PROPOSED PROBLEM

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4919. If $A, B \in M_6(\mathbb{R})$ are matrices such that:

$$A^2 + B^2 = AB + A + B - I_6$$

then

$$\det(BA - AB) \geq 0$$

Solution 1 by editors.
Let $\omega = \exp(\frac{2\pi i}{3})$; then we have $\omega^3 = 1, 1 + \omega + \omega^2 = 0, \overline{\omega} = \omega^2$. Observe that $I_6 + \omega^2 A + \omega B = \overline{I_6 + \omega A + \omega^2 B}$; it follows that

$$\det(I_6 + \omega A + \omega^2 B)\det(I_6 + \omega^2 A + \omega B)$$

$$= \det(I_6 + \omega A + \omega^2 B) \cdot \overline{\det(I_6 + \omega A + \omega^2 B)}$$

$$= |\det(I_6 + \omega A + \omega^2 B)|^2 \geq 0$$

On the other hand, by the hypothesis, we calculate

$$(I_6 + \omega A + \omega^2 B)(I_6 + \omega^2 A + \omega B)$$

$$= I_6 + A^2 + B^2 + (\omega + \omega^2)(A + B) + \omega BA + \omega^2 AB$$

$$= AB + A + B - (A + B) + \omega BA + \omega^2 AB = \omega(BA - AB).$$

Now the conclusion follows from taking the determinant on both sides of the above equation. □

Solution 2 by editors.
Write $I = I_6$. Let $X = A - I$ and $Y = B - I$. Then the hypothesis implies that

$$(X + I)^2 + (Y + I)^2 = (X + I)(Y + I) + (X + I) + (Y + I) - I,$$

that is, $X^2 + Y^2 = XY$. Furthermore, note that $BA - AB = YX - XY$ and

$$(X + Y)^2 + 3(X - Y)^2 = 4(X^2 + Y^2) - 2(XY + YX) = 2(XY - YX).$$

Thus,

$$\det(BA - AB) = \frac{\det((X + Y)^2 + 3(X - Y)^2)}{(-2)^6}$$

$$= \frac{\det(X + Y + i\sqrt{3}(X - Y))\det(X + Y - i\sqrt{3}(X - Y))}{2^6}$$

$$= \frac{|\det(X + Y - i\sqrt{3}(X - Y)|^2}{2^6} \geq 0$$

□

Solution 3 by proposer.
Let be $\omega_{1,2} = \frac{-1\pm i\sqrt{3}}{2} \in \mathbb{C}$ and $\omega \in \{\omega_1, \omega_2\}$.

$$\omega^2 + \omega + 1 = 0 \text{ and}$$

$$\omega \neq 1; \omega^3 = 1 \tag{1}$$

$$\omega^4 = \omega^3 \cdot \omega = 1 \cdot \omega = \omega$$

$$\omega^4 = \omega \tag{2}$$

$$|\omega| = 1; \omega \cdot \overline{\omega} = (|\omega|)^2 = 1$$

$$\overline{\omega} = \frac{1}{\omega} = \frac{\omega^2}{\omega^3} = \frac{\omega^2}{1} = \omega^2$$

$$\overline{\omega} = \omega^2 \tag{3}$$

$$\overline{\omega^2} = \frac{1}{\omega^2} = \frac{\omega}{\omega^3} = \frac{\omega}{1} = \omega$$

$$\overline{\omega^2} = \omega \tag{4}$$

$$\omega^2 + \omega + 1 = 0 \Rightarrow \omega^2 + 1 = -\omega$$

$$\omega^2 + 1 = -\omega \tag{5}$$

Let be: $C = A + \omega B + \omega^2 I_6$

$$\overline{C} = \overline{A + \omega B + \omega^2 I_6} = A + \overline{\omega} B + \overline{\omega^2} I_6$$

By (3); (4):

$$\overline{C} = A + \omega^2 B + \omega I_6$$

$$\det(C \cdot \overline{C}) = \det C \cdot \det(\overline{C}) = \det C \cdot \overline{\det C} = (|\det C|)^2 \geq 0$$

$$\det((A + \omega B + \omega^2 I_6)(A + \omega^2 B + \omega I_6)) \geq 0$$

$$\det(A^2 + \omega^2 AB + \omega A + \omega BA + \omega^3 B^2 + \omega^2 B + \omega^2 A + \omega^4 B + \omega^3 I_6) \geq 0$$

By (1); (2):

$$\det(A^2 + \omega^2 AB + \omega A + \omega BA + B^2 + \omega^2 B + \omega^2 A + \omega B + I_6) \geq 0$$

$$\det(A^2 + B^2 + I_6 + \omega(BA + A + B) + \omega^2(AB + A + B)) \geq 0$$

By hypothesis:

$$\det(AB + A + B + \omega(BA + A + B) + \omega^2(AB + A + B)) \geq 0$$

$$\det((1 + \omega^2)(AB + A + B) + \omega(BA + A + B)) \geq 0$$

By (3):

$$\det((-\omega)(AB + A + B) + \omega(BA + A + B)) \geq 0$$

$$\det(\omega(BA + A + B - AB - A - B)) \geq 0$$

$$\det(\omega(BA - AB)) \geq 0$$

$$\omega^6 \det(BA - AB) \geq 0 \Rightarrow (\omega^3)^2 \cdot \det(BA - AB) \geq 0$$

By (1):

$$1^2 \cdot \det(BA - AB) \geq 0$$

$$\det(BA - AB) \geq 0$$

□

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