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Let $A, B \in M_3(\mathbb{C}); C, D \in M_5(\mathbb{C}); E, F \in M_7(\mathbb{C})$. Show that:

$$|rank(AB) - rank(BA)| + |rank(CD) - rank(DC)| + |rank(EF) - rank(FE)| \leq 6$$

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Solution 1 by proposer.
Lemma: If $U, V \in M_n(\mathbb{C}); n \in \mathbb{N}; n \geq 2$ then:

$$|rank(UV) - rank(VU)| \leq \left[\frac{n}{2}\right] \tag{1}$$

$[*]$ - great integer function.

Proof.
Without lost of generalization suppose that:

$$rank(UV) \geq rank(VU)$$

$$rank(UV) - rank(VU) \leq rank(UV) \leq rank(U)$$

$$rank(UV) - rank(VU) \leq rank(U) \tag{2}$$

By Sylvester's inequality:

$$rank(VU) \geq rank(U) + rank(V) - n$$

$$-rank(VU) \leq n - rank(U) - rank(V)$$

$$rank(UV) - rank(VU) \leq rank(UV) + n - rank(U) -$$

$$-rank(V) \leq rank(V) + n - rank(U) - rank(V) = n - rank(U)$$

$$rank(UV) - rank(VU) \leq n - rank(U) \tag{3}$$

By adding $(2); (3)$:

$$2(rank(UV) - rank(VU)) \leq n$$

$$rank(UV) - rank(VU) \leq \frac{n}{2}$$

$$\text{But } rank(UV) - rank(VU) \in \mathbb{N} \Rightarrow$$

$$\Rightarrow rank(UV) - rank(VU) \leq \left[\frac{n}{2}\right]$$

□

Back to the problem. By (1):

$$|rank(AB) - rank(BA)| \leq \left[\frac{3}{2}\right] = 1 \tag{4}$$

$$|rank(CD) - rank(DC)| \leq \left[\frac{5}{2}\right] = 2 \tag{5}$$

$$|rank(EF) - rank(FE)| \leq \left[\frac{7}{2}\right] = 3 \tag{6}$$

By adding $(4); (5); (6)$:

$$|rank(AB) - rank(BA)| + |rank(CD) - rank(DC)| +$$
$$+|rank(EF) - rank(FE)| \leq 1 + 2 + 3 = 6$$

□

Solution 2 by Oliver Geupel.
Let n be a positive integer and $M, N \in M_n(\mathbb{C})$. We will show that

$$|rank(MN) - rank(NM)| \leq \frac{n}{2} \tag{1}$$

Since the left hand side is an integer the result then follows immediately. We know that $rank(MN)$ and $rank(NM)$ are at most $\min(rank(M), rank(N))$. If $\min(rank(M), rank(N)) \leq \frac{n}{2}$, then (1) follows immediately. So suppose that $rank(M)$ and $rank(N)$ are both greater than $\frac{n}{2}$. Applying Sylvester's inequality, we obtain

$$\min(rank(MN), rank(NM)) \geq rank(M) + rank(N) - n.$$

Therefore

$$\begin{aligned} -\frac{n}{2} < rank(M) - n &= (rank(M) + rank(N) - n) - rank(N) \\ &\leq rank(MN) - rank(NM) \\ &\leq rank(M) - (rank(M) + rank(N) - n) \\ &= n - rank(N) < \frac{n}{2} \end{aligned}$$

It follows that $|rank(MN) - rank(NM)| \leq \frac{n}{2}$ and we are done. □

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