SSMA - MATH CHALLENGES -(VI)

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5716. Prove:

$$\text{If } x, y \in \mathbb{R}, \text{ then } |\cos x \cos y \sin(x+y)| \le \frac{3\sqrt{3}}{8}$$

Daniel Sitaru

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
We need to prove that $f(x,y) := \cos^2 x \cos^2 y \sin^2(x+y) \le \frac{27}{64}, f(x,y)$ is periodic with respect to x and y. Hence the extrema of $f(x,y)$ are assumed at points where the partial derivatives with respect to x and y vanish. We find

$$\frac{\partial}{\partial x} f(x,y) = -2\sin x \cos x \cos^y \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$\frac{\partial}{\partial y} f(x,y) = -2\sin y \cos y \cos^2 x \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$0 = \frac{\partial}{\partial x} f(x,y) - \frac{\partial}{\partial y} f(x,y) = 2\cos x \cos y \sin(y-x)\sin^2(x+y)$$

So either $x \equiv \frac{\pi}{2} (\mod \pi)$, or $y \equiv \frac{\pi}{2} (\mod \pi)$, or $x \equiv -y (\mod \pi)$, or $x \equiv y (\mod \pi)$.
The first three alternatives lead to $f(x,y) = 0$, while the last one leads to

$$0 = -2\sin x \cos^3 x \sin^2(2x) + 2\cos^4 x \sin(2x)\cos(2x) =$$

$$= 2\cos^3 x\Big(-\sin x(4\sin^2 x \cos^2 x) + \cos x(2\sin x \cos x)\cos(2x)\Big) =$$

$$= 4\cos^5 x \sin x(2\cos(2x) - 1).$$

So either $x \equiv \frac{\pi}{2} (\mod \pi)$, or $x \equiv 0 (\mod \pi)$, or $x \equiv \pm\frac{\pi}{6} (\mod \pi)$. When combined with $y \equiv x (\mod \pi)$ we get indeed $f(x,y) \le \cos^2(\frac{\pi}{6})\cos^2(\frac{\pi}{6})\sin^2(\frac{\pi}{3}) = \frac{27}{64}$. □

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Using well-known trigonometric formulas we obtain

$$f(x,y) := |\cos x \cos y \sin(x+y)| = \frac{1}{2}|\cos(x+y) + \cos(x-y)|\sqrt{1-\cos^2(x+y)}$$

$$\le \frac{1}{2}|z+1|\sqrt{1-z^2}$$

where $z = |\cos(x+y)|$. Hence,

$$f^2(x,y) \le \frac{1}{4}(z+1)^2(1-z^2) = \frac{27}{64} - \left(z - \frac{1}{2}\right)^2\left(\frac{11}{16} + \frac{3}{4}z + \frac{1}{4}z^2\right)$$

Since $0 \leq z \leq 1$, we conclude that

$$f(x,y) \leq \sqrt{\frac{27}{64}} = \frac{3\sqrt{3}}{8}$$

Remark: The inequality is sharp. Equality occurs if $x = y = \frac{\pi}{6}$ $\square$

Solution 3 by Brian Bradie, Department of Mathematics, Chrisopher Newport University, Newport News, VA.

Let $f(x,y) = \cos x \cos y \sin(x+y)$. Then

$$\frac{\partial f}{\partial x} = -\sin x \cos y \sin(x+y) + \cos x \cos y \cos(x+y)$$

$$= \cos y(\cos x \cos(x+y) - \sin x \sin(x+y)) = \cos y \cos(2x+y)$$

which is equal to 0 when

$$y = \left(n + \frac{1}{2}\right)\pi \text{ or } 2x + y = \left(n + \frac{1}{2}\right)\pi,$$

for some integer n. Similarly,

$$\frac{\partial f}{\partial y} = \cos x \cos(x+2y),$$

which is equal to 0 when

$$x = \left(m + \frac{1}{2}\right)\pi \text{ or } x + 2y = \left(m + \frac{1}{2}\right)\pi,$$

for some integer m. It follows that f has four categories of critical points:
1. $((m+\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ for any integers m and n
2. $((m+\frac{1}{2})\pi, (n-2m-\frac{1}{2})\pi)$, for any integers m and n
3. $((m-2n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$, for any integers m and n
4. $(\frac{1}{3}(2n-m+\frac{1}{2})\pi, \frac{1}{3}(2m-n+\frac{1}{2})\pi)$, for any integers m and n

When evaluate at any critical point from the first three categories, f is equal to 0. For the critical points in the fourth category, note

$$2m - n = 2n - m + 3(m-n) \Rightarrow 2m - n \equiv 2n - m (\mod 3).$$

This leads to three cases to consider:

Case 1: $2n - m \equiv 0 (\mod 3)$

Then

$$x = j\pi + \frac{\pi}{6}, \quad y = k\pi + \frac{\pi}{6}, \quad \text{and} \quad x + y = (j+k)\pi + \frac{\pi}{3}$$

for some integers j and k, and

$$f(x,y) = \pm\frac{3\sqrt{3}}{8}.$$

Case 2: $2n - m \equiv (\mod 3)$

Then

$$x = j\pi + \frac{\pi}{2}, \quad y = k\pi + \frac{\pi}{2}, \quad \text{and} \quad x + y = (j+k+1)\pi$$

for some integers j and k, and

$$f(x,y) = 0.$$

Case 3: $2n - m \equiv 2 (\mod 3)$

Then

$$x = j\pi + \frac{5\pi}{6}, \quad y = k\pi + \frac{5\pi}{6}, \quad \text{and} \quad x + y = (j+k)\pi + \frac{5\pi}{3}$$

for some integers j and k, and

$$f(x,y)=\pm\frac{3\sqrt{3}}{8}$$

Thus, for all $x,y\in\mathbb{R}$,

$$-\frac{3\sqrt{3}}{8}\le f(x,y)\le\frac{3\sqrt{3}}{8},$$

or

$$|f(x,y)|\le\frac{3\sqrt{3}}{8}.$$

□

Solution 4 by David Huckaby, Angelo State University, San Angelo, TX.
Let $f(x,y)=\cos x\cos y\sin(x+y)$. Note that $f(x+\pi,y)=\cos(x+\pi)\cos y\sin(x+\pi+y)=-\cos x\cos y[-\sin(x+y)]=f(x,y)$. Similarly, $f(x,y+\pi)=f(x,y)$. So we need only consider the square $[-\frac{\pi}{2},-\frac{\pi}{2}]\times[\frac{\pi}{2},\frac{\pi}{2}]$.
We first note that since $\cos(-\frac{\pi}{2})=\cos(\frac{\pi}{2})=0$, $f(x,y)=0$ for every point on the boundary of $[-\frac{\pi}{2},-\frac{\pi}{2}]\times[\frac{\pi}{2},\frac{\pi}{2}]$.
To find extrema for f in the interior of $[-\frac{\pi}{2},-\frac{\pi}{2}]\times[\frac{\pi}{2},\frac{\pi}{2}]$, we compute $\frac{\partial f}{\partial x}=-\sin x\cos y\sin(x+y)+\cos x\cos y\cos(x+y)=\cos y[\cos x\cos(x+y)-\sin x\sin(x+y)]$. From the symmetry of $f(x,y)$ in x and y, $\frac{\partial f}{\partial y}=\cos x[\cos y\cos(x+y)-\sin y\sin(x+y)]$.
Setting $\frac{\partial f}{\partial x}=0$ gives $\cos y=0$ or $\cos x\cos(x+y)-\sin x\sin(x+y)=0$. Since $\cos y\neq 0$ in the interior of $[-\frac{\pi}{2},-\frac{\pi}{2}]\times[\frac{\pi}{2},\frac{\pi}{2}]$, we have $\cos x\cos(x+y)-\sin x\sin(x+y)=0$. Now

$$\cos x\cos(x+y)-\sin x\sin(x+y)$$

$$=\cos x[\cos x\cos y-\sin x\sin y]-\sin x[\sin x\cos y+\cos x\sin y]$$

$$=\cos^2 x\cos y-\sin^2 x\cos y-2\cos x\sin x\sin y$$

$$=\cos 2x\cos y-\sin 2x\sin y$$

$$=\cos(2x+y).$$

So $\frac{\partial f}{\partial x}=0$ implies $\cos(2x+y)=0$. By Symmetry, $\frac{\partial f}{\partial y}=0$ implies $\cos(x+2y)=0$. Now $\cos(2x+y)=0$ when $2x+y=\frac{\pi}{2}+\pi n$ for any integer n. Solving for y gives $y=-2x+\frac{\pi}{2}+\pi n$. Similarly, $\cos(x+2y)=0$ when $x+2y=\frac{\pi}{2}+\pi n$ for some integer n. Solving for y gives $y=\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi n}{2}$. Setting these two values of y equal to each other yields $-2x+\frac{\pi}{2}+\pi n=-\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi n}{2}$, whence $x=\frac{\pi}{6}+\frac{\pi n}{3}$.
The only values of $x=\frac{\pi}{6}+\frac{\pi n}{3}$ in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ are $x=\pm\frac{\pi}{6}$. So any point (x,y) yielding an extremum of f in the interior of $[-\frac{\pi}{2},-\frac{\pi}{2}]\times[\frac{\pi}{2},\frac{\pi}{2}]$ must lie on $(\frac{\pi}{6},y)$ or $(-\frac{\pi}{6},y)$. By symmetry, any extremum must also lie on $(x,\frac{\pi}{6})$ or $(x,-\frac{\pi}{6})$. So there are only four possible points that could yield an extremum.
Note that if $x+y=0$, then $\sin(x+y)=0$ so that $f(x,y)=0$. So we need only check two points: $f(\frac{\pi}{6},\frac{\pi}{6})=\frac{3\sqrt{3}}{8}$ and $f(-\frac{\pi}{6},-\frac{\pi}{6})=-\frac{3\sqrt{3}}{8}$. (Note that rather than using direct calculation, the latter can be obtained from the former by noting that $f(-x,-y)=\cos(-x)\cos(-y)\sin(-(x+y))=\cos x\cos y[-\sin(x+y)]=-f(x,y)$.) So f attains a maximum value of $\frac{3\sqrt{3}}{8}$ and a minimum value of $-\frac{3\sqrt{3}}{8}$. Thus $|\cos x\cos y\sin(x+y)|=|f(x,y)|\le\frac{3\sqrt{3}}{8}$. □

Solution 5 by Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $f(x,y) = \cos x \cos y \sin(x+y)$, and consider $g(x) = f(x,x) = \cos^2 x \sin(2x)$, which has period π. Since $g'(x) = (2\cos(2x)-1)(\cos(2x)+1)$, by the first derivative test we see that g achieves its maximum value of $\frac{3\sqrt{3}}{8}$ at $x = \frac{\pi}{6} + n\pi$ and its minimum value of $-\frac{3\sqrt{3}}{8}$ at $x = -\frac{\pi}{6} + n\pi$, where n is an integer. Thus

$$f\Big(\frac{\pi}{6}+n\pi, \frac{\pi}{6}+n\pi\Big) = \frac{3\sqrt{3}}{8} \text{ and } f\Big(-\frac{\pi}{6}+n\pi, -\frac{\pi}{6}+n\pi\Big) = -\frac{3\sqrt{3}}{8}.$$

Since $f(x,y)$ attains the two values above, in searching for absolute extreme values of $f(x,y)$ we may assume $f(x,y) \neq 0$; that is, we assume $\cos x, \cos y$ and $\sin(x+y)$ are all nonzero.

Since the partial derivatives of $f(x,y) = \cos x \cos y \sin(x+y)$ are

$$f_x(x,y) = \cos y(\cos x \cos(x+y) - \sin x \sin(x+y)) \text{ and}$$

$$f_y(x,y) = \cos x(\cos y \cos(x+y) - \sin y \sin(x+y)),$$

then any critical points with $f(x,y) \neq 0$ must satisfy

$$\sin x \cos y \sin(x+y) = \cos x \cos y \cos(x+y) = \cos x \sin y \sin(x+y),$$

and $\tan x = \tan y$. Thus, $y = x + n\pi$, where n is an integer, and since $\cos^2(n\pi) = 1$, then

$$f(x,y) = \cos x \cos(x+n\pi)\sin(2x+n\pi) = \cos^2 x \cos^2(n\pi)\sin(2x) = \cos^2 x \sin(2x) = g(x).$$

From the analysis of $g(x)$ above, $f(x,y)$ must achieve its maximum at $\frac{3\sqrt{3}}{8}$ and its minimum at $-\frac{3\sqrt{3}}{8}$. □

Solution 6 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Note that

$$|\cos x \cos y \sin(x+y)| \leq \frac{3\sqrt{3}}{8} \Leftrightarrow (\cos x \cos y \sin(x+y))^2 \leq \frac{27}{64}$$

$$(\sin(x+y) + \sin x + \sin y)^2 \leq \frac{27}{4},$$

which must be proved.

Let $f(x,y) = \sin(x+y) + \sin x + \sin y$, over $x, y \in \mathbb{R}$. It is enough to show that $f(x,y)^2 \leq \frac{27}{4}$.

Observe that $f(x,y) = f(2a\pi + x, 2b\pi + y), \forall a, b \in \mathbb{Z}$; so, WLOG, $x, y \in [0, 2\pi]$.

CASE 1: If $x, y \in [0, \pi]$.

We have

$$(1) \qquad -1 \leq \sin(x+y) \leq f(x,y) \leq \sin(x+y) + 2\sin\Big(\frac{x+y}{2}\Big)$$

Consider the function $f_1(x) = \sin 2x + 2\sin x, \forall x \in [0, \pi]$. Then, $f_1'(x) = 2(2\cos x1)(\cos x + 1)$; f_1 is increasing when $x \in [0, \frac{\pi}{3}]$ and decreasing when $x \in [\frac{\pi}{3}, \pi]$. Therefore, $\sin(x+y) + 2\sin(\frac{x+y}{2}) \leq f(\frac{2\pi}{3}) = \frac{3\sqrt{3}}{2}$. By (1), we get $-1 \leq f(x,y) \leq \frac{3\sqrt{3}}{2}$ and thus, $f(x,y)^2 \leq \frac{27}{4}$.

CASE 2: If $x, y \in [\pi, 2\pi]$.

Let $x = \pi + x_1$ and $y = \pi + y_1$ where $x_1, y_1 \in [0, \pi]$. Then, $f(x,y) = \sin(x_1 + y_1) -$

$\sin x_1 - \sin y_1$.
We have

$$\sin(x_1+y_1) - 2\sin\Big(\frac{x_1+y_1}{2}\Big) \le f(x,y) \le \sin(x_1+y_1) \le 2 \tag{2}$$

Consider the function $f_2(x) = \sin 2x - 2\sin x, \forall x \in [0,\pi]$. Then, $f_2'(x) = 2(2\cos x + 1)(\cos x - 1)$; f_2 is decreasing for $x \in [0, \frac{2\pi}{3}]$ and increasing for $x \in [\frac{2\pi}{3}, \pi]$. Therefore, $\sin(x_1+y_1) - 2\sin(\frac{x_1+y_1}{2}) \ge f_2(\frac{2\pi}{3}) = \frac{-3\sqrt{3}}{2}$. By (2), we get $\frac{-3\sqrt{3}}{2} \le f(x,y) \le 1$ and thus, $f(x,y)^2 \le \frac{27}{4}$.
CASE 3: If one of x and y is in $[0,\pi]$ while another one is in $[\pi, 2\pi]$. By symmetry, WLOG $x \in [0,\pi]$ and $y \in [\pi, 2\pi]$.
We have $-1 \le \sin(x+y) \le 1, 0 \le \sin x \le 1$, and $-1 \le \sin y \le 0$. Summing up these 3 inequalities give us $-2 \le f(x,y) \le 2$, so $f(x,y)^2 \le 4 < \frac{27}{4}$.
All 3 cases above yield that $f(x,y)^2 \le \frac{27}{4}$ and the result follows. □

Solution 7 by Michael C. Fleski, Delta College, University Center, MI.
Let P be the product in question. We want to maximize the quantity
$P = \cos(x)\cos(y)\sin(x+y)$. So, we take derivatives of the expression finding

$$\frac{\partial P}{\partial y} = -\cos(x)\sin(y)\sin(x+y) + \cos(x)\cos(y)\cos(x+y) = 0$$

$$\cos(x)(-\sin(y)\sin(x+y) + \cos(y)\cos(x+y)) = 0$$

$$\cos(x)\cos(x+2y) = 0 \to x = \frac{(2p+1)\pi}{2}; x+2y = \frac{(2n+1)\pi}{2}$$

and

$$\frac{\partial P}{\partial x} = -\sin(x)\cos(y)\sin(x+y) + \cos(x)\cos(y)\cos(x+y) = 0$$

$$\cos(y)(-\sin(x)\sin(x+y) + \cos(x)\cos(x+y)) = 0$$

$$\cos(y)\cos(2x+y) = 0 \to y = \frac{(2q+1)\pi}{2}; 2x+y = \frac{(2m+1)\pi}{2}$$

with $m,n,p,q \in \mathbb{Z}$
We analyze the results by cases.
CASE 1: $\cos(x) = 0$ or $\cos(y) = 0$
Arbitrarily choosing the case of $\cos(x) = 0$ leads to

$$P = (1)\cos(y)\sin(y) = \frac{1}{2}\sin(2y)$$

The maximum value of $\sin(2y) = 1$ leading to $|P| = \frac{1}{2} < \frac{3\sqrt{3}}{8}$.
For the other conditions, by taking the difference in the equations gives

$$y - x = (n-m)\pi = r\pi \to y = x + r\pi \qquad r \in \mathbb{Z}$$

Because of the periodicty involved with the proble, we can restrict $r = 0, 1$.
By adding the expressions, one find

$$y + x = \frac{1}{3}(n+m)\pi + \frac{\pi}{3}$$

Combining our relations together allows for solutions to the angles of x and y as

$$y = \frac{\pi}{6} + \frac{\pi}{3}(2n-m) \qquad x = \frac{\pi}{6} + \frac{\pi}{3}(2m-n)$$

CASE 2: $n - m = r = 0$
This restriction makes $x = y = \frac{\pi}{6} + \frac{n\pi}{3}$. Hence,

n	$x = y$	$\|P\|$
0	$\frac{\pi}{6}$	$\|\cos(\frac{\pi}{6})\cos(\frac{\pi}{6})\sin(\frac{2\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
1	$\frac{3\pi}{6}$	$\|\cos(\frac{3\pi}{6})\cos(\frac{3\pi}{6})\sin(\frac{6\pi}{6})\| = \|(0)(0)(0)\| = 0$
2	$\frac{5\pi}{6}$	$\|\cos(\frac{5\pi}{6})\cos(\frac{5\pi}{6})\sin(\frac{10\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
3	$\frac{7\pi}{6}$	$\|\cos(\frac{7\pi}{6})\cos(\frac{7\pi}{6})\sin(\frac{14\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
4	$\frac{9\pi}{6}$	$\|\cos(\frac{9\pi}{6})\cos(\frac{9\pi}{6})\sin(\frac{18\pi}{6})\| = \|(0)(0)(0)\| = 0$
5	$\frac{11\pi}{6}$	$\|\cos(\frac{11\pi}{6})\cos(\frac{11\pi}{6})\sin(\frac{22\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$

CASE 3: $n - m = r = 1$
Since $x + y = \frac{1}{3}(m+n)\pi + \frac{\pi}{3}$ and $m + n$ must be odd, we restrict $m + n = 1, 3, 5$ as $\sin(2\pi + x) = \sin x$.
Therefore, we have cases: $n = 1, m = 0; n = 2, m = 1$; and $n = 3, m = 2$ to consider.

n	m	x	y	$\|P\|$
1	0	$\frac{-\pi}{6}$	$\frac{5\pi}{6}$	$\|\cos(\frac{-\pi}{6})\cos(\frac{5\pi}{6})\sin(\frac{4\pi}{6})\| = \frac{3\sqrt{3}}{8}$
2	1	$\frac{\pi}{6}$	$\frac{7\pi}{6}$	$\|\cos(\frac{\pi}{6})\cos(\frac{7\pi}{6})\sin(\frac{8\pi}{6})\| = \frac{3\sqrt{3}}{8}$
3	2	$\frac{3\pi}{6}$	$\frac{9\pi}{6}$	$\|\cos(\frac{3\pi}{6})\cos(\frac{9\pi}{6})\sin(\frac{12\pi}{6})\| = 0$

Consequently, there is no value of $|P| > \frac{3\sqrt{3}}{8}$. This means that
If $x, y \in \mathbb{R}$, then $||\cos(x)\cos(y)\sin(x+y)|| \leq \frac{3\sqrt{3}}{8}$. □

Solution 8 by Michel Bataille, Rouen, France.
We have

$$\cos x \cos y \sin(x+y) = \sin x \cos x \cos^2 y + \sin y \cos y \cos^2 x$$
$$= \frac{1}{4}((1 + \cos 2y)\sin 2x + (1 + \cos 2x)\sin 2y)$$
$$= \frac{1}{4}(\sin 2x + \sin 2y + \sin(2x + 2y)),$$

hence the problem boils down to proving that $|f(x, y)| \leq \frac{3\sqrt{3}}{2}$ for all $x, y \in \mathbb{R}$ where

$$f(x, y) = \sin x + \sin y + \sin(x + y).$$

Note that due to periodicity it suffices to prove the inequality for $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$.
Now, if $(u, v) \in \mathbb{R}^2$ and $f(u, v)$ is a local extremum of f, we must have $\frac{\partial f}{\partial x}(u, v) = \frac{\partial f}{\partial y}(u, v) = 0$, that is, $\cos u + \cos(u + v) = \cos v + \cos(u + v) = 0$ or equivalently: $(u = v(\mod 2\pi)$ and $\cos 2u + \cos u = 0)$ or $(u = -v(\mod 2\pi)$ and $\cos u = -1)$. Thus, the candidates for and extremum in $[-\pi, \pi] \times [-\pi, \pi]$ are $(\frac{\pi}{3}, \frac{\pi}{3}), (-\frac{\pi}{3}, -\frac{\pi}{3}), (\pi, \pi), (-\pi, -\pi), (-\pi, \pi), (\pi, -\pi)$. Being continuous on the compact set $[-\pi, \pi] \times [-\pi, \pi]$, the function f attains its (absolute) maximum and minimum on this set (and or $\mathbb{R}^2$) at one of these pairs. However, we have $f(\pi, \pi) = f(-\pi, -\pi) = f(-\pi, \pi) = f(\pi, -\pi) = 0$ while $f(\frac{\pi}{4}, \frac{\pi}{4}) > 0$ and $f(-\frac{\pi}{4}, -\frac{\pi}{4}) < 0$, hence no extremum is attained at $(\pi, \pi), (-\pi, -\pi), (-\pi, \pi)$ or $(\pi, -\pi)$. It follows

that the maximum and the minimum of f are $f(\frac{\pi}{3},\frac{\pi}{3})=\frac{3\sqrt{3}}{2}$ and $f(-\frac{\pi}{3},-\frac{\pi}{3})=-\frac{3\sqrt{3}}{2}$. Thus we have

$$-\frac{3\sqrt{3}}{2}\le f(x,y)\le\frac{3\sqrt{3}}{2}$$

for all $(x,y)\in[-\pi,\pi]\times[-\pi,\pi]$ (and all $(x,y\in\mathbb{R}^2)$. The result follows. □

Solution 9 by Moti Levy, Rehovot, Israel.
Since

$$|\cos(x)|=\left|\sin\left(\frac{\pi}{2}-x\right)\right|=\left|\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right)\right|=\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right),$$
$$|\sin(x)|=\sin(x\mod\pi),$$

the original inequality can be rewritten as follows:

$$\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right)\sin((x+y)\mod\pi)\le\frac{3\sqrt{3}}{8}.$$

By AM-GM inequality:

$$\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right)\sin\left(\left(\frac{\pi}{2}-y\right)\mod\pi\right)\sin((x+y)\mod\pi)$$
$$\le\left(\frac{\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right)+\sin\left(\left(\frac{\pi}{2}-y\right)\mod\pi\right)+\sin((x+y)\mod\pi)}{3}\right)^3.$$

By Jensen's inequality

$$\left(\frac{\sin\left(\left(\frac{\pi}{2}-x\right)\mod\pi\right)+\sin\left(\left(\frac{\pi}{2}-y\right)\mod\pi\right)+\sin((x+y)\mod\pi)}{3}\right)^3$$
$$\le\left(\sin\left(\frac{\left(\frac{\pi}{2}-x\right)\mod\pi+\left(\frac{\pi}{2}-y\right)\mod\pi+\sin((x+y)\mod\pi)}{3}\right)\right)^3$$
$$=\sin^3\left(\frac{\left(\left(\frac{\pi}{2}-x\right)+\left(\frac{\pi}{2}-y\right)+(x+y)\right)\mod\pi}{3}\right)$$
$$=\sin^3\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{8}.$$

□

Solution 10 by Perfetti Paolo, dipatimento di matematica, Universita de "Tor Vergata", Roma, Italy.

It is equivalent

$$F(x,y)=(\cos x)^2(\cos y)^2(\sin(x+y))^2\frac{27}{64}$$

$F(x,y)$ is π - periodic both in x and y.
We search the maximum of $F(x,y)$ which exists because $F(x,y)$ is continuous and periodic hence it suffices to search the maximum in $[0,\pi]\times[0,\pi]$ which is compact. Let's observe that $F(0,y)\equiv F(x,0)=0$ and $F(\pi,y)=\frac{(\sin(2y))^2}{4}$, $F(x,\pi)=\frac{(\sin(2x))^2}{4}$ thus on the boundary of the square $[0,\pi]\times[0,\pi]$ the functions does not exceed the value $\frac{1}{4}$.

$$F_x=(-2\sin(2x)(\sin(x+y))^2+(\cos x)^2\sin2(x+y))(\cos y)^2=0$$
$$F_y=(-2\sin(2x)(\sin(x+y))^2+(\cos y)^2\sin2(x+y))(\cos x)^2=0$$

$$F_x = (-2\sin x \sin(x+y) + 2\cos x \cos(x+y))\cos x(\cos y)^2 \sin(x+y) = 0$$
$$F_y = (= 2\sin y \sin(x+y) + 2\cos y \cos(x+y))\cos y(\cos x)^2 \sin(x+y) = 0$$

$(x,y) = (\frac{\pi}{2}, y), y \in \mathbb{R}$ and $(x,y) = (x, \frac{\pi}{2}), x \in \mathbb{R}$ all are critical points .Moreover $\{(x,y) \in [0,\pi] \times [0,\pi] : x+y = k\pi, k = 0,1,2,\}$ also are critical points. Since $F(x,y)$ annihilates on each of the above points, no one of them can be point of maximum. Actually the are all point of minimum.
Based on that we can write

$$(1) \qquad F_x = -\sin x \sin(x+y) + \cos x \cos(x+y) = 0 \Rightarrow \operatorname{cotg}(x+y) = \tan x$$

$$(2) \qquad F_y = -\sin y \sin(x+y) + \cos y \cos(x+y) \Rightarrow \operatorname{cotg}(x+y) = \tan y$$

hence $\tan x = \tan y, y = x$. It follows

$$\tan x = \frac{1}{\tan(2x)} \Leftrightarrow \tan x = \frac{1-(\tan x)^2}{2\tan x} \Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6} + k\pi$$

Clearly by periodicity of $F(x,y)$ it suffices to consider $x = \frac{\pi}{6}$ and then $y = \frac{\pi}{6}$

$$F\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{27}{64} > \frac{1}{4}$$

and then $(\frac{\pi}{6}, \frac{\pi}{6})$ is the point of the searched maximum. □

Solution 11 by proposer. First, we prove that for $x, y \in \mathbb{R}$

$$(1) \qquad \cos^2 x + \cos^2 y + \sin^2(x+y) \le \frac{9}{4}$$

$$\frac{1+\cos 2x}{2} + \frac{1+\cos 2y}{2} + 1 - \cos^2(x+y) \le \frac{9}{4}$$
$$2 + 2\cos 2x + 2 + 2\cos 2y + 4 - 4\cos^2(x+y) \le 9$$
$$2(\cos 2x + \cos 2y) - 4\cos^2(x+y) \le 1$$
$$2 \cdot 2\cos\frac{2x+2y}{2}\cos\frac{2x-2y}{2} - 4\cos^2(x+y) \le 1$$
$$4\cos(x+y)\cos(x-y) - 4\cos^2(x+y) \le 1$$
$$4\cos(x+y)[\cos(x-y) - \cos(x+y)] \le 1$$

Denote $x+y = u; x-y = v$

$$4\cos u(\cos v - \cos u) \le 1$$
$$4\cos u \cos v - 4\cos^2 u \le 1$$
$$4\cos^2 u - 4\cos u \cos v + \cos^2 v + \sin^2 v \ge 0$$
$$(2\cos u - \cos v)^2 + \sin^2 v \ge 0$$

By AM-GM:

$$\sqrt[3]{\cos^2 x \cos^2 y \sin^2(x+y)} \le \frac{\cos^2 x + \cos^2 y + \sin^2(x+y)}{3} \overset{(1)}{\le} \frac{\frac{9}{4}}{3} = \frac{3}{4}$$
$$\cos^2 x \cos^2 y \sin^2(x+y) \le \frac{27}{64}$$
$$|\cos x \cos y \sin(x+y)| \le \frac{3\sqrt{3}}{8}$$

Equality holds for $x = y = \frac{\pi}{6}$. □

5727. If $f:(0,\infty)\to(0,\infty)$ is a continuous function and $\int_a^b f(x)dx = 5(b-a)$ where $0<a\le b$, then

$$\int_a^b\Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx\le 9(b-a)$$

Daniel Sitaru

Solution 1 by Ivan Hadinata, Senior High School 1 Jember, Jember, Idonesia.
Note that

$$(1)\quad \frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}=18-2\Big(\frac{4^2}{f(x)+7}+\frac{5^2}{f(x)+9}+\frac{6^2}{f(x)+11}\Big)$$

By Titu's lemma,

$$(2)\qquad 2\Big(\frac{4^2}{f(x)+7}+\frac{5^2}{f(x)+9}+\frac{6^2}{f(x)+11}\Big)\ge\Big(\frac{15^2}{3f(x)+27}\Big)\ge\frac{144}{f(x)+9}$$

By AM-GM inequality,

$$(3)\qquad \frac{144}{f(x)+9}+f(x)+9\ge 2\sqrt{\Big(\frac{144}{f(x)+9}\Big)(f(x)+9)}=24$$

Combining $(1),(2)$ and (3) gives us

$$\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\le 18-(15-f(x))=f(x)+3.$$

Then,

$$\int_a^b\Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx\le\int_a^b f(x)dx+\int_a^b 3dx=8(b-a)\le 9(b-a).$$

proven. Equality holds if and only if $a=b$. □

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The function

$$g(x):=\frac{5x+3}{x+7}+\frac{6x+4}{x+9}+\frac{7x+5}{x+11}=18-\frac{32}{x+7}-\frac{50}{x+9}-\frac{72}{x+11}$$

is concave on $(0,\infty)$ since $g''(x)<0$. Therefore, it can be estimated from above by its tangent in the point $(5,g(5))$, i.e., we have

$$g(x)\le g(5)+g'(5)(x-5).$$

We infer that

$$\int_a^b\Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx=\int_a^b g(f(x))dx$$

$$\le\int_a^b\big(g(5)+g'(5)(f(x)-5)\big)dx=g(5)(b-a),$$

since by assumption $\int_a^b(f(x)-5)dx=0$. Now the inequality follows since $g(5)=\frac{305}{42}\approx 7.2619<9$.
Remark: The inequality shown above is sharp. Equality occurs if $f(x)=5$ on $(0,\infty)$. □

Solution 3 by Albert Stadler, Herrliberg, Switzerland.
We will prove the stronger inequality

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx \le \frac{305}{42}(b-a)$$

Clearly,

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\Big)dx =$$

$$=\int_a^b \Big(5-\frac{32}{f(x)+7}+6-\frac{50}{f(x)+9}+7-\frac{72}{f(x)+11}\Big)dx =$$

$$=18(b-a)-\int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx.$$

We need to prove that

$$(*) \qquad \frac{451}{42}(b-a) \le \int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx$$

Let $r>0$. By the Cauchy-Schwarz inequality for integrals,

$$(b-a)^2=\left(\int_a^b dx\right)^2 \le \int_a^b (f(x)+r)dx \int_a^b \Big(\frac{1}{f(x)+r}\Big)dx = (5+r)(b-a)\int_a^b \Big(\frac{1}{f(x)+r}\Big)dx$$

which implies

$$\int_a^b \Big(\frac{1}{f(x)+r}\Big)dx \ge \frac{b-a}{5+r}.$$

We conclude that

$$\int_a^b \Big(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\Big)dx \ge (b-a)\Big(\frac{32}{5+7}+\frac{50}{5+9}+\frac{72}{5+11}\Big) = \frac{451}{42}(b-a)$$

which is (*). □

Solution 4 by Michel Bataille, Rouen, France.
Let $g(x)=\frac{5x+3}{x+7}, h(x)=\frac{6x+4}{x+9}, k(x)=\frac{7x+5}{x+11}$. It is easily checked that g,h,k are non-decreasing and concave on $(0,\infty)$.
We want to prove that $\int_a^b \phi(x)dx \le 9(b-a)$ where $\phi(x)=g(f(x))+h(f(x))+k(f(x))$..
Let m and M be the minimum and the maximum of the continuous function f on the interval $[a,b]$.
Then, $0<m\le M$ and since $m(b-a)\le \int_a^b f(x)dx \le M(b-a)$, the hypothesis gives $m\le 5\le M$.
From the concavity of g on the interval $[m,M]$, the curve $y=g(x)$ is under its tangent at $(5,g(5))$.
The equation of this tangent is $y-\frac{7}{3}=\frac{2}{9}(x-5)$ (note that $g'(x)=\frac{32}{(x+7)^2}$), that is, $y=\frac{2x}{9}+\frac{11}{9}$ and therefore $g(f(x))\le \frac{2f(x)}{9}+\frac{11}{9}$ for $x\in[a,b]$.
Similar calculations lead to $h(f(x))\le \frac{25f(x)}{98}+\frac{113}{98}$ and $k(f(x))\le \frac{9f(x)}{32}+\frac{35}{32}$ and we deduce that for $x\in[a,b]$,

$$\phi(x)\le\Big(\frac{2}{9}+\frac{25}{98}+\frac{9}{32}\Big)\cdot f(x)+\frac{11}{9}+\frac{113}{98}+\frac{35}{32}=\frac{10705}{14112}\cdot f(x)+\frac{48955}{14112}.$$

Integrating yields

$$\int_a^b \phi(x)dx \leq \frac{10705}{14112}\int_a^b f(x)dx + \frac{48955}{14112}(b-a),$$

that is,

$$\int_a^b \phi(x)dx \leq \Big(\frac{53525}{14112} + \frac{48955}{14112}\Big)(b-a) = \frac{6405}{882}(b-a).$$

Since $\frac{64405}{882} < 9$, we obtain a sharper result than the required one. □

Solution 5 by proposer.

$$\frac{5f(x)+3}{f(x)+7} \leq \frac{f(x)+1}{2} \Leftrightarrow 10f(x)+6 \leq$$

$$\leq f^2(x)+8f(x)+7 \Leftrightarrow f^2(x)-2f(x)+1 \geq 0$$

$$\Leftrightarrow (f(x)-1)^2 \geq 0$$

(1)
$$\frac{5f(x)+3}{f(x)+7} \geq \frac{f(x)+1}{2}$$

$$\frac{6f(x)+4}{f(x)+9} \leq \frac{f(x)+1}{2} \Leftrightarrow 12f(x)+8 \leq f^2(x)+10f(x)+9$$

$$\Leftrightarrow f^2(x)-2f(x)+1 \geq 0 \Leftrightarrow (f(x)-1)^2 \geq 0$$

(2)
$$\frac{6f(x)+4}{f(x)+9} \leq \frac{f(x)+1}{2}$$

$$\frac{7f(x)+5}{f(x)+11} \leq \frac{f(x)+1}{2} \Leftrightarrow 14f(x)+10 \leq f^2(x)+12(x)+11$$

$$\Leftrightarrow f^2(x)-2f(x)+1 \geq 0 \Leftrightarrow (f(x)-1)^2 \geq 0$$

(3)
$$\frac{7f(x)+5}{f(x)+11} \leq \frac{f(x)+1}{2}$$

By adding (1);(2);(3):

$$\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11} \leq \frac{3}{2}(f(x)+1)$$

$$\int_a^b \Big(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11}\Big)dx \leq$$

$$\leq \frac{3}{2}\Big(\int_a^b f(x)dx + \int_a^b dx\Big) = \frac{3}{2}(5(b-a)+(b-a)) =$$

$$= 9(b-a)$$

Equality holds for $a = b$. □

5739. Prove that for any triangle ΔABC:

$$\frac{h_a^3}{h_b^3} + \frac{h_b^3}{h_c^3} + \frac{h_c^2}{h_a^3} \geq \frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C}$$

where h_a, h_b, h_c are the altitudes respectively issued from the vertices A, B, C.

Daniel Sitaru

Solution 1 by Michel Bataille, Rouen, France.
Let F and R be the area and the circumradius of the triangle. Let $a = BC$, $b = CA, c = AB$. Since $ah_a = bh_b = ch_c = 2F$ and $2R\sin A = a$, $2R\sin B = b, 2R\sin C = c$, the inequality is equivalent to

$$\frac{b^3}{a^3} + \frac{c^3}{b^3} + \frac{a^3}{c^3} \geq \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \tag{1}$$

From an inequality of means, we have

$$\frac{b^3}{a^3} + \frac{c^3}{b^3} + \frac{a^3}{c^3} \geq \frac{1}{\sqrt{3}}\Big(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\Big)^{\frac{3}{2}} \tag{2}$$

and from AM-GM, we have

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \geq 3\Big(\frac{b^2}{a^2} \cdot \frac{c^2}{b^2} \cdot \frac{a^2}{c^2}\Big)^{\frac{1}{3}} = 3$$

hence

$$\Big(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\Big)^{\frac{3}{2}} = \Big(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\Big)^{\frac{1}{2}}\Big(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\Big) \geq \sqrt{3}\Big(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\Big).$$

Combining with (2), the desired inequality (1) follows. □

Solution 2 by Albert Stadler, Herrliberg, Switzerland.
By law of sines,

$$\frac{h_a}{h_b} = \frac{b}{a} = \frac{\sin B}{\sin A}, \frac{h_b}{h_c} = \frac{c}{b} = \frac{\sin C}{\sin B}, \frac{h_c}{h_a} = \frac{a}{c} = \frac{\sin A}{\sin C}.$$

So

$$\frac{h_a^3}{h_b^3} + \frac{h_b^3}{h_c^3} + \frac{h_c^3}{h_a^3} = \frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}$$

By Hölder's inequality,

$$\frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C} \leq \Big(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\Big)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}}$$

It remains to prove that

$$\Big(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\Big)^{\frac{2}{3}}(1+1+1)^{\frac{1}{3}} \leq \frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}$$

which is equivalent to $\frac{sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C} \geq 3$. However this inequality follows from the AM-GM inequality:

$$\frac{1}{3}\Big(\frac{\sin^3 B}{\sin^3 A} + \frac{\sin^3 C}{\sin^3 B} + \frac{\sin^3 A}{\sin^3 C}\Big) \geq \frac{\sin B}{\sin A} \cdot \frac{\sin C}{\sin B} \cdot \frac{\sin A}{\sin C} = 1.$$

□

Solution 3 by proposer.
First we prove that:

$$\sum_{cyc} \frac{h_a^3}{h_b^3} \geq \sum_{cyc} \frac{h_a^2}{h_b^2} \tag{1}$$

By multiplying (1) with $(h_a h_b h_c)^3$:

$$\sum_{cyc} h_a^6 h_c^3 \geq \sum_{cyc} h_a^5 h_b h_c^3 \tag{2}$$

We prove (2):

$$\sum_{cyc} h_a^6 h_c^3 = \sum_{cyc} \frac{9h_a^6 h_c^3}{9} =$$

$$= \sum_{cyc} \frac{7h_a^6 h_c^3 + h_a^6 h_c^3 + h_a^6 h_c^3}{9} =$$

$$= \frac{1}{9} \sum_{cyc} (7h_a^6 h_c^3 + h_b^6 h_a^3 + h_c^6 h_b^3) \overset{\text{AM-GM}}{\geq}$$

$$\geq \frac{1}{9} \cdot 9 \sum_{cyc} \sqrt[9]{(h_a^6 h_c^3)^7 \cdot h_b^6 h_a^3 \cdot h_c^6 h_b^3} =$$

$$= \sum_{cyc} \sqrt[9]{h_a^{45} \cdot h_b^9 \cdot h_c^{27}} = \sum_{cyc} h_a^5 h_b h_c^3$$

Result (2) is true. Result (1) is true.

$$\sum_{cyc} \frac{h_a^3}{h_b^3} \geq \sum_{cyc} \frac{h_a^2}{h_b^2} = \sum_{cyc} \frac{(\frac{2F}{a})^2}{(\frac{2F}{b})^2} =$$

$$= \sum_{cyc} \left(\frac{4F^2}{a^2} \cdot \frac{b^2}{4F^2} \right) = \sum_{cyc} \frac{b^2}{a^2} =$$

$$\overset{\text{sine-law}}{=} \sum_{cyc} \frac{(2R\sin B)^2}{(2R\sin A)^2} = \sum_{cyc} \frac{4R^2 \sin^2 B}{4R^2 \sin^2 A} =$$

$$= \sum_{cyc} \frac{\sin^2 B}{\sin^2 A} = \frac{\sin^2 B}{\sin^2 A} + \frac{\sin^2 C}{\sin^2 B} + \frac{\sin^2 A}{\sin^2 C}$$

Equality holds for an equilateral triangle: $a = b = c$.

□

5751. Show that if $0 < a \leq b < \frac{\pi}{2}$, then:

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2 \tan(x)dx \leq 0$$

Daniel Sitaru

Solution 1 by Albert Stadler, Herrliberg, Switzerland.
It is sufficient to prove that:

$$\tan x \geq \frac{3x}{3 - x^2}, 0 \leq x < \frac{x}{2},$$

for then

$$0 \geq 2\int_a^b \left((x^2 - 3)\tan(x) + 3x \right) dx = 6\log(\cos(x)) \Big|_{x=a}^{x=b} + 3x^2 \Big|_{x=a}^{x=b} + 2\int_a^b x^2 \tan(x)dx =$$

$$= 6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2 \tan(x)dx.$$

To prove initially stated inequality we start from the product representation of the cosine function:

$$\cos x = \prod_{n=1}^{\infty}\Big(1 - \frac{4x^2}{\pi^2(2n-1)^2}\Big).$$

Logarithmic differentiation then gives

$$\tan x = -\frac{d}{dx}\log(\cos(x)) = 2\sum_{n=1}^{\infty}\frac{\frac{4x}{\pi^2(2n-1)^2}}{1-\frac{4x^2}{\pi^2(2n-1)^2}} = 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}(2n-1)^{2k}} =$$

$$= 2\sum_{k=1}^{\infty}\frac{4^k x^{2k-1}}{\pi^{2k}}\left(\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}\right) = 2\sum_{k=1}^{\infty}\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}}x^{2k-1}.$$

Thus, if $0 \le x < \frac{\pi}{2}$,

$$\tan x - \frac{3x}{3-x^2} = 2\sum_{k=1}^{\infty}\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}}x^{2k-1} - \sum_{k=1}^{\infty}\frac{1}{3^{k-1}}x^{2k-1} =$$

$$= \sum_{k=3}^{\infty}\left(2\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}}\right)x^{2k-1} \ge 0,$$

taking into account that $\frac{\pi}{2} < \sqrt{3}, (2) = \frac{\pi^2}{6}, (4) = \frac{\pi^4}{90}, (2k) > 1, k \ge 1, (2\pi)^2 < 40$ so that

$$2\frac{4^k(4^k-1)(2k)}{(2\pi)^{2k}} - \frac{1}{3^{k-1}} > 2\frac{4^k-1}{10^k} - \frac{1}{3^{k-1}} > 0, k \ge 3.$$

□

Solution 2 by Michel Bataille, Rouen, France.
The inequality is equivalent to

$$-3\int_a^b \tan(x)dx + 3\int_a^b xdx + \int_a^b x^2\tan(x)dx \le 0,$$

that is, to $\int_a^b f(x)dx \ge 0$ where

$$f(x) = 3\tan(x) - 3x - x^2\tan(x).$$

Thus, it suffices to prove that $f(x) \ge 0$ for $x \in [0, \frac{\pi}{2})$. Since $f(0) = 0$, it is even sufficient to prove that $f'(x) \ge 0$.
A simple calculation gives $f'(x) = \frac{1}{\cos^2(x)}, g(x)$ where $g(x) = 3\sin^2(x) - x\sin(2x) - x^2$.
Now, for $x \in [0, \frac{\pi}{2})$, we obtain

$$g'(x) = 6\sin(x)\cos(x) - \sin(2x) - 2x\cos(2x) - 2x = 2\sin(2x) - 2x(1+\cos(2x)) = 4\cos^2(x)(\tan(x) - x);$$

since $\tan(x) \ge x$, we have $g'(x) \ge 0$, hence $g(x) \ge g(0)$ and consequently $f'(x) \ge 0$, as desired. □

Solution 3 by Moti Levy, Rehovot, Israel.
We rewrite the problem statement as follow:

$$(1) \qquad \int_a^b x^2\tan(x)dx \le -3\ln(\cos(b)) - \frac{3}{2}b^2 + 3\ln(\cos(a)) + \frac{3}{2}a^2$$

Let

$$F(x) := -\left(3\ln(\cos(x)) + \frac{3}{2}x^2\right) \tag{2}$$

The inequality is equivalent to

$$\int_a^b x^2 \tan(x)dx \le F(b) - F(a), \tag{3}$$

but

$$F(b) - f(a) = \int_a^b 3(\tan(x) - x)dx.$$

Hence the original inequality is equivalent to

$$\int_a^b x^2 \tan(x)dx \le \int_a^b 3(\tan(x) - x)dx,$$

or to

$$\int_a^b \left((x^2 - 3)\tan(x) + x\right)dx \le 0. \tag{4}$$

We now prove (4) by showing that the integrand is negative in (a, b) where $0 < a \le b < \frac{\pi}{2}$

$$(x^2 - 3)\tan(x) + x \le 0. \tag{5}$$

Inequality (5) is equivalent to

$$\frac{\tan(x)}{x} \ge \frac{1}{3 - x^2} \tag{6}$$

The series expansion of $\frac{\tan(x)}{x}$ implies that

$$\frac{\tan(x)}{x} \ge 1 + \frac{1}{3}x^2 + \frac{2}{15}x^4. \tag{7}$$

One can check that

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3 - x^2} \ge 0,$$

since the function $30 - 2x^4 - x^6$ is concave in $0 < x < \frac{\pi}{2}$ then

$$1 + \frac{1}{3}x^2 + \frac{1}{15}x^4 - \frac{1}{3 - x^2} = \frac{30 - 2x^4 - x^6}{15(3 - x^2)} \ge 0 \text{ for } 0 < x < \frac{\pi}{2}, \tag{8}$$

It follows from (7) and (8) that the inequality $\frac{\tan(x)}{x} \ge \frac{1}{3-x^2}$ holds for $0 < x < \frac{\pi}{2}$. □

Solution 4 by Perfetti Paolo, dipatimento di matematica Universita di "Tor Vergata", Roma, Italy.

$$\frac{d}{da}\left(6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2\tan(x)dx\right) = 2((3 - a^2)\tan a - 3a)$$

$$3 - a^2 \ge 3 - \frac{\pi^2}{4} \text{ and } \tan a \ge a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^5}{315} \text{ thus}$$

$$(3 - a^2)\tan a - 3a \ge (3 - a^2)\left(a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^5}{315}\right) - 3a \ge$$

$$= \frac{a^5}{315}(21 + 9a^2 - 17a^4) \geq 0 \text{ for } a \leq \left(\frac{9 + \sqrt{1509}}{34}\right)^{\frac{1}{2}} \sim 1.186$$

Thus for $a \leq 1.18$ the inequality is proved.
Now let's define $b = \frac{\pi}{2} - a$. The inequality $(3 - a^2)\tan a - 3a$ becomes

$$(1) \qquad \left(3 - \left(\frac{\pi}{2} - b\right)^2\right)\frac{\cos b}{\sin b} - 3\left(\frac{\pi}{2} - b\right) \geq \left(3 - \left(\frac{3}{2} - b4\right)^2 \frac{1 - \frac{b^2}{2}}{b} - 3\left(\frac{\pi}{2} - b\right)\right.$$

for $0 \leq b \leq \frac{\pi}{2} - 1.18 \sim 0.3907$ and $\cos b \geq 1 - \frac{b^2}{2}$, and $\sin b \leq b$. The r.h.s of (1)

$$\frac{24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4}{8b} \geq 0, \quad 0 \leq b \leq \frac{2}{5}$$

$$f(b) = 24 + 4b^2 - 2\pi^2 + \pi^2 b^2 - 4\pi b - 4\pi b^3 + 4b^4 \text{ and}$$

$$f'(b) = 16b^3 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leq 16b^2 + 8b + 2\pi^2 b - 4\pi - 32\pi b^2 \leq 0$$

if and only if

$$2\pi - b(4 + \pi^2) + (16\pi - 8)b^2 \geq 0 \text{ (true by) } (4 + \pi^2)^2 - 8\pi(16\pi - 8) \sim -1011 < 0$$

This implies that $f(b)$ decreases and since

$$f\left(\frac{2}{5}\right) = \frac{15464}{625} - \frac{46\pi^2}{25} - \frac{232\pi}{125} \sim 0.75 \Rightarrow f(b) > 0$$

and this in turn implies that through (1) the inequality $(3 - a^2)\tan a - 3a > 0$ also for $1.18 \leq a \leq \frac{\pi}{2}$. This implies that

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b \tan(x)dx$$

increases with a and then the maximum value is attained when $a = b$ thus proving the inequality. □

Solution 5 by proposed by G.C. Greubel, Newport News, VA.
Using the series

$$\ln(\cos(x)) = -\frac{1}{2}\sum_{k=1}^{\infty} \frac{a_{2k}x^{2k}}{k}$$

$$\tan(x) = \sum_{k=1}^{\infty} a_{2k}x^{2k-1},$$

and integral

$$\int x^2 \tan(x)dx = \sum_{k=2}^{\infty} \frac{a_{2k-2}x^{2k}}{2k},$$

where

$$a_{2k} = \frac{4^k(4^k - 1)|B_{2k}|}{(2k)!}$$

with B_n being the Bernoulli numbers, then

$$S = 6\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2 - a^2) + 2\int_a^b x^2 \tan(x)dx$$

$$= -3\sum_{k=1}^{\infty} \frac{a_{2k}}{k}(b^{2k} - a^{2k}) + 3(b^2 - a^2) + 2\sum_{k=2}^{\infty} \frac{a_{2k-2}}{2k}(b^{2k} - a^{2k})$$

$$= -3\sum_{k=2}^{\infty}\frac{a_{2k}}{k}(b^{2k}-a^{2k}) + \sum_{k=2}^{\infty}\frac{a_{2k-2}}{k}(b^{2k}-a^{2k})$$

$$= -\sum_{k=2}^{\infty}\frac{3a_{2k}-a_{2k-2}}{k}(b^{2k}-a^{2k}).$$

Since $a_2 = 1$ and $a_4 = 1$ then:

$$S = -\sum_{k=3}^{\infty}\frac{3a_{2k}-a_{2k-2}}{k}(b^{2k}-a^{2k})$$

It is evident that $3a_{2n} > a_{2n-2}$ for $n \geq 3$ and leads to

$$6\ln\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + \int_a^b x^2\tan(x)dx \leq 0$$

for $b \geq a$. Equality occurs when $b = a$. □

Solution 6 by proposer.

$$(1+x^2)(1+y^2) \overset{\text{AM-GM}}{\geq} 2x(1+y^2)$$

$$(1+x^2)(1+y^2) \overset{\text{AM-GM}}{\geq} 2y(1+x^2)$$

By adding:

$$2(1+x^2)(1+y^2) \geq 2x(1+y^2) + 2y(1+x^2)$$

$$(1+x^2)(1+y^2) \geq x(1+y^2) + y(1+x^2)$$

$$\frac{1}{x(1+y^2)+y(1+x^2)} \geq \frac{1}{(1+x^2)(1+y^2)}$$

$$\int_a^b\int_a^b \frac{dxdy}{x(1+y^2)+y(1+x^2)} \geq \int_a^b\int_a^b \frac{dxdy}{(1+x^2)(1+y^2)} =$$

$$= \left(\int_a^b \frac{dx}{1+x^2}\right)\left(\int_a^b \frac{dy}{1+y^2}\right) = (\arctan b - \arctan a)^2$$

Equality holds for $a = b$. □

5757. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 f(x)dx = \frac{1}{2}$$

Show that

$$2 + \int_0^1 f^2(x)dx \geq 6\int_0^1 xf(x)dx$$

Daniel Sitaru

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
We prove the more general inequality

$$\frac{1}{4}\int_0^1 f^2(x)dx + 2\left(\int_0^1 f(x)dx\right)^2 \geq 3\int_0^1 f(x)dx \cdot \int_0^1 xf(x)dx \tag{1}$$

Then substituting the given integral value and clearing fractions gives us the desired inequality.
Now set $\int_0^1 f(x)dx = t$ and consider the quadratic polynomial

$$t^2 - 3\left(\int_0^1 \left(x - \frac{1}{3}\right) f(x)\right) t + \frac{1}{4}\int_0^1 f^2(x)dx \tag{2}$$

The discriminant of this polynomial is

$$D = 9\left(\int_0^1 \left(x - \frac{1}{3}\right) f(x)dx\right)^2 - \int_0^1 f^2(x)dx$$

The CBS inequality yields

$$D \le 9 \cdot \int_0^1 \left(x - \frac{1}{3}\right)^2 dx \cdot \int_0^1 f^2(x)dx - \int_0^1 f^2(x)dx$$
$$= \int_0^1 f^2(x)dx - \int_0^1 f^2(x)dx = 0.$$

Since $D \le 0$ and the coefficient of t^2 in (2) is positive, we see that the quadratic is nonnegative for all values of t. Therefore

$$\left(\int_0^1 f(x)dx\right)^2 + \frac{1}{4}\int_0^1 f^2(x)dx \ge 3\left(\int_0^1 \left(x - \frac{1}{3}\right) f(x)dx\right) \cdot \int_0^1 f(x)dx$$
$$= 3\left(\int_0^1 xf(x)dx - \frac{1}{3}\int_0^1 f(x)dx\right) \cdot \int_0^1 f(x)dx$$
$$= 3\int_0^1 xf(x)dx \cdot \int_0^1 f(x)dx - \left(\int_0^1 f(x)dx\right)^2.$$

which gives us (1). □

Solution 2 by Perfetti Paolo, dipartimento de matematica Universita di "Tor Vergata", Roma, Italy.

$$\int_0^1 (f - 3x + a)^2 dx = \int_0^1 (f^2 - 6xf + 9x^2 + a^2 + 2af - 6xa)dx \ge 0$$

Thus

$$\int_0^1 (f^2 - 6xf)dx \ge -3 - a^2 - a + 3a \ge -2 \Leftrightarrow (a-1)^2 \le 0 \Leftrightarrow a = 1$$

and this concludes the proof. □

Solution 3 by Albert Stadler, Herrliberg, Switzerland.
Suppose $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x)dx = \frac{1}{2}$. Show that

$$2 + \int_0^1 f^2(x)dx \ge 6\int_0^1 xf(x)dx.$$

Solution of the problem
We have

$$0 \le \int_0^1 (f(x) - 3x + 1)^2 dx = \int_0^1 (f^2(x) + 9x^2 + 1 - 6xf(x) + 2f(x) - 6x)dx =$$

$$= \int_0^1 f^2(x)dx + 3 + 1 - 6\int_0^1 xf(x)dx + 1 - 3$$

which implies

$$2 + \int_0^1 f^2(x)dx \geq 6\int_0^1 xf(x)dx.$$

□

Solution 4 by Moti Levy, Rehovot, Israel.
Let $F(x) := \int_0^x f(t)dt$. After integrations by parts,

$$\int_0^1 xf(x)dx = xF(x)\Big]_0^1 - \int_0^1 F(x)dx = \frac{1}{2} - \int_0^1 F(x)dx \tag{3}$$

Substituting (3) in the original inequality we get

$$2 + \int_0^1 \Big(F'(x)\Big)^2 dx \geq 3 - 6\int_0^1 F(x)dx \int_0^1 xf(x)dx,$$

or,

$$\int_0^1 \Big(6F(x) + \Big(F'(x)\Big)^2\Big)dx \geq 1$$

Let

$$J(F) := \int_0^1 \Big(6F(x) + \Big(F'(x)\Big)^2\Big)dx \geq 1,$$

then the original inequality is equivalent to the statement that the functional $J(F)$ is grater than or equal to 1 for every differentiable function $F(x)$, which satisfies the boundary conditions $F(0) = 0$ and $F(1) = \frac{1}{2}$.
Every differentiable function $F(x)$, which satisfies the boundary conditions $F(0) = 0$ and $F(1) = \frac{1}{2}$ can be expressed as $F(x) = \frac{3}{2}x^2 - x + \eta(x)$, where $\eta(x)$ is defferentiable function in the interval $(0,1)$ and $\eta(0) = \eta(1) = 0$.
Then

$$J\Big(\frac{3}{2}x^2 - x + \eta(x)\Big) = \int_0^1 \Big(6\Big(\frac{3}{2}x^2 - x + \eta(x)\Big) + \Big(3x - 1 + \eta(x)\Big)^2\Big)dx$$

$$= \int_0^1 \Big(6\Big(\frac{3}{2}x^2 - x\Big) + (3x-1)^2\Big)dx + \int_0^1 6\eta(x) + 2(3x-1)\eta'(x) + (\eta'(x))^2 dx$$

$$= 1 + \int_0^1 6\eta(x) + 2(3x-1)\eta(x) + (\eta'(x))^2 dx$$

Applying integration by parts, we obtain

$$J\Big(\frac{3}{2}x^2 - x + \eta(x)\Big) = 1 + \int_0^1 (\eta'(x))^2 dx$$

It follows that $JF(x)) \geq 1$ for every differentialbe function $F(x)$ which satisfies $F(0) = 0$ and $F(1) = \frac{1}{2}$. The functional $J(F)$ attains its minimum when $\eta'(x) = 0$ which implies (together with the boundary conditions $\eta(0) = \eta(1) = 0$ that $\eta(x) = 0$ in $(0,1)$. □

Solution 5 by Michel Bataille, Rouen, France.
Let $I = \int_0^1 (3x-1)f(x)dx$. Then, we have

$$6\int_0^1 xf(x)dx = 2I + 2\int_0^1 f(x)dx = 2I + 1.$$

On the other hand, since $\int_0^1 (3x-1)^2 dx = \int_0^1 (9x^2-6x+1)dx = 1$, the Cauchy-Schwarz inequality gives

$$\int_0^1 f^2(x)dx = \left(\int_0^1 (3x-1)^2 dx\right)\left(\int_0^1 f^2(x)dx\right) \geq \left(\int_0^1 (3x-1)f(x)dx\right)^2 = I^2$$

As a result, we obtain

$$2 + \int_0^1 f^2(x)dx - 6\int_0^1 xf(x)dx \geq 2 + I^2 - 2I - 1 = (I-1)^2 \geq 0$$

and the desired inequality follows. □

Solution 6 by proposer.

$$2 + \int_0^1 f^2(x)dx \geq 6\int_0^1 xf(x)dx$$

$$\int_0^1 f^2(x)dx - 6\int_0^1 xf(x)dx + 1 + 3 - 3 + 1 \geq 0$$

$$\int_0^1 f^2(x)dx - 6\int_0^1 xf(x)dx + 2\cdot\frac{1}{2} + 9\cdot\frac{1}{3} - 6\cdot\frac{1}{2} + 1 \geq 0$$

$$\int_0^1 f^2(x)dx - 6\int_0^1 xf(x)dx + 2\int_0^1 f(x)dx + \int_0^1 (9x^2-6x+1)dx \geq 0$$

$$\int_0^1 f^2(x)dx - 2\int_0^1 (3x-1)f(x)dx + \int_0^1 (3x-1)^2 dx \geq 0$$

$$\int_0^1 (f(x)-(3x-1))^2 dx \geq 0$$

Equality holds for $f(x) = 3x - 1$.

$$LHS = 2 + \int_0^1 (3x-1)^2 dx = 2 + 9\cdot\frac{1}{3} - 6\cdot\frac{1}{2} + 1 = 6 - 3 = 3$$

$$RHS = 6\int_0^1 x(3x-1)dx = 18\cdot\frac{1}{3} - 6\cdot\frac{1}{2} = 6 - 3 = 3$$

□

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