>0\text{ and }x<\frac{1}{2}\right)\therefore a+2=$$

$$\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

$$\text{Similarly, we set}:\frac{1}{2+b}=\frac{1}{2}-y\text{ and }\frac{1}{2+c}=\frac{1}{2}-z\therefore 1\overset{\text{via }(m)}{=}$$

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(i)$$

$$\therefore(1)\text{ and }(i)\Rightarrow a=\frac{2x}{y+z}\text{ and analogously}, b=\frac{2y}{z+x}\text{ and }c=\frac{2z}{x+y}$$

$$\text{and hence}:\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})=\sqrt{\frac{2x}{y+z}\cdot\frac{2y}{z+x}\cdot\frac{2z}{x+y}}\cdot\sqrt{\frac{2x}{y+z}+\frac{2y}{z+x}+\frac{2z}{x+y}}$$

$$\overset{CBS}{\le}4\cdot\sqrt{\frac{xyz}{\prod_{cyc}(x+y)}}\cdot\sqrt{\sum_{cyc}x}\cdot\sqrt{\sum_{cyc}\frac{1}{y+z}}=4\cdot\sqrt{\frac{xyz\sum_{cyc}x}{\prod_{cyc}(x+y)^2}}\cdot\sqrt{\left(\sum_{cyc}x\right)^2+\sum_{cyc}xy}$$

$$\le 4\cdot\sqrt{\frac{xyz\sum_{cyc}x}{\left(\frac{8}{9}(\sum_{cyc}x)(\sum_{cyc}xy)\right)^2}}\cdot\sqrt{\frac{4}{3}\left(\sum_{cyc}x\right)^2}\overset{\text{via }(i)}{=}4\cdot\sqrt{\frac{81xyz\sum_{cyc}x}{16(\sum_{cyc}xy)^2}}\cdot\sqrt{\frac{4}{3}\cdot\frac{1}{4}}$$

$$\le\sqrt{\frac{27xyz\sum_{cyc}x}{3xyz\sum_{cyc}x}}=3\therefore\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})\le 3$$

$$\forall a,b,c>0\mid ab+bc+ca+abc=4,''=''\text{ iff }a=b=c=1\text{ (QED)}$$

1466. If $a,b,c>0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{2}(a+b+c)=\frac{15}{2}$, then prove that :

$$a^2+b^2+c^2\le 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, } \left(\sum_{cyc} ab\right)^3 \overset{A-G}{\geq} 27a^2b^2c^2 \Rightarrow abc \leq \sqrt{\frac{\left(\sum_{cyc} ab\right)^3}{27}} \to (1) \text{ and also,}$$

$$\frac{15}{2} = \sum_{cyc}\left(\frac{1}{a} + a\right) + \frac{1}{2}\left(\sum_{cyc} a\right) \overset{A-G}{\geq} 6 + \frac{1}{2}\left(\sum_{cyc} a\right) \Rightarrow \sum_{cyc} a \leq 3 \to (2)$$

$$\text{Now, we assume : } \sum_{cyc} a^2 > 3 \text{ and we have : } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{3}{2}(a+b+c) = \frac{15}{2}$$

$$\Rightarrow \sum_{cyc} ab + \frac{3}{2}abc\left(\sum_{cyc} a\right) = \frac{15}{2}abc \Rightarrow \sum_{cyc} ab = \frac{3}{2}abc\left(5 - \sum_{cyc} a\right)$$

$$\overset{via\ (1)}{\leq} \frac{3}{2}\cdot\sqrt{\frac{\left(\sum_{cyc} ab\right)^3}{27}}\cdot\left(5 - \sum_{cyc} a\right)\left(\because \text{via } (2), \sum_{cyc} a \leq 3 < 5 \Rightarrow 5 - \sum_{cyc} a > 0\right)$$

$$= \frac{1}{2}\cdot\left(\sum_{cyc} ab\right)\cdot\sqrt{\frac{\sum_{cyc} ab}{3}}\cdot\left(5 - \sqrt{\sum_{cyc} a^2 + 2\sum_{cyc} ab}\right)$$

$$\overset{via\ assumption}{<} \frac{1}{2}\cdot\left(\sum_{cyc} ab\right)\cdot\sqrt{\frac{\sum_{cyc} ab}{3}}\cdot\left(5 - \sqrt{3 + 2\sum_{cyc} ab}\right)$$

$$\Rightarrow \boxed{\sqrt{\frac{t}{3}}\cdot\left(5 - \sqrt{3+2t}\right) - 2 > 0} \to (*) \left(t = \sum_{cyc} ab\right)$$

$$\text{Now, } t = \sum_{cyc} ab \leq \frac{1}{3}\left(\sum_{cyc} a\right)^2 \overset{via\ (2)}{\leq} \frac{1}{3}\cdot 9 \Rightarrow \boxed{0 < t \leq 3} \to (3) \text{ and we denote}$$

$$f(t) = \sqrt{\frac{t}{3}}\cdot\left(5 - \sqrt{3+2t}\right) - 2\ \forall\ t \in (0,3] \text{ and then : } f'(t) = \frac{5.\sqrt{2t+3} - (4t+3)}{2.\sqrt{3t(2t+3)}}$$

$$= \frac{25(2t+3) - (4t+3)^2}{2.\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3} + (4t+3)\right)} = \frac{-2(8t^2 - 13t - 33)}{2.\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3} + (4t+3)\right)}$$

$$= \frac{(3-t)(8t+11)}{\sqrt{3t(2t+3)}.\left(5.\sqrt{2t+3} + (4t+3)\right)} \overset{via\ (3)}{\geq} 0 \Rightarrow f'(t) \geq 0\ \forall\ t \in (0,3]$$

$$\Rightarrow f(t) \text{ is } \uparrow \text{ on } (0,3] \Rightarrow f(t) \leq f(3) = 0 \Rightarrow \boxed{\sqrt{\frac{t}{3}}\cdot\left(5 - \sqrt{3+2t}\right) - 2 \leq 0}$$

which contradicts $(*)$ and hence, we conclude that our assumption is incorrect

$$\Rightarrow a^2+b^2+c^2 \le 3, " = " \text{ iff } a=b=c=1 \text{ (QED)}$$

1467. If $x, y, z \ge 0$ and $x^2+y^2+z^2=3$, then prove that :

$$\frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}}+\frac{xy+yz+zx+1}{x+y+z} \ge \frac{28}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly two variables equal to zero and WLOG we may assume

$$y=z=0\ \left(x=\sqrt{3}\right) \text{ and then : LHS} = 16+\frac{1}{\sqrt{3}} > \frac{28}{3}$$

$\boxed{\text{Case 2}}$ Exactly one variable equals to zero and WLOG we may assume

$x=0$ with $y, z>0$ such that : $y^2+z^2=3$ and then :

$$\text{LHS} = \frac{16}{\sqrt{y^2z^2+1}}+\frac{yz+1}{y+z} \to (1)$$

$$\text{Now, } yz \overset{A-G}{\le} \frac{y^2+z^2}{2} = \frac{3}{2} \therefore y^2z^2+1 \le \frac{13}{4} \Rightarrow \frac{16}{\sqrt{y^2z^2+1}} \ge \frac{32}{\sqrt{13}} \to (\text{i})$$

$$\text{and also, } \frac{yz+1}{y+z} \overset{?}{>} \frac{1}{2} \Leftrightarrow 2yz+2 \overset{?}{>} y+z \Leftrightarrow y(2z-1)+2-z \overset{?}{>} 0$$

$$\Leftrightarrow \underbrace{\sqrt{3-z^2}.(2z-1)+2-z \overset{?}{>} 0}_{(\blacksquare)}$$

$$(\blacksquare) \text{ is true when } z \ge \frac{1}{2}\ \left(\because 2>\sqrt{3}>z \text{ as } y^2+z^2=3\right) \text{ and so, we now}$$

$$\text{consider the case when : } z<\frac{1}{2} \text{ and then : } (\blacksquare) \Leftrightarrow 2-z \overset{?}{>} \sqrt{3-z^2}.(1-2z)$$

$$\Leftrightarrow (2-z)^2 \overset{?}{>} \left(3-z^2\right)(1-2z)^2 \Leftrightarrow 4z^4-4z^3-10z^2+8z+1 \overset{?}{>} 0$$

$$\Leftrightarrow z^2(1-2z)^2+8z(1-2z)+5z^2+1 \overset{?}{>} 0 \to \text{true} \because 0<z<\frac{1}{2}$$

$$\Rightarrow (\blacksquare) \text{ is true and so, } \frac{yz+1}{y+z} > \frac{1}{2} \to (\text{ii}) \forall\, y, z>0 \,\middle|\, y^2+z^2=3 \therefore (\text{i})+(\text{ii}) \Rightarrow$$

$$\frac{16}{\sqrt{y^2z^2+1}}+\frac{yz+1}{y+z} > \frac{32}{\sqrt{13}}+\frac{1}{2} \approx 9.3752 > \frac{28}{3} \overset{\text{via (1)}}{\Rightarrow} \text{LHS} > \frac{28}{3}$$

$\boxed{\text{Case 3}}$ $x, y, z>0$ and assigning $y+z=a, z+x=b, x+y=c \Rightarrow a+b-c=2z$ $>0, b+c-a=2x>0$ and $c+a-b=2y>0 \Rightarrow a+b>c, b+c>a, c+a>b$ $\Rightarrow a, b, c$ form sides of a triangle with semiperimeter, circumradius and inradius

$$= s, R, r\ (say) \text{ yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b,$$

$$z = s - c \therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and}, \sum_{cyc} xy = \sum_{cyc} (s-a)(s-b) = 4Rr + r^2$$

$$\Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and also}, \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2 \text{ and also}, \sum_{cyc} x^2y^2 =$$

$$\left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{=} (4Rr + r^2)^2 - 2r^2s^2$$

$$\Rightarrow \sum_{cyc} x^2y^2 \overset{(\bullet\bullet\bullet\bullet\bullet)}{=} r^2((4R + r)^2 - 2s^2)$$

$$\text{Now, LHS} \overset{x^2+y^2+z^2=3}{=} \frac{16}{\sqrt{\sum_{cyc} x^2y^2 + \frac{(\sum_{cyc} x^2)^2}{9}}} + \frac{\sum_{cyc} xy + \frac{\sum_{cyc} x^2}{3}}{(\sum_{cyc} x).\sqrt{\frac{\sum_{cyc} x^2}{3}}} \overset{x^2+y^2+z^2=3}{=}$$

$$\frac{16\sum_{cyc} x^2}{\sqrt{(\sum_{cyc} x^2)^2 + 9\sum_{cyc} x^2y^2}} + \frac{(\sum_{cyc} x)^2 + \sum_{cyc} xy}{(\sum_{cyc} x).\sqrt{3\sum_{cyc} x^2}}$$

$$\Rightarrow LHS^2 = \frac{256(\sum_{cyc} x^2)^2}{(\sum_{cyc} x^2)^2 + 9\sum_{cyc} x^2y^2} + \frac{\left((\sum_{cyc} x)^2 + \sum_{cyc} xy\right)^2}{(\sum_{cyc} x)^2.3\sum_{cyc} x^2}$$

$$+ \frac{32\sum_{cyc} x^2}{\sqrt{(\sum_{cyc} x^2)^2 + 9\sum_{cyc} x^2y^2}} . \frac{(\sum_{cyc} x)^2 + \sum_{cyc} xy}{(\sum_{cyc} x).\sqrt{3\sum_{cyc} x^2}}$$

$$\overset{CBS}{\geq} \frac{256(\sum_{cyc} x^2)^2}{(\sum_{cyc} x^2)^2 + 9\sum_{cyc} x^2y^2} + \frac{\left((\sum_{cyc} x)^2 + \sum_{cyc} xy\right)^2}{(\sum_{cyc} x)^2.3\sum_{cyc} x^2}$$

$$+ \frac{32\sum_{cyc} x^2}{\sqrt{(\sum_{cyc} x^2)^2 + 3(\sum_{cyc} x^2)^2}} . \frac{(\sum_{cyc} x)^2 + \sum_{cyc} xy}{\sqrt{3\sum_{cyc} x^2}.\sqrt{3\sum_{cyc} x^2}} \overset{?}{\geq} \frac{784}{9}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{768s^2(s^2 - 8Rr - 2r^2)^3 + (s^2 + 4Rr + r^2)(17s^2 + 4Rr + r^2)\begin{pmatrix} s^2(s^2 - 8Rr - 2r^2)^2 + \\ 9r^2((4R + r)^2 - 2s^2) \end{pmatrix}}{3s^2(s^2 - 8Rr - 2r^2)(s^2(s^2 - 8Rr - 2r^2)^2 + 9r^2((4R + r)^2 - 2s^2))} \overset{?}{\geq} \frac{784}{9}$$

$$\Leftrightarrow 1571s^8 - (37080Rr - 3924r^2)s^6 + r^2(186144R^2 - 23712Rr - 17562)s^4 +$$

$$r^3(169088R^3+125952R^2r+31272Rr^2+2588r^3)s^2+39r^4(4R+r)^4 \overset{?}{\underset{(*)}{\geq}} 0 \text{ and}$$

$$\because \text{via Gerretsen}, P = 1571(s^2-16Rr+5r^2)^4 + 4r(15866R-6874r)(s^2-16Rr+5r^2)^3 + 4r^2(204840R^2-196830Rr+39807)(s^2-16Rr+5r^2)^2 \geq 0 \therefore \text{in order to prove } (*), \text{it suffices to prove : LHS of } (*) \geq P$$

$$\Leftrightarrow (105860R^3-181650R^2r+80709Rr^2-9781r^3)s^2 \overset{(**)}{\geq} r(1648584R^4-3282264R^3r+2056212R^2r^2-524997Rr^3+47673r^4)$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (105860R^3-181650R^2r+80709Rr^2-9781r^3)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(1648584R^4-3282264R^3r+2056212R^2r^2-524997Rr^3+47673r^4)$$

$$\Leftrightarrow 22588t^4-76718t^3+71691t^2-17522t+616 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(22588t^2+13634t+35875)+71442\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}} + \frac{xy+yz+zx+1}{x+y+z} \geq \frac{28}{3}$$

under case (3) and combining all cases,

$$\frac{16}{\sqrt{x^2y^2+y^2z^2+z^2x^2+1}} + \frac{xy+yz+zx+1}{x+y+z} \geq \frac{28}{3}$$

$$\forall x,y,z \geq 0 \mid x^2+y^2+z^2=3, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

1468. If $a,b,c>0$ and $a+b+c=abc$, then prove that :

$$\frac{3\sqrt{3}}{4} \leq \frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)} \leq \frac{a+b+c}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)} \leq \frac{a+b+c}{4} \overset{a+b+c=abc}{\Leftrightarrow}$$

$$\sum_{cyc} \frac{bc}{a\left(\frac{abc}{a+b+c}+bc\right)} \leq \frac{a+b+c}{4} \Leftrightarrow (a+b+c).\sum_{cyc}\frac{1}{a(2a+b+c)} \leq \frac{a+b+c}{4}$$

$$\overset{a+b+c=abc}{\Leftrightarrow} \frac{abc}{a+b+c}.\sum_{cyc}\frac{1}{a(2a+b+c)} \leq \frac{1}{4} \Leftrightarrow \sum_{cyc}\frac{bc}{(a+b)+(c+a)} \overset{(*)}{\leq} \frac{a+b+c}{4}$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$$

$$\therefore abc = r^2 s \to (2) \text{and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3) \therefore \sum_{cyc} \frac{bc}{(a+b)+(c+a)} = \sum_{cyc} \frac{(s-y)(s-z)}{y+z}$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc} \left((s-y)(s-z)(z+x)(x+y)\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc} \left((s-y)(s-z)\left(x^2 + \sum_{cyc} xy\right)\right)$$

$$= \frac{1}{2s(s^2+2Rr+r^2)} \cdot \begin{pmatrix} \sum_{cyc} \left(x^2(-s^2+sx+yz)\right) \\ +(s^2+4Rr+r^2) \cdot \sum_{cyc} (s-y)(s-z) \end{pmatrix}$$

$$= \frac{-2s^2(s^2-4Rr-r^2) + 2s^2(s^2-6Rr-3r^2) + 8Rrs^2 + (s^2+4Rr+r^2)(4Rr+r^2)}{2s(s^2+2Rr+r^2)}$$

$$\overset{?}{\leq} \frac{a+b+c}{4} \overset{\text{via (1)}}{=} \frac{s}{4} \Leftrightarrow 2r\left((8R-3r)s^2 + r(4R+r)^2\right) \overset{?}{\leq} s^2(s^2+2Rr+r^2)$$

$$\Leftrightarrow s^4 - (14Rr - 7r^2)s^2 - 2r^2(4R+r)^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (2Rr + 2r^2)s^2 - 2r^2(4R+r)^2 \overset{\text{Gerretsen}}{\geq}$$

$$(2Rr+2r^2)(16Rr-5r^2) - 2r^2(4R+r)^2 = 3r^3(R-2r) \overset{\text{Euler}}{\geq} 0 \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore \frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)} \leq \frac{a+b+c}{4}$$

$$\text{Again,} \frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)} \overset{a+b+c=abc}{=} \sum_{cyc} \frac{bc}{a + \sum_{cyc} a}$$

$$= \sum_{cyc} \frac{b^2c^2}{2abc + b^2c + bc^2} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} ab\right)^2}{6abc + \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc} \overset{?}{\geq} \frac{3\sqrt{3}}{4} \overset{a+b+c=abc}{=}$$

$$\frac{3\sqrt{3}}{4} \cdot \sqrt{\frac{abc}{a+b+c}} \overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} \frac{r^4(4R+r)^4}{\left(3r^2s + s(4Rr+r^2)\right)^2} \overset{?}{\geq} \frac{27}{16} \cdot \frac{r^2 s}{s}$$

$$\Leftrightarrow (4R+r)^4 \boxed{\underset{(***)}{\overset{?}{\geq}}} 27s^2(R+r)^2$$

$$\text{Now, } (4R+r)^4 \overset{\text{Trucht or Doucet}}{\geq} 3s^2(4R+r)^2 \overset{?}{\geq} 27s^2(R+r)^2 \Leftrightarrow 4R + r \overset{?}{\geq} 3R + 3r$$

$$\Leftrightarrow R \overset{?}{\geq} 2r \to \text{true via Euler} \Rightarrow (***) \text{ is true} \therefore \frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)}$$

$$\geq \frac{3\sqrt{3}}{4} \text{ and so, } \frac{3\sqrt{3}}{4} \leq \frac{bc}{a(1+bc)} + \frac{ca}{b(1+ca)} + \frac{ab}{c(1+ab)} \leq \frac{a+b+c}{4}$$

$$\forall a,b,c>0\,\middle|\,a+b+c=abc,''='' \text{ iff } a=b=c=\sqrt{3}\ (\text{QED})$$

1469.If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\ge 3\sqrt{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}\left((2+b)(2+c)\right)-(2+a)(2+b)(2+c)=4-(ab+bc+ca+abc)$$

$$=0\ \therefore\sum_{\text{cyc}}\frac{1}{2+a}=1\to(\text{m})$$

$$\text{Now},\frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\ \therefore\ \text{we can set}:\frac{1}{2+a}=\frac{1}{2}-x\left(x>0\text{ and }x<\frac{1}{2}\right)$$

$$\therefore a+2=\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

$$\text{Similarly, we set}:\frac{1}{2+b}=\frac{1}{2}-y\text{ and }\frac{1}{2+c}=\frac{1}{2}-z$$

$$\therefore 1\overset{\text{via (m)}}{=}\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(\text{i})$$

$$\therefore(1)\text{ and (i)}\Rightarrow a=\frac{2x}{y+z}\text{ and analogously, }b=\frac{2y}{z+x}\text{ and }c=\frac{2z}{x+y}\text{ and hence}:$$

$$\frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\ge 3.\sqrt{abc}\text{ transforms into}:$$

$$2\sum_{\text{cyc}}\sqrt{\frac{x+y}{2z}}-\sum_{\text{cyc}}\left(\sqrt{\frac{x+y}{2z}}.\sqrt{\frac{2x}{y+z}.\frac{2y}{z+x}}\right)\ge 3\sqrt{\frac{8xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{\text{cyc}}\sqrt{\frac{x+y}{z}}-\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}.\sum_{\text{cyc}}\frac{x+y}{z}\ge 6\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{\text{cyc}}\sqrt{\frac{x+y}{z}}\ge\left(6+\sum_{\text{cyc}}\frac{x+y}{z}\right).\sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}$$

$$\Leftrightarrow\sum_{\text{cyc}}\frac{x+y}{z}+2\sum_{\text{cyc}}\sqrt{\frac{(y+z)(z+x)}{xy}}\ge\frac{xyz}{(y+z)(z+x)(x+y)}.\left(6+\sum_{\text{cyc}}\frac{x+y}{z}\right)^2$$

$$\Leftrightarrow\frac{\left(\sum_{\text{cyc}}x\right)\left(\sum_{\text{cyc}}xy\right)}{xyz}-3+2\sum_{\text{cyc}}\sqrt{\frac{(y+z)(z+x)}{xy}}$$

$$\overset{(*)}{\geq}\frac{xyz}{(y+z)(z+x)(x+y)}\cdot\left(\frac{(\sum_{cyc}x)(\sum_{cyc}xy)}{xyz}+3\right)^2$$

Assigning $y+z=A, z+x=B, x+y=C \Rightarrow A+B-C=2z>0, B+C-A=2x>0$ and $C+A-B=2y>0 \Rightarrow A+B>C, B+C>A, C+A>B \Rightarrow A,B,C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}A=2s \Rightarrow \sum_{cyc}x=s \to (1) \Rightarrow x=s-A, y=s-B, z=s-C$

$$\Rightarrow xyz=r^2s \to (2) \text{ and } \sum_{cyc}xy=\sum_{cyc}(s-A)(s-B)=4Rr+r^2 \to (3)$$

$$\therefore \text{ via } (1),(2),(3),(*) \Leftrightarrow \frac{s(4Rr+r^2)}{r^2s}-3+2\sum_{cyc}\sqrt{\frac{AB}{(s-A)(s-B)}}$$

$$\geq \frac{r^2s}{4Rrs}\cdot\left(\frac{s(4Rr+r^2)}{r^2s}+3\right)^2 \Leftrightarrow \frac{4R-2r}{r}+2\sum_{cyc}\operatorname{cosec}\frac{\alpha}{2}\geq\frac{r^2(4R+4r)^2}{4Rr.r^2}$$

$$(\alpha,\beta,\gamma \to \text{angles of } \Delta \text{ with sides } A,B \text{ and } C) \Leftrightarrow \frac{2R-r}{r}+\sum_{cyc}\operatorname{cosec}\frac{\alpha}{2}\overset{(**)}{\geq}\frac{2(R+r)^2}{Rr}$$

$$\text{Now, LHS of } (**) \overset{\text{Jensen}}{\geq} \frac{2R-r}{r}+3\operatorname{cosec}\frac{\pi}{6}=\frac{2R-r}{r}+6=\frac{2R+5r}{r}\overset{?}{\geq}\frac{2(R+r)^2}{Rr}$$

$$\Leftrightarrow 2R^2+5Rr\overset{?}{\geq}2R^2+4Rr+2r^2 \Leftrightarrow Rr\overset{?}{\geq}2r^2 \to \text{true via Euler} \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true } \therefore \frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\geq 3.\sqrt{abc}\ \forall\, a,b,c>0$$

$$\left|\, ab+bc+ca+abc=4, ''=''\ \text{ iff } a=b=c=1 \text{ (QED)}\right.$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morroco

By AM − GM inequality, we have

$$(2+\sqrt{ab})(2-\sqrt{ab})=4-ab=bc+ca+abc\geq 2c\sqrt{ab}+abc=c\sqrt{ab}(2+\sqrt{ab})$$

$$\Rightarrow\ 2-\sqrt{ab}\geq c\sqrt{ab}\ \Rightarrow\ \frac{2-\sqrt{ab}}{\sqrt{c}}\geq\sqrt{abc}\ \text{ (and analogs)}$$

Therefore

$$\frac{2-\sqrt{ab}}{\sqrt{c}}+\frac{2-\sqrt{bc}}{\sqrt{a}}+\frac{2-\sqrt{ca}}{\sqrt{b}}\geq 3\sqrt{abc}.$$

Equality holds iff $a=b=c=1$.

1470. If $a,b,c>0$ and $a^2+b^2+c^2+abc=4$ then prove that:

$$2(a+b+c)+ab+bc+ca-abc\le 8$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$4-a^2=b^2+c^2+abc\ge 2bc+abc=bc(2+a)\Rightarrow 2-a$$
$$\ge bc\overset{a<2}{\Rightarrow}(2-a-bc)(2-a)\ge 0$$

$$\Rightarrow 4\ge 4a+2bc-a^2-abc\ \ (and\ analogs)$$

Adding this inequality with similar ones, we get

$$12\ge 4(a+b+c)+2(ab+bc+ca)-(a^2+b^2+c^2+abc)-2abc$$

$$=4(a+b+c)+2(ab+bc+ca)-4-2abc$$

$$\Leftrightarrow 8\ge 2(a+b+c)+ab+bc+ca-abc.$$

Equality holds iff $a=b=c=1$.

1471. If $a,b,c\in[0,4]$ and $a+b+c=6$, then prove that :

$$a^2+b^2+c^2+ab+bc+ca\le 28$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{Case\ 1}$ $b\ge 2\ (b\le 4)$ and then : $a^2+b^2+c^2+ab+bc+ca$

$$=b(b+a+c)+a^2+c^2+ca\overset{a+b+c=6}{=}6b+(c+a)^2-ca\overset{a+b+c=6}{=}$$
$$6b+(6-b)^2-ca\overset{-ca\le 0}{\le}b^2-6b+36=28+b^2-6b+8$$
$$=28+(b-2)(b-4)\le 28\because 2\le b\le 4\Rightarrow(b-2)(b-4)\le 0$$

$$\therefore a^2+b^2+c^2+ab+bc+ca\le 28$$

$\boxed{Case\ 2}$ $b\le 2\ (b\ge 0)$ and if $c,a<2$, then : $c+a<4\overset{a+b+c=6}{\Rightarrow}6-b<4$
$\Rightarrow b>2$, a contradiction $\therefore$ when $b\le 2$, then, at least one of c,a must be ≥ 2

• When $a\ge 2\ (a\le 4)$, then : $a^2+b^2+c^2+ab+bc+ca=a(a+b+c)$
$$+b^2+c^2+bc\overset{a+b+c=6}{=}6a+(b+c)^2-bc\overset{a+b+c=6}{=}6a+(6-a)^2-bc$$

$$\stackrel{-bc\le 0}{\le} a^2-6a+36=28+a^2-6a+8=28+(a-2)(a-4)\le 28 \because 2\le a\le 4$$
$$\Rightarrow (a-2)(a-4)\le 0 \therefore a^2+b^2+c^2+ab+bc+ca\le 28$$

• When $c\ge 2$ $(c\le 4)$, then : $a^2+b^2+c^2+ab+bc+ca=c(c+b+a)$
$$+a^2+b^2+ab \stackrel{a+b+c=6}{=} 6c+(a+b)^2-ab \stackrel{a+b+c=6}{=} 6c+(6-c)^2-ab$$
$$\stackrel{-ab\le 0}{\le} c^2-6c+36=28+c^2-6c+8=28+(c-2)(c-4)\le 28 \because 2\le c\le 4$$

$\Rightarrow (c-2)(c-4)\le 0 \therefore a^2+b^2+c^2+ab+bc+ca\le 28$ and so, combining *all* scenarios, $a^2+b^2+c^2+ab+bc+ca\le 28$

$$\forall\, a,b,c\in[0,4]\,\big|\, a+b+c=6,\text{"="} \text{ iff } (a=0,b=2,c=4)$$

or $(a=0,b=4,c=2)$ or $(b=0,c=2,a=4)$ or $(b=0,c=4,a=2)$

or $(c=0,a=2,b=4)$ or $(c=0,a=4,b=2)$ (QED)

1472. If $x,y,z>-1$, then prove that :

$$\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We have } y\le\frac{1+y^2}{2} \text{ and analogs } \forall\, y\in\mathbb{R}$$
$$\Rightarrow \frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\ge$$
$$\ge\frac{1+x^2}{1+\frac{1+y^2}{2}+z^2}+\frac{1+y^2}{1+\frac{1+z^2}{2}+x^2}+\frac{1+z^2}{1+\frac{1+x^2}{2}+y^2}$$
$$=2\left(\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\right)$$

$$(a=1+x^2, b=1+y^2, c=1+z^2; a,b,c\ge 1>0)$$

$$=2\left(\frac{a^2}{ab+2ca}+\frac{b^2}{bc+2ab}+\frac{c^2}{ca+2bc}\right)\stackrel{\text{Bergstrom}}{\ge}\frac{2\left(\sum_{\text{cyc}} a\right)^2}{3\sum_{\text{cyc}} ab}\ge 2$$

$$\therefore \frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\ge 2$$

$$\forall\, x,y,z>-1,''='' \text{ iff } x=y=z=1\ (\text{QED})$$

1473. If $a,b,c>0$ and $ab+bc+ca+abc=4$, then prove that :

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\sqrt{abc}+\frac{8}{\sqrt{abc}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\big((2+b)(2+c)\big)-(2+a)(2+b)(2+c)=$$

$$=4-(ab+bc+ca+abc)=0\ \therefore\sum_{cyc}\frac{1}{2+a}=1\to(m)$$

$$\text{Now, }\frac{1}{2+a}\overset{a>0}{<}\frac{1}{2}\ \therefore \text{ we can set : }\frac{1}{2+a}=\frac{1}{2}-x\ \left(x>0\text{ and }x<\frac{1}{2}\right)$$

$$\therefore a+2=\frac{2}{1-2x}\Rightarrow a=\frac{2}{1-2x}-2=\frac{2x}{\frac{1}{2}-x}\to(1)$$

$$\text{Similarly, we set : }\frac{1}{2+b}=\frac{1}{2}-y\text{ and }\frac{1}{2+c}=\frac{1}{2}-z\ \therefore 1\overset{\text{via }(m)}{=}$$

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{1}{2}-x+\frac{1}{2}-y+\frac{1}{2}-z\Rightarrow x+y+z=\frac{1}{2}\to(\text{i})$$

$$\therefore(1)\text{ and }(\text{i})\Rightarrow a=\frac{2x}{y+z}\text{ and analogously, }b=\frac{2y}{z+x}\text{ and }c=\frac{2z}{x+y}\text{ and hence :}$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\sqrt{abc}+\frac{8}{\sqrt{abc}}\Leftrightarrow\sqrt{abc}.\sum_{cyc}\sqrt{a}+2\sum_{cyc}\sqrt{ab}$$

$$\le abc+8\overset{ab+bc+ca+abc=4}{\Leftrightarrow}\Leftrightarrow\sqrt{abc}.\sum_{cyc}\sqrt{a}+2\sum_{cyc}\sqrt{ab}\le 4-\sum_{cyc}ab+8$$

$$\Leftrightarrow\sqrt{abc}.\sum_{cyc}\sqrt{a}+2\sum_{cyc}\sqrt{ab}+\sum_{cyc}ab\le 12$$

$$\Leftrightarrow \sqrt{\frac{xyz}{(y+z)(z+x)(x+y)}}\cdot\sum_{cyc}\sqrt{\frac{x}{y+z}}+\sum_{cyc}\sqrt{\frac{xy}{(y+z)(z+x)}}$$

$$+\sum_{cyc}\frac{xy}{(y+z)(z+x)} \overset{(*)}{\le} 3$$

Assigning $y+z=A, z+x=B, x+y=C \Rightarrow A+B-C=2z>0, B+C-A=2x>0$ and $C+A-B=2y>0 \Rightarrow A+B>C, B+C>A, C+A>B \Rightarrow A, B, C$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}A=2s \Rightarrow \sum_{cyc}x=s \Rightarrow x=s-A, y=s-B, z=s-C$

$\Rightarrow xyz=r^2s$ and via such substitutions, LHS of $(*)$ becomes :

$$\sqrt{\frac{r^2s}{\underbrace{16R^2r^2s^2}_{(y+z)^2(z+x)^2(x+y)^2}}}\cdot\sum_{cyc}\sqrt{BC(s-A)}+\sum_{cyc}\sqrt{\frac{(s-A)(s-B)}{AB}}+\sum_{cyc}\frac{(s-A)(s-B)}{AB}$$

$$\overset{CBS}{\le}\sqrt{\frac{1}{16R^2s}}\cdot\sqrt{\sum_{cyc}(s-A)}\cdot\sqrt{\overbrace{s^2+4Rr+r^2}^{\sum_{cyc}BC}}+\sum_{cyc}\sin\frac{\alpha}{2}+\sum_{cyc}\sin^2\frac{\alpha}{2}$$

$(\alpha, \beta, \gamma \to$ angles of triangle with sides A, B and $C)$ $\overset{\substack{CBS\\ and\\ Jensen}}{\le}$

$$\frac{\sqrt{4R^2+8Rr+4r^2}}{4R}+\frac{3}{2}+\frac{2R-r}{2R}=\frac{R+r+3R+2R-r}{2R}=3 \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\sqrt{abc}+\frac{8}{\sqrt{abc}}$$

$\forall\, a, b, c>0 \mid ab+bc+ca+abc=4, "=" $ iff $a=b=c=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$(2+\sqrt{bc})(2-\sqrt{bc})=4-bc=ab+ca+abc\ge 2a\sqrt{bc}+abc=a\sqrt{bc}(2+\sqrt{bc})$$

$$\Rightarrow\ 2-\sqrt{bc}\ge a\sqrt{bc}\ \overset{bc\le 4}{\Rightarrow}\ (2-\sqrt{bc}-a\sqrt{bc})(2-\sqrt{bc})\ge 0$$

$$\Rightarrow 4+bc+abc\ge 2\sqrt{bc}(a+2)\ \Rightarrow\ \frac{4+bc}{2\sqrt{abc}}+\frac{\sqrt{abc}}{2}\ge\sqrt{a}+\frac{2}{\sqrt{a}}\ \text{(and analogs)}$$

Adding this inequality with similar ones, we get

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+2\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)\le\frac{12+ab+bc+ca}{2\sqrt{abc}}+\frac{3\sqrt{abc}}{2}=$$

$$=\frac{16-abc}{2\sqrt{abc}}+\frac{3\sqrt{abc}}{2}=\sqrt{abc}+\frac{8}{\sqrt{abc}},$$

as desired. Equality holds iff $a=b=c=1$.

1474. If $a,b,c\ge 0$ and $a^2+b^2+c^2+abc=4$ then prove that:

$$a+b+c\le\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We will prove that :

$$\sqrt{2-a}\ge\frac{b+c}{2}\quad(1)$$

We have

$$(1)\ \Leftrightarrow 4(2-a)\ge b^2+c^2+2bc=4-a^2-abc+2bc=(2-a)(2+a+bc)$$

$$\Leftrightarrow (2-a)(2-a-bc)\ge 0$$

By AM − GM inequality, we have

$$(2-a)(2+a)=4-a^2=b^2+c^2+abc\ge 2bc+abc=bc(2+a)\Rightarrow 2-a\ge bc,$$

then (1) is true. Adding this inequality with similar ones yields the desired inequality.

Equality holds iff $a=b=c=1$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{Firstly, }(4-b^2),(4-c^2)=(a^2+c^2+abc),(a^2+b^2+abc)\overset{a,b,c\ge 0}{\ge}0$$

$$\text{and we have}: a^2+b^2+c^2+abc=4\Rightarrow a^2+a.bc+(b^2+c^2-4)=0$$

$$\Rightarrow a=\frac{-bc\pm\sqrt{b^2c^2-4(b^2+c^2-4)}}{2}=\frac{-bc\pm\sqrt{b^2(c^2-4)-4(c^2-4)}}{2}$$

$$=\frac{-bc\pm\sqrt{(b^2-4)(c^2-4)}}{2}$$

$$\Rightarrow a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}\text{ or } a=\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}$$

$$\boxed{\text{When } a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}}\text{, then : } b,c\ge 0$$

$$\Rightarrow a=\frac{-bc-\sqrt{(4-b^2)(4-c^2)}}{2}\le 0\Rightarrow a=0\ (\because a\ge 0)$$

$$\therefore \overbrace{bc}^{\ge 0}+\overbrace{\sqrt{(4-b^2)(4-c^2)}}^{\ge 0}=0\Rightarrow(b=0,c=2)\text{ or }(c=0,b=2)$$

$$\text{and in either case : }\boxed{a+b+c=2}\text{ and }\boxed{\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}=\sqrt{2}+\sqrt{2}}$$

$$=2\sqrt{2}\Rightarrow a+b+c<\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}$$

$$\boxed{\text{When } a=\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}}\text{, then : } 2-a=$$

$$2-\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}=\frac{8+2bc-2\sqrt{(4-b^2)(4-c^2)}}{4}$$

$$\overset{A-G}{\ge}\frac{8+2bc-(8-b^2-c^2)}{4}=\frac{b^2+c^2+2bc}{4}=\frac{(b+c)^2}{4}$$

$$\Rightarrow\sqrt{2-a}\ge\frac{b+c}{2}\text{ and analogously, }\sqrt{2-b}\ge\frac{c+a}{2}\text{ and }\sqrt{2-c}\ge\frac{a+b}{2}$$

$$\text{and summing up, we get : }\sqrt{2-a}+\sqrt{2-b}+\sqrt{2-c}\ge a+b+c\ \forall a,b,c\ge 0,$$

$$''='' \text{ iff } a=b=c=1\text{ (QED)}$$

1475. If $a,b,c\ge 0$, then prove that :

$$(a+bc)^2+(b+ca)^2+(c+ab)^2\ge\sqrt{2}(a+b)(b+c)(c+a)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 Exactly 2 variables equal to zero and WLOG we may assume :
$b = c = 0$ and then : $LHS - RHS = a^2 > 0$

Case 2 Exactly 1 variable equals to 0 and WLOG we may assume

$$a = 0 \text{ and then : } LHS = b^2c^2 + b^2 + c^2 = \left(\frac{b^2c^2}{2} + b^2\right) + \left(\frac{b^2c^2}{2} + c^2\right)$$

$$\overset{A-G}{\geq} 2b^2.\sqrt{\frac{c^2}{2}} + 2c^2.\sqrt{\frac{b^2}{2}} = \sqrt{2}bc(b+c) = RHS \Rightarrow LHS \geq RHS,$$

$'' ='' $ iff $\left(a = 0, b = c = \sqrt{2}\right)$ and permutations

Case 3 $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y$
$\Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say); so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y,$

$c = s - z \therefore abc = r^2s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3) \text{ and } \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4) \text{ and also,}$$

$$\sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2) \rightarrow (5)$$

$$\text{Now, } (a + bc)^2 + (b + ca)^2 + (c + ab)^2 = \sum_{cyc} a^2 + \sum_{cyc} a^2b^2 + 6abc$$

$$\overset{A-G}{\geq} 2\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} + 6abc \overset{?}{>} \sqrt{2}(a + b)(b + c)(c + a)$$

$$\Leftrightarrow 4\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) + 36a^2b^2c^2 + 24abc.\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}$$

$$\overset{?}{>} 2(a + b)^2(b + c)^2(c + a)^2 \overset{\text{via (2),(4) and (5)}}{\Leftrightarrow}$$

$$12r^2s.\sqrt{r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2)} \overset{?}{>}$$

$$16R^2r^2s^2 - 18r^4s^2 - 2r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2) \Leftrightarrow$$

$$3s.\sqrt{r^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)}\underbrace{\overset{?}{>}}_{(*)}$$

$s^4-s^2(4R^2+12Rr+7r^2)+r(4R+r)^3$ and $\because$ LHS of $(*)>0 \therefore$ when : RHS of $(*)\le 0$, then : $(*)$ is true and so, we now focus on the scenario when : $s^4-s^2(4R^2+12Rr+7r^2)+r(4R+r)^3>0$ and then : $(*)\Leftrightarrow$

$$9r^2s^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)\overset{?}{>}$$
$$(s^4-s^2(4R^2+12Rr+7r^2)+r(4R+r)^3)^2$$
$$\Leftrightarrow (s^4-s^2(4R^2+12Rr+7r^2)+r(4R+r)^3)^2-$$
$$9r^2s^2(s^2-8Rr-2r^2)((4R+r)^2-2s^2)\underbrace{\overset{?}{<}}_{(**)}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$
$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(i)}{\le}0\Rightarrow$$
$$(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\begin{pmatrix}s^4-s^2(4R^2+12Rr+7r^2)\\+r(4R+r)^3\end{pmatrix}\le 0$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)<$

$$(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\begin{pmatrix}s^4-s^2(4R^2+12Rr+7r^2)\\+r(4R+r)^3\end{pmatrix}$$
$$\Leftrightarrow (8R+9r)s^4-(32R^3+204R^2r+164Rr^2-18r^3)s^2$$
$$+r(512R^4+960R^3r+528R^2r^2+116Rr^3+9r^4)\overset{(***)}{<}0 \text{ and via (i)},$$

$(8R+9r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\le 0 \therefore$ in order to prove $(***)$, it suffices to prove : LHS of $(***)<$ $(8R+9r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\Leftrightarrow -8R^2rs^2<0\to$ true $\therefore (***)\Rightarrow(**)\Rightarrow(*)$ is true $\therefore (a+bc)^2+(b+ca)^2+(c+ab)^2>\sqrt{2}(a+b)(b+c)(c+a)$ and so, combining *all* cases, $(a+bc)^2+(b+ca)^2+(c+ab)^2\ge\sqrt{2}(a+b)(b+c)(c+a)\ \forall a,b,c\ge 0$, " = " iff $\left(a=0, b=c=\sqrt{2}\right)$ or $\left(b=0, c=a=\sqrt{2}\right)$ or $\left(c=0, a=b=\sqrt{2}\right)$ (QED)

1476. If $a,b,c>0$, then :

$$\sum_{cyc}\frac{\left(\frac{c+a}{a+b}\right)^2+\left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2+\frac{c^2}{a^2}(b+c)^2}\ge\frac{9}{4(a^2+b^2+c^2)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now}, \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have: }\frac{\left(\frac{c+a}{a+b}\right)^2+\left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2+\frac{c^2}{a^2}(b+c)^2}+\frac{\left(\frac{a+b}{b+c}\right)^2+\left(\frac{c+a}{b+c}\right)^2}{\frac{c^2}{b^2}(a+b)^2+\frac{a^2}{b^2}(c+a)^2}$$

$$+\frac{\left(\frac{b+c}{c+a}\right)^2+\left(\frac{a+b}{c+a}\right)^2}{\frac{a^2}{c^2}(b+c)^2+\frac{b^2}{c^2}(a+b)^2}$$

$$=\frac{\frac{a^2}{(a+b)^2(b+c)^2}+\frac{a^2}{(a+b)^2(c+a)^2}}{\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}}+\frac{\frac{b^2}{(b+c)^2(c+a)^2}+\frac{b^2}{(b+c)^2(a+b)^2}}{\frac{c^2}{(c+a)^2}+\frac{a^2}{(a+b)^2}}$$

$$+\frac{\frac{c^2}{(c+a)^2(a+b)^2}+\frac{c^2}{(c+a)^2(b+c)^2}}{\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}}$$

$$=\frac{\frac{a^2}{(a+b)^2}\cdot\left(\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)}{\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}}+\frac{\frac{b^2}{(b+c)^2}\cdot\left(\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right)}{\frac{c^2}{(c+a)^2}+\frac{a^2}{(a+b)^2}}$$

$$+\frac{\frac{c^2}{(c+a)^2}\cdot\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}\right)}{\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}}$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(\begin{array}{l}x=\frac{a^2}{(a+b)^2},y=\frac{b^2}{(b+c)^2},z=\frac{c^2}{(c+a)^2},\\ A=\frac{1}{(a+b)^2},B=\frac{1}{(b+c)^2},C=\frac{1}{(c+a)^2}\end{array}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{1}{(a+b)^2}\cdot\frac{1}{(b+c)^2}\right)}$$

$$\overset{\text{Radon}}{\geq}\sqrt{3}.\sqrt{\frac{(1+1+1)^3}{\left(\sum_{cyc}((a+b)(b+c))\right)^2}}=9.\sqrt{\frac{1}{\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)^2}}$$

$$\geq 9.\sqrt{\frac{1}{\left(3\sum_{cyc}a^2+\sum_{cyc}a^2\right)^2}}=\frac{9}{4(a^2+b^2+c^2)}$$

$$\therefore\frac{\left(\frac{c+a}{a+b}\right)^2+\left(\frac{b+c}{a+b}\right)^2}{\frac{b^2}{a^2}(c+a)^2+\frac{c^2}{a^2}(b+c)^2}+\frac{\left(\frac{a+b}{b+c}\right)^2+\left(\frac{c+a}{b+c}\right)^2}{\frac{c^2}{b^2}(a+b)^2+\frac{a^2}{b^2}(c+a)^2}+\frac{\left(\frac{b+c}{c+a}\right)^2+\left(\frac{a+b}{c+a}\right)^2}{\frac{a^2}{c^2}(b+c)^2+\frac{b^2}{c^2}(a+b)^2}$$

$$\geq\frac{9}{4(a^2+b^2+c^2)}\ \forall a,b,c>0''='' \text{ iff } a=b=c\ (QED)$$

1477. If $a,b,c>0$ then:

$$\sum\frac{b^2\left(\frac{a+c}{a+b}\right)^2+c^2\left(\frac{b+c}{a+b}\right)^2}{(a+c)^2+(b+c)^2}\geq\frac{9(abc)^{\frac{2}{3}}}{4(a^2+b^2+c^2)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$(a+c)^2+(b+c)^2\overset{CBS}{\leq}2(a^2+c^2)+2(b^2+c^2)=2(a^2+b^2+2c^2)\ \ (1)$$

$$b^2\left(\frac{a+c}{a+b}\right)^2+c^2\left(\frac{b+c}{a+b}\right)^2\overset{AM-GM}{\geq}\frac{2bc(a+c)(b+c)}{(a+b)^2}\ \ (2)$$

$$\sum\frac{b^2\left(\frac{a+c}{a+b}\right)^2+c^2\left(\frac{b+c}{a+b}\right)^2}{(a+c)^2+(b+c)^2}\overset{(1)\&(2)}{\geq}\sum\frac{\frac{2bc(a+c)(b+c)}{(a+b)^2}}{2(a^2+b^2+2c^2)}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{a^2b^2c^2}{\prod a^2+b^2+2c^2}}\overset{AM-GM}{\geq}\frac{3(abc)^{\frac{2}{3}}}{\frac{\sum(a^2+b^2+2c^2)}{3}}=\frac{9(abc)^{\frac{2}{3}}}{4(a^2+b^2+c^2)}$$

Equality holds for $a=b=c$

1478. If $a,b,c>0$ and $abc=1$, then :

$$\frac{a^4+a^2c^2}{a^2b^4+b^2c^4}+\frac{b^4+a^2b^2}{b^2c^4+c^2a^4}+\frac{c^4+b^2c^2}{c^2a^4+a^2b^4}\geq 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall$ A, B, C >0, $(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\ \therefore\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have : }\frac{a^4+a^2c^2}{a^2b^4+b^2c^4}+\frac{b^4+a^2b^2}{b^2c^4+c^2a^4}+\frac{c^4+b^2c^2}{c^2a^4+a^2b^4}$$

$$=\frac{x^2+xz}{xy^2+yz^2}+\frac{y^2+xy}{yz^2+zx^2}+\frac{z^2+yz}{zx^2+xy^2}\ (x=a^2,y=b^2,c=z^2)\overset{xyz=1}{=}$$

$$\frac{\frac{x}{yz}+\frac{1}{y}}{\frac{xy}{zx}+\frac{yz}{xy}}+\frac{\frac{y}{zx}+\frac{1}{z}}{\frac{yz}{xy}+\frac{zx}{yz}}+\frac{\frac{z}{xy}+\frac{1}{x}}{\frac{zx}{yz}+\frac{xy}{zx}}=\frac{\frac{1}{y}\left(\frac{x}{z}+1\right)}{\frac{y}{z}+\frac{z}{x}}+\frac{\frac{1}{z}\left(\frac{y}{x}+1\right)}{\frac{z}{x}+\frac{x}{y}}+\frac{\frac{1}{x}\left(\frac{z}{y}+1\right)}{\frac{x}{y}+\frac{y}{z}}$$

$$=\frac{\frac{x}{y}\left(\frac{1}{z}+\frac{1}{x}\right)}{\frac{y}{z}+\frac{z}{x}}+\frac{\frac{y}{z}\left(\frac{1}{x}+\frac{1}{y}\right)}{\frac{z}{x}+\frac{x}{y}}+\frac{\frac{z}{x}\left(\frac{1}{y}+\frac{1}{z}\right)}{\frac{x}{y}+\frac{y}{z}}$$

$$=\frac{X}{Y+Z}(B+C)+\frac{Y}{Z+X}(C+A)+\frac{Z}{X+Y}(A+B)$$

$$\left(X=\frac{x}{y},Y=\frac{y}{z},Z=\frac{z}{x},A=\frac{1}{y},B=\frac{1}{z},C=\frac{1}{x}\right)$$

$$=\frac{X}{Y+Z}.\sqrt{B+C}^2+\frac{Y}{Z+X}.\sqrt{C+A}^2+\frac{Z}{X+Y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB.\frac{\sqrt{3}}{2}}=\sqrt{3\sum_{cyc}\frac{1}{yz}}$$

$$\overset{A-G}{\geq}\sqrt{3}.\sqrt{3.\sqrt[3]{\frac{1}{x^2y^2z^2}}}\overset{xyz=1}{=}3\ \therefore\ \frac{a^4+a^2c^2}{a^2b^4+b^2c^4}+\frac{b^4+a^2b^2}{b^2c^4+c^2a^4}+\frac{c^4+b^2c^2}{c^2a^4+a^2b^4}\geq 3$$

$$\forall\, a,b,c>0\mid abc=1,''=''\text{ iff } a=b=c=1\ (QED)$$

1479. If $a,b,c>0$ *with* $a^2+b^2+c^2+abc=4$, then :

$$a+b+c+ab+bc+ca\leq 6$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-India

$$\text{Now, } a^2+b^2+c^2+abc=4\Rightarrow 4-a^2\overset{A-G}{\geq}2bc+abc=bc(2+a)$$

$$\Rightarrow 2-a\geq bc>0\Rightarrow(2-a-bc)(2-a)\geq 0\Rightarrow 4-4a+a^2-2bc+abc\geq 0$$

$$\Rightarrow 4a+2bc\leq 4+a^2+abc\ \text{and analogs}\Rightarrow 4\sum_{cyc}a+2\sum_{cyc}ab\leq$$

$$12+\sum_{cyc}a^2+3abc\Rightarrow 4\sum_{cyc}a+4\sum_{cyc}ab\leq 12+\sum_{cyc}a^2+2\sum_{cyc}ab+3abc$$

$$\leq 12+3\sum_{cyc}a^2+3abc\overset{a^2+b^2+c^2+abc=4}{=}12+3(4-abc)+3abc=24$$

$$\Rightarrow a+b+c+ab+bc+ca\leq 6\ \forall\, a,b,c>0\mid a^2+b^2+c^2+abc=4,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

Solution 2 by Tapas Das-India

$$a^2+b^2+c^2+abc=4 \text{ or } 4=a^2+b^2+c^2+abc \overset{AM-GM}{\geq} 4\sqrt[4]{a^3b^3c^3} \text{ or}$$

$$\sqrt[4]{a^3b^3c^3} \leq 1 \text{ or}, abc \leq 1 \ (1)$$

$$a^2+b^2+c^2+abc=4 \text{ can be written as}$$

$$\frac{a}{2a+bc}+\frac{b}{2b+ac}+\frac{c}{2c+ab}=1$$

$$\text{Now } 1=\frac{a}{2a+bc}+\frac{b}{2b+ac}+\frac{c}{2c+ab}$$

$$=\frac{a^2}{2a^2+abc}+\frac{b^2}{2b^2+abc}+\frac{c^2}{2c^2+abc} \overset{Berstrom}{\geq}$$

$$\geq \frac{(a+b+c)^2}{2(a^2+b^2+c^2)+3abc} \overset{a^2+b^2+c^2+abc=4}{=} \frac{(a+b+c)^2}{2(4-abc)+3abc}=$$

$$=\frac{(a+b+c)^2}{8+abc} \overset{(1)}{\geq} \frac{(a+b+c)^2}{8+1}=\frac{(a+b+c)^2}{9}$$

$$\text{or } 1 \geq \frac{(a+b+c)^2}{9} \text{ or } 9 \geq (a+b+c)^2$$

$$\text{or } a+b+c \leq 3 \ (2)$$

$$a+b+c+ab+bc+ca \overset{\forall x,y,z>0 \ (\sum x)^2 \geq 3\sum xy}{\leq} a+b+c+\frac{(a+b+c)^2}{3} \overset{2}{\leq} 3+\frac{3^2}{3}=6$$

$$\text{Equality holds for } a=b=c=1$$

1480. If $a,b,c>0,\ ab+bc+ca=1$ **then:**

$$\sum \frac{(b+\lambda c)^2}{a^2+a(b+c)} \geq \frac{(\lambda+1)^2}{a^2+b^2+c^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{(b+\lambda c)^2}{a^2+a(b+c)} \overset{Bergstrom}{\geq} \frac{\left(a+b+c+\lambda(a+b+c)\right)^2}{a^2+b^2+c^2+a(b+c)+b(c+a)+c(a+b)}=$$

$$=\frac{(\lambda+1)^2(a+b+c)^2}{a^2+b^2+c^2+2ab+2bc+2ca}=\frac{(\lambda+1)^2(a+b+c)^2}{(a+b+c)^2}=$$

$$=\frac{(\lambda+1)^2}{1}=\frac{(\lambda+1)^2}{ab+bc+ca} \ (as\ ab+bc+ca=1) \geq \frac{(\lambda+1)^2}{a^2+b^2+c^2}$$

$$Equality\ holds\ for\ \ a=b=c=\frac{1}{\sqrt{3}}$$

1481. If $x,y,z\ge 1$ and $\lambda\ge 0$, then :

$$\sum_{cyc}\frac{x^3+3x}{3(y^2+\lambda z^2)+\lambda+1}\ge\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$x^3+3x\overset{?}{\ge}3x^2+1\Leftrightarrow(x-1)^3\overset{?}{\ge}0\to\text{true}\because x\ge 1$$
$$\therefore x^3+3x\ge 3x^2+1\ and\ analogs$$

$$\therefore\sum_{cyc}\frac{x^3+3x}{3(y^2+\lambda z^2)+\lambda+1}\ge\sum_{cyc}\frac{3x^2+1}{3y^2+1+\lambda(3z^2+1)}=$$

$$=\sum_{cyc}\frac{a^2}{ab+\lambda ca}\ (a=3x^2+1,b=3y^2+1,c=3z^2+1)\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a\right)^2}{(\lambda+1)\left(\sum_{cyc}ab\right)}\ge$$

$$\ge\frac{\left(\sum_{cyc}a\right)^2}{(\lambda+1)\left(\frac{\left(\sum_{cyc}a\right)^2}{3}\right)}\therefore\sum_{cyc}\frac{x^3+3x}{3(y^2+\lambda z^2)+\lambda+1}\ge\frac{3}{\lambda+1}$$

$$\forall x,y,z\ge 1\ and\ \lambda\ge 0,''=''\ \text{ iff } x=y=z=1\ (\text{QED})$$

1482. If $a,b,c>0$, $ab+bc+ca=3$ and $\lambda\ge 1$, then:

$$\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}\ge 3(\lambda+1)$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}=\sum_{cyc}\sqrt{[(\lambda-1)a+(a+b)][(\lambda-1)a+(a+c)]}\ge$$

$$\geq \sum_{cyc}\left((\lambda-1)a+\sqrt{(a+b)(a+c)}\right)=(\lambda-1)\sum_{cyc}a+\sum_{cyc}\sqrt{a^2+3}\geq$$

$$\geq(\lambda-1)\sum_{cyc}a+\sum_{cyc}\frac{a+3}{\sqrt{1+3}}=\left(\lambda-\frac{1}{2}\right)\sum_{cyc}a+\frac{9}{2}\geq\left(\lambda-\frac{1}{2}\right)\sqrt{3\sum_{cyc}bc}+\frac{9}{2}=3(\lambda+1).$$

Equality holds iff $a=b=c=1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The desired inequality is successively equivalent to

$$2\sum_{cyc}\sqrt{(\lambda a+b)(\lambda a+c)}\geq 2(\lambda+1)\sqrt{3(ab+bc+ca)}$$

$$2(\lambda+1)\left(a+b+c-\sqrt{3(ab+bc+ca)}\right)$$
$$\geq\sum_{cyc}\left[(\lambda a+b)+(\lambda a+c)-2\sqrt{(\lambda a+b)(\lambda a+c)}\right]$$

$$\frac{2(\lambda+1)(a^2+b^2+c^2-ab-bc-ca)}{a+b+c+\sqrt{3(ab+bc+ca)}}\geq\sum_{cyc}\left(\sqrt{\lambda a+b}-\sqrt{\lambda a+c}\right)^2$$

$$\frac{(\lambda+1)[(a-b)^2+(b-c)^2+(c-a)^2]}{a+b+c+\sqrt{3(ab+bc+ca)}}\geq\sum_{cyc}\left(\frac{b-c}{\sqrt{\lambda a+b}+\sqrt{\lambda a+c}}\right)^2,$$

$$\sum_{cyc}\left(\frac{\lambda+1}{a+b+c+\sqrt{3(ab+bc+ca)}}-\frac{1}{\left(\sqrt{\lambda a+b}+\sqrt{\lambda a+c}\right)^2}\right)(b-c)^2\geq 0,$$

which is true because

$$(\lambda+1)\left(\sqrt{\lambda a+b}+\sqrt{\lambda a+c}\right)^2\geq 2\left(\sqrt{a+b}+\sqrt{a+c}\right)^2\geq 2(a+b+c)$$
$$\geq a+b+c+\sqrt{3(ab+bc+ca)}\ \text{(and analogs)}$$

Equality holds iff $a=b=c=1$.

1483. **If $a,b,c>0$ and $a^2+b^2+c^2=1$, then prove that :**

$$(1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)\leq\frac{9}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

If $(1+9abc-a-b-c)\leq 0$, then : LHS ≤ 0

$$\left(\because \sum_{cyc} a^2 = 1 \wedge a,b,c > 0 \Rightarrow a,b,c < 1 \Rightarrow (1-ab),(1-bc),(1-ca) > 0 \atop \Rightarrow \frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca} > 0\right)$$

$< \frac{9}{2}$ and so, we now concentrate on the scenario when : $(1+9abc-a-b-c)$

$$> 0 \text{ and then : } (1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)$$

$$= (1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{a^2+b^2+c^2-bc}\right) \le$$

$$(1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{a^2+\frac{(b+c)^2}{4}}\right) \overset{A-G}{\le}$$

$$(1+9abc-a-b-c)\left(\sum_{cyc}\frac{1}{\frac{1}{2}\left(a+\frac{b+c}{2}\right)^2}\right)$$

$$= 8(1+9abc-a-b-c).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2} \overset{?}{\le} \frac{9}{2} \overset{a^2+b^2+c^2=1}{\Leftrightarrow}$$

$$8\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2} \boxed{\overset{?}{\underset{(*)}{\le}}}$$

$$8\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)-9abc\right).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}+\frac{9}{2}$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a > 0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$$\Rightarrow abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (2) \text{ and } \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (2)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (3)$$

Now, via (1), (2) and (3), $8\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)-9abc\right).\sum_{cyc}\frac{1}{\left((c+a)+(a+b)\right)^2}$

$$= 8(s(s^2-8Rr-2r^2)-9r^2s).\sum_{cyc}\frac{1}{(y+z)^2} \geq (s^2-8Rr-11r^2).\sum_{cyc}\frac{1}{(y+z)^2}$$

$$\left(\begin{array}{c} \because s^2-8Rr-11r^2 = s^2-16Rr+5r^2+8r(R-2r) \overset{\substack{\text{Gerretsen}\\ and\\ \text{Euler}}}{\geq} 0 \text{ and} \\ s = \sum_{cyc} a = \sqrt{\sum_{cyc} a^2 + 2\sum_{cyc} ab} > \sqrt{\sum_{cyc} a^2} \overset{a^2+b^2+c^2=1}{=} 1 \end{array}\right)$$

$$= 8(s^2-8Rr-11r^2).\left(\left(\sum_{cyc}\frac{1}{y+z}\right)^2 - 2\sum_{cyc}\frac{1}{(y+z)(z+x)}\right)$$

$$= 8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2}{4s^2(s^2+2Rr+r^2)^2} - \frac{2.4s}{2s(s^2+2Rr+r^2)}\right)$$

$$= 8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$\therefore \text{RHS of } (*) - \text{LHS of } (*) \overset{a^2+b^2+c^2=1}{\geq}$$

$$\frac{9}{2} + 8(s^2-8Rr-11r^2).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$-8\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)\left(\sum_{cyc} a^2\right) \overset{via\,(3)}{=}$$

$$\frac{9}{2} + 8\left(s^2-8Rr-11r^2-(s^2-8Rr-2r^2)\right).\left(\frac{(5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)}{4s^2(s^2+2Rr+r^2)^2}\right)$$

$$\overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{1}{2} \overset{?}{\geq} \frac{2r^2\left((5s^2+4Rr+r^2)^2-16s^2(s^2+2Rr+r^2)\right)}{s^2(s^2+2Rr+r^2)^2}$$

$$\Leftrightarrow s^6 + (4Rr-34r^2)s^4 + r^2s^2(4R^2-28Rr+25r^2) - 4r^4(4R+r)^2 \boxed{\overset{?}{\underset{(**)}{\geq}}} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (20Rr-39r^2)s^4 + r^2s^2(4R^2-28Rr+25r^2)$$

$$-4r^4(4R+r)^2 \overset{\text{Gerretsen}}{\geq} \left((20Rr-39r^2)(16Rr-5r^2)+r^2(4R^2-28Rr+25r^2)\right)s^2$$

$$-4r^4(4R+r)^2 = 4r^2\left((81R^2-188Rr+55r^2)s^2 - r^2(4R+r)^2\right) \overset{\text{Gerretsen}}{\geq}$$

$$4r^2\left((81R^2-188Rr+55r^2)(16Rr-5r^2) - r^2(4R+r)^2\right)$$

$$\left(\because 81R^2-188Rr+55r^2 = (R-2r)(81R-26r)+3r^2 \overset{\text{Euler}}{\geq} 3r^2 > 0\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow 432t^3 - 1143t^2 + 604t - 92 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(432t^2-279t+46) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore (1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)\le\frac{9}{2}$$

$$\forall\, a,b,c>0\,\middle|\, a^2+b^2+c^2=1,''=''\text{ iff } a=b=c=\frac{1}{\sqrt{3}}\text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$a+b+c=(a^2+b^2+c^2)(a+b+c)\ge 3\sqrt[3]{a^2b^2c^2}.3\sqrt[3]{abc}=9abc$$

$$\Rightarrow\ 1+9abc-a-b-c\le 1.$$

Now, let $p:=a+b+c, q:=ab+bc+ca\le 1, r:=abc$. We have $p^2=1+2q$.

By Schur's inequality, we have

$$r\ge\frac{p(4q-p^2)}{9}=\frac{(1+2q)(2q-1)}{p.9}=\frac{4q^2-1}{9p}\quad(1)$$

$$\sum_{cyc}\frac{1}{1-bc}=\sum_{cyc}\left(2-\frac{1-2bc}{1-bc}\right)$$

$$=6-\sum_{cyc}\frac{1-2bc}{1-bc}\overset{CBS\ \&\ 1\ge 2bc}{\le}6-\frac{\left(\sum_{cyc}(1-2bc)\right)^2}{\sum_{cyc}(1-2bc)(1-bc)}$$

$$=6-\frac{(3-2q)^2}{3-3q+2(q^2-2pr)}\overset{(1)}{\le}6-\frac{(3-2q)^2}{3-3q+2q^2-\frac{4(4q^2-1)}{9}}=\frac{3(35-18q-8q^2)}{31-27q+2q^2}$$

$$=\frac{9}{2}-\frac{3(1-q)(23-22q)}{31-27q+2q^2}\overset{q\le 1}{\le}\frac{9}{2}.$$

Therefore

$$(1+9abc-a-b-c)\left(\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\right)\le 1.\frac{9}{2}=\frac{9}{2}.$$

$$\text{Equality holds iff } a=b=c=\frac{\sqrt{3}}{3}.$$

1484. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5\ge(a+b)(b+c)(c+a)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{a.1.1} \overset{\text{GM-HM}}{\geq} \frac{3a}{2a+1} \text{ and analogs} \Rightarrow \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5 \geq$$

$$\sum_{\text{cyc}} \frac{3a^2}{2a^2+a}+5 \overset{\text{Bergstrom}}{\geq} \frac{3\left(\sum_{\text{cyc}} a\right)^2}{2\sum_{\text{cyc}} a^2+\sum_{\text{cyc}} a}+5 \overset{a+b+c=3}{=}$$

$$\frac{\frac{3}{27}\left(\sum_{\text{cyc}} a\right)^5}{2\sum_{\text{cyc}} a^2+\frac{\left(\sum_{\text{cyc}} a\right)^2}{3}}+\frac{5}{27}\cdot\left(\sum_{\text{cyc}} a\right)^3 = \frac{\left(\sum_{\text{cyc}} a\right)^5}{18\sum_{\text{cyc}} a^2+3\left(\sum_{\text{cyc}} a\right)^2}+\frac{5}{27}\cdot\left(\sum_{\text{cyc}} a\right)^3$$

$$\overset{?}{\geq} (a+b)(b+c)(c+a)$$

$$\Leftrightarrow \frac{27\left(\sum_{\text{cyc}} a\right)^5}{18\sum_{\text{cyc}} a^2+3\left(\sum_{\text{cyc}} a\right)^2}+5\left(\sum_{\text{cyc}} a\right)^3 \boxed{\overset{?}{\underset{(*)}{\geq}}} 27(a+b)(b+c)(c+a)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{\text{cyc}} a = \sum_{\text{cyc}} x = 2s \Rightarrow \sum_{\text{cyc}} a = s \rightarrow (1) \Rightarrow a = s-x, b=s-y, c=s-z$ and

such substitutions $\Rightarrow \sum_{\text{cyc}} ab = \sum_{\text{cyc}} (s-x)(s-y) \Rightarrow \sum_{\text{cyc}} ab = 4Rr+r^2 \rightarrow (2)$ and

$$\sum_{\text{cyc}} a^2 = \left(\sum_{\text{cyc}} a\right)^2 - 2\sum_{\text{cyc}} ab \overset{\text{via (1) and (2)}}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{\text{cyc}} a^2 = s^2-8Rr-2r^2 \rightarrow (3)$$

Now, via (1) and (3), $(*) \Leftrightarrow \frac{27s^5}{18(s^2-8Rr-2r^2)+3s^2}+5s^3 \geq 27.4Rrs$

$$\Leftrightarrow 11s^4-(249Rr+15r^2)s^2+324Rr^2(4R+r) \boxed{\overset{(**)}{\geq}} 0 \text{ and } \because$$

$11(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0, \therefore$ in order to prove $(**)$, it suffices to prove :

$$\text{LHS of } (**) \geq 11(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (103R-125r)s^2 \boxed{\overset{(***)}{\geq}} r(1520R^2-2084Rr+275r^2)$$

Again, $(103R-125r)s^2 \overset{\text{Gerretsen}}{\geq} (103R-125r)(16Rr-5r^2) \overset{?}{\geq}$

$$r(1520R^2-2084Rr+275r^2) \Leftrightarrow 128R^2-431Rr+350r^2 \overset{?}{\geq} 0$$

$\Leftrightarrow (R-2r)(128R-175r) \overset{?}{\geq} 0 \rightarrow$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true

$$\therefore \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}+5 \geq (a+b)(b+c)(c+a)$$

$$\forall a,b,c>0 \mid a+b+c=3, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

$$a^3 + 3\sqrt[3]{a} \geq 4\sqrt[4]{a^3.\left(\sqrt[3]{a}\right)^3} = 4a \;\Rightarrow\; \sqrt[3]{a} \geq \frac{4a - a^3}{3} \quad \text{(and analogs)}$$

Then

$$\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + 5 \geq \frac{4(a+b+c) - (a^3+b^3+c^3)}{3} + 5$$

$$= \frac{4.3 - (a+b+c)^3 + 3(a+b)(b+c)(c+a)}{3} + 5$$

$$= (a+b)(b+c)(c+a),$$

Equality holds iff $a = b = c = 1$.

1485. If $x, y, z > 0$ and $xy + yz + zx \leq 3$, then prove that :

$$\frac{2}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

$$\frac{2}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} = \frac{1}{\sqrt{xyz}} + \frac{1}{\sqrt{xyz}} + \frac{27}{(2x+y)(2y+z)(2z+x)} \geq$$

$$\geq 3\sqrt[3]{\left(\frac{1}{\sqrt{xyz}}\right)^2 \cdot \frac{27}{(2x+y)(2y+z)(2z+x)}} = \frac{9}{\sqrt[3]{z(2x+y).x(2y+z).y(2z+x)}} \geq$$

$$\geq \frac{3.9}{z(2x+y) + x(2y+z) + y(2z+x)} = \frac{9}{xy+yz+zx} \geq 3.$$

Equality holds iff $x = y = z = 1$.

1486. If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}\ge\frac{1}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}=$$

$$=\sum_{cyc}\frac{1}{(1+1+x)\left(1+(1+x)^2-(1+x)\right)}=\frac{1}{2}\sum_{cyc}\frac{(2+x)-x}{(x+2)(x^2+x+1)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{1}{x^2+x+1}-\frac{1}{2}\sum_{cyc}\frac{x}{(x+2)(x^2+x+1)}\overset{AM-GM}{\ge}$$

$$\ge\frac{1}{2}+\frac{\sum_{cyc}\left((y^2+y+1)(z^2+z+1)\right)-\prod_{cyc}(x^2+x+1)}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{2}\sum_{cyc}\frac{x}{(x+2)(3x)}=$$

$$=\frac{1}{2}+\frac{\sum_{cyc}x^2+\sum_{cyc}x+2-xyz-x^2y^2z^2-xyz\sum_{cyc}x-xyz\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\sum_{cyc}\frac{1}{x+2}\overset{xyz=1}{=}$$

$$=\frac{1}{2}+\frac{\sum_{cyc}x^2+\sum_{cyc}x+2-1-1-\sum_{cyc}x-\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\cdot\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}=$$

$$=\frac{1}{3}+\frac{1}{6}+\frac{\sum_{cyc}x^2-\sum_{cyc}xy}{2(x^2+x+1)(y^2+y+1)(z^2+z+1)}-\frac{1}{6}\cdot\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\ge$$

$$\ge\frac{1}{3}+\frac{1}{6}\left(1-\frac{\sum_{cyc}xy+4\sum_{cyc}x+12}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\right)=$$

$$=\frac{1}{3}+\frac{\sum_{cyc}xy-3}{9+2\sum_{cyc}xy+4\sum_{cyc}x}\ge\frac{1}{3}\left(\because\sum_{cyc}xy\overset{AM-GM}{\ge}3.\sqrt[3]{x^2y^2z^2}\overset{xyz=1}{=}3\right)$$

$$\therefore\frac{1}{1+(1+x)^3}+\frac{1}{1+(1+y)^3}+\frac{1}{1+(1+z)^3}\ge\frac{1}{3}$$

$$\forall\, x,y,z>0\,\big|\, xyz=1,''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

1487. If $a, b, c \in \mathbb{R}$ and $a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3)\ge 8$

then prove that : $a^2+b^2+c^2\ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Firstly}, \forall\, b, c \in \mathbb{R}, (b^2+c^2)^5 \overset{?}{\geq} 8b^2c^2(b^3+c^3)^2$$

$$\Leftrightarrow \frac{(b^2+c^2)^5}{c^{10}} \overset{?}{\geq} \frac{8b^2c^2(b^3+c^3)^2}{c^{10}} \Leftrightarrow (x^2+1)^5 \overset{?}{\geq} 8x^2(x^3+1)^2 \left(x = \frac{b}{c}\right)$$

$$\Leftrightarrow x^{10} - 3x^8 + 10x^6 - 16x^5 + 10x^4 - 3x^2 + 1 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-1)^2\left(x^8 + 2x^7 - 2x^5 + 6x^4 - 2x^3 + 2x + 1\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-1)^2\left(4x^4 + \frac{4\left(x^8+2x^7-2x^5+x^4\right)}{4} + \frac{4(x^4-2x^3+2x+1)}{4}\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-1)^2\left(4x^4 + \frac{x^4\left((2x-1)^2\left(\left(x+\frac{3}{2}\right)^2+\frac{1}{2}\right)+\frac{5}{4}\right)}{4} + \frac{(2x+1)^2\left(\left(x-\frac{3}{2}\right)^2+\frac{1}{2}\right)+\frac{5}{4}}{4}\right) \overset{?}{\geq} 0 \to \text{true } \forall\, x \in \mathbb{R}$$

$$\therefore b^2c^2(b^3+c^3)^2 \leq \frac{(b^2+c^2)^5}{8} \text{ and analogs } \forall\, a, b, c \in \mathbb{R} \to (1)$$

$$\text{Now, } a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3) \geq 8$$

$$\Rightarrow 64 \leq a^4b^4c^4(a^3+b^3)^2(b^3+c^3)^2(c^3+a^3)^2 =$$

$$= \prod_{cyc}\left(b^2c^2(b^3+c^3)^2\right) \overset{\text{via }(1)}{\leq} \prod_{cyc}\frac{(b^2+c^2)^5}{8}$$

$$\Rightarrow 2^{15} \leq \prod_{cyc}(b^2+c^2)^5 \Rightarrow 8 \leq \prod_{cyc}(b^2+c^2)$$

$$\Rightarrow 2 \leq \sqrt[3]{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \overset{A-G}{\leq} \frac{2(a^2+b^2+c^2)}{3}$$

$$\therefore a^2+b^2+c^2 \geq 3\ \forall\, a, b, c \in \mathbb{R} \,\middle|\, a^2b^2c^2(a^3+b^3)(b^3+c^3)(c^3+a^3) \geq 8$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1488. **If $0 < b < a \leq 4$ and $2ab \leq 3a + 4b$, then prove that :**

$$a^2 + b^2 \leq 25$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } b \le 3, \text{then}: a^2 + b^2 \le 4^2 + 3^2 = 25 \text{ and so, we now focus on the}$$

$$\text{scenario when}: \boxed{b > 3} \text{ and then}: b > \frac{3}{2} \Rightarrow 2b - 3 > 0 \therefore 2ab \le 3a + 4b$$

$$\Rightarrow a(2b-3) \le 4b \Rightarrow 0 < b < a \overset{(*)}{\le} \frac{4b}{2b-3} \Rightarrow 2b - 3 < 4 \Rightarrow b \overset{(**)}{<} \frac{7}{2}$$

$$\text{Now, via } (*), a^2 + b^2 \le \frac{16b^2}{(2b-3)^2} + b^2 \overset{?}{<} 25$$

$$\Leftrightarrow 4b^4 - 12b^3 - 75b^2 + 300b - 225 \overset{?}{<} 0 \Leftrightarrow (b-3)(4b^3 - 75b + 75) \overset{?}{<} 0$$

$$\Leftrightarrow (b-3)\big((b-3)(2b-7)(2b+13) + 26(2b-7) - 16\big) \overset{?}{<} 0 \to \text{true}$$

$$\because (b-3) > 0 \text{ and } (b-3)(2b-7)(2b+13) + 26(2b-7) - 16 < 0 \text{ via } (**)$$

$$\therefore a^2 + b^2 < 25 \text{ and combining all cases}, a^2 + b^2 \le 25$$

$$\text{for } 0 < b < a \le 4 \wedge 2ab \le 3a + 4b, '' ='' \text{ iff } (a = 4, b = 3) \text{ (QED)}$$

1489. **If $a + b + c = a^3 + b^3 + c^3 - 3abc = 2$, then prove that:**

$$max\{a,b,c\} - min\{a,b,c\} \le \frac{2\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have: } 2 = a^3 + b^3 + c^3 - 3abc$$
$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca), \text{then}$$
$$a^2+b^2+c^2-ab-bc-ca = 1,$$
$$\text{WLOG, we assume that } a \ge b \ge c. \text{ We will prove that}$$

$$a - c \le 2\sqrt{\frac{a^2+b^2+c^2-ab-bc-ca}{3}}. \quad (1)$$

$$RHS_{(1)} = \sqrt{\frac{2}{3}[(a-b)^2 + (b-c)^2 + (c-a)^2]} \ge \sqrt{\frac{2}{3}\left[\frac{[(a-b)+(b-c)]^2}{2} + (c-a)^2\right]}$$
$$= a - c,$$

$$\text{Equality holds iff } a = \frac{2+\sqrt{3}}{3}, b = \frac{2}{3}, c = \frac{2-\sqrt{3}}{3} \text{ and permutation.}$$

1490.

Let $a, b, c \ge 0, ab + bc + ca > 0$. Prove that:

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{6(a^3+b^3+c^3)+9abc}{(a+b+c)^2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(a+b+c)^2\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right)=\sum_{cyc}[(b+c)^2+2a(b+c)+a^2].\frac{b^2+c^2}{b+c}$$

$$=\sum_{cyc}(b+c)(b^2+c^2)+2\sum_{cyc}a(b^2+c^2)+\sum_{cyc}a^2.\frac{(b+c)^2-2bc}{b+c}$$

$$=2\sum_{cyc}a^3+3\sum_{cyc}a^2(b+c)+\sum_{cyc}\left(a^2(b+c)-2abc.\frac{a}{b+c}\right)$$

$$\overset{Nesbitt}{\le}2\sum_{cyc}a^3+4\sum_{cyc}a^2(b+c)-2abc.\frac{3}{2}\overset{Schur}{\le}2\sum_{cyc}a^3+4\left(\sum_{cyc}a^3+3abc\right)-3abc$$

$$=6(a^3+b^3+c^3)+9abc,$$

as desired. Equality holds iff $(a=b=c>0)$ and

$(a=b>0,c=0)$ and permutation.

1491. If $a,b,c>0,then:$

$$\frac{a^5+a^2c^3}{a^3b^4c+ab^2c^5}+\frac{b^5+b^2a^3}{ab^3c^4+a^5bc^2}+\frac{c^5+c^2b^3}{a^4bc^3+a^2b^5c}\ge\frac{3}{abc}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ *and analogs*$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with *area* F (*say*) *and* $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \to (1)$$

$$\text{Now, } \forall\, x, y, z > 0,\ \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc} \frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have: } \frac{a^5 + a^2c^3}{a^3b^4c + ab^2c^5} + \frac{b^5 + b^2a^3}{ab^3c^4 + a^5bc^2} + \frac{c^5 + c^2b^3}{a^4bc^3 + a^2b^5c}$$

$$= \frac{a^2(a^3 + c^3)}{ab^2c(a^2b^2 + c^4)} + \frac{b^2(b^3 + a^3)}{abc^2(b^2c^2 + a^4)} + \frac{c^2(c^3 + b^3)}{a^2bc(c^2a^2 + b^4)}$$

$$= \frac{\frac{a^2}{b^2}\cdot\left(\frac{a^3 + c^3}{ac}\right)}{a^2b^2c^2\left(\frac{1}{c^2} + \frac{c^2}{a^2b^2}\right)} + \frac{\frac{b^2}{c^2}\cdot\left(\frac{b^3 + a^3}{ab}\right)}{a^2b^2c^2\left(\frac{1}{a^2} + \frac{a^2}{b^2c^2}\right)} + \frac{\frac{c^2}{a^2}\cdot\left(\frac{c^3 + b^3}{bc}\right)}{a^2b^2c^2\left(\frac{1}{b^2} + \frac{b^2}{c^2a^2}\right)}$$

$$= \frac{\frac{a^2}{b^2}\cdot\left(\frac{a^3 + c^3}{a^3c^3}\right)}{\frac{b^2}{c^2} + \frac{c^2}{a^2}} + \frac{\frac{b^2}{c^2}\cdot\left(\frac{b^3 + a^3}{a^3b^3}\right)}{\frac{c^2}{a^2} + \frac{a^2}{b^2}} + \frac{\frac{c^2}{a^2}\cdot\left(\frac{c^3 + b^3}{b^3c^3}\right)}{\frac{a^2}{b^2} + \frac{b^2}{c^2}}$$

$$= \frac{\frac{a^2}{b^2}\cdot\left(\frac{1}{c^3} + \frac{1}{a^3}\right)}{\frac{b^2}{c^2} + \frac{c^2}{a^2}} + \frac{\frac{b^2}{c^2}\cdot\left(\frac{1}{a^3} + \frac{1}{b^3}\right)}{\frac{c^2}{a^2} + \frac{a^2}{b^2}} + \frac{\frac{c^2}{a^2}\cdot\left(\frac{1}{b^3} + \frac{1}{c^3}\right)}{\frac{a^2}{b^2} + \frac{b^2}{c^2}}$$

$$= \frac{x}{y+z}(B + C) + \frac{y}{z+x}(C + A) + \frac{z}{x+y}(A + B)$$

$$\left(x = \frac{a^2}{b^2}, y = \frac{b^2}{c^2}, z = \frac{c^2}{a^2}, A = \frac{1}{b^3}, B = \frac{1}{c^3}, C = \frac{1}{a^3}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B + C}^2 + \frac{y}{z+x}\cdot\sqrt{C + A}^2 + \frac{z}{x+y}\cdot\sqrt{A + B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc} \frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3\sum_{cyc}\left(\frac{1}{b^3}\cdot\frac{1}{c^3}\right)} \overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{1}{a^6b^6c^6}}} \therefore \frac{a^5 + a^2c^3}{a^3b^4c + ab^2c^5} + \frac{b^5 + b^2a^3}{ab^3c^4 + a^5bc^2} + \frac{c^5 + c^2b^3}{a^4bc^3 + a^2b^5c} \geq \frac{3}{abc}$$

$$\forall\, a, b, c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

1492.

If $a,b,c>0$, then:

$$\frac{a^5}{a^4+b^4}+\frac{b^5}{b^4+c^4}+\frac{c^5}{c^4+a^4}+\frac{1}{2}\left(\frac{a^2}{c}+\frac{b^2}{a}+\frac{c^2}{b}\right)\geq a+b+c$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$\frac{a^5}{a^4+b^4}+\frac{b^2}{2a}=a-\frac{ab^4}{a^4+b^4}+\frac{b^2}{2a}\geq a-\frac{ab^4}{2a^2b^2}+\frac{b^2}{2a}=a.$$

Similarly, we get

$$\frac{b^5}{b^4+c^4}+\frac{c^2}{2b}\geq b\quad\text{and}\quad\frac{c^5}{c^4+a^4}+\frac{a^2}{2c}\geq c.$$

Adding these inequalities yields the desired inequality.

Equality holds iff $a=b=c$.

1493. If $x,y,z>0$ then:

$$\sum\frac{x^2+2\lambda+1}{y+\lambda}\geq 6$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{x^2+2\lambda+1}{y+\lambda}=\frac{(x^2+1)+2\lambda}{y+\lambda}\overset{AM-GM}{\geq}\frac{(2x)+2\lambda}{y+\lambda}=\frac{2(x+\lambda)}{y+\lambda}\ (1)$$

$$\sum\frac{x^2+2\lambda+1}{y+\lambda}\overset{(1)}{\geq}\sum\frac{2(x+\lambda)}{y+\lambda}=2\sum\frac{(x+\lambda)}{y+\lambda}\overset{AM-GM}{\geq}$$

$$\geq 6\sqrt[3]{\frac{(x+\lambda)}{y+\lambda}\frac{(y+\lambda)}{z+\lambda}\frac{(z+\lambda)}{x+\lambda}}=6$$

Equality holds for $x = y = z$

1494. If $a, \mathrm{b} > 0$ with $a\mathrm{b} = 1$ and $0 \le \lambda \le 10$, then :

$$\frac{1}{a^3(\mathrm{b}+\lambda)} + \frac{1}{\mathrm{b}^3(a+\lambda)} + \frac{1}{a+\mathrm{b}} \ge \frac{\lambda+5}{2(\lambda+1)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3(\mathrm{b}+\lambda)} + \frac{1}{\mathrm{b}^3(a+\lambda)} + \frac{1}{a+\mathrm{b}} \overset{ab=1}{=} \frac{\mathrm{b}^3}{\mathrm{b}+\lambda} + \frac{a^3}{a+\lambda} + \frac{1}{a+\mathrm{b}} \overset{ab=1}{=}$$

$$= \frac{\mathrm{b}^2}{1+\lambda a} + \frac{a^2}{1+\lambda \mathrm{b}} + \frac{1}{a+\mathrm{b}} \overset{ab=1}{=} \frac{\lambda\left((a+\mathrm{b})^3 - 3(a+\mathrm{b})\right) + (a+\mathrm{b})^2 - 2}{1+\lambda(a+\mathrm{b})+\lambda^2} + \frac{1}{a+\mathrm{b}}$$

$$= \frac{\lambda(\mathrm{t}^4 - 3\mathrm{t}^2) + \mathrm{t}^3 - 2\mathrm{t} + 1 + \lambda^2 + \lambda\mathrm{t}}{1+\lambda^2+\lambda\mathrm{t}} \ (\mathrm{t} = a+\mathrm{b}) \boxed{\overset{?}{\ge}} \frac{\lambda+5}{2(\lambda+1)}$$

$$\Leftrightarrow -\lambda^3(\mathrm{t}-2) + \lambda^2(2\mathrm{t}^4 - 7\mathrm{t}^2 - 3\mathrm{t} + 2) + \lambda(2\mathrm{t}^4 + 2\mathrm{t}^3 - 11\mathrm{t}^2 - 3\mathrm{t} + 2) + 2\mathrm{t}^3 - 9\mathrm{t} + 2 \ge$$
$$\ge -10\lambda^2(\mathrm{t}-2) + \lambda^2(\mathrm{t}-2)(2\mathrm{t}^3 + 4\mathrm{t}^2 + \mathrm{t} - 1) + \lambda(\mathrm{t}-2)(2\mathrm{t}^3 + 6\mathrm{t}^2 + \mathrm{t} - 1)$$
$$+(\mathrm{t}-2)(2\mathrm{t}^2 + 4\mathrm{t} - 1) \left(\because \mathrm{t} = a+\mathrm{b} \overset{A-G}{\ge} 2\sqrt{a\mathrm{b}} \overset{ab=1}{=} 2 \text{ and } -\lambda^3 \overset{\lambda \le 10}{\ge} -10\lambda^2\right) =$$

$$= (\mathrm{t}-2)(\lambda^2(2\mathrm{t}^3 + 4\mathrm{t}^2 + \mathrm{t} - 11) + \lambda(2\mathrm{t}^3 + 6\mathrm{t}^2 + \mathrm{t} - 1) + 2\mathrm{t}^2 + 4\mathrm{t} - 1)$$

$$= (\mathrm{t}-2)\left(\begin{matrix}\lambda^2(2\mathrm{t}^3 + 4(\mathrm{t}^2 - 4) + \mathrm{t} + 5) + \lambda(2\mathrm{t}^3 + 6\mathrm{t}^2 + (\mathrm{t}-2) + 1) \\ +2\mathrm{t}^2 + 3\mathrm{t} + (\mathrm{t}-2) + 1\end{matrix}\right) \overset{\substack{\mathrm{t}\ge 2 \\ and \\ \lambda \ge 0}}{\ge} 0$$

$$\therefore \frac{1}{a^3(\mathrm{b}+\lambda)} + \frac{1}{\mathrm{b}^3(a+\lambda)} + \frac{1}{a+\mathrm{b}} \ge \frac{\lambda+5}{2(\lambda+1)}$$

$$\forall a, \mathrm{b} > 0 \,\middle|\, a\mathrm{b} = 1 \text{ and } 0 \le \lambda \le 10, '' ='' \text{ iff } a = \mathrm{b} = 1 \text{ (QED)}$$

1495. Let $a, b, c \ge 0, ab + bc + ca > 0$. Prove that :

$$\frac{a^2+b^2-c^2}{a+b} + \frac{b^2+c^2-a^2}{b+c} + \frac{c^2+a^2-b^2}{c+a} \le \frac{a+b+c}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Multiplying the both sides of the desired inequality by $2(a+b)(b+c)(c+a)$, we get the equivalent inequality

$$2\sum_{cyc}(b^2+c^2-a^2)(a+b)(a+c) \le (a+b+c)(a+b)(b+c)(c+a),$$

which, after expanding and simplifying, is equivalent to

$$\sum_{cyc} a^3(b+c) + 2\sum_{cyc}(bc)^2 \le 2\sum_{cyc} a^4 + 2abc\sum_{cyc} a$$

$$\Leftrightarrow\ 0 \le 2\sum_{cyc} a^2(a-b)(a-c) + \sum_{cyc} bc(b-c)^2,$$

which is true by fourth degree Schur's inequality.

Equality holds iff $(a=b=c>0)$ and $(a=b>0, c=0)$ and permutation.

1496. If $a, \mathrm{b}, \mathrm{c} > 0$, *then* :

$$\frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right)+\frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right)+\frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right)\ge 3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall\, A, B, C > 0, (A+B), (B+C), (C+A)$ **form sides of a triangle**

$(\because (A+B)+(B+C) > (C+A)$ *and analogs*$) \Rightarrow \sqrt{A+B}, \sqrt{B+C}, \sqrt{C+A}$ **form sides of a triangle with *area* F (*say*) *and* $16F^2 =$**

$$2\sum_{cyc}(A+B)(B+C) - \sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc} AB + B^2\right) - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB$$

$$= 6\sum_{cyc} AB + 2\sum_{cyc} A^2 - 2\sum_{cyc} A^2 - 2\sum_{cyc} AB \Rightarrow 4F = 2\sqrt{\sum_{cyc} AB} \rightarrow (1)$$

$$\text{Now, } \forall\, x, y, z > 0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\ge} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\ge}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc} xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc} xy + z^2\right)\right)} = \frac{\left(\sum_{cyc} xy\right)^2}{\left(\sum_{cyc} xy\right)^2 + xyz\sum_{cyc} x}$$

$$\overset{?}{\geq} \frac{3}{4} \Leftrightarrow \left(\sum_{cyc} xy\right)^2 \overset{?}{\geq} 3xyz\sum_{cyc} x \to \text{true} \therefore \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \to (2)$$

$$\text{We have}: \frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right) + \frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right) + \frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right)$$

$$= \frac{a^3b^3}{c^3a^3+b^3c^3}\cdot\left(\frac{b^2c^2+a^4}{a^2b^2}\right) + \frac{b^3c^3}{a^3b^3+c^3a^3}\cdot\left(\frac{c^2a^2+b^4}{b^2c^2}\right)$$

$$+ \frac{c^3a^3}{b^3c^3+a^3b^3}\cdot\left(\frac{a^2b^2+c^4}{c^2a^2}\right)$$

$$= \frac{a^3b^3}{b^3c^3+c^3a^3}\cdot\left(\frac{c^2}{a^2}+\frac{a^2}{b^2}\right) + \frac{b^3c^3}{c^3a^3+a^3b^3}\cdot\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}\right) + \frac{c^3a^3}{a^3b^3+b^3c^3}\cdot\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)$$

$$= \frac{x}{y+z}(B+C) + \frac{y}{z+x}(C+A) + \frac{z}{x+y}(A+B)$$

$$\left(x = a^3b^3, y = b^3c^3, z = c^3a^3, A = \frac{b^2}{c^2}, B = \frac{c^2}{a^2}, C = \frac{a^2}{b^2}\right)$$

$$= \frac{x}{y+z}\cdot\sqrt{B+C}^2 + \frac{y}{z+x}\cdot\sqrt{C+A}^2 + \frac{z}{x+y}\cdot\sqrt{A+B}^2 \overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{\text{via (1) and (2)}}{\geq} 2\sqrt{\sum_{cyc} AB}\cdot\frac{\sqrt{3}}{2} = \sqrt{3\sum_{cyc}\left(\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\right)}$$

$$\overset{A-G}{\geq} 3.\sqrt{\sqrt[3]{\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}\cdot\frac{a^2}{c^2}}} \therefore \frac{ab}{a^3+b^3}\cdot\left(\frac{b^2}{c}+\frac{a^4}{c^3}\right) + \frac{bc}{b^3+c^3}\cdot\left(\frac{c^2}{a}+\frac{b^4}{a^3}\right)$$

$$+ \frac{ca}{c^3+a^3}\cdot\left(\frac{a^2}{b}+\frac{c^4}{b^3}\right) \geq 3\ \forall\ a,b,c > 0, '' ='' \text{ iff } a = b = c\ (QED)$$

1497. If $a, b > 0$ and $n \in \mathbb{N}$, then :

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{8n(a+b)}{a+b+2} \geq 2(2n+1)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } n=0, \text{LHS}-\text{RHS}=0 \Rightarrow \text{LHS}=\text{RHS} \text{ and so, we now focus on } n\in\mathbb{N}^*$$

$$\text{and } \because n\geq 1 \therefore \text{via Bernoulli}, \frac{1}{a^n}+\frac{1}{b^n}+\frac{8n(a+b)}{a+b+2}$$

$$=\left(1+\frac{1}{a}-1\right)^n+\left(1+\frac{1}{b}-1\right)^n+\frac{8n(a+b)}{a+b+2}\geq$$

$$2+n\left(\frac{1}{a}-1+\frac{1}{b}-1\right)+\frac{8n(a+b)}{a+b+2}\overset{?}{\geq}2(2n+1)\Leftrightarrow\frac{a+b-2ab}{ab}+\frac{8(a+b)}{a+b+2}\overset{?}{\geq}4$$

$$\Leftrightarrow (a+b)^2-2ab(a+b)+2(a+b)-4ab+8ab(a+b)\overset{?}{\geq}4ab(a+b)+8ab$$

$$\Leftrightarrow (a+b)^2+2(a+b)+2ab(a+b-6)\boxed{\underset{(*)}{\overset{?}{\geq}}}0$$

$\boxed{\text{Case 1}}$ $a+b-6\geq 0$ and then : LHS of $(*)\geq(a+b)^2+2(a+b)>0$
$\Rightarrow (*)$ is true (strict inequality)

$\boxed{\text{Case 2}}$ $a+b-6<0$ and then : LHS of $(*)\geq(a+b)^2+2(a+b)+$
$$\frac{(a+b)^2}{2}.(a+b-6)\left(\because 2ab\overset{A-G}{\leq}\frac{(a+b)^2}{2}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow 2t+4+t(t-6)\overset{?}{\geq}0\Leftrightarrow t^2-4t+4\overset{?}{\geq}0\Leftrightarrow(t-2)^2\overset{?}{\geq}0\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\therefore \text{combining both cases}, (*) \text{ is true } \forall a,b>0 \therefore \frac{1}{a^n}+\frac{1}{b^n}+\frac{8n(a+b)}{a+b+2}\geq 2(2n+1)$$

$$\forall a,b>0 \text{ and } n\in\mathbb{N}, ''='' \text{ iff } n=0 \text{ or } a=b \text{ (QED)}$$

1498. **If $x,y,z\in[-2,2]$, then prove that :**

$$2(x^6+y^6+z^6)-x^4y^2-y^4z^2-z^4x^2\leq 192$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$-2\leq x\leq 2\Rightarrow(x+2)(x-2)\leq 0\Rightarrow x^2-4\leq 0\Rightarrow 4-x^2\geq 0 \text{ and}$$
$$\text{similarly}, 4-y^2\geq 0 \text{ and } 4-z^2\geq 0 \text{ and setting } a=4-x^2, b=4-y^2,$$
$$c=4-z^2, \text{we notice that}$$
$$:\boxed{0\leq a,b,c\leq 4}(\because a=4-x^2\leq 4 \text{ and analogs}) \text{ and}$$
$$x^2=4-a, y^2=4-b, z^2=4-c \text{ and via such substitutions},$$
$$2(x^6+y^6+z^6)-x^4y^2-y^4z^2-z^4x^2\leq 192$$
$$\Leftrightarrow 2\sum_{cyc}(4-a)^3-\sum_{cyc}\left((4-a)^2(4-b)\right)-192\leq 0$$

$$\Leftrightarrow \sum_{cyc} a^2b + 20\sum_{cyc} a^2 \overset{(*)}{\leq} 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a$$

$$\text{Now, } (a-4)^2 \geq 0 \Rightarrow a^2 + 16 \geq 8a \Rightarrow 2a^3 + 32a$$

$$\geq 16a^2\ (\because a \geq 0)\ and\ analogs$$

$$\Rightarrow 2\sum_{cyc} a^3 + 32\sum_{cyc} a \geq 16\sum_{cyc} a^2 \to (1)$$

$$\text{Also, } 16\sum_{cyc} a \geq 4\sum_{cyc} a^2$$

$$\to (2)\ (\because 4 \geq a \Rightarrow 4a \geq a^2\ (\because a \geq 0)\ and\ analogs)\ and$$

$$\text{moreover, } 4\sum_{cyc} ab \geq a^2b + b^2c + c^2a\ (\because 4 \geq a, b, c\ and\ ab, bc, ca \geq 0)$$

$$\Rightarrow 8\sum_{cyc} ab \geq \sum_{cyc} a^2b + 4\sum_{cyc} ab \overset{ab,bc,ca \geq 0 \Rightarrow \sum_{cyc} ab \geq 0}{\geq} \sum_{cyc} a^2b \to (3)$$

$$\therefore (1) + (2) + (3) \Rightarrow 2\sum_{cyc} a^3 + 8\sum_{cyc} ab + 48\sum_{cyc} a \geq \sum_{cyc} a^2b + 20\sum_{cyc} a^2$$

$$\Rightarrow (*) \text{ is true } \therefore 2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \leq 192\ \forall\, x, y, z \in [-2, 2],$$

$$'' ='' \ (x = 2, y = 2, z = 2) \text{ or } (x = -2, y = -2, z = -2)$$

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$$\text{or } (x = -2, y = 2, z = -2) \text{ or } (x = -2, y = -2, z = 2)\ (\text{QED})$$

1499. *If $a, b, c \in \mathbb{R}$ and $abc(16a^4 + 1)(16b^4 + 1)(16c^4 + 1) \geq 4913$,*

then prove that :

$$a^2 + b^2 + c^2 \geq 3$$

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$$abc(16a^4 + 1)(16b^4 + 1)(16c^4 + 1) \geq 4913 = 17^3$$

$$\Rightarrow a^2b^2c^2(16a^4 + 1)^2(16b^4 + 1)^2(16c^4 + 1)^2 \geq 17^6$$

$$\Rightarrow \sqrt[3]{a^2b^2c^2(16a^4 + 1)^2(16b^4 + 1)^2(16c^4 + 1)^2} \geq 289$$

$$\Rightarrow \sqrt[3]{xyz(16x^2 + 1)^2(16y^2 + 1)^2(16z^2 + 1)^2} \geq 289\ (x = a^2, y = b^2, z = c^2)$$

$$\Rightarrow \ln\left(\left(\sqrt[3]{x(16x^2 + 1)^2}\right)\left(\sqrt[3]{y(16y^2 + 1)^2}\right)\left(\sqrt[3]{z(16z^2 + 1)^2}\right)\right) \geq \ln 289$$

$$\Rightarrow \sum_{cyc} \ln \sqrt[3]{x(16x^2+1)^2} \geq \ln 289 \ (\because a,b,c \neq 0 \Rightarrow x,y,z > 0)$$

$$\Rightarrow \sum_{cyc} \ln(x(16x^2+1)^2) \overset{(*)}{\geq} 3\ln 289$$

$$\text{Now, } f(x) = \ln(x(16x^2+1)^2) \text{ is concave as } f''(x) = -\frac{1280x^4-32x^2+1}{x^2(16x^2+1)^2} < 0$$

$$\left(\begin{array}{c}\because \text{discriminant of } (1280x^4-32x^2+1) = 1024-5120 < 0 \\ \Rightarrow 1280x^4-32x^2+1 > 0\end{array}\right)$$

$$\therefore \sum_{cyc} \ln(x(16x^2+1)^2) \overset{\text{Jensen}}{\leq} 3\ln(t(16t^2+1)^2) \left(t = \frac{1}{3}\sum_{cyc} x > 0\right)$$

$$\overset{\text{via } (*)}{\Rightarrow} 3\ln(t(16t^2+1)^2) \geq 3\ln 289 \Rightarrow t(256t^4+32t^2+1) \geq 289 \Rightarrow$$
$$256t^5+32t^3+t-289 \geq 0 \Rightarrow (t-1)(256t^4+256t^3+288t^2+288t+289) \geq 0$$

$$\Rightarrow t = \frac{1}{3}\sum_{cyc} x \geq 1 \Rightarrow x+y+z \geq 3 \therefore a^2+b^2+c^2 \geq 3$$

$$\forall a,b,c \in \mathbb{R} \,\big|\, abc(16a^4+1)(16b^4+1)(16c^4+1) \geq 4913,$$

$$'' ='' \text{ iff } (a=b=c=1) \text{ or } (a=1, b=c=-1) \text{ or } (a=b=-1, c=1)$$

$$\text{or } (a=-1, b=1, c=-1) \text{ (QED)}$$

1500. **If $x,y,z > 0$ and $3xyz \geq x+y+z$, then prove that :**

$$\frac{xy+yz+zx-1}{\sqrt{3x^2+1}+\sqrt{3y^2+1}+\sqrt{3z^2+1}} \geq \frac{1}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3xyz \geq x+y+z \Rightarrow \left(\sum_{cyc} xy\right)^2 \geq 3xyz\left(\sum_{cyc} x\right) \geq \left(\sum_{cyc} x\right)^2 \geq 3\sum_{cyc} xy$$

$$\therefore \sum_{cyc} xy \geq \sum_{cyc} x \to (1) \text{ and } \sum_{cyc} xy \geq 3 \to (2)$$

$$\text{Now, } 3xyz \geq x+y+z \Rightarrow 1 \leq \frac{3xyz}{\sum_{cyc} x} \Rightarrow \sqrt{3x^2+1} \leq \sqrt{3x^2+\frac{3xyz}{\sum_{cyc} x}}$$

$$\therefore \sqrt{3x^2+1} \le \sqrt{\frac{3}{\sum_{cyc} x}}.\sqrt{x}.\sqrt{x^2+\sum_{cyc} xy} \text{ and analogs}$$

$$\Rightarrow \sqrt{3x^2+1}+\sqrt{3y^2+1}+\sqrt{3z^2+1} \overset{CBS}{\le} \sqrt{\frac{3}{\sum_{cyc} x}}.\sqrt{\sum_{cyc} x}.\sqrt{\sum_{cyc} x^2+3\sum_{cyc} xy}$$

$$= \sqrt{3}.\sqrt{\left(\sum_{cyc} x\right)^2+\sum_{cyc} xy} \overset{via\ (1)}{\le} \sqrt{3}.\sqrt{\left(\sum_{cyc} xy\right)^2+\sum_{cyc} xy}$$

$$\Rightarrow \frac{xy+yz+zx-1}{\sqrt{3x^2+1}+\sqrt{3y^2+1}+\sqrt{3z^2+1}} \ge \frac{\sum_{cyc} xy-1}{\sqrt{3}.\sqrt{\left(\sum_{cyc} xy\right)^2+\sum_{cyc} xy}}$$

$$\left(\because \text{via } (2), \sum_{cyc} xy \ge 3 > 1 \Rightarrow xy+yz+zx-1 > 0\right) \overset{?}{\ge} \frac{1}{3}$$

$$\Leftrightarrow \frac{t^2-2t+1}{3(t^2+t)} \overset{?}{\ge} \frac{1}{9}\left(t = \sum_{cyc} xy\right) \Leftrightarrow 2t^2-7t+3 \overset{?}{\ge} 0 \Leftrightarrow (2t-1)(t-3) \overset{?}{\ge} 0$$

$$\to \text{true} \because t = \sum_{cyc} xy \ge 3, \text{via } (2) \therefore \frac{xy+yz+zx-1}{\sqrt{3x^2+1}+\sqrt{3y^2+1}+\sqrt{3z^2+1}} \ge \frac{1}{3}$$

$$\forall\, x,y,z > 0 \,\middle|\, 3xyz \ge x+y+z, ''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru