CALCULUS MARATHON

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In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal. Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897). In three years the website of RMM was visited by over 5,000,000 people from all over the world. With over 6000 proposed problems posted, over 10,000 solutions and many math articles and math notes RMM is a big chance for young mathematicians from whole world to be known as great proposers and solvers. This book is a small part of RMM-Interactive Journal. Many thanks to RMM-Team for proposed problems and solutions (few of these appear even in this book).The authors are grateful to: ***Henry Ricardo, Nassim Nicholas Taleb(USA), Nguyen Van Nho, Ngo Minh Ngoc Bao,Hung Nguyen Viet, Khanh Hung Vu,Tran Hong,Hoang Le Nhat Tung, Duong Viet Thong, Nguyen Thanh Nho,Quang Minh Tran (Vietnam),Seyran Ibrahimov, Adil Abdullayev, Rovsen Pirguliyev,Togrul Ehmedov (Azerbaijan),D.M.Bătinețu-Giurgiu,Neculai Stanciu, Marin Chirciu, Koczinger Eva, Kovacs Bela,Vasile Mircea Popa, Radu Butelca, Geanina Tudose,Serban George Florin, Mihaly Bencze, Remus Florin Stanca, Andrew Okukura,Zaharia Burghelea (Romania),Amit Dutta, Soumitra Mandal, Ravi Prakash, Subhajit Chattopadhyay, Rajeev Rastogi, Soumava Chakraborty, Sagar Kumar, Srinivasa Raghava, Nirapada Pal, Shivam Sharma, Nishant Kumar, Saptak Bhattacharya, Nitin Gurbani, Vidyamanohar Sharma Astakala, Kartick Chandra Betal (India), Jhoaw Carlos (Bolivia),Regragui El Khammal, Abdelhak Maoukouf, Anas Adlany (Morocco),Kevin Soto Palacios (Peru), Santos Martins Junior (Belgium), Carlos Suarez (Ecuador),Boris Colakovic, Artan Ajredini (Serbie),Chris Kyriazis, Michael Sterghiou, Dimitris Kastriotis, George Apostolopoulos,Lazaros Zachariadis (Greece),Abdallah El Farissi (Algerie),Omran Kouba, Abdallah Almalih (Syria),Naren Bhandari(Nepal), Shafiqur Rahman, Arafat Rahman Akib (Bangladesh), Myagmarsuren Yadamsuren(Mongolia), Ahmed Albaw, Khalef Ruhemi, Nader Al Homsi (Jordan),Francis Fregeau (Canada),Yubian Bedoya Henao (Columbia),Yen Tung Chung (Taiwan), Abdul Mukhtar, Ekpo Samuel,Eliezer Okeke, Ibrahim Abdulazeez ,Tobi Joshua (Nigeria),Igor Soposki (Macedonia),Kays Tomy (Tunisia), Vural Ozap (Turkey), Max Wong (Hong Kong), Ruanghaw Chaoka, Sanong Huayrerai (Thailand), Nawar Alasadi (Iraq),Sameer Shihab (Saudi Arabia)***

CONTENT

CHAPTER 1

EQUATIONS.SYSTEMS-PROBLEMS

1.1 *Solve for natural numbers:*

$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=3025$$

1.2 *Find $x, y \in \mathbb{Z}$ such that:*

$$x^2+6xy+5y^2=6y\sqrt{x^2+2xy}$$

1.3 *Find $(a_n) \subset \mathbb{N}$ such that:*

$$\sum_{k=0}^{n} a_k \binom{n}{k}\binom{n+1}{k+1}=(n+1)\binom{2n}{n}, n \in \mathbb{N}$$

1.4 *Solve for natural numbers:*

$$1-\sum_{k=1}^{n}\frac{k}{\sqrt{k!}\left(k+1+\sqrt{k+1}\right)} \le \frac{1}{12\sqrt{5}}$$

1.5 *Find $x \in \mathbb{N}$ such that:*

$$\sum_{k=1}^{n}\frac{x+k}{3k+2}=\sum_{k=1}^{n}\frac{k+1}{x+3k+1}$$

1.6 *Solve for natural numbers:*

$$\sum_{k=3}^{x}\binom{\binom{k}{2}}{2} \le 168$$

1.7 *Solve for natural numbers:*

$$\frac{1^2\cdot 2!+2^2\cdot 3!+\cdots+n^2(n+1)!-2}{(n+1)!}=108$$

1.8 *Find* $n \in \mathbb{N}, n > 1$ *such that:*

$$\frac{2!\,(2^3-1)+3!\,(3^3-1)+\cdots+n!\,(n^3-1)-2}{n^2-2}=40320$$

1.9 *Solve for real numbers:*

$$\begin{cases} 2^y+2^z+tan^{-1}z=9 \\ |3\,sin\,x-4\,cos\,x|=y^2-6y+14 \end{cases}$$

1.10 *Find* $n \in \mathbb{N}, n \geq 3$ *such that:*

$$\sum_{k=3}^{n}\binom{n}{k}\binom{k-1}{2}=21(2^{n-2}-1)$$

1.11 *Solve for real numbers:*

$$\sqrt{x^3-2x^2+2x}+3\cdot\sqrt[3]{x^2-x+1}+2\cdot\sqrt[4]{4x-3x^4}=\frac{x^4-3x^3}{2}+7$$

1.12 *Solve for real numbers:* $arcsin[x]\cdot arccos[x]=\frac{\pi x}{2}-x^2$

1.13 *Find* $x, y, z \in \mathbb{R}^*$ *such that:*

$$\frac{x^2}{1+x^2}+\frac{y^2}{(1+x^2)(1+y^2)}+\frac{z^2}{(1+x^2)(1+y^2)(1+z^2)}+\frac{1}{8xyz}=1$$

1.14 *Find* $x, y, z, t \in \mathbb{R}$ *such that:*

$$5x^2+5y^2+5z^2+5t^2-5xy-5yz-5zt-5t+2=0$$

1.15 *Solve for real numbers:*

$$\begin{cases} sin[x]+cos(x-[x])=\frac{\sqrt{3}}{2} \\ sin(x-[x])+cos[x]=\frac{3}{2} \end{cases},\ [*]-great\ integer\ function$$

1.16 *Solve for $x > 0$ the equation:*

$$e^x + \pi^x + \frac{1}{e^x} + \frac{1}{\pi^x} = \frac{1}{cot^{-1}(e^x)} + \frac{1}{cot^{-1}(\pi^x)}$$

1.17 *Solve for real numbers:*

$$x\left(3\sin\sqrt{x} - 2\sqrt{x}\right) = \sin^3\sqrt{x}$$

1.18 *Solve for real positive numbers:*

$$\begin{cases} 27\sqrt{\left(x^2 + \frac{1}{y^2}\right)\left(y^2 + \frac{1}{z^2}\right)\left(z^2 + \frac{1}{x^2}\right)} = 8(x + y + z)^3 \\ x + y + z = \frac{1}{xyz} \end{cases}$$

1.19 *Solve the equation in $\mathbb{R}$:*

$$\sqrt[10]{x^2 + 4x - 9} + (2x - 10)\sqrt{x^2 + 1} = \sqrt[5]{|x + 1|}.$$

1.20 *Solve for real numbers:*

$$|x^2 - 1|^{[x]} + |x^2 - 2|^{[x]} + |x^2 - 3|^{[x]} = \tan\frac{\pi[x]}{4} + \cot\frac{\pi[x]}{4},$$ $[*]$ - *great integer function*

1.21 *Solve for real numbers:*

$$\begin{cases} 27^x + 2 = 3^{y+1} \\ 27^y + 2 = 3^{z+1} \\ 27^z + 2 = 3^{x+1} \end{cases}$$

1.22 *Solve for real numbers:*

$$(x + \sin x + \cos x)^3 = (x + \sin x - \cos x)^3 + (x + \cos x - \sin x)^3 + (\sin x + \cos x - 3)^3$$

1.23 *Solve for real numbers:*

$$\frac{1}{1 + 8^x} + \frac{1}{1 + 27^x} + \frac{1}{1 + 64^x} = \frac{3}{1 + 24^x}$$

1.24 If $z \in \mathbb{C}$, $\alpha \geq 2$ then:

$$|Rez|^{\alpha} + |Imz|^{\alpha} \geq 2^{1-\frac{\alpha}{2}} \cdot |z|^{\alpha}$$

1.25 *For* $z_1, z_2 \in \mathbb{C}$, *satisfy:* $|z_1 + z_2| = |z_1| + |z_2|$.

Prove: $|z_1 - z_2| = max\{|z_1|; |z_2|\} - min\{|z_1|; |z_2|\}$

1.26 *If* $z_1, z_2, z_3 \in \mathbb{C}^*, |z_1| = |z_2| = |z_3| = 1, (z_1 + z_2)(z_2 + z_3)(z_3 + z_1) \neq 0$,

$A(z_1), B(z_2), C(z_3), \sum \frac{z_1^2}{(z_1+z_2)(z_1+z_3)} = 3$ *then* $AB = BC = CA$

1.27 *If* $A(z_1), B(z_2), C(z_3), z_1, z_2, z_3 \in \mathbb{C}^*, |z_1| = |z_2| = |z_3| = 1$,

$\sum \frac{1}{2+|z_1+z_2|} = 1$ *then:* $AB = BC = CA$

1.28 *Solve for real numbers:*

$[tan\, x] \cdot (cot\, x - [cot\, x]) = (tan\, x - [tan\, x]) \cdot [cot\, x]$, $[*]$ – *great integer function*

1.29 *Solve for* $x \in (0{,}2\pi)$: $sin(1 + x) + sin(1 + 2x) + \cdots + sin(1 + 10x) = 0$

1.30 *Find* $A, B, C \in (0, \pi), A + B + C = \pi$ *such that:*

$$\begin{cases} cos\, A\, |cos\, B| + cos\, B\, |cos\, A| = 1 + cos\, 2C \\ cos\, B\, |cos\, C| + cos\, C\, |cos\, B| = 1 + cos\, 2A \\ cos\, C\, |cos\, A| + cos\, A\, |cos\, C| = 1 + cos\, 2B \end{cases}$$

1.31 *Solve for real numbers:*

$$\begin{cases} \frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy \\ x^5 + y^5 = 8xy \end{cases}$$

1.32 *Solve for real numbers:*

$$\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1 \\ x^2 + y^2 = \left(\dfrac{x^2}{5} + \dfrac{y^2}{4}\right)^2 \end{cases}$$

1.33 *Find* $x, y, z \in \mathbb{R}$ *such that:*

$$\begin{cases} x + y + z = 3 \\ 4(max(x, y, z) - min(x, y, z))^2 \geq 3\sum |x - y|^2 \end{cases}$$

1.34 *Find* $x, y, z \in (0, \infty)$ *such that:*

$$\begin{cases} x + y + z = xyz \\ \dfrac{x}{y^3 z^2} + \dfrac{y}{z^3 x^2} + \dfrac{z}{x^3 y^2} = \dfrac{1}{3} \end{cases}$$

1.35 *Solve for real numbers:*

$$\begin{cases} \begin{vmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{vmatrix} = 0 \\ x + y - \sqrt{xy} = \sqrt{\dfrac{x^2 + y^2}{2}} \end{cases}$$

1.36 *Solve for real numbers:*

$$\begin{cases} \tan x \tan y \tan z = 6 \\ \tan x \tan y + \tan x \tan z + \tan y \tan z = 11 \\ x + y + z = \pi \end{cases}$$

1.37 *Solve for real numbers*

$$:\begin{cases} 1+2\sqrt{y}=2\sqrt{x+1} \\ \frac{2\sqrt{y}}{12y+1}+\frac{\sqrt{x+1}}{x+4}+\frac{2\sqrt{y(x+1)}}{3x+4y+3}=\frac{3}{4} \end{cases}$$

1.38 *Find $z\in\mathbb{C}$ such that:*

$$\begin{cases} |z-7-i|=3\sqrt{2} \\ |z-1-7i|\le 3\sqrt{2} \end{cases}$$

1.39. *Solve for real numbers:*

$$\begin{cases} \cos 2x+\cot 3y=\tan 5z \\ \cot 3y+\cot 5z=\tan 2x \\ \cot 5z+\cot 2x=\tan 3y \end{cases}$$

1.40 *Solve for real numbers:*

$$\begin{cases} [y^{2017}]+\left[\frac{x}{1}\right]\cdot\left[\frac{x}{2}\right]\cdot\left[\frac{x}{3}\right]\cdot\ldots\cdot\left[\frac{x}{2017}\right]=y^{2017} \\ [x^{2017}]+\left[\frac{y}{1}\right]\cdot\left[\frac{y}{2}\right]\cdot\left[\frac{y}{3}\right]\cdot\ldots\cdot\left[\frac{y}{2017}\right]=x^{2017} \end{cases},\ [*]-great\ integer\ function$$

1.41 *Find $x,y,z\ge 2, t\ge 1$ such that:*

$$\begin{cases} \sqrt{x-2}+\sqrt[3]{y-2}+\sqrt[4]{z-2}=3\sqrt{t} \\ x+y+z=6+3t \end{cases}$$

1.42 *Find all functions $f:(0,\infty)\to\mathbb{R}$ such that:*

$$f\left(x\sqrt{yz}\right)+f\left(y\sqrt{zx}\right)+f\left(z\sqrt{xy}\right)=f\left(\sqrt{xyz}\right)$$

1.43 *If $f:[2,\infty)\to\mathbb{R}, f(x)+f\left(\frac{1}{1-x}\right)=x, \forall\, x\ge 2$ then $\forall x,y,z\ge 2$:*

$$2f(x)+\frac{1}{x}+2f(y)+\frac{1}{y}+2f(z)+\frac{1}{z}\ge 3\sqrt[3]{\frac{x^2y^2z^2}{(x-1)(y-1)(z-1)}}$$

1.44 *Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:*

$$x + y \leq 2^x f(x) + 2^y f(y) \leq 2^{x+y} f(x + y), \forall x, y \in \mathbb{R}$$

1.45 *Find $f: \mathbb{R} \to \mathbb{R}$ such that:*

$$(x - y)^2 \left(f^2(x) - f^2(y)\right) = (x + y) f^3(x - y)$$

1.46 *Find all continuous functions:*

$$f: \mathbb{R} \to \mathbb{R}, f(x^3) - f(y^3) = (x^2 + xy + y^2) f(x - y), \forall x, y \in \mathbb{R}$$

1.47 *Find all continuous functions $f: \mathbb{R} \to (0, \infty)$ such that:*

$$f(x) f(2x) f(4x) = 2^x, \forall\, x \in \mathbb{R}$$

1.48 *Solve for real numbers:*

$$\begin{cases} 0 < x, y, z < 1 \\ \dfrac{x}{1 - x^2} + \dfrac{y}{1 - y^2} + \dfrac{z}{1 - z^2} = \dfrac{4xyz}{(1 - x^2)(1 - y^2)(1 - z^2)} \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

1.49 *Solve for real numbers:*

$$\frac{1}{(x + 1)^3} + \frac{1}{(x + 1)^2} + \frac{1}{x + 1} = 156 + log_5(x + 1)$$

1.50 $f, g: (0, \infty) \to \mathbb{R}\,, f(x) = sin\left(ln \frac{x(x+1)}{x+2}\right) + sin\left(ln \frac{(x+1)(x+2)}{x}\right)$

$$g(x) = sin\left(ln\left(x(x + 1)(x + 2)\right) - sin\left(ln \frac{x(x+2)}{x+1}\right)\right).$$

Solve the equation: $f(x) = g(x)$.

1.51 *Solve for real numbers:*

$$\begin{cases} 3^x + 3^y + 3^z + 3^t = 24 \\ log_z x + log_z t = y \\ \dfrac{x}{x^4 + y^2} + \dfrac{y}{x^2 + y^4} = \dfrac{1}{xy} \end{cases}$$

1.52 *Solve the system of equations:*

$$\begin{cases} \sqrt{x} + \sqrt{y} + \sqrt{z} + 1 = 4\sqrt{xyz} \\ xy + yz + zx + 3 = 2 \cdot \left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \end{cases}$$

1.53 *Solve for integers:*

$$\begin{cases} x(y + z) = y^2 + z^2 - 6 \\ y(z + x) = z^2 + x^2 - 6 \\ z(x + y) = x^2 + y^2 - 6 \end{cases}$$

1.54 *Find* $x, y, z \in (0, \infty)$ *such that:*

$$\begin{cases} x^3 - y^3 = ln\left(\dfrac{y}{x}\right) \\ y^5 - z^5 = ln\left(\dfrac{z}{y}\right) \\ 2x^y + 3y^z + 5z^x = 10 \end{cases}$$

1.55 *Solve over the set of real numbers the following system of equations written on base – 42 numeral system:*

$$a_a^2 + a_3^2 + a_4^2 + \cdots + a_{2018}^2 = 4 \cdot (97a_1 - 1)$$

$$a_1^2 + a_3^2 + a_4^2 + \cdots + a_{2018}^2 = 4 \cdot (97a_2 - 1)$$

$$a_1^2 + a_2^2 + a_4^2 + \cdots + a_{2018}^2 = 4 \cdot (97a_3 - 1)$$

… ….

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{2017}^2 = 4 \cdot (97a_{2018} - 1)$$

1.56. *Solve for real numbers:*

$$x^3 + x^2 + 2x + log(x^3 + 2x + 1) = y, \quad y^3 + y^2 + 2y + log(y^3 + 2y + 1) = x$$

1.57 *Prove that:*

$$csc\left(\frac{\pi}{14}\right) - 4\,cos\left(\frac{2\pi}{7}\right) = 2$$

1.58 *Solve for complex numbers:*

$$\left|z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right|^2 + \left|z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right|^2 + |z - 1|^2 - 3|z|^2 = z$$

1.59 *Solve for $\mathbb{R}$:*

$$log_2(2^{cos\,x} + 1) + log_3(3^{cos\,x} + 2) + log_4(4^{cos\,x} + 3) = \sqrt[3]{27 - cos\,x}$$

1.60 *Find $x, y, z \geq 0$ such that:*

$$\frac{2x^2 + 4}{z^2 + 2y + 3} + \frac{2y^2 + 4}{x^2 + 2z + 3} + \frac{2z^2 + 4}{y^2 + 2x + 3} = 3$$

1.61 *Solve for real numbers:*

$$\frac{1}{x + 1} + \frac{1}{x(x + 2)} + \frac{1}{x(x + 1)(x + 3)} + \cdots + \frac{1}{x(x + 1) \cdot \ldots \cdot (x + 99)(x + 101)} =$$

$$= \frac{1}{3} - \frac{1}{x(x + 1)(x + 2) \cdot \ldots \cdot (x + 100)(x + 101)}$$

1.62 *Solve for real numbers:*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^x & e^{-x} & e & e^{-1} \\ e^{2x} & e^{-2x} & e^2 & e^{-2} \\ e^{4x} & e^{-4x} & e^4 & e^{-4} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ e^x & e^{-x} & e & e^{-1} \\ e^{3x} & e^{-3x} & e^3 & e^{-3} \\ e^{4x} & e^{-4x} & e^4 & e^{-4} \end{vmatrix} = 0$$

1.63 *Solve for real numbers:*

$$\frac{\left(e^{\pi x^{2018}}+1\right)\left(e^{2\pi x^{2018}}+1\right)\left(e^{4\pi x^{2018}}+1\right)\left(e^{8\pi x^{2018}+1}\right)\ldots\left(e^{2^{n}\pi x^{2018}}+1\right)}{\left(\pi^{\frac{2e}{x}}+1\right)\left(\pi^{\frac{4e}{x}}+1\right)\left(\pi^{\frac{8e}{x}}+1\right)\left(\pi^{\frac{16e}{x}}+1\right)\ldots\left(\pi^{\frac{2^{n+1}e}{x}}+1\right)}=\frac{e^{2^{n+1}\pi x^{2018}-1}}{\pi^{\frac{2^{n+2}e}{x}}-1}$$

1.64. *Solve for real numbers:*

$$cos^{12}x+4\,cos^{8}x\,sin\,2x+2\,sin^{2}2x\,(3\,cos^{4}x-4)+4\,sin^{3}2x-3\,cos\,x+19=0$$

1.65 *Find all continuous functions:* $f, g, h: \mathbb{R} \to \mathbb{R}$ *such that:*

$$f\left(\frac{x+y}{2}\right)=\frac{g(x)+h(y)}{2}, \forall x, y \in \mathbb{R}$$

1.66 *Solve for real numbers:*

$$\begin{cases} 2x\sqrt{1-y^2}+2y\sqrt{1-x^2}=\sqrt{3} \\ 2y\sqrt{1-z^2}+2z\sqrt{1-y^2}=\sqrt{3} \\ 2z\sqrt{1-x^2}+2x\sqrt{1-z^2}=\sqrt{3} \end{cases}$$

1.67 *Solve for real numbers:*

$$\begin{cases} 0<x,y,z<1 \\ \dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)} \\ x^2+y^2+z^2=1 \end{cases}$$

1.68 *Solve for real numbers:*

$$x^3+x^2+2x+log(x^3+2x+1)=y$$

$$y^3+y^2+2y+log(y^3+2y+1)=x$$

1.69 *Find all continuous functions* $f: \mathbb{R} \to \mathbb{R}$ *such that:*

$$f(x)+f(y)+x^2y+xy^2=f(x+y), \forall x, y \in \mathbb{R}$$

1.70 *Solve for real positive numbers:*

$$\begin{cases} \frac{x+y}{1+xy} + \frac{xy}{1+x} + \frac{xy}{1+y} + \frac{x+y+2xy}{(1+x)(1+y)xy} = 3 \\ \sin x = \cos y \end{cases}$$

1.71 *Find all real numbers x satisfying the following equation:*

$$(x + \{x\})^2 - (x + \{x\}) = 6\lfloor x \rfloor \{x\} - 1$$

where $\lfloor x \rfloor$ and $\{x\}$ denote the integer part and fractional part of x, respectively.

1.72 *Solve the following equation in set of real numbers:*

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125$$

CHAPTER 2

MATRIX.DETERMINANTS-PROBLEMS

2.1 $A \in M_4(\mathbb{R}), det(A^2 + 3I_4) = det(A^2 + 2A + 2I_4) = 0.$

Find: $\Omega = det\, A.$

2.2 *If* $A \in M_3(\mathbb{R})$ *then:*

$$det(I_3 + A^2) = 0 \Leftrightarrow Tr\, A = det\, A \text{ and } Tr\, A^* = 1.$$

2.3 $A \in M_2(\mathbb{R}), det\, A = tr\, A = 1.$

Solve for real numbers:

$$det(A^4 + I_2) + 10\, det(A^2 + I_2) + x = 4\, det(A^3 + I_2) + 16\, det(A + I_2)$$

2.4 *If* $A \in M_2(\mathbb{R})$ *then:*

$$det(A^2 + 2A + 2I_2) \geq (2 + Tr\, A)^2$$

2.5 $A, B \in M_2(\mathbb{R}), det\, A \neq 0, det\, B \neq 0, Tr(AB^{-1}) = det(AB^{-1}) = 1$

Find: $\Omega = det(I_2 + A^{-1}B)$

2.6 *If* $A, B \in M_5(\mathbb{R}), A^3 + 7I_5 = A^2, B^3 + 9I_5 = B^2$ *then:*

$$det(AB) > 0$$

2.7 *If* $A, B \in M_5(\mathbb{R}), A^3 - 2I_5 = A^2, B^3 - 3I_5 = B^2$ *then:*

$$det(AB) > 0$$

2.8 *Find* $A, B \in M_2(\mathbb{R})$ *such that:*

$$det\, A < 0, det(A - B) > 0, det(A + B) < 0, det(2A + B) > 0$$

2.9 *If* $A \in M_4(\mathbb{C}), det\, A \neq 0, Tr\, A = 0$ *then:*

$$Tr\,(A^3) = 3(det\, A)(Tr\, A^{-1})$$

2.10 *GENERALIZATION FOR A DAN RADU SECLAMAN'S INEQUALITY*

If $A, B \in M_n(\mathbb{R}), n \geq 2, p \geq 1, n, p \in \mathbb{N}, A^{2p+1} + B^{2p} = I_n, A^{4p+1} = A^{2p}$ *then:* $det(I_n + A^{2p} + B^{2p}) \geq 0$

2.11 *If* $A, B \in M_2(\mathbb{C}), det(A + B) = 1$ *then:*

$$det(A \cdot det\, B + B \cdot det\, A) = det(AB)$$

2.12 *If* $A \in M_2(\mathbb{R}), B \in M_3(\mathbb{R}), C \in M_4(\mathbb{R}),$

$A^2 - A = I_2, B^2 - B = I_3, C^2 - C = I_4$ *then:* $|det\, A + det\, B + det\, C| < 28$

2.13 *If* $A, B, C, D \in M_n(\mathbb{C}), n \in \mathbb{N}, n \geq 2, det(ABCD) \neq 0$ *then:*

$$rank(AB \cdot det(CD) + CD \cdot det(AB)) = rank\left(\frac{1}{det\, C \cdot det\, D} B^{-1}A^{-1} + \frac{1}{det\, A \cdot det\, B} D^{-1}C^{-1}\right)$$

2.14 *If* $a, b, c > 0$*, different in pairs,* $ab + bc + ca = 1$ *then:*

$$\begin{vmatrix} a+b & ab & 0 & 0 \\ 1 & a+b & ab & 0 \\ 0 & 1 & a+b & ab \\ 0 & 0 & 1 & a+b \end{vmatrix} + \begin{vmatrix} b+c & bc & 0 & 0 \\ 1 & b+c & bc & 0 \\ 0 & 1 & b+c & bc \\ 0 & 0 & 1 & b+c \end{vmatrix} + \begin{vmatrix} c+a & ca & 0 & 0 \\ 1 & c+a & ca & 0 \\ 0 & 1 & c+a & ca \\ 0 & 0 & 1 & c+a \end{vmatrix} > 3$$

2.15 $A \in M_n(\mathbb{R}), det\, A \neq 0, \alpha \in (-1,1), A^2 + A^{-2} = \alpha(A + A^{-1})$

Find: $|det\, A|$

2.16 *Solve for real numbers:*

$$\begin{vmatrix} 1 & 3 + sin\, x & 2 + 3\, sin\, x & 2\, sin\, x \\ 1 & 2 + sin\, x + cos\, x & 2\, sin\, x + sin\, x\, cos\, x & sin\, 2x \\ 1 & 1 + sin\, x + cos\, x & sin\, x + cos\, x + sin\, x\, cos\, x & sin\, x\, cos\, x \\ 1 & 3 + cos\, x & 2 + 3\, cos\, x & 2\, cos\, x \end{vmatrix} = 0$$

2.17 *If* $A, B, C \in M_n(\mathbb{Z}), n \geq 3, (A^*B^*)^* = BA, (B^*C^*)^* = CB$ *then:*

$$det\, A + det\, B + det\, C < \sqrt{10}$$

2.18 *If* $X, Y, Z \in M_n(\mathbb{R}), n \geq 2, n \in \mathbb{N}, XY = YX, YZ = ZY, ZX = XZ$ *then:*

$$det(9X^2 + 5Y^2 + 5Z^2 + 12XY + 6YZ + 12ZX) \geq 0$$

2.19 $A, B \in M_2(\mathbb{R}), Tr((AB)^2) = Tr(A^2B^2), n \in \mathbb{N}, n \geq 2$. *Find:*

$$\Omega = Tr[(AB - BA)^n]$$

2.20 *If* $A \in M_2(\mathbb{Z})$ *then:*

$$\Omega = det(A + A^T + A^*) + det(-A + A^T + A^*) + det(A - A^T + A^*) + det(A + A^T - A^*)$$

is divisible with 12.

2.21 $A \in M_2(\mathbb{R}), det\, A = tr\, A = 1$. *Solve for real numbers:*

$$det(A^4 + I_2) + 10\, det(A^2 + I_2) + x = 4\, det(A^3 + I_2) + 16\, det(A + I_2)$$

2.22 *If* $A, B \in M_2(\mathbb{C}), det\, A \neq 0, det\, B \neq 0$ *then:*

$$det(A\, det\, B + B\, det\, A) + det\left(\frac{A}{det\, A} + \frac{B}{det\, B}\right) = det(A + B)\left(det(AB) + \frac{1}{det(AB)}\right)$$

2.23 *In* $\Delta\, ABC$ *the following relationship holds:*

$$\frac{2\begin{vmatrix} s + a^2 & ab & ac \\ ab & s + b^2 & bc \\ ac & bc & s + c^2 \end{vmatrix}}{s^2} \geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

2.24 *If* $a, b, c \in (0, \infty)$ *then:*

$$\sum \begin{vmatrix} 1 + a & 1 & 1 \\ 1 & 1 + b & 1 \\ 1 & 1 & 1 + c \end{vmatrix} \geq \frac{48abc}{1 + a + b + c}$$

2.25 *If* $a, b, c, d, e, f \in (0, \infty)$ *then:*

$$\begin{vmatrix} a & \sqrt{ad} & \sqrt{ae} \\ \sqrt{ad} & b + d & \sqrt{de} + \sqrt{bf} \\ \sqrt{ae} & \sqrt{de} + \sqrt{bf} & c + e + f \end{vmatrix} > 0$$

2.26 *If* $\Delta(x) = \begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix}, a, b, c \in (0, \infty)$ *then:*

$$\frac{\Delta'(a+b+c)}{\Delta(a+b+c)} \le \frac{1}{6\sqrt[3]{abc}} + \frac{1}{2}\sum \frac{1}{\sqrt{ab}}$$

2.27 $A_{2n+1} = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ 0 & a & b & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & b \\ b & 0 & 0 & \cdots & 0 & a \end{pmatrix} \in M_{2n+1}(\mathbb{R}_+^*), n \in \mathbb{N}^*,\ \Omega_{2n+1} = det(A_{2n+1})$

Prove that: $\frac{\Omega_{2n+7}}{\Omega_{2n+5}} \ge \frac{\Omega_{2n+3}}{\Omega_{2n+1}}$

2.28 *If* $a,b,c,d,e,f > 0$ *then:*

$$64\begin{vmatrix} 1 & b & b & b & b \\ a & c & 0 & 0 & 0 \\ a & 0 & d & 0 & 0 \\ a & 0 & 0 & e & 0 \\ a & 0 & 0 & 0 & f \end{vmatrix} \le (a+f)^2(b+e)^2(c+d)^2\left(\frac{1}{ab} - \frac{1}{c} - \frac{1}{d} - \frac{1}{e} - \frac{1}{f}\right)$$

2.29 *In* $\Delta\ ABC$ *the following relationship holds:*

$$\begin{vmatrix} a & 0 & c & b \\ 0 & a & b & c \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} \ge 432r^4$$

2.30 *If* $a,b,c \in (0,\infty), a \ne b \ne c \ne a$

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}$$

then:

$$\frac{\Delta_1 - \Delta_2}{(b-a)(a-c)(b-c)} \ge 12\sqrt[6]{(abc)^5}$$

2.31 *If* $A,B \in M_n(\mathbb{R}), AB = BA$ *then:*

$$det\big((A^2+AB+2B^2)(A^2+2AB+3B^2)(A^2+3AB+4B^2)\big) \ge 0$$

2.32 *If* $a, b, c \geq 0, a + b + c = 3$ *then:*

$$\begin{vmatrix} a^2 + b^2 + c^2 & a^3 + b^3 + c^3 & a^4 + b^4 + c^4 \\ a^3 + b^3 + c^3 & a^4 + b^4 + c^4 & a^5 + b^5 + c^5 \\ a^4 + b^4 + c^4 & a^5 + b^5 + c^5 & a^6 + b^6 + c^6 \end{vmatrix} \leq (b - a)^2(c - a)^2(c - b)^2$$

2.33 *If* $A, B, C \in M_n(\mathbb{R}), AB = BA = BC = CB = CA = AC = O_3$ *then:*

$$det(I_3 + 2A + 3B + 4C + 4A^2 + 9B^2 + 16C^2) \geq 0$$

2.34 *If* $a, b, c \geq 0$ *then:*

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2 + b^2 & a^2 + c^2 & 1 \\ b^2 & a^2 + b^2 & 0 & b^2 + c^2 & 1 \\ c^2 & a^2 + c^2 & b^2 + c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \leq \frac{1}{8}\prod(a + b)^2$$

2.35 *If* $a, b, c \in [0,1]$ *then:*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \frac{a^2}{1 + a^2} & \frac{b^2}{1 + b^2} & \frac{c^2}{1 + c^2} \end{vmatrix} \leq \frac{1}{2}$$

2.36 $A(a, b, c), B(d, e, f), C(g, h, i)$ *belongs to* $S: x^2 + y^2 + z^2 = R^2$.

Prove that: $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 \leq R^6$$

2.37 *In* $\Delta\ ABC$ *the following relationship holds:*

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} \leq 4abcR\sqrt{\left(\sum sin^2 A\right)\left(\sum cos^2 A\right)}$$

2.38 *If* $A, B, C \in M_n(\mathbb{R}), AB = BA, AC = CA, BC = CB, n \in \mathbb{N}, n \geq 2$ *then:*

$$det(A^2 - 6AB + 10B^2 + 16BC + 10C^2 - 6AC) \geq 0$$

2.39 $\Omega = \begin{vmatrix} sin^2 x & sin^2 y \cdot cos^2 x & cos^2 y \cdot cos^2 x \\ cos^2 y \cdot cos^2 x & sin^2 x & sin^2 y \cdot cos^2 x \\ sin^2 y \cdot cos^2 x & cos^2 y \cdot cos^2 x & sin^2 x \end{vmatrix}, x, y \in \mathbb{R}$

Prove that: $|\Omega| \leq 1$.

2.40 *If* $A, B \in M_2(\mathbb{R}), AB = BA, det\, A = \alpha > 0, det(A + i\alpha B) = 0$ *then find:*

$$\Omega = det(A^2 - \alpha AB + \alpha^2 B^2)$$

2.41 *Find* $A, B \in M_2(\mathbb{R})$ *such that:*

$$det\, A < 0, det(A - B) > 0, det(A + B) < 0, det(2A + B) > 0$$

CHAPTER 3

LIMITS.SERIES-PROBLEMS

3.1 *Find:*

$$\Omega = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)(i+3)}$$

3.2 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{n\int_1^{n^2}\left[\sqrt{x}\right]dx}{\int_1^{n^3}\left[\sqrt{x}\right]dx}, [*] - great\ integer\ function$$

3.3 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}e^{-\left(\frac{k}{n}\right)^2}}$$

3.4 *Calculate:*

$$\Omega = \lim_{n\to\infty}\left\{\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\dots\binom{n+1}{n+1}}}{e^{\frac{(n+1)}{2}}(n+1)^{-\frac{3}{2}}} - \frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{3}{2}}}\right\}$$

3.5 *If* $a, b, c \in \mathbb{N}^*$,

$$\Omega(a) = \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)\cdot\ldots\cdot(n+a)}$$

then:

$$\left(\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c}\right)^{a+b+c} \geq \frac{1}{a^b\cdot b^c\cdot c^a\cdot(a!)^b\cdot(b!)^c\cdot(c!)^a}$$

3.6 *Prove that:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{1!\,2!\,3!\cdot\ldots\cdot k!}\right)^{\frac{1}{k}} \leq e(e-1)$$

3.7 *Find:*

$$L = \lim_{n\to\infty}\left(\int_{0}^{\left(\frac{\pi}{2}\right)^{n}} \sin\left(\sqrt[n]{x}\right)dx\right)$$

3.8 *If:*

$$\Omega_n = \lim_{x\to 0}\left(\frac{1}{(2^x-1)^n} - \frac{1}{\left(\frac{x\log 2}{1!}+\frac{(x\log 2)^2}{2!}+\cdots+\frac{(x\log 2)^n}{n!}\right)^n}\right); n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\Omega_k$$

3.9 $A = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, A^n = \begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}, n \in \mathbb{N}^*, \Omega = \lim_{n\to\infty}\left(\frac{a_n+b_n}{c_n+d_n}\right).$

Prove that: $\Omega < 1$

3.10 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}}$$

3.11 *Find:*

$$\Omega = \lim_{n\to\infty}\left(\left(\frac{1}{1+n^{2016}}+\frac{1}{2+n^{2016}}+\cdots+\frac{1}{n^{4032}}\right)\cdot \ln n\right)$$

3.12 *Find:*

$$\Omega = \lim_{n\to\infty}\left(n\cdot \tan\left(\frac{\pi e n!}{5}\right)\right)$$

3.13 *Let* $a_n, b_n \in (0,\infty)$ *and* $n \geq 1$, $\lim_{n\to\infty}(a_{n+1}-a_n) = a$ *and*

$b_n = \lim_{n\to\infty}\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\ldots\sqrt[n]{n!}}$. *Find:* $\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1}-\frac{a_n\cdot b_n}{n}\right)$

3.14 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n^2]{2!!\sqrt{3!!}\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}$$

3.15 *Find:*

$$\Omega = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(\frac{1}{(25k^2+5k-6)(n-k+1)^2}\right)\right)$$

3.16 *If* $a,b \in \mathbb{N}, a < b$, $\Omega(a,b) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(a+i)(b+i)}$

then:

$$\Omega(a,b) \geq \left(\frac{a!}{b!}\right)^{\frac{1}{b-a}}$$

3.17 *If:*

$$a,b,c > 0, a+b+c = 1, \Omega(a,b) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{\sqrt[n]{b}+\sqrt[n]{a+1}}\right)^n$$

then:

$$\sum \sqrt{b(a+1)} \cdot \Omega(a,b) \leq 2$$

3.18 *Find:*

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}\right)$$

3.19 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{1}{n!}\int_0^1 ((1-x)^n + \cos nx)e^x dx$$

3.20 *If :*

$$a,b > 0, |x| < 1, |y| < 1, \Omega(x) = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-x)^i}{i+1}$$

then:

$$(a+b)\Omega\left(\frac{ax+by}{a+b}\right) \leq a\Omega(x) + b\Omega(y)$$

3.21 *Find:*

$$L = \lim_{n\to\infty}\left(\sum_{k=1+100n}^{200n}\frac{1}{k} + \sum_{k=1+200n}^{400n}\frac{1}{k} + \sum_{k=1+300n}^{400n}\frac{1}{k}\right)$$

3.22 *If:*

$$\mathbf{\Omega}(x,y) = \sum_{n=1}^{\infty}\frac{2n^2 + (2x+2y+5)n + 2xy + 6x - y}{3^n(n+y)(n+y+1)(n+y+2)}, x,y > 0$$

Prove that:

$$\Omega(x,y)\cdot\Omega(y,x)\le\frac{1}{9\sqrt[3]{xy}}$$

3.23 *Evaluate:*

$$\lim_{x\to0}\frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x}$$

3.24 *Evaluate:*

$$\lim_{n\to\infty}\int_n^{n+1}e^{\frac{1}{x}}\,dx$$

3.25 *If:*

$$\Omega(a)=\lim_{n\to\infty}n^2\left(\sqrt[n+5]{e^{a^2+a+1}}-\sqrt[n+7]{e^{a^2+a+1}}\right),a>0$$

then:

$$\frac{\Omega(a)}{b+c}+\frac{\Omega(b)}{c+a}+\frac{\Omega(c)}{a+b}>a+b+c+3$$

3.26 *Find:*

$$\Omega=\lim_{n\to\infty}\frac{\sum_{k=2}^{n}\left(k^2\cdot\sqrt[k]{\binom{2k}{k}}\right)}{n(n+1)(2n+1)}$$

3.27 *Find:*

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(n-k+1)e^{-k^2}}{1+2+\cdots+n}$$

3.28 *If:*

$$a_n>0,n\ge1,\lim_{n\to\infty}a_n=a,b,c>0$$

then find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{a_k}{b + ca_k}$$

3.29 *Find:*

$$\Omega = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{k}{(k+1)!} \right) \left(\sum_{k=1}^{n} \frac{k(k+2)}{\left((k+1)!\right)^2} \right) \left(\sum_{k=1}^{n} \frac{k(k^2+3k+3)}{\left((k+1)!\right)^3} \right)$$

3.30 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \sum_{p=m+1}^{n} p \right)$$

3.31 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{n(n+1)(n+2)\cdot \ldots \cdot (2n-2)\, arctan \frac{\pi}{2^n}}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}$$

3.32 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{3n+1} \sum_{k=2}^{n} \frac{1}{\sqrt[k]{k!}}$$

3.33 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{(e+\sqrt{2})\cdot(e+\sqrt{3})\cdot \ldots \cdot (e+\sqrt{n})}{(\pi+\sqrt{2})\cdot(\pi+\sqrt{3})\cdot \ldots \cdot (\pi+\sqrt{n})}$$

3.34 *If* $a, b, c, x, y, z > 0, a + b + c = 1$

$$\Omega(a) = \lim_{n\to\infty} n \left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}-1}} - 1 \right)$$

then: $\Omega(ax + by + cz) \geq \Omega(x^a y^b z^c)$

3.35 *Find:*

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1 + 6 + 11 + 16 + \cdots + (10k - 9)}{2k - 1}$$

3.36 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n + 3} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{k!}}$$

3.37 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n^5} \sum_{k=1}^{n} k^4 \, arctan\left(\frac{k}{n}\right)^5$$

3.38 *Find:*

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{m=1}^{n} \left(\left(1 + \frac{1}{m}\right) \sum_{p=1}^{m} p!\,(1 + p^2)\right)}$$

3.39 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1 \cdot n + 3 \cdot (n - 1) + 5 \cdot (n - 2) + \cdots + (2n - 1) \cdot 1}{(n + 1)^4 - n^4}$$

3.40 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \sqrt[n]{7^i \cdot 5^{n-i}}$$

3.41 *If:*

$$\Omega_n \in [1, \infty), n \geq 1, lim_{n\to\infty} \Omega_n = \Omega \in \mathbb{R}.$$

Find:

$$\lim_{n\to\infty} e^{\sqrt[n]{(1+\ln\Omega_1)(1+\ln\Omega_2)\cdot\ldots\cdot(1+\ln\Omega_n)}-1}$$

3.42 *Find:*

$$\Omega = \lim_{n\to\infty} n\sqrt[n]{\frac{\left((2n)!!\right)^2}{(2n)!}}$$

3.43 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{13\cdot 25\cdot 37\cdot\ldots\cdot(12n-11)}{7\cdot 19\cdot 31\cdot\ldots\cdot(12n-5)}$$

3.44 *Find:*

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\tan^2\frac{x}{2^k}\right)\right)$$

3.45 *Compute:*

$$\lim_{n\to\infty}\left(\sqrt[3n+3]{(n+1)!}-\sqrt[3n]{n!}\right)\cdot\sqrt[3]{n^2}$$

3.46 *Compute:*

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[2n+2]{(2n+1)!!}}-\frac{n}{\sqrt[2n]{(2n-1)!!}}\right)^{\sqrt{n}}$$

3.47 *Find:*

$$\Gamma = \lim_{n\to\infty}\left(\Omega(n)-\Omega(n+1)\right), \Omega(n) = \int_1^e \frac{dx}{x(1+x^3)^n}, n\in\mathbb{N}^*$$

3.48 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!}$$

3.49 *Find:*

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n} arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right)$$

3.50 *In $\Delta\, ABC, A, B, C$ – angles. Find:*

$$\Omega(x) = \lim_{n\to\infty}\sum_{k=1}^{n}\begin{vmatrix} sin\,A & sin\,B & sin\,C \\ sin(A+kx) & sin(B+kx) & sin(C+kx) \\ cos(A+kx) & cos(B+kx) & cos(C+kx) \end{vmatrix}, x \in \mathbb{R}$$

3.51 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{cos\frac{1}{n^2} + 2cos\frac{4}{n^2} + 3cos\frac{9}{n^2} + \cdots + ncos1}{n^2}$$

3.52 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{1}{(n+1)^2 4^n}\cdot\sum_{k=0}^{n}(2n+1-k)\binom{2n+1}{k}$$

3.53 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{1}{2^n}\sum_{k=3}^{n}\frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-k+3)}{1\cdot 2\cdot 3\cdot\ldots\cdot k}$$

3.54 *If $a, b, c > 0$ find:*

$$\Omega = \lim_{x\to 0}\frac{log_a\left(log_b\left(log_c\left(\frac{c^b}{cos\,x}\right)\right)\right)}{log_c\left(log_b\left(log_a\left(\frac{a^b}{cos\,x}\right)\right)\right)}$$

3.55 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[3n]{(3n)!}}{(3n)!}$$

3.56 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\binom{k}{i}\right)}{\sqrt[n]{n!}}$$

3.57 *Find:*

$$L = lim_{n\to\infty} \sqrt[n]{p! + \frac{(p+1)!}{1!} + \frac{(p+2)!}{2!} + \cdots + \frac{(p+n)!}{n!}}, p \in \mathbb{N}^*, p - \textit{fixed}$$

3.58 *If:*

$$(x_n)_{n\geq 0}, x_0 > 0, x_{n+1} = x_n + \frac{1}{x_n^p}, p \in \mathbb{N}^*, p - \textit{fixed}$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n\sqrt[p+1]{n}}$$

3.59 *Find:*

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2} + \sum_{k=1}^{n}\frac{k^p}{k^{4p} + k^{2p} + 1}\right)^{n^{3p-1}}, p \in \mathbb{N}^*$$

3.60 *If:*

$$x_0 > 0, x_{n+1} = x_n + \frac{a}{x_n}, a > 0, n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} x_i x_j}{n^3}$$

3.61 *Find:*

$$\Omega = \lim_{n\to\infty}(n + 1)!\left(e - \sum_{k=0}^{n}\frac{1}{k!}\right)$$

3.62 *If:*

$$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{a_{k+1}\sqrt{k+1}}\right)$$

3.63 *Compute:*

$$\lim_{n\to\infty} \sqrt[4]{n!} \prod_{k=1}^{n} \left((k+1)^{\frac{3}{4}} - k^{\frac{3}{4}}\right)$$

3.64 *If:*

$$\Omega_n = \int_0^{\frac{\pi}{4}} (4x^2 - \pi x + 4n^2)\, ln(1 + tan\, x)\, dx, n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\Omega_n}{1 + 2 + 3 + \cdots + n}$$

3.65 *If α, β and γ are three distinct real values such that*

$$\frac{sin\, \alpha + sin\, \beta + sin\, \gamma}{sin(\alpha+\beta+\gamma)} = \frac{cos\, \alpha + cos\, \beta + cos\, \gamma}{cos(\alpha+\beta+\gamma)} = 2 \text{ and}$$

$cos(\alpha + \beta) + cos(\beta + \gamma) + cos(\alpha + \gamma) = a$. *Then find the value of*

$$\lim_{x\to a} \frac{\sqrt{x^2 - a^2}}{\sqrt{x - a} + \sqrt{x} - \sqrt{a}}$$

3.66 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \sqrt[n]{e^{2na+(i+j)b}}, a, b \ge 0$$

3.67 *If:*

$$x_n = \left(1 + \frac{1}{n}\right)^n, y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!},$$

$$z_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{2x_n + 3y_n + 5z_n}{5(e + \gamma)}\right)^n$$

3.68 *If* $a, b \in \mathbb{N}$ *then:*

$$\left(\sqrt{a} + \sqrt{b}\right)\sqrt[4]{ab} \le \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \sqrt[n]{a^k b^{n-k}} \le \left(\sqrt{a} + \sqrt{b}\right)^2$$

3.69 *If:*

$$\lim_{n\to\infty} \frac{a_n \cdot a_{n+2}^6 \cdot a_{n+4}}{a_{n+1}^4 \cdot a_{n+3}^4} = 10, a_n > 0, n \in \mathbb{N}, n \ge 2$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n^4]{a_n}$$

3.70 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{2^{2n}(n + 1)} \sum_{k=0}^{n-1} (n - k) \binom{2n + 1}{2n - 2k}$$

3.71 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{(n-1)\cdot\frac{1}{n} + (n-2)\cdot\left(\frac{1}{n}+\frac{1}{n-1}\right) + \cdots + 1\cdot\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}\right)}{(n+1)^3 - n^3}$$

3.72 *Find:*

$$\Omega = \lim_{n\to\infty}\left((n+1)\cdot\sqrt[5n+5]{(5n+5)\cos\frac{\pi}{n+1}} - n\cdot\sqrt[5n]{5n\cos\frac{\pi}{n}}\right)$$

3.73

$$\Omega(x) = \frac{1}{x^{n+2}}\left(\prod_{k=1}^{n} \tan^{-1}(nx) - \prod_{k=1}^{n} \sin(nx)\right), n \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{x\to 0} \Omega(x)$$

3.74

$$\Omega_n = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} x^{n-1} \sin^n dx + \int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} x^{n-1} \arcsin^n x\, dx; n \in \mathbb{N}, n \geq 1$$

Find:

$$\lim_{n\to\infty}(n^2\Omega_n)$$

3.75 *Find:*

$$\Omega = \lim_{n\to\infty}\frac{1}{n}\left(\tan^{-1} n + \frac{1}{2}\tan^{-1}(n-1) + \frac{1}{3}\tan^{-1}(n-2) + \cdots + \frac{1}{n}\tan^{-1} 1\right)$$

3.76 *Find:*

$$\Omega = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}}$$

3.77 *Find:*

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n+1} + \sum_{k=0}^{n} \left(\frac{1}{k+1}\binom{n}{k} \sum_{i=1}^{k} \binom{i}{k}\right)}$$

3.78 *Find:*

$$\Omega = \lim_{n\to\infty} \left(n! \cdot \prod_{k=1}^{n} (tan^{-1}(k+1) - tan^{-1} k)\right)$$

3.79 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{e^{1+n^2}} + \frac{2}{e^{2+(n-1)^2}} + \frac{3}{e^{3+(n-2)^2}} + \cdots + \frac{n}{e^{n+1}}\right)$$

3.80 *Find:*

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(k+n)^5}{7 + tan^{-1}(k+n) + (k+n)^6}$$

3.81 *Find:*

$$\Omega_n = \lim_{x\to 0} \frac{ln(1 + sinh^n x) - ln^n(1 + sinh x)}{x^{n+1}}, n \in \mathbb{N}, n \geq 2$$

3.82 *Find:*

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{sin\left(\frac{k}{n}\right)}{\sqrt[7]{k + n^7}}$$

3.83 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{\sin n}{n}\left(\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}} + \left(1-\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}\right)$$

3.84

$$\Omega(x) = \lim_{n\to\infty} \sum_{k=1}^{n} 2^{k-1} \tan\frac{x}{2^{k-1}} \tan^2\frac{x}{2^k}$$

Prove that in acute $\Delta\, ABC$ the following relationship holds:

$$\Omega(A) + \Omega(B) + \Omega(C) > A \cdot B \cdot C - \pi$$

3.85

$$\Omega(a,b,c) = \sum_{n=1}^{\infty} \frac{an^2 + bn + c}{n!}, a, b, c > 0$$

Prove that:

$$\Omega(a,b,c) + \Omega(b,c,a) + \Omega(c,a,b) \geq 3(4e-1)\sqrt[3]{abc}$$

3.86 *Find:*

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right)$$

3.87 *If:*

$$x_n, y_n > 0, x_n \neq y_n, n \in \mathbb{N}, \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = p > 0$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n^{y_n} - y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}} - \left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}}$$

3.88 *Find:*

$$\Omega = \lim_{k\to\infty} \left(2k \sum_{n=k+1}^{\infty} \frac{1}{(n-k)(n+k)} - \log\left(\frac{k^2}{k+1}\right) \right)$$

3.89 *Find:*

$$\Omega = \lim_{n\to\infty} \left(\sqrt[7]{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)(\log n)^6} - \log n \right)$$

3.90 *Find:*

$$\Omega_n(a) = \sum_{k=0}^{n} (k^2 - a^2 + 1)(a + k)!, a, n \in \mathbb{N}$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\Omega_n(a) - (a+1)!}$$

3.91 *Find:*

$$\Omega = \lim_{p\to\infty} \sqrt[p]{\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2 + kp - 1}{(p + k + 1)!} \right)}$$

3.92 *Find:*

$$\Omega(a) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3^k} \sin^3(3^k \sin a)$$

If $a, b, c \in \left[0, \frac{\pi}{2}\right)$ *then:*

$$4\left(b\Omega(a) + c\Omega(b) + a\Omega(c)\right) \leq 3(a^2 + b^2 + c^2)$$

3.93 *Find:*

$$\Omega_n = \sum_{k=1}^{n} \left(\int_{-\frac{1}{k}}^{\frac{1}{k}} \left((2x^8 + 3x^6 + 1) \cdot cos^{-1}(kx)\right) dx \right)$$

Find:

$$\Omega = \lim_{n\to\infty} (\Omega_n - \pi \cdot H_n)$$

3.94 *Find:*

$$\Omega = \lim_{n\to\infty} \left(\int_{\pi}^{2\pi} \left(\frac{|sin(nx)|}{x^2} \right) dx \right)$$

3.95 *Find:*

$$\Omega = \sum_{n=1}^{\infty} \left(log\left(\frac{n^2 + 3n + 2}{n^2 + 3n}\right) + \frac{1}{4} log\left(\frac{(n-1)^2 + 3(n-1) + 2}{(n-1)^2 + 3(n-1)}\right) + \cdots + \frac{1}{n^2} log\frac{3}{2} \right)$$

3.96 *Let:*

$$\Omega_n(x) = \int_{1}^{x} \left(\frac{t^n - 1}{t - 1}\right) dt, n \in \mathbb{N}, n \geq 1$$

Find:

$$\Omega = \lim_{x\to 1} \frac{tan^{-1}(nx - n) - \Omega_n(x)}{(x-1)^2}$$

3.97 *Find:*

$$\Omega = \lim_{n\to\infty} n\left(1 - \frac{log\left(1 + \frac{\sqrt[n]{10}}{n}\right)^{n+1}}{log\left(1 + \frac{\sqrt[n+1]{10}}{n+1}\right)^{n}}\right)$$

3.98 *Find:*

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}}\sum_{k=1}^{n}\left(3^k \cdot tan^{-1}\left(\frac{3}{k^2 - k - 1}\right)\right)\right)$$

3.99 *Let:*

$$\Omega(x) = -\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{n + x}{(n+1)(n+2)(n+3)}, x \in \mathbb{R}$$

If $a \in (0,1), b > 1$ *then:*

$$\left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < 1 + \Omega(a) \cdot \Omega(b)$$

3.100 *Let:*

$$x_1 = 1, x_2 = 3, x_n = x_{n-2} + 2x_{n-1}, n \geq 3$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{(-1)^{n+1}(x_n^2 - 2x_n x_{n-1} - x_{n-1}^2)}{n}\right)$$

3.101 *Let:*

$$\alpha(x) = \frac{4}{3}\sum_{n=1}^{\infty}\left(\left(-\frac{1}{3}\right)^n \cdot cos^3(3^{n-1}x)\right), x \in \left(0, \frac{\pi}{2}\right), \beta(x) = \alpha\left(\frac{\pi}{2} - x\right)$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to 0\\ x>0}} \left(\frac{\beta(x)\beta(3x)\cdot \ldots \cdot \beta((2n-1)x)}{\beta(2x)\beta(4x)\cdot \ldots \cdot \beta(2nx)}\right)}$$

3.102 *Let:*

$$a, b, c > 0, 2e(a+b+c) = 3e+2, \Omega(a) = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a} - e\right)$$

Prove that:

$$\Omega(a)\cdot\Omega(b)\cdot\Omega(c) \le \frac{1}{27}$$

3.103 *Find:*

$$\Omega = \lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{1}{x}\left(\frac{\cos 3x}{1+3x} - \frac{\cos 2x}{1+2x}\right) dx$$

3.104 *Find:*

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}}$$

3.105 *If:*

$$\Omega(n) = \sum_{k=1}^{\infty} \frac{2k^2 + 2nk + k - 1}{(2k+2n+2)!!}$$

Find:

$$\Omega = \lim_{n\to\infty} \left(n!\cdot\Omega(n)\right)$$

3.106 *If:*

$a_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, n = 2,3,\ldots$ *Find the limit*

$$\lim_{n\to\infty}\left[\frac{1}{(n-2)(a_n+a_{n-2})}\right]^{2n}$$

3.107 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{1}{n}\sum_{k=2}^{n}\frac{1+k^2}{(k-1)k(k+1)!}}$$

3.108 *Find:*

$$\Omega = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{n}+\frac{2k}{n^2}+\frac{k^2}{n^3}\right)$$

3.109 *Find:*

$$\Omega = \lim_{n\to\infty}\left(n^7\cdot\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}\right)$$

3.110 *If:*

$$\Omega_n = \binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots+(n-6)\binom{7}{7}, n\geq 7$$

Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\Omega_n}$$

3.111

Find:

$$\Omega = \lim_{n\to\infty}\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}}$$

3.112 *Find:*

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)\right)$$

3.113 *Find:*

$$\Omega = \lim_{n\to\infty} n^2\left(\sqrt[n+5]{7}-\sqrt[n+8]{7}\right)$$

3.114 *If:*

$$\omega_n = sin\left(\frac{1}{n+1}\right)+sin\left(\frac{1}{n+2}\right)+\cdots+sin\left(\frac{1}{2n}\right), n\in\mathbb{N}, n\geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)\cdot\omega_{n+1}^{n+1}}-\sqrt[n]{n\cdot\omega_n^n}\right)$$

3.115

$$H_n = \sum_{k=1}^{n}\frac{1}{k}, G_n = \sum_{k=1}^{n}\frac{1}{k^2}, T_n = \frac{H_n}{G_n}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{log(1+2^{T_n})^{log(1+7^{T_n})}}{log(1+3^{T_n})^{log(1+5^{T_n})}}\right)$$

3.116

Find:

$$\Omega = \lim_{n\to\infty}\left(1-log\,2+\sum_{k=1}^{n}sin\frac{1}{n+k}\right)^n$$

3.117

Find:

$$\Omega = \lim_{n\to\infty} \frac{\left(1 + 2^p \cdot \sqrt{2} + 3^p \cdot \sqrt[3]{3} + \cdots + n^p \cdot \sqrt[n]{n}\right)^{q+1}}{(1^q + 3^q + 5^q + \cdots + (2n-1)^q)^{p+1}}, p, q \in \mathbb{N}, p, q \geq 1$$

3.118

Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=0}^{2n} (-1)^k \cdot \frac{1}{\binom{2n}{k}}$$

3.119

Find:

$$\Omega = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \left(\frac{n^2 + n + k^2}{n^2 + k^2} \right) \right)$$

3.120

Find:

$$\Omega = \lim_{n\to\infty} \left(\int_0^{\frac{1}{2}} \left(\frac{arcsec(nx) \cdot log(1-x)}{2x^2 - 2x + 1} \right) dx \right)$$

3.121

If:

$$x_0 > 0, x_{n+1} = \frac{1}{x_1^p} + \frac{1}{x_2^p} + \cdots + \frac{1}{x_n^p}, p \in \mathbb{N}^*$$

Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{\sqrt[p+1]{n^{p+2}}} \cdot \sqrt{\sum_{1\leq i<j\leq n} x_i x_j} \right)$$

3.122

$f:[-1,1] \to \mathbb{R}, f(x) = (sin^{-1} x)^2, f^{(n)} - n'th\ derivative.$ *Find:*

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[2n]{f^{(2n)}(0)}}{n}$$

3.123 *Find:*

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2^n} \prod_{k=1}^{n} \left(2 - \frac{3}{3k-1} \right) \right)$$

3.124 *If:*

$$x_0 > 0, \sqrt[p]{x_n} = \frac{1 + x_n - x_{n+1}}{x_{n+1} - x_n}, n \in \mathbb{N}^*, p \in \mathbb{N}^*, p \geq 2$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{x_n^{p+1}}{n^p}$$

3.125 *If:*

$$x_0 \in (0,1), x_{n+1} = x_n \sqrt[p]{1 - x_n}, y_0 > 0, y_{n+1} = y_n + \frac{1}{y_n^{p-1}}, n, p \in \mathbb{N}, p \geq 2.$$

Find:

$$\Omega = \lim_{n\to\infty} \left(y_n \sqrt[p]{x_n} \right)$$

3.126 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \sqrt[p]{\frac{n^2}{i \cdot j}}, p \in \mathbb{N}, p \geq 2$$

3.127 *Find:*

$$\Omega = \lim_{n\to\infty} n^8 \int_0^{\frac{1}{n^5}} \frac{x^2 + 1}{x^4 + x^2 + 1} dx$$

3.128 *Find:*

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2^n} \sum_{k=1}^{n} \left(\binom{n}{k} \cdot \sin^{-1}\left(\frac{k}{n}\right) \right) \right)$$

CHAPTER 4

INTEGRALS-PROBLEMS

4.1 *Find:*

$$\Omega = \int \left(x^{10} + \sqrt{1 + x^{20}}\right)^{\frac{21}{10}} dx, x \in \mathbb{R}$$

4.2 *If:*

$$a, b, c > 0, 2e(a + b + c) = 3e + 2, \Omega(a) = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a} - e\right)$$

Prove that: $\Omega(a) \cdot \Omega(b) \cdot \Omega(c) \leq \frac{1}{27}$

4.3 *Let be:*

$$\Omega = \lim_{n\to\infty} sin\left(\frac{1}{n}\int_{n}^{n+1} \frac{dx}{\sqrt{(x - n)(n + 1 - x)}}\right)$$

True or false: $-1 < \Omega < 1$?

4.4 *True or false:*

If $m \in \mathbb{N}^*$ *then:*

$$I_m = \int_0^1 \frac{\sqrt{e^x}(x + 3)}{(x + 1)\sqrt{e^x} + m} dx < 1$$

4.5 *In* $\Delta\ ABC$ *the following relationship holds:*

$$abc\prod\left(\int_A^B x\sqrt[3]{cos\ x}\, dx\right) \leq 8Ss\prod sin\frac{B - A}{2}$$

4.6 *If:*

$$\Omega = \int \frac{(x^2+1)^2}{\left(x^{\frac{113}{25}} + \frac{11}{3}x^{\frac{63}{25}} + 11x^{\frac{13}{25}}\right)^5} dx = u(x^a + ex^b + fx^c)^k + Q$$

Find: $\vartheta = -\frac{5}{k}(a+b+c)$

4.7 *Find:*

$$\Omega = \lim_{x\to\infty} \int_0^x \frac{x^4}{(1+x^3)^2} dx$$

4.8 *If* $a, b, c > 0$

$$I(a) = \int_{-\pi}^{\pi} \frac{\cos^2 x}{x + a + \sqrt{x^2 + a^2}} dx$$

then:

$$I(a) + I(b) + I(c) \geq \frac{9\pi}{2(a+b+c)}$$

4.9 *If* $a \in \left(0, \frac{\pi}{2}\right)$ *find:*

$$\Omega = \int_{\tan a}^{\cot a} \frac{\ln x}{1+x^2} dx$$

4.10 *If* $a > 0, f: \mathbb{R} \to \mathbb{R}$ *continuous*, $f(x) + f(-x) = a\cos x, \forall x \in \mathbb{R}$ *then find:*

$$\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x)}{\cos^3 x} dx$$

4.11 *Find:*

$$I = \int \frac{x^2 \cos x + x + \sin x \cos x}{x \sin x\,(x + \cos x)} dx$$

4.12 *Find:*

$$\Omega = \int \frac{\cot x \cot 2x\, dx}{(\cot^2 x - \tan^2 x) \sin^3 2x}$$

4.13 *Find:*

$$\Omega = \int \frac{\cos 2x \cot x\, dx}{(\cot^2 x - \tan^2 x) \sin^3 2x}, x \in \left(0, \frac{\pi}{4}\right)$$

4.14 *Find:*

$$\Omega = \int_a^b \tan(\arccos(\sin(\arctan x)))\, dx, 0 < a < b < \frac{\pi}{2}$$

4.15 *Find:*

$$\Omega = \int \frac{\sin 3x \cdot \cos^2 x}{\cos 2x \cdot \cos 3x} dx, x \in \left(0, \frac{\pi}{6}\right)$$

4.16 *Compute the following integral:*

$$\int \frac{(x^2(1 - e^{-x}) - 1)dx}{x^4 + 2x^3 e^{-x} + (3 + e^{-2x})x^2 + 2xe^{-x} + 1}$$

4.17 *Find:*

$$\Omega = \int_{n+1}^{n+2} \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{x - j} \right) \right) dx$$

4.18 *If* $1 < a \leq c \leq b$ *then:*

$$\int_a^b \sum ln(x+b)^{\frac{1}{x+c}} \cdot ln(x+c)\,dx \geq ln\frac{b}{a} \cdot ln(a+b) \cdot ln(a+c)$$

4.19 *If* $0 < a < b$ *and* $x > 0$, *then compute:*

$$\int \frac{b^x - a^x + (x+1)(b^x \ln a - a^x \ln b) + ln\frac{a}{b}}{(x+1)^2(ab)^x + (x+1)(a^x + b^x) + 1}\,dx$$

4.20 *Find:*

$$\int_0^1 \int_0^2 \dots \int_0^{2018} \left\{\sum_{k=1}^{2018} x_{2018}\right\} dx_1 dx_2 \dots dx_{2018}$$

where $\{x\}$ *represents the fractional part of* x

4.21 *If* $f: [1,6] \to \mathbb{R}, f(1) = 3, f(6) = 18$ *is integrable and invertible then:*

$$\Omega = \int_1^6 [f^2(x) + 18xf^{-1}(3x)]\,dx = 1935$$

4.22 *For* $m, n \in \mathbb{N}^* \wedge m, n \geq 2$. *Prove:*

$$\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{n-1}(\cos x)^{m+1}}{m}\,dx = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{n+1}(\cos x)^{m-1}}{n}\,dx$$

4.23 *Calculate:*

$$\Omega = \int_{e^{e^{e^{e^2}}}}^{e^{e^{e^{e^3}}}} \frac{(ln(ln(ln(\ln x))) + 100)^{99}}{x(\ln x)(ln(\ln x))(ln(ln(\ln x)))}\,dx$$

4.24 *Find:*

$$\Omega = \int_{a}^{\frac{\pi}{2}} \left(\frac{\sin 7x}{\sin x}\right)^2 dx, 0 < a \leq \frac{\pi}{2}$$

4.25 *Find:*

$$\Omega = \int \frac{x(\tan x + 2\tan 2x + 4\tan 4x)}{\cot x - 8\cot 8x} dx$$

4.26 *Find:*

$$\Omega = \int \tan^2 x\,(\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x)dx, x \in \left(0, \frac{\pi}{2}\right)$$

4.27 *Find:*

$$\Omega = \int \frac{\tanh^2 x + \tanh^2 x\,(1 + \tanh^2 x)^2}{(1 + \tanh^2 x)^2} dx, x \in \mathbb{R}$$

4.28 *Find:*

$$\Omega = \int_{0}^{\frac{\pi}{6}} \frac{\tan^4 x\,(\tan^2 x - 2)}{(1 - \tan^2 x)^2} dx$$

4.29 *Find:*

$$\Omega = \int \frac{\tan^2 x \cdot \tan^4 x \cdot \tan^6 x \cdot \tan^8 x \cdot \ldots \cdot \tan^{2n} x}{\sin^2 x\sqrt{1 - \tan^{n^2+n+1} x} - \sqrt{1 - \tan^{n^2+n+1} x} - \cos^2 x\sqrt{1 - \tan^{n^2+n+1} x}} dx$$

4.30 *Find:*

$$\Omega = \int \frac{2x^4 + 5x^3 + 6x^2 + 6x + 12}{(x^2 + 2x + 2)\sqrt{x^2 + 2x + 2}} dx, x \in \mathbb{R}$$

4.31 *Find:*

$$\Omega = \int \frac{tanh(x)}{1 + e^{3x}} dx, x \in \mathbb{R}$$

4.32 *Find:*

$$\Omega = \int_{1}^{16} tan^{-1}\left(\sqrt[4]{x-1}\right) dx$$

4.33 *Find:*

$$\int \left(\frac{\sqrt{1 + ln^2 x}\,(1 + ln\, x) + ln^2 x}{\sqrt{1 + ln^2 x}\,(ln\, x) + ln^2 x + 1}\right) dx$$

4.34 *Find:*

$$\Omega = \int \frac{x^4 e^x dx}{(x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x)^2}$$

4.35 *Find:*

$$\Omega = \int \frac{tanh(x)}{1 + e^{3x}} dx, x \in \mathbb{R}$$

4.36 *Find:*

$$\Omega = \int tan^{-1}\left(\sqrt{x + \sqrt{x^2 + 1}}\right) dx, x \in \mathbb{R}$$

4.37 *Find:*

$$\Omega = \int \frac{242(x+2)^5 - (x+1)^5 - (x+3)^5}{26(x+2)^3 - (x+1)^3 - (x+3)^3} dx, x > 0$$

4.38 *Find:*

$$\Omega = \lim_{n\to\infty} n^2 \int_0^1 \frac{x^2 + \arctan x}{e^{nx}} dx$$

4.39 *Find:*

$$\int_0^{\frac{\pi}{2}} \frac{10\tan^3 x - 19\tan^{\frac{8}{3}} x - 36\tan^2 x + 10\tan x - 19\tan^{\frac{2}{3}} x - 36}{2\tan^{\frac{8}{3}} x + 35\tan^{\frac{5}{3}} x + 108\tan^{\frac{2}{3}} x} dx$$

4.40 *Find:*

$$\Omega = \lim_{x\to 0} \frac{\int_0^x \frac{\sqrt[3]{1+2x}\cdot\sqrt[5]{1+3x}-1}{x} dx}{\int_0^x \frac{\sqrt[3]{1+3x}\cdot\sqrt[5]{1+2x}-1}{x} dx}$$

4.41 *Find:*

$$\Omega = \lim_{a\to\infty} \int_0^a \frac{2x+3}{x(x+1)(x+2)(x+3)+a} dx$$

4.42 *Find:*

$$\Omega = \lim_{n\to\infty} (n+1) \left(\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx \right)^n$$

4.43 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k \sin\left(\frac{k^2+k}{n^2+n}\right)$$

4.44 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{1}{n(n+1)(2n+1)} \sum_{k=1}^{n} k^2 \tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

4.45 *If:*

$$\Omega(n) = \int_{-1}^{1} x \log(1 + n^{3x})\, dx, n \in \mathbb{N}^*$$

Prove that:

$$9\left(1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1 + e^{\Omega(n)}\right)$$

4.46 *Let be:*

$$\omega(n) = \int_{-1}^{1} \frac{(1 + 2x + x^2)^n (1 - 2x + x^2)^n}{(1 - x^2)(1 + 2x^2 + x^4)^n} dx, n \in \mathbb{N}, n \geq 2$$

Find:

$$\Omega = \lim_{n\to\infty} \frac{\omega(n+2) - \omega(n+1)}{\omega(n+1) - \omega(n)}$$

4.47 *Find:*

$$\Omega = \lim_{n\to\infty} \frac{\int_0^1 \left(\frac{2x}{1+x^2}\right)^{n+1} dx}{\int_0^1 \left(\frac{2x}{1+x^2}\right)^{n} dx}$$

4.48 *Let* $f: [a, b] \to (0, \infty)$ *be a continuos function.*

If $m = \min f(x), M = \max f(x)$ *then:*

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{1}{f(x)}\,dx\right) \le \frac{\left((m+M)(b-a)\right)^2}{4mM}$$

4.49 *Prove that:*

$$\int_0^{\frac{\pi}{2}} \left(\frac{ln\left(\frac{1-sin(x)}{1+sin(x)}\right)\sqrt{cos(x)}}{(1+sin(x))\sqrt{1-sin(x)}}\right)dx = -8$$

4.50 *If* $[a,b] \subset \left(0,\frac{\pi}{2}\right)$ *then:*

$$\int_a^b sin\,x\,dx > \sqrt{b^2+1} - \sqrt{a^2+1}$$

4.51 *If* $f,g:[a,b] \to (0,\infty)$ *integrable, such that* $f(x)+g(x) \le 8$ *then:*

$$\int_a^b \frac{f(x)\sqrt{g(x)} + g(x)\sqrt{f(x)}}{f(x) - \sqrt{f(x)g(x)} + g(x)}\,dx \le 4(b-a)$$

4.52 *If* $f:[0,1] \to (0,\infty), f$ *derivable,* f' *continuous,*

$$f'(x) = f'(1-x), \forall\, x \in [0,1] \textit{ then:}$$

$$\int_0^1 f(x)\,dx \ge \sqrt{f(0)\cdot f(1)}$$

4.53 *If* $n \in \mathbb{N}, n \ge 2, n -$ *fixed,* $f:[1,n] \to (0,\infty), f -$ *integrable,* $i \in \overline{1,n-1}$

$$\Omega = \int_1^n f(x)\,dx, 0 \le \alpha \le min\left(\int_i^{i+1} f(x)\,dx\right) \le max\left(\int_i^{i+1} f(x)\,dx\right) \le \beta$$

then:

$$(n-1)\alpha\beta + \sum_{i=1}^{n-1}\left(\int_{i}^{i+1} f(x)\,dx\right)^2 \le (\alpha+\beta)\Omega$$

4.54 *If* $f:[a,b]\to(0,\infty), a < b$, *continuous,*

$m = \min f(x), M = \max f(x), n \in \mathbb{N}^*$ *then:*

$$\left(\frac{m}{M}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n} \le \prod_{k=1}^{n}\left(\int_a^b f^k(x)\,dx\right)\left(\int_a^b \frac{1}{f^k(x)}dx\right) \le \left(\frac{M}{m}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n}$$

4.55 *If* $e \le a \le c \le b \le e^2$

then:

$$(b-a)\int_a^c \frac{x}{\log x}dx \le (c-a)\int_a^b \frac{x}{\log x}dx$$

4.56 *If* $0 < a < b < 1$ *then:*

$$\frac{1}{b-a}\int_a^b \left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)dx \ge \left(1+\frac{4}{\pi}\right)^2$$

4.57 *If:*

$$a,b,c \in (2,\infty), \Omega(a) = \int_0^1 \frac{1-x^2}{1+ax^2+x^4}\,dx$$

then:

$$2bc\Omega(a) + 2c\Omega(b) + 2ab\Omega(c) < a^2+b^2+c^2$$

4.58 *Find:*

$$\int_0^1 log^2(1+e^x)\,dx < \left(\int_0^1 log(1+e^x)\,dx\right)^2 + \frac{1}{12}$$

4.59 *Prove that:*

$$\frac{\pi}{8} \le \int_0^1 arctan\,x\,dx \le arctan\frac{1}{2}$$

4.60 *Find:*

$$\Omega = \int \frac{\sqrt{3+cos\,2x}}{cos\,x}dx, x \in \left(0,\frac{\pi}{2}\right)$$

4.61 *If* $a,b,c \in (0,\infty)$ *then:*

$$\left(1+\frac{b}{a}\right)^c \left(\int_0^1 e^{-x^2}\,dx\right)^{c+1} + \left(1+\frac{a}{b}\right)\left(\int_0^1 e^{x^2}\,dx\right)^{c+1} \ge \left(\int_0^1 \left(e^{-x^2}+e^{x^2}\right)dx\right)^{c+1}$$

4.62 *Let* $f:[0,1] \to (-1,1)$ *be a continuous function such that*

$$\int_0^1 f(x)\,dx \notin \{-1,1\}$$

Prove that:

$$\frac{e^{\int_0^1 f(x)dx}}{1+e^{\int_0^1 f(x)dx}} \le \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx$$

4.63 *If* $a,b \in \mathbb{R}$ *then:*

$$2(b-a)^2\sqrt{e^{a+b}} \le 8\left(\sqrt{e^a}-\sqrt{e^b}\right)^2 \le (e^a+e^b)(b-a)^2$$

4.64 *If* $0 < a < b < \frac{\pi}{2}$ *then:*

$$7\int_a^b x^3 \tan x \left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x}\, dx > b^7 - a^7$$

4.65 *Let* $f: \mathbb{R}^+ \to \mathbb{R}^+$ *be integrable and satisfying*

$f(xt + (1-t)y) \le tf(x) + (1-t)f(y)$ *where* $x, y \in \mathbb{R}^+$ *and* $t \in (0,1)$

Prove that:

$$\frac{1}{\ln a}\int_1^a f(x^4)\, dx + \frac{1}{\ln b}\int_1^b f(x^4)\, dx \ge 2\int_0^1 (ab)^{\frac{x}{2}} f(a^{4x} - a^{3x}b^x + a^{2x}b^{2x} - a^x b^{3x} + b^{4x})dx$$

where $a, b > 0$.

4.66 *If* $0 < a < b < \frac{\pi}{2}$ *then:*

$$\int_a^b \frac{\cos x}{\sin x + 4x \cos x} dx < \frac{1}{5}\left(\log\frac{b}{a}\right)^{\frac{4}{5}} \cdot \sqrt[5]{\log\left(\frac{\sin b}{\sin a}\right)}$$

4.67 *For* $a, b, c \in (0, \infty)$; $a < b < c$; $f: [0, a] \to [0, b]$; $g: [0, b] \to [0, c]$

continuous, bijectifs and strictly increasing functions prove that:

$$\frac{1}{c}\int_0^a (g \circ f)^2\,(x)dx + \frac{1}{a}\int_0^c (f^{-1} \circ g^{-1})^2\,(x)dx \le ac$$

4.68 *Find:*

$$\Omega = \lim_{n\to\infty}\int_0^{2018} \left|\left[nx + \frac{1}{2}\right] - nx\right| dx,$$

$[*]$ - *great integer function*

4.69 *Find:*

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\varepsilon}^{\frac{\pi}{2}} \left(tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)dx$$

4.70 *If* $0 < a < 1$, $f:\left[a,\frac{1}{a}\right]\to\mathbb{R}$, f *convexe and increasing function then:*

$$\frac{1-a^2}{a}f\left(\frac{1+a^2}{2a}\right) \le \int_{a}^{\frac{1}{a}} \left(\frac{1+x^2}{2x^2}\right)f(x)dx \le \frac{1-a^2}{2a}\left(f(a)+f\left(\frac{1}{a}\right)\right)$$

4.71 *If* $a,b,c \in \left(0,\frac{\pi}{4}\right)$ *then:*

$$0 \le \int_0^a\left(\int_0^b\left(\int_0^c\left(\sum(tan\,x + 2\,tan\,y\,tan\,z) + 4\prod tan\,x\right)dx\right)dy\right)dz \le abc$$

4.72 *If* $a,b,c > 0$ *then:*

$$e^{\sum a\int_b^c \frac{x^4+1}{x^6+1}dx} \le \left(\frac{c}{b}\right)^a \cdot \left(\frac{a}{c}\right)^b \cdot \left(\frac{b}{a}\right)^c$$

4.73 *Prove that:*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(arctan\,x\,arctan\frac{1}{x}\right)dx \le \frac{\pi^3}{96}$$

4.74 *Let* $f:[a,b]\to\mathbb{R}$ *be continuously differentiable function and*

$$\int_a^b f(x)\,dx = 0.$$

Prove that

$$\left|\int_a^b xf(x)dx\right| \leq \frac{(b-a)^3}{12} max\{f'(x): x \in [a,b]\}$$

4.75 *Let* $a, b, c > 0$*. Prove that*

$$a^2 \int_0^b \frac{arctan\, x}{x} dx + b^2 \int_0^c \frac{arctan\, x}{x} dx + c^2 \int_0^a \frac{arctan\, x}{x} dx < a^3 + b^3 + c^3$$

4.76 *If* $0 < a < b < \frac{\pi}{2}$ *then:*

$$\frac{1}{2}\int_a^b (sin(cos\, x) + tan(cos\, x))\, dx > sin\, b - sin\, a$$

4.77 $a.\, 2\left(\int_3^4 e^{-x^2}\, dx\right)^2 + 2\left(\int_2^3 e^{-x^2}\, dx\right)^2 \geq \left(\int_2^4 e^{-x^2}\, dx\right)^2$

$$b.\left(\int_2^4 e^{-x^2}\, dx\right)^2 \geq \left(\int_3^4 e^{-x^2}\right)^2 + \left(\int_2^3 e^{-x^2}\, dx\right)^2$$

4.78 *If* $a, b, c \geq 0$

$$\Omega(a) = \int_0^a \sqrt{\frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}}\, dx$$

then:

$$(\Omega(a) + \Omega(b) + \Omega(c))^3 \geq \prod log\left(\sqrt{a^3+1} + \sqrt{a^3}\right)$$

4.79 *Find:*

$$\int_0^1 \sqrt{x}\, sin\, x\, dx > \frac{49}{135}$$

4.80 $a, b \in \left(0, \frac{\pi}{2}\right), a < b$

Prove that exists $\alpha, \beta \in (a, b)$ *such that:*

$$\left(\int_a^b \tan x \, dx\right)\left(\int_a^b \cot x \, dx\right) \leq \frac{(b-a)^2}{\sin \beta \cos \alpha}$$

4.81 $0 < p < q < \frac{\pi}{2}$; $f: \mathbb{R} \to \mathbb{R}$ *a continuous function,*

$$\int_{p \sin t}^{q \cos t} f(x) \, dx \leq \int_{\frac{p}{\sqrt{2}}}^{\frac{q}{\sqrt{2}}} f(x) \, dx, \forall t \in \left(0, \frac{\pi}{2}\right)$$

Prove that: $qf\left(\frac{q}{\sqrt{2}}\right) + pf\left(\frac{p}{\sqrt{2}}\right) = 0$

4.82

$$\int_{-1}^{1} \sqrt{1-x^2} \cos^{-1} x \, dx > \frac{e^2}{4}$$

4.83 *If* $a, b, c > 0$ *then:*

$$\int_{-a}^{a} \frac{e^{x^2} + e^{-x^2}}{2^x + 1} dx + \int_{-b}^{b} \frac{e^{x^2} + e^{-x^2}}{3^x + 1} + \int_{-c}^{c} \frac{e^{x^2} + e^{-x^2}}{5^x + 1} dx > 6\sqrt[3]{abc}$$

4.84 *If* $a > 0, f: [0, a] \to \mathbb{R}, f'(a) = f(a) = 0,$

$f \in C^2([0, a]), f'f'' \in C^1([0, a])$ *then:*

$$60\left(\int_0^a f(x) \, dx\right)^4 \leq a^8 \left(\int_0^a \left(f'(x)\right)^2 dx\right)\left(\int_0^a \left(f''(x)\right)^2 dx\right)$$

4.85 *If* $a_1, a_2, \dots, a_n \in (0,\infty), n \in \mathbb{N}^*$ *then:*

$$\sum_{k=1}^{n} \int_{-1}^{1} \frac{e^{x^2}}{a_k^x + 1} dx < ne$$

4.86 *If* $a, b \in \mathbb{R}, a < b, f: [a,b] \to \mathbb{R}$, *continuous,* $f'(x) > 0$ *then:*

$$\sum_{k=2}^{n} \int_{a}^{\frac{a+(k-1)b}{k}} f(x)dx \le \left(\sum_{k=2}^{n} \frac{k-1}{k}\right) \int_{a}^{b} f(x)\, dx$$

4.87 *If* $a, b, c, d > 0, a + b + c + d = \pi$

$$\Omega(a) = \int_{a}^{2a} \frac{arctan(x+1)}{x} dx$$

then: $\Omega(a) + \Omega(b) + \Omega(c) + \Omega(d) < \pi(1 + log\, 2)$

4.88 *If* $f: [0,1] \to (0,\infty)$ *is a continuous function such that*

$\int_0^1 f(x)\, dx = 1$ *, then*

$$\left(\int_0^1 \sqrt[3]{f(x)}\, dx\right)\left(\int_0^1 \sqrt[5]{f(x)}\, dx\right)\left(\int_0^1 \sqrt[7]{f(x)}\, dx\right) \le 1$$

4.89 *If* $[0,1] \to (0,\infty)$ *continuous;* $\int_0^1 f^3(x)\, dx = \sqrt[7]{2}$ *then:*

$$\left(\int_0^1 f^5(x)\, dx\right)\left(\int_0^1 f^7(x)\, dx\right)\left(\int_0^1 f^9(x)\, dx\right) \ge 2$$

4.90 *In all $\Delta\ ABC$:*

$$\sum_{cyc} \int_0^{\frac{h_a^2}{h_a h_b + h_b h_c + h_a h_c}} e^{-t^2}\, dt \le 3\, tan^{-1} \frac{R}{6r}$$

4.91 *If* $x, y, z \in (0, \infty)$

$$\Omega(x) = \lim_{n\to\infty} \left(\frac{(x+3)^{\frac{1}{n}} + x^{\frac{1}{n}}}{(x+2)^{\frac{1}{n}} + (x+1)^{\frac{1}{n}}} \right)^n$$

then:

$$\Omega^2(x) + \Omega^2(y) + \Omega^2(z) < 3 + 2\sum \frac{1}{x+2}$$

4.92 *If* $m, n \in \mathbb{N}, m \ge 2, n \ge 2$ *then:*

$$\left(\int_0^{\frac{\pi}{2}} \sqrt[m]{tan\, x}\, dx \right) \left(\int_0^{\frac{\pi}{2}} \sqrt[n]{tan\, x}\, dx \right) \ge \frac{\pi^2}{\left(cos\frac{\pi}{2m} + cos\frac{\pi}{2n} \right)^2}$$

4.93

$$\int_1^{\sqrt{3}} sin^{-1}\left(\frac{2x}{1+x^2}\right)(tan^{-1} x)^2 dx < \frac{\pi^3}{27}(\sqrt{3} - 1)$$

4.94 *Prove that if* $a \in \mathbb{R}$ *then:*

$$\int_{a+8}^{a+11} e^{x^2}\, dx + \int_{a+4}^{a+7} e^{x^2} dx \le \int_a^{a+3} e^{x^2} dx + \int_{a+12}^{a+15} e^{x^2} dx$$

4.95 *Prove:*

$$\int_0^1 ln^2\left(1+\sqrt{sin\,x}\right)dx < \frac{1}{2}$$

4.96

$$I(a,b) = \int_a^b \left(arctan\left(\frac{a\,sin\,x}{b+a\,cos\,x}\right) + arctan\left(\frac{b\,sin\,x}{a+b\,cos\,x}\right)\right)dx,$$

$$0 < a < b < c < \frac{\pi}{2}$$

Prove that:

$$\frac{2}{b-a}I(a,b) + \frac{2}{c-b}I(b,c) + \frac{2}{c-a}I(a,c) \geq \sum\left(\sqrt{ab} + \sqrt{\frac{a^2+b^2}{2}}\right)$$

4.97 *If* $a,b,c > 0, a+b+c = \pi$ *then:*

$$2\sum\int_0^a \frac{arctan^2\,x}{x}dx + log(1+a^2)\,log(1+b^2)\,log(1+c^2) < \pi^2$$

4.98 *If* $0 < a < b$ *then:*

$$\frac{2}{\pi}ln\left(\frac{b}{a}\right) + b - a < \frac{\pi}{2}\int_a^b \frac{dx}{arctan\,x} < \frac{\pi}{2}ln\left(\frac{b}{a}\right) + b - a$$

4.99 *If* $0 < a < b$ *then:*

$$\frac{\int_a^b e^{x^2}dx}{\int_a^b x^5 e^{x^2}\,dx} < \frac{1}{4}\left(\frac{1}{ab^4} + \frac{1}{a^2b^3} + \frac{1}{a^3b^2} + \frac{1}{a^4b}\right)$$

4.100 *If* $a, b, c \geq 0, m, n \geq 2$

$$\Omega(a) = \sqrt[n]{\int_0^a \sqrt[m]{e^{(m+n)x^2}}\, dx} \cdot \sqrt[m]{\int_0^a \frac{dx}{\sqrt[n]{e^{(m+n)x^2}}}}$$

then: $\Omega^2(a) + \Omega^2(b) + \Omega^2(c) \geq ab + bc + ca$

4.101 *If* $a > 0$ *then:*

$$\left(\int_0^a e^{3x^2} dx\right)\left(\int_0^a e^{-3x^2} dx\right) > \frac{1}{a^4}\left(\int_0^a e^{x^2} dx\right)^3 \left(\int_0^a e^{-x^2} dx\right)^3$$

4.102

$$\int_0^{\frac{\pi}{4}} \left(\frac{1 - \sin^2 x}{1 + \sin^2 x} + \frac{1 - \cos^2 x}{1 + \cos^2 x}\right) \ln(1 + \tan x)\, dx > \frac{\pi \ln 2}{12}$$

4.103 *If* $a > 1$ *then:*

$$\int_a^{2a} \frac{e^x}{x^3} dx \leq \frac{3e^a(e^a - 1)}{8a^3}$$

4.104 *If* $1 < a < b$ *then:*

$$\int_1^a \log^2 x\, dx + \int_1^b \log^2 x\, dx \geq \int_1^{\frac{3a+b}{4}} \log^2 x\, dx + \int_1^{\frac{a+3b}{4}} \log^2 x\, dx$$

4.105 *If* $0 < a < b; 0 < c < d; f, g$ *integrable functions*

$f, g: [a, b] \to [c, d]$ *then:*

$$cd\left(\int_a^b \frac{f(x)}{g(x)} dx + \int_a^b \frac{g(x)}{f(x)} dx\right) < (c^2 + d^2)(b - a)$$

4.106 *If* $f:[a,b]\to\mathbb{R}, f-$ *continuous,* $f-$ *increasing then:*

$$\left(\sqrt{a}+\sqrt{b}\right)\int_{a}^{\sqrt{ab}} f(x)dx \le \sqrt{a}\int_{a}^{b} f(x)dx$$

4.107 *For acute triangle* ABC*:*

$$\textit{If: } \zeta(A)=\int_0^A \frac{1}{\sqrt{\cos x + x\left(1+\frac{2}{\pi}\right)}}dx$$

$$\textit{Prove: } \zeta(A)+\zeta(B)+\zeta(C)\le 2\sqrt{3(\pi+3)}-6$$

4.108 *Let* $f:[1,13]\to\mathbb{R}$ *be a convexe and integrable function. Prove that*

$$\int_{1}^{3} f(x)\,dx + \int_{11}^{13} f(x)\,dx \ge \int_{5}^{9} f(x)\,dx$$

4.109 *If* $0<a<b$ *then:*

$$\int_{a}^{b}\frac{dx}{(x^3+1)^2} > \frac{5}{9(b^5-a^5)}\ln^2\left(\frac{b^3+1}{a^3+1}\right)$$

4.110 *If* $m,n,p\ge 2$

$$\Omega(n) = 4^{n-1}\int_{\frac{1}{2}}^{\frac{3}{2}}\left(1+\frac{1}{\sqrt{2-x}}\right)\left(1+\frac{1}{\sqrt[n]{x}}\right)dx$$

then:

$$\Omega(n)\Omega(m)\Omega(p)\ge 64\sqrt[3]{mnp}$$

4.111 *If* $a\in\mathbb{R}, f:[a,a+2]\to\mathbb{R}, f\in C^2([a,a+2]), 6\le f''(x)\le 12$*, then:*

$$1+f(a+1)\le \frac{1}{2}\int_{a}^{a+2} f(x)dx \le 2+f(a+1)$$

4.112 *If* $a, b > 0$ *then:*

$$2\int_0^{\sqrt{ab}} e^x \, ln(x+1)\, dx \le \int_0^a e^x \, ln(x+1)\, dx + \int_0^b e^x \, ln(x+1)\, dx$$

4.113 *If* $a, b, c > 0, abc = 1$,

$$\Omega(a) = \int_0^a \frac{x^4 + 7x^3 - 25x^2 + 37x + 4}{x^4 - 3x^3 + 5x^2 - 3x + 4} dx$$

then:

$$\Omega(a) + \Omega(b) + \Omega(c) \ge 3 + 5ln\prod(1 + a^2)$$

4.114 *If* $a, b, c > 0, abc = 1$

$$\Omega(a, b) = \int_0^1 \sqrt{(x+a)(x+b)}\, dx$$

then:

$$(a+b)\Omega(a, b) + (b+c)\Omega(b, c) + (c+a)\Omega(c, a) > 6$$

4.115 *Prove that:*

$$\int_0^{\frac{\pi}{2}} \left((sin^2\theta)^{cos^2\theta}(cos^2\theta)^{sin^2\theta} + (sin^2\theta)^{sin^2\theta}(cos^2\theta)^{cos^2\theta}\right) d\theta > \frac{\pi}{4}$$

4.116 *Prove that:*

$$\int_0^{\frac{\pi}{2}} \left(\frac{\pi^2}{4}\sqrt{cosx} - \pi x\sqrt{cosx} + x^2\sqrt{cosx}\right) dx > \frac{\pi^3}{28}$$

4.117 *Prove that:*

$$\int_0^1 e^{2x^2}\, dx - \left(\int_0^1 e^{x^2}\, dx\right)^2 < \frac{e^2}{3}$$

4.118 *If* $1 < a \leq b$ *then:*

$$\int_a^b \left(\frac{ln(x+a)}{x+b} + \frac{ln(x+b)}{x+a}\right) dx \leq \frac{1}{a} ln(a+b)^{b-a}$$

4.119 *If* $0 < a \leq b < 1$ *then:*

$$\int_a^b e^{-\sqrt{x}} sin\left(\frac{\sqrt{x}+100}{100}\right) dx \leq \frac{2}{3}\left(b^{\frac{3}{4}} - a^{\frac{3}{4}}\right)$$

4.120 *If* $a, b, c > 1$ *then:*

$$2\sum \int_0^{\frac{a+b}{a^2+b^2}} \left((sin\, x)^{2\, cos^2 x} \cdot (cos\, x)^{2\, sin^2 x}\right) dx \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

4.121 *For* $a < 0$*. Prove:*

$$11^{-1} \int_1^{10} \left(x + \frac{10}{x}\right)^a dx \geq 9$$

4.122 *Prove that:*

$$\frac{\pi(e-1)}{2\left(e^{\frac{\pi}{2}} - 1\right)} \leq \frac{1}{e^{\frac{\pi}{2}} - 1} \int_0^{\frac{\pi}{2}} e^{sinx} dx \leq 1$$

4.123 *If* $0 < a \leq b, f : [a, b] \to \mathbb{R}, f'(x) > 0, \forall x \in [a, b]$ *then:*

$$\int_a^b x^2 f(x)\, dx \geq ab \int_a^b f(x)\, dx$$

4.124 *Find:*

$$\Omega = \lim_{n\to\infty} \int_0^1 \frac{e^{x-[x]} + e^{2x-[2x]} + \cdots + e^{nx-[nx]}}{nx} dx,$$

$[*]$ *- great integer function*

4.125 *If* $0 < a \leq b, f:[a,b] \to \mathbb{R}, f'(x) > 0, \forall x \in [a,b]$ *then:*

$$\int_a^b x^2 f(x)dx \geq ab \int_a^b f(x)\,dx$$

4.126 *Prove:*

$$\int_0^1 \sqrt{x + \sqrt{x + \sqrt{x}}}\,dx \geq \frac{2}{3}\sqrt{1 + \sqrt{2}}$$

4.127 *Let* $b > a > 1$ *and* n *be a positive integer. Prove that:*

$$\int_{\ln a}^{\ln b} \frac{\sqrt{e^{nx}}}{e^{nx} + e^{(n-1)x} + \cdots + e^{2x} + e^x + 1}\,dx \leq \ln \sqrt[n+1]{\frac{b}{a}} \quad (x \in \mathbb{R}).$$

4.128 *Prove that:*

$$\int_0^{\frac{\pi}{2}} \left(2^{\sin x} + 2^{\cos x}\right) dx > \pi + 2\ln 2$$

4.129 *If* $f:[0,1] \to \mathbb{R}$*, continuous then:*

$$1 + \left(\int_0^1 f^2(x)\,dx\right)^3 > \left(\int_0^1 f(x)dx\right)^3$$

4.130 *If* $\frac{\pi}{6} \leq a < \frac{\pi}{2}$ *then:*

$$\frac{125\pi}{24} + \int_{\frac{\pi}{6}}^{a} \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 dx + \int_{\frac{\pi}{6}}^{a} \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3 dx > \frac{125a}{6}$$

4.131 *If* $f:[0,1]\to[0,\infty)$, f *– continuous, then:*

$$\int_0^1 f^3(x)dx + 25\int_0^1 f(x)\,dx \geq 10\left(\int_0^1 f(x)\,dx\right)^2$$

4.132 *If* $0 < a \leq b \leq a+3$, $f:[a,b]\to(0,\infty)$, $f'(x) > 0, \forall x\in[a,b]$ *then:*

$$\int_a^b xf(x)dx \leq bf(b)\sqrt{a^2+ab+b^2}$$

4.133 *If* $0 < a \leq b < \frac{\pi}{2}$ *then:*

$$\int_a^b\left(\sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}}\right)dx \geq \frac{4}{3}\left(\sqrt[3]{(b+1)^2}-\sqrt[3]{(a+1)^2}\right)$$

4.134 *If* $p,q,r,s:\mathbb{R}\to(0,\infty)$ *continuous functions,* $a\geq 0$ *then:*

$$\int_0^a s(x)\,dx \geq 4\int_0^a \sqrt[4]{p(x)q(x)r(x)s(x)}\,dx - 3\int_0^a \sqrt[3]{p(x)q(x)r(x)}\,dx$$

4.135 *If* $a,b,c\in\mathbb{N}^*$ *then:*

$$\frac{1}{3}\sum\int_0^1 \sin^{-1}(x^a(1-x)^b)\,dx \geq \sqrt[3]{\prod\frac{(a!)^2}{(b+a+1)!}}$$

4.136 *If* $0 < a < b < 1$ *then:*

$$\frac{1}{2}+\frac{2}{a+b} < \frac{2(\ln b-\ln a)}{b-a}+\frac{1}{b-a}\int_a^b\frac{dx}{\ln(1-x)}$$

4.137 *Find:*

$$\Omega = \int\frac{\tanh^6 x + 2\tanh^4 x}{(1+\tanh^2 x)^2}dx, x\in\mathbb{R}$$

4. 138 *Evaluate:*

$$\Omega = \lim_{n\to\infty} \int_{2n}^{3n} \frac{tan^{-1}(x)}{x} dx$$

4.139 *Find:*

$$\Omega = \lim_{\substack{\varepsilon\to 0 \\ \varepsilon<0}} \left(\int_{-1}^{\varepsilon} \sqrt{\frac{1+e^x}{1-e^x}} dx \right)$$

4.140 *If* $0 < a < b < 1$ *then:*

$$\frac{2(ln\, b - ln\, a)}{b-a} + \frac{1}{b-a} \int_a^b \frac{dx}{ln(1-x)} < 1 + \frac{1}{\sqrt{ab}}$$

4.141 *If* $0 < a \leq b$ *then:*

$$2\int_a^b \frac{1}{1+e^{2x^2}} dx \geq \frac{tan^{-1}\left(e^{b^2}\right) - tan^{-1}\left(e^{a^2}\right)}{be^{b^2}}$$

4.142 *If* $a, b, c \in \left(0, \frac{1}{2}\right)$ *then:*

$$b\int_a^{2a} \frac{15x+2}{36x^3+1} dx + c\int_a^{2b} \frac{15x+2}{36x^3+1} dx + a\int_a^{2c} \frac{15x+2}{36x^3+1} dx \leq (a+b+c)\, ln\, 2$$

4.143 *If* $0 \leq a < b$ *then:*

$$(1 + ab - a^2)e^{a^2} < \frac{1}{b-a}\int_a^b e^{x^2} dx < (1 - ab + b^2)\, e^{b^2}$$

4.144 *If* $0 < a < \frac{\pi}{2}$ *then:*

$$a\pi + \pi \int_a^{\frac{\pi}{2}} \frac{sin\, x}{x} dx > (\pi - 2)(1 + a - sin\, a)$$

4.145 *If:*

$$a, b, c \in \mathbb{N}^*, \Omega(a) = \lim_{n\to\infty} n \int_0^1 \frac{x^n}{(x+\pi)^a} dx$$

Prove that:

$$(1+\pi)^b \Omega(a) + (1+\pi)^c \Omega(b) + (1+\pi)^a \Omega(c) \geq 3$$

4.146 *If* $0 < a \leq b < \frac{\pi}{2}$ *then:*

$$\int_a^b \frac{\cos x \cdot \sin^2(\sin x)}{\sin^2 x} dx \geq \tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \cdot \sin b}\right)$$

4.147 *For* $0 < a < b$. *Prove:*

$$\int_a^b \sin\left(\sqrt[3]{x}\right) dx \leq (b-a)\sqrt[3]{b}$$

4.148

$$F(a) = \int_0^a \frac{\cos^7 x}{(\cos 6x + 6\cos 4x + 15\cos 2x + 10)} dx, a \in \left[0, \frac{\pi}{2}\right]$$

Prove that:

$$F(a)F(b)F(c)[F(a) + F(b) + F(c)] \leq 2^{-20}(a^4 + b^4 + c^4)$$

4.149 *Prove that:*

$$\frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,(x)^{\frac{3}{2}} dx \leq \frac{\pi^2}{8}$$

4.150 *Prove:*

$$\int_0^{\frac{\pi}{2}} \left(\frac{9^{\sin x}}{4^{\sin x} + 5^{\sin x}} + \frac{16^{\sin x}}{3^{\sin x} + 5^{\sin x}} + \frac{25^{\sin x}}{3^{\sin x} + 4^{\sin x}}\right)^3 dx \geq \frac{1593}{8\ln 60}$$

4.151 *For* $n \in \mathbb{N}^* \wedge n \geq 2$. *Prove:*

$$\int_0^1 \left(\sum_{k=1}^{n} e^{x^k}\right) dx > n + \frac{n}{\left((n+1)!\right)^{\frac{1}{n}}}.$$

4.152 *If:*

$$-1 < a, b, c < 1, \Omega(a) = \int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x} dx$$

Prove that:

$$\frac{1}{\pi^2}\left(\Omega^2(a) + \Omega^2(b) + \Omega^2(c)\right) \geq \sum (\sin^{-1} a \cdot \sin^{-1} b)$$

4.153 *If:*

$$\Omega(n) = \int_{-1}^{1} x \log(1 + n^{3x})\, dx, n \in \mathbb{N}^*$$

Prove that:

$$9\left(1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}\right)^2 > 4e^{2\Omega(n)}\left(1 + e^{\Omega(n)}\right)$$

4.154 *If* $x < a$, $a \in \mathbb{R}$ *then:*

$$(1 + ax - x^2)e^{x^2} < \frac{1}{a - x}\int_x^a e^{x^2}\, dx < (a^2 - ax)e^{a^2} + e^{x^2}$$

4.155 *If* $f: [0,1] \to \mathbb{R}$, $f(1) = 3$, f *– continuous,* f *– convexe then:*

$$\int_{\frac{1}{2}}^{1} f(x)\, dx < 1 + \frac{1}{3}\int_0^{\frac{1}{2}} f(x)\, dx$$

4.156 *If:*

$$a,b,c \geq \frac{\pi}{4}, \Omega(a) = \int_{\frac{\pi}{4}}^{a} \left(\frac{x^2 + 1 + tan^{-1} x}{x^4 + 2x^2 + 1 + (tan^{-1} x)^2} \right) dx$$

Prove that:

$$\left(1 + 2\Omega(a)\right)b^2 + \left(1 + 2\Omega(b)\right)c^2 + \left(1 + 2\Omega(c)\right)a^2 \leq a^4 + b^4 + c^4$$

4.157 *If:*

$$0 < a,b,c \leq 1, \Omega(a) = \frac{8}{\pi} \int_0^a \frac{x\, cos(10\, cos^{-1} x)\, dx}{(x^2 + a^2)(cos(11\, cos^{-1} x) + cos(9\, cos^{-1} x))}$$

Prove that:

$$\left(\Omega(a) + \Omega(b) + \Omega(c)\right)\left(6 + \frac{1}{\Omega^3(a)} + \frac{1}{\Omega^3(b)} + \frac{1}{\Omega^3(c)}\right) \geq 27$$

4.158 *If:*

If $f, g, h: [0,1] \to (0, \infty), f, g, h$ *- continuous then:*

$$27e^{\int_0^1 \left(log\left(f(x)\cdot g(x)\cdot h(x)\right)\right)dx} \leq \left(\int_0^1 \left(f(x) + g(x) + h(x)\right)dx \right)^3$$

4.159

Prove without softs:

$$\left(\int_0^1 (\gamma^x \cdot e^{1-x})dx \right)\left(\int_0^1 (e^x \cdot \pi^{1-x})dx \right) < e \int_0^1 (\gamma^x \cdot \pi^{1-x})\, dx$$

4.160

If $0 < a \leq b < \frac{\pi}{2}$ *then:*

$$\int_a^b \left(\frac{\cos^2 x + \tan x}{\cos^4 x + \tan^2 x}\right)^2 dx \leq \log\left|\frac{\tan b}{\tan a}\right|$$

4.161 *If* $f:[a,b] \to (0,\infty), a \leq 5 < 7 \leq b, f$ *continuous then:*

$$\frac{\left(\int_a^b f^5(x)dx\right)^7}{\left(\int_a^b f^7(x)dx\right)^5} < (b-a)^2$$

4.162

If $a,b,c > 0, a+b+c = 2$ *then:*

$$b^3\Omega(a) + c^3\Omega(b) + a^3\Omega(c) \geq \frac{8}{25}, \Omega(a) = \lim_{n\to\infty} n\int_0^1 \frac{x^n}{(x+a)^2}dx$$

4.163

$$\Omega(k) = \int_0^1 \frac{x^k - kx^{k-1} - x^{k-1} + 1}{x^{2k+1} + 2x^{k+1} + x^k + x + 1}dx, k \in \mathbb{N}, k \geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\log 2 \cdot \log n - n \cdot \log\left(\frac{3}{2}\right) + \sum_{k=1}^{n} \Omega(k)\right)$$

4.164

$$\int \frac{x \sec x\,(1 + \sin x)}{x + \sin x - \cos x - 1}dx$$

4.165 *If* $f:\mathbb{R} \to (0,\infty), f$ *continuos then:*

$$\int_0^a\int_0^a\int_0^a \left(\frac{f^4(x) + f^4(y) + f^4(z)}{f^3(x) + f^3(y) + f^3(z)}\right)^5 dx\,dy\,dz \geq a^2\int_0^a f^5(x)\,dx$$

4.166 *Find:*

$$\Omega = \int \frac{x^6 \cdot \log x}{(3 + x^7)^5} dx, x > 0$$

4.167 *Find:*

$$\int \frac{(2e^x + 1)\sin x - \cos x - 1}{e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1} dx$$

4.168 *Find:*

$$\int \frac{e^x \ln(1 + e^x) - e^{2x}}{(1 + e^x)^2 \ln^2(1 + e^x)} dx; x \in \mathbb{R}$$

4.169 *Prove that:*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx \leq \frac{\pi}{24}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \frac{\sqrt{2} - 1}{4} + \frac{152}{\pi^3}$$

4.170 *Prove that:*

$$1 \leq \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} \leq \frac{\pi}{2}$$

CHAPTER 5

ADVANCED CALCULUS-PROBLEMS

5.1 *If* $0 < a \leq b$ *then:*

$$\int_a^b \int_a^b \int_a^b \left(\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}} \right) dx\, dy\, dz \leq 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

5.2 *Find:*

$$\Omega = \int_0^\infty \frac{1-\cos x}{8 - 4x \sin x + x^2(1-\cos x)} dx$$

5.3 $f: \mathbb{R} \to [a, b], a < b$

Find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(n-k+1)^2 f(k)}{k(1^2 + 2^2 + \cdots + n^2)}$$

5.4. *Find:*

$$S = \sum_{n=0}^{\infty} \frac{1}{16^n (2n+1)^3} \binom{2n}{n}$$

5.5. *Find:*

$$\Omega = \int_0^\infty \frac{x^2+2}{x^6+1} dx$$

5.6 *Find:*

$$\Omega = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{\tan x + \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x + \frac{2\pi}{3}\right)}{\tan 3x \tan 3y} dx\, dy$$

5.7 *Find:*

$$\Omega = \int_1^2 \left(\int_1^2 \left(\int_1^2 \left(\frac{x^5 + y^5 + z^5 - (x+y+z)^5}{x^3 + y^3 + z^3 - (x+y+z)^3} \right) dx \right) dy \right) dz$$

5.8 *Prove that:*

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dx, Re\,(z) > 0$$

5.9 *Find:*

$$\Omega = \int_1^2 \left(\int_1^2 \left(\frac{(x^3 + y^3 - (x+y)^3)(x^7 + y^7 - (x+y)^7)}{((x+y)^5 - x^5 - y^5)^2} \right) dy \right) dx$$

5.10 *Calculate:*

$$\Omega = \int_0^1 \int_0^1 \frac{(xy)\big(xy(x^3 + y^3) - (x^4 + y^4)\big) + x^5 + y^5 + (x+y)(xy + 2 - x - y) - 2}{(x^2 - x + 1)(y^2 - y + 1)} dxdy$$

5.11 *Let n be a positive integer. Evaluate:*

$$\int_0^\infty \frac{(n-1)e^{nx} - ne^{(n-1)x} + 1}{xe^{nx}(e^x - 1)} dx$$

5.12 *Evaluate*

$$\frac{\pi}{2}\left(1+\frac{1}{2}\left(1+\frac{3}{4}\left(1+\frac{5}{6}(1+\cdots)\right)\right)\right)-\left(1+\frac{2}{3}\left(1+\frac{4}{5}\left(1+\frac{6}{7}(1+\cdots)\right)\right)\right)$$

5.13 *Find:*

$$\Omega = \prod_{n=0}^{\infty}\left(1+\left(\frac{1}{e}\right)^{3^n}+\left(\frac{1}{e}\right)^{2\cdot 3^n}\right)$$

5.14 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}}$$

5.15 *Find:*

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\int_0^{1}x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx}$$

5.16 *Find:*

$$\Omega = \int_0^{\infty}\frac{x\,ln^2\,x}{e^x-1}dx$$

5.17 *Find:*

$$\Omega = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} ln(sin(x+y))\,dx\,dy$$

5.18 *Let $n \geq 1$ be positive integer then prove that:*

$$\int_0^1 \int_0^1 \left\{\frac{x}{y}\right\}^n dx\,dy = \frac{1}{2(n+1)} + \sum_{n=1}^{\infty} \frac{\zeta(k+1)-1}{2\binom{n+k}{k}}$$

Where $\{.\}$ denotes the fractional part function and ζ – zeta function

5.19

$$\int_0^1 \int_0^1 ln\,\Gamma\,(x+y+1)\,dx\,dy$$

5.20 *Evaluate:*

$$\int_0^{\infty} \frac{1}{(1+y^{(20n)!!})(1+y^2)}\,dy, n \in \mathbb{N}$$

5.21 *Prove that:*

$$\int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}}\,dx = \frac{\beta(p,q)}{a^q(a+b)^{p+q}}$$

5.22 *For any complex number m & $Re(m) > -\frac{1}{2}$*

$$H_m = H_{m-\frac{1}{2}} + \frac{d}{dm}\left(\int_0^1 \frac{x^{2m}-x}{ln(x)} - \frac{dx}{1+x}\right)$$

Where H_m – Harmonic Number.

5.23 *Evaluate:*

$$\int \frac{\tan x + \ln(1-x)^{\ln x}}{x} dx$$

5.24 *If*

$$\int_0^1 \left(\int_0^t \frac{\ln(x)}{1-x^5} dx \right) dt = \frac{A\Psi_1\left(\frac{1}{5}\right) + B\Psi_1\left(\frac{2}{5}\right)}{C}$$

Then prove that: $25(A-B) + 2C = 0$ *where* $\Psi_n(x)$ *is Poly – Gamma function.*

5.25 *Find:*

$$\Omega = \int_0^\infty \frac{1}{(1+x^2)(3-\cos x)} dx$$

5.26 *Find:*

$$\Omega = \int \left(\sum_{n=1}^\infty 3^n \sinh^3 \frac{x}{3^n} \right) dx$$

5.27 *If:*

$$\Psi(m) = \int_0^1 x^2 \ln(x) \ln(x+m)\, dx$$

then show that

$$\int_0^\infty \frac{\Psi(m)}{1+m^2} dm = \frac{1}{9} + \frac{\pi G}{6} - \frac{G}{9} - \frac{\zeta(3)}{16} - \frac{5\pi}{27} + \frac{5\pi^2}{432} - \frac{\pi \ln(2)}{36}$$

where G is Catalan's constant.

5.28 *Find:*

$$\Omega = \int_{-\beta}^{\beta} \frac{dx}{\sqrt[3]{(\beta - x)(\beta^2 - x^2)}}, \beta > 0$$

5.29 *Find:*

$$\Omega = \int_0^\infty \left(\frac{1}{x} \cdot tan^{-1}\left(\frac{3x^2}{4x^4 + 5x^2 + 2}\right)\right) dx$$

5.30 *Find:*

$$\Omega = \lim_{n\to\infty} n\left(\sum_{k=1}^{n} e^{\frac{k}{n^2}} - n - \frac{1}{2}\right)$$

5.31 *Find:*

$$\Omega = \sum_{k=1}^{\infty}\left(\sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{1}{n^2 - k^2}\right)$$

5.32 *Find:*

$$\Omega = \lim_{n\to\infty}\left((n + 1)\int_0^1 \left(\frac{2x}{1 + x^2}\right)^{n+1} dx - n\int_0^1 \left(\frac{2x}{1 + x^2}\right)^{n} dx\right)$$

5.33

$$\omega(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{2^n} tanh\left(\frac{x}{2^n}\right), x > 0$$

Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to 0 \\ x>0}} \left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)}$$

5.34 *Find:*

$$\Omega = \int_0^{\infty} \frac{x\cdot tan^{-1} x}{x^4 + x^2 + 1} dx$$

5.35 *Let's define the function* $\chi(m)$ *for any complex number* $m, Re(m) > 0$

$$\chi(m) = \int_0^1 x\, ln(x)\, ln(x+m)\, dx$$

then we have the following,

$$\int_0^1 \chi(m)\, dm = \frac{\pi^2 - 16\, ln(2) + 4}{72}$$

5.36 *Find:*

$$\Omega = \int_0^{\infty} \frac{\pi x - 2\log x}{\left(\frac{\pi^2}{4} + log^2 x\right)(1+x^2)^2} dx$$

5.37

$$If\ 0 \le x \le a, 0 \le y \le b, c > 0$$

$$\Omega_1 = \int_0^c \left(\int_0^c \sqrt{x^2 + y^2 - 2ax + a^2}\, dx \right) dy\ ,$$

$$\Omega_2 = \int_0^c \left(\int_0^c \sqrt{x^2 + y^2 - 2by + b^2} dx \right) dy$$

$$then\ \Omega_1 + \Omega_2 \le (a + b)c^2$$

5.38

$$If\ 0 \le x \le 3, 0 \le y \le 4, a > 0$$

$$\Omega_1 = \int_0^a \left(\int_0^a \sqrt{x^2 + y^2 - 6x + 9}\, dx \right) dy$$

$$\Omega_2 = \int_0^a \left(\int_0^a \sqrt{x^2 + y^2 - 8y + 16}\, dx \right) dy$$

$$then:\ \Omega_1 + \Omega_2 > 5a^2$$

5.39 *Prove that:*

$$\int_0^1 \int_0^1 (x^2 + 34y^2 - 10xy - 6y + 2)^2 dx\, dy \ge 1$$

5.40 *If:*

$$\Omega = \int_0^1 \frac{ln(1 - x^2)^2\, ln(1 - x)}{x} dx$$

Prove that: $\Omega > \frac{5}{2}\zeta(3)$

5.41 *Prove that:*

$$1 < \int_0^1 \int_0^1 (x+y)^4 dxdy < \frac{16}{5}$$

$$\int_a^b \int_a^b (x+y)^4 \, dxdy \le \int_a^b \int_a^b \int_0^1 (tx + (1-t)y)^4 \, dxdydt, a < b$$

5.42 *If* $a, b, p, q \in \mathbb{R}, a < b, p > 1, p + q = pq$ *then:*

$$(b-a)^2\sqrt{e^{a+b}} \le \int_a^b \int_a^b e^{\frac{qx+py}{p+q}} \, dx \, dy \le (b-a)(e^b - e^a)$$

5.43 *For* $a_i \in (0,1], \forall \, i \in [1, n]$

Prove:

$$\frac{1}{\pi^n} \cdot \prod_{i=1}^{n} a_i^2 \le \int_o^{a_n} \dots \int_o^{a_1} \left(\prod_{i=1}^{n} \sin x_i \right) dx_1 \dots dx_n \le \frac{1}{2^n} \cdot \left(\prod_{i=1}^{n} a_i \right)^2$$

5.44 *If* $a, b, p, q \in \mathbb{R}, a < b, p > 1, p + q = pq$ *then:*

$$\frac{a^2 + 2ab + b^2}{4} < \frac{\int_a^b \int_a^b (px + qy)^2 \, dx \, dy}{(b-a)^2(p+q)^2} < \frac{a^2 + ab + b^2}{3}$$

5.45 *If* $0 < a < b$ *then:*

$$\frac{1}{b-a} a \int_a^b \int_a^b \frac{dxdy}{x+y} < \frac{13}{25} log\left(\frac{b}{a}\right)$$

5.46 *If:*

$$\Omega(a) = \iint\limits_{(x,y)=(0,0)}^{(x,y)=(a,a)} \left(\sqrt{x^2 + 2xy} + \sqrt{y^2 + 2xy}\right) dxdy, a > 0$$

then:

$$\frac{\Omega(a)}{b^3} + \frac{\Omega(b)}{c^3} + \frac{\Omega(c)}{a^3} \geq 2\sqrt{3}$$

5.47 *If* $n \in \mathbb{N}^*$ *then:*

$$\int_0^1 \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} \left(1 + x_i^2\right) dx_i + \int_0^1 \int_0^1 \dots \int_0^1 \prod_{i=1}^{n} \left(1 - x_i^2\right) dx_i \leq 2^n$$

5.48 *Prove that:*

$$\int_0^1 \int_0^1 \int_0^1 \left(x\sqrt{x^2 + z^2} + y\sqrt{y^2 + z^2}\right) dxdydz \leq 1$$

5.49 *If* $a, b \geq 1$ *then:*

$$2\int_1^b \left(y \int_1^a log\frac{x}{y} dx\right) dy \leq (a-1)(b-1)(a-b)$$

5.50

If $a, b, c > 0, \alpha \in \left(0, \frac{\pi}{2}\right)$

$$\Omega(a,b) = \int_0^b \left(\int_0^a (x\sin^2\alpha + y\cos^2\alpha)(x\cos^2\alpha + y\sin^2\alpha)\, dx\right) dy$$

then:

$$4\Omega(b,c) + 4\Omega(c,a) + 4\Omega(a,b) \geq abc(a+b+c)$$

5.51 *If* $x, y, z \in (0,1]$,

$$\Omega(x) = \int_0^x \frac{ln(1+ax)}{1+a^2} da$$

then:

$$2\big(\Omega(x) + \Omega(y) + \Omega(z)\big) \geq 3\, ln\, 2 + ln(xyz)$$

5.52 *If* $a, b, c > 0$ *then:*

$$\int_a^{2a} \left(\int_b^{2b} \left(\int_c^{2c} \left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) dx \right) dy \right) dz \leq ln \sqrt{2^{ab+bc+ca}}$$

5.53 *Find:*

$$\Omega = \lim_{n\to\infty} \left((\pi + H_n)^{\frac{1+H_n}{H_n}} - (H_n)^{\frac{1+\pi+H_n}{\pi+H_n}} \right)$$

5.54 *Find:*

$$\Omega = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\zeta(n+1) - 1\right)$$

5.55 *Prove that:*

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\sqrt[3]{xyz} + \sqrt[3]{yzt} + \sqrt[3]{ztx} + \sqrt[3]{txy} \right) dxdydzdt \leq 2$$

5.56

If $a, b, c \geq 1$,

$$\Omega(a,b) = \int_{a}^{2a} \left(\int_{b}^{2b} \big((ln(x+y) - lnx)(ln(x+y) - lny)\big)dy \right) dx$$

then:

$$\frac{1}{ln2}(\Omega(a,b) + \Omega(b,c) + \Omega(c,b)) < ln2^{a^2+b^2+c^2}$$

5.57 *If* $0 < a, b, c \leq 1$,

$$\Omega(a,b) = \int_{0}^{1} \left(\int_{0}^{1} \sqrt{(x^2 + a^2 + b^2)(y^2 + a^2 + b^2)}dy \right) dx$$

then:

$$\Omega(a,b) + \Omega(b,c) + \Omega(c,a) \geq 6\sqrt{abc}$$

5.58 *If* $a, b, c > 0$,

$$\Omega(a,b) = \int_{a}^{2a} \left(\int_{b}^{2b} \frac{(x^2 + xy + y^2)^2}{(2x + y)(x + 2y)} dy \right) dx$$

then:

$$\frac{\Omega(a,b)}{a^2b^2} + \frac{\Omega(b,c)}{b^2c^2} + \frac{\Omega(c,a)}{c^2a^2} \geq \frac{27}{4}$$

5.59 *Prove that:*

$$\Omega = \int_1^2 \int_1^2 \int_1^2 \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{x}{z} + \frac{y}{x} + \frac{z}{y}\right) dxdydz < \frac{15}{2}$$

5.60 *If* $a, b, c \geq 0$

$$\Omega(a,b) = \int_a^{2a} \left(\int_b^{2b} |sin(x-y)\, cos(x+y) - sin(x+y)| dy \right) dx$$

then:

$$\Omega(a,b) + \Omega(b,c) + \Omega(c,a) \leq \sqrt{2}(a^2 + b^2 + c^2)$$

5.61 *If* $0 < a \leq b$ *then :*

$$\int_a^b \int_a^b \frac{x^4 + y^4 + x^2y^2}{x^2 + y^2 + xy} dxdy \geq \frac{(b^2 - a^2)^2}{4}$$

5.62 *If* $a \geq 0$ *then:*

$$\int_0^a \int_0^a \int_0^a \sqrt{x^2 + xy + yz + zx}\, dx\, dy\, dz \geq \frac{8a^4}{9}$$

5.63 *If* $1 \leq a \leq b \leq 2$ *then:*

$$\sqrt{2}(a-b)^2 \leq \int_a^b \int_a^b \sqrt{\frac{x}{y} + \frac{y}{x}} dxdy \leq \frac{\sqrt{10}}{2}(a-b)^2$$

5.64 *If* $0 < a < b < 1$ *then:*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b \frac{(x^y + y^x)(x+y)^{x+y}}{(2x)^y(2y)^x}\,dx\,dy > 1$$

5.65 *If* $a, b, c \geq 1$ *then:*

$$\frac{1}{2abc}\int_a^{2a}\left(\int_b^{2b}\left(\int_c^{2c}\left(\frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}}\right)dx\right)dy\right)dz \geq 1$$

5.66 *Prove that:*

$$\Omega_1 = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\sum \tan x\right)dxdydz, \Omega_2 = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\sum \tan x \tan y\right)dxdydz,$$

$$\Omega_3 = \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\prod \tan x\right)dxdydz$$

Prove that: $\Omega_1 - 2\Omega_2 + 4\Omega_3 \leq \frac{\pi^3}{64}$

5.67 *If* $a, b, c \geq 1$ *then:*

$$2\int_1^a\int_1^b\int_1^c\left(\sum \frac{x}{x^2+yz}\right)dxdydz \leq ln\left(\prod a^{(b-1)(c-1)}\right)$$

5.68 *If* $0 < a \leq b$ *then:*

$$\int_a^b\int_a^b \sqrt{\frac{x+y+\sqrt{xy}}{\sqrt{x}+\sqrt{y}+\sqrt[4]{xy}}}\,dxdy \geq \frac{64}{81}\left(b^{\frac{9}{8}} - a^{\frac{9}{8}}\right)^2$$

5.69 *If* $0 < a \leq b$ *then* :

$$\int_a^b \int_a^b \left(\frac{e^x + e^y - 2}{\sqrt{xy}}\right)^{10} dxdy \geq 2^{10}(b-a)^2$$

5.70 *For* $1 \leq a \leq b \leq 2$. *Prove:*

$$2(b-a)^2 \leq \int_a^b \int_a^b \sqrt{\frac{x^2 + xy}{y^2} + \frac{y^2 + xy}{x^2}}\, dxdy \leq \frac{3\sqrt{3}}{2}(b-a)^2$$

5.71 *If* $0 \leq a, b, c, d \leq 1$ *then:*

$$\int_0^a \int_0^b \int_0^c \int_0^d \frac{\left(\sqrt{x} + \sqrt{y}\right)\left(\sqrt{z} + \sqrt{t}\right)}{1 + \sqrt{xyzt}}\, dxdydzdt \leq 2abcd$$

5.72 *Prove that:*

$$\int_0^1 \int_0^1 \int_0^1 sin^{10}(x^4 + y^4 + z^4)\, dxdydz \leq \frac{3}{5}$$

5.73

If $0 < a \leq b$ *then:*

$$\int_a^b \int_a^b \left((x+y)^4 - \frac{x^5 + y^5}{x + y}\right) dxdy \geq \frac{5}{3}(b^3 - a^3)^2$$

5.74 *If* $0 < a \leq b$ *then:*

$$\int_a^b \int_a^b \left(\sqrt{\frac{x + a}{(x + b)(y + b)}} + \sqrt{\frac{x + b}{(x + a)(y + a)}}\right) dxdy \geq \frac{\sqrt[4]{8}(b-a)^2}{\sqrt[4]{b(a + b)}}$$

5.75 *Prove:*

$$\int_0^1\int_0^1\int_0^1\int_0^1 \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{t}}{16+x+y+z+t}\,dxdydzdt \le \frac{1}{5}$$

5.76 *Prove:*

$$\frac{1}{16}\int_0^1\int_0^1\int_0^1 \left(\frac{x^4+y^4+z^4+5}{xy+yz+zx+1}\right)dxdydz \ge 1$$

5.77 *For* $a > 0$. *Prove:*

$$\frac{1}{a^4}\int_a^{2a}\int_a^{2a}\int_a^{2a}\int_a^{2a} \frac{x+y+x+t}{\sqrt[5]{x^5+y^5+z^5+t^5}}\,dxdydzdt \le \sqrt[5]{256}$$

5.78 *For* $a > 0$. *Prove:*

$$\frac{1}{a^4}\int_a^{2a}\int_a^{2a}\int_a^{2a} \left(\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}\right)dxdydz \ge \frac{3}{2}$$

5.79 *Prove that:*

$$\int_1^2\int_1^2\int_1^2 \frac{xyz}{(x^2+y^2+z^2)(x+y+z)}\,dxdydz \le \frac{1}{9}$$

5.80 *If* $a > 0, z \in \mathbb{R}$ *then:*

$$\frac{1}{a^3}\int_0^a\int_0^a \frac{xydxdy}{x\cos^2 z + y\sin^2 z} \le 1$$

5.81 *If* $f:[0,1]\to(0,\infty)$ *continuous then:*

$$\int_0^1\int_0^1 \frac{dxdy}{1+f(x)f(y)} \leq \frac{1}{2}\int_0^1 \frac{dx}{f(x)}$$

5.82 *If* $a,b>1$ *then:*

$$\int_1^a\int_1^b \frac{x+y}{x^2+y^2}dxdy \leq ln\sqrt{\frac{a^b\cdot b^a}{ab}}$$

5.83 *Prove that:*

$$\int_1^2\int_1^2\int_1^2\int_1^2 \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{t}+6}{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{z}}dxdydzdt \geq \frac{5}{2}$$

5.84 *Prove:*

$$\int_1^2\int_1^2\int_1^2 \left(\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}\right)^2 dxdydz < \frac{21}{4}$$

5.85 *Prove that:*

$$\int_2^3\int_2^3 \left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5 dxdy \geq 32$$

5.86 *Prove that:*

$$\int_2^3 ln\,x\,ln(x^2-1)\,dx < \frac{35}{8}+ln\frac{3}{2}$$

5.87 *If* $a, b, c > 0$

$$\Omega(a) = \int_a^{2a}\int_a^{2a} \frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}\,dx\,dy \Rightarrow \Omega(a)+\Omega(b)+\Omega(c) \ge ab+bc+ca$$

5.88 *If* $f:[0,a]\to[0,\infty), a\ge 0, f$ *– continuous then:*

$$\int_0^a\int_0^a \sqrt{f^2(x)+f^2(y)}\,dx\,dy + \int_0^a\int_0^a \sqrt{2f(x)f(y)}\,dx\,dy \le 2a\sqrt{2}\int_0^a f(x)\,dx$$

5.89 *If* $a, b, c \ge 1$ *then:*

$$\int_1^a\int_1^b\int_1^c \left(\frac{x}{x^2+yz}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy}\right)dx\,dy\,dz \le ln\sqrt{\prod a^{(b-1)(c-1)}}$$

5.90 *If* $0 \le a < b \le 1, n\in\mathbb{N}, n\ge 2$

$$\Omega_1 = \int_a^b\int_a^b \dots \int_a^b (1+x_1^2)(1+x_2^2)\dots(1+x_n^2)dx_1dx_2\dots dx_n$$

$$\Omega_2 = \int_a^b\int_a^b \dots \int_a^b (1-x_1^2)(1-x_2^2)\dots(1-x_n^2)dx_1dx_2\dots dx_n$$

then: $\sqrt[n^2]{\Omega_1+\Omega_2}+\frac{1}{n} < 1+\frac{2(b-a)}{n}$

5.91 *If* $a > 0$ *then:*

$$\frac{1}{16}\int_0^a\int_0^a (x+y)^4 dx\,dy \le \int_0^{2a}\int_0^{2a}\int_0^1 (tx+(1-t)y)^4\,dt\,dy\,dx$$

5.92 *If* $a \in (-1,1]$ *then:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^k \cdot \cos(k\cos^{-1}a)}{k} \le \log\left(\sqrt{2e^a}\right)$$

5.93 *If* $a > 0$ *then:*

$$\int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}}\,dx\,dy\,dz > 2a^2\log 2$$

5.94 *If* $a > 0$ *then:*

$$a^2 \cdot \sqrt[4]{e^{9a^2}} < \int_a^{2a}\int_a^{2a}\sqrt[16]{e^{(x+3y)^2}}\,dx\,dy < a \cdot \int_a^{2a} e^{x^2}\,dx$$

5.95 *If* $f: \mathbb{R} \to (0,\infty)$, f *continuous then:*

$$\int_0^a\int_0^a\int_0^a\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz \ge a^2\int_0^a f^5(x)\,dx$$

5.96 *If* $a, b, c > 0$ *then:*

$$2 \cdot \int_a^{2a}\int_a^{2b}\int_a^{2c}\frac{(2x+y)(2y+z)(2z+x)}{(x+y+z)^2}\,dx\,dy\,dz \le 3abc(a+b+c)$$

5.97 *If* $0 < a < \frac{\pi}{4}$ *then:*

$$a^2\cot\left(\frac{3a}{2}\right) \le \int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy \le \log(2\cos a)^a$$

5.98 *If* $a, b, c > 0$ *then:*

$$3\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{x+y+z}{x^2+y^2+z^2}\right)dx\,dy\,dz \le (ab+bc+ca)\ln 2$$

5.99 *If* $f:[0,1]\to(0,\infty)$, f *– continuous,* $\int_0^1 f(x)\,dx = 1$ *then:*

$$\int_0^1\int_0^1 f(x)f(y)\,dx\,dy \le \int_0^1 f^{10}(x)\,dx$$

5.100 *Let be* $\Omega:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$,

$$\Omega(x) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\int_\varepsilon^x \frac{\sin t}{t}\,dt$$

Prove that:

$$(a+b+c)\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right) \ge a\Omega(a)+b\Omega(b)+c\Omega(c)$$

5.101 *If* $0 < a \le b < \frac{\pi}{4}$ *then:*

$$\int_a^b\int_a^b\int_a^b\left(\frac{(x+y)^2}{(y+z)\sin(z+x)}+\frac{(y+z)^2}{(z+x)\sin(x+y)}+\frac{(z+x)^2}{(x+y)\sin(y+z)}\right)dxdydz \ge 3(b-a)^3$$

5.102 *If* $0 \le a,b,c,d,e,f,x,y,z \le 1$ *then:*

$$108\int_0^1\int_0^1\int_0^1 \frac{(3abcdefxyz-1)dxdydz}{(a^2+b^2+x^2+3)(c^2+d^2+y^2+3)(e^2+f^2+z^2+3)} \le 1$$

5.103 *Let* $0 < a \leq b < \frac{\pi}{4}$. *Prove:*

$$\frac{1}{a+b}\int_a^b \int_a^b \sin(\sin(\sin(x+y)))\,dxdy \leq (b-a)^2$$

5.104

$$si\,(x) = -\int_x^\infty \left(\frac{\sin t}{t}\right) dt, x > 0$$

$$\Omega_1 = \int_\gamma^e \left(\frac{1}{x}\left(si\,(e^2x) - si(\pi x)\right)\right) dx, \Omega_2 = \int_\pi^{e^2} \left(si(ex) - si(\gamma x)\right) dx$$

$$A \cdot \Omega_1 < \Omega_2, B \cdot \Omega_1 = \Omega_2, C \cdot \Omega_1 > \Omega_2$$

5.105 *If* $1 < a \leq b$ *then:*

$$\int_a^b \int_a^b \log\left(\frac{x+y}{2}\right)^{x+y} dx\,dy \geq (b-a)^2 \log\left(\frac{a+b}{2}\right)^{a+b}$$

5.106 $f: [a,b] \to (0,\infty)$ f – *continuous,* $0 < a \leq b$

$$\Omega(x,y,z) = \left(\frac{f^2(x)+f^2(y)}{f(x)+f(y)}\right)^3 + \left(\frac{f^2(y)+f^2(z)}{f(y)+f(z)}\right)^3 + \left(\frac{f^2(z)+f^2(x)}{f(z)+f(x)}\right)^3$$

Prove that:

$$\int_a^b \int_a^b \int_a^b \Omega(x,y,z)\,dx\,dy\,dz \geq 3(b-a)^2 \int_a^b f^3(x)\,dx$$

5.107 *If* $0 < a \leq b$ *then:*

$$\frac{3}{2}\int_a^b\int_a^b\left(\frac{x^2+y^2}{x^4+x^2y^2+y^4}\right)dxdy \leq \left(\log\left(\frac{b}{a}\right)\right)^2$$

5.108 *Find:*

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_\varepsilon^1\left(\frac{\log(1+x)}{x(1+x^2)}\right)dx\right)$$

5.109

If $n \in \mathbb{N}, n \geq 1$ *then:*

$$2+\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\cdot\ldots\cdot(1+H_k)}{H_1H_2\cdot\ldots\cdot H_{k+1}} < \left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}$$

CHAPTER 6

EQUATIONS.SYSTEMS-SOLUTIONS

1.1.

$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=3025 \Leftrightarrow x>0 \Rightarrow \frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=(55)^2$$

Desde que: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$. *Si:* $n=10 \Leftrightarrow 1^3+2^3+3^3+\cdots+10^3=(55)^2$

$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=1^3+2^3+3^3+\cdots+10^3$$

$$\left(\frac{1}{x}-1\right)+\left(\frac{16}{x+1}-8\right)+\left(\frac{81}{x+2}-27\right)+\cdots+\left(\frac{10000}{x+9}-1000\right)=0$$

$$\left(\frac{1-x}{x}\right)+\frac{8(1-x)}{x+1}+\frac{27(1-x)}{(x+2)}+\frac{1000(1-x)}{(x+9)}=0$$

$$(1-x)\left(\frac{1}{x}+\frac{8}{x+1}+\frac{27}{x+2}+\cdots+\frac{1000}{x+9}\right)=0 \Leftrightarrow \left(\frac{1}{x}+\frac{8}{x+1}+\frac{27}{x+2}+\cdots+\frac{1000}{x+9}\right)>0,\ \textit{ya que: } x>0$$

Por la tanto: $x=1$

1.2.

$(x,y)=(0,0)$ *is the trivial solution*

$(x^2+6xy+5y^2)=(x+y)(x+5y)$ *and* $(x^2+2xy)=(x+y)^2-y^2$

Equation becomes: $(x+y)(x+5y)=6y\cdot\sqrt{(x+y)^2-y^2}$, *Squarring:* $(x+y)^2(x+5y)^2=36y^2((x+y)^2-y^2)$

$36y^4=(x+y)^2\{36y^2-(x+5y)^2\}$, $\left\{\frac{6y^2}{x+y}\right\}^2=(y-x)(11y+x)$ *(1)*

Let $\frac{6y^2}{x+y}=k$ *(2) where* k *is integer* $\neq 0$ *since RHS is integer* $\Rightarrow \frac{6y^2}{k}-y=x$ *(3)*

(1) because: $k^2=(y-x)(11y+x) \Rightarrow k^2\left(2y-\frac{6y^2}{k}\right)\left(\frac{6y^2}{k}+10y\right)$

$$\Rightarrow k^2 = \left\{2y\left(1-\frac{3y}{k}\right)\right\}\left\{2y\left(\frac{3y}{k}+5\right)\right\} \Rightarrow \left(\frac{k}{y}\right)^2 = 4\left(1-\frac{3y}{k}\right)\left(\frac{3y}{k}+5\right)$$

$$Let\,\frac{y}{k} = T \Rightarrow \frac{1}{T^2} = 4(1-3T)(3T+5) \Rightarrow 36T^4 + 48T^3 - 20T^2 + 1 = 0$$

where roots are complex or irrational numbers $\Rightarrow T = \left(\frac{y}{k}\right)$ *is complex or irrational.*

But from (3): $6y\cdot\left(\frac{y}{k}\right) = x + y$ *meaning that LHS is not interger* $\Rightarrow$ *no integer solutions.*

1.3.

$$\binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!\,(n-k)!} = \frac{n+1}{k+1}\cdot\frac{n!}{k!\,(n-k)!} = \frac{n+1}{k+1}\binom{n}{k} \Rightarrow \binom{n}{k}\binom{n+1}{k+1} =$$

$$= (n+1)\cdot\frac{1}{k+1}\binom{n}{k}\binom{n}{k} \Rightarrow \sum_{k=0}^{n} a_k\binom{n}{k}\binom{n+1}{k+1} = (n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2$$

$$We\ know\binom{2n}{n} = \textstyle\sum_{k=0}^{n}\binom{n}{k}^2$$

$$\therefore \sum_{k=0}^{n} a_k\binom{n}{k}\binom{n+1}{k+1} = (n+1)\binom{2n}{n} \Rightarrow (n+1)\sum_{k=0}^{n}\frac{a_k}{k+1}\binom{n}{k}^2 = (n+1)\sum_{k=0}^{n}\binom{n}{k}^2$$

Thus a possible sequence is $a_k = k+1, \forall k \in \mathbb{N}.$

1.4.

$$1-\sum_{k=1}^{n}\frac{k}{\sqrt{k!}\left(\sqrt{k+1}\right)\left(\sqrt{k+1}+1\right)} \le \frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{k\left(\sqrt{k+1}-1\right)}{\sqrt{(k+1)!}\left(\sqrt{k+1}+1\right)\left(\sqrt{k+1}-1\right)} \le \frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{k\left(\sqrt{k+1}-1\right)}{\sqrt{(k+1)!}\,(k)} \le \frac{1}{12\sqrt{5}} \Rightarrow 1-\sum_{k=1}^{n}\left(\frac{\sqrt{k+1}}{\sqrt{(k+1)k!}}-\frac{1}{\sqrt{(k+1)!}}\right) \le \frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{1}{\sqrt{k!}}+\sum_{k=1}^{n}\frac{1}{\sqrt{(k+1)!}} \le \frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\frac{1}{\sqrt{1!}}+\frac{1}{\sqrt{2!}}-\frac{1}{\sqrt{2!}}+\frac{1}{\sqrt{3!}}-\cdots+\frac{1}{\sqrt{(k+1)!}}\leq\frac{1}{12\sqrt{5}}$$

$$\Rightarrow \frac{1}{\sqrt{(k+1)!}}\leq\frac{1}{12\sqrt{5}}\rightarrow\sqrt{(k+1)!}\geq 12\sqrt{5}\rightarrow(k+1)!\geq 720\rightarrow(k+1)!\geq 6!\rightarrow k\geq 5$$

1.5

$$\frac{x+1}{5}+\frac{x+2}{8}+\frac{x+3}{11}+\cdots+\frac{x+n}{3n+2}=\frac{2}{x+4}+\frac{3}{x+7}+\frac{4}{x+10}+\cdots+\frac{n+1}{x+1+3n}$$

$$\Rightarrow\left(\frac{x+1}{5}-\frac{2}{x+4}\right)+\left(\frac{x+2}{8}-\frac{3}{x+7}\right)+\left(\frac{x+3}{11}-\frac{4}{x+10}\right)+\cdots+\left(\frac{x+n}{3n+2}-\frac{n+1}{x+1+3n}\right)=0$$

$$\Rightarrow\frac{x^2+5x-6}{5(x+4)}+\frac{x^2+9x-10}{8(x+7)}+\frac{x^2+13x-14}{11(x+10)}+\cdots+\frac{(x+n)(x+1+3n)-(n+1)(3n+2)}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow\frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\cdots+\frac{x^2+x+4nx+n+3n^2-3n^2-5n-2}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow\frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\cdots+\frac{x^2+x(4n+1)-4n-2}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow\frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\cdots+\frac{(x-1)(x+4n+2)}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow(x-1)\left(\frac{x+6}{5(x+4)}+\frac{x+10}{8(x+7)}+\frac{x+14}{11(x+10)}+\cdots+\frac{x+4n+2}{(3n+2)(x+1+3n)}\right)=0$$

$\Rightarrow$ Por la tanto: $x=1$, ya que:

$$\left(\frac{x+6}{5(x+4)}+\frac{x+10}{8(x+7)}+\frac{x+14}{11(x+10)}+\cdots+\frac{x+4n+2}{(3n+3)(x+1+3n)}\right)>0\Leftrightarrow x\wedge n>0$$

1.6.

$$\binom{k}{2} = \tfrac{1}{2}k(k-1) = m \quad \textit{(say)} \quad \therefore \binom{\binom{k}{2}}{2} = \binom{m}{2} = \tfrac{1}{2}m(m-1)$$

$$= \frac{1}{8}k(k-1)[k^2-k-2] = \frac{1}{8}k(k-1)(k+1)(k-2) =$$

$$= \frac{3}{24}(k+1)k(k-1)(k-2) = 3\binom{k+1}{4}$$

$$\sum_{k=3}^{x}\binom{\binom{k}{2}}{2} = 3\sum_{k=3}^{x}\binom{k+1}{4} = 3\left[\binom{4}{4}+\binom{5}{4}+\cdots+\binom{x+1}{4}\right] =$$

$$= 3\left[\binom{5}{5}+\binom{5}{4}+\cdots+\binom{x+1}{4}\right] = 3\left[\binom{6}{5}+\binom{6}{4}+\cdots+\binom{x+1}{4}\right] =$$

$$= 3\left[\binom{7}{5}+\binom{7}{4}+\cdots+\binom{x+1}{4}\right] = \cdots = 3\binom{x+2}{5}$$

Thus, $3\binom{x+2}{5} \le 168 \Rightarrow \binom{x+2}{5} \le 56$. *As* $\binom{n}{r} = 0$ *for* $n < r$, $x = 1,2,3,4,5,6$

1.7

For $r \ge 1$, *write* $r^2 \equiv (r+3)(r+2) + A(R+2) + B$

Put $r = -2, 4 = B$

Put $r = -3, 9 = -A + B \Rightarrow A = -5$

$$\therefore r^2 \equiv (r+3)(r+2) - 5(r+2) + 4 \Rightarrow r^2(r+1)! = (r+3)! - 5(r+2)! + 4(r+1)!$$

$$= \big((r+3)! - (r+2)!\big) - 4\big((r+2)! - (r+1)!\big)$$

$$\Rightarrow \sum_{r=1}^{n} r^2(r+1)! = \big((n+3)! - 3!\big) - 4\big((n+2)! - 2!\big) = (n+3)! - 4(n+2)! + 2$$

$$\therefore \sum_{k=1}^{n} k^2(k+1)! - 2 = (n+2)!\,(n+3-4) = (n+2)!\,(n-1)$$

$$\therefore \frac{\sum_{k=1}^{n}(k^2)\,(k+1)! - 2}{(n+1)!} = 108$$

$$\Rightarrow (n+2)(n-1) = 108 \Rightarrow n^2 + n - 110 = 0 \Rightarrow (n+11)(n-10) = 0$$

As $n \in \mathbb{N}, n = 10$

1.8

$$\sum_{k=1}^{n} (k^3 - 1)k! = (n^2 - 2)(n+1)! + 2\,, \frac{(n^2-2)(n+1)! + 2 - 2}{(n^2 - 2)} = 40320$$

$$\frac{(n^2-2)(n+1)!}{(n^2-2)} = 40320, \qquad (n+1)! = 40320, \qquad n = 7$$

1.9

$\rightarrow \because$ *we know that*

$$-\sqrt{a^2 + b^2} \le a\cos x + b\sin x \le \sqrt{a^2 + b^2} \Rightarrow -\sqrt{3^2 + 4^2} \le 3\sin x - 4\cos x \le \sqrt{3^2 + 4^2}$$

$$\Rightarrow -5 \le 3\sin x - 4\cos x \le 5 \Rightarrow 0 \le |3\sin x - 4\cos x| \le 5$$

$$\because |3\sin x - 4\cos x| = y^2 - 6y + 14$$

$$|3\sin x - 4\cos x| = (y-3)^2 + 5 \Rightarrow LHS \le 5, RHS \ge 5 \Rightarrow LHS = RHS = 5 \Rightarrow$$

$$\Rightarrow (y-3)^2 = 0 \Rightarrow y = 3$$

$$|3\sin x - 4\cos x| = 5 \Rightarrow 3\sin x - 4\cos x = \pm 5 \Rightarrow \frac{3}{5}\sin x - \frac{4}{5}\cos x = \pm 1$$

$$\frac{3}{5}\sin x - \frac{4}{5}\cos x = \pm 1$$

$$\sin(x - \alpha) = \sin\left(\pm\frac{\pi}{2}\right), \tan\alpha = \frac{4}{3} \Rightarrow x - \alpha = n\pi + (-1)^n\left(\pm\frac{\pi}{2}\right)$$

$$x = n\pi \pm (-1)^n\left(\frac{\pi}{2}\right) + \tan^{-1}\left(\frac{4}{3}\right), n \in I$$

Now, $2^y + 2^z + \tan^{-1} z = 9 \therefore y = 3 \Rightarrow 8 + 2^z + \tan^{-1} z = 9 \Rightarrow 2^z = \tan^{-1} z = 1$

Let $b(z) = 2^z + \tan^{-1} z - 1; b'(z) = 2^z \ln 2 + \frac{1}{1+z^2} > 0 \Rightarrow b'(z) > 0 \Rightarrow b(z)$ *is increasing function. So,* $b(z)$ *can have atmost one root* $\because b(0) = 0 \Rightarrow z = 0$ *is the only possible solution.*

$$\begin{cases} x = n\pi \pm (-1)^n \left(\frac{\pi}{2}\right) + tan^{-1}\left(\frac{4}{3}\right), n \in I \\ y = 3 \\ z = 0 \end{cases}$$

1.10

$$\binom{k-1}{2} = \frac{1}{2}(k-1)(k-2) = \frac{1}{2}[k(k-1) - 2k + 2] = \frac{1}{2}k(k-1) - k + 1$$

$$\therefore \sum_{k=3}^{n} \binom{n}{k}\binom{k-1}{2}$$

$$= \sum_{k=3}^{n} \binom{n}{k}\left[\frac{1}{2}k(k-1) - k + 1\right] = \frac{1}{2}\sum_{k=3}^{n} k(k-1)\binom{n}{k} - \sum_{k=3}^{n} k\binom{n}{k} + \sum_{k=3}^{n} \binom{n}{k}$$

$$= \frac{1}{2}n(n-1)\sum_{k=3}^{n} \binom{n-2}{k-2} - n\sum_{k=3}^{n} \binom{n-1}{k-1} + \sum_{k=3}^{n} \binom{n}{k}$$

$$= \frac{1}{2}n(n-1)[2^{n-2} - 1] - n\big(2^{n-1} - 1(n-1)\big) + \left[2^n - 1 - n - \frac{1}{2}n(n-1)\right]$$

$$= n(n-1)2^{n-3} - \frac{1}{2}n(n-1) - n\big(2^{n^{-1}}\big) + n + n(n-1) + 2^n - 1 - n - \frac{1}{2}n(n-1)$$

$$= n(n-1)2^{n-3} - (n-2)2^{n-1} - 1$$

$$\therefore n(n-1)2^{n-3} - (n-2)2^{n-1} - 1 = 21(2^{n-1} - 1)$$

$$\Rightarrow n(n-1)2^{n-3} - (n-2)2^{n-1} - 21(2^{n-2}) + 20 = 0$$

$$\Rightarrow n(n-1) - 4(n-2) - 42 + 20(2^{3-n}) = 0$$

$$\Rightarrow n^2 - 5n - 34 + 5(2^{7-n}) = 0 \Rightarrow 5(2^{7-n}) = 34 + 5n - n^2$$

As RHS is an integer, and $n \geq 3$, $3 \leq n \leq 7$.

But $n = 3,4,5,6,7$ *do not satisfy it . So, no solution.*

1.11

* *We have:* $\begin{cases} x^3 - 2x^2 + 2x \geq 0 \\ 4x - 3x^4 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x(x^2 - 2x + 2) \geq 0 \\ x(3x^3 - 4) \leq 0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x((x-1)^2+1) \geq 0 \\ 0 \leq x \leq \sqrt[3]{\frac{4}{3}} \end{cases} \Leftrightarrow 0 \leq x \leq \sqrt[3]{\frac{4}{3}}$$

** Because:* $x^2 - x + 1 = \left(x^2 - x + \frac{1}{4}\right) + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$

- Therefore, since inequality AM – GM for 2,3,4 *real numbers:*

$$\sqrt{x^3 - 2x^2 + 2x} + 3 \cdot \sqrt[3]{x^2 - x + 1} + 2 \cdot \sqrt[4]{4x - 3x^4}$$

$$= \sqrt{x(x^2 - 2x + 2)} + 3 \cdot \sqrt[3]{(x^2 - x + 1) \cdot 1 \cdot 1} + 2 \cdot \sqrt[4]{x(4 - 3x^3) \cdot 1 \cdot 1} \leq$$

$$\leq \frac{x + x^2 - 2x + 2}{2} + (x^2 - x + 1) + 1 + 1 + \frac{2(x + (4 - 3x^3) + 1 + 1)}{4}$$

$$\Rightarrow \sqrt{x^3 - 2x^2 + 2x} + 3 \cdot \sqrt[3]{x^2 - x + 1} + 2 \cdot \sqrt[4]{4x - 3x^4} \leq$$

$$\leq \frac{x^2 - x + 2}{2} + x^2 - x + 3 + \frac{-3x^3 + x + 6}{2}$$

$$\Leftrightarrow \sqrt{x^3 - 2x^2 + 2x} + 3 \cdot \sqrt[3]{x^2 - x + 1} + 2 \cdot \sqrt[4]{4x - 3x^4} \leq \frac{-3x^3 + 3x^2 - 2x + 14}{2} \quad \textit{(2)}$$

- *Since (1), (2):*

$$\Rightarrow \frac{x^4 - 3x^3}{2} + 7 \leq \frac{-3x^3 + 3x^2 - 2x + 14}{2} \Leftrightarrow \frac{x^4 - 3x^3 + 14}{2} \leq \frac{-3x^3 + 3x^2 - 2x + 14}{2}$$

$$\Leftrightarrow x^4 - 3x^3 + 14 \leq -3x^3 + 3x^2 - 2x + 14 \Leftrightarrow x^4 - 3x^2 + 2x \leq 0$$

$$\Leftrightarrow x(x^3 - 3x + 2) \leq 0$$

$$\Leftrightarrow x\left(x^2(x-1) + x(x-1) - 2(x-1)\right) \leq 0 \Leftrightarrow x(x-1)(x^2 + x - 2) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow x(x+2)(x-1)^2 \leq 0 \quad \textit{(3)}$$

- *Other,* $x \geq 0, x(x+2) \geq 0$*. That* $(x-1)^2 \geq 0; \forall x \in R$ *therefore*

$$x(x+2)(x-1)^2 \geq 0 \quad \textit{(4)}$$

** Since (3), (4)*: $\Rightarrow x(x+2)(x-1)^2 = 0 \Leftrightarrow \begin{cases} x = x^2 - 2x + 2 \\ x^2 - x + 1 = 1 \\ x = 4 - 3x^3 = 1 \\ x(x+2)(x-1)^2 = 0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} (x-1)(x-2) = 0 \\ x(x-1) = 0 \\ 3x^3 + x - 4 = 0; x = 1 \\ x(x+2)(x-1)^2 = 0 \end{cases} \Leftrightarrow x = 1$$

1.12

If $[x]$ = *greatest integer then,* $[x] = -1,0,1$

1. $[x] = -1, -1 \le x < 0$, *the equation becomes,*

$$\left(-\frac{\pi}{2}\right)\pi = \frac{\pi}{2}x - x^2 \Rightarrow x^2 - \frac{\pi}{2}x - \frac{\pi^2}{2} = 0 \Rightarrow x = \frac{\frac{\pi}{2} \pm \sqrt{\frac{\pi^2}{4} + 2\pi^2}}{2} = \frac{\pi \pm 3\pi}{4} = \pi, -\frac{\pi}{2}. \text{ Not possible}$$

2. *For* $[x] = 0, 0 \le x < 1$ *The equation becomes* $0 = \frac{\pi}{2}x - x^2 \Rightarrow x = 0$ *or* $x = \frac{\pi}{2}$
3. *For* $[x] = 1, 1 \le x < 2$ *The equation becomes*

$$0 = \frac{\pi}{2}x - x^2 \Rightarrow x = 0 \text{ or } x = \frac{\pi}{2} \therefore \text{ in this case solution is } \left\{0, \frac{\pi}{2}\right\}$$

1.13

$$\frac{x^2(1+y^2)+y^2}{(1+x^2)(1+y^2)} + \frac{z^2}{(1+x^2)(1+y^2)(1+z^2)} + \frac{1}{8xyz} = 1$$

$$\Leftrightarrow \frac{(x^2y^2+x^2+y^2)(z^2+1)+z^2}{(1+x^2)(1+y^2)(1+z^2)} + \frac{1}{8xyz} = 1 \Leftrightarrow \frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2+1)(y^2+1)(z^2+1)} = 1 - \frac{1}{8xyz}$$

$$\Leftrightarrow \frac{1}{(x^2+1)(y^2+1)(z^2+1)} = \frac{1}{8xyz} \Leftrightarrow (x^2+1)(y^2+1)(z^2+1) = 8xyz$$

By AM-GM $(x^2+1)(y^2+1)(z^2+1) \ge 2x \cdot 2y \cdot 2z = 8xyz$

$\Rightarrow$ *Equality occurs if* $\Leftrightarrow x = y = z = 1$

1.14

$$5x^2 + 5y^2 + 5z^2 + 5t^2 - 5xy - 5yz - 5zt - 5t + 2 = 0$$

or, $5\left(x - \frac{y}{2}\right)^2 + \frac{15y^2}{4} + 5z^2 + 5t^2 - 5yz - 5zt - 5t + 2 = 0$

or, $5\left(x - \frac{y}{2}\right)^2 + 5\left(\frac{\sqrt{3}y}{2} - \frac{z}{\sqrt{3}}\right)^2 + \frac{10z^2}{3} - 5zt + 5t^2 - 5t + 2 = 0$

or, $5\left(x - \frac{y}{2}\right)^2 + 5\left(\frac{\sqrt{3}y}{2} - \frac{z}{\sqrt{3}}\right)^2 + 5\left(\frac{\sqrt{2}z}{\sqrt{3}} - \frac{\sqrt{3}t}{2\sqrt{2}}\right)^2 + \frac{25t^2}{8} - 5t + 2 = 0$

or, $5\left(x - \frac{y}{2}\right)^2 + 5\left(\frac{\sqrt{3}y}{2} - \frac{z}{\sqrt{3}}\right)^2 + 5\left(\frac{\sqrt{2}z}{\sqrt{3}} - \frac{\sqrt{3}t}{2\sqrt{2}}\right)^2 + \left(\frac{5t}{2\sqrt{2}} - \sqrt{2}\right)^2 = 0$

$$t, x, y, z \in \mathbb{R} \Rightarrow x = \frac{y}{2}; \frac{\sqrt{3}y}{2} = \frac{z}{\sqrt{3}}$$

$$\frac{\sqrt{2}z}{\sqrt{3}} = \frac{\sqrt{3}t}{2\sqrt{2}}; \frac{5t}{2\sqrt{2}} = \sqrt{2} \Rightarrow t = \frac{4}{5}, z = \frac{3}{5}; y = \frac{2}{5}; x = \frac{1}{5}$$

1.15

Squaring and adding: $sin^2[x] + cos^2(x - [x]) + 2sin[x]cos(x - [x]) +$

$$+sin^2(x - [x]) + cos^2[x] + 2cos[x]sin(x - [x]) = \frac{3}{4} + \frac{9}{4}$$

$$2 + 2sin([x] + x - [x]) = 3,\ sinx = \frac{1}{2} \rightarrow x = \frac{\pi}{6}$$

1.16

We know, for $x > 0$, $tan^{-1} x < x \Rightarrow tan^{-1}(a^{-x}) < a^{-x}; (a > 1)$

$$\Rightarrow cot^{-1}(a^x) < a^{-x} \therefore \frac{1}{cot^{-1}(a^x)} > a^x\ \forall x > 0, a > 1$$

Thus, $\frac{1}{cot^{-1}(\pi^x)} + \frac{1}{cot^{-1}(e^x)} > \pi^x + e^x > \pi^x + e^x + \pi^{-x} + e^{-x}$

Hence, given equation has no solution

1.17

Given equation can be written as

$$\left(sin\sqrt{x}\right)^3 + \left(\sqrt{x}\right)^3 + \left(\sqrt{x}\right)^3 = 3 \cdot \sqrt{x} \cdot \sqrt{x} \cdot sin\sqrt{x}$$

which is of the form $a^3 + b^3 + c^3 = 3abc \Rightarrow$ *either* $a = b = c$ *or* $a + b + c = 0$

$\Rightarrow$ *either* $sin\sqrt{x} = \sqrt{x}$ *or* $sin\sqrt{x} = -2\sqrt{x}$ *which has only one solution* $x = 0$.

1.18

$$27\sqrt{\left(x^2 + \frac{1}{y^2}\right)\left(y^2 + \frac{1}{z^2}\right)\left(z^2 + \frac{1}{x^2}\right)} \overset{(1)}{=} 8(x + y + z)^3$$

$x + y + z \overset{(2)}{=} \frac{1}{xyz}$, *LHS of (1)* $= \frac{27}{xyz}\sqrt{(x^2y^2 + 1)(y^2z^2 + 1)(z^2x^2 + 1)}$

$$\overset{(a)}{=} \frac{27}{xyz}\sqrt{\{x^2y^2 + xyz(x + y + z)\}\{y^2z^2 + xyz(x + y + z)\}\{z^2x^2 + xyz(x + y + z)\}}$$

$$\left(\because 1 = xyz(x + y + z)\right)$$

Now, $x^2y^2 + xyz(x + y + z) = xy(xy + zx + yz + z^2) \overset{(b)}{=} xy(y + z)(z + x)$

Similarly, $y^2z^2 + xyz(x + y + z) \overset{(c)}{=} yz(x + y)(z + x)$ &

$z^2x^2 + xyz(x + y + z) \overset{(d)}{=} zx(x + y)(y + z)$

(a), (b), (c), (d) $\Rightarrow LHS \overset{(i)}{=} 27(x + y)(y + z)(z + x)$

Now, $\sum x = \frac{1}{2}\{(x + y) + (y + z) + (z + x)\} \overset{A-G}{\geq} \frac{3}{2}\sqrt[3]{(x + y)(y + z)(z + x)}$

$$\Rightarrow \left(2\sum x\right)^3 \geq 27(x + y)(y + z)(z + x) \Rightarrow 8\left(\sum x\right)^3 \overset{(ii)}{\geq} 27(x + y)(y + z)(z + x)$$

(i), (ii) $\Rightarrow$ *RHS of (1)* $\geq LHS$ *of (1), with equality occuring when* $x = y = z$.

But LHS of (1) = RHS of (1) $\therefore x = y = z$ $\therefore$ *using (2),* $3x = \frac{1}{x^3} \Rightarrow x^4 = \frac{1}{3} \Rightarrow x = \frac{1}{\sqrt{\sqrt{3}}}$

$\therefore$ *only possible solution is:* $(x, y, z) = \left(\frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}, \frac{1}{\sqrt{\sqrt{3}}}\right)$ *(answer)*

1.19

We know that $|k|^2 = k^2$ *for any real* k

Equation becomes: $(x^2 + 4x - 9)^{\left(\frac{1}{10}\right)} + (2x - 10) \cdot \sqrt{(x^2 + 1)} = \{(x + 1)^2\}^{\left(\frac{1}{10}\right)}$

$(x^2 + 4x - 9)^{\left(\frac{1}{10}\right)} - \{(x + 1)^2\}^{\left(\frac{1}{10}\right)} = -(2x - 10) \cdot \sqrt{(x^2 + 1)}$ *(1)*

Also we must have that $(x^2 + 4x - 9) \geq 0$*; we also have* $(x + 1)^2 \geq 0$

Let $(x^2 + 4x - 9) = m^{10}$ *(2) and* $(x + 1)^2 = n^5$ *(3) where* $m, n \geq 0$

Then $(2x - 10) = (x^4 + 4x - 9) - (x + 1)^2 \Rightarrow (2x - 10) = (m^{10} - n^5)$ *(4)*

Equation (1) becomes: $\left(m - \sqrt{n}\right) = -(m^{10} - n^5) \cdot \sqrt{(x^2 + 1)}$

$$\left(m-\sqrt{n}\right) = -\left(m-\sqrt{n}\right)\left(m+\sqrt{n}\right)(m^8+nm^6+n^2m^4+m^2n^3+n^4)\cdot\sqrt{(x^2+1)}$$

1) $m=\sqrt{n}$ *is one solution* ⇒ *from (4):* $(2x-10)=0 \Rightarrow x=5$

Otherwise $1=-\left(m+\sqrt{n}\right)(m^8+nm^6+n^2m^4+m^2n^3+n^4)\cdot\sqrt{(x^2+1)}$

⇒ *IMPOSSIBLE since* $m\geq 0, n\geq 0, \sqrt{(x^2+1)}>0$ *meaning that RHS is* ≤ 0

1.20

$$|x^2-1|^{[x]}+|x^2-2|^{[x]}+|x^2-3|^{[x]} = tan\left(\frac{\pi}{4}[x]\right)+cot\left(\frac{\pi}{4}[x]\right) \quad (1)$$

Note that $LHS>0$*, and RHS is not defined if* $[x]=4m, 4m+2$ *and RHS is negative for* $[x]=4m+3$*, where* $m\in\mathbb{Z}$*. Also, RHS is equal to* $=2$ *if* $[x]=4m+1, m\in\mathbb{Z}$

For $m=0, [x]=1$ *and (1) becomes,* $|x^2-1|+|x^2-2|+|x^2-3|=2$ *(2)*

Let $x^2-2=t$ *(2) becomes* $f(t)=|t+1|+|t|+|t-1|=2 \Rightarrow t=0$

$$\therefore x=\sqrt{2} \qquad [\because m=0]$$

1.21

$$\begin{cases}3^{3x}+2=3^{y+1}\\3^{3y}+2=3^{z+1}\\3^{3z}+2=3^{x+1}\end{cases} \text{Substitutions } 3^x=a; 3^y=b, 3^z=c: \begin{cases}a^3+2=3b\\b^3+2=3c\\c^3+2=3a\end{cases}$$

$$a^3+1+1 \overset{AM-GM}{\geq} 3\sqrt[3]{a^3}=3a, \quad b^3+1+1 \overset{AM-GM}{\geq} 3\sqrt[3]{b^3}=3b$$

$c^3+1+1 \overset{AM-GM}{\geq} 3\sqrt[3]{c^3}=3a.$ *Equality holds for* $a=1; b=1; c=1$

$$3^x=1 \Rightarrow x=0, 3^x=1 \Rightarrow y=0, 3^z=1 \Rightarrow z=0$$

1.22

Let $(x+sin\,x-cos\,x)=a; (x+cos\,x-sin\,x)=b; (sin\,x+cos\,x-x)=c$

Hence, given equation reduces to $(a+b+c)^3=a^3+b^3+c^3 \Rightarrow$

$$\Rightarrow (a+b+c)^3-a^3-b^3-c^3=0 \Rightarrow 3(a+b)(b+c)(c+a)=0$$

Putting the values of a,b,c: $3(2x)(2\,cos\,x)(2\,sin\,x)=0$

$$x = 0 \quad cosx = 0 \left| \begin{matrix} cos\,x = 0 \\ x = (2n+1)\frac{\pi}{2} \\ n \in I \end{matrix} \right| \begin{matrix} sin\,x = 0 \\ x = m\pi \\ m \in I \end{matrix}$$

Real solutions are $x = 0, x = (2n+1)\frac{\pi}{2}; x = m\pi$

1.23

Let be $f: [0,\infty) \to \mathbb{R}, f(x) = \frac{1}{1+e^x}, f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3} \geq 0, f - convexe$

If $u, v, w \geq 0$ *then by Jensen's inequality:*

$$f\left(\frac{u+v+w}{3}\right) \leq \frac{1}{3}(f(u)+f(v)+f(w)), \qquad \frac{1}{1+e^{\frac{u+v+w}{3}}} \leq \frac{1}{3}\left(\frac{1}{1+e^u}+\frac{1}{1+e^v}+\frac{1}{1+e^w}\right)$$

Denote $a = e^u, b = e^v, c = e^w$*:* $\frac{1}{1+\sqrt[3]{abc}} \leq \frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)$

$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \leq \frac{3}{1+\sqrt[3]{abc}}$. *Equality holds if* $a = b = c$. *Denote* $a = 8^x, b = 27^x, c = 64^x$

$$\frac{1}{1+8^x}+\frac{1}{1+27^x}+\frac{1}{1+64^x} \leq \frac{3}{1+\sqrt[3]{8^x \cdot 27^x \cdot 64^x}} = \frac{3}{1+24^x}$$

Equality holds for $8^x = 27^x = 64^x \to x = 0$

1.24

$$\sqrt[\alpha]{\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha} \overset{POWER\ MEANS}{\geq} \sqrt{\frac{1}{2}|Rez|^2+\frac{1}{2}|Imz|^2}, (\alpha \geq 2)$$

$$\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha \geq \frac{1}{2^{\frac{\alpha}{2}}}(|Rez|^2+|Imz|^2)^\alpha = \frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha$$

$$\frac{1}{2}|Rez|^\alpha+\frac{1}{2}|Imz|^\alpha \geq \frac{1}{2^{\frac{\alpha}{2}}}|z|^\alpha \to |Rez|^\alpha+|Imz|^\alpha \geq 2^{1-\frac{\alpha}{2}}|z|^\alpha$$

1.25

$|z_1+z_2| = |z_1|+|z_2| \Rightarrow z_1 = kz_2$ *for some* $k \geq 0$.

Now, $|z_1-z_2| = |(k-1)z_2| = |(k-1)z_2|$

If $k \geq 1$*, then* $|z_1| = k|z_2| \geq |z_2|$,

and $|z_1 - z_2| = (k-1)|z_2| = k|z_2| - |z_2| = |z_1| - |z_2| = max\{|z_1|, |z_2|\} - min\{|z_1|, |z_2|\}$

If $0 \leq k < 1$, $|z_1| = k|z_2| < |z_2|$ *and*

$$|z_1 - z_2| = |k-1||z_2| = (1-k)|z_2| = |z_2| - k|z_2| = |z_2| - |z_1| = max\{|z_1|, |z_2|\} - min\{|z_1|, |z_2|\}$$

1.26

It's easy to see that: $\frac{z_1^2}{(z_1+z_2)(z_1+z_3)} = 1 - \frac{z_1z_2+z_2z_3+z_3z_1}{(z_1+z_2)(z_1+z_3)}$*. So,*

$$\sum \frac{z_1^2}{(z_1+z_2)(z_1+z_3)} = 3 \Leftrightarrow -(z_1z_2 + z_2z_3) + z_3z_1 \cdot \frac{2(z_1+z_2+z_3)}{(z_1+z_1)(z_1+z_2)(z_2+z_3)} = 0 \quad (1)$$

But $z_1 + z_2 + z_3 = \frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}} = \frac{\overline{z_1+z_2+z_3}}{\overline{z_1z_2z_3}}$

So, (1) $\Rightarrow z_1z_2 + z_2z_3 + z_3z_1 = 0$ *and (equivalent)*

$$z_1 + z_2 + z_3 = 0$$

We have $z_1(z_2 + z_3) + z_2z_3 = 0 \Leftrightarrow -z_1^2 + z_2z_3 = 0 \Rightarrow z_1^2 = z_2z_3 \Leftrightarrow$

$$z_1^2 - z_2^2 = z_2z_3 - z_2^2 \Rightarrow (z_1 - z_2)(z_1 + z_2) = z_2(z_3 - z_2) \Rightarrow |-z_3| = 1$$

$$\Rightarrow |z_1 - z_2| \cdot \left|\underbrace{z_1 + z_2}_{-z_3}\right| = |z_2| \cdot |z_3 - z_2| \Rightarrow |z_2| = 1$$

$|z_2 - z_2| = |z_3 - z_2| \Leftrightarrow (AB) = (BC)$. *Working just the same*

$$z_2^2 = z_1z_3 \Leftrightarrow z_2^2 - z_1^2 = z_1z_3 - z_1^2 \Leftrightarrow (z_2 - z_1)(z_2 + z_1) = z_1(z_3 - z_1) \Rightarrow |-z_3| = 1$$

$$|z_2 - z_1| \cdot \left|\underbrace{z_2 + z_1}_{-z_3}\right| = |z_1| \cdot |z_3 - z_1| \Rightarrow |z_1| = 1$$

$|z_2 - z_1| = |z_3 - z_1| \Leftrightarrow (AB) = (AC)$*. So,* $(AB) = (AC) = (BC)$

1.27

We know that $z_1 + z_2 + z_3 = z_4 \Leftrightarrow z_1 + z_2 = z_4 - z_3$

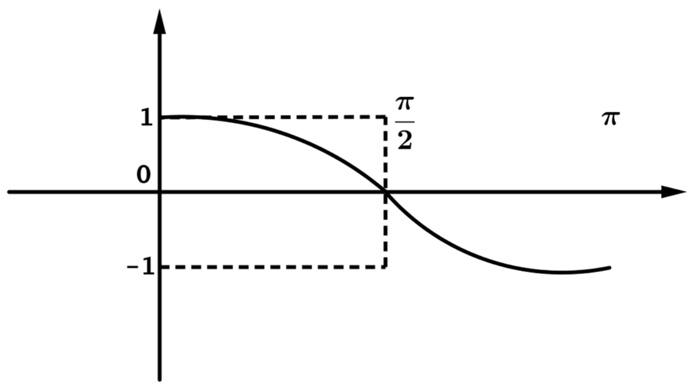
$\Leftrightarrow |z_1 + z_2| = |z_4 - z_3| \Leftrightarrow |z_1 + z_2| = HC$ *(1)*

$$1 = \sum \frac{1}{2 + |z_1 + z_2|} \overset{(1)}{=} \sum \frac{1}{2 + HC} \overset{HC=2R\cos C}{=} \sum \frac{1}{2(1 + R\cos C)} =$$

$$\left.\begin{matrix} = \frac{1}{2}\sum \frac{1}{1 + R\cos C} \Rightarrow \sum \frac{1}{1 + R\cos C} = 2 \\ |z_1| = |z_2| = |z_3| = 1 \Rightarrow R = 1 \end{matrix}\right\} \Rightarrow \sum \frac{1}{1 + \cos c} = 2$$

$$2 = \sum \frac{1}{1 + \cos C} \overset{Cauchy}{\geq} \frac{9}{3 + \sum \cos A} \Leftrightarrow 2\left(3 + \sum \cos A\right) \geq 9 \Leftrightarrow$$

$$\Leftrightarrow 6 + 2\sum \cos A \geq 9 \Leftrightarrow 2\sum \cos A \geq 3 \Leftrightarrow \sum \cos A \geq \frac{3}{2} \quad (2)$$



$f: (0,\pi), f(x) = \cos x$ *it's a concave function*

$$\left.\begin{matrix} \frac{\sum \cos A}{3} \geq \cos\frac{\sum A}{3} = \cos 60° = \frac{1}{2} \Rightarrow \sum \cos A \geq \frac{3}{2} \\ (2) \end{matrix}\right\} \Rightarrow \sum \cos A = \frac{3}{2},$$ *equality holds when $\Delta\ ABC$ is equilateral*

1.28

$$[\tan x](\cot x - [\cot x]) = (\tan x - [\tan x])[\cot x] \quad (1)$$

For $0 < x < \frac{\pi}{4}, 0 < \tan x < 1, [\tan x] = 0$*and* $[\cot x] \geq 1$.*Now (1) becomes:* $0 = (\tan x)[\cot x] \neq 0$ $\therefore$ *(1) has no solution for* $0 < x < \frac{\pi}{4}$.*For* $x = \frac{\pi}{4}$, *(1) becomes*

$1(1-1) = (1-1)(1)$ *which is clearly holds.*

For $\frac{\pi}{4} < x < \frac{\pi}{2}, [\tan x] \geq 1$ *and* $[\cot x] = 0$. *Now (1) becomes*

$[\tan x]\cot x = 0$.*i.e.* $0 = [\tan x]\cot x \neq 0$

∴ (1) has no solution for $\frac{\pi}{4} < x < \frac{\pi}{2}$*. Next, let* $-\frac{\pi}{4} < x < 0, [\tan x] = -1, [\cot x] \leq -2$

Write (1) as $(-1)(\cot x - [\cot x]) = (\tan x + 1)[\cot x] \Rightarrow -\cot x = (\tan x)[\cot x]$ *(2)*

Let $[\cot x] = k$*, then* $k \leq -2$*and* $\cot x \leq k$ LHS *of (2)* $\geq -k$ *and* RHS *of (2)* $< -k$

Thus, (1) has not solution for $-\frac{\pi}{4} < x < 0$*. For* $x = -\frac{\pi}{4}$*, (1) is satisfied*

Similarly, (1) has no solution for $-\frac{\pi}{2} < x < -\frac{\pi}{4}$

As $\tan x$ *and* $\cot x$ *are periodic with period* π*, we get solution set to be*

$(2k+1)\frac{\pi}{4}$ *where* k *is an integer.*

1.29

$$sen(1+x) + sen(1+10x) + sen(1+2x) + sen(1+9x) + \cdots + sen(1+5x) + sen(1+6x)$$

$$2\,sen\left(\frac{2+11x}{2}\right)cos\left(\frac{9x}{2}\right) + 2\,sen\left(\frac{2+11x}{2}\right)cos\left(\frac{7x}{2}\right) + \cdots + 2\,sen\left(\frac{2+11x}{2}\right)cos\left(\frac{x}{2}\right) = 0$$

$$2\,sen\left(\frac{2+11x}{2}\right)\left[cos\frac{9x}{2} + cos\frac{7x}{2} + cos\frac{5x}{2} + cos\frac{3x}{2} + cos\frac{x}{2}\right] = 0$$

$$2\,sen\left(\frac{2+11x}{2}\right)\left[cos\frac{9x}{2} + cos\frac{x}{2} + cos\frac{7x}{2} + cos\frac{3x}{2} + cos\frac{5x}{2}\right] = 0$$

$$2\,sen\left(\frac{2+11x}{2}\right)\left[2\,cos\frac{5x}{2} + cos\frac{4x}{2} + 2\,cos\frac{5x}{2}cos\frac{2x}{2} + cos\frac{5x}{2}\right] = 0$$

$$2\,sen\left(\frac{2+11x}{2}\right)cos\frac{5x}{2}[2\,cos\,2x + 2\,cos\,x + 1] = 0 \rightarrow$$
$$\rightarrow 2\,sen\left(\frac{2+11x}{2}\right)cos\frac{5x}{2}[4\,cos^2 x + 2\,cos\,x - 1] = 0$$

Si: $sen\left(\frac{2+11x}{2}\right) = 0 \rightarrow \frac{2+11x}{2} = \pi k \Leftrightarrow x = \frac{2\pi k-2}{11} \rightarrow$ *Válido para* $k = 1,2,3,4,5,6,7,8,9,10,11 \rightarrow$
$k \in \mathbb{Z}$*. Si:* $cos\frac{5x}{2} = 0 \rightarrow \frac{5x}{2} = (2n+1)\frac{\pi}{2} \Leftrightarrow x = \frac{(2n+1)\pi}{5} \rightarrow$ *Válido para*

$n = 0,1,2,3,4 \rightarrow n \in \mathbb{Z} \Rightarrow 4\,cos^2 x + 2\,cos\,x - 1 = 0 \rightarrow \left(2\,cos\,x + \frac{1}{2}\right)^2 = \frac{5}{4} \rightarrow$

$cos\,x = \frac{\sqrt{5}-1}{4}$ V $cos\,x = \frac{-\sqrt{5}-1}{4}$ *Si:* $cos\,x = \frac{\sqrt{5}-1}{4} \rightarrow x = \frac{2\pi}{5}, \frac{8\pi}{5}$, *Si:* $cos\,x = \frac{-\sqrt{5}-1}{4} \rightarrow x = \frac{4\pi}{5}, \frac{6\pi}{5}$

1.30

$$cos\,A\,|cos\,B| + cos\,B\,|cos\,A| = 1 + cos\,2C \quad (1)$$

$$cos\,B\,|cos\,C| + cos\,C\,|cos\,B| = 1 + cos\,2A \quad (2)$$

$$cos\,C\,|cos\,A| + cos\,A\,|cos\,C| = 1 + cos\,2B \quad (3)$$

Assume C *is obtuse, then* $|cos\,C| = -\,cos\,C$, *(2) becomes*$0 = 2\,cos^2\,A$. *Not possible.*

If $C = \frac{\pi}{2}$, *then (2) become* $0 = 2\,cos^2\,A$. *Not possible.*

Thus, A, B, C *must be all acute angles.From (1)*

$$2\,cos\,A\,cos\,B = 2\,cos^2\,C \Rightarrow 2\,cos^2\,C = cos(A+B) + cos(A-B)$$

$$\Rightarrow 2\,cos^2\,C + cos\,C \leq 1 \Rightarrow (2\,cos\,C - 1)(cos\,C + 1) \leq 0$$

$\Rightarrow 0 < cos\,C \leq \frac{1}{2} \Rightarrow \frac{\pi}{3} \leq C < \frac{\pi}{2}$ *Similarly, from (2), (3)* $\frac{\pi}{3} \leq A, B < \frac{\pi}{2}$

As $A + B + C = \pi_{\phi}$ *and* $\frac{\pi}{3} \leq A, B, C < \frac{\pi}{2}$ *we get* $A = B = C = \frac{\pi}{3}$

1.31

$$\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy \quad (1),\ x^5 + y^5 = 8xy \ (2)$$

$x, y > 0$ *: (1)* $\Rightarrow \frac{1}{x\sqrt{x}} + \frac{1}{y\sqrt{y}} = 1 \Rightarrow a^3 + b^3 = 1$, $\left(\frac{1}{\sqrt{x}} = a, \frac{1}{\sqrt{y}} = b\right)$, $x = \frac{1}{a^2}$ *and* $y = \frac{1}{b^2}$
$(a, b > 0)$ *(2)* $\Rightarrow \frac{1}{a^{10}} + \frac{1}{b^{10}} = \frac{8}{a^2b^2} \Rightarrow a^{10} + b^{10} = 8a^8b^8$, $AM \geq GM \Rightarrow a^{10} + b^{10} \geq 2a^5b^5$

$$\Rightarrow 8a^8b^8 \geq 2a^5b^5 \Rightarrow a^3b^3 \geq \frac{1}{4} \Rightarrow 4a^3b^3 \geq 1$$

$$\Rightarrow 4a^3b^3 \geq (a^3 + b^3)^2 \Rightarrow 0 \geq (a^3 - b^3)^2 \Rightarrow (a^3 - b^3)^2 \leq 0$$

But $(a^3 - b^3)^2 \geq 0 \Rightarrow (a^3 - b^3)^2 = 0 \Rightarrow a^3 = b^3 \Rightarrow a = b$, $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{y}} \Rightarrow x = y$

$$\frac{2}{x\sqrt{x}} = 1 \Rightarrow \sqrt{x} = \sqrt[3]{2} \Rightarrow x = y = \sqrt[3]{4}$$

1.32

$$\frac{x^2}{25}+\frac{y^2}{16}=1, x^2+y^2=\left(\frac{x^2}{5}+\frac{y^2}{4}\right)^2 \Rightarrow x^2=a \quad y^2=b, \; 16a+25b=400$$

$$a+b=\frac{a^2}{25}+\frac{b^2}{16}+\frac{ab}{10}=\frac{16a^2+25b^2+40ab}{400}=\frac{(4a+5b)^2}{400}$$

$$400(a+b)=(4a+5b)^2, \qquad (16a+25b)(a+b)=(4a+5b)^2$$

$$16a^2+16ab+25ab+25b^2=16a^2+25b^2+40ab$$

$$ab=0 \Rightarrow a=0 \quad x=0 \Rightarrow 25b=400 \quad b=16 \quad y=\pm 4$$

answer $(0;4)$ *and* $(0;-4) \Rightarrow b=0 \quad y=0 \Rightarrow \quad 16a=400 \quad a=25 \quad x=\pm 5$

answer $(5;0)$ *and* $(-5;\ 0)$

1.33

Let $x \geq y \geq z$, *then* $max\{x,y,z\}=x, min\{x,y,z\}=z$

$$4(max(x,y,z)-min(x,y,z))^2 \geq 3\sum|x-y|^2$$

$$\Rightarrow 4(x-z)^2 \geq 3[(x-y)^2+(y-z)^2+(z-x)^2] \Rightarrow (x-z)^2 \geq 3(x-y)^2+3(y-z)^2$$

$$\Rightarrow [(x-y)+(y-z)]^2 \geq 3(x-y)^2+3(y-z)^2$$

$$\Rightarrow (x-y)^2+(y-z)^2+2(x-y)(y-z) \geq 3(x-y)^2+3(y-z)^2$$

$$\Rightarrow (x-y)^2+(y-z)^2-(x-y)(y-z) \leq 0 \Rightarrow \left[x-y-\frac{1}{2}(y-z)\right]^2+\frac{3}{4}(y-z)^2 \leq 0$$

$$\Rightarrow x-\frac{3}{2}y+\frac{1}{2}z=0, y=z \Rightarrow x=y=z, \quad \therefore x=y=z=1$$

1.34

$$x,y,z \in <0,\infty> \quad x=\tan A, y=\tan B, z=\tan C \Leftrightarrow A+B+C=\pi$$

$$\frac{\tan A}{\tan^3 B\tan^2 C}+\frac{\tan B}{\tan^3 C\tan^2 A}+\frac{\tan C}{\tan^3 \tan^2 B}=\frac{1}{3},$$

$$MA \geq MG \Leftrightarrow \tan A, \tan B, \tan C > 0$$

$$\frac{\tan A}{\tan^3 B\tan^2 C}+\frac{\tan B}{\tan^3 C\tan^2 A}+\frac{\tan C}{\tan^3 A\tan^2 B} \geq 3\sqrt[3]{\cot^4 A\cot^4 B\cot^4 C}$$

$$\left(\frac{1}{9}\right)^3 \geq cot^4 A\, cot^4 B\, cot^4 C \Rightarrow \left(\frac{1}{3}\right)^6 \geq cot^4 A\, cot^4 B\, cot^4 C\,, \frac{1}{3\sqrt{3}} \geq cot\, A\, cot\, B\, cot\, C$$

La cual es cierto en un $\Delta\ ABC$ equilátero, la igualdad se alcanza cuando: $x = y = z = \sqrt{3}$

1.35

As $\sqrt{xy}$ is involved, either both $x, y \leq 0$ or both $x, y \geq 0$. If $x, y < 0$, then

$x + y - \sqrt{xy} < 0$ *and* $\sqrt{\frac{x^2+y^2}{2}} > 0 \therefore$ *both $x, y \nless 0$. Thus, $x, y \geq 0$. Let* $\Delta = \begin{vmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3 + C_4$, we get $\Delta = (x + y + 5)\Delta_1$, where

$$\Delta_1 = \begin{vmatrix} 1 & y & 2 & 3 \\ 1 & x & 3 & 2 \\ 1 & 3 & x & y \\ 1 & 2 & y & x \end{vmatrix} = \begin{vmatrix} 1 & y & 2 & 3 \\ 0 & x-y & 1 & -1 \\ 0 & 3-y & x-2 & y-3 \\ 0 & 2-y & y-2 & x-3 \end{vmatrix} = \begin{vmatrix} x-y & 1 & -1 \\ 3-y & x-2 & y-3 \\ 2-y & y-2 & x-3 \end{vmatrix}$$

$$C_1 \to C_1 + C_3, C_2 \to C_2 + C_3 \text{ gives } \Delta_1 = \begin{vmatrix} x-y-1 & 0 & -1 \\ 0 & x+y-5 & y-3 \\ x-y-1 & x+y-5 & x-3 \end{vmatrix} =$$

$$= (x-y-1)(x+y-5)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & y-3 \\ 1 & 1 & x-3 \end{vmatrix} = (x-y-1)(x+y-5)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & y-3 \\ 1 & 1 & x-2 \end{vmatrix} =$$

$$= (x-y-1)(x+y-5)(x-2-y+3) = (x-y-1)(x-y+1)(x+y-5)$$

Now, $\Delta = 0 \Rightarrow (x + y + 5)\Delta_1 = 0$. As $x, y \geq 0$, we get $\Delta_1 = 0 \Rightarrow$

$\Rightarrow (x-y-1)(x-y+1)(x+y-5) = 0 \Rightarrow x - y = 1$ *or* $x - y = -1$ *or* $x + y = 5$.

Case 1: $x - y = 1$. Let $x = t + \frac{1}{2}, y = t - \frac{1}{2}, t \geq \frac{1}{2}$. The second equation becomes

$$2t - \sqrt{t^2 - \frac{1}{4}} = \sqrt{t^2 + \frac{1}{4}} \Rightarrow 2t = \sqrt{t^2 - \frac{1}{4}} + \sqrt{t^2 + \frac{1}{4}}$$

$\Rightarrow 4t^2 = t^2 + \frac{1}{4} + t^2 - \frac{1}{4} + 2\sqrt{t^4 - \frac{1}{4}} \Rightarrow t^2 = \sqrt{t^4 - \frac{1}{16}} < t^2$. *Not possible.*

Simillarly, $y - x = 1$ is not possible. Thus, we consider Case 2: $x + y = 5$

$$Let\ x = \frac{5}{2} - t, y = \frac{5}{2} + t \text{ where } -\frac{5}{2} \leq t \leq \frac{5}{2}$$

Second equation now becomes: $5 - \sqrt{\frac{25}{4} - t^2} = \sqrt{\frac{25}{4} + t^2} \Rightarrow t = 0$. *Thus,* $x = \frac{5}{2}, y = \frac{5}{2}$.

1.36

$$\tan x \tan y \tan z = 6 \ldots (A)$$

$$\tan x \tan y + \tan y \tan z + \tan z \tan x = 1 \ldots (B)$$

Si: $x + y + z = \pi$

$$\tan x + \tan y + \tan z = \tan x \tan y \tan z \Rightarrow \tan x + \tan y + \tan z = 6 \ldots (I)$$

Desde que tenemos la suma y el producto, se puede construir una ecuación cúbica:

$$(x - \tan x)(x - \tan y)(x - \tan z) = 0 \Leftrightarrow cuyas\ raices\ son: \tan x, \tan y, \tan z$$

$$x^3 - x^2(\tan x + \tan y + \tan z) + x(\tan x \tan y + \tan y \tan z + \tan z \tan x) -$$

$$- \tan x \tan y \tan z = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0 \rightarrow (x-1)(x-2)(x-3) = 0$$

Un posible coso es cuando: $\tan x = 1, \tan y = 2 \wedge \tan z = 3 \Rightarrow$

$$x = 45°, y = 63{,}5° \wedge z = 71{,}5°$$

1.37

$$1 + 2\sqrt{y} = 2\sqrt{x+1} \quad (1), \quad \frac{2\sqrt{y}}{12y+1} + \frac{\sqrt{x+1}}{x+4} + \frac{2\sqrt{y(x+1)}}{3x+4y+3} = \frac{3}{4} \quad (2)$$

$$(2) \Rightarrow \frac{\sqrt{y}}{12y+1} + \frac{1+2\sqrt{y}}{13+4\sqrt{y}+4y} + \frac{2\sqrt{y}+4y}{3+12\sqrt{y}+28y} = \frac{3}{8}$$

$$\Rightarrow \frac{t}{12t^2+1} + \frac{1+2t}{13+4t+4t^2} + \frac{2t+4t^2}{3+12t+28t^2} \overset{(using\ (1))}{=} \frac{3}{8} \quad (t = \sqrt{y})$$

$$\Rightarrow 2496t^6 - 2816t^5 + 3440t^4 - 1728t^3 + 148t^2 - 160t + 93 = 0$$

$$\Rightarrow (2t-1)^2 \underbrace{(624t^4 - 80t^3 + 624t^2 + 212t + 93)}_{e} = 0 \text{ Now, } 624t + 624t^2 \overset{A-G}{\geq} 1248t^3 > 80t^3$$

(when $t > 0$ and $t = \sqrt{y} \geq 0$) $\Rightarrow e > 0, \forall\, t > 0$ For $t = 0, e = 93 > 0 \Rightarrow \forall\, t \geq 0, e > 0$

$\therefore 2t = 1 \Rightarrow t = \frac{1}{2} \Rightarrow \sqrt{y} = \frac{1}{2} \Rightarrow y = \frac{1}{4} \Rightarrow x = 0$ $\therefore$ only solution is $(x, y) = \left(0, \frac{1}{4}\right)$

At $t = \frac{1}{2}$ $f''(t) = -\frac{3}{8} < 0$ $\therefore$ at $t = \frac{1}{2}, f(t)$ attains a maxima,

and $\because f(t)$ never attains a minima $\forall\, t \geq 0$, $\therefore f(t) \leq f\left(\frac{1}{2}\right) = \frac{3}{8}$

But (1) $\Rightarrow f(t) = \frac{3}{8} \therefore t = \frac{1}{2} \Rightarrow \sqrt{y} = \frac{1}{2} \Rightarrow y = \frac{1}{4}$

Putting, $y = \frac{1}{4}$ in (1), $\sqrt{x+1} = 1 \Rightarrow x = 0$ $\therefore$ only solution is $(x, y) = \left(0, \frac{1}{4}\right)$

1.38

Two circles: $|z - 7 - i| = 3\sqrt{2}$ and $|z - 1 - 7i| = 3\sqrt{2}$

touch each other externally as $|(7 + i) - (1 + 7i)| = 6\sqrt{2} = 3\sqrt{2} + 3\sqrt{2}$

Thus, $|z - 7 - i| = 3\sqrt{2}$ (1)

and $|z - 1 - 7i| \leq 3\sqrt{2}$ meet exactly at one point viz. the mid – point $4 + 4i$ of segment AB where $A(7 + i), B(1 + 7i)$

1.39

Let $\cot 2x = a, \cot 3y = b, \cot 5z = c, \tan 2x, \tan 3y, \tan 5z$ are defined, $\cos 2x, \cos 3y, \cos 5z \neq 0 \Rightarrow a, b, c \neq 0$

$a + b = \frac{1}{c}$ (1), $b + c = \frac{1}{a}$ (2), $c + a = \frac{1}{b}$ (3)

(1) – (2) $\Rightarrow a - c = \frac{1}{c} - \frac{1}{a} \Rightarrow (a - c)\left(1 - \frac{1}{ac}\right) = 0$ (4)

(2) – (3) $\Rightarrow b - a = \frac{1}{a} - \frac{1}{b} \Rightarrow (b - a)\left(1 - \frac{1}{ab}\right) = 0$ (5)

(3) – (1) $\Rightarrow c - b = \frac{1}{b} - \frac{1}{c} \Rightarrow (c - b)\left(1 - \frac{1}{bc}\right) = 0$ (6)

If $1 = \frac{1}{ac}$, then $\frac{1}{c} = a \Rightarrow a + b = a$ (form (1)) $\Rightarrow b = 0$

If $1 = \frac{1}{ab}$, then $\frac{1}{b} = a \Rightarrow c + a = a$ (from (3)) $\Rightarrow c = 0$

$$If\ 1 = \frac{1}{bc},\ then\ \frac{1}{c} = b \Rightarrow a + b = b\ (from\ (1)) \Rightarrow a = 0$$

$$But\ a, b, c \neq 0,\quad 1 \neq \frac{1}{ac}, 1 \neq \frac{1}{ab}, 1 \neq \frac{1}{bc},\ (4), (5), (6) \Rightarrow a = b = c$$

$$Putting\ b = c = a\ in\ (1), a + a = \frac{1}{a} \Rightarrow a^2 = \frac{1}{2} \Rightarrow a = \pm\frac{1}{\sqrt{2}}$$

$$(a, b, c) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\ or\ \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$cot\, 2x = cot\, 3y = cot\, 5z = \frac{1}{\sqrt{2}} \Rightarrow tan\, 2x = tan\, 3y = tan\, 5z = \sqrt{2}$$

$$\Rightarrow 2x = tan^{-1}(\sqrt{2}) + n\pi \Rightarrow x = \frac{1}{2}\left(tan^{-1}\sqrt{2}\right) + \frac{n\pi}{2}$$

$$y = \frac{1}{3}tan^{-1}(\sqrt{2}) + \frac{n'\pi}{3},\qquad z = \frac{1}{5}tan^{-1}(\sqrt{2}) + \frac{n''\pi}{5}$$

$$Similarly,\ cot\, 2x = cot\, 3y = cot\, 5z = -\frac{1}{\sqrt{2}} \Rightarrow x = -\frac{1}{2}tan^{-1}(\sqrt{2}) + \frac{n\pi}{2}$$

$$y = -\frac{1}{3}tan^{-1}(\sqrt{2}) + \frac{n'\pi}{3},\qquad z = -\frac{1}{5}tan^{-1}(\sqrt{2}) + \frac{n''\pi}{5}$$

$$solutions\ are: \begin{cases} x = \pm\frac{1}{2}tan^{-1}(\sqrt{2}) + \frac{n\pi}{2} \\ y = \pm\frac{1}{3}tan^{-1}(\sqrt{2}) + \frac{n'\pi}{3} \\ z = \pm\frac{1}{5}tan^{-1}(\sqrt{2}) + \frac{n''\pi}{5} \end{cases}$$

1.40

$$y^{2017} = [y^{2017}] + \alpha, 0 \leq \alpha < 1,\ x^{2017} = [x^{2017}] + \beta, 0 \leq \beta < 1,$$

$$\begin{cases} \left[\frac{x}{1}\right] \cdot \left[\frac{x}{2}\right] \cdot \left[\frac{x}{3}\right] \cdot \ldots \cdot \left[\frac{x}{2017}\right] = \alpha \\ \left[\frac{y}{1}\right] \cdot \left[\frac{y}{2}\right] \cdot \left[\frac{y}{3}\right] \cdot \ldots \cdot \left[\frac{y}{2017}\right] = \beta \end{cases},$$

$$\left[\frac{x}{1}\right] \cdot \left[\frac{x}{2}\right] \cdot \left[\frac{x}{3}\right] \cdot \ldots \cdot \left[\frac{x}{2017}\right] \in \mathbb{Z}, 0 \leq \alpha < 1 \to \left[\frac{x}{1}\right] \cdot \left[\frac{x}{2}\right] \cdot \left[\frac{x}{3}\right] \cdot \ldots \cdot \left[\frac{x}{2017}\right] = 0$$

$$\left[\frac{y}{1}\right] \cdot \left[\frac{y}{2}\right] \cdot \left[\frac{y}{3}\right] \cdot \ldots \cdot \left[\frac{y}{2017}\right] \in \mathbb{Z}, 0 \leq \beta < 1 \to \left[\frac{y}{1}\right] \cdot \left[\frac{y}{2}\right] \cdot \left[\frac{y}{3}\right] \cdot \ldots \cdot \left[\frac{y}{2017}\right] = 0$$

$$Solutions: x, y \in \left\{0, 1, 2^{\frac{1}{2017}}, \ldots, (2017^{2017} - 1)^{\frac{1}{2017}}\right\}$$

1.41

$$\text{First if } \begin{matrix} x < 3 \\ y < 3 \\ z < 3 \end{matrix} \Rightarrow x + y + t < 9, \text{ but } t \geq 1 \Rightarrow 6 + 3t \geq 9 \Rightarrow \text{its false} \Rightarrow x, y, z \geq 3.$$

$$\text{Let } x - 2 = a, y - 2 = b, z - 2 = c, a, b, c \geq 1$$

$$\begin{cases} \sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} = 3\sqrt{t} \\ a + b + c = 3t \end{cases} \Rightarrow \sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} = \sqrt{3(a + b + c)} \quad (1)$$

$$\text{Because } a, b, c \geq 1 \Rightarrow \sqrt[3]{b} \leq \sqrt{b} \text{ and } \sqrt[4]{c} \leq \sqrt{c} \text{ with equality for } b = c = 1 \Rightarrow$$

$$\sqrt{a} + \sqrt[3]{b} + \sqrt[4]{c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \quad (2). \text{ From } (1)+(2) \Rightarrow$$

$$\sqrt{3(a + b + c)} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \Rightarrow 3(a + b + c) \leq \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \quad (3)$$

$$\text{From Cauchy's inequality } 3(a + b + c) \geq \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2 \quad (4)$$

$$\text{From } (3)+(4) \Rightarrow \text{in Cauchy's inequality we have equality} \Rightarrow a = b = c = 1 \Rightarrow \begin{matrix} x = 3 \\ y = 3 \\ z = 3 \end{matrix} \Rightarrow t = 1.$$

1.42

$$\text{Let } x = y = \sqrt[3]{t^2}, z = 1$$

$$f\left(\sqrt[3]{t^2} \cdot \sqrt{\sqrt[3]{t^2} \cdot 1}\right) + f\left(\sqrt[3]{t^2} \cdot \sqrt{1 \cdot \sqrt[3]{t^2}}\right) + f\left(1 \cdot \sqrt{\sqrt[3]{t^2} \cdot \sqrt[3]{t^2}}\right) = f\left(\sqrt{\left(\sqrt[3]{t^2}\right)^3}\right)$$

$$f(t) + f(t) + f\left(\sqrt[3]{t^2}\right) = f\left(\sqrt[3]{t^2}\right), \qquad 2f(t) = 0 \to f \equiv 0$$

1.43

$$\text{If } f: [2; +\infty) \to \mathbb{R}, f(x) + f\left(\frac{1}{1-x}\right) = x \text{ (1) } \forall x \geq 2 \text{ then } \forall x, y, z \geq 2:$$

$$2f(x) + \frac{1}{x} + 2f(y) + \frac{1}{y} + 2f(z) + \frac{1}{z} \geq 3 \cdot \sqrt[3]{\frac{x^2y^2z^2}{(x-1)(y-1)(z-1)}}$$

$$\text{Substitute } x \text{ to } \frac{1}{1-x}, \text{ we have (1)} \Rightarrow f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} \quad (2)$$

Substitute x *to* $\frac{x-1}{x}$, *we have (1)* $\Rightarrow f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x}$ *(3)*

(2)+(3) $\Rightarrow f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) + f\left(\frac{x-1}{x}\right) + f(x) = \frac{1}{1-x} + \frac{x-1}{x} \Rightarrow$

$\Rightarrow x + 2f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} + \frac{x-1}{x} \Rightarrow f\left(\frac{x-1}{x}\right) = \frac{-x^3+2x^2-3x+1}{2x(x-1)}$ *(4)*

Substitute x *to* $\frac{1}{1-x}$, *we have (4)* $\Rightarrow f(x) = \frac{x^3-x+1}{2x(x-1)} \Rightarrow 2f(x) + \frac{1}{x} = \frac{x^2}{x-1}\ \forall\, x \geq 2$

We have $2f(x) + \frac{1}{x} + 2f(y) + \frac{1}{y} + 2f(z) + \frac{1}{z} = \frac{x^2}{x-1} + \frac{y^2}{y-1} + \frac{z^2}{z-1}$

By AM-GM inequality, we have $\frac{x^2}{x-1} + \frac{y^2}{y-1} + \frac{z^2}{z-1} \geq 3\sqrt[3]{\frac{x^2y^2z^2}{(x-1)(y-1)(z-1)}}$

So, $2f(x) + \frac{1}{x} + 2f(y) + \frac{1}{y} + 2f(z) + \frac{1}{z} \geq 3\sqrt[3]{\frac{x^2y^2z^2}{(x-1)(y-1)(z-1)}}$ *(QED)*

The equality occurs when $x = y = z$.

1.44

Let's set $g(x) = 2^x f(x), \forall x \in \mathbb{R}$. *Then the given inequality*

$x + y \leq g(x) + g(y) \leq g(x + y)$. *For* $x = y = 0$ *we have that:*

$0 \leq 2g(0) \leq g(0) \Rightarrow g(0) = 0$. *For* $y = -x$, *we have that:*

$0 \leq g(x) + g(-x) \leq g(0) \Rightarrow 0 \leq g(x) + g(-x) \leq g(0), \forall x \in \mathbb{R} \Rightarrow$

$g(-x) = -g(x), \forall x \in \mathbb{R}$ *(1)*

For $y = 0$: $x \leq g(x) \leq g(x), \forall x \in \mathbb{R}$ *(2)*

We set $x \to -x$ *and …* $-x \leq g(-x) \overset{(1)}{\Rightarrow} -x \leq -g(x) \Rightarrow x \geq g(x), \forall x \in \mathbb{R}$ *(3)*

So, (by (2)+(3)) $g(x) = x, \forall x \in \mathbb{R}$. *Then* $f(x) = \frac{x}{2^x}, \forall x \in \mathbb{R}$

which is acceptable because it verifies the given conditions.

1.45

Find $f: \mathbb{R} \to \mathbb{R}$ *such that:* $(x - y)^2[f^2(x) - f^2(y)] = (x + y)f^3(x - y)$ *(1)*

$$y \to x\text{: } (1) \Rightarrow 0 = 2xf^3(0) \Rightarrow f(0) = 0$$

$$y \to 0\text{: } (1) \Rightarrow x^2 f^2(x) = xf^3(x) \Rightarrow xf^2(x)[x - f(x)] = 0 \quad (*)$$

Suppose that $\exists a, b \neq 0$ *such that* $f(a) = 0$ *and* $f(b) = b$

$$x \to a, y \to b\text{: } (1) \Rightarrow (a-b)^2(0^2 - b^2) = (a+b)f^3(a-b) \quad (2)$$

Case 1: $f(a-b) = 0$*: (2)* $\Rightarrow b^2(a-b)^2 = 0 \Rightarrow a = b \Rightarrow b = 0$ *(Absurd)*

Case 2: $f(a-b) = a-b$*: (2)* $\Rightarrow (a-b)^2(0^2 - b^2) = (a+b)(a-b)^3 \Rightarrow$

$-b^2 = a^2 - b^2 \Rightarrow a = 0$ *(Absurd)*

So, (*) $\Rightarrow f(x) = 0\ \forall\, x \neq 0$ *or* $f(x) = x\ \forall x \neq 0$

On the other hand, we have $f(0) = 0$

Then $f(x) = 0\ \forall x \in \mathbb{R}$ *or* $f(x) = x\ \forall x \in \mathbb{R}$

1.46

Consider a continuous function f *satisfying the proposed property. Let* $P(x, y)$ *be the property* $f(x^3) - f(y^3) = (x^2 + xy + y^2)f(x-y)$

From $P(1,1)$ *we conclude that* $f(0) = 0$*.*

From $P(x, 0)$ *we conclude that* $f(x^3) = x^2 f(x)$ *for every* x

From $P(tx, x)$ *for* $x \neq 0$ *we get*

$$t^2 f(tx) - f(x) = (t^2 + t + 1)f\big((t-1)x\big) \qquad (1)$$

Which is also true when $x = 0$ *according to the first point.*

Setting $t = 0$ *in (1) we conclude that* f *is odd.*

Setting $t = 2$ *in (1) we conclude that* $f(2x) = 2f(x)$ *for all* x*.*

Now suppose that $f(nx) = nf(x)$ *for some positive integer* n *and for all* x*. Applying (1) with* $t = n + 1$ *we get*

$$(n+1)^2 f\big((n+1)x\big) = f(x) + (n^2 + 3n + 3)nf(x) = (n+1)^3 f(x)$$

that is $f\big((n+1)x\big) = (n+1)f(x)$ *for all* x*. Thus, since* f *is odd, we have proved that*

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, f(nx) = nf(x) \qquad (2)$$

Applying (2) with positive n and $\frac{x}{n}$ instead of x we get also

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{N}^*, f\left(\frac{x}{n}\right) = \frac{1}{n}f(x) \qquad (3)$$

Combining (2) and (3) we get for $n \in \mathbb{N}^*, m \in \mathbb{Z}$ and $x \in \mathbb{R}$ the following

$$f\left(\frac{m}{n}x\right) = \frac{1}{n}f(mx) = \frac{m}{n}f(x) \qquad (4)$$

Thus $f(r) = f(1)r$ for all $r \in \mathbb{Q}$.

Now, the continuity of f shows that $f(x) = f(1)x$ for all real x.

Conversely, any function of the form $x \to \alpha x$ satisfies the proposed functional equation.

1.47

We put $x = 0$ in $f(x)f(2x)f(4x) = 2^x$, we get $f^3(0) = 1$ then $f(0) = 1$.

We have $\begin{cases} f\left(\frac{x}{2}\right)f(x)f(2x) = 2^{\frac{x}{2}} \\ f(x)f(2x)f(4x) = 2^x \end{cases}$ *it follows that $f(4x) = 2^{\frac{x}{2}}f\left(\frac{x}{2}\right)$ then $f(x) = 2^{\frac{x}{8}}f\left(\frac{x}{8}\right)$ by induction we get $f(x) = 2^{\frac{x}{8}}2^{\frac{x}{8^2}}\cdots 2^{\frac{x}{8^n}}f\left(\frac{x}{8^n}\right) = 2^{\frac{x}{8}\left(\frac{1-\left(\frac{1}{8}\right)^n}{1-\frac{1}{8}}\right)}f\left(\frac{x}{8^n}\right)$ for all $n \in \mathbb{N}$ then*

$$f(x) = \lim_{n\to+\infty} f(x) = \lim_{n\to+\infty} 2^{\frac{x}{8}\left(\frac{1-\left(\frac{1}{8}\right)^n}{1-\frac{1}{8}}\right)}f\left(\frac{x}{8^n}\right) = 2^{\frac{x}{7}}f(0) = 2^{\frac{x}{7}}$$

1.48

$$\sum x^2 \overset{(1)}{=} 1 \;\&\; \frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \overset{(2)}{=} \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

$$(2) \Rightarrow x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz$$

$$\Rightarrow x(1-(1-x^2)+y^2z^2) + y(1-(1-y^2)+z^2x^2) + z(1-(1-z^2)+x^2y^2) =$$

$$= 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x^3 + xyz\left(\sum xy\right) = 4xyz \Rightarrow 3xyz + \sum x\left(1-\sum xy\right) +$$

$$+xyz\left(\sum xy\right) = 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x\left(1-\sum xy\right) = xyz\left(1-\sum xy\right) = 0$$

$$\Rightarrow (1 - \sum xy)(\sum x - xyz) = 0 \quad (1)$$

Now, $\sum x^2 \overset{A-G}{\geq} 3\sqrt[3]{x^2y^2z^2} \Rightarrow 1 \geq 27x^2y^2z^2 \Rightarrow 27x^2y^2z^2 \overset{(2)}{\leq} 1$*. If* $\sum x = xyz$*, then,*

$$\left(\sum x\right)^2 = x^2y^2z^2 \Rightarrow 27\left(1 + 2\sum xy\right) = 27x^2y^2z^2 \left(\because \sum x^2 = 1\right) \overset{by\ (2)}{\leq} 1 \Rightarrow$$

$\Rightarrow 54\sum xy \leq -26$*. But* $54\sum xy > 0$ $(\because x, y, z > 0) \therefore \sum x \neq xyz \therefore$ *(1)* $\Rightarrow 1 = \sum xy$

Now, $\sum x^2 \geq \sum xy$ *(equality when* $x = y = z$*)* $\Rightarrow 1 \geq \sum xy$*,*

with equality when $x = y = z$*.*

$\& \because 1 = \sum xy \therefore x = y = z \;\&\because \sum x^2 = 1, \therefore x^2 = \frac{1}{3} \Rightarrow x = y = z = \frac{1}{\sqrt{3}}$ *(ans)*

1.49

Denote $x + 1 = t$*, then* $\frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{t} = 156 + log_5 t$ *(1)*

domain the equation (1) $t > 0$*,* $f(t) = \frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{t} \downarrow$ *in* $(0; +\infty)$

$g(t) = 156 + log_5 t \uparrow$ *in* $(0; +\infty)$*, and has at most one root*

$$t = \frac{1}{5} \Rightarrow x + 1 = \frac{1}{5} \Rightarrow x = -\frac{4}{5}$$

1.50

Let $A = ln(x), B = ln(x + 1), C = ln(x + 2)$

$$\therefore f(x) = sin(A + B - C) + sin(B + C - A) = 2\sin B \cos(A - C)$$

$$g(x) = sin(A + B + C) - sin(A - B + C)$$

$$= sin(B + A + C) + sin(B - (A + C)) = 2\sin B\cos(A + C)$$

$$f(x) = g(x)$$

$\Rightarrow \sin B = 0$ *or* $cos(A - C) = cos(A + C) \Rightarrow \sin B = 0$ *or* $\sin A \sin C = 0$

$\Rightarrow \sin A = 0$ *or* $\sin B = 0$ *or* $\sin C = 0 \Rightarrow A, B, C = n\pi, n \in \mathbb{Z}$

$\Rightarrow x = e^{n\pi}$ *or* $e^{n\pi} - 1$ *or* $e^{n\pi} - 2$ *for same* $n \in \mathbb{Z}$

$\Rightarrow sin(ln\,x) = 0$ *or* $sin(ln(x+1)) = 0$ *or* $sin(ln(x+2)) = 0 \Rightarrow x = e^{n\pi}, e^{n\pi}, e^{n\pi} - 2$

1.51

Clearly, $x > 0, t > 0, z > 0, z \neq 1$*. From 3rd equation:*

$$\left(\frac{1}{2xy} - \frac{x}{x^4 + y^2}\right) + \left(\frac{1}{2xy} - \frac{y}{x^2 + y^4}\right) = 0$$

$$\Rightarrow \frac{(x^4 + y^2 - 2x^2y)}{2xy(x^4 + y^2)} + \frac{(x^2 + y^4 - 2xy^2)}{2x(x^2 + y^4)} = 0 \Rightarrow \frac{(x^2 - y)^2}{(x^4 + y^2)} + \frac{(x - y^2)^2}{x^2 + y^4} = 0$$

$$\Rightarrow x^2 - y = 0, x - y^2 = 0 \Rightarrow y^4 = x^2 = y \Rightarrow y(y^3 - 1) = 0$$

$$\Rightarrow y = 1 \quad [\because y \in \mathbb{R}, y \neq 0] \quad \therefore x = 1$$

From 2nd equation: $log_z t = y = 1\ [\because log_z x = 0] \Rightarrow t = z$

From first equation: $3 + 3 + 3^z + 3^z = 24 \Rightarrow 2(3^z) = 18 \Rightarrow z = 2$

$$\therefore x = 1, y = 1, z = 2, t = 2$$

1.52.

Solve in $\mathbb{R}$*:*

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + 1 = 4\sqrt{xyz} \quad (1)$$

$$xy + yz + zx + 3 = 2\left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \quad (2)$$

If $x = 0$, *(1)* $\Rightarrow \sqrt{y} + \sqrt{z} + 1 = 0$

But $\sqrt{y} + \sqrt{z} + 1 \geq 1 \Rightarrow \sqrt{y} + \sqrt{z} + 1 = 0$ *is impossible,* $\Rightarrow x \neq 0$

Similalry, it can be concluded that $y, z \neq 0$

$$\therefore x, y, z \neq 0 \therefore x, y, z > 0$$

Let $\sqrt[4]{x} = a, \sqrt[4]{y} = b, \sqrt[4]{z} = c; a, b, c > 0$

Then (1) $\Rightarrow a^2 + b^2 + c^2 + 1 = 4a^2b^2c^2$ *(3)*

(2) $\Rightarrow a^4b^4 + b^4c^4 + c^4a^4 + 3 = 2(a + b + c)$ *(4)*

Now, $a^2 + b^2 + c^2 + 1 \overset{A-G}{\geq} 4\sqrt[4]{a^2b^2c^2}$

$\therefore$ *(3)* $\Rightarrow 4a^2b^2c^2 \geq 4\sqrt[4]{a^2b^2c^2} \Rightarrow a^8b^8c^8 \geq a^2b^2c^2 \Rightarrow a^6b^6c^6 \geq 1 \Rightarrow abc \geq 1$

Now, $2(a + b + c) \leq 2abc(a + b + c) \overset{(5)}{\leq} 2(a^2b^2 + b^2c^2 + c^2a^2)$

($\because$ $\alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \gamma\alpha$, *where* $\alpha = ab, \beta = bc, \gamma = ca$*)*

(4), (5) $\Rightarrow a^4b^4 + b^4c^4 + c^4a^4 + 3 \leq 2(a^2b^2 + b^2c^2 + c^2a^2)$

$\Rightarrow 2(a^2b^2 + b^2c^2 + c^2a^2) \geq ((a^2b^2)^2 + (b^2c^2)^2 + (c^2a^2)^2) + 3$

$\geq \left(\frac{1}{3}(a^2b^2 + b^2c^2 + c^2a^2)^2\right) + 3$

($\because 3(u^2 + v^2 + w^2) \geq (u + v + w)^2$, *where* $u = a^2b^2, v = b^2c^2, w = c^2a^2$*)*

$\Rightarrow 2t \geq \frac{t^2}{3} + 3$ *(where* $t = a^2b^2 + b^2c^2 + c^2a^2$*)*

$\Rightarrow 6t \geq t^2 + 9 \Rightarrow (t - 3)^2 \leq 0$ *. But* $(t - 3)^2 \geq 0$

$\therefore (t - 3)^2 = 0 \Rightarrow t = 3 \Rightarrow a^2b^2 + b^2c^2 + c^2a^2 = 3$ *(6)*

But $\sum a^2b^2 \overset{A-G}{\geq} 3\sqrt[3]{a^4b^4c^4} \geq 3$ $(\because abc \geq 1)$,

equality when $a = b = c \therefore \sum a^2b^2 = 3 \Rightarrow a = b = c$ *(7)*

$\therefore 3a^4 = 3$ *(from (6), (7))* $\Rightarrow a = 1 \Rightarrow a = b = c = 1 \Rightarrow x = y = z = 1$ *is the only solution*

1.53

$x(y + z) = y^2 + z^2 - 6$ *(1)*

$y(z + x) = z^2 + x^2 - 6$ *(2)*

$z(x + y) = x^2 + y^2 - 6$ *(3)*

Adding we get

$2(xy + yz + zx) = 2(x^2 + y^2 + z^2 - 9) \Rightarrow xy + yz + zx = x^2 + y^2 + z^2 - 9$ *(4)*

From (1), (4): $yz = x^2 - 3 \Rightarrow x^2 - yz = 3$ *(5)*

Similarly : $y^2 - zx = 3$ *(6)*, $z^2 - xy = 3$ *(7)*

$\therefore x^2 - y^2 - yz + zx = 0 \Rightarrow (x-y)(x+y+z) = 0$, *etc.*

When $x + y + z \neq 0$, *we get* $\Rightarrow x = y = z$

Not possible in view of (5): $x + y + z = 0$

From (1), we get: $-x^2 = y^2 + z^2 - 6 \Rightarrow x^2 + y^2 + z^2 = 6$

and $xy + yz + zx = -3$ *[from (4)]*

Also, $x^2 + y^2 + (-x-y)^2 = 6 \Rightarrow x^2 + xy + y^2 = 3 \Rightarrow x^2 + xy + y^2 - 3 = 0$

$$\Rightarrow x = \frac{-y \pm \sqrt{y^2 - 4(y^2 - 3)}}{2} = \frac{1}{2}\left(-y \pm \sqrt{3(4 - y^2)}\right)$$

Thus, $-2 \leq y \leq 2$. *Similarly,* $-2 \leq x \leq 2, -2 \leq z \leq 3$

As x, y, z *are integers*

$(x, y, z) = (-2,1,1), (2,-1,-1), (-1,2,-1), (1,-2,1), (1,1,-2), (-1,-1,2)$

1.54

The system can be rewritten

$$\begin{cases} x^3 + \ln x = y^3 + \ln y \\ y^5 + \ln y = z^5 + \ln z \\ 2x^y + 3y^z + 5z^x = 10 \end{cases}$$

Let $f: (0, \infty) \to R, f(x) = x^3 + \ln x$ *a sum of strictly increasing functions, so* f *strictly increasing* $\Rightarrow f$ *one to one* $\Rightarrow x = y$

$g: (0, \infty) \to \mathbb{R}, g(x) = x^5 + \ln x$ *similarly we have* $y = z$

The last equation becomes: $5x^x + 5x^x = 10 \Leftrightarrow x^x = 1 \Leftrightarrow x = 1$

Note for $x > 1 \Rightarrow x^x > 1$ *so* $x^x = 1$ *doesn't have solutions for*

$0 < x < 1, x = \frac{1}{a}, a > 1 \Rightarrow \left(\frac{1}{a}\right)^{\frac{1}{a}} = \frac{1}{a^{\frac{1}{a}}} < 1$, *again with no solution*

Hence $x = y = z = 1$.

1.55

$$2017_{42} = 2 \cdot 42^3 + 42 + 7 = 148225 = 385^2 = (9 \cdot 42 + 7)^2 = (97_{42})^2$$

Taking into account: $2017_{42} = (97_{42})^2$, *adding the equations;*

$$\sum_{k=1}^{2018} (2017a_k^2 - 4 \cdot 97a_k + 4) = 0 \Leftrightarrow \sum_{k=1}^{2018} (97^2 \cdot a_k^2 - 4 \cdot 97a_k + 4) = 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{2018} (97a_k - 2)^2 = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_{2018} = \frac{2}{97}$$

So, the solution to the equation system is: $\left(\frac{2}{97}, \frac{2}{97}, \frac{2}{97}, \dots, \frac{2}{97}\right)$.

Or, rewriting the equations into decimal numeral system:

$$a_2^2 + a_3^2 + a_4^2 + \cdots + a_{148226}^2 = 4 \cdot (385a_1 - 1)$$

$$a_1^2 + a_3^2 + a_4^2 + \cdots + a_{148226}^2 = 4 \cdot (385a_2 - 1)$$

$$a_1^2 + a_2^2 + a_4^2 + \cdots + a_{148226}^2 = 4 \cdot (385a_3 - 1)$$

… ..

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{148225}^2 = 4 \cdot (385a_{148226} - 1)$$

Adding then, we get:

$$\sum_{k=1}^{148226} (148225a_k^2 - 4 \cdot 385a_k + 4) = 0 \Leftrightarrow \sum_{k=1}^{148226} (385^2 \cdot a_k^2 - 4 \cdot 385a_k + 4) = 0 \Leftrightarrow$$

$$\sum_{k=1}^{148226} (385a_k - 2)^2 = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_{148226} = \frac{2}{385}$$

So, the solution to the equation system is: $\left(\frac{2}{385}, \frac{2}{385}, \frac{2}{385}, \dots, \frac{2}{385}\right)$.

1.56

Let $f(x) = x^3 + x^2 + 2x + log(x^3 + 2x + 1)$

$f'(x) = \frac{3x^2+2}{x^3+2x+1} + 3x^2 + 2x + 2$. *We must have* $x^3 + 2x + 1 > 0$ *for* $log(x^3 + 2x + 1)$ *to be defined on* $\mathbb{R} \Rightarrow x^3 + 2x + 1 > 0 \therefore f'(x) = \frac{3x^2+2}{x^3+2x+1} + (x+1)^2 + 2x^2 + 1 > 0 \Rightarrow$

$\Rightarrow f(x)$ *is an increasing* f^n. *Let us assume* $x \geq y$. *Then* $f(x)$ *is an increasing* f^n,

$\therefore f(x) \geq f(y) \Rightarrow y \geq x(\because y = f(x)\, \& x = f(y)) \therefore x \geq y \& y \geq x$, *we must have* $x = y$. *If we assume* $x \leq y$, *then* $f(y) \geq f(x) \Rightarrow x \geq y \Rightarrow x = y$. *So, we conclude combining both cases that* $x = y \therefore x^3 + x^2 + x + log(x^3 + 2x + 1) = 0$. *Let* $g(x) = x^3 + x^2 + x +$

$$+ log(x^3 + 2x + 1)\,; g'(x) = \frac{3x^2 + 2}{x^3 + 2x + 1} + (x+1)^2 + 2x^2 > 0 \Rightarrow g(x)$$

is an increasing f^n *&*

$\because g(0) = 0, \therefore g(x) = 0$ *iff* $x = 0 \therefore$ *only possible pair satisfying given equation is*

$$\begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix} \text{ (answer)}$$

1.57

$$Let\ \theta = \frac{\pi}{14} \text{ and } a = sin\left(\frac{\pi}{14}\right) = sin\,\theta$$

$cos\frac{2\pi}{7} = sin\left(\frac{\pi}{2} - \frac{2\pi}{7}\right) = sin\left(\frac{3\pi}{14}\right) = sin\,3\theta$. *Also,* $7\theta = \frac{\pi}{2} \Rightarrow 4\theta = \frac{\pi}{2} - 3\theta \Rightarrow sin(4\theta) = cos\,3\theta \Rightarrow$

$$\Rightarrow 4\,sin\,\theta\,cos\,\theta\,cos\,2\theta = 4\,cos^3\theta - 3\,cos\,\theta \Rightarrow 4\,sin\,\theta\,(1 - 2\,sin^2\theta) = 4(1 - sin^2\theta) - 3 \Rightarrow$$

$\Rightarrow 4a(1 - 2a^2) = 4(1 - a^2) - 3 \Rightarrow 8a^3 - 4a^2 - 4a + 1 = 0$. *Now,* $csc\,\theta = \frac{1}{a} = 4 + 4a - 8a^2$

$$cos\frac{2\pi}{7} = sin\,3\theta = 3a - 4a^3$$

$$LHS = csc\left(\frac{\pi}{14}\right) - 4\,cos\frac{2\pi}{7} = 4 + 4a - 8a^2 - 4(3a - 4a^3) = 16a^3 - 8a^2 - 8a + 4 =$$

$$= 2(8a^3 - 4a^2 - 4a + 1) + 2 = 2(0) + 2 = 2$$

1.58

$$Let\ \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$|z-\omega^2|^2+|z-\omega|^2+|z-1|^2-3|z|^2=z$$

$$\Rightarrow |z|^2+|\omega^2|^2-\bar{z}\omega^2-z\bar{\omega}^2+|z|^2+|\omega|^2-\bar{z}\omega-z\bar{\omega}+|z|^2+1-\bar{z}-z-3|z|^2=z$$

$$\Rightarrow 3|z|^2+3-\bar{z}(\omega^2+\omega+1)-z(\omega+\omega^2+1)-3|z|^2=z\Rightarrow z=3$$

1.59

If $0<\cos x<1$,

$$2^{\cos x}+1>2\Rightarrow log_2(2^{\cos x}+1)>1$$

$$3^{\cos x}+2>3\Rightarrow log_3(3^{\cos x}+2)>1$$

$$4^{\cos x}+3>4\Rightarrow log_4(4^{\cos x}+3)>1$$

$$\Rightarrow log_2(2^{\cos x}+1)+log_3(3^{\cos x}+2)+log_4(4^{\cos x}+3)>3$$

and $(27-\cos x)^{\frac{1}{3}}<3$*. Similarly, if* $-1<\cos x<0$*, then*

$LHS<3$ *and* $RHS>3$*. Thus, only possible solution is*

$$\cos x=0\Rightarrow x=(2n+1)\frac{\pi}{2},n\in\mathbb{Z}$$

1.60

$$2y\le y^2+1\Rightarrow z^2+2y+3\le z^2+y^2+4\Rightarrow$$

$$E=\sum\frac{2x^2+4}{z^2+y^2+4}=3\Rightarrow E=\sum\frac{x^2+2}{y^2+2+z^2+2}=\frac{3}{2}\quad (1)$$

Let $x^2+2=a,y^2+2=b,z^2+2=c\Rightarrow$ *(1) becomes*

$$\sum\frac{a}{b+c}=\frac{3}{2}\quad (2)$$

But $\sum\frac{a}{b+c}\ge\frac{3}{2}$ *(3)*

From (2)+(3)$\Rightarrow a=b=c\Rightarrow x^2=y^2=z^2\Rightarrow x=y=z=1$.

1.61

$$\overset{LHS}{=}\sum_{r=1}^{101}\frac{(x-1)!\,(x+r-1)}{(x+r)!}\Rightarrow (x-1)!\sum_{r=1}^{101}\left(\frac{1}{(x+r-1)!}-\frac{1}{(x+r)!}\right)\Rightarrow$$

$$\Rightarrow \frac{1}{x} - \frac{(x-1)!}{(x+101)!} \quad (1)$$

$$\overset{RHS}{=} \frac{1}{3} - \frac{(x-1)!}{(x+101)!} \quad (2)$$

From (1) and (2)

$x = 3$ *is the only solution.*

1.62

$$\text{Let } a = e^x, b = e. \text{ Put } \Delta_1 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & \frac{1}{a} & b & \frac{1}{b} \\ a^2 & \frac{1}{a^2} & b^2 & \frac{1}{b^2} \\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4} \end{vmatrix} = \frac{1}{a^4b^4}\Delta_2$$

$$\Delta_3 = \begin{vmatrix} 1 & a^4 & 1 & b^4 \\ a & a^3 & b & b^3 \\ a^2 & a^2 & b^2 & b^2 \\ a^4 & 1 & b^4 & 1 \end{vmatrix} C_1 \to C_1 - C_2, C_3 \to C_3 - C_4$$

$$\Delta_2 = (1-a^2)(1-b^2)\Delta_3 \text{ where } \Delta_4 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ 0 & a^2 & 0 & b^2 \\ -(1+a^2) & 1 & -(1+b^2) & 1 \end{vmatrix} \text{Expand along } R_3$$

$$\Delta_4 = -a^2 \begin{vmatrix} 1+a^2 & 1+b^2 & b^4 \\ a & b & b^3 \\ -(1+a^2) & -(1+b^2) & 1 \end{vmatrix} - b^2 \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 \\ a & a^3 & b \\ -(1+a^2) & 1 & -(1+b^2) \end{vmatrix} R_3 \to R_3 + R_1$$

$$\Delta_4 = -a^2 \begin{vmatrix} 1+a^2 & 1+b^2 & b^4 \\ a & b & b^3 \\ 0 & 0 & 1+b^4 \end{vmatrix} - b^2 \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 \\ a & a^3 & b \\ 0 & 1+a^4 & 0 \end{vmatrix} =$$

$$= -a^2(1+b^4)[(1+a^2)b - (1+b^2)a] + b^2(1+a^4)[(1+a^2)b - (1+b^2)a]$$

$$= [(b-a) - ab(b-a)][b^2 - a^2 - a^2b^2(b^2-a^2)] =$$

$$= (b-a)(1-ab)(b^2-a^2)(1-a^2b^2) = (b-a)^2(b+a)(1-ab)^2(1+ab)$$

$$\text{Thus, } \Delta_1 = \frac{(1+a)}{(ab)^4}(1-b^2)(a+b)(1+ab)(1-a)(b-a)^2(1-ab)$$

Next, put $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{a} & b & \frac{1}{b} \\ a^3 & \frac{1}{a^3} & b^3 & \frac{1}{b^3} \\ a^4 & \frac{1}{a^4} & b^4 & \frac{1}{b^4} \end{vmatrix} = \frac{1}{a^4b^4}\Delta_5$ *where* $\Delta_5 = \begin{vmatrix} 1 & a^4 & 1 & b^4 \\ a & a^3 & b & b^3 \\ a^3 & a & b^3 & b \\ a^4 & 1 & b^4 & 1 \end{vmatrix}$

Use $C_1 \to C_1 - C_2, C_3 \to C_3 - C_4,\ \Delta_5 = (1-a^2)(1-b^2)\Delta_6$ *where*

$$\Delta_6 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ -a & a & -b & b \\ -(1+a^2) & 1 & -(1+b^2) & 1 \end{vmatrix} \quad R_4 \to R_4 + R_1, R_3 \to R_3 + R_2$$

$$\Delta_6 = \begin{vmatrix} 1+a^2 & a^4 & 1+b^2 & b^4 \\ a & a^3 & b & b^3 \\ 0 & a+a^3 & 0 & b+b^3 \\ 0 & 1+a^4 & 0 & 1+b^4 \end{vmatrix} = (1+a^2)\begin{vmatrix} a^3 & b & b^3 \\ a+a^3 & 0 & b+b^3 \\ 1+a^4 & 0 & 1+b^4 \end{vmatrix} -$$

$$-a\begin{vmatrix} a^4 & 1+b^2 & b^4 \\ a+a^3 & 0 & b+b^3 \\ 1+a^4 & 0 & 1+b^4 \end{vmatrix} = -(1+a^2)b[(1+a^3)(1+b^4) - (1+a^4)(b+b^3)] +$$

$$+a(1+b^2)[(a+a^3)(1+b^4) - (1+a^4)(b+b^3)] =$$

$$= (a-b)(1-ab)[(a-b)(1-a^3b^3) + (1-ab)(a^3-b^3)]$$

$$= (a-b)^2(1-ab)^2[1+ab+a^2b^2+a^2+b^2+ab]$$

Thus, $\Delta = \frac{1}{(ab)^8}(1-a^2)^2(1-b^2)^2(b-a)^4(1-ab)^4(a+b)(1+ab)$

$$(1+2ab+a^2b^2+a^2+b^2)$$

As $a, b > 0, b \neq 1,\ \Delta = 0 \Leftrightarrow a^2 - 1$ *or* $b = a$ *or* $ab = 1 \Leftrightarrow e^x = 1$ *or* $e^x = e, e^{x+1} = 1 \Leftrightarrow$

$$\Leftrightarrow x = 0, x = 1, x = -1.$$

1.63

Put $e^{\pi x^{2018}} = t, \pi^{\frac{2e}{x}} = u$

Numerator of LHS$= (t+1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right)$

$$= \frac{1}{t-1}(t^2-1)(t^2+1)(t^4+1)\dots\left(t^{2^n}+1\right) = \dots = \frac{1}{t-1}\left(t^{2^{n+1}}-1\right)$$

Denominator of RHS

$$= (u+1)(u^2+1)\dots\left(u^{2^n}+1\right) = \frac{u^{2^{n+1}}-1}{u-1}$$

$$\therefore LHS = \frac{u-1}{t-1}\cdot\frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1} \quad \text{(1). Also, } RHS=\frac{t^{2^{n+1}}-1}{u^{2^{n+1}}-1} \quad \text{(2)}$$

From (1), (2), we get $u-1 = t-1 \Rightarrow u = t$

$$\Rightarrow e^{\pi x^{2018}} = \pi^{\frac{2e}{x}} \Rightarrow \pi x^{2018} = \frac{2e}{x}\ln\pi \Rightarrow x^{2019} = \frac{2e\ln\pi}{\pi} \Rightarrow x = \left(\frac{2e\ln\pi}{\pi}\right)^{\frac{1}{2019}}$$

1.64

$$\cos^{12}x + 4\cos^8 x\sin 2x + (3\cos^4 x - 4)(2\sin^2 2x) + 4\sin^3 2x - 3\cos x + 19 = 0$$

$$\Rightarrow \cos^{12}x + 8\cos^9 x\sin x + 24\cos^6 x\sin^2 x + 32\cos^3 x\sin^3 x - 32\cos^2 x\sin^2 x -$$

$$-3\cos x + 19 = 0 \Rightarrow$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16\sin^4 x - 32\cos^2 x\sin^2 x - 3\cos x + 19 = 0$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 - 16(\sin^2 x + \cos^2 x)^2 + 16\cos^4 x - 3\cos x + 19 = 0$$

$$\Rightarrow (\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 3(\cos x - 1)$$

LHS ≥ 0 *and* $RHS \leq 0$

Equality when $LHS = 0, RHS = 0$

$$(\cos^3 x + 2\sin x)^4 + 16\cos^4 x = 0, \cos x - 1 = 0$$

$$\Rightarrow \cos^3 x + 2\sin x = 0, \cos x = 0 \text{ and } \cos x = 1$$

Thus, no solution.

1.65

Let's set $y = 0: f\left(\frac{x}{2}\right) = \frac{g(x)+h(0)}{2} \Rightarrow g(x) = 2f\left(\frac{x}{2}\right) = h(0)$ *(1)*

Set $x = 0: f\left(\frac{y}{2}\right) = \frac{g(0)+h(y)}{2} \Rightarrow h(y) = 2f\left(\frac{y}{2}\right) - g(0)$ *(2)*

Using (1), (2) we have: $f\left(\frac{x+y}{2}\right) = f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right) - \frac{h(0)+g(0)}{2}$ *or*

$f(a+b) = f(a) + f(b) - \frac{h(0)+g(0)}{2}$ *where* $a: \frac{x}{2}, b: \frac{y}{2}, a, b \in \mathbb{R}$. *Now let's set*

$k(a) = f(a) - \frac{h(0)+g(0)}{2}$. *Then* $k(a+b) = k(a) + k(b), \forall a, b \in \mathbb{R}$. *So* k *is a Cauchy function and continuos. So* $k(x) = cx, c \in \mathbb{R} \Rightarrow f(x) = cx - \frac{h(0)+g(0)}{2}, \forall x \in \mathbb{R}$ *and*

$$g(x) = cx - h(0) - \frac{h(0)+g(0)}{2} \Rightarrow$$

$$\Rightarrow g(x) = cx - \frac{3h(0)+g(0)}{2}, h(x) = cx - \frac{3g(0)-h(0)}{2};$$

and similarly these functions satisfy the equation.

1.66

$$\begin{cases} 2y\sqrt{1-x^2} + 2y\sqrt{1-y^2} = \sqrt{3} \\ 2y\sqrt{1-z^2} + 2y\sqrt{1-y^2} = \sqrt{3} \\ 2z\sqrt{1-x^2} + 2z\sqrt{1-x^2} = \sqrt{3} \end{cases}$$

These systems of equations can be further expressed as:

$$\begin{cases} sin^{-1} x + sin^{-1} y = \frac{\pi}{3}, \frac{2\pi}{3} & (1) \\ sin^{-1} y + sin^{-1} z = \frac{\pi}{3}, \frac{2\pi}{3} & (2) \\ sin^{-1} x + sin^{-1} z = \frac{\pi}{3}, \frac{2\pi}{3} & (3) \end{cases}$$

Adding these equations, we have that:

$$sin^{-1} x + sin^{-1} y + sin^{-1} z = \frac{\pi}{2}, \pi \quad (4)$$

Notice that: $\frac{-\pi}{2} \leq \phi \leq \frac{\pi}{2} \Rightarrow -1 \leq sin\phi \leq 1$ *this implies the inequality above hold true for the following cases ie.*

(1). If $x = y = z$ *then:* $sin^{-1} x = sin^{-1} y = sin^{-1} z = \frac{\pi}{6}, \frac{\pi}{3}$ *which implies that*

$x = y = z = \frac{1}{2}, \frac{\sqrt{3}}{2}$ *from equation (4)*

(2). If any two of them are equal and third one is different. WLOG, let $x = y$ which directly implies $z = 0, x = y = 1$ which further follows as either $x = y = \frac{\sqrt{3}}{2}$ or $x = y = \frac{1}{2}$ and the same value will corresponds either $x = 0$ case or $y = 0$. Thus, the solutions are $0, 1, \frac{1}{2}, \frac{\sqrt{3}}{2}$.

$$\text{Thus, } x = 0, y = z = \frac{\sqrt{3}}{2} \quad x = y = z = \frac{1}{2}$$

$$x = 1, y = z = \frac{1}{2} \quad y = 1, x = z = \frac{1}{2}$$

$$z = 1, x = y = \frac{1}{2} \quad y = 0, x = z = \frac{\sqrt{3}}{2}$$

$$x = 0, x = y = \frac{\sqrt{3}}{2}$$

1.67

$$\sum x^2 \overset{(1)}{=} 1 \ \& \ \frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \overset{(2)}{=} \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

$$(2) \Rightarrow x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) = 4xyz$$

$$\Rightarrow x(1-(1-x^2)+y^2z^2) + y(1-(1-y^2)+z^2x^2) + z(1-(1-z^2)+x^2y^2) =$$

$$= 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x^3 + xyz\left(\sum xy\right) = 4xyz \Rightarrow 3xyz + \sum x\left(1-\sum xy\right) +$$

$$+xyz\left(\sum xy\right) = 4xyz\left(\because \sum x^2 = 1\right) \Rightarrow \sum x\left(1-\sum xy\right) = xyz\left(1-\sum xy\right) = 0$$

$$\Rightarrow (1-\sum xy)(\sum x - xyz) = 0 \quad (1)$$

Now, $\sum x^2 \overset{A-G}{\geq} 3\sqrt[3]{x^2y^2z^2} \Rightarrow 1 \geq 27x^2y^2z^2 \Rightarrow 27x^2y^2z^2 \overset{(2)}{\leq} 1$. If $\sum x = xyz$, then,

$$\left(\sum x\right)^2 = x^2y^2z^2 \Rightarrow 27\left(1+2\sum xy\right) = 27x^2y^2z^2\left(\because \sum x^2 = 1\right) \overset{by\,(2)}{\leq} 1 \Rightarrow$$

$$\Rightarrow 54\sum xy \leq -26. \text{ But } 54\sum xy > 0 \ (\because x, y, z > 0) \therefore \sum x \neq xyz \therefore (1) \Rightarrow 1 = \sum xy$$

$$\text{Now, } \sum x^2 \geq \sum xy \ \text{ (equality when } x = y = z) \Rightarrow 1 \geq \sum xy,$$

with equality when $x = y = z$.

$\& \because 1 = \sum xy \therefore x = y = z \& \because \sum x^2 = 1, \therefore x^2 = \frac{1}{3} \Rightarrow x = y = z = \frac{1}{\sqrt{3}}$ *(ans)*

1.68

$$\text{Let } f(x) = x^3 + x^2 + 2x + log(x^3 + 2x + 1)$$

$f'(x) = \frac{3x^2+2}{x^3+2x+1} + 3x^2 + 2x + 2$. *We must have* $x^3 + 2x + 1 > 0$ *for* $log(x^3 + 2x + 1)$ *to be defined on* $\mathbb{R} \Rightarrow x^3 + 2x + 1 > 0 \therefore f'(x) = \frac{3x^2+2}{x^3+2x+1} + (x+1)^2 + 2x^2 + 1 > 0 \Rightarrow$

$\Rightarrow f(x)$ *is an increasing* f^n. *Let us assume* $x \geq y$. *Then* $f(x)$ *is an increasing* f^n,

$\therefore f(x) \geq f(y) \Rightarrow y \geq x(\because y = f(x) \& x = f(y)) \therefore x \geq y \& y \geq x$, *we must have* $x = y$. *If we assume* $x \leq y$, *then* $f(y) \geq f(x) \Rightarrow x \geq y \Rightarrow x = y$. *So, we conclude combining both cases that* $x = y \therefore x^3 + x^2 + x + log(x^3 + 2x + 1) = 0$. *Let* $g(x) = x^3 + x^2 + x +$

$$+ log(x^3 + 2x + 1)\,; g'(x) = \frac{3x^2 + 2}{x^3 + 2x + 1} + (x+1)^2 + 2x^2 > 0 \Rightarrow g(x)$$

is an increasing f^n *&*

$\because g(0) = 0, \therefore g(x) = 0$ *iff* $x = 0 \therefore$ *only possible pair satisfying given equation is*

$$\begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix} \text{ (answer)}$$

1.69

$$f(x) + f(y) = f(x+y) - xy(x+y) \Rightarrow f(x) - \frac{x^3}{3} + f(y) - \frac{y^3}{3} = f(x+y) =$$

$$= \frac{x^3}{3} - \frac{y^3}{3} - xy(x+y) \Rightarrow f(x) - \frac{x^3}{3} + f(y) - \frac{y^3}{3} = f(x+y) - \frac{1}{3}(x+y)^3 \quad (1)$$

Now, let $g(x) = f(x) - \frac{x^3}{3}$, g *continuous (2)*

From (1)+(2) $\Rightarrow g(x) + g(y) = g(x+y) \Rightarrow g(x) = ax, a \in \mathbb{R}$ *(3) (from Cauchy equation).* *From (2)+(3)* $\Rightarrow f(x) - \frac{x^3}{3} = ax \Rightarrow f(x) = \frac{x^3}{3} + ax$

1.70

$$\frac{x+y}{1+xy} + \frac{xy}{1+x} + \frac{xy}{1+y} + \frac{x+y+2xy}{(1+x)(1+y)xy} = 3$$

$$\frac{(x+1)(y+1)}{xy+1}+xy\left(\frac{1}{1+x}+\frac{1}{1+y}\right)+\frac{x(y+1)+y(x+1)}{xy(1+x)(1+y)}=4$$

AM-GM:

$$LHS \geq \frac{(x+1)(y+1)}{xy+1}+\frac{2xy}{\sqrt{(1+x)(1+y)}}+\frac{2\sqrt{xy(1+x)(1+y)}}{xy(1+x)(1+y)}$$

$$=\frac{(x+1)(y+1)}{xy+1}+\frac{2\left(\sqrt{(xy)^3}+1\right)}{\sqrt{xy(1+x)(1+y)}}$$

$$\geq 2\sqrt{\frac{(x+1)(y+1)\cdot 2\left(\sqrt{(xy)^3}+1\right)}{(xy+1)\sqrt{xy(x+1)(y+1)}}}=2\sqrt{\frac{2\left(\sqrt{(xy)^3}+1\right)\sqrt{(x+1)(y+1)}}{(xy+1)\sqrt{xy}}} \quad (1)$$

$$\sqrt{(xy)^3}+1 \geq \sqrt{\frac{(xy+1)^3}{2}}=(xy+1)\sqrt{\frac{xy+1}{2}} \geq (xy+1)\cdot\sqrt[4]{xy} \quad (2)$$

$$(1),(2) \Rightarrow LHS \geq 2\sqrt{\frac{2(xy+1)\sqrt[4]{xy}\cdot\sqrt{2\sqrt{x}\cdot 2\sqrt{y}}}{(xy+1)\sqrt{xy}}}=2\sqrt{\frac{4\cdot\sqrt[4]{xy}\cdot\sqrt[4]{xy}}{\sqrt{xy}}}=2\sqrt{4}=4$$

$\Rightarrow$ *Equality occurs if* $x=y=1 \Rightarrow \sin x=\sin 1 \neq 0>1=0>y \rightarrow$ *(absurd)*

$\Rightarrow$ *system has no solution.*

1.71

Let $\lfloor x\rfloor = I$ *and* $\{x\}=f\in[0,1)$, $x=I+f$

$$(I+f+f)^2-(I+f+f)=6If-1 \Rightarrow (I+2f)^2-(I+2f)=6If-1$$

$$\Rightarrow I^2+4f^2-2If-I-2f+1=0 \Rightarrow I^2-I(2f+1)+4f^2-2f+1=0 \quad (1)$$

$$\Delta=(2f+1)^2-4(4f^2-2f+1)=-3(4f^2-4f+1)=-3(2f-1)^2\leq 0$$

But, $I\in\mathbb{Z}\in\mathbb{R}, \Delta\geq 0$. *So,* $\Delta\leq 0$ *and* $\Delta\geq 0 \Rightarrow \Delta=0 \Rightarrow 2f-1=0\Rightarrow f=\frac{1}{2}$

$\Delta=0$, *from (1), we get,* $I=\frac{2f+1}{2}=\frac{2\cdot\frac{1}{2}+1}{2}=1$, $x=I+f=1+\frac{1}{2}=\frac{3}{2}$

1.72

For $x\neq 0$, *let* $f(x)=8^x+27^{\frac{1}{x}}+2^{x+1}3^{\frac{x+1}{x}}+2^x3^{\frac{2x+1}{x}}=2^{3x}+3^{\frac{3}{x}}+(15)\left(2^x3^{\frac{1}{x}}\right)$

$$For\ x < 0, f(x) < 1 + 1 + 15(1) < 125$$

$$For\ 0 < x < 1\ \ f'(x) = (2^{3x})(3\ln 2) + 3^{\frac{3}{x}}\left(-\frac{3}{x^2}\ln 3\right) + 15\left[2^x 3^{\frac{1}{x}}\ln 2 + 2^x 3^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln 3\right)\right]$$

$$= (3)2^x\left[2^{2x}\ln 2 - \frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3\right] + \left(3^{\frac{1}{x}}\right)(3)\left[5(2^x)\ln 2 - \frac{3^{\frac{2}{x}}}{x^2}\ln 3\right]$$

$$For\ 0 < x < 1, 0 < 2^{2x} < 4,\ \ln 2 < 0.7 \Rightarrow 0 < 2^{2x}\ln 2 < 2\cdot 8$$

$$For\ 0 < x < 1, \frac{1}{x^2} > 1,\ \ 3^{\frac{1}{x}} > 3, \ln 3 > 1$$

$$2^{2x}\ln 2 - \frac{5}{x^2}\left(3^{\frac{1}{x}}\right)\ln 3 < 0\ \ Also,\ for\ 0 < x < 1,$$

$$5(2^x)\ln 2 < (10)(0.7) = 7\ and\ \frac{3^{\frac{2}{x}}\ln 3}{x^2} > 9\ \Rightarrow 5(2^x)\ln 2 - \frac{3^{\frac{2}{x}}}{x^2}\ln 3 < 0$$

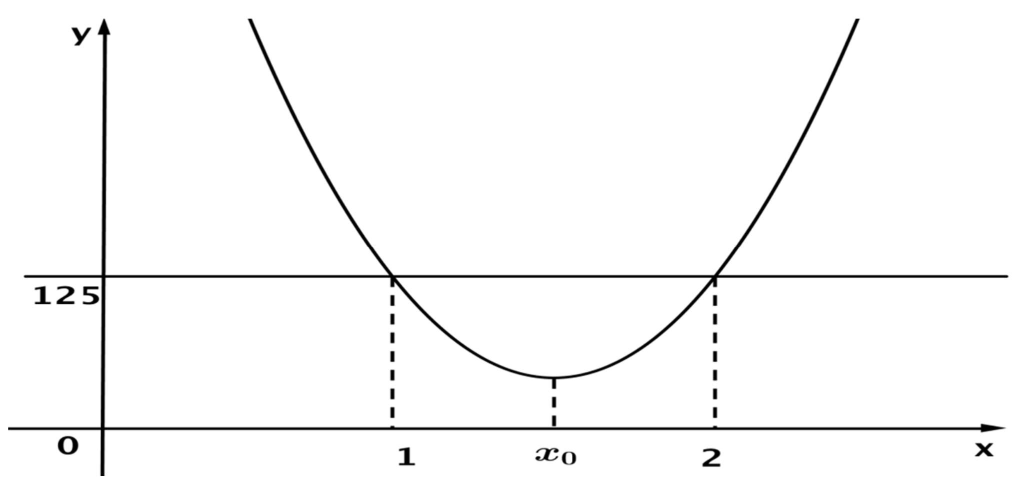
Thus, $f'(x) < 0$ *for* $0 < x \leq 1$, $f(x)$ *is strictly decreasing* $(0,1]$

Also, note that $f'(x)$ *is continuous for* $x \geq 1$. $f'(1) = 24\ln 2 - 81\ln 3 + 90\ln 2 - 90\ln 3 < 0$

$$f'(2) = 192\ln 2 - 15\sqrt{3}\ln 3 + 60\sqrt{3}\ln 2 - \frac{9}{4}\sqrt{3}\ln 3 > 0$$

Thus, $\exists$ *some* $x_0 \in (1,2)$ *such that* $f'(x_0) = 0$. *For* $x \geq 2, 2^{3x} \geq 64, 27^{\frac{1}{x}} > 1, 2^x 3^{\frac{1}{x}} > 4$

$f(x) > 64 + 1 + 60 = 125\ \forall x \geq 2$. *Graph of* $y = f(x), x > 0$ *is as follow.*



Thus, $f(x) = 125$ *has two solutions,* $\alpha = 1$ *and* β *where* $1 < \beta < 2$.

CHAPTER 7

MATRIX.DETERMINANTS-SOLUTIONS

2.1

$$P(x) = det(A - xI_4) = x^4 + ax^3 + bx^2 + cx + d$$

$$det\left(A + \sqrt{3}iI_4\right) \cdot det\left(A - \sqrt{3}iI_4\right) = 0 \Rightarrow P\left(i\sqrt{3}\right) = 0$$

$$\left(i\sqrt{3}\right)^4 + a\left(i\sqrt{3}\right)^3 + b\left(i\sqrt{3}\right)^2 + ci\sqrt{3} + d = 0$$

$$9 - 3a\sqrt{3}i - 3b + ci\sqrt{3} + d = 0, 9 - 3b + d + i\sqrt{3}(c - 3a) = 0$$

$$\Rightarrow c - 3a = 0, 9 - 3b + d = 0$$

$$x^2 + 2x + 2 = 0, \Delta = 4 - 8 = -4, x_1 = \frac{-2+2i}{2} = -1 + i, x_2 = -1 - i$$

$$det(A(-1 + i)I_4) \cdot det(A - (-1 - i)I_4) = 0$$

$$\Rightarrow P(-1 + i) = (-1 + i)^4 + a(-1 + i)^3 + b(-1 + i)^2 + c(-1 + i) + d = 0$$

$$(1 - 2i + i^2)^2 + a(-1 + 3 \cdot i + 3(-1) \cdot i^2 + i^3) + b(1 - 2i + i^2) + c(-1 + i) + d = 0$$

$$-4 + a(-1 + 3i + 3 - i) - 2bi - c + ci + d = 0$$

$$-4 + 2a + 2ai - 2bi - c + ci + d = 0$$

$$(2a - c + d - 4) + i(2a - 2b + c) = 0$$

$$\begin{cases} c = 3a \\ d - 3b = -9 \\ 2a - c + d = 4 \\ 2a - 2b + c = 0 \end{cases} \Rightarrow \begin{cases} d - 3b = -9 \\ 2a - 3a + d = 4 \\ 2a - 2b + 3a = 0 \end{cases} \begin{cases} d - 3b = -9, d = 3b - 9 \\ -a + d = 4 \\ 5a - 2b = 0 \end{cases}$$

$$\begin{cases} -a + 3b - 9 = 4 \\ 5a - 2b = 0 \end{cases} \begin{cases} -a + 3b = 13| \cdot 5 \\ 5a - 2b = 0 \end{cases} \Rightarrow 13b = 13 \cdot 5, b = 5$$

$$5a = 10, a = 2, d = 3b - 9 = 3 \cdot 5 = 9 = 6, c = 6$$

$$\Omega = det\, A = P(0) = d = 6$$

2.2

$$\textit{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \textit{ characteristic equation of } A \textit{ is}$$

$$f(t) = det(A - tI_3) = \begin{vmatrix} a_{11} - t & a_{12} & a_{13} \\ a_{21} & a_{22} - t & a_{23} \\ a_{31} & a_{32} & a_{33} - t \end{vmatrix} = (a_{11} - t)(a_{22} - t)(a_{33} - t) +$$

$$+a_{12}a_{23}a_{31} + a_{13} + a_{21}a_{32} - a_{31}a_{13}(a_{22} - t) - a_{23}a_{32}(a_{11} - t) - a_{12}a_{21}(a_{33} - t)$$

$$= -t^3 + (a_{11} + a_{22} + a_{33})t^2 - t(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{31}a_{13} - a_{23}a_{32} - a_{12}a_{21})$$

$$+ det(A) = -t^3 + (tr\,A)t^2 - t[A_{11} + A_{22} + A_{33}] + det(A)$$

$$det(A^2 + I_3) = 0$$

$$\Rightarrow det\big((A + iI)(A - iI)\big) = 0 \Rightarrow det(A + iI)\,\overline{(A + \imath I)} = 0 \Rightarrow |det(A + iI)|^2 = 0 \Rightarrow$$

$\Rightarrow det(A + iI) = 0 \Rightarrow -i$ *is an eigen value of* A*. Thus,*

$$-(-i)^3 + \big(tr(A)\big)(-i)^2 + i(A_{11} + A_{22} + A_{33}) + det(A) = 0$$

$$\Rightarrow (A_{11} + A_{22} + A_{33} - 1)i - tr(A) + det(A) = 0.$$

Equating real and imaginary parts, we get $det(A) = tr\,(A)$ *and* $tr\,(adjA) = 1$*.*

2.3

Characteristic eq[2] *of 2 by 2 matrix* A *is* $\lambda^2 - \lambda Tr(A) + det(A) = 0 \Rightarrow \lambda^2 - \lambda + 1 = 0$

$\therefore A^2 - A + 1 = 0 \Rightarrow A^3 + I = 0$ *and Characteristic roots are* $\lambda_1 = -\omega, \lambda_1 = -\omega^2$*.*

$det(A^n + I_2) = (\lambda_1^n + 1)(\lambda_2^n + 1) = (1 + (-\omega)^n)(1 + (-\omega^2)^n)$ *&* $det(A^3 + I_2) = 0$

Now, $det(A^4 + I_2) + 10\,det(A^2 + I_2) + x = 4\,det(A^3 + I_2) + 16\,det(A + I_2)$

$$\Rightarrow (1 + \omega^4)(1 + \omega^8) + 10(1 + \omega^2) + x = 4 \times 0 + 16(1 - \omega)(1 - \omega^2) \Rightarrow$$

$$\Rightarrow 1 + 10 + x = 16 \times 3 \therefore x = 37$$

2.4

$$\textit{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R}$$

$$A^2 + 2A + 2I_2 = (A + I_2)^2 + I_2 = (A + I_2 + iI_2)(A + I_2 - iI_2) = (A + I_2 + iI_2)(\overline{A + I_2 + \imath I_2})$$

$$det(A^2 + 2A + 2I_2) = det(A + (1 + i)I_2)\left(\overline{A + (1 + \imath)I_2}\right)$$

$$= det(A + (1 + i)I_2)\,\overline{det(A + (1 + \imath)I_2)} = |det(A + (1 + i)I_2)|^2 =$$

$$= \left\|\begin{matrix} a + (1 + i) & b \\ c & d + (1 + i) \end{matrix}\right\|^2 = |(1 + i)^2 + (a + d)(1 + i) + ad - bc|^2$$

$$= |(a + d + ad - bc) + (2 + a + d)i|^2 \geq \left(2 + (a + d)\right)^2 = (2 + tr\,A)^2$$

2.5

Let $X = AB^{-1}$. *As* $tr(X) = 1$, *we take* $X = \begin{pmatrix} a & b \\ c & 1 - a \end{pmatrix}$

$$1 = det(X) = a(1 - a) - bc$$

$$det(I + AB^{-1}) = det(I + X) = \begin{vmatrix} a + 1 & b \\ c & 2 - a \end{vmatrix} = (a + 1)(2 - a) - bc$$

$$= 2 + a - a^2 - bc = 3$$

Now $det(I + A^{-1}B) = det\{A^{-1}(AB^{-1} + I)B\} =$

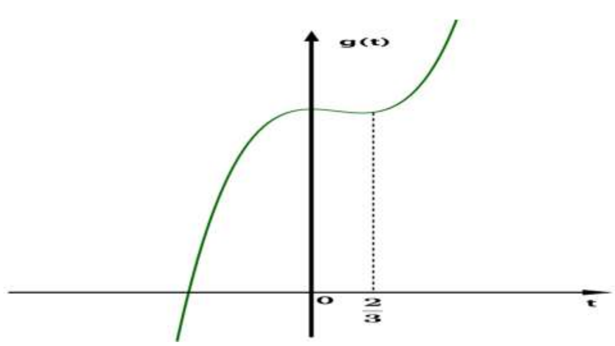
$$= det(A^{-1}\,det(AB^{-1} + I))\,det(B) = (det(A))^{-1}(det\,B)\,det(X)$$

$$= [det(A)\,(det(B))^{-1}]^{-1}(3) = (det(AB^{-1}))^{-1}(3) = (1)(3) = 3$$

2.6

A polynomial satisfied by A *is* $g(t) = t^3 - t^2 + 7$

Graph of $g(t)$ *is as follows:*



This shows $g(t)$ has a negative root and two imaginary roots

Let α be negative root and $\beta + i\gamma, \beta - i\gamma, \beta, \gamma \in \mathbb{R}$ be imaginary roots.

Note tat $\alpha, \beta + i\gamma, \beta - i\gamma$ are distinct eigen values of A.

Also, $g(t) = (t-\alpha)(\alpha - \beta - i\gamma)(t - \beta + i\gamma)$ is minimal polynomial of A.

As minimal polynomial and characteristic polynomials have same zeros, and k is real.

or $k = \alpha(\beta + i\gamma)^2(\beta - i\gamma)^2 < 0$. Also, $det(A) = k$

$\therefore det(A) < 0$ *Similarly* $det(B) < 0 \Rightarrow det(AB) = det(A)\,det(B) > 0$.

2.7

$$A^3 \cdot A^2 = 2I_5 \Rightarrow A^2(A - I_5) = 2I_5 \Rightarrow (det\,A)^2 \cdot det(A - I_5) = 2^5 \Rightarrow det\,A \neq 0 \quad (1)$$

$$A^3 = A^2 + 2I_5 \Rightarrow det\,A^3 = det(A^2 + 2I_5) \Rightarrow (det\,A)^3 = det\left(A + \sqrt{2}iI_5\right)\left(A - \sqrt{2}iI_5\right) \Rightarrow$$

$$(det\,A)^3 = det\left(A + \sqrt{2}iI_5\right) \cdot \overline{det\left(A + \sqrt{2}\imath I_5\right)} \geq 0 \quad (2)$$

From (1)+(2)⇒ $det\,A > 0$ *(3)*

$$B^3 - B^2 = 3I_5 \Rightarrow B^2(B - I_5) = 3I_5 \Rightarrow (det\,B)^2 \cdot det(B - I_5) = 3^5 \Rightarrow det\,B \neq 0 \quad (4)$$

$$B^3 = B^2 + 3I_5 \Rightarrow det\,B^3 = det(B^2 + 3I_5) \Rightarrow (det\,B)^3 = det\left(B + \sqrt{3}iI_5\right)\left(B - \sqrt{3}iI_5\right) \Rightarrow$$

$$\Rightarrow (det\,B)^3 = det\left(B + \sqrt{3}iI_5\right) \cdot \overline{det\left(B + \sqrt{3}\imath I_5\right)} \geq 0 \quad (5)$$

From (4)+(5)⇒ $det(B) > 0$ *(6). From (3)+(6)* $\Rightarrow det(AB) > 0$

2.8

Suppose that A and B satisfy the proposed conditions. Let $C = A^{-1}B$ and let

$$\chi(\lambda) = det(\lambda I_2 - C) = \lambda^2 - tr\,(A)\lambda + det(C)$$

be the characteristic polynomial of C. The proposed inequalities yields

$$\chi(1) = \frac{det(A - B)}{det\,A} < 0$$

$$\chi(-1) = \frac{det(-A-B)}{det\,A} = \frac{det(A+B)}{det\,A} > 0$$

$$\chi(-2) = \frac{det(-2A-B)}{det\,A} = \frac{det(2A+B)}{det\,A} < 0$$

But $\chi(\lambda)$ is positive for large $|\lambda|$, so the above conditions imply the second degree polynomial χ has at least 4 zeros and this is absurd. Thus, no such matrices exist.

2.9

Let $A = (a_{ij})_{4\times 4} \in M_4(\mathbb{C})$ *and* $Tr\,(A) = 0, det(A) \neq 0$.

Let $f(t) = t^4 - \alpha t^3 + \beta t^2 - \gamma t + \delta$ *be the characteristic polynomial of* A.

Then $\alpha = Tr\,(A) = 0$ *and* $\delta = det(A) \neq 0$.

$$\therefore f(t) = t^4 + \beta t^2 - \gamma t + \delta$$

We have

$A^4 = -\beta A^2 + \gamma A - \delta I_4$ *(1)* $\Rightarrow A^3 = -\beta A - \gamma I - \delta A^{-1}$

$Tr\,(A^3) = -\beta Tr(A) + 4\gamma - \delta Tr\,(A^{-1}) = 4\gamma - \delta Tr(A^{-1})$ *(1)*

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *be eigen values of* A, *then*

$$\sum \lambda_i = 0, \sum \lambda_i \lambda_j = B$$

Let λ *be an eigenvalue of* $A \Rightarrow \exists a\ x \neq 0$ *such that* $Ax = \lambda x \Rightarrow$

$$\Rightarrow A^2(x) = A(Ax) = A(\lambda x) = \lambda Ax = \lambda(\lambda x) = \lambda^2 x$$

Similarly, $A^3 = \lambda^3 x \Rightarrow \lambda^3$ *is an eigenvalue of* A^3. *If* A^{-1} *exists, then*

$$A^{-1}(Ax) = A^{-1}(\lambda x) \Rightarrow \lambda^{-1}x = A^{-1}x$$

$\therefore \lambda^{-1}$ *is an eigenvalue of* A^{-1}.

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *eigenvalues of* A, *then* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = Tr\,(A) = 0$.

Now, $Tr(A^3) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 = (\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4)^3 -$

$$-3\lambda_3\lambda_4(\lambda_3 + \lambda_4)$$

$$= (-\lambda_3 - \lambda_4)^3 + 3\lambda_1\lambda_2(\lambda_3 + \lambda_4) + (\lambda_3 + \lambda_4)^3 + 3\lambda_3\lambda_4(\lambda_1 + \lambda_2)$$

$$[\because \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0]$$

$$= 3\lambda_1\lambda_2\lambda_3\lambda_4\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right) = 3\,det(A)\,Tr\,(A^{-1})$$

$$\left[\because \lambda_1\lambda_2\lambda_3\lambda_4 = det(A)\ \ and\ Tr(A^{-1}) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right]$$

$$\sum \lambda_i\lambda_j\lambda_k = \gamma,\ \lambda_1\lambda_2\lambda_3\lambda_4 = \delta$$

Note

$$\gamma = \delta\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right) = det(A)\,Tr\,(A^{-1})\ \ (2)$$

From (1), (2):

$$Tr(A^3) = 3\,det(A)\,Tr\,(A^{-1})$$

2.10

$$A^{2p+1} + B^{2p} = I_n| \cdot A^{2p} \Rightarrow A^{4p+1} + B^{2p} \cdot A^{2p} = A^{2p} \Rightarrow A^{2p} + B^{2p}A^{2p} = A^{2p} \Rightarrow$$

$$\Rightarrow B^{2p}A^{2p} = O_n\ \ (1)$$

$$A^{2p}|A^{2p+1} + B^{2p} = I_n \Rightarrow A^{4p+1} + A^{2p}B^{2p} = A^{2p} \Rightarrow A^{2p}B^{2p} = O_n\ \ \ (2)$$

From (1)+(2) we must show:

$$det(I_nA^{2p} + B^{2p} + A^{2p} \cdot B^{2p}) \geq 0 \Leftrightarrow det[(I_n + A^{2p})(I_n + B^{2p})] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow det(I_n + A^{2p})\,det(I_n + B^{2p}) \geq 0\ \ \ (3)$$

But $det(I_n + A^{2p}) = det(I_n^2 + 2A^{2p}) = det[(I_n + iA^p)(I_n - iA^p)] =$

$$= det[(I_n + iA^p)(\overline{I_n + \imath A^p})] \geq 0\ \ \ (4)$$

Similarly: $det(I_n + B^{2p}) \geq 0\ \ \ (5)$

From (4)+(5)$\Rightarrow det(I_n + A^{2p})\,(I_n + B^{2p}) \geq 0 \Rightarrow$ *(3) is true.*

2.11

The polynomial $P(X,Y) = det(XA + YB)$ *is homogenous of degree* 2, *so it has the form*

$P(X,Y) = aX^2 + bXY + cY^2$. *Testing* $(X,Y) \in \{(1,0),(0,1),(1,1)\}$ *and using the hypothesis*

$det(A+B) = 1$ *we see that* $a = det(A) \triangleq \beta, b = 1 - \alpha - \beta$. *It follows that*

$$det(\beta A + \alpha B) = P(\beta,\alpha) = \alpha\beta^2 + (1-\alpha-\beta)\alpha\beta + \beta\alpha^2 = \alpha\beta = det(ABC)$$

2.12

Let $f(x) = x^2 - x - 1, f(x) = 0 \Rightarrow x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. *Now the own values for* A *is* $\lambda_1, \lambda_2 \Rightarrow$ *from McCoy theorem* $\Rightarrow \lambda_1, \lambda_2 \in \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\} \Rightarrow |\lambda_i| \le \frac{1+\sqrt{5}}{2}, i = 1,2 \Rightarrow$

$$|det\,A| = |\lambda_1\lambda_2| = |\lambda_1| \cdot |\lambda_2| \le \left(\frac{1+\sqrt{5}}{2}\right)^2 \quad (1)$$

Let $\lambda_1, \lambda_2, \lambda_3$ *the own values for* $B \Rightarrow$ *from McCoy theorem* $\Rightarrow \{\lambda_1, \lambda_2, \lambda_3\} \in \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$

$$\Rightarrow |\lambda_i| \le \frac{1+\sqrt{5}}{2}, i = 1,2,3 \Rightarrow |det\,B| = |\lambda_1||\lambda_2||\lambda_3| \le \left(\frac{1+\sqrt{5}}{2}\right)^3 \quad (2)$$

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *the own values for* $C \Rightarrow \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \in \left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\} \Rightarrow |\lambda_i| \le \frac{1+\sqrt{5}}{2}$,

$$i = 1,2,3,4 \Rightarrow |det\,C| = |\lambda_1| \cdot |\lambda_2| \cdot |\lambda_3| \cdot |\lambda_4| \le \left(\frac{1+\sqrt{5}}{2}\right)^4 \quad (3)$$

From (1)+(2)+(3)$\Rightarrow |det\,A + det\,B + det\,C| \le |det\,A| + |det\,B| + |det\,C| \le$

$$\le \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1+\sqrt{5}}{2}\right)^3 + \left(\frac{1+\sqrt{5}}{2}\right)^4 = 7 + 3\sqrt{5} < 28$$

2.13

We use two properties:

(1) $rank\,(\alpha \cdot A) = rank\,A, \forall \alpha \ne 0$ *(obvious)*

(2) $rank\,(A) = rank\,(A \cdot B^{-1}), \forall B =$ *invertible (from Sylvester)*

$$rank\,(AB \cdot det(CD) + CD \cdot det(AB)) = rank(B\,det(CD) + A^{-1}C \cdot D\,det(AB)) =$$

$$= rank\,(det(CD)\,I_n + B^{-1}A^{-1}C \cdot D\,det(AB)) = rank(det(CD)\,D^{-1} + B^{-1}A^{-1}C\,det(AB)) =$$

$$= rank(det(CD)\,D^{-1} \cdot C^{-1} + B^{-1}A^{-1} \cdot det(AB)) =$$

$$= rank(det\,D \cdot D^{-1}\,det\,C \cdot C^{-1} + det\,B^{-1} \cdot det\,A \cdot A^{-1})$$

$$= rank\ (D^*C^* + B^*A^*)\ \ (3)$$

$$Now,\ rank\ \left(\frac{1}{det\,C \cdot det\,D} B^{-1}A^{-1} + \frac{1}{det\,A \cdot det\,B} D^{-1}C^{-1}\right)$$

$$= rank\left(\frac{1}{det\,A\,det\,B\,det\,C\,det\,D} B^*A^* + \frac{1}{det\,A\,det\,B\,det\,C\,det\,D} D^*C^*\right) =$$

$$= rank(B^*A^* + D^*C^*)\ \ (4)$$

From (3) + (4) ⇒ relation from hypothesis.

2.14

$$\Delta_1 = \begin{vmatrix} a+b & ab & 0 & 0 \\ 1 & a+b & ab & 0 \\ 0 & 1 & a+b & ab \\ 0 & 0 & 1 & a+b \end{vmatrix}$$

$$C_4 \to C_4 - (a+b)C_3$$

$$= \begin{vmatrix} a+b & ab & 0 & 0 \\ 1 & a+b & ab & -ab(a+b) \\ 0 & 1 & a+b & ab-(a+b)^2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} a+b & ab & 0 \\ 1 & a+b & -ab(a+b) \\ 0 & 1 & ab-(a+b)^2 \end{vmatrix} =$$

$$= -(a+b)^2[ab-(a+b)^2] - ab(a+b)^2 + ab[ab-(a+b)^2]$$

$$\Delta_1 = (a+b)^4 - 3ab(a+b)^2 + a^2b^2$$

$$\Delta_2 = (b+c)^4 - 3bc(b+c)^2 + b^2c^2$$

$$\Delta_3 = (c+a)^4 - 3ca(c+a)^2 + c^2a^2$$

As $b, c \to a, \Delta_1 + \Delta_2 + \Delta_3 = (16a^4 - 12a^4 + a^4)(3) = 15a^4$. *Also, in this case,*

$$a^2 + a^2 + a^2 = 1 \Rightarrow a^2 = \frac{1}{3} \therefore \Delta_1 + \Delta_2 + \Delta_3 \to 15\left(\frac{1}{3}\right)^2 = \frac{5}{3} < 3 = RHS$$

$$\Delta_1 = \begin{vmatrix} a+b & ab & 0 & 0 \\ 1 & a+b & ab & 0 \\ 0 & 1 & a+b & ab \\ 0 & 0 & 1 & a+b \end{vmatrix}$$

$$C_4 \to C_4 - (a+b)C_3$$

$$= \begin{vmatrix} a+b & ab & 0 & 0 \\ 1 & a+b & ab & -ab(a+b) \\ 0 & 1 & a+b & ab-(a+b)^2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} a+b & ab & 0 \\ 1 & a+b & -ab(a+b) \\ 0 & 1 & ab-(a+b)^2 \end{vmatrix} =$$

$$= -(a+b)^2[ab-(a+b)^2] - ab(a+b)^2 + ab[ab-(a+b)^2]$$

$$\Delta_1 = (a+b)^4 - 3ab(a+b)^2 + a^2b^2 = (a+b)^2(a^2+b^2-4b) + a^2b^2 =$$

$$= (a+b)(a^3+b^3) + a^2b^2 > \left(2\sqrt{ab}\right)\left(2(ab)^{\frac{3}{2}}\right) + a^2b^2 = 5a^2b^2$$

$$\therefore \Delta_1 + \Delta_2 + \Delta_3 > 5a^2b^2 + 5b^2c^2 + 5c^2a^2 \geq \frac{5}{3}(ab+bc+ca)^2 = \frac{5}{3}$$

2.15

Let $A \in M_n(\mathbb{R})$ be an invertible matrix with

$$A^2 + A^{-2} = \alpha(A + A^{-1}), \text{ for some } \alpha \in (-1,1) \qquad (H)$$

Find $|det(A)|$

Step 1. If $\alpha \in (-1,1)$ then all the complex roots of the polynomial

$$P(x) = X^4 - \alpha X^3 - \alpha X + 1$$

belong to the unit circle.

Indeed, $P(z) = 0$ is equivalent to $z^3 = \frac{az-1}{z-\alpha}$ thus

$$|z|^6 - 1 = \left|\frac{az-1}{z-\alpha}\right|^2 - 1 = \frac{(1-\alpha^2)(1-|z|^2)}{|z-\alpha|^2}$$

and consequently

$$(|z|^2 - 1)\underbrace{\left[1 - |z|^2 + |z|^4 + \frac{1-\alpha^2}{|z-\alpha|^2}\right]}_{positive} = 0$$

Thus, $|z| = 1$

Step 2 $|det\, A| = 1$

Consider A as a complex matrix. If $\lambda \in \mathbb{C}$ is an eigenvalue of A then according to (H), λ satisfies

$$\lambda^2 + \frac{1}{\lambda^2} = \alpha\left(\lambda + \frac{1}{\lambda}\right)$$

Equivalently $P(\lambda) = 0$, hence $|\lambda| = 1$ according to Step 1. But $det\,A$ is the product of all the eigenvalues of A, (each one is repeated according to its multiplicity), so $|det\,A| = 1$.

2.16

After simplification we have:

$$\left|\begin{pmatrix} 1 & \sin x + 3 & 3\sin x + 2 & 2\sin x \\ 1 & \cos x + \sin x + 2 & \sin x\cos x + 2\cos x + 2\sin x & \sin 2x \\ 1 & \cos x + \sin x + 1 & \sin x\cos x + \cos x + \sin x & \sin x\cos x \\ 1 & \cos x + 3 & 3\cos x + 2 & 2\cos x \end{pmatrix}\right| =$$

$-\frac{1}{4}(\sin x - 2)(\sin x + \cos x - 1)^2(4\sin x + \cos 2x - 2(\sin x + 2)\cos x + 1)$. *Solve for x:*

$$-\frac{1}{4}(\sin x - 2)(-1 + \cos x + \sin x)^2(1 + \cos 2x + 4\sin x - 2\cos x)(\sin x + 2) = 0$$

Multiply both sides by a constant to simplify the equation. Multiply both sides by -4:

$$(\sin x - 2)(-1 + \cos x + \sin x)^2(1 + \cos 2x + 4\sin x - 2\cos x)(\sin x + 2) = 0$$

Find the roots of each term in the product separately. Split into three equations:

$$\sin x - 2 = 0 \text{ or } (-1 + \cos x + \sin x)^2 = 0 \text{ or}$$

$$1 + \cos 2x + 4\sin x - 2\cos x\,(\sin x + 2) = 0$$

Isolate terms with x to the left hand side. Add 2 to both sides: $\sin x = 2$ or

$(-1 + \cos x + \sin x)^2 = 0$ *or* $1 + \cos 2x + 4\sin x - 2\cos x\,(\sin x + 2) = 0$. *After solving each equation separately and some calculations we have the following solutions*

$$x = \pi\left(\frac{n-7}{4}\right), x = 2\pi n, x = 2\pi n + \frac{\pi}{2}, x = 2\pi n + \frac{\pi}{4}, x = 2\pi n - \frac{3\pi}{4}$$

$$x = 2\pi n - 2i\tanh^{-1}\frac{1}{\sqrt{3}}, x = 2\pi n + 2i\tanh^{-1}\frac{1}{\sqrt{3}}, x = 2\pi n + \pi - \sin^{-1} 2$$

2.17

If $det\,A=0$ *or* $det\,B=0$ *or* $det\,C=0$ *obvious. Let* $det\,A\neq 0, det\,B\neq 0, det\,C\neq 0$.

Lemma 1: $(AB)^*=B^*A^*$ *(1)*

Lemma 2: $(A^*)^*=(det\,A)^{n-2}A$ *(2)*

From $(A^*B^*)^*=BA^{(1)}\Rightarrow((BA)^*)^*=BA\overset{(2)}{\Rightarrow}$

$$\left.\begin{array}{c}(det\,BA)^{n-2}BA=BA\\ BA\ invertible\end{array}\right\}\Rightarrow(det\,BA)^{n-2}=1\Rightarrow$$

$$\Rightarrow\left.\begin{array}{c}det\,BA=\pm1\Rightarrow det\,A\cdot det\,B=\pm1\\ but\ det\,A\ and\ det\,B\in\mathbb{Z}\end{array}\right\}\Rightarrow det\,A, det\,B\in\{-1,1\}\quad (3)$$

Similarly: $det\,B, det\,C\in\{-1,1\}$ *(4)*

From (3)+(4)$\Rightarrow det\,A+det\,B+det\,C\leq 3<\sqrt{10}$

2.18

$$9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX=$$

$$=[3X+(2+i)Y+(2-i)Z][3X+(2-i)Y+(2+i)Z]\Rightarrow$$

$$\Rightarrow det(9X^2+5Y^2+5Z^2+12XY+6YZ+12ZX)=$$

$$=det[(3X+(2+i)Y+(2-i)Z)(3X+(2-i)Y+(2+i)Z)]$$

$$=det\left[(3X+(2+i)Y+(2-i)Z)\overline{(3X+(2+\imath)Y+(2-\imath)Z)}\right]=$$

$$=(det[3X+(2+i)Y+(2-i)Z])\left(det\left(\overline{3X+(2+i)Y}+\overline{(2-i)Z}\right)\right)$$

$$=(det(3X+(2+i)Y+(2-i)Z))\,\overline{det(3X+(2+\imath)Y+(2-\imath)Z)}$$

$$=|det(3X+(2+i)Y+(2-i)Z|^2\geq 0$$

2.19

If X *and* Y *are two* $n\times n$ *matrices, then:* $Tr(XY)=Tr(YX)$

$Tr(X\pm Y)=Tr(X)\pm Tr(Y)$. *We are given:* $Tr((AB)^2)=Tr(A^2B^2)\Rightarrow$

$$\Rightarrow Tr\{ABAB-AABB\}=0\Rightarrow Tr\{A(BA-AB)B\}=0\Rightarrow Tr\{BA(BA-AB)\}=0\quad (1)$$

$$\Rightarrow Tr((BA)^2) = Tr(BA^2B) = Tr(BBA^2) = Tr(B^2A^2) \Rightarrow$$

$$\Rightarrow Tr\{BABA - BBAA\} = 0 \Rightarrow Tr\{B(AB - BA)A\} = 0 \Rightarrow Tr\{AB(AB - BA)\} = 0 \quad (2)$$

Now, $Tr\{(AB - BA)^2\} = Tr\{AB(AB - BA) + BA(BA - AB)\} =$

$= Tr\big(AB(AB - BA)\big) + Tr\big(BA(BA - AB)\big) = 0 + 0 = 0$ *[from (1), (2)]*

Let $x = AB - BA$, *then* $Tr(x) = Tr(AB) - Tr(BA) = 0$.

Also, $Tr(X^2) = 0$ *[Prove above]*

Let $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ $[\because Tr(X) = 0]$

$$X^2 = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix}$$

$$Tr(X^2) = 0 \Rightarrow 2(a^2 + bc) = 0 \Rightarrow a^2 + bc = 0$$

$$\therefore X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow Tr(X^n) = 0 \ \forall n \geq 2$$

2.20

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, a, b, c, d \in \mathbb{Z}$

$A + A^T + A^* = \begin{pmatrix} 2a + d & c \\ b & a + 2d \end{pmatrix} = B_1$ *(say)*

$-A + A^T + A^* = \begin{pmatrix} d & c - 2b \\ b - 2c & a \end{pmatrix} = B_2$ *(say)*

$A - A^T + A^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = B_2$ *(say)*

$A + A^T - A^* = \begin{pmatrix} 2a - d & c + 2b \\ 2c + b & 2d - a \end{pmatrix} = B_4$ *(say)*

$$\therefore det(B_1) + det(B_2) + det(B_3) + det(B_4) =$$

$$= (2a + d)(a + 2d) - bc + ad - (b - 2c)(c - 2b) + ad - bc$$

$$+(2a - d)(2d - a) - (c + 2b)(2c + b)$$

$$= 2a^2 + 5ad + 2d^2 - bc + ad - (5bc - 2c^2 - 2b^2) + ad - bc$$

$$+5ad - 2d^2 - 2a^2 - (2c^2 + 5bc + 2b^2)$$

$= 12(ad - bc)$ *which is divisible by* 12.

2.21

Characteristic eq² of 2 by 2 matrix A *is* $\lambda^2 - \lambda Tr(A) + det(A) = 0 \Rightarrow \lambda^2 - \lambda + 1 = 0$

$\therefore A^2 - A + 1 = 0 \Rightarrow A^3 + I = 0$ *and Characteristic roots are* $\lambda_1 = -\omega, \lambda_1 = -\omega^2$.

$det(A^n + I_2) = (\lambda_1^n + 1)(\lambda_2^n + 1) = (1 + (-\omega)^n)(1 + (-\omega^2)^n)$ & $det(A^3 + I_2) = 0$

Now, $det(A^4 + I_2) + 10\,det(A^2 + I_2) + x = 4\,det(A^3 + I_2) + 16\,det(A + I_2)$

$$\Rightarrow (1 + \omega^4)(1 + \omega^8) + 10(1 + \omega^2) + x = 4 \times 0 + 16(1 - \omega)(1 - \omega^2) \Rightarrow$$

$$\Rightarrow 1 + 10 + x = 16 \times 3 \therefore x = 37$$

2.22

Let $A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ *and* $\alpha, \beta \in \mathbb{C}$

$$\alpha A + \beta B = \begin{pmatrix} \alpha a_1 + \beta a_2 & \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 & \alpha d_1 + \beta d_2 \end{pmatrix}$$

$$det(\alpha A + \beta B) = (\alpha a_1 + \beta a_2)(\alpha d_1 + \beta d_2) - (\alpha b_1 + \beta b_2)(\alpha c_1 + \beta c_2)$$

$$= \alpha^2(a_1 d_1 - b_1 c_1) + \alpha\beta(a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1) + \beta^2(a_2 d_2 - b_2 c_2)$$

Let $det(A) = a = a_1 d_1 - b_1 c_1$; $det(B) = b = a_2 d_2 - b_2 c_2$

$$\therefore det(bA + aB) = b^2 a + ab(a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1) + a^2 b$$

and $det\left(\frac{1}{a}A + \frac{1}{b}B\right) = \frac{1}{a^2}(a) + \frac{1}{ab}(a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1) + \frac{1}{b^2}(b)$

$$= \frac{1}{a} + \frac{1}{b} + \frac{1}{ab}(a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1)$$

Thus, $det(det(A)B + det(B)\,A) + det\left(\frac{1}{det(A)}A + \frac{1}{det(B)}B\right)$

$$= a^2 b + ab^2 + \frac{1}{a} + \frac{1}{b} + \left(ab + \frac{1}{ab}\right)(a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1) \quad (1)$$

Also, $det(A + B) = 1^2 a + 1^2 b + (a_2 d_1 + a_1 d_2 - b_1 c_2 - b_2 c_1) =$

$$= a + b + (a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1)$$

$$\left[\frac{1}{det(AB)} + det(AB)\right] det(A + B) =$$

$$= \left(\frac{1}{ab} + ab\right)[a + b + (a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1)]$$

$$= a^2b + ab^2 + \frac{1}{a} + \frac{1}{b} + \left(ab + \frac{1}{ab}\right)[a_2d_1 + a_1d_2 - b_1c_2 - b_2c_1] \quad (2)$$

From (1), (2): $det(det(A)B + det(B)A) + det\left(\frac{1}{det(A)}A + \frac{1}{det(B)}B\right) =$

$$= \left(det(AB) + \frac{1}{det(AB)}\right) det(A + B)$$

2.23

$$\frac{2}{s^2}\left((s + a^2)\{(s + b^2)(s + c^2) - b^2c^2\} + ab\{abc^2 - ab(s + c^2)\} + \{acb^2 - ac(s + b^2)\}\right) \geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

$$LHS = \frac{2}{s^2}\left((s + a^2)(s^2 + sb^2 + sc^2) + a^2b^2(-s) + a^2c^2(-s)\right)$$

$$= \frac{2}{s^2}\left(s^3 + s^2(a^2 + b^2 + c^2)\right) = 2s + 2(a^2 + b^2 + c^2)$$

$$= (a + b + c) + 2(a^2 + b^2 + c^2)$$

Now, $a + b + c \geq 3\sqrt[3]{abc}\ (AM \geq GM) = 3\sqrt[3]{4RS} \quad (1)$

Now, $2(a^2 + b^2 + c^2) \geq 8\sqrt{3}S \Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 48S^2 = 48s(s - a)(s - b)(s - c)$

$$\Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 48\left(\frac{a + b + c}{2}\right)\left(\frac{b + c - a}{2}\right)\left(\frac{c + a - b}{2}\right)\left(\frac{a + b - c}{2}\right)$$

$$\Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 3((a + b)^2 - c^2)(c^2 - (a - b)^2)$$

$$\Leftrightarrow (a^2 + b^2 + c^2)^2 \geq 3(c^2(a + b)^2 - (a^2 - b^2)^2 - c^4 + c^2(a - b)^2)$$

$$\Leftrightarrow a^4 + b^4 + c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2 \geq 3(2c^2(a^2 + b^2) + 2a^2b^2 - a^4 - b^4 - c^4)$$

$$\Leftrightarrow 4a^4 + 4b^4 + 4c^4 - 4a^2b^2 - 4b^2c^2 - 4c^2a^2 \geq 0$$

$$\Leftrightarrow 2a^4 + 2b^4 + 2c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 \geq 0$$

$$\Leftrightarrow (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2) \geq 0 \text{ which is true } 2(a^2 + b^2 + c^2) \geq 8\sqrt{3}S \quad (2)$$

$$(1) + (2) \Rightarrow a + b + c + 2(a^2 + b^2 + c^2) \geq 8\sqrt{3}S + 3\sqrt[3]{4RS}$$

2.24

$$\sum(1+a)\begin{bmatrix} 1+b & 1 \\ 1 & 1+c \end{bmatrix} - \sum 1\begin{bmatrix} 1 & 1 \\ 1 & 1+c \end{bmatrix} + \sum 1\begin{bmatrix} 1 & 1+b \\ 1 & 1 \end{bmatrix} \geq \frac{48abc}{1+a+b+c}$$

$$\Rightarrow \sum(1+a)[(1+b)(1+c) - 1] - \sum 1[1(1+c) - 1] + \sum 1[1 - (1+b)] \geq \frac{48abc}{1+a+b+c}$$

$$\Rightarrow \sum(1+a)(b+c+bc) - \sum c - \sum b = \sum bc + \sum ab + \sum ac + \sum abc =$$

$$= 3ab + 3ac + 3bc + 3abc \Rightarrow 3(ab + ac + bc + abc) \geq \frac{48abc}{1+a+b+c}$$

Desde que: $a, b, c > 0$, *dividimos la expersión* $\div (abc)$*:*

$$\Rightarrow 3\left(\frac{1}{c} + \frac{1}{b} + \frac{1}{a} + 1\right)(1+a+b+c) \geq 48$$

Por: $MA \geq MH \rightarrow \frac{1+a+b+c}{4} \geq \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1} \rightarrow 3\left(\frac{1}{c} + \frac{1}{b} + \frac{1}{a} + 1\right)(1+a+b+c) \geq 48$

La igualdad se alcanza cuando: $a = b = c = 1$

2.25

$$a\begin{bmatrix} b+d & \sqrt{de}+\sqrt{bf} \\ \sqrt{de}+\sqrt{bf} & c+e+f \end{bmatrix} - \sqrt{ad}\begin{bmatrix} \sqrt{ad} & \sqrt{de}+\sqrt{bf} \\ \sqrt{ae} & c+e+f \end{bmatrix} + \sqrt{ae}\begin{bmatrix} \sqrt{ad} & b+d \\ \sqrt{ae} & \sqrt{de}+\sqrt{bf} \end{bmatrix}$$

$$a\left((b+d)(c+e+f) - \left(\sqrt{de}+\sqrt{bf}\right)^2\right) - \sqrt{ad}\left(\sqrt{ad}(c+e+f) - \sqrt{ae}\left(\sqrt{de}+\sqrt{bf}\right)\right) +$$

$$+\sqrt{ae}\left(\left(\sqrt{ad}\right)\left(\sqrt{de}+\sqrt{bf}\right) - \sqrt{ae}(b+d)\right)$$

$$T_1 = a\left((b+d)(c+e+f) - \left(\sqrt{de}+\sqrt{bf}\right)^2\right)$$

$$T_1 = a\left(bc + be + bf + dc + de + df - de - bf - 2\sqrt{bdfe}\right)$$

$$\Rightarrow T_1 = abc + abe + adc + adf - 2a\sqrt{bdfe},$$

$$T_2 = -\sqrt{ad}\left(\sqrt{ad}(c + e + f) - \sqrt{ae}\left(\sqrt{de} + \sqrt{bf}\right)\right)$$

$$T_2 = -\sqrt{ad}\left(\sqrt{ad}c + \sqrt{ad}e + \sqrt{ad}f - e\sqrt{ad} - \sqrt{aebf}\right) \Rightarrow T_2 = -adc - adf + a\sqrt{bdfe}$$

$$T_3 = \sqrt{ae}\left(\left(\sqrt{ad}\right)\left(\sqrt{de} + \sqrt{bf}\right) - \sqrt{ae}(b + d)\right)$$

$$T_3 = \sqrt{ae}\left(d\sqrt{ae} + \sqrt{abdf} - b\sqrt{ae} - d\sqrt{ae}\right) \Rightarrow T_3 = a\sqrt{bdfe} - abe \rightarrow T_1 + T_2 + T_3 = abc > 0 \Leftrightarrow a, b, c, d, e, f \in <0, \infty>.$$

2.26

$$\Delta\,(x) = \begin{vmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{vmatrix} = (x + a + b + c)\begin{vmatrix} 1 & a & b & c \\ 1 & x & b & c \\ 1 & b & x & c \\ 1 & b & c & x \end{vmatrix} =$$

$$= (x + a + b + c)\begin{vmatrix} 1 & a & b & c \\ 0 & x - a & 0 & 0 \\ 0 & b - a & x - b & 0 \\ 0 & b - a & c - b & x - c \end{vmatrix}$$

$$\Delta(x) = (x + a + b + c)(x - a)(x - b)(x - c)$$

$$ln\,\Delta\,(x) = ln(x + a + b + c) + ln(x - a) + ln(x - b) + ln(x - c)$$

$$\frac{\Delta'(x)}{\Delta(x)} = \frac{1}{x + a + b + c} + \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

$$\frac{\Delta'(a + b + c)}{\Delta(a + b + c)} = \frac{1}{2(a + b + c)} + \frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \leq \frac{1}{6(abc)^{\frac{1}{3}}} + \frac{1}{2}\sum\frac{1}{\sqrt{ab}}$$

2.27

$$det(A_{2n+1}) = a^{2n+1} + b^{2n+1} = \Omega_{2n+1}$$

$$\left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2 \geq 2\ \textit{(True)} \Rightarrow \left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2 - 1 \geq 1 \,|\cdot \left(\frac{a}{b} + \frac{b}{a} > 0\right)$$

$$\left(\frac{a}{b}\right)^3 + \left(\frac{b}{a}\right)^3 \geq \frac{a}{b} + \frac{b}{a} \,|\cdot (x \cdot y > 0)\ \ \left(\frac{a}{b}\right)^3 \cdot xy + \left(\frac{b}{a}\right)^3 \cdot xy \geq \left(\frac{a}{b} + \frac{b}{a}\right)xy$$

$$(x^2+y^2)+\left(\frac{a}{b}\right)^3\cdot xy+\left(\frac{b}{a}\right)^3\cdot xy\geq(x^2+y^2)+\frac{a}{b}\cdot xy+\left(\frac{b}{a}\right)\cdot xy$$

$$(a^3\cdot x+b^3\cdot y)\left(\frac{1}{a^3}\cdot x+\frac{1}{b^3}\cdot y\right)\geq(ax+by)\cdot\left(\frac{1}{a}\cdot x+\frac{1}{b}\cdot y\right)$$

$$\frac{a^3\cdot x+b^3\cdot y}{a\cdot x+b\cdot y}\geq\frac{\frac{1}{a}\cdot x+\frac{1}{b}\cdot y}{\frac{1}{a^3}\cdot x+\frac{1}{b^3}\cdot y}\quad\begin{bmatrix}x=a^{2n+4}\\y=b^{2n+4}\end{bmatrix}$$

$$\frac{a^{2n+7}+b^{2n+7}}{a^{2n+5}+b^{2n+5}}\geq\frac{a^{2n+3}+b^{2n+3}}{a^{2n+1}+b^{2n+1}}\Rightarrow\frac{\Omega_{2n+7}}{\Omega_{2n+5}}\geq\frac{\Omega_{2n+3}}{\Omega_{2n+1}}$$

2.28

Let $P=64\begin{vmatrix}1&b&b&b&b\\a&c&0&0&0\\a&0&d&0&0\\a&0&0&e&0\\a&0&0&0&f\end{vmatrix}$*. Expanding this determinant, we get*

$$P=64(cdef-abdef-abcef-abcdf-abcde).$$

$$P=64abcdef\left(\frac{1}{ab}-\frac{1}{c}-\frac{1}{d}-\frac{1}{e}-\frac{1}{f}\right)$$

$P=(4af)(4be)(4cd)\left(\frac{1}{ab}-\frac{1}{c}-\frac{1}{d}-\frac{1}{e}-\frac{1}{b}\right)$*. By AM-GM:* $\sqrt{af}\leq\frac{a+f}{2}\Rightarrow 4af\leq(a+f)^2$

$$\sqrt{bc}\leq\frac{b+e}{2}\Rightarrow 4be\leq(b+e)^2;\sqrt{cd}\leq\frac{c+d}{2}\Rightarrow 4cd\leq(c+d)^2\Rightarrow$$

$$\Rightarrow P\leq(a+f)^2(b+e)^2(c+d)^2\left(\frac{1}{ab}-\frac{1}{c}-\frac{1}{d}-\frac{1}{e}-\frac{1}{f}\right)$$

2.29

$$LHS=a\times\begin{vmatrix}a&b&c\\c&0&a\\b&a&0\end{vmatrix}+c\times\begin{vmatrix}0&a&c\\b&c&a\\c&b&0\end{vmatrix}-b\times\begin{vmatrix}0&a&b\\b&c&0\\c&b&a\end{vmatrix}=$$

$$=a\{a(-a^2)-b(-ab)+c\cdot ca\}+c\{-a(-ac)+c(b^2-c^2)\}-$$

$$-b\{-a(ab)+b(b^2-c^2)\}=a(-a^3+ab^2+ac^2)+c(a^2c+b^2c-c^3)+$$

$$+b(-a^2b+b^3-bc^2)=a^2(b^2+c^2-a^2)+c^2(a^2+b^2-c^2)+b^2(c^2+a^2-b^2)$$

$$= 2a^2bc\cos A + 2c^2abc\cos C + 2b^2ca\cos B = 2abc\left(\sum a\cos A\right) =$$

$$= 2Rabc(\sin 2A + \sin 2B + \sin 2C) = 2Rabc \cdot 4\sin A\sin B\sin C = 2R \cdot 4Rrs\left(4\frac{abc}{8R^3}\right)$$

$$= 16\frac{R^2rs \cdot Rrs}{R^3} = 16r^2s^2 \overset{s\geq 3\sqrt{3}r}{\geq} 16 \cdot 27r^4 = 432r^4$$

2.30

$$\Delta_2 = \begin{vmatrix} a^2 & b^2 & c^2 \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \\ bc & ca & ab \end{vmatrix}$$

$$R_2 \to R_2 + R_1$$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ bc & ca & ab \end{vmatrix} = \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ abc & abc & abc \end{vmatrix} =$$

$$= (a^2+b^2+c^2)\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = -(a^2+b^2+c^2)\Delta_1$$

$$\Delta_1 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix} =$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a^2+b^2+ab \\ 1 & b^2+c^2+cb \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 1 & a^2+b^2+ab \\ 0 & c^2-a^2+b(c-a) \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(c+a+b)$$

$$\therefore \frac{\Delta_1 - \Delta_2}{(a-b)(b-c)(c-a)}$$

$$= (a+b+c) + (a+b+c)(a^2+b^2+c^2) \geq$$

$$\geq 3(abc)^{\frac{1}{3}} + 3(abc)^{\frac{1}{3}}(a^2+b^2+c^2) = 3(abc)^{\frac{1}{3}}(4)(a^2b^2c^2)^{\frac{1}{4}} = 12(abc)^{\frac{5}{6}}$$

2.31

Let

$X = (x_{ij})_{n\times n}$ *where* $x_{ij} \in \mathbb{C}$

$$\bar{X} = (\bar{x}_{ij})_{n \times n}$$

$$det(X\bar{X}) = det(X)\,det(\bar{X}) = det(X)\,\overline{det(X)} = |det(x)|^2 \geq 0$$

Now,

$$A^2 + AB + 2B^2 = (A + \alpha B)(A + \bar{\alpha}B) = (A + \alpha B)(\overline{A + \alpha B})$$

Where $\alpha = \frac{-1+\sqrt{7}i}{2}$

$\therefore det(A^2 + AB + 2B^2) = |det(A + \alpha B)|^2 \geq 0$. *Similarly*

$$det(A^2 + 2AB + 3B^2) \geq 0$$

and $det(A^2 + 3AB + 4B^2) \geq 0$

$$\therefore det\{(A^2 + AB + 2B^2)(A^2 + 2AB + 3B^2)(A^2 + 3AB + 4B^2)\} \geq 0$$

2.32

$$\begin{vmatrix} \sum a^2 & \sum a^3 & \sum a^4 \\ \sum a^3 & \sum a^4 & \sum a^5 \\ \sum a^4 & \sum a^5 & \sum a^6 \end{vmatrix} = \sum_{cyc} \begin{vmatrix} a^2 & a^3 & a^4 \\ b^3 & b^4 & b^5 \\ c^4 & c^5 & c^6 \end{vmatrix} + \sum_{cyc} \begin{vmatrix} a^2 & a^3 & a^4 \\ c^3 & c^4 & c^5 \\ b^4 & b^5 & b^6 \end{vmatrix}$$

[All the other vanishes after splitting]

$$= a^2b^2c^2(a-b)(b-c)(c-a)\sum bc^2 - a^2b^2c^2(a-b)(b-c)(c-a)\sum b^2c$$

$$= a^2b^2c^2(a-b)(b-c)(c-a)\sum(bc^2 - b^2c)$$

$$= a^2b^2c^2(a-b)^2(b-c)^2(c-a)^2$$

Now, $a^2b^2c^2 \overset{GM-AM}{\leq} \left[\frac{2a+2b+2c}{6}\right]^3 = 1$ *As* $a + b + c = 3$

$$\therefore \begin{vmatrix} \sum a^2 & \sum a^3 & \sum a^4 \\ \sum a^3 & \sum a^4 & \sum a^5 \\ \sum a^4 & \sum a^5 & \sum a^6 \end{vmatrix} \leq (a-b)^2(b-c)^2(c-a)^2$$

2.33

Let $z = \left(a_{ij}\right)_{3\times 3}$ *where* $a_{ij} \in \mathbb{C}$ *and* $\overline{z} = \left(\overline{a_{ij}}\right)_{3\times 3}$*, then*

$$det(z\overline{z}) = det(z)\,det(\overline{z}) = det(z)\,\overline{det(z)} = |det(z)|^2 \geq 0$$

Assuming A, B, C *commute so that*

$$BA = AB = CA = AC = BC = CB = 0$$

Now, let $D = 2A + 3B + 4C$

$$\Rightarrow D^2 = (2A + 3B + 4C)^2 = 4A^2 + 9B^2 + 16C^2$$

Let $E = I + 2A + 3B + 4C + 4A^2 + 9B^2 + 16C^2 = I + D + D^2$

$$= (D - \omega I)(D - \omega^2 I), \omega = \frac{1}{2}\left(-1 + \sqrt{3}i\right) = (D - \omega I)(\overline{D - \omega I})$$

$$det(E) = |det(D - \omega I)|^2 \geq 0$$

2.34

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & -a^2 & b^2 & c^2 & 1 \\ b^2 & a^2 & -b^2 & c^2 & 1 \\ c^2 & a^2 & b^2 & -c^2 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} =$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix} = a^2b^2c^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{vmatrix} =$$

$$= a^2b^2c^2 \cdot \begin{vmatrix} 2 & 2 & 2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 8a^2b^2c^2 \leq \frac{1}{8}(a + b)^2(b + c)^2(a + c)^2$$

$$\Rightarrow 8abc \leq (a + b)(b + c)(a + c)$$

$$a + b \geq 2\sqrt{ab}$$

$$b + c \geq 2\sqrt{bc}$$

$$a + c \geq 2\sqrt{ac}$$

$$\Rightarrow (a+b)(b+c)(a+c) \geq 8abc$$

2.35

Put $a = \tan\alpha, b = \tan\beta, c = \tan\gamma$

$$0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{4}$$

The given determinant becomes

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \tan\alpha & \tan\beta & \tan\gamma \\ \sin^2\alpha & \sin^2\beta & \sin^2\gamma \end{vmatrix}$$

$$= (\sin^2\alpha)(\tan\gamma - \tan\beta) + \sin^2\beta\,(\tan\alpha - \tan\gamma) + \sin^2\gamma\,(\tan\beta - \tan\alpha)$$

As $(\tan\gamma - \tan\beta) + (\tan\alpha - \tan\gamma) + (\tan\beta - \tan\alpha) = 0$

Either two of them non-negative and one is non-positive or one of them is non-negative and two are non-positive.

Case 1

$\tan\alpha - \tan\gamma \geq 0, \tan\beta - \tan\alpha \geq 0, \tan\gamma - \tan\alpha \leq 0$ *then*

$$\Delta \leq \sin^2\beta\,(\tan\alpha - \tan\gamma) + \sin^2\gamma(\tan\beta - \tan\alpha)$$

$$\leq \frac{1}{2}(\tan\alpha - \tan\gamma + \tan\beta - \tan\alpha) \left[\because 0 \leq \beta, \gamma \leq \frac{\pi}{4}\right] \Rightarrow \Delta \leq \frac{1}{2}(\tan\beta - \tan\gamma) \leq \frac{1}{2}$$

$\left[\because 0 \leq \beta, \gamma \leq \frac{\pi}{4}\right]$ *Similarly for other such cases.*

Case 2 $\tan\alpha - \tan\gamma \geq 0, \tan\beta - \tan\alpha \leq 0, \tan\gamma - \tan\alpha \leq 0$

$\therefore \Delta \leq \sin^2\beta\,(\tan\alpha - \tan\gamma) \leq \frac{1}{2}$ *Similarly for other such cases.*

2.36

Let $\hat{a}$ *be unit vector along* $\overrightarrow{OA}$, $\hat{b}$ *along* $\overrightarrow{OB}$ *and* $\hat{c}$ *along* $\overrightarrow{OC}$, *then*

$$\overrightarrow{OA} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = R\hat{a};\ \overrightarrow{OB} = d\hat{\imath} + e\hat{\jmath} + f\hat{k} = R\hat{b};\ \overrightarrow{OC} = g\hat{\imath} + h\hat{\jmath} + i\hat{k} = R\hat{c}$$

Now, $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 = \left[\overrightarrow{OA}\,\overrightarrow{OB}\,\overrightarrow{OC}\right]^2 = \left[R\hat{a}\;R\hat{b}\;R\hat{c}\right]^2 = R^6\left[\hat{a}\hat{b}\hat{c}\right]^2$*. But* $\left[\hat{a}\hat{b}\hat{c}\right] = \pm$ *volume of*

parallelipiped with sides $\hat{a}, \hat{b}, \hat{c} \Rightarrow \left[\hat{a}\hat{b}\hat{c}\right]^2 \le 1 \therefore \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}^2 \le R^6$

2.37

$$\begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & a^2 & b^2 \\ 0 & 1 & 1 & 1 \\ 0 & a^2 & -a^2 & c^2-b^2 \\ 0 & b^2 & c^2-a^2 & -b^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & -a^2 & c^2-b^2 \\ b^2 & c^2-a^2 & -b^2 \end{vmatrix} =$$

$$a^2b^2 + a^2c^2 - a^4 + b^2c^2 - b^4 + a^2b^2 + a^2b^2 - (c^2-a^2)(c^2-b^2) =$$

$$= 2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \quad (1)$$

From (1) we must show this:

$$2(a^2b^2 + b^2c^2 + a^2c^2) - (a^4 + b^4 + c^4) \le 4abcR\sqrt{\left(\sum \sin 2A\right)\sum \cos^2 A} \quad (2)$$

From Cauchy inequality $\Rightarrow \sqrt{\sum \sin^2 A} \ge \frac{1}{\sqrt{3}}\left(\sum \sin A\right)$ *and* $\sqrt{\sum \cos^2 A} \ge \frac{1}{\sqrt{3}}\left(\sum \cos A\right)$ *(3)*

From (2)+(3) we must show this:

$$2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \ge \frac{4}{3}abcR\left(\sum \sin A\right)\left(\sum \cos A\right) \; (4)$$

But $\sum \sin A = \frac{a+b+C}{2R}$ *(5)*

$$\sum \cos A = \sum \frac{b^2+c^2-a^2}{2bc} = \frac{\sum a(b^2+c^2-a^2)}{2abc} \quad (6)$$

From (4)+(5)+(6) we must show this: $2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4) \ge$

$$\ge \frac{1}{3}(a+b+c)(ab^2 + ac^2 - a^3 + ba^2 + bc^2 - b^3 + ca^2 + cb^2 - c^3) \Leftrightarrow$$

$$6(a^2b^2 + a^2c^2 + b^2c^2) - 3(a^4 + b^4 + c^4) \ge -a^4 - b^4 - c^4 + a^3(b+c) +$$

$$+b^3(a+c) + c^3(a+b) - a(b^3+c^3) - b(a^3+c^3) - c(a^3+b^3) +$$

$$+a^2(b^2+c^2) + b^2(a^2+c^2) + c^2(a^2+b^2) + abc(b+c) + abc(a+c) + abc(a+b)$$
$$\Leftrightarrow 2(a^4 + b^4 + c^4) - 4(a^2b^2 + b^2c^2 + a^2c^2) + 2abc(a+b+c) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow a^4 + b^4 + c^4 - 2(a^2b^2 + b^2c^2 + a^2c^2) + abc(a + b + c) \geq 0 \quad (7)$$

$$\Leftrightarrow a^4 + b^4 + c^4 + abc(a + b + c) \geq 2(a^2b^2 + b^2c^2 + a^2c^2) \quad (8)$$

By Schur's inequality we have:

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \quad (9)$$

From (8)+(9) we must show:

$$ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 2(a^2b^2 + b^2c^2 + a^2c^2) \quad (10)$$

But $ab(a^2 + b^2) \geq 2a^2b^2 \Leftrightarrow a^2 + b^2 \geq 2ab$ *which its true. Similarly:*

$bc(b^2 + c^2) \geq 2bc$ and $ac(a^2 + c^2) \geq 2a^2c^2 \Rightarrow$ *(10) its true.*

2.38 *We make a generalization:*

Lemma 1: Let $P \in R[x], p(x) = x^2 + ax + b, \Delta = b^2 - 4b < 0$. *Then* $\forall A, B \in M_n(\mathbb{R})$ *the following statement is true:* $det[(A + x_1B + x_2C)(A + x_2B + x_1C)] \geq 0, x_1, x_2$ *being the roots of* p

Demonstration: $If\ \Delta < 0 \Rightarrow x_1, x_2 \in \mathbb{C}, x_2 = x_1$ *and using* $det(x \cdot \bar{x}) \geq 0$,

$$\forall x \in M_n(\mathbb{R}) \Rightarrow$$

$$det[(A + x_1B + x_2C)(A + x_2B + x_1C)] = det[(A + x_1B + x_2C)(\overline{A + x_1B + x_2C})] \geq 0$$

Lemma 2. If $AB = BA, AC = CA, BC = CB$ *then the conclusion of this theorem can be written this way:* $det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 - 2b)BC] \geq 0$

Demonstration: $det[(A + x_1B + x_2C)(A + x_2B + x_1C)] =$

$$= det[A^2 + x_1x_2(B^2 + C^2) + (x_1 + x_2)(AB + AC) + (x_1^2 + x_2^2)BC] =$$

$$= det[A^2 + b(B^2 + C^2) - a(AB + AC) + (a^2 - 2b)BC] \geq 0$$

(we used $AB = BA, AC = CA, BC = CB$ *and Viéte relations)*

Now, in our case: $a = 6, b = 10$. *Done.*

2.39

$$\Omega = sin^2 x\,(sin^4 x - cos^4 x\, sin^2 y\, cos^2 y) -$$

$$- sin^2 y\, cos^2 x\,(sin^2 x\, cos^2 x\, cos^2 y - cos^4 x\, sin^4 y) +$$

$$+ \cos^2 x \cos^2 y \left(\cos^4 x \cos^4 y - \sin^2 x \cos^2 x \sin^2 y\right) =$$

$$\overset{(1)}{=} \sin^6 x + \cos^6 x \sin^6 y + \cos^6 x \cos^6 y - 3 \sin^2 x \cos^4 x \sin^2 y \cos^2 y =$$

$$= a^3 + b^3 + c^3 - 3abc(a = \sin^2 x\,, b = \cos^2 x \sin^2 y\,, c = \cos^2 x \cos^2 y)$$

$$= \frac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$$

$$= \frac{1}{2}(\sin^2 x + \cos^2 x \sin^2 y + \cos^2 x \cos^2 y)\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \geq 0$$

$\therefore \Omega \overset{(a)}{=} 0$. *Also,* $\because 3 \sin^2 x \cos^4 x \sin^2 y \cos^2 y \geq 0, \therefore$ *by (1),*

$$\Omega \leq \sin^6 x + \cos^6 x \sin^6 y + \cos^6 x \cos^6 y = \sin^6 x + \cos^6 x \left(\sin^6 y + \cos^6 y\right) \leq$$

$$\leq \sin^6 x + \cos^6 x \ \ (\because \sin^6 y + \cos^6 y \leq \sin^2 y + \cos^2 y = 1 \ \& \cos^6 x \geq 0)$$

$$\leq \sin^2 x + \cos^2 x = 1 \therefore \Omega \overset{(1)}{\leq} 1$$

(a), (b) $\Rightarrow 0 \leq \Omega \leq 1 \Rightarrow -1 < \Omega \leq 1 \Rightarrow |\Omega| \leq 1$ *(Proved)*

2.40

$$x^2 - \alpha x + \alpha^2 = 0, \Delta = \alpha^2 - 4\alpha^2 = -3\alpha^2 < 0$$

$$x_1 = \frac{\alpha + \alpha\sqrt{3}i}{2}, x_2 = \frac{\alpha - \alpha\sqrt{3}i}{2}$$

$$\det(A + Bx) = x^2 \det B + u \cdot x + \det A = x^2 \det B + ux + \alpha$$

$$\det(A + i\alpha B) = (i\alpha)^2 \det B + ui\alpha + \alpha = (-\alpha^2 \det B + \alpha) + ui\alpha = 0 \Rightarrow u\alpha = 0 \Rightarrow u = 0$$

$$\Rightarrow -\alpha^2 \det B + \alpha = 0 \Rightarrow \alpha \cdot \det B = 1$$

$$\Omega = \det(A^2 - \alpha AB + \alpha^2 B^2) = \det(A - x_1 B) \cdot \det(A - x_2 B)$$

$$\Omega = (x_1^2 \det B + \alpha) \cdot (x_2^2 \det B + \alpha) = (x_1 x_2)^2 \cdot \det{}^2 B + \alpha^2 + \alpha \det B \left(x_1^2 + x_2^2\right) =$$

$$= \alpha^4 \cdot \det^2 B + \alpha^2 + 1(-\alpha^2) = \alpha^2 \cdot (\alpha \det B)^2 = \alpha^2 \cdot 1^2 = \alpha^2$$

Vieté relationships $x_1 + x_2 = \alpha, x_1 x_2 = \alpha^2, x_1^2 + x_2^2 = S^2 - 2P = \alpha^2 - 2\alpha^2 = -\alpha^2$

2.41

We will use the following formula:

$det(A + xB) = ax^2 + bx + c$, *when:* $a = det\,B$, $b = tr(AB^*)$, $c = det\,A$

We will note $p(x) = det(A + xB)$. *Because* p *is a polygon of second degree,*

it's obvious that it can be at most two changes in the value of $sgn\,(p(x))$. *But:*

$p(-1) > 0, p(0) < 0, p\left(\frac{1}{2}\right) > 0, p(1) < 0 \Rightarrow 3$ *changes of sign.*

That means there are no matrices with the properties in the hypothesis. Observation:

$$det(2A + B) = 4\,det\left(A + \frac{1}{2}B\right) = 4p\left(\frac{1}{2}\right) > 0 \Rightarrow p\left(\frac{1}{2}\right) > 0$$

CHAPTER 8

LIMITS.SERIES-SOLUTIONS

3.1

$$\frac{1}{3}\frac{(i+3)-(i)}{i(i+1)(i+2)(i+3)}=\frac{1}{3}\left(\frac{1}{i(i+1)(i+2)}-\frac{1}{(i+1)(i+2)(i+3)}\right)$$

$$Si: i=1\rightarrow\frac{1}{3}\left(\frac{1}{1\times2\times3}-\frac{1}{2\times3\times4}\right)$$

$$\Rightarrow i=2\rightarrow\frac{1}{3}\left(\frac{1}{2\times3\times4}-\frac{1}{3\times4\times5}\right)$$

$$\Rightarrow i=3\rightarrow\frac{1}{3}\left(\frac{1}{3\times4\times5}-\frac{1}{4\times5\times6}\right)$$

....

$$\Rightarrow i=n\rightarrow\frac{1}{3}\left(\frac{1}{k(k+1)(k+2)}-\frac{1}{(k+1)(k+2)(k+3)}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)(i+3)}=\frac{1}{3}\left(\frac{1}{6}-\frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$\Rightarrow\frac{1}{18}-\lim_{n\to\infty}\frac{1}{3}\frac{1}{(n+1)(n+2)(n+3)}=\frac{1}{18}$$

3.2

Let β be a natural number. $\beta^2<x\le(\beta+1)^2\Rightarrow\left[\sqrt{x}\right]=\beta+1$

Let n be a perfect square, and $1\le x\le n^2$*. Divide the interval* $(1,n^2)$ *into the partition:*

$$(1,4]\cup(4,9]\cup\ldots\cup(k^2,(k+1)^2]\cup\ldots\cup((n-1)^2,n^2)$$

Now:

$$\int_{j^2}^{(j+1)^2}\left[\sqrt{x}\right]dx=\int_{j2}^{(j+1)^2}(j+1)\,dx=(2j+1)(j+1)=2j^2+3j+1$$

$$\int_{1}^{n^2} [\sqrt{x}]\, dx = \sum \int_{j^2}^{(j+1)^2} [\sqrt{x}]\, dx; 1 \leq j \leq n-1$$

Let $n - 1 = m$

$$\sum_{1}^{m} \int_{j^2}^{(j+1)^2} [\sqrt{x}]\, dx = \sum_{1}^{m} 2j^2 + 3j + 1 = \frac{2m^3 + 3m^2 + m}{3} + \frac{3m(m+1)}{2} + m$$

$$= \frac{4m^3 + 15m^2 + 17m}{6}$$

$$n \int_{1}^{n^2} [\sqrt{x}]\, dx = (m+1) \cdot \frac{4m^3 + 15m^2 + 17m}{6} = n \cdot \frac{4(n-1)^3 + 15(n-1)^2 + 17(n-1)}{6}$$

But n^3 *is also a perfect square since* n *is a perfect square.*

$$\Rightarrow \int_{1}^{n^3} [\sqrt{x}]\, dx = \sum 2j^2 + 3j + 1\, ; 1 \leq j \leq n^{\frac{3}{2}} - 1$$

$$\int_{1}^{n^3} [\sqrt{x}]\, dx = \frac{4\left(n^{\frac{3}{2}} - 1\right)^3 + 15\left(n^{\frac{3}{2}} - 1\right)^2 + 17\left(n^{\frac{3}{2}} - 1\right)}{6}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n \cdot \int_1^{n^2} [\sqrt{x}]\, dx}{\int_1^{n^3} [\sqrt{x}]\, dx} = \lim_{n\to\infty} \frac{n\big(4(n-1)^3 + 15(n-1)^2 + 17(n-1)\big)}{4\left(n^{\frac{3}{2}} - 1\right)^3 + 15\left(n^{\frac{3}{2}} - 1\right)^2 + 17\left(n^{\frac{3}{2}} - 1\right)}$$

Applying De l'Hôpital's rule four times yields:

$$\lim_{n\to\infty} \frac{n \cdot \int_1^{n^2} [\sqrt{x}]\, dx}{\int_1^{n^3} [\sqrt{x}]\, dx} = 0$$

when n *is a perfect square.*

But for every real number R*, there exists a perfect square* n *such that* $R < n$*.*

$$\lim_{R\to\infty}\frac{R\cdot\int_1^{R^2}\left[\sqrt{x}\right]dx}{\int_1^{R^3}\left[\sqrt{x}\right]dx}=\lim_{n\to\infty}\frac{n\cdot\int_1^{n^2}\left[\sqrt{x}\right]dx}{\int_1^{n^3}\left[\sqrt{x}\right]dx}=0$$

3.3

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}e^{-\left(\frac{k}{n}\right)^2}}=\lim_{n\to\infty}\sqrt[n]{e^{-\left(\left(\frac{1}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right)}}=$$

$$=\lim_{n\to\infty}\frac{1}{e^{\left(\left(\frac{1}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right)\cdot\frac{1}{n}}}=\frac{1}{e^{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}}=\frac{1}{e^{\lim_{n\to\infty}\frac{1}{n}}\cdot\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2}=\frac{1}{e^{\int_0^1 x^2dx}}=\frac{1}{\sqrt[3]{e}}$$

3.4

$$Let\ P_n=\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{3}{2}}}\Rightarrow\Omega=\lim_{n\to\infty}(P_{n+1}-P_n)$$

$$=\lim_{n\to\infty}\frac{P_{n+1}-P_n}{n+1-n}\overset{Stolz}{=}\lim_{n\to\infty}\frac{P_n}{n}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\dots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}}=\lim_{n\to\infty}\frac{\sqrt{n}\,n!\,(n!)^{\frac{1}{n}}}{e^{\frac{n}{2}}(1!\,2!\dots n!)^{\frac{2}{n}}}=\lim_{n\to\infty}\sqrt[n]{\frac{\left(\sqrt[n]{n}\right)^n n!\,(n!)^n}{e^{\frac{n^2}{2}}(1!\,2!\dots n!)^2}}=$$

$$\overset{Cauchy-D'Alembert}{=}\lim_{n\to\infty}\frac{\dfrac{\left(\sqrt{n+1}\right)^{n+1}(n+1)!\,[(n+1)!]^{n+1}}{e^{\frac{(n+1)^2}{2}}(1!\,2!\dots(n+1)!)^2}}{\dfrac{\sqrt{n}^{\,n}n!\,(n!)^n}{e^{\frac{n^2}{2}}(1!\,2!\dots n!)^2}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{n+1}\left(\sqrt{1+\frac{1}{n}}\right)^n(n+1)(n+1)!\,(n+1)^n}{e^{\frac{n+1}{2}}(n+1)!^2}=\lim_{n\to\infty}\frac{e^{\frac{1}{2}}\sqrt{n+1}(n+1)^n}{e^{\frac{1}{2}}e^n n!}\cdot 1$$

(Stirling from $1=\lim_{n\to\infty}\frac{n!}{\sqrt{2n\pi n}n^n e^{-n}}$*)*

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}(n+1)^n}{e^n n!} \cdot \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \sqrt{1+\frac{1}{n}} \left(1+\frac{1}{n}\right)^n = \frac{e}{\sqrt{2\pi}}$$

3.5

$$S_a = \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)(n+2)\cdot \ldots \cdot (n+a)}\right) \Rightarrow$$

$$\Rightarrow \frac{1}{a}\left(\frac{1}{n(n+1)\ldots(n+a-1)} - \frac{1}{(n+1)\ldots(n+a)}\right)$$

$\Rightarrow \frac{1}{a}\left(S_{a-1} - \sum_{n=1}^{\infty}\left(\frac{1}{(n+1)\ldots(n+a)}\right)\right)$*.Replace,* $n+1 = k \Rightarrow$

$$\Rightarrow \frac{1}{a}\left(S_{a-1} - \left(\sum_{k=1}^{\infty}\left(\frac{1}{k(k+1)-(k+a-1)}\right) - \frac{1}{a!}\right)\right) \Rightarrow \frac{1}{a}\left(S_{a-1} - S_{a-1} + \frac{1}{a!}\right)$$

$$S_a = \frac{1}{a\cdot a!} \quad \Omega(a) = S_a = \frac{1}{a\cdot a!}$$

Now, applying this initial problem, and applying A.M.-G.M., we get,

$$\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c} = \left(\frac{b\left(\frac{1}{a\cdot a!}\right) + c\left(\frac{1}{b\cdot b!}\right) + a\left(\frac{1}{c\cdot c!}\right)}{a+b+c}\right)^{a+b+c} \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{a^b b^c c^a (a!)^b (b!)^c (c!)^a}$$

$$\left(\frac{b\Omega(a) + c\Omega(b) + a\Omega(c)}{a+b+c}\right)^{a+b+c} \geq \frac{1}{a^b b^c c^a (a!)^b (b!)^c (c!)^a}$$

3.6

Let $a_k = \left(\frac{1}{1!2!3!\cdot\ldots\cdot k!}\right)^{\frac{1}{k}}$

We show that

$$a_k \leq \frac{2^k - 1}{k!} \;\forall\, k \geq 1$$

For $k = 1$, $a_k = 1 \leq \frac{2^1-1}{1!}$

Assume that

$$a_k = \frac{1}{k!}(2^k - 1) \textit{ for some } k \in \mathbb{N}$$

$$\Rightarrow \frac{1}{1!\,2!\cdot\ldots\cdot k!} \le \left(\frac{1}{k!}\right)^k (2^k-1)^k = \frac{1}{1!\,2!\cdot\ldots\cdot k!\,(k+1)!}$$

$$\le \left(\frac{1}{k!}\right)^k (2^k-1)^k \frac{1}{(k+1)!}$$

We now show that

$$\left(\frac{1}{k!}\right)^k (2^k-1)^k \frac{1}{(k+1)!} \le \left(\frac{1}{(k+1)!}\right)^{k+1} (2^{k+1}-1)^{k+1}$$

$$\Leftrightarrow \left(\frac{(k+1)!}{k!}\right)^k (2^k-1)^k \le (2^{k+1}-1)^{k+1} \Leftrightarrow (k+1)^k(2^k-1)^k \le (2^{k+1}-1)^{k+1} \quad (1)$$

$$LHS$$

$$(k+1)^k(2^k-1) \le (2^{k+1}-1)^k(2^{k+1}-1) = (2^{k+1}-1)^{k+1}$$

$$[\therefore k+1 \le 2^{k+1}-1\ \forall k \in \mathbb{N}]$$

$\therefore$ (1) *is true*

Thus, $a_k \le \frac{1}{k!}(2^k-1) \Rightarrow \sum_{k=1}^n a_k \le \sum_{k=1}^n \frac{1}{k!}(2^k-1)$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n} a_k \le e^2 - e = e(e-1)$$

3.7.

$$x - \frac{x^3}{6} < \sin x\,,\ \sqrt[n]{x} - \frac{\sqrt[n]{x^3}}{6} < \sin\left(\sqrt[n]{x}\right)$$

$$\left.\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right|_0^{\left(\frac{\pi}{2}\right)^n} - \frac{1}{6}\cdot\left.\frac{x^{\frac{3}{n}+1}}{\frac{3}{n}+1}\right|_0^{\left(\frac{\pi}{2}\right)^n} < \int_0^{\left(\frac{\pi}{2}\right)^n} \sin\left(\sqrt[n]{x}\right)dx$$

$$\frac{\frac{\left(\frac{\pi}{2}\right)^{n+1}}{\frac{n+1}{n}}}{}-\frac{1}{6}\cdot\frac{\left(\frac{\pi}{2}\right)^{n+3}}{\frac{n+3}{n}}<\int_0^{\left(\frac{\pi}{2}\right)^n} sin\left(\sqrt[n]{x}\right)dx$$

$$\infty = \lim_{n\to\infty}\left(\frac{\pi}{2}\right)^{n+1}\cdot\left(\frac{1}{\frac{n+1}{n}}-\frac{\left(\frac{\pi}{2}\right)^2}{\frac{n+3}{n}}\right)\leq L \Rightarrow L = \infty$$

3.8. *Let:*

$$a = \frac{x\,log\,2}{1!}+\frac{x^2(log\,2)^2}{2!}+\cdots+\frac{x^n(log\,2)^n}{n!} \quad , b = \frac{x^{n+1}(log\,2)^{n+1}}{(n+1)!}+\cdots$$

$$a+b=\sum_{k=1}^{\infty}\frac{(x\,log\,2)^k}{k!}=e^{x\,log\,2}-1=2^x-1$$

$$\Rightarrow (a+b)^n = (2^x-1)^n,\ \Omega_n = lim_{x\to 0}\left[\frac{1}{(a+b)^n}-\frac{1}{a^n}\right] = lim_{x\to 0}\left[\frac{-\binom{n}{1}a^{n-1}b-\binom{n}{2}a^{n-2}b^2-\cdots-b^n}{(a+n)^n a^n}\right]$$

Coefficient of x^{2n} *in* $a^{n-1}b$ *in the numerator:*

$$-\frac{(log\,2)^{n-1}(log\,2)^{n+1}}{(n+1)!}$$

Coefficient of x^{2n} *in* $(a+b)^n a^n$ *is:*

$$\frac{(log\,2)^n(log\,2)^n}{(1!)^n(1!)^n}$$

Also, on the terms except first in the numerator involve x^{3n} *and higher powers of* x*.*

$$\Omega_n = -\binom{n}{1}\frac{1}{(n+1)!}\ \forall\, n\geq 1, \qquad \Omega_n = -\frac{(n+1-1)}{(n+1)!}$$

$$\Omega_n = -\left(\frac{1}{n!}-\frac{1}{(n+1)!}\right)\Rightarrow\sum_{n=1}^{\infty}\Omega_n = -\left(\frac{1}{1!}-\frac{1}{2!}\right)-\left(\frac{1}{2!}-\frac{1}{3!}\right)\ldots\Rightarrow\Omega = -1$$

3.9

Let be the sequence $f_0 = 0; f_1 = 1; f_{n+2} = f_n + f_{n+1}$ *with the characteristic equation* $\lambda^2-\lambda-1=0$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}; f_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_0 = \alpha + \beta = 0; f_1 = \alpha \cdot \frac{1+\sqrt{5}}{2} + \beta \cdot \frac{1-\sqrt{5}}{2} = 1$$

$$\alpha = \frac{2}{\sqrt{5}}; \beta = -\frac{2}{\sqrt{5}};\ f_n = \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$A = \begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix}$ *and we prove by induction that:* $A^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}; n \geq 1$

$$A^{n+1} = A^n \cdot A = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_n & f_{n-1} + f_n \\ f_{n-1} + f_n & f_n + f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix}$$
$$= \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

$$L = \lim_{n\to\infty}\left(\frac{a_n + b_n}{c_n + d_n}\right) = \lim_{n\to\infty}\frac{f_n + f_{n+1}}{f_{n+1} + f_{n+2}} = \lim_{n\to\infty}\frac{f_{n+2}}{f_{n+3}} =$$

$$= \lim_{n\to\infty}\frac{\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3}} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2} < 1$$

3.10

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}} = \lim_{n\to\infty}\sqrt[n]{e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^2}\right]}} =$$

$$= \lim_{n\to\infty} e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^3}\right]} = e^{-\lim_{n\to\infty}\left(\left[\frac{n(n+1)(2n+1)}{6n^3}\right]\right)} = e^{-\frac{1}{3}}$$

3.11

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \cdots + \frac{1}{n^{4032}}\right)\ln n$$

We have, $\frac{1}{1+n^{2016}} < 1, \frac{1}{2+n^{2016}} < \frac{1}{2}, \ldots, \frac{1}{n^{4032}} < \frac{1}{n}$

$$\textit{Similarly, } -1 < \frac{1}{1+n^{2016}}, -\frac{1}{2} < \frac{1}{2+n^{2016}}, \dots, -\frac{1}{n} < \frac{1}{n^{4032}}$$

$$\therefore -\frac{1}{n} - \dots - \frac{1}{2} - 1 < \frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \dots + \frac{1}{n^{4032}} < 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

$$\therefore -\ln n\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) < \ln\left(\frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \dots + \frac{1}{n^{4032}}\right) < \ln n\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$$

$$-\ln n\,(\gamma_n + \ln n) < \Omega < \ln n\,(\gamma_n + \ln n) \textit{ where } \gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$$

$|\Omega - \ln n\,(\ln n + \gamma_n)| < \varepsilon$ *will hold only if there exists a* k *such that*

$k = [\ln^2 n] + [\ln n \cdot \gamma_n]$ *where* $[x]$ *=gif function*

$$\therefore \lim_{n\to\infty} \Omega = \lim_{n\to\infty} \ln n\,(\ln n + \gamma_n) = \lim_{n\to\infty} (\ln n)^2 + \gamma \lim_{n\to\infty} \ln n$$

where γ *= Euler's Constant*

$$\therefore \lim_{n\to\infty}\left(\frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \dots + \frac{1}{n^{4082}}\right)\ln n =$$

$$= \lim_{n\to\infty} (\ln n)^2 + \gamma \lim_{n\to\infty} \ln n = +\infty$$

3.12

$$\Omega = \lim_{n\to\infty} n\tan\left(e \cdot \frac{\pi}{5} \cdot n!\right) = \lim_{n\to\infty} n\tan\left(\frac{\pi}{5} \cdot n! \cdot \sum_{k=0}^{n+1} \frac{1}{k!} + \frac{\pi}{5} \cdot n! \cdot \sum_{k=n+2}^{\infty} \frac{1}{k!}\right) =$$

$$= \lim_{n\to\infty} n\tan\left(m\pi + \frac{\pi}{5(n+1)}\right) = \lim_{n\to\infty} n\tan\left(\frac{\pi}{5(n+1)}\right) = \frac{\pi}{5}$$

Observation:

$$0 < n! \sum_{k=n+2}^{\infty} \frac{1}{k!} = n!\left(\frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots\right) =$$

$$= \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots < \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \dots$$

$$< \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \dots < \frac{2}{n+1}$$

$$L = \frac{\pi}{5}$$

3.13

$$\lim_{n\to\infty}(a_{n+1} - a_n) = a$$

and

$$b_n = \lim_{n\to\infty} \sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right) = \lim_{n\to\infty}\frac{c_{n+1} - c_n}{n+1-n}$$

where $c_n = \frac{a_n\cdot b_n}{n}$ *for all* $n \geq 1$

$$= \lim_{n\to\infty}\frac{c_n}{n}\ [applying\ Reverse\ Cesaro - Stolz] = \lim_{n\to\infty}\frac{a_n\cdot b_n}{n^2}$$

$$= \left(\lim_{n\to\infty}\frac{a_n}{n}\right)\left(\lim_{n\to\infty}\frac{b_n}{n}\right) = \left(\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}\right)\left(\lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}}\right)$$

$$= \lim_{n\to\infty}(a_{n+1}-a_n)\cdot\lim_{n\to\infty}\left(\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n+1]{(n+1)!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}\right)$$

Applying Cauchy D – Alembert's Theorem

$$= a\cdot\lim_{n\to\infty}\left(\frac{n^n}{(n+1)^{n+1}}\cdot\sqrt[n+1]{(n+1)!}\right)$$

$$= a\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}}{\left(1+\frac{1}{n}\right)^n} = \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{(n+2)!}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right)$$

(Cauchy D-Alembert's Theorem)

$$= \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n+2}\cdot\frac{(n+1)^{n+1}}{(n+2)^{n+1}}\right) = \frac{a}{e}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{a}{e^2}$$

3.14

$$\textit{We know } (2n+1)!! = \frac{(2n+1)!}{2^n \cdot n!}$$

$$\Omega = \lim_{n\to\infty} \sqrt[n^2]{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{c_n}$$

where

$$c_n = \sqrt[n]{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n]{(2n-1)!!}}$$

$$\therefore \Omega = \lim_{n\to\infty}\sqrt[n]{c_n} = \lim_{n\to\infty}\frac{c_{n+1}}{c_n}\ [Cauchy\ D-Alembert] = \lim_{n\to\infty}\frac{\sqrt[n+1]{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n]{(2n+1)!!}}}{\sqrt[n]{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n]{(2n-1)!!}}}$$

$$= \lim_{n\to\infty}\left(\frac{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n+2]{(2n+3)!!}}{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n+1]{(2n+1)!!}}\cdot\frac{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n]{(2n-1)!!}}{2!!\sqrt{3!!}\sqrt[3]{5!!}\ldots\sqrt[n+1]{(2n+1)!!}}\right)$$

[Cauchy D-Alembert]

$$= \lim_{n\to\infty}\frac{\sqrt[n+2]{(2n+3)!!}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\left(\frac{(2n+5)!!}{(2n+3)!!}\cdot\frac{(2n+1)!!}{(2n+3)!!}\right)$$

[Cauchy D-Alembert]

$$= \lim_{n\to\infty}\left(\frac{\frac{(2n+5)!}{2^{n+2}(n+2)!}}{\frac{(2n+3)!}{2^{n+1}(n+1)!}}\cdot\frac{\frac{(2n+1)!}{2^n n!}}{\frac{(2n+3)!}{2^{n+1}(n+1)!}}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\frac{(n+2)(2n+3)}{(n+1)(2n+1)} = 1$$

3.15

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\left[\frac{1}{25k^2+5k-6}\cdot\frac{1}{(n-k+1)^2}\right]\right) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{25k^2+5k-6}\cdot\frac{1}{(n-k+1)^2}\right]$$

$$= \sum_{k=1}^{\infty}\frac{1}{25k^2+5k-6}\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{25k^2 + 5k - 6} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{6} \cdot \frac{1}{5} \lim_{m\to\infty} \sum_{k=1}^{m} \left(\frac{1}{5n-2} - \frac{1}{5n+3}\right)$$

$$= \frac{\pi^2}{30} \cdot \lim_{m\to\infty} \left(\frac{1}{3} - \frac{1}{8} + \frac{1}{8} - \cdots - \frac{1}{5m-2} + \frac{1}{5m-2} - \frac{1}{5m+3}\right)$$

$$= \frac{\pi^2}{30} \cdot \lim_{m\to\infty} \left(\frac{1}{3} - \frac{1}{5m+3}\right) = \frac{\pi^2}{90}$$

3.16

$$\sum_{i=1}^{n} \frac{1}{(a+i)(b+i)} = \frac{1}{b-a} \sum_{i=1}^{n} \left(\frac{1}{a+i} - \frac{1}{b+i}\right)$$

$$= \frac{1}{b-a}\left[\frac{1}{a+1} + \frac{1}{a+2} + \cdots + \frac{1}{a+(b-a)} - \frac{1}{n+a+1} - \frac{1}{n+a+2} + \cdots + \frac{1}{b+n}\right]$$

$$= \frac{1}{b-a}\left[\frac{n}{(a+1)(n+a+1)} + \frac{n}{(a+2)(n+a+2)} + \cdots + \frac{n}{b(b+n)}\right]$$

$$= \frac{1}{b-a} \sum_{k=1}^{b-a} \left[\frac{1}{(a+k)\left(1 + \frac{a+k}{n}\right)}\right]$$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(a+i)(b+i)} = \lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{b-a} \frac{1}{(a+k)\left(1 + \frac{a+k}{n}\right)}$$

$$= \frac{1}{b-a} \sum_{k=1}^{b-a} \frac{1}{(a+k)} > \left(\prod_{k=1}^{b-a} \frac{1}{a+k}\right)^{\frac{1}{b-a}} = \left(\frac{a!}{b!}\right)^{\frac{1}{b-a}}$$

3.17

LEMMA: let $a_i, i = 1,2,\ldots,n$ be positive real numbers, then

$$\lim_{x\to\infty} \left(\frac{\sqrt[x]{a_1} + \sqrt[x]{a_2} + \cdots + \sqrt[x]{a_n}}{n}\right)^{nx} = a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

$$\therefore \Omega(a,b) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{\sqrt[n]{b}+\sqrt[n]{a+1}}\right)^n = \lim_{n\to\infty}\left(\frac{\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{2}}{\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{2}}\right)^n = \frac{\sqrt{a(b+1)}}{\sqrt{b(a+1)}}$$

$$\therefore \sum_{cyc}\sqrt{b(a+1)}\cdot\Omega(a,b) = \sum_{cyc}\sqrt{a(b+1)} \overset{Cauchy-Schwarz}{\leq}$$

$$\leq \sqrt{(a+b+c)(a+b+c+3)} = 2 \text{ (proved)}$$

3.18

Let $E_n = \sum_{k=0}^{n}\frac{1}{k!}$. *Then* $\lim_{n\to\infty} E_n = e$

$$\therefore \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{Cauchy-D'Alembert}{=} \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right) = \frac{1}{e}$$

$$\therefore \lim_{n\to\infty}\sqrt[n]{E_n} \overset{Cauchy-D'Alembert}{=} \lim_{n\to\infty}\frac{E_{n+1}}{E_n} = \frac{e}{e} = 1$$

$$\therefore \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!\,E_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\sqrt[n]{E}\cdot\frac{u_n-1}{\ln u_n}\cdot \ln u_n^n\right)$$

where $u_n = \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!E_n}}$ *for all* $n \in \mathbb{N}$

$$\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{1}{\frac{\sqrt[n]{n!}}{n}}\cdot\frac{1}{\sqrt[n]{E_n}}\right)$$

$$= \left(\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)\left(\lim_{n\to\infty}\frac{n+1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{\frac{\sqrt[n]{n!}}{n}}\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[n]{E_n}}\right) = 1$$

$\therefore$ *as* $u_n \to 1$ *then* $\frac{u_n-1}{\ln u_n} \to 1$ *as* $n \to \infty$

Now, $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{1}{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\cdot\frac{1}{n+1}\cdot\frac{(n+1)!}{n!}\cdot\frac{1}{E_n}\right) = 1$. *So,*

$$\lim_{n\to\infty} \ln u_n^n = \ln\left(\lim_{n\to\infty} u_n^n\right) = 0$$

$$\therefore \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}\right) = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\sqrt[n]{E_n}\cdot\frac{u_n-1}{\ln u_n}\cdot \ln u_n^n\right) = 0$$

3.19

$$\text{For } 0 \le x \le 1, 0 \le 1-x \le 1 \Rightarrow 0 \le (1-x)^n \le 1, |cos(nx)| \le 1$$

$$\therefore |(1-x)^n + cos(nx)| \le |(1-x)^n| + |cos(nx)| \le 2$$

$$\Rightarrow \left|\int_0^1 \{(1-x)^n + cos(nx)\}e^x\right| \le \int_0^1 2e^x dx = 2(e-1)$$

$$\Rightarrow \left|\frac{1}{n!}\int_0^1 \{(1-x)^n + cos(nx)\}e^x dx\right| \le \frac{2}{n!}(e-1)$$

As $\frac{2}{n!}(e-1) \to 0$ *as* $n \to \infty$, $lim_{n\to\infty}\frac{1}{n!}\int_0^1[(1-x)^n + cos(nx)]\, e^x dx = 0$

3.20

$$\Omega(x) = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-x)^i}{i+1} = \lim_{n\to\infty}\left(1-\frac{x}{2}+\frac{x^2}{3}+\cdots+\frac{(-1)^n x^n}{n+1}\right)$$

$$\therefore x\Omega(x) = \lim_{n\to\infty}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{(-1)^n x^{n+1}}{n+1}\right) = log(1+x)$$

$$\therefore \Omega(x) = \frac{log(1+x)}{x} \Rightarrow \Omega'(x) = \frac{1}{x^2+x} - \frac{log(1+x)}{x^2}$$

$$\Omega''(x) = -\frac{2x+1}{(x^2+x)^2} - \frac{1}{x^2(1+x)} + \frac{2\,log(1+x)}{x^3} =$$

$$= \frac{2(1+x)^2\,log(1+x) - (3x^2+2x)}{x^3(1+x)^2}$$

Let $f(x) = 2(1+x^2)\,log(1+x) - (3x^2+2x)$ *for all* $x \in [0,1]$

$$f'(x) = 2(1+x) + 4(1+x)\,log(1+x) - 6x - 2 =$$

$$= (1+x)\left(log(1+x) - \frac{x}{1+x}\right) \geq 0$$

$\left[\because log(1+x) \geq \frac{x}{1+x}\ for\ all\ x \geq 0\right]$. *So, f is increasing* $f(x) \geq f(0) = 0$

$\therefore \Omega''(x) \geq 0$ *for all* $|x| < 1$. *Hence, Ω is convex. So,*

$$\Omega\left(\frac{ax+by}{a+b}\right) \leq \frac{a\Omega(x)}{a+b} + \frac{b\Omega(y)}{a+b}$$

$$\therefore a\Omega(x) + b\Omega(y) \geq \Omega\left(\frac{ax+by}{a+b}\right)$$

3.21

Let be $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$L = \lim_{n\to\infty}(a_{200n} - a_{100n} + a_{400n} - a_{200n} + a_{400n} - a_{300n}) =$$

$$= \lim_{n\to\infty}(2a_{400n} - a_{100n} - a_{300n}) =$$

$$= \lim_{n\to\infty}\left(2a_{400n} - 2\,log\,400n - a_{100n} + log\,100n - a_{300n} + log\,300n + log\frac{400n\cdot 400n}{100n\cdot 300n}\right)$$

$$= 2\gamma - \gamma - \gamma + log\frac{16}{3} = log\frac{16}{3}$$

3.22

$$\Omega(x,y) = \sum_{n}^{\infty}\frac{2n^2 + (2x+2y+5)n + 2xy + 6x - y}{3^n(n+y)(n+y+1)(n+y+2)} =$$

$$= \sum_{n=1}^{\infty}\left(\frac{n+x}{3^{n-1}(n+y)(n+y+1)} - \frac{n+x+1}{3^n(n+y+1)(n+y+2)}\right) = \frac{x+1}{(y+1)(y+2)} \Rightarrow$$

$$\Rightarrow \Omega(x,y)\cdot\Omega(y,x) = \frac{1}{x+1+1}\cdot\frac{1}{y+1+1} \leq \frac{1}{3\sqrt[3]{x}}\cdot\frac{1}{3\sqrt[3]{y}}$$

$$\Omega(x,y)\cdot\Omega(y,x) \leq \frac{1}{9\sqrt[3]{xy}}$$

3.23

$$l = \lim_{x\to 0}\frac{2\sqrt{1+x}+2\sqrt{2^2+x}+\cdots+2\sqrt{n^2+x}-n(n+1)}{x} = \frac{0}{0}$$

$$l = 2\lim_{x\to 0}\frac{\sqrt{1+x}+\sqrt{2^2+x}+\cdots+\sqrt{n^2+x}-\frac{n(n+1)}{2}}{x}, 1+2+\cdots+n = \frac{n(n+1)}{2}$$

$$l = 2\cdot\lim_{x\to 0}\frac{\left(\sqrt{1+x}-1\right)+\left(\sqrt{2^2+x}-2\right)+\cdots+\left(\sqrt{n^2+x}-n\right)}{x}$$

$$L_n = \lim_{x\to 0}\frac{\sqrt{n^2+x}-n}{x} = \frac{0}{0} = \lim_{x\to 0}\frac{n^2+x-n^2}{x\left(\sqrt{n^2+x}+n\right)} = \lim_{x\to 0}\frac{x}{x\left(\sqrt{n^2+x}+n\right)}$$

$$L_n = \lim_{x\to 0}\frac{1}{\sqrt{n^2+x}+n} = \frac{1}{n+n} = \frac{1}{2n}, l = 2\cdot(L_1+L_2+\cdots+L_n)$$
$$= 2\cdot\left(\frac{1}{2\cdot 1}+\frac{1}{2\cdot 2}+\cdots+\frac{1}{2n}\right), \qquad l = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$$

3.24

For $n \le x \le n+1$

$$\frac{1}{n+1} \le \frac{1}{x} \le \frac{1}{n} \Rightarrow e^{\frac{1}{(n+1)}} \le e^{\frac{1}{x}} \le e^{\frac{1}{n}}$$

$$\Rightarrow \int_n^{n+1} e^{\frac{1}{(n+1)}}\,dx \le \int_n^{n+1} e^{\frac{1}{x}}\,dx \le \int_n^{n+1} e^{\frac{1}{n}}\,dx \Rightarrow e^{\frac{1}{n+1}} \le \int_n^{n+1} e^{\frac{1}{x}}\,dx \le e^{\frac{1}{n}}$$

Since $e^{\frac{1}{n}} \to e^0 = 1$ *as* $n\to\infty$, $e^{\frac{1}{(n+1)}} \to e^0 = 1$ *as* $n\to\infty$

we get

$$\lim_{n\to\infty}\int_n^{n+1} e^{\frac{1}{x}}\,dx = 1$$

3.25

Let $f(x) = e^{x(a^2+a+1)}$ *for all* $x \in \left[\frac{1}{n+7},\frac{1}{n+5}\right]$

$\therefore$ *by Lagrange's Mean Value Theorem;*

$$\frac{\sqrt[n+5]{e^{a^2+a+1}}-\sqrt[n+7]{e^{a^2+a+1}}}{\frac{1}{n+5}-\frac{1}{n+7}} = (a^2+a+1)e^{\xi_n(a^2+a+1)} \text{ where } \xi \in \left[\frac{1}{n+7}, \frac{1}{n+5}\right]$$

$$\sqrt[n+5]{e^{a^2+a+1}}-\sqrt[n+7]{e^{a^2+a+1}} = \frac{2(a^2+a+1)}{(n+5)(n+7)}e^{\xi_n(a^2+a+1)}$$

$$\text{Now, } \frac{1}{n+7} \le \xi_n \le \frac{1}{n+5} \Rightarrow \frac{a^2+a+1}{n+7} \le \xi_n(a^2+a+1) \le \frac{a^2+a+1}{n+5}$$

$$\sqrt[n+7]{e^{a^2+a+1}} \le e^{\xi_n(a^2+a+1)} \le \lim_{n\to\infty} \sqrt[n+5]{e^{a^2+a+1}}$$

$$\lim_{n\to\infty} \sqrt[n+7]{e^{a^2+a+1}} \le e^{\xi_n(a^2+a+1)} \le \lim_{n\to\infty} \sqrt[n+5]{e^{a^2+a+1}}$$

So, by Sandwich Theorem,

$$\lim_{n\to\infty} e^{\xi_n(a^2+a+1)} = 1$$

$$\therefore \lim_{n\to\infty} n^2\left(\sqrt[n+5]{e^{a^2+a+1}}-\sqrt[n+7]{e^{a^2+a+1}}\right) = \lim_{n\to\infty} \frac{2(a^2+a+1)}{\left(1+\frac{5}{n}\right)\left(1+\frac{7}{n}\right)} \cdot \lim_{n\to\infty} e^{\xi_n(a^2+a+1)}$$

$$= 2(a^2+a+1)$$

$$\therefore \sum_{cyc} \frac{\Omega(a)}{b+c} = 2\sum_{cyc} \frac{a^2}{b+c} + \sum_{cyc} \frac{2a}{b+c} + 2\sum_{cyc} \frac{1}{b+c}$$

$$\ge a+b+c+3+\frac{9}{a+b+c} > a+b+c+3$$

3.26

$$\text{Let } a_n = \binom{2n}{n}$$

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n\to\infty} \binom{2n+2}{n+1}\Big/\binom{2n}{n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4\lim_{n\to\infty}\left(-\frac{1}{2n+2}\right) = 4$$

Let $0 < \epsilon < 1$, *there exists a positive integer* m *such that*

$$\left|(a_n)^{\frac{1}{n}} - 4\right| < \varepsilon \quad \forall n > m \Rightarrow 4 - \varepsilon < (a_n)^{\frac{1}{n}} < 4 + \varepsilon \ \ \forall n > m$$

$$Let\ b_k = k^2 \binom{2k}{k}^{\frac{1}{k}} = k^2 (a_k)^{\frac{1}{k}}$$

$$Let\ A = b_1 + b_2 + \cdots + b_m - (1^2 + 2^2 + \cdots + m^2)(4 - \epsilon)$$

and

$$B = b_1 + b_2 + \cdots + b_m - (1^2 + 2^2 + \cdots + m^2)(4 + \epsilon)$$

Now, for $n > m$

$$(2^2 + 3^2 + \cdots + n^2)(4 - \varepsilon) + A <$$

$$< b_2 + b_3 + \cdots + b_n < (2^2 + \cdots + n^2)(4 + \varepsilon) + B$$

$$\Rightarrow \frac{\left[\frac{1}{6} n(n+1)(2n+1) - 1\right](4 - \varepsilon) + A}{n(n+1)(2n+1)}$$

$$< \frac{\sum_{k=2}^{n} b_k}{n(n+1)(2n+1)} < \frac{\left(\frac{1}{6} n(n+1)(2n+1) - 1\right)(4 + \epsilon) + B}{n(n+1)(2n+1)}$$

Taking limit as $n \to \infty$*, we get*

$$\frac{1}{6}(4 - \varepsilon) \le \lim_{n\to\infty} \frac{\sum_{k=2}^{n} b_k}{n(n+1)(2n+1)} \le \frac{1}{4}(4 + \varepsilon) \Rightarrow \frac{2}{3} - \varepsilon \le \Omega \le \frac{2}{3} + \varepsilon$$

Its true for each $\varepsilon > 0$,

$$\therefore \Omega = \frac{2}{3}$$

3.27

$$Let\ a_k = \frac{(n-k+1)e^{-k^2}}{1+2+\cdots+n} = \frac{2}{n(n+1)}[(n+1) - k]e^{-k^2} = \frac{2}{n}\left(1 - \frac{k}{n+1}\right)e^{-k^2}$$

$$Let\ b_k = e^{-k^2}, c_k = ke^{-k^2}$$

$$\lim_{n\to\infty} b_k = 0, \lim_{n\to\infty} c_k = 0 \Rightarrow \lim_{n\to\infty} \frac{b_1 + b_2 + \cdots + b_n}{n} = 0$$

$$\text{and } \lim_{n\to\infty} \frac{c_1+c_2+\cdots+c_n}{n} = 0$$

Now,

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = 2 \lim_{n\to\infty} \frac{b_1 + b_2 + \cdots + b_n}{n} - \lim_{n\to\infty} \frac{2}{n+1} \lim_{n\to\infty} \frac{c_1 + c_2 + \cdots + c_n}{n} = 0$$

3.28

By Cesaro's limit theorem

$$\lim_{n\to\infty} A_n = A \Rightarrow \lim_{n\to\infty} \frac{A_1 + A_2 + A_3 + \cdots + A_n}{n} = A$$

Now, we have $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lim_{n\to\infty} f(n)}{\lim_{n\to\infty} g(n)}$ *provided* $\lim_{n\to\infty} g(n) \neq 0$

Given $\lim_{n\to\infty} a_n = a$ *So* $\lim_{n\to\infty} \frac{a_n}{b+ca_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty}(b+ca_n)} = \frac{a}{b+ca}$

So by Cesaro's limit theorem we get $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{b+ca_k} = \lim_{n\to\infty} \frac{a_n}{b+ca_n} = \frac{a}{b+ca}$

3.29

$$s_1 = \sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \frac{k+1-1}{(k+1)!} = \sum_{k=1}^{n} \frac{1}{k!} - \frac{1}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$

$$s_2 = \sum_{k=1}^{n} \frac{k(k+2)}{[(k+1)!]^2} = \sum_{k=1}^{n} \frac{(k+1)^2 - 1^2}{[(k+1)!]^2} = \sum_{k=1}^{n} \frac{1}{(k!)^2} - \frac{1}{[(k+1)!]^2} = 1 - \frac{1}{[(n+1)!]^2}$$

$$s_3 = \sum_{k=1}^{n} \frac{k(k^2+3k+3)}{[(k+1)!]^3} = \sum_{k=1}^{n} \frac{(k+1)^3 - 1^3}{[(k+1)!]^3} = \sum_{k=1}^{n} \frac{1}{(k!)^3} - \frac{1}{[(k+1)!]^3} = 1 - \frac{1}{[(n+1)!]^3}$$

$$\Omega = \lim_{n\to\infty} \left[1 - \frac{1}{(n+1)!}\right]\left[1 - \frac{1}{[(n+1)!]^2}\right]\left[1 - \frac{1}{[(n+1)!]^3}\right] = \lim_{t\to 0}[1-t][1-t^2][1-t^3] = 1$$

3.30

$$\Omega = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \sum_{p=m+1}^{n} p \right) = \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \left(\sum_{p=1}^{n} p - \sum_{p=1}^{m} p \right) \right)$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} \left(m \left(\frac{n(n+1)}{2} - \frac{m(m+1)}{2} \right) \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{1}{n^4} \left(n(n+1) \sum_{m=1}^{n-1} m - \sum_{m=1}^{n-1} m^3 - \sum_{m=1}^{n-1} m^2 \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{n(n+1)n(n-1)}{2} - \frac{n^2(n-1)^2}{4} \right)$$

As

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{m=1}^{n-1} m^2 = 0$$

$$= \frac{1}{8} \lim_{n\to\infty} \frac{1}{n^4} n^2(n-1)(n+3) = \frac{1}{8} \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)\left(1 + \frac{3}{n}\right) = \frac{1}{8}$$

3.31

$$\Omega = \lim_{n\to\infty} \frac{n(n+1)(n+2)\cdots(2n-2)\tan^{-1}\frac{\pi}{2^n}}{1\cdot 3\cdot 5\cdot \ldots \cdot 2n-3}$$

$$= \lim_{n\to\infty} \frac{(2\cdot 4\cdot 6\cdot \ldots \cdot (2n-2))\big(n(n+1)(n+2)\cdots(2n-2)\big)\tan^{-1}\frac{\pi}{2^n}}{(1\cdot 3\cdot 5\cdot \ldots \cdot 2n-3)\big(2\cdot 4\cdot 6\cdot \ldots \cdot (2n-2)\big)}$$

$$= \lim_{n\to\infty} \frac{2^{n-1}(2n-2)!\tan^{-1}\frac{\pi}{2^n}}{(2n-2)!} = \frac{1}{2} \underbrace{\lim_{n\to\infty} 2^n \tan^{-1}\frac{\pi}{2^n}}_{let\ t=\frac{1}{2^n}} = \frac{1}{2} \lim_{t\to 0^+} \frac{\tan^{-1}\pi t}{t}$$

$$= \frac{1}{2} \underbrace{\lim_{t\to 0^+} \frac{\frac{\pi}{1+\pi^2 t^2}}{1}}_{L'Hospital\ Rule} = \frac{1}{2}\cdot\pi = \frac{\pi}{2}$$

3.32

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{Cauchy-D'Alembert}{\triangleq} \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e}$$

$$\Omega = \lim_{n\to\infty}\frac{1}{3n+1}\sum_{k=2}^{n}\frac{1}{\sqrt[k]{k!}} = \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{1}{\sqrt[k]{k!}} - \sum_{k=2}^{n}\frac{1}{\sqrt[k]{k!}}}{3n+4-3n-1} = \frac{1}{3}\lim_{n\to\infty}\frac{1}{\sqrt[n+1]{(n+1)!}}$$

$$= \frac{1}{3}\lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{1}{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\right) = \frac{e}{3}\lim_{n\to\infty}\frac{1}{n+1} = 0$$

3.33

$$\ln\Omega = \lim_{n\to}\frac{\ln\left(e+\sqrt{2}\right)^{n} + \ln\left(e+\sqrt{3}\right)^{n} + \cdots + \ln\left(e+\sqrt{n}\right)^{n}}{n} -$$

$$-\lim_{n\to\infty}\frac{\ln\left(\pi+\sqrt{2}\right)^{n} + \ln\left(\pi+\sqrt{3}\right)^{n} + \cdots + \ln\left(\pi+\sqrt{n}\right)^{n}}{n} \overset{Stolz-Cesaro}{\cong}$$

$$= \lim_{n\to\infty}\ln\left(e+\sqrt{n}\right)^{n} - \lim_{n\to\infty}\ln\left(\pi+\sqrt{n}\right)^{n} = \lim_{n\to\infty}\ln\left(\frac{e+\sqrt{n}}{\pi+\sqrt{n}}\right)^{n} = \lim_{n\to\infty}\frac{\ln\left(\frac{e+\sqrt{n}}{\pi+\sqrt{n}}\right)}{\frac{1}{n}} \overset{L'Hospital}{\cong}$$

$$= \lim_{n\to\infty}\frac{\frac{\pi+\sqrt{n}}{e+\sqrt{n}}\cdot\frac{\pi-e}{2\sqrt{n}\left(\pi+\sqrt{n}\right)^{2}}}{-\frac{1}{n^{2}}} = -\frac{\pi-e}{2}\lim_{n\to\infty}\frac{n^{\frac{3}{2}}}{\left(e+\sqrt{n}\right)\left(\pi+\sqrt{n}\right)} =$$

$$= -\frac{\pi-e}{2}\cdot\frac{3}{2}\lim_{n\to\infty}\frac{n^{\frac{1}{2}}}{\frac{1}{2}n^{-\frac{1}{2}}\left(\pi+\sqrt{n}\right) + \frac{1}{2}n^{-\frac{1}{2}}\left(e+\sqrt{n}\right)} =$$

$$= -\frac{3}{2}(\pi-e)\lim_{n\to\infty}\frac{n}{\pi+e+2\sqrt{n}} = -\frac{3}{2}(\pi-e)\lim_{n\to\infty}\frac{1}{2\cdot\frac{1}{2}n^{-\frac{1}{2}}} =$$

$$= -\frac{3}{2}(\pi-e)\lim_{n\to\infty}\sqrt{n} = -\infty \Rightarrow \ln\Omega = -\infty \Rightarrow -\Omega = \frac{1}{e^{\infty}} = 0$$

3.34

$$We\ have\ \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2}\cdot\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\cdot\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \Rightarrow \sqrt[n]{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}} - 1 > \frac{1}{n} \Rightarrow n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1} - 1\right) < n^{\left(a^{\frac{1}{n}-1}\right)}\ (1)$$

$$Lemma: 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e$$

On the other hand, using the lemma, we have $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < e \Rightarrow$

$$\Rightarrow \sqrt[n]{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}} - 1 < \sqrt[n]{e} - 1 \Rightarrow n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1} - 1\right) > n\left(a^{\sqrt[n]{e}-1} - 1\right)\ (2)$$

$$(1)\ and\ (2) \Rightarrow n\left(a^{\sqrt[n]{e}-1} - 1\right) < n\left(a^{\sqrt[n]{1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}}-1} - 1\right) < n\left(a^{\frac{1}{n}-1}\right)\ (3)$$

On the other hand, we have

$$lim_{n\to\infty} n\left(a^{\frac{1}{n}} - 1\right) = lim_{n\to\infty} \frac{e^{\frac{\ln a}{n}} - 1}{\frac{\ln a}{n}} \cdot \ln a = \ln a\ \ (4)$$

$$\lim_{n\to\infty} n\left(a^{\sqrt[n]{e}} - 1\right) = \lim_{x\to 0} \frac{a^x - 1}{\ln(x+1)} = \lim_{x\to 0} \frac{a^x - 1}{x} \cdot \frac{x}{\ln(x+1)} =$$

$$= \ln a \cdot 1 = \ln a\ \ (5)$$

$$(3),\ (4)\ and\ (5) \Rightarrow \Omega(a) = \ln a$$

By Bernoulli inequality, we have $x^a < e^{a(x-1)}$

Similarly, we have $y^b < e^{b(y-1)}$ *and* $c^z < e^{c(z-1)}$

$$\Rightarrow x^a y^b c^z < e^{a(x-1)+b(y-1)+c(z-1)} \Rightarrow x^a y^b z^c < e^{ax+by+cz-1} \Rightarrow$$

$$\ln(x^a y^b z^c) < ax + by + cz - 1\ \ (6)$$

By Bernoulli inequality, we have $e^{ax+by+cz} \geq ax + by + cz + 1$ *(7)*

(6) and (7) ⇒ QED

3.35

$$\Omega = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1+6+11+16+\cdots+(10k-9)}{2k-1}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(2k-1)\big(2+10(k-1)\big)}{2\times(2k-1)} = \lim_{n\to\infty}\sum_{k=1}^{n}(5k-4) = \lim_{n\to\infty}\frac{5n^2-3n}{2} = \infty$$

3.36

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}} \overset{Cauchy}{=} \lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{1}{e}$$

$$\lim_{n\to\infty}\frac{1}{n+3}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} \overset{Cesaro-Stolz}{=} \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{k}{\sqrt[k]{k!}}-\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}}}{n+4-(n+3)} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} = e$$

3.37

$$\Omega = \lim_{n\to+\infty}\frac{1}{n^5}\sum_{k=1}^{n}k^4\,arctan^5\left(\frac{k}{n}\right) = \lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4 arctan^5\left(\frac{k}{n}\right)$$

$$= \int_0^1 x^4\,arctan^5 x\,dx = \left[\frac{x^5}{5}\cdot arctan^5 x\right]_0^1 - \int_0^1\frac{x^5}{5}\cdot\frac{5x^4}{x^{10}+1}dx$$

$$= \frac{\pi}{20}-\int_0^1\frac{x^9}{x^{10}+1}dx = \frac{\pi}{20}-\left[\frac{1}{10}ln|x^{10}+1|\right]_0^1 = \frac{\pi}{20}-\frac{ln\,2}{10}$$

3.38

First let's compute the sum

$$\sum_{p=1}^{m}\big(p!\,(1+p^2)\big)$$

$$p!\,(1+p^2) = p!\,[(p+2)(p+1)-3(p+1)+2] = (p+2)!-3(p+1)!+2p!$$

$$= [(p+2)! - (p+1)!] - 2[(p+1)! - p!]$$

$$\sum_{p=1}^{m} p!\,(1+p^2) = \sum_{p=1}^{m}[(p+2)! - (p+1)!] - 2\sum_{p=1}^{m}[(p+1)! - p!]$$

$$= (m+2)! - 2! - 2\big((m+1)! - 1!\big) = (m+2)! - 2 - 2(m+1)! + 2$$

$$= (m+2)! - 2(m+1)! = (m+1)!\,[m+2-2] = m(m+1)!$$

Let $a_n = \sum_{m=1}^{n}\left(1+\frac{1}{m}\right)\sum_{p=1}^{m} p!\,(1+p^2)$*. Then* $a_n = \sum_{m=1}^{n}\left(1+\frac{1}{m}\right)(m(m+1)!)$

$$= \sum_{m=1}^{n}\frac{(m+1)}{m} m\,(m+1)! = \sum_{m=1}^{n}(m+1)(m+1)!$$

$$= \sum_{m=1}^{n}[(m+2)! - (m+1)!] = (n+2)! - 2! = (n+2)! - 2$$

We want to compute $lim_{n\to\infty}\sqrt[n]{a_n}$ *We know that* $lim_{n\to\infty}\sqrt[n]{a_n} = lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

So $lim_{n\to\infty}\sqrt[n]{a_n} = lim_{n\to\infty}\frac{a_{n+1}}{a_n} = lim_{n\to\infty}\frac{(n+3)!-2}{(n+2)!-2} = lim_{n\to\infty}\frac{n+3-\frac{2}{(n+2)!}}{1-\frac{2}{(n+2)!}} = \infty$

3.39

$$\Omega = \lim_{n\to\infty}\left[\frac{\sum_{r=1}^{n}(2r-1)(n-r+1)}{(n+1)^4 - n^4}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{\sum_{r=1}^{n}(2rn - 2r^2 + 2r - n + r - 1)}{(n+1)^4 - n^4}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{2n\sum_{r=1}^{n} r - 2\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r - (1+n)\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} r}{(n+1)^4 - n^4}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{\frac{2n\cdot n(n+1)}{2} - \frac{2n(n+1)(2n+1)}{6} + 3\sum_{r=1}^{n} r - n(n+1)}{(n+1)^4 - n^4}\right]$$

$$\Omega = \lim_{n\to\infty}\left[\frac{n^2(n+1) - \frac{2n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} - n(n+1)}{[(n+1)^2 + n^2][(n+1)^2 - n^2]}\right]$$

$$\Omega = \lim_{n\to\infty} \frac{n(n+1)\left[n - \frac{(2n+1)}{3} + \frac{3}{2} - 1\right]}{[(n+1)^2 + n^2][(2n+1)]}, \Omega = \lim_{n\to\infty} \left[\frac{n(n+1)(2n+1)}{6(2n+1)(2n^2+2n+1)}\right]$$

$$\Omega = \lim_{n\to\infty} \left[\frac{(n^2+n)}{6(2n^2+2n+1)}\right], \Omega = \lim_{n\to\infty} \left[\frac{1}{6\left[\frac{2(n^2+n)+1}{n^2+n}\right]}\right], \Omega = \lim_{n\to\infty} \left[\frac{1}{6\left[2 + \frac{1}{n^2+n}\right]}\right] = \frac{1}{12}$$

3.40

$$\sum_{i=0}^{n} \sqrt[n]{7^i 5^{n-i}} = 5\sum_{i=0}^{n} \left(\sqrt[n]{\frac{7}{5}}\right)^i = 5\frac{\left(\sqrt[n]{\frac{7}{5}}\right)^n - 1}{\left(\sqrt[n]{\frac{7}{5}}\right) - 1} = \frac{2}{\left(\sqrt[n]{\frac{7}{5}}\right) - 1}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \sqrt[n]{7^i 5^{n-i}} = \lim_{n\to\infty} \frac{2}{ln\frac{7}{5}} \cdot \frac{\frac{ln\frac{7}{5}}{n}}{e^{\frac{ln\frac{7}{5}}{n}} - 1} = \frac{2}{ln\,7 - ln\,5}$$

$$\Omega = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} \sqrt[n]{7^i 5^{n-i}} = \lim_{n\to\infty} \frac{5}{n}\sum_{i=0}^{n} \left(\frac{7}{5}\right)^{\frac{i}{n}} = 5\int_0^1 \left(\frac{7}{5}\right)^x dx =$$

$$= 5\int_0^1 e^{x\,ln\frac{7}{5}} dx = 5\left[\frac{e^{x\,ln\frac{7}{5}}}{ln\frac{7}{5}}\right]_0^1 = \frac{2}{ln\,7 - ln\,5}$$

3.41

$$\lim_{n\to\infty} \Omega_n = \Omega \Leftrightarrow \lim_{n\to\infty} ln\,\Omega_n = ln\,\Omega$$

$$\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} (1 + ln\,\Omega_k)} = \lim_{n\to\infty} e^{\frac{1}{n}ln\prod_{k=1}^{n}(1 + ln\,\Omega_k)} = \lim_{n\to\infty} e^{\frac{1}{n}\sum_{k=1}^{n} ln(1 + ln\,\Omega_k)}$$

$$\overset{Cesaro}{\triangleq} \lim_{n\to\infty} e^{ln(1+ln\,\Omega_k)} = 1 + ln\,\Omega \Rightarrow \lim_{n\to\infty} e^{\sqrt[n]{\prod_{k=1}^{n}(1+ln\,\Omega_k)}-1} = e^{1+ln\,\Omega-1} = \Omega$$

3.42

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{\left((2n)!!\right)^2}{(2n)!}} = \lim_{n\to\infty} \sqrt[n]{\frac{\left(2^n(n!)\right)^2}{(2n)!}} = \lim_{n\to\infty} \sqrt[n]{\frac{\left((2^n\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)^2}{\left(\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}\right)}} = \lim_{n\to\infty} \sqrt[n]{\sqrt{\pi n}}$$

$$\to \infty$$

3.43

$U_n = \prod_{k=1}^{n}\left(\frac{12k-11}{12k-5}\right)$ *we have* $0 < U_n < 1$; *So we put* $\lim U_n = l$

$$U_n = \frac{12n-11}{12-5}U_{n-1} \Leftrightarrow (12(n+1)-11)U_n - (12n-11)U_{n-1} = 6U_n$$

$$\Rightarrow \sum_{k=2}^{n}\left((12(k+1)-11)U_k - (12k-11)U_{k-1}\right) = \sum_{k=2}^{n} 6U_k$$

$$\Leftrightarrow (12(n+1)-11)U_n + 5U_1 = 6\sum_{k=1}^{n} U_k \Leftrightarrow 12U_n + \frac{U_n + 5U_1}{n} = 6 \underbrace{\frac{1}{n}\sum_{k=1}^{n} U_k}_{Cesaro's\ Lemma}$$

$$\overset{n\to\infty}{\Longrightarrow} 12l + 0 = 6l \Leftrightarrow l = 0 \to \lim U_n = 0$$

3.44

$$\Omega = \lim_{x\to 0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\tan^2\frac{x}{2^k}\right)\right) = \lim_{x\to 0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\cos^2\frac{x}{2^k}-\sin^2\frac{x}{2^k}}{\cos^2\frac{x}{2^k}}\right)\right) =$$

$$= \lim_{x\to 0}\left(\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{\cos\frac{x}{2^{k-1}}}{\cos^2\frac{x}{2^k}}\right)\right) = \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{1}{\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right) =$$

$$= \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{1}{\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right) = \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{2^n\sin\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}\prod_{k=1}^{n}\cos\frac{x}{2^k}}\right) =$$

$$= \lim_{x\to 0}\left(\lim_{n\to\infty}\frac{\cos x}{\cos\frac{x}{2^n}}\cdot\frac{\sin\frac{x}{2^n}}{\frac{2}{2^n}}\cdot\frac{x}{\sin x}\right) = 1$$

3.45

$$x_n = \sqrt[3n]{n!} \Rightarrow ln\, x_n = \frac{1}{3n} ln\, n!$$

Using Stirling's formula, $ln\, n! = ln\sqrt{2\pi} + \left(n + \frac{1}{2}\right) ln\, n - n + O\left(\frac{1}{n}\right)$

$$ln\, x_n = \frac{1}{3n} ln\, n! = \frac{1}{3n} ln\sqrt{2\pi} + \frac{1}{3}\left(1 + \frac{1}{2n}\right) ln\, n - \frac{1}{3} + \frac{1}{3} O\left(\frac{1}{n^2}\right)$$

$$ln\, x_{n+1} = \frac{1}{3(n+1)} ln\sqrt{2\pi} + \frac{1}{3}\left(1 + \frac{1}{2(n+1)}\right) ln(n+1) - \frac{1}{3} + \frac{1}{3} O\left(\frac{1}{(n+1)^2}\right)$$

$$ln\, x_{n+1} - ln\, x_n = -\frac{ln\, 2\pi}{6n(n+1)} + \frac{1}{3} ln\left(1 + \frac{1}{n}\right) + \frac{1}{6(n+1)} ln(n+1) - \frac{1}{6n} ln\, n + O\left(\frac{1}{n^2}\right)$$

Using Lagrange's Mean Value Theorem:

$$ln\, x_{n+1} - ln\, x_n = (x_{n+1} - x_n)\frac{1}{c_n} \text{ where } c_n \in (x_n, x_{n+1})$$

Also,

$$\lim_{n\to\infty} c_n = \frac{1}{e}$$

$$\sqrt[3]{n^2}(ln\, x_{n+1} - ln\, x_n) = -\frac{\sqrt[3]{n^2}\, ln\, 2\pi}{6n(n+1)} + \frac{\sqrt[3]{n^2}}{3} ln\left(1 + \frac{1}{n}\right) + \frac{\sqrt[3]{n^2}}{6(n+1)} ln(n+1) - \frac{\sqrt[3]{n^2}}{6n} ln + \sqrt[3]{n^2}\, O\left(\frac{1}{n^2}\right)$$

Now,

$$\lim_{n\to\infty} \sqrt[3]{n^2}\,(ln\, x_{n+1} - ln\, x_n) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\,(x_{n+1} - x_n) = \frac{1}{e}\lim_{n\to\infty} \sqrt[3]{n^2}\,(ln\, a_{n+1} - ln\, x_n) = 0$$

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3n+3]{(n+1)!} - \sqrt[3n]{n!}\right) = 0$$

3.46

Proposition: Let be the sequence:

$$(a_n)_{n\geq 1}, a_n > 0, \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1, \lim_{n\to\infty}\frac{a_n}{n} = a \in (0,\infty), \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = b,$$

then

$$\lim_{n\to\infty}(a_{n+1} - a_n) = a\, ln\, b$$

Solution:

$$\left(\frac{a_{n+1}}{a_n}\right)^n = \left[(1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)}$$

$$ln\left(\frac{a_{n+1}}{n}\right)^n = ln\left[1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right]^{\frac{n}{a_n}(a_{n+1}-a_n)},$$

$$\frac{a_n}{n} ln\left(\frac{a_{n+1}}{a_n}\right)^n = (a_{n+1} - a_n)\, ln\left[1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} - \lim_{n\to\infty} \frac{a_n}{a_n} = 1 - 1 = 0,$$

$$\lim_{n\to\infty} ln\left[(1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right] = ln\, e = 1,$$

$$\lim_{n\to\infty} \frac{a_n}{n} ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty} (a_{n+1} - a_n)\, ln\left[(1 + \left(\frac{a_{n+1} - a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right],$$

$$\lim_{n\to\infty} \frac{a_n}{n} ln\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty} (a_{n+1} - a_n) = a(ln\, b).$$

$$l = \lim_{n\to\infty} \left(\frac{n + 1}{\sqrt[2n+2]{(2n + 1)!!}} - \frac{n}{\sqrt[2n]{(2n - 1)!!}}\right)^{\sqrt{n}} =$$

$$= \lim_{n\to\infty} \left(\frac{(n + 1)\sqrt{n + 1}}{\sqrt[2n+2]{(2n + 1)!!}} \cdot \frac{\sqrt{n}}{\sqrt{n + 1}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n - 1)!!}}\right)^{\sqrt{n}} \cdot \left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n + 1}} = 1,$$

$$l = \lim_{n\to\infty} \left(\frac{(n + 1)\sqrt{n + 1}}{\sqrt[2n+2]{(2n + 1)!!}} - \frac{n\sqrt{n}}{\sqrt[2n]{(2n - 1)!!}}\right)^{\sqrt{n}} \cdot \left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} =$$

$$= \lim_{n\to\infty} (a_{n+1} - a_n) \cdot \left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}},$$

$$a_n = \frac{n\sqrt{n}}{\sqrt[2n]{(2n-1)!!}}, \lim_{n\to\infty}\frac{a_n}{n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt[2n]{(2n-1)!!}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{(n+1)^n(n+1)}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n}},$$

$$\lim_{n\to\infty}\frac{a_n}{n} = \sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\frac{n+1}{2n+1}} = \sqrt{\frac{e}{2}},$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n} = \sqrt{\frac{e}{2}}\cdot\sqrt{\frac{2}{e}}\cdot 1 = 1,$$

$$b = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n = \lim_{n\to\infty}\left(\frac{(n+1)\sqrt{n+1}}{\sqrt[2n+2]{(2n+1)!!}}\cdot\frac{\sqrt[2n]{(2n-1)!!}}{n\sqrt{n}}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt[2n+2]{(2n+1)!!}}\right)^n, \lim_{n\to\infty}\frac{(n+1)\sqrt{n+1}}{n\sqrt{n}} = 1,$$

$$b = \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n-1)!!}}{\sqrt{2n+1}}\right)^{\frac{n}{n+1}} = \sqrt{\lim_{n\to\infty}\left(\sqrt[n]{\frac{(2n-1)!!}{(2n+1)^n}}\right)^{\frac{n}{n+1}}} =$$

$$= \sqrt{\lim_{n\to\infty}\frac{(2n+1)!!}{(2n+3)^n(2n+3)}\cdot\frac{(2n+1)^n}{(2n-1)!!}},$$

$$b = \sqrt{\lim_{n\to\infty}\left(\frac{2n+1}{2n+3}\right)^n\cdot\frac{2n+1}{2n+3}} = \sqrt{\lim_{n\to\infty}\left(1+\frac{-2}{2n+3}\right)^{\frac{2n+3}{-2}\cdot\frac{-2n}{2n+3}}} = \sqrt{e^{-1}} = e^{-\frac{1}{2}},$$

$$\lim_{n\to\infty}\frac{2n+1}{2n+3} = 1,$$

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a\cdot\ln b = \sqrt{\frac{e}{2}}\cdot\ln e^{-\frac{1}{2}} = \frac{-1}{2}\sqrt{\frac{e}{2}},$$

$$l = \lim_{n\to\infty}(a_{n+1}-a_n)\cdot\left(\frac{1}{\sqrt{n}}\right)^{\sqrt{n}} = \frac{-1}{2}\sqrt{\frac{e}{2}}\cdot 0 = 0.$$

3.47

$$\Omega(n) = \int_{1}^{e} \frac{dx}{x(1+x^3)^n} = \frac{1}{3}\int_{1}^{e} \frac{3x^2}{x^3(1+x^3)^n}dx$$

Put $1 + x^3 = t, 3x^2dx = dt \therefore \Omega(n) = \frac{1}{3}\int_2^{1+e^3} \frac{dt}{(t-1)t^n}$. *For* $n \geq 2$

$$\Omega(n) - \Omega(n-1) = \frac{1}{3}\int_{2}^{1+e^3} \frac{1}{t-1}\left(\frac{1}{t^n} - \frac{1}{t^{n-1}}\right)dt = \frac{1}{3}\int_{2}^{1+e^3} \frac{1}{t-1}\cdot\frac{(1-t)}{t^n}dt =$$

$$= -\frac{1}{3}\int_{2}^{1+e^3} t^{-n}\,dt = -\frac{1}{3(-n+1)}[t^{-n+1}]_2^{1+e^3} = \frac{1}{3(n-1)}\left[\frac{1}{(1+e^3)^{n-1}} - \frac{1}{2^{n-1}}\right]$$

$$\therefore \Gamma = \lim_{n\to\infty}[\Omega(n) - \Omega(n-1)] = 0$$

For $n \geq 2, 1 \leq x \leq e \Rightarrow 2 \leq 1 + x^3 \leq 1 + e^3 \Rightarrow 2^n \leq (1+x^3)^n \leq (1+e^3)^n$

$$\Rightarrow 2^n \leq x(1+x^3)^n \leq e(1+e^3)^n \Rightarrow \frac{1}{e(1+e^3)^n} \leq \frac{1}{x(1+x^3)^n} \leq \frac{1}{2^n} \Rightarrow$$

$$\Rightarrow \frac{e-1}{e(1+e^3)n} \leq \int_{1}^{e} \frac{dx}{x(1+x^3)^n} \leq \frac{e-1}{2^n}$$

As $lim_{n\to\infty} \frac{e-1}{e(1+e^3)^n} = 0 = lim_{n\to\infty} \frac{e-1}{2^n}$

$$\therefore \lim_{n\to\infty}\int_{1}^{e} \frac{1}{x(1+x^3)^n}dx = 0 \Rightarrow \lim_{n\to\infty}\Omega(n) = 0 \Rightarrow \lim_{n\to\infty}\big(\Omega(n) - \Omega(n-1)\big) = 0$$

3.48

$\Omega_n = lim_{n\to\infty} \frac{\left(\sqrt[n]{n!}+n\right)^n}{(2n)!}$ *we have* $\sqrt[n]{n!} \leq n$ *since* $n! \leq n^n$.

Hence $\Omega_n \leq \frac{(n+n)^n}{(2n)!} = \frac{(2n)^n}{(2n)!} = a_n$

$$\frac{a_{n+1}}{a_n} = \frac{\big(2(n+1)\big)^{n+1}}{(2n+2)!}\cdot\frac{(2n)!}{(2n)^n} = \frac{(2n+2)^{n+1}}{(2n+1)(2n+2)}\cdot\frac{1}{(2n)^n} =$$

$$= \frac{1}{2n+1} \cdot \left(\frac{2n+2}{2n}\right)^n = \frac{1}{2n+1} \cdot \left(1 + \frac{1}{n}\right)^n \to 0$$

$$Since\ \frac{a_{n+1}}{a_n} \to 0 < 1 \Rightarrow \left.\begin{matrix} lim_{n\to\infty} a_n = 0 \\ 0 \le \Omega_n \le a_n \end{matrix}\right\} \Rightarrow lim\, \Omega_n = 0$$

3.49

$$arctan\, x - arctan\, y = arctan \frac{x-y}{1+xy}$$

$$arctan \frac{9}{9+(3k+5)(3k+8)} = arctan \frac{1}{1 + \frac{3k+5}{3} \cdot \frac{3k+8}{3}} =$$

$$= arctan \frac{\frac{3k+8}{3} - \frac{3k+5}{3}}{1 + \frac{3k+5}{3} \cdot \frac{3k+8}{3}} = arctan \frac{3k+8}{3} - arctan \frac{3k+5}{3}$$

$$\sum_{k=1}^{n} arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right) = \sum_{k=1}^{n}\left(arctan \frac{3k+8}{3} - arctan\left(\frac{3k+5}{3}\right)\right)$$

$$= arctan \frac{11}{3} - arctan \frac{8}{3} + arctan \frac{14}{3} - arctan \frac{11}{3} + \cdots + arctan\left(\frac{3n+8}{3}\right) - arctan\left(\frac{3n+5}{3}\right) = arctan\left(\frac{3n+8}{3}\right) - arctan \frac{8}{3}$$

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} arctan\left(\frac{9}{9+(3k+5)(3k+8)}\right) =$$

$$= \Omega = \lim_{n\to\infty}\left(arctan\left(\frac{3n+8}{3}\right) - arctan \frac{8}{3}\right) = \frac{\pi}{2} - arctan \frac{8}{3}$$

3.50

$$Let\ \Delta(k,x) = \begin{vmatrix} sin\, A & sin\, B & sin\, C \\ sin(A+kx) & sin(B+kx) & sin(C+kx) \\ cos(A+kx) & cos(B+kx) & cos(C+kx) \end{vmatrix}$$

Using $R_2 \to R_2 - (cos\, kx)R_1;\ R_3 \to R_3 + (sin\, kx)R_1$, *we get*

$$\Delta(k,x) = \begin{vmatrix} sin\, A & sin\, B & sin\, C \\ cos\, A\, sin\, kx & cos\, B\, sin\, kx & cos\, C\, sin\, kx \\ cos\, A\, cos\, kx & cos\, B\, cos\, kx & cos\, C\, cos\, kn \end{vmatrix} = 0$$

[$\therefore$ R_2 and R_3 are proportional]. Now,

$$\sum_{k=1}^{n} \Delta(k,x) = 0 \Rightarrow \Omega(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta(k,x) = 0$$

3.51

$$\Omega = \lim_{n\to\infty} \frac{cos\frac{1}{n^2} + 2cos\frac{4}{n^2} + 3cos\frac{9}{n^2} + \cdots + ncos1}{n^2} =$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum kcos\left(\frac{k^2}{n^2}\right) = \lim_{n\to\infty} \frac{1}{n} \sum \frac{k}{n} cos\left(\frac{k}{n}\right)^2 = \int_0^1 xcosx^2 dx = \frac{1}{2} sin1$$

3.52

$$a_n = \sum_{k=0}^{n} (2n+1-2k)\binom{2n+1}{k} = \sum_{k=0}^{n} \left[(2n+1-k)\binom{2n+1}{k} - k\binom{2n+1}{k}\right] =$$

$$= \sum_{k=0}^{n} \left[(2n+1-k)\binom{2n+1-k}{k} - k\binom{2n+1}{k}\right]$$

$$= \sum_{k=0}^{n} (2n+1-k)\binom{2n+1-k}{k} - \sum_{k=0}^{n} k\binom{2n+1}{k} =$$

$$= \sum_{k=0}^{n} (2n+1)\binom{2n}{k} - \sum_{k=0}^{n} (2n+1)\binom{2n}{k} = 0$$

$$\Omega = \lim_{n\to\infty} \frac{1}{(n+1)^2 4^n} \cdot a_n = \lim_{n\to\infty} \frac{1}{(n+1)^2 4^n} \cdot 0 = 0$$

3.53

First we note that $F_n(t) = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^{n-k}$*. Now we find that*

$$\int_0^x F_n(t)\,dt = \frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} x^{n-k+1}.$$

Integrating with respect to x *from* 0 *to* 1 *gives us*

$$\frac{2^{n+2}}{(n+2)(n+1)}-\frac{1}{(n+2)(n+1)}-\frac{1}{n+1}=\sum_{k=0}^{n}\frac{1}{(n-k+2)(n-k+1)}\binom{n}{k}$$

$$=\sum_{k=0}^{n}\frac{n\cdot(n-1)\cdot(n-2)\ldots(n-k+3)}{1\cdot 2\cdot 3\cdot\ldots\cdot k}$$

Dividing by 2^n, we see that

$$\Omega=\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=3}^{n}\frac{n\cdot(n-1)(n-2)\ldots(n-k+3)}{1\cdot 2\cdot 3\cdot\ldots\cdot k}=\lim_{n\to\infty}\frac{4}{(n+2)(n+1)}+O(n)=0.$$

3.54

$$\Omega=\lim_{x\to 0}\frac{log_a\left(log_b\left(log_c\left(\frac{c^b}{cos\,x}\right)\right)\right)}{log_c\left(log_b\left(log_a\left(\frac{a^b}{cos\,x}\right)\right)\right)}=$$

$$=\lim_{x\to 0}\frac{(log_a e)\left(\frac{log_b e}{log_b\left(log_c\left(\frac{c^b}{cos\,x}\right)\right)}\right)\frac{(log_c e)(tan\,x)}{log_c\left(\frac{c^b}{cos\,x}\right)}}{(log_c e)\left(\frac{log_b e}{log_b\left(log_a\left(\frac{a^b}{cos\,x}\right)\right)}\right)\frac{(log_a e)(tan\,x)}{log_a\left(\frac{a^b}{cos\,x}\right)}}=\frac{(log_a e)\left(\frac{log_b e}{1}\right)\cdot\frac{log_c e}{b}}{(log_c e)\left(\frac{log_b e}{1}\right)\cdot\frac{log_a e}{b}}=1$$

3.55

We know for $k>1$

$$(k!)^{\frac{1}{k}}<\frac{1+2+\cdots+k}{k}=\frac{k+1}{2}$$

$$\therefore (2!)^{\frac{1}{2}}(3!)^{\frac{1}{3}}\cdot\ldots\cdot(3n!)^{\frac{1}{3n}}<\left(\frac{2+1}{2}\right)\left(\frac{3+1}{2}\right)\cdot\ldots\cdot\left(\frac{3n+1}{2}\right)=\frac{1}{2}\cdot\frac{(3n+1)!}{2^{3n-1}}$$

$$\Rightarrow\frac{(2!)^{\frac{1}{2}}\cdot(3!)^{\frac{1}{3}}\cdot\ldots\cdot(3n)!^{\frac{1}{3n}}}{(3n)!}<\frac{(3n+1)!}{(3n)!}\cdot\frac{1}{2^{3n}}\Rightarrow 0<\frac{(2!)^{\frac{1}{2}}(3!)^{\frac{1}{3}}\cdot\ldots\cdot(3n)!^{\frac{1}{3n}}}{(3n)!}<\frac{3^{n+1}}{2^{3n}}$$

As $lim_{n\to\infty}\frac{3^{n+1}}{2^{3n}}=0$, we get $lim_{n\to\infty}\frac{(2!)^{\frac{1}{2}}(3!)^{\frac{1}{3}}\cdot\ldots\cdot((3n)!)^{\frac{1}{3n}}}{(3n)!}=0$

3.56

For $n \geq 2$

$$\sum_{k=1}^{n}\left(\sum_{k=1}^{k}\binom{k}{i}\right) = \sum_{k=1}^{n}(2^k - 1) = (2^{n+1} - 2) - n$$

$$= 2^{n+1} - (n+2) > {}^{n+1}C_1 + {}^{n+1}C_2 - (n+2) = \frac{1}{2}(n+1)n - 1 = \frac{1}{2}(n-1)(n+2)$$

Also, $(n!)^{\frac{1}{n}} < \frac{1+2+\cdots+n}{n} = \frac{n+1}{2}$

$$\therefore \frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\binom{k}{i}\right)}{(n!)^{\frac{1}{n}}} > \frac{(n-1)(n+2)}{n+1} > n-1, \forall n \geq 2 \Rightarrow \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\left(\sum_{i=1}^{k}\binom{k}{i}\right)}{(n!)^{\frac{1}{n}}} = \infty$$

3.57

$$p! + \frac{(p+1)!}{1!} + \frac{(p+2)!}{2!} + \cdots + \frac{(p+n)!}{n!} = p!\left[1 + \frac{(p+1)!}{p!\,1!} + \frac{(p+2)!}{p!\,2!} + \cdots + \frac{(p+n)!}{p!\,n!}\right]$$

$$= p!\left[{}^{p+1}C_0 + {}^{p+1}C_1 + {}^{p+2}C_2 + {}^{p+3}C_3 + \cdots + {}^{p+n}C_n\right] = p!\left[{}^{p+2}C_1 + {}^{p+2}C_2 + \cdots + {}^{p+n}C_n\right]$$

$$= p!\left[{}^{p+3}C_2 + {}^{p+3}C_3 + \cdots + {}^{p+n}C_n\right] = p!\left[{}^{p+4}C_3 + \cdots + {}^{p+n}C_n\right] = \cdots$$

$$= p!\,{}^{p+n+1}C_n = \frac{(p+n+1)!}{(p+1)!\,n!}p!$$

$$\frac{(p+n+1)(p+n)\ldots(n+1)}{p+1} = \frac{n^{n+1}}{(p+1)}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{p}{n}\right)\left(1+\frac{p+1}{n}\right)$$

$$L = \lim_{n\to\infty}\frac{\left(n^{\frac{1}{n}}\right)^{p+1}}{(p+1)^{\frac{1}{n}}}\left(1+\frac{1}{n}\right)^{\frac{1}{n}}\left(1+\frac{2}{n}\right)^{\frac{1}{n}}\ldots\left(1+\frac{p}{n}\right)^{\frac{1}{n}}\left(1+\frac{p+1}{n}\right)^{\frac{1}{n}}$$

$$= \frac{1}{1}e\cdot e^2\cdot e^3\cdot\ldots\cdot e^{p+1} = e^{\frac{(p+1)(p+2)}{2}}$$

3.58

We know, $lim_{u\to 0}\frac{(1+u)^r - 1}{u} = r$ *now* $x_{n+1} - x_n = \frac{1}{x_n^p} > 0$ *for all* $n \in \mathbb{N}$

Hence the sequence is increasing, implying its bounded

then let $lim_{n\to\infty} x_n = l \Rightarrow l = l + \frac{1}{l^p} \Rightarrow l \to \infty$, *which is a contradiction*

$\therefore lim_{n\to\infty} x_n = \infty$ *let* $L = lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n \sqrt[p+1]{n}} = \left(lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k}{n}\right)\left(lim_{n\to\infty} \frac{1}{\sqrt[p+1]{n}}\right)$

$$\overset{\substack{Caesaro\\Stolz}}{\cong} \left(\lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} x_k - \sum_{k=1}^{n} x_k}{n+1-n}\right)\left(\lim_{n\to\infty} \frac{1}{\sqrt[p+1]{n}}\right) = \lim_{n\to\infty} \frac{x_{n+1}}{\sqrt[p+1]{n}}$$

$$\Rightarrow L^{p+1} = \lim_{n\to\infty} \frac{x_{n+1}^{p+1}}{n} \overset{\substack{Caesaro\\Stolz}}{\cong} \lim_{n\to\infty} \frac{x_{n+2}^{p+1} - x_{n+1}^{p+1}}{n+1-n} = \lim_{n\to\infty}\left(x_{n+2}^{p+1} - x_{n+1}^{p+1}\right)$$

$$= \lim_{n\to\infty}\left\{\left(x_{n+1} + \frac{1}{x_{n+1}^{p}}\right)^{p+1} - x_{n+1}^{p+1}\right\} = \lim_{n\to\infty} \frac{\left(1 + \frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}}$$

$$= \lim_{x_{n+1}\to\infty} \frac{\left(1 + \frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}} = p + 1 \Rightarrow L = \sqrt[p+1]{p+1}$$

3.59

$$k^{4p} + k^{2p} + 1 = (k^{2p} + 1)^2 - k^{2p} = (k^{2p} - k^p + 1)(k^{2p} + k^p + 1)$$

$$\therefore \frac{k^p}{k^{4p} + k^{2p} + 1} = \frac{1}{2}\left[\frac{1}{k^{2p} - k^p + 1} - \frac{1}{k^{2p} + k^p + 1}\right]$$

$$\Rightarrow \sum_{k=1}^{n} \frac{k^p}{k^{4p} + k^{2p} + 1} = \frac{1}{2}\sum_{k=1}^{n}\left[\frac{1}{k^{2p} - k^p + 1} - \frac{1}{k^{2p} + k^p + 1}\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{n^p + n^p + 1}\right]$$

$$\Rightarrow \left(\frac{1}{2} + \sum_{k=1}^{n} \frac{k^p}{k^{4p} + k^{2p} + 1}\right)^{n^{3p-1}} = \left(1 - \frac{1}{2(n^{2p} + n^p + 1)}\right)^{n^{3p-1}}$$

$$= \left[\left(1 - \frac{1}{2(n^{2p}+n^p+1)}\right)^{-(2n^{2p}+2n^p+1)}\right]^m \textit{ where } m = -\left(\frac{n^{3p-1}}{2n^{2p}+2n^p+1}\right)$$

$$\textit{As } n \to \infty, m \to -\infty \therefore \Omega = e^{-\infty} = 0$$

3.60

$$\textit{We know } lim_{u\to 0} \frac{(1+u)^r - 1}{u} = r \textit{ and } x_{n+1} - x_n = \frac{a}{x_n} > 0 \textit{ for all } n \in \mathbb{N}$$

hence the sequence is increasing, implying is bounded let

$$lim_{n\to\infty} x_n = l \textit{ then } l = l + \frac{a}{l} \Rightarrow l \to \infty \textit{ which is a contradiction}$$

$$\therefore lim_{n\to\infty} x_n = \infty \textit{ now, } \Omega = lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} x_i x_j}{n^3}$$

$$= \frac{1}{2}\left\{\lim_{n\to\infty} \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^3} - \lim_{n\to\infty} \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n^3}\right\} = \frac{L_1 - L_2}{2}$$

$$L_1 = \lim_{n\to\infty} \frac{(x_1 + x_2 + \cdots + x_n)^2}{n^3} \Rightarrow \sqrt{L_1} = \lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n^{\frac{3}{2}}}$$

$$\overset{CESARO\ STOLZ}{\cong} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}}} = \left(\lim_{n\to\infty} \frac{x_{n+1}}{\sqrt{n}}\right)\left(\lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^{\frac{3}{2}} - 1}{\frac{1}{n}}}\right) = \frac{2}{3} \lim_{n\to\infty} \frac{x_{n+1}}{\sqrt{n}}$$

$$= \frac{2}{3}\sqrt{\lim_{n\to\infty} \frac{x_{n+1}^2}{n}} \overset{CESARO\ STOLZ}{\cong} \frac{2}{3}\sqrt{\lim_{n\to\infty}(x_{n+2}^2 - x_{n+1}^2)} = \frac{2}{3}\sqrt{\lim_{n\to\infty}\left\{\left(x_{n+1} + \frac{a}{x_{n+1}}\right)^2 - x_{n+1}^2\right\}}$$

$$= \frac{2}{3}\sqrt{\lim_{x_n\to\infty} \frac{\left(1 + \frac{a}{x_n^2}\right)^2 - 1}{\frac{1}{x_n^2}}} = \frac{2\sqrt{2a}}{3} \Rightarrow L_1 = \frac{8a}{9}$$

$$L_2 = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k^2}{n^3} \overset{\substack{CESARO\\STOLZ}}{\cong} \lim_{n\to\infty} \frac{x_{n+1}^2}{(n+1)^3 - n^3} = \left(\lim_{n\to\infty} \frac{x_{n+1}^2}{n^2}\right)\left(\lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^3 - 1}{\frac{1}{n}}}\right)$$

$$\overset{\substack{CESARO\\STOLZ}}{\cong} \frac{1}{3} \lim_{n\to\infty} \frac{x_{n+2}^2 - x_{n+1}^2}{n} \cdot \lim_{n\to\infty} \frac{1}{\frac{\left(1+\frac{1}{n}\right)^2 - 1}{\frac{1}{n}}}$$

$$\overset{\substack{CESARO\\STOLZ}}{\cong} \frac{1}{6} \lim_{n\to\infty} \{(x_{n+3}^2 - x_{n+2}^2) - (x_{n+2}^2 - x_{n+1}^2)\} = 0$$

$$\therefore L_2 = 0 \text{ then } \lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} x_i x_j}{n^3} = \frac{4a}{9} \quad (Ans:)$$

3.61

Using Cesaro – Stolz from $\frac{0}{0}$:

$$\Omega = \lim_{n\to\infty} \frac{e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)}{\frac{1}{(n+1)!}}$$

Let $a_n = e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right), b_n = \frac{1}{(n+1)!}$. *Then:*

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

b) b_n *is strict decreasing*

c) $\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{(n+1)!}\right) - e + \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)}{\frac{1}{(n+2)!} - \frac{1}{(n+1)!}} =$

$$= \lim_{n\to\infty} \frac{-\frac{1}{(n+1)!}}{\frac{(n+1)! - (n+2)!}{(n+1)!\,(n+2)!}} = \lim_{n\to\infty} \frac{-(n+2)!}{(n+1)!\,(1-n-2)} = \lim_{n\to\infty} \frac{-(n+2)}{-(n+1)} = 1$$

From Cesaro – Stolz $\Rightarrow \Omega = 1$.

3.62

Let $a_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$. *Find:*

$$\Omega = \lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{a_{k+1}\sqrt{k+1}}\right)$$

The answer is $\Omega = \frac{1}{2}$.

First note that

$$\frac{1}{\sqrt{n}} - 2\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{1}{\sqrt{n}} - \frac{2}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\left(\sqrt{n+1} + \sqrt{n}\right)^2 \sqrt{n}} = \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right)$$

and since the series $\sum \frac{1}{n^{\frac{3}{2}}}$ *is convergent we conlude that there exists a real number* ℓ *such that* $lim_{n\to\infty}\left(a_n - 2\sqrt{n+1}\right) = \ell$. *In particular,*

$$lim_{n\to\infty} \frac{a_n}{\sqrt{n}} = 2 \qquad (1)$$

On the other hand,

$$1 - \frac{1}{a_{k+1}\sqrt{k+1}} = \frac{1}{a_{k+1}}\left(a_{k+1} - \frac{1}{\sqrt{k+1}}\right) = \frac{a_k}{a_{k+1}}$$

Thus

$$\sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{a_{k+1}\sqrt{k+1}}\right) = \sqrt{n} \prod_{k=1}^{n} \frac{a_k}{a_{k+1}} = \frac{a_1\sqrt{n}}{a_{n+1}} \qquad (2)$$

Combining (1) and (2) we get

$$\Omega = \lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{a_{k+1}\sqrt{k+1}}\right) = \frac{1}{2}.$$

3.63

$$a_n = \sqrt[4]{1 \cdot 2 \cdot \ldots \cdot n} \prod_{k=1}^{n} \left(\sqrt[4]{(k+1)^3} - \sqrt[4]{k^3}\right) = \prod_{k=1}^{n} \left[\sqrt[k]{k}\left(\sqrt[4]{(k+1)^3} - \sqrt[k]{k^3}\right)\right] \ (1)$$

Let $f : [k, k+1] \to \mathbb{R}, f(x) = \sqrt[4]{x^3}$

From Lagrange Theorem $\Rightarrow \exists c_k \in (k, k+1)$ *such that*

$$\frac{f(k+1)-f(k)}{k+1-k} = f'(c_k) \Rightarrow \sqrt[4]{(k+1)^3} - \sqrt[4]{k^3} = \frac{3}{4} \cdot \frac{1}{\sqrt[4]{c}} \quad (2)$$

But $c \in (k, k+1) \Rightarrow k < c < k+1 \Rightarrow \sqrt[4]{k} < \sqrt[4]{c} < \sqrt[4]{k+1} \Rightarrow \sqrt[4]{\frac{k}{c}} < 1$ *(3)*

From (2) $\Rightarrow \sqrt[4]{k}\left(\sqrt[4]{(k+1)^3} \cdot \sqrt[4]{k^3}\right) = \frac{3}{4}\sqrt[4]{\frac{k}{c}} \overset{(3)}{<} \frac{3}{4}$ *(4)*

From (1)+(4)$\Rightarrow 0 < a_n < \left(\frac{3}{4}\right)^n \Rightarrow \left(lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0\right)$, $lim_{n\to\infty} a_n = 0$

3.64

$$\left.\begin{array}{c} \Omega_n = \int\limits_0^{\frac{\pi}{4}} (4x^2 - \pi x + 4n^2)\, ln(1 + tan\, x)\, dx \\ x = \frac{\pi}{4} - y \Rightarrow dx = -dy; x = 0 \Rightarrow y = \frac{\pi}{4} \wedge x = \frac{\pi}{4} \Rightarrow y = 0 \end{array}\right\} \Rightarrow$$

$$\Omega_n = \int\limits_{\frac{\pi}{4}}^{0} \left[4\left(\frac{\pi}{4} - y\right)^2 - \pi\left(\frac{\pi}{4} - y\right) + 4n^2\right] ln\left(1 + tan\left(\frac{\pi}{4} - y\right)\right)(-dy)$$

$$= \int\limits_0^{\frac{\pi}{4}} \left(\frac{\pi^2}{4} - 2\pi y - 4y^2 - \frac{\pi^2}{4} + \pi y + 4n^2\right) ln\left(1 + \frac{1 - tan\, y}{1 + tan\, y}\right) dy$$

$$= \int\limits_0^{\frac{\pi}{4}} (4y^2 - \pi y + 4n^2)\,(ln\, 2 - ln(1 + tan\, y))dy =$$

$$= ln\, 2 \int\limits_0^{\frac{\pi}{4}} (4y^2 - \pi y + 4n^2)\, dy - \int\limits_0^{\frac{\pi}{4}} (4y^2 - \pi y + 4n^2)\, ln(1 + tan\, y)\, dy \Rightarrow$$

$$\Omega = ln\, 2\left(4\frac{y^3}{3}\Big|_0^{\frac{\pi}{4}} - \pi\frac{y^2}{2}\Big|_0^{\frac{\pi}{4}} + 4n^2 y\Big|_0^{\frac{\pi}{4}}\right) - \Omega_n \Rightarrow$$

$$2\Omega_n = ln\, 2\left(\frac{4}{3} \cdot \frac{\pi^3}{64} - \frac{\pi}{2} \cdot \frac{\pi^2}{16} + 4n^2 \cdot \frac{\pi}{n}\right) \Rightarrow$$

$$\Omega_n = \frac{\ln 2}{2}\left(\frac{\pi^3}{48} - \frac{\pi^3}{32} + n^2\pi\right) \Rightarrow \Omega_n = \frac{\ln 2}{2}\left(n^2\pi - \frac{\pi^3}{96}\right)$$

$$\Omega = \lim_{n\to\infty} \frac{\frac{\ln 2}{2}\left(n^2\pi - \frac{\pi^3}{96}\right)}{\frac{n(n+1)}{2}} = \pi \ln 2$$

3.65

$$\begin{matrix} \sin\alpha + \sin\beta + \sin\gamma = 2\sin(\alpha+\beta+\gamma) \\ \cos\alpha + \cos\beta + \cos\gamma = 2\cos(\alpha+\beta+\gamma) \end{matrix} \quad \textit{(1)}$$

Let $z_1 = \cos\alpha + i\sin\alpha, z_2 = \cos\beta + i\sin\beta, z_3 = \cos\gamma + i\sin\gamma, z_1, z_2, z_3 \in \mathbb{C}$ *with*

$$|z_1| = |z_2| = |z_3| = 1$$

From (1) $\Rightarrow z_1 + z_2 + z_3 = 2(\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma))$ *(1)*

But $z_1 \cdot z_2 \cdot z_3 = \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)$ *(2)*

From (1)+(2)$\Rightarrow z_1 + z_2 + z_3 = 2z_1 \cdot z_2 \cdot z_3$ *(3)*

But $|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1 \cdot \overline{z_1} = 1, z_2\overline{z_2} = 1, z_3\overline{z_3} = 1$

From (3) $\Rightarrow \overline{z_1 + z_2 + z_3} = 2\overline{z_1 z_2 z_3} \Rightarrow \overline{z_1} + \overline{z_2} + \overline{z_3} = 2\overline{z_1}\,\overline{z_2}\,\overline{z_3} \Rightarrow$

$$\Rightarrow \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{2}{z_1 z_2 z_3} \Rightarrow z_1 z_2 + z_2 z_3 + z_1 z_3 = 2 \Rightarrow$$

$$\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) + i(\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)) = 2$$

$$\left.\begin{matrix} \Rightarrow \cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 2 \\ \text{But } \cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = a \end{matrix}\right\} \Rightarrow a = 2$$

$$\lim_{x\to a} \frac{\sqrt{x^2 - a^2}}{\sqrt{x-a} + \sqrt{x} - \sqrt{a}} = \lim_{x\to 2} \frac{\sqrt{x^2 - 4}}{\sqrt{x-2} + \sqrt{x} - \sqrt{2}}$$

$\sqrt{x-2} = t, t \ge 0 \Rightarrow x - 2 = t^2 \Rightarrow x = t^2 + 2, t \to 0$ *because* $x \to 2$

$$= \lim_{\substack{t\to 0 \\ t>0}} \frac{\sqrt{(t^2+2)^2 - 4}}{t + \sqrt{t^2+2} - \sqrt{2}} = \lim_{\substack{t\to 0 \\ t>0}} \frac{\sqrt{t^4 + 4t^2 + 4 - 4}\left(t + \sqrt{t^2+2} + \sqrt{2}\right)}{t^2 + 2 + \sqrt{t^2+2} + t^2 + 2 - 2}$$

$$= \lim_{\substack{t\to 0\\ t>0}} \frac{\sqrt{t^4+4+2}\left(t+\sqrt{t^2+2}+\sqrt{2}\right)}{2t\left(t+\sqrt{t^2+2}\right)} =$$

$$\lim_{\substack{t\to 0\\ t>0}} \frac{t\sqrt{t^4+4}\left(t+\sqrt{t^2+2}+\sqrt{2}\right)}{2t\left(t+\sqrt{t^2+2}\right)} = \frac{2\cdot\sqrt{2}}{2\cdot\sqrt{2}} = 2$$

3.66

If $a = 0$ *its easy* $\Rightarrow \Omega = e^{2a}$

$$\Omega = lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n} e^{\frac{2na+(i+j)b}{n}} = lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n} e^{2a}\cdot e^{\frac{(i+j)b}{n}} =$$
$$e^{2a}\, lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n} e^{\frac{bi}{n}}\cdot e^{\frac{bi}{n}} \quad (1)$$

But $lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n} e^{\frac{bi}{n}}\cdot e^{\frac{bj}{n}} = \frac{1}{2}\left(lim_{n\to\infty} \frac{1}{n^2}\left(\sum_{k=1}^{n} e^{\frac{bk}{n}}\right)\right)^2 - \frac{1}{n^2}\sum_{k=1}^{n} e^{\frac{2bj}{n}}$ *(2)*

$$lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} e^{\frac{bk}{n}}\right)^2 = \left(\int_0^1 e^{bx}\,dx\right)^2 = \left(\frac{e^{bx}}{b}\Big|_0^1\right)^2 = \left(\frac{e^b-1}{b}\right)^2 \quad (3)$$

$$lim_{n\to\infty} \frac{1}{n^2}\cdot\sum_{k=1}^{n} e^{\frac{2bi}{n}} = lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} e^{\frac{2bi}{n}}\right) = 0 \quad (4)$$

Because $\frac{1}{n}\sum_{k=1}^{n} e^{\frac{2bi}{4}}$ *its an convergent sequence.*

Form (1) + (2) + (3) + (4) $\Rightarrow \Omega = e^{2a}\cdot\frac{1}{2}\cdot\left(\frac{e^b-1}{b}\right)^2$

3.67

$$\Omega = \lim_{n\to\infty}\left(\frac{2x_n+3y_n+5z_n}{5(e+\gamma)}\right)^n =$$

$$= lim_{n\to\infty}\left(1+\frac{2x_n+3y_n+5z_n-5(e+\gamma)}{5(e+\gamma)}\right)^n = e^{\frac{1}{5(e+\gamma)} lim_{n\to\infty} n[2(x_n-e)+3(y_n-e)+5(z_n-\gamma)]} \quad (1)$$

$$\lim_{n\to\infty} n[2(x_n-e)] = 2\lim_{n\to\infty}\frac{x_n-e}{\frac{1}{n}} = 2\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n - e}{\frac{1}{n}} \overset{(Heine)}{=} 2\lim_{n\to\infty}\frac{\left(1+\frac{1}{x}\right)^x - e}{\frac{1}{x}} =$$

$$= 2\lim_{n\to\infty}\frac{(1+t)^{\frac{1}{t}}-e}{t} = 2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^{\frac{1}{t}}\left[-\frac{1}{t^2}\cdot ln(1+t)+\frac{1}{t}\cdot\frac{1}{t+1}\right]$$

$$= 2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\left[\frac{-(t+1)\,ln(t+1)+t}{t^3+t^2}\right] = 2\lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\left[-\frac{ln(t+1)-1+1}{3t^2+2t}\right] =$$

$$= 2\,lim_{n\to\infty}\left(1+\frac{1}{t}\right)^t\cdot\frac{-\,ln(t+1)}{t(2+3t)} = 2\cdot e\cdot\left(-\frac{1}{2}\right) = -e \quad (2)$$

Now, using Cesaro – Stolz for $\frac{0}{0}$

$$\lim_{n\to\infty} n[3(yn-e)] = 3\lim_{n\to\infty}\frac{y_n-e}{\frac{1}{n}} = 3\lim_{n\to\infty}\frac{y_{n+1}-e-y_n+e}{\frac{1}{n+1}-\frac{1}{n}} = 3\lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{-\frac{1}{n(n+1)}} =$$

$$= 3\,lim_{n\to\infty} -\frac{1}{(n-1)!} = 0 \quad (2)$$

$$\lim_{n\to\infty} n[5(z_n-\gamma)] = 5\lim_{n\to\infty}\frac{z_n-\gamma}{\frac{1}{n}} = 5\lim_{n\to\infty}\frac{z_{n+1}-\gamma-z_n+\gamma}{\frac{1}{n+1}-\frac{1}{n}} =$$

$$= 5\lim_{n\to\infty}\frac{\frac{1}{n+1}-ln(n+1)+ln\,n}{-\frac{1}{n(n+1)}}$$

$$= 5\lim_{n\to\infty}\frac{\frac{1}{n+1}-ln\left(\frac{n+1}{n}\right)}{-\frac{1}{n(n+1)}} = 5\lim_{n\to\infty}\frac{1-(n+1)\,ln\left(1+\frac{1}{n}\right)}{-\frac{1}{n}}$$

$$= 5\lim_{n\to\infty}\frac{1-(x+1)\,ln\left(1+\frac{1}{x}\right)}{-\frac{1}{x}} = 5\lim_{n\to\infty}\frac{1-\left(\frac{1}{t}+1\right)ln(1+t)}{-t}$$

$$= 5\lim_{t\to 0}\frac{t-(1+t)\,ln(1+t)}{-t^2} = 5\lim_{t\to 0}\frac{1-ln(1+t)-1}{-2t}$$

$$= 5\,lim_{n\to\infty}\frac{ln(1+t)}{2t} = \frac{5}{2} \quad (3)$$

From (1)+(2)+(3)⇒ $\Omega = e^{\frac{1}{5(e+\gamma)}\left(-e+\frac{5}{2}\right)}$

3.68

Let $a > b \geq 1$ *now*

$$\left(\sqrt{a}+\sqrt{b}\right)\sqrt[4]{ab} \le \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\sqrt[n]{a^k b^{n-k}} \le \left(\sqrt{a}+\sqrt{b}\right)^2$$

$$\Leftrightarrow \left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}} \le \lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{a}{b}\right)^{\frac{k}{n}} \le \left(1+\sqrt{\frac{a}{b}}\right)^2$$

$$\Leftrightarrow \left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}} \le 2\int_0^1 \left(\frac{a}{b}\right)^x dx \le \left(1+\sqrt{\frac{a}{b}}\right)^2 \Leftrightarrow \left(1+\sqrt{\frac{a}{b}}\right)\sqrt[4]{\frac{a}{b}} \le 2\frac{\frac{a}{b}-1}{\log\left(\frac{a}{b}\right)} \le \left(1+\sqrt{\frac{a}{b}}\right)^2$$

$$\Leftrightarrow (1+m)\sqrt{m} \le \frac{m^2-1}{\log m} \le (1+m)^2 \text{ where } m=\sqrt{\frac{a}{b}}$$

$$\text{Let } f(m)=\sqrt{m}-\frac{1}{\sqrt{m}}-\log m \text{ for all } m\ge 1$$

$$f'(m)=\frac{1}{2\sqrt{m}}+\frac{1}{2m^{\frac{3}{2}}}-\frac{1}{m}=\frac{m-2\sqrt{m}+1}{2m^{\frac{3}{2}}}=\frac{\left(\sqrt{m}-1\right)^2}{2m^{\frac{3}{2}}}\ge 0$$

$$\therefore f(m)\ge f(1)=0 \Rightarrow (1+m)\sqrt{m}\le\frac{m^2-1}{\log m}$$

$$\text{Let } \varphi(m)=\log m-\frac{m-1}{m+1} \text{ for all } m\ge 1, \varphi'(m)=\frac{1}{m}-\frac{2}{(m+1)^2}=\frac{m^2+1}{m(m+1)^2}>0$$

$$\therefore \varphi(m)\ge\varphi(1)=0\Rightarrow\frac{m^2-1}{\log m}\le(1+m)^2$$

3.69

We use Cesaró – Stolz ⇒

$$\ln\Omega=\ln\sqrt[n^4]{a_n}=\frac{\ln a_n}{n^4}\Rightarrow\lim_{n\to\infty}\ln\Omega=\lim_{n\to\infty}\frac{\ln a_n}{n^4}=\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{(n+1)^4-n^4}=$$

$$=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{4n^3+6n^2+4n+1}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{n^3}\cdot\frac{n^3}{4n^3+6n^2+4n+1}\quad (1)$$

$$\lim_{n\to\infty}\frac{\ln\frac{a_{n+1}}{a_n}}{n^3}=\lim_{n\to\infty}\frac{\ln\frac{a_{n+2}}{a_{n+1}}-\ln\frac{a_{n+1}}{a_n}}{(n+1)^3-n^3}=\lim_{n\to\infty}\frac{\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{3n^2+3n+1}=\lim_{n\to\infty}\frac{\ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{n^2}\cdot\frac{n^2}{3n^2+3n+1}\quad (2)$$

$$\lim_{n\to\infty}\frac{ln\frac{ln\, a_n\cdot a_{n+2}}{a_{n+1}^2}}{n^2}=\lim_{n\to\infty}\frac{ln\frac{a_{n+1}\cdot a_{n+3}}{a_{n+2}^2}\cdot ln\frac{a_n\cdot a_{n+2}}{a_{n+1}^2}}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{ln\frac{a_{n+1}^3\cdot a_{n+3}}{a_n\cdot a_{n+2}^3}}{2n+1}=$$

$$=lim_{n\to\infty}\frac{ln\frac{a_{n+2}^3\cdot a_{n+4}}{a_{n+1}\cdot a_{n+3}}-ln\frac{a_{n+1}^3\cdot a_{n+3}}{a_n\cdot a_{n+2}^3}}{(2n+3)-(2n+1)}=lim_{n\to\infty}\frac{ln\frac{a_{n+2}^6\cdot a_n\cdot a_{n+4}}{a_{n+3}^4\cdot a_{n+1}^4}}{2}=\frac{10}{2}=5\ \textit{(3)}$$

From (1)+(2)+(3)$\Rightarrow lim_{n\to\infty} ln\,\Omega=\frac{5}{12}\Rightarrow lim_{n\to\infty}\Omega=e^{\frac{5}{12}}=\sqrt[12]{e^5}$

3.70

$$\sum_{k=0}^{n-1}(n-k)C_{2n+1}^{2n-2k}=\sum_{k=0}^{n-1}(n-k)\frac{(2n+1)!}{(2n-2k)!\,(2k+1)!}=$$

$$=\sum_{k=0}^{n-1}(n-k)\cdot\frac{(2n+1)(2n)!}{(2n-2k)(2n-2k-1)!\,(2k+1)!}=$$

$$=\frac{2n+1}{2}\sum_{k=0}^{n-1}C_{2n}^{2k+1}=\frac{2n+1}{2}(C_{2n}^1+C_{2n}^3+\cdots+C_{2n}^{2n-1})=$$

$$=\frac{2n+1}{2}\cdot 2^{2n-1}=(2n+1)\cdot 2^{2n-2}\quad\textit{(1)}$$

From (1) $\Rightarrow\Omega=lim_{n\to\infty}\frac{(2n+1)\cdot 2^{2n-2}}{2^{2n}(n+1)}=2\cdot 2^{-2}=\frac{1}{2}$

3.71

Numerator $=(n-1)\frac{1}{n}+(n-2)\left(\frac{1}{n}+\frac{1}{n-1}\right)+\cdots+\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{2}\right)=$

$$=\frac{1}{n}[(n-1)+(n-2)+\cdots+1]+\frac{1}{n-1}[(n-2)+(n-3)+\cdots+1]+$$

$$+\frac{1}{n-2}[(n-3)+\cdots+1]+\cdots+\frac{1}{2}\ \textit{(1)}$$

$$=\frac{1}{n}\cdot\frac{n(n-1)}{2}+\frac{1}{n-1}\cdot\frac{(n-1)(n-2)}{2}+\frac{1}{n-2}\cdot\frac{(n-2)(n-3)}{2}+\cdots+\frac{1}{3}\cdot\frac{3\times 2}{2}+\frac{1}{2}$$

$$=\frac{1}{2}(n-1)+\frac{1}{2}(n-2)+\frac{1}{2}(n-3)+\cdots+\frac{1}{2}(2)+\frac{1}{2}=$$

$= \frac{1}{2}[(n-1)+(n-2)+\cdots+1] = \frac{n(n-1)}{4}$. *Also,* $(n+1)^3 - n^3 = 3n^2+3n+1$

$$\therefore \Omega = \lim_{n\to\infty} \frac{\frac{1}{4}n(n-1)}{3n^2+3n+1} = \lim_{n\to\infty} \frac{1}{4}\left[\frac{\left(1-\frac{1}{n}\right)}{3+3\left(\frac{1}{n}\right)+\frac{1}{n^2}}\right] = \frac{1}{4}\cdot\frac{1-0}{3+0+0} = \frac{1}{12}$$

3.72

$$\text{Let } f:[n,n+1]\to\mathbb{R}, f(x) = x\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}.$$

From Lagrange theorem $\Rightarrow \exists c \in (n,n+1)$ *then:* $f(n+1)-f(n) = f'(c) \Rightarrow (n+1)\sqrt[5n+5]{(5n+5)\cos\frac{\pi}{a+1}} - n\sqrt[5n]{5n\cos\frac{\pi}{4}} = f'(c) \Rightarrow$

$$lim_{n\to\infty}\left[(n+1)\sqrt[5n+5]{(5n+5)\cos\frac{\pi}{n+1}} - n\sqrt[5n]{5n\cos\frac{\pi}{4}}\right] = lim_{n\to\infty} f'(c) \quad (1)$$

Because $c \in (n,n+1)$ *and* $n\to\infty$ *we can assume, WLOG,*

$$lim_{n\to\infty} f'(c) = lim_{n\to\infty} f'(n) \quad (2)$$

$$f'(x) = \left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}} + x\left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}\left[-\frac{1}{5x^2}\ln\left(5x\cos\frac{\pi}{x}\right)+\frac{1}{5x}\cdot\frac{5\cos\frac{\pi}{x}+5x\sin\frac{\pi}{x}\cdot\frac{\pi}{x^2}}{5x\cos\frac{\pi}{x}}\right]$$

$$\Rightarrow f'(x) = \left(5x\cos\frac{\pi}{x}\right)^{\frac{1}{5x}}\left[1-\frac{\ln\left(5x\cos\frac{\pi}{x}\right)}{5x}+\frac{1}{5}\cdot\frac{5\cos\frac{\pi}{x}+5\sin\frac{\pi}{x}\cdot\frac{\pi}{x}}{5x\cos\frac{\pi}{x}}\right] \quad (3)$$

From (1)+(2)+(3) $\Rightarrow$

$$\Omega = lim_{n\to\infty}\left(5n\cos\frac{\pi}{n}\right)^{\frac{1}{5n}}\left[1-\frac{\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n}+\frac{1}{5}\cdot\frac{5\cos\frac{\pi}{n}+5\sin\frac{\pi}{n}\cdot\frac{\pi}{n}}{5n\cos\frac{\pi}{n}}\right] \quad (4)$$

$$lim_{n\to\infty}\left(5n\cos\frac{\pi}{n}\right)^{\frac{1}{5n}} = \sqrt[5]{lim_{n\to\infty}\sqrt[n]{5n\cos\frac{\pi}{n}}} = \sqrt[5]{lim_{n\to\infty}\frac{(5n+5)\cos\frac{\pi}{n+1}}{5n\cos\frac{\pi}{n}}} = 1 \quad (5)$$

$$\lim_{n\to\infty}\frac{\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n} = \lim_{n\to\infty}\frac{\ln\left((5n+5)\cos\frac{\pi}{5n+5}\right)-\ln\left(5n\cos\frac{\pi}{5n}\right)}{5n+5-5n} =$$

$$= lim_{n\to\infty}\ln\left(\frac{(5n+5)\cos\frac{\pi}{n+1}}{5n\cos\frac{\pi}{n}}\right) = \ln 1 = 0 \quad (6)$$

$$lim_{n\to\infty}\frac{5\,cos\frac{\pi}{n}+5\,cos\frac{\pi}{n}\cdot\frac{\pi}{n}}{5n\,cos\frac{\pi}{n}} = 0 \quad (7)$$

From (4)+(5)+(6)+(7)⇒ $\Omega = 1$.

3.73

For $x \neq 0, |x| < \frac{1}{n}$,

$$tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots \qquad sin\, x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

∴ *For* $x \neq 0, |x| < \frac{1}{n}, 1 \leq k \leq n$

$$tan^{-1}(kx) = kx - \frac{1}{3}k^3x^3 + \frac{1}{5}k^5x^5 - \cdots,\ sin(kx) = kx - \frac{1}{6}k^3x^3 + \frac{1}{120}k^5x^5 - \cdots$$

$$\Rightarrow \prod_{k=1}^{n} tan^{-1}(kx) = \prod_{k=1}^{n}\left[kx - \frac{1}{3}k^3x^3 + \frac{1}{5}k^5x^5 \ldots\right]$$

$$= n!\,x^n\prod_{k=1}^{n}\left[1 - \frac{1}{3}k^2x^2 + \frac{1}{5}k^4x^4 - \cdots\right] = n!\,x^n\left[1 - \frac{1}{3}x^2\sum_{k=1}^{n}k^2 + 0(x^4)\right] =$$

$$= n!\,x^m\left[1 - \frac{1}{3}x^2\cdot\frac{n(n+1)(2n+1)}{6} + 0(x^4)\right] =$$

$$= n!\,x^n\left[1 - \frac{1}{18}x^2n(n+1)(2n+1) + 0(x^4)\right]$$ *Similarly,*

$$\prod_{k=1}^{n} sin(kx) = n!\,x^n\left[1 - \frac{1}{36}x^2n(n+1)(2n+1) + 0(x^4)\right]$$

$$\prod_{k=1}^{n} tan^{-1}(kx) - \prod_{k=1}^{n} sin(kx) = n!\,x^n\left[-\frac{1}{36}x^2n(n+1)(2n+1) + 0(x^4)\right]$$

∴ *For* $x \neq 0, |x| < \frac{1}{n}; \frac{1}{x^{n+2}}\left[\prod_{k=1}^{n} tan^{-1}(kx) - \prod_{k=1}^{n} sin(kx)\right] =$

$= -\frac{1}{36}n!\,n(n+1)(2n+1) + 0(x^2)$. *Taking limit as* $x \to 0^+$, *we get*

$$\Omega = -\frac{1}{36}(n+1)!\,(2n^2 + n)$$

3.74

$$\int_{1}^{\frac{\sqrt{2}}{2}} x^{n-1} arcsin^n x\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} sin^{n-1} t \cdot t^n \cdot cos\, t\, dt = arcsin\, x = t \Rightarrow x = sin\, t \Rightarrow$$

$$\Rightarrow dx = cos\, t\, dt; x = \frac{1}{2} \Rightarrow t = \frac{\pi}{6} \wedge x = \frac{\sqrt{2}}{2} \Rightarrow t = \frac{\pi}{4}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{sin^n t}{n}\right)' t^n dt = \frac{1}{n} \cdot sin^n t \cdot t^n \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{sin^n t}{n} \cdot nt^{n-1} dt$$

$$= \frac{1}{n}\left(\left(\frac{\sqrt{2}}{2} \cdot \frac{\pi}{4}\right)^n - \left(\frac{1}{2} \cdot \frac{\pi}{6}\right)^n\right) - \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} sin^n t \cdot t^{n-1} dt \quad (1)$$

From (1) $\Rightarrow \Omega_n = \frac{1}{n}\left[\left(\frac{\sqrt{2}\pi}{8}\right)^n - \left(\frac{\pi}{12}\right)^n\right] \Rightarrow lim_{n\to\infty} n^2\, \Omega_n = lim_{n\to\infty} n\left[\left(\frac{\sqrt{2}n}{8}\right)^n - \left(\frac{\pi}{12}\right)^n\right] = 0$

because $lim_{n\to\infty} n^\alpha a^n = 0, \alpha > 1, n \in \mathbb{N}^*; a \in (-1,1)$

3.75

Let $a_n = \frac{1}{n}\left(arctan\, n + \frac{1}{2} arctan(n-1) + \cdots + \frac{1}{n} arctan\, 1\right)$

$$|a_n| = \frac{1}{n}\left|arctan\, n + \frac{1}{2} arctan(n-1) + \cdots + \frac{1}{n} arctan\, 1\right| \le$$

$$\le \frac{1}{n}|arctan\, n| + \frac{1}{2}|arctan(n-1)| + \cdots + \frac{1}{n}|arctan\, 1| \le \frac{\pi}{2} \frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{2} \quad (1)$$

But $lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} \overset{C.S}{=} lim_{n\to\infty} \frac{1}{n+1} = 0$ *(2)*

From (1) + (2) $\Rightarrow \Omega = 0$.

3.76

Let $a_n = \prod_k^n \left(1 + \frac{k^4}{n^5}\right)^{1+\frac{k^4}{n^5}} \Rightarrow ln\, a_n = \sum_{k=1}^n \left(1 + \frac{k^4}{n^5}\right) ln\left(1 + \frac{k^4}{n^5}\right)$. *Now we show:*

$$x \le (1+x)\, ln(1+x) \le x + \frac{x^2}{2}, \forall x \ge 0 \quad (1)$$

$$f(x) = (1+x)\,ln(1+x) - x - \frac{x^2}{2}; f:[0,+\infty] \to \mathbb{R}; f'(x) = ln(1+x) - x,$$

$f''(x) = \frac{-x}{1+x} \le 0, f'(0) = 0, f(0) = 0 \Rightarrow f(x) \le 0, \forall x \ge 0$. *Similarly for left side. From (1)*

$$\Rightarrow \frac{k^4}{n^5} \le \left(1 + \frac{k^4}{n^5}\right) ln\left(1 + \frac{k^4}{n^5}\right) \le \frac{k^4}{n^5} + \frac{k^8}{2n^{10}} \Rightarrow$$

$$\sum_{k=1}^{n} \frac{k^4}{n^5} \le ln\, a_n \le \sum_{k=1}^{n} \frac{k^4}{n^5} + \frac{1}{2}\sum_{k=1}^{n} \frac{k^8}{n^{10}} \quad (2)$$

But $lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5} = lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left(\frac{k}{n}\right)^4 = \int_0^1 x^4 dx = \left.\frac{x^5}{5}\right|_0^1 = \frac{1}{5}$ *(3)*

$$lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^8}{n^{10}} \overset{C.S.}{=} lim_{n\to\infty} \frac{(n+1)^8}{(n+1)^{10} - n^{10}} = lim_{n\to\infty} \frac{(n+1)^8}{n^9 + \cdots + 1} = 0 \quad (4)$$

From (2)+(3)+(4)$\Rightarrow lim_{n\to\infty} ln\, a_n = \frac{1}{5} \Rightarrow \Omega = lim_{n\to\infty} a_n = e^{\frac{1}{5}}$

3.77

$\sum_{i=i}^{k} \binom{i}{k} = \binom{1}{k} + \binom{2}{k} + \cdots + \binom{k-1}{k} + \binom{k}{k} = 1$ *[rest are zeros]*

$$\therefore \sum_{k=0}^{k} \frac{1}{k+1}\binom{n}{k}\left(\sum_{i=1}^{k}\binom{i}{k}\right) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1} =$$

$$= \frac{1}{n+1}(2^{n+1} - 1) \Rightarrow \frac{1}{n+1} + \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\left(\sum_{i=1}^{k}\binom{i}{k}\right) = 2^{n+1} \Rightarrow$$

$$\Rightarrow \Omega = \lim_{n\to\infty}\left[\frac{1}{n+1} + \sum_{k=0}^{n} \frac{1}{k+1}\left(\sum_{i=1}^{k}\binom{i}{k}\binom{n}{k}\right)\right]^{\frac{1}{n}} = \lim_{n\to\infty} 2^{1+\frac{1}{n}} = 2$$

3.78

For $1 \le k \le n; tan^{-1}(k+1) - tan^{-1}(k) = tan^{-1}\left(\frac{k+1-k}{1+(k+1)k}\right) =$

$$= tan^{-1}\left(\frac{1}{1+k+k^2}\right) < \frac{1}{1+k+k^2} < \frac{1}{k^2}$$

$$\therefore \prod_{k=1}^{n}[tan^{-1}(k-1) - tan^{-1}(k)] < \prod_{k=1}^{n}\frac{1}{k^2} = \left(\frac{1}{n!}\right)^2 \Rightarrow$$

$$\Rightarrow 0 < n!\prod_{k}^{n}[tan^{-1}(k+1) - tan^{-1}(k)] < \frac{1}{n!}$$

As $lim_{n\to\infty}\frac{1}{n!} = 0$*, by the sandwich theorem, we get:*

$$\lim_{n\to\infty}\left[n!\prod_{k=1}^{n}(tan^{-1}(k+1) - tan^{-1}(k))\right] = 0$$

3.79

$$\Omega = lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{e\cdot e^{n^2}} + \frac{2}{e^2\cdot e^{(n-2)^2}} + \cdots + \frac{n}{e^n\cdot e}\right) \quad (1)$$

$$Let\ a_n = \frac{1}{n}\left(\frac{1}{e\cdot e^{n^2}} + \frac{2}{e^2\cdot e^{(n-1)^2}} + \cdots + \frac{n}{e^n\cdot e}\right)$$

$$|a_n| = \frac{1}{n}\left(\frac{1}{e\cdot e^{n^2}} + \frac{2}{e^2\cdot e^{(n-1)^2}} + \cdots + \frac{n}{e^n\cdot e}\right) \leq \frac{1}{e}\cdot\frac{1}{n}\left(\frac{1}{e} + \frac{2}{e^2} + \cdots + \frac{n}{e^n}\right) \quad (2)$$

$$\left.\begin{array}{l} But\ \lim_{n\to\infty}\dfrac{\frac{1}{e} + \frac{2}{e^2} + \cdots + \frac{n}{e^n}}{n} \overset{C.S.}{=} \lim_{n\to\infty}\dfrac{n+1}{e^{n+1}} \\ \lim_{n\to\infty}\dfrac{x+1}{e^{x+1}} \overset{L'H}{=} \lim_{n\to\infty}\dfrac{1}{e^{x+1}} = 0 \end{array}\right\} \Rightarrow$$

$$lim_{n\to\infty}\frac{\frac{1}{e}+\frac{2}{e^2}+\cdots+\frac{n}{e^n}}{n} = 0 \quad (3)$$

From (2) + (3) $\Rightarrow \Omega = 0$.

3.80

$$Let\ a_n = \sum_{k=1}^{n}\frac{(k+n)^5}{7+arctan(k+n)+(k+n)^6}.$$

Because $-\frac{\pi}{2} < arctan\,x < \frac{\pi}{2}, \forall x \in \mathbb{R} \Rightarrow$

$$\Rightarrow \sum_{k=1}^{n}\frac{(k+n)^5}{7+\frac{\pi}{2}+(k+n)^6} < a_n < \sum_{k=1}^{n}\frac{(k+n)^5}{7-\frac{\pi}{2}+(k+n)^6} \quad (1)$$

Now we want to show this:

$$lim_{n\to\infty}\sum_{k=1}^{n}\frac{(k+n)^5}{\alpha+(k+n)^6} = lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(k+n)} \quad (2), \forall\alpha > 0$$

(2) $\Leftrightarrow lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{(k+n)^5}{\alpha+(k+n)^6}-\frac{1}{(k+n)}\right]=0 \Leftrightarrow lim_{n\to\infty}\sum_{k=1}^{n}\left[\frac{(k+n)^6-\alpha-(k+n)^6}{(\alpha+(k+n)^6)(k+n)}\right]=0 \Leftrightarrow$

$\Leftrightarrow lim_{n\to\infty} -\alpha\sum_{k=1}^{n}\frac{1}{(\alpha+(k+1)^6)(k+n)}=0$*, which, obvious its true. But*

$$lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k+n}=lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{1}{1+x}dx=ln(1+x)\Big|_0^1=ln\,2 \quad (3)$$

From (1)+(2)+(3)$\Rightarrow \Omega=ln\,2$.

3.81

For $|x|<1, ln(1+x)=x-\frac{x^2}{2}+\cdots$*. Also,*

$$sinh\,x=\frac{1}{2}(e^x-e^{-x})\Rightarrow ln(1+(sinh\,x)^n)=\left(\frac{1}{2}(e^x-e^{-x})\right)^n-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)^{2n}+\cdots$$

$=\left(\frac{1}{2}(e^x-e^{-x})\right)^n\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)^n+\cdots\right]$ *and* $ln(1+sinh\,x)=\frac{1}{2}(e^x-e^{-x})-$

$$-\frac{1}{2}\left[\frac{1}{2}(e^x-e^{-x})\right]^2+\cdots=\left(\frac{1}{2}(e^x-e^{-x})\right)\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)+\cdots\right]\Rightarrow$$

$$\Rightarrow(ln(1+sinh\,x))^n=\left(\frac{1}{2}(e^x-e^{-x})\right)^n\left[1-\frac{1}{2}\left(\frac{1}{2}(e^x-e^{-x})\right)+\cdots\right]^n=$$

$$=\frac{1}{2^n}(e^x-e^{-x})^n\left[1-\frac{n}{4}(e^x-e^{-x})+\cdots\right]$$

For, sufficiently small x

$$\therefore ln(1+(sinh\,x)^n)-(ln(1+sinh\,x))^n=-\frac{1}{2^{2n+1}}(e^x-e^{-x})^{2n}+$$

$+(e^x-e^{-x})^{3n}$ *and higher power* $+\frac{n}{2^{n+2}}(e^x-e^{-x})^{n+1}+(e^x-e^{-x})^{n+2}$ *and higher power*

$\Rightarrow\therefore$ *for* $x\neq 0$*,* x *sufficiently small,* $\frac{ln(1+(sinh\,x)^n)-(ln(1+sinh\,x))^n}{x^{n+1}}=$

$=-\frac{1}{2^{2n+1}}\left(\frac{e^x-e^{-x}}{x}\right)^{2n}x^{n-1}+\frac{n}{2^{n+2}}\left(\frac{e^x-e^{-x}}{x}\right)^{n+1}+\left(\frac{e^x-e^{-x}}{x}\right)^{n+2}x$ *and similar expressions.*

But $lim_{x\to0}\frac{e^x-e^{-x}}{x}=lim_{x\to0}\left[\frac{e^x-1}{x}+\frac{e^{-x}-1}{(-x)}\right]=1+1=2$

$$\therefore \lim_{x\to 0} \frac{\ln[1+(\sinh x)^n]-(\ln(1+\sinh x))^n}{x^{n+1}} = 0 + \frac{n}{2^{n+2}} \cdot 2^{n+1} = \frac{n}{2} \text{ if } n > 1.$$

3.82

$$\frac{1}{\sqrt[7]{n^7+n}} \le \frac{1}{\sqrt[7]{n^7+k}} \le \frac{1}{\sqrt[7]{n^7+1}}, \forall k = \overline{1,n} \Rightarrow \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+n}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+k}} \le \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+n}} \le \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+k}} \le \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} \quad (1)$$

But $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin\frac{k}{n}}{\sqrt[7]{n^7+n}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{\sqrt[7]{n^7+n}} \cdot \frac{1}{n} \cdot \sin\frac{k}{n} = \lim_{n\to\infty} \frac{n}{\sqrt[7]{n^7+n}} \cdot \frac{1}{n} \sum_{k=1}^{n} \sin\left(\frac{k}{n}\right) =$

$$= \int_0^1 \sin x\, dx = -\cos\Big|_0^1 = 1 - \cos 1 \quad (2)$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin\left(\frac{k}{n}\right)}{\sqrt[7]{n^7+1}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{\sqrt[7]{n^7+1}} \cdot \frac{1}{n} \cdot \sin\frac{k}{n} = \lim_{n\to\infty} \frac{n}{\sqrt[7]{n^7+1}} \cdot \frac{1}{n} \sum_{k=1}^{n} \sin\frac{k}{n} =$$

$$= \int_0^1 \sin x\, dx = 1 - \cos 1 \quad (3)$$

From (1) + (2) + (3) $\Rightarrow \Omega = 1 - \cos 1.$

3.83

$$1 < \left(1 + \frac{n^{\frac{1}{n}}}{n}\right) = 1 + \frac{1}{n^{1-\frac{1}{n}}} \le 2 \text{ and } \left(1 - \frac{n^{\frac{1}{n}}}{n}\right) \le 1$$

$$1 < \left(1 + \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}} + \left(1 - \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}} \le 3$$

Also, $\sin n \in [-1,1] \ \therefore\ -3 < (\sin n)\left[\left(1 + \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}} + \left(1 - \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right] < 3$

$$\Rightarrow -\frac{3}{n} < \frac{\sin n}{n}\left[\left(1 + \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}} + \left(1 - \frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right] < \frac{3}{n}$$

As $lim_{n\to\infty}\left(-\frac{3}{n}\right) = lim_{n\to\infty}\frac{3}{n} = 0$*. By the Sandwich theorem*

$$\lim_{n\to\infty}\frac{\sin n}{n}\left[\left(1+\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}+\left(1-\frac{n^{\frac{1}{n}}}{n}\right)^{\frac{1}{n}}\right] = 0$$

3.84

$$\frac{\tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)}{1+\tan\left(\frac{x}{2^{k-1}}\right)\cdot\tan\left(\frac{x}{2^k}\right)} = \tan\left(\frac{x}{2^{k-1}}-\frac{x}{2^k}\right) = \tan\left(\frac{x}{2^k}\right)\text{; let } \tan\left(\frac{x}{2^{k-1}}\right)\tan\left(\frac{x}{2^k}\right) = \alpha$$

$$\frac{\tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)}{1+\alpha} = \tan\left(\frac{x}{2^k}\right) \Leftrightarrow \alpha\tan\left(\frac{x}{2^k}\right)+\tan\left(\frac{x}{2^k}\right) = \tan\left(\frac{x}{2^{k-1}}\right)-\tan\left(\frac{x}{2^k}\right)$$

$$\alpha = \frac{\tan\left(\frac{x}{2^{k-1}}\right)-2\tan\left(\frac{x}{2^k}\right)}{\tan\left(\frac{x}{2^k}\right)} = \frac{\tan\left(\frac{x}{2^{k-1}}\right)}{\tan\left(\frac{x}{2^k}\right)}-2 \text{ so we have: } \tan\left(\frac{x}{2^{k-1}}\right)\tan\left(\frac{x}{2^k}\right) = \frac{\tan\left(\frac{x}{2^{k-1}}\right)}{\tan\left(\frac{x}{2^k}\right)}-2$$

$$\Rightarrow \tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right) = \tan\left(\frac{x}{2^{k-1}}\right)-2\tan\left(\frac{x}{2^k}\right) \Rightarrow 2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right) =$$

$$= 2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)-2^k\tan\left(\frac{x}{2^k}\right)$$

$$\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right) = \sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)-2^k\tan\left(\frac{x}{2^k}\right)$$

Let $a_k = 2^k\tan\left(\frac{x}{2^k}\right), k = \overline{1,n} \Rightarrow \sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right) = \sum_{k=1}^{n}a_{k-1}-a_k =$

$$= a_0-a_n = \tan(x)-2^n\tan\left(\frac{x}{2^n}\right)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}2^{k-1}\tan\left(\frac{x}{2^{k-1}}\right)\tan^2\left(\frac{x}{2^k}\right) = \lim_{n\to\infty}\left(\tan x-2^n\tan\left(\frac{x}{2^n}\right)\right) = \tan x-\lim_{n\to\infty}2^n\tan\left(\frac{x}{2^n}\right) =$$

$$= \tan x-\lim_{n\to\infty}\frac{\tan\left(\frac{x}{2^n}\right)}{\frac{1}{2^n}} = \tan x-\lim_{n\to\infty}\frac{\tan\left(\frac{x}{2^n}\right)}{\frac{x}{2^n}}\cdot x = \tan x-x \Rightarrow$$

$$\Rightarrow \Omega(x) = \lim_{n\to\infty} \sum_{k=1}^{n} 2^{k-1} \tan\left(\frac{x}{2^{k-1}}\right) \tan^2\left(\frac{x}{2^k}\right) = \tan x - x$$

$$\Omega(A) + \Omega(B) + \Omega(C) > ABC - \pi \Leftrightarrow \tan A + \tan B + \tan C - A - B - C > ABC - \pi \Leftrightarrow$$

$$\Leftrightarrow \tan(A) + \tan(B) + \tan(C) > ABC \quad (1)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A + B) = \tan(\pi - C) = \frac{\tan \pi - \tan C}{1 + \tan \pi \tan C} = \frac{-\tan C}{1 + 0} = -\tan C$$

$$\Leftrightarrow \tan A + \tan B + \frac{\tan A + \tan B}{\tan A \tan B - 1} > ABC \Leftrightarrow$$

$$(\tan A + \tan B) \cdot \frac{\tan A \tan B}{\tan A \tan B - 1} > ABC >$$

$$-\tan(A + B) \tan A \tan B > ABC > \tan C \tan A \tan B > ABC$$

$$f(x) = \tan x - x > f(x) = \frac{1}{\cos^2 x} - 1 \, |\cos x| < 1 > \cos^2 x < 1 > \frac{1}{\cos^2 x} > 1 >$$

$> \frac{1}{\cos^2 x} - 1 > 0 > f(x) > 0 > f(x)$ *is an increasing function*

Let $x = 0 > f(0) = 0 > f(x) > \tan x > x > \tan A \tan B \tan C > ABC$ *(what we needed to prove)*

$\Omega(A) + \Omega(B) + \Omega(C) > ABC - \pi$ *(Q.E.D.)*

3.85

$$\Omega(a, b, c) = \sum_{n=1}^{\infty} \frac{an^2 + bn + c}{n!} \Rightarrow \Omega(a, b, c) + \Omega(b, c, a) + \Omega(c, a, b) =$$

$$= (a + b + c) \sum_{n=1}^{\infty} \frac{n^2 + n + 1}{n!} =$$

$$= (a + b + c)\left(\sum_{n=1}^{\infty} \frac{1}{(n-2)!} + 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=1}^{\infty} \frac{1}{n!}\right) = (a + b + c)(e + 2e + e - 1) \leq$$

$$\leq 3\sqrt[3]{abc}(4e - 1)$$

$$\therefore \Omega(a, b, c) + \Omega(b, c, a) + \Omega(c, a, b) \leq 3(4e - 1)\sqrt[3]{abc}$$

3.86

$$\Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right) =$$

$$= \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{n+1} - \frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+2}{(n+1)!}}{n+2}\right) + \sum_{n=0}^{\infty}\frac{1}{(n+1)!} =$$

$$= 1 + (e-1)$$

$$\therefore \Omega = \sum_{n=0}^{\infty}\left(\frac{1+\frac{2}{1!}+\frac{3}{2!}+\cdots+\frac{n+1}{n!}}{(n+1)(n+2)}\right) = e$$

3.87

$$\Omega = \lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}}-\left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{\substack{x_n\to p\\ y_n\to p}}\frac{x_n^{y_n}-y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}}-\left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{x_n\to p}\lim_{y_n\to p}\frac{x_n^{y_n}-y_n^{x_n}}{(x_n)^{-\frac{1}{y_n}}-(y_n)^{-\frac{1}{x_n}}} =$$

$$= \lim_{x_n\to p}\frac{x_n^{p}-p^{x_n}}{(x_n)^{-\frac{1}{p}}-p^{-\frac{1}{x_n}}} = \lim_{x_n\to p}\frac{\frac{x_n^p-p^p}{x_n-p}-p^p\cdot\frac{p^{(x_n-p)}-1}{(x_n-p)}}{\frac{(x_n)^{-\frac{1}{p}}-p^{-\frac{1}{p}}}{x_n-p}-\frac{p^{-\frac{1}{p}}}{px_n}\cdot\frac{p^{-\left(\frac{1}{x_n}-\frac{1}{p}\right)}-1}{-\left(\frac{1}{x_n}-\frac{1}{p}\right)}} =$$

$$= \frac{p\cdot p^{p-1}-p^p\ln p}{-\frac{1}{p}p^{-\frac{1}{p}-1}-p^{-\frac{1}{p}-2}\ln p} = p^{p+\frac{1}{p}+p}\,\frac{\ln p-1}{\ln p+1}$$

Similarly $\lim_{\substack{x_n\to p\\ y_n\to p}}\frac{x_n^{y_n}-y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}}-\left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = \lim_{y_n\to p}\lim_{x_n\to p}\frac{x_n^{y_n}-y_n^{x_n}}{(x_n)^{-\frac{1}{y_n}}-(y_n)^{-\frac{1}{x_n}}} = \lim_{y_n\to p}\frac{p^{y_n}-y_n^{p}}{p^{-\frac{1}{y_n}}-(y_n)^{-\frac{1}{p}}} =$ $p^{p+\frac{1}{p}+2}\,\frac{\ln p-1}{\ln p+1}$

$$\therefore \Omega = \lim_{n\to\infty}\frac{x_n^{y_n}-y_n^{x_n}}{\left(\frac{1}{x_n}\right)^{\frac{1}{y_n}}-\left(\frac{1}{y_n}\right)^{\frac{1}{x_n}}} = p^{p+\frac{1}{p}+2}\,\frac{\ln p-1}{\ln p+1}$$

3.88

$$\sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)} = \lim_{p\to\infty}\sum_{n=k+1}^{p}\frac{2k}{(n-k)(n+k)}$$

$$\sum_{n=k+1}^{p}\frac{2k}{(n-k)(n+k)} = \sum_{n=k+1}^{p}\frac{n+k-(n-k)}{(n-k)(n+k)} = \sum_{n=k+1}^{p}\frac{1}{n-k}-\frac{1}{n+k} = 1+\frac{1}{2}+\cdots+$$

$$+\frac{1}{p-k}-\left(\frac{1}{2k+1}+\frac{1}{2k+2}+\cdots+\frac{1}{p+k}\right) = 1+\frac{1}{2}+\cdots+\frac{1}{p-k}-ln(p-k)+$$

$$+ln(p-k)-\left(1+\frac{1}{2}+\cdots+\frac{1}{p+k}-ln(p+k)+ln(p+k)-\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}\right)\right)$$

$$\Rightarrow \lim_{p\to\infty}\sum_{n=k+1}^{p}\frac{2k}{(n+k)(n-k)} = \lim_{p\to\infty}\left(ln(p-k)-ln(p+k)+1+\frac{1}{2}+\cdots+\frac{1}{2k}\right) =$$

$$= 1+\frac{1}{2}+\cdots+\frac{1}{2k} \succ \sum_{n=k+1}^{\infty}\frac{2k}{(n-k)(n+k)} = 1+\frac{1}{2}+\cdots+\frac{1}{2k}$$

$$\Rightarrow \Omega = \lim_{k\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}-ln\left(\frac{k^2}{k+1}\right)\right) =$$

$$= \lim_{k\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}-ln(2k)+ln(2k)-ln\left(\frac{k^2}{k+1}\right)\right) =$$

$$= \gamma + \lim_{k\to\infty} ln\frac{2k^2+2k}{k^2} = \gamma + ln(2)$$

$$\Omega = \gamma + ln(2)$$

3.89

$$\Omega = \lim_{n\to\infty} ln(n)\left(\sqrt[7]{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}\,log_n^e - 1\right) =$$

$$= \lim_{n\to\infty}\frac{\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right)\,log_n^e} - 1}{log_n^e} =$$

$$= \lim_{n\to\infty} \frac{\dfrac{\left(1+\cdots+\frac{1}{n}\right) log_n^e \ -1}{\left(\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right) log_n^e}\right)^0 + \cdots + \left(\sqrt[7]{\left(1+\cdots+\frac{1}{n}\right) log_n^e}\right)^6}}{log_n^e}$$

$$\lim_{n\to\infty} \frac{1+\cdots+\frac{1}{n}}{ln(n)} \overset{Stolz\ Cesaro}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{ln\frac{n+1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}\cdot\frac{n}{n+1}}{ln\left(1+\frac{1}{n}\right)} = 1$$

$$\succ \Omega = \frac{1}{7}\lim_{n\to\infty} ln(n)\left(\left(1+\cdots+\frac{1}{n}\right) log_n^e \ -1\right) =$$

$$\frac{1}{7}\lim_{n\to\infty}\left(1+\cdots+\frac{1}{n} - ln(n)\right) = \frac{\gamma}{7} \succ \Omega = \frac{\gamma}{7}$$

3.90

$$\Omega_n(a) = \sum_{k=0}^{n}(k^2 - a^2 + 1)\,(a+k)! = \sum_{k=0}^{n}(k-a)(k+a)(a+k)! + (a+k)!$$

$$= \sum_{k=0}^{n}(k-a)(k+a+1-1)(a+k)! + (a+k)! =$$

$$= \sum_{k=0}^{n}(k-a)(a+k+1)! - (k-a)(a+k)! + (a+k)! =$$

$$= \sum_{k=0}^{n}(k-a)(a+k+1)! - (k-a-1)(a+k)! =$$

$$= (n-a)(n+1+a)! + (a+1)! \Rightarrow$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{(n-a)(n+1+a)! + (a+1)! - (a+1)!} =$$

$$= \lim_{n\to\infty} \sqrt[n]{(n-a)(n+1+a)!} = \lim_{n\to\infty} \frac{(n+1-a)(n+2+a)!}{(n-a)(n+1+a)!} =$$

$$= \lim_{n\to\infty} \frac{(n+1-a)(n+2+a)}{n-a} = \infty$$

3.91

$$\lim_{p\to\infty}\sqrt[p]{\lim_{k=1}\sum_{k=1}^{n}\frac{k^2+kp-1}{(p+k+1)!}}=\lim_{p\to\infty}\sqrt[p]{\sum_{k}^{\infty}\frac{k}{(p+k)!}-\frac{k+1}{(p+k+1)!}}=\lim_{p\to\infty}\sqrt[p]{\frac{1}{(p+1)!}}$$

$$=\lim_{p\to\infty}\frac{(p+1)!}{(p+2)!}=\lim_{p\to\infty}\frac{1}{p+2}$$

$$\therefore\lim_{p\to\infty}\sqrt[p]{\lim_{n\to\infty}\sum_{k}^{n}\frac{k^2+kp-1}{(p+k+1)!}}=0$$

3.92

We have $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \Rightarrow \sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha) \Rightarrow$

$$\Rightarrow \sin^3(3^k\sin a) = \frac{1}{4}(3\sin(3^k\sin a) - \sin(3^{k+1}\sin a))$$

$$k=1 \Rightarrow \sin^3(3\sin a) = \frac{1}{4}(3\sin(3\sin a) - \sin(3^2\sin a))\,\Big|\frac{1}{3}$$

$$k=2 \Rightarrow \sin^3(3^2\sin a) = \frac{1}{4}(3\sin(3^2\sin a) - \sin(3^3\sin a))\,\Big|\frac{1}{3^2}$$

$$\vdots$$

$$k=n \Rightarrow \sin^3(3^n\sin a) = \frac{1}{4}(3\sin(3^n\sin a) - \sin(3^{n+1}\sin a))\,\Big|\frac{1}{3^n}$$

$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3^k}\sin^3(3^k\sin a) = \lim_{n\to\infty}\frac{1}{4}\left(\sin(3\sin a) - \frac{1}{3^n}\sin(3^{n+1}\sin a)\right)$ *(1)*

$\left|\frac{1}{3^n}\cdot\sin(3^{n+1}\sin a)\right| \le \frac{1}{3^n}\to 0$ *(2) . From (1)+(2)* $\Rightarrow \Omega(a) = \frac{1}{4}\sin(3\sin a)$ *(3)*

But $\sin\alpha \le \alpha, \forall\alpha \ge 0 \Rightarrow \sin(3\sin a) \le 3\sin a \le 3a$ *(4)*

From (3)+(4)$\Rightarrow \Omega(a) \le \frac{3}{4}(a) \Rightarrow$ *inequality becomes:*

$$4\big(b\Omega(\alpha) + c\Omega(b) + a\Omega(c)\big) \le 3(ab+ac+bc) \Rightarrow$$

We must show:

$$3(ab + ac + bc) \leq 3(a^2 + b^2 + c^2) \ \textit{true.}$$

3.93

$$\text{Let } I_k = \int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\cos^{-1}(kx)\,dx. \text{ Put } x = -t.$$

$$I_k = \int_{\frac{1}{k}}^{-\frac{1}{k}} (2t^8 + 3t^6 + 1)\cos^{-1}(-kt)\,(-1)dt = \int_{\frac{1}{k}}^{-\frac{1}{k}} (2x^8 + 3x^6 + 1)\,[\pi - \cos^{-1}(kx)]dx$$

$$\Rightarrow 2I_k = \pi \int_{-\frac{1}{k}}^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\,dx \Rightarrow I_k = \pi \int_0^{\frac{1}{k}} (2x^8 + 3x^6 + 1)\,dx$$

$$= \pi\left(\frac{2}{9}x^9 + \frac{3}{7}x^7 + x\right)\Big]_0^{\frac{1}{k}} = \pi\left(\frac{2}{9k^9} + \frac{3}{7k^7} + \frac{1}{k}\right)$$

$$\Rightarrow \sum_{k=1}^{n} I_k = \frac{2\pi}{9}\left(\sum_{k=1}^{n}\frac{1}{k^9}\right) + \frac{3\pi}{7}\sum_{k=1}^{n}\frac{1}{k^7} + \pi H_n \Rightarrow \Omega_n - \pi H_n = \frac{2\pi}{9}\sum_{k=1}^{n}\frac{1}{k^9} + \frac{3\pi}{7}\sum_{k=1}^{n}\frac{1}{k^7}$$

$$\lim_{n\to\infty}(\Omega_n - \pi H_n) = \frac{2\pi}{9}\zeta(9) + \frac{3\pi}{7}\zeta(7)$$

3.94

$$I := \lim_{n\to\infty}\left(\int_{\pi}^{2\pi}\frac{|\sin(nx)|dx}{x^2}\right) \quad (*)$$

$$|\sin(nx)| = \frac{2}{\pi} - \frac{4}{\pi}\cdot\sum_{j=1}^{\infty}\frac{\cos(2jnx)}{4j^2 - 1}$$

$$\therefore I = \lim_{n\to\infty}\left(\frac{2}{\pi}\cdot\int_{\pi}^{2\pi}\frac{dx}{x^2} - \frac{4}{\pi}\cdot\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\cdot\int_{\pi}^{2\pi}\frac{\cos(2jnx)}{x^2}dx\right)$$

$$\text{But } \int_{\pi}^{2\pi}\frac{dx}{x^2} = \frac{1}{x}\Big|_{2\pi}^{\pi} = \frac{1}{\pi} - \frac{1}{2\pi} = \frac{1}{2\pi}$$

$$\text{And } \int_{\pi}^{2\pi}\frac{\cos(2jnx)dx}{x^2} = \left(\frac{1}{x^2}\right)\cdot\frac{\sin(2jnx)}{(2jn)}\Big|_{\pi}^{2\pi} + \frac{7}{2jn}\int_{\pi}^{2\pi}\frac{\sin(2jnx)}{x^3}dx = \frac{1}{jn}\int_{\pi}^{2\pi}\frac{\sin(2jnx)dx}{x^3}$$

$$\therefore I = \left(\frac{2}{\pi}\right)\left(\frac{1}{2\pi}\right) - \left(\frac{4}{\pi}\right) lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\frac{1}{(4j^2-1)j}\cdot\int_{\pi}^{2\pi}\frac{sin(2jnx)dx}{n\cdot x^3}\right) \quad (1)$$

Since $\int_{\pi}^{2\pi}\frac{sin(2jnx)dx}{x^3}$ *exists for every* $j, n \in \mathbb{N}$

$$\Rightarrow \int_{\pi}^{2\pi}\frac{dx}{x^3} \geq \int_{\pi}^{2\pi}\frac{sin(2jnx)\,dx}{x^3} \geq \int_{\pi}^{2\pi}\frac{-dx}{x^3}$$

$$\therefore \frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{dx}{x^3} \geq \frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{sin(2jnx)dx}{x^3} \geq \frac{1}{n}\int_{\pi}^{2\pi}\frac{-dx}{x^3}$$

Since $\int_{\pi}^{2\pi}\frac{dx}{x^3}$ *exists, then*

$$\lim_{n\to\infty}\left(\frac{1}{n}\cdot\int_{\pi}^{2\pi}\frac{dx}{x^3}\right) \geq \lim_{n\to\infty}\left(\frac{1}{n}\int_{\pi}^{2\pi}\frac{sin(2jnx)\,dx}{x^3}\right) \geq \lim_{n\to\infty}\left(-\frac{1}{n}\int_{\pi}^{2\pi}\frac{dx}{x^3}\right)$$

Then $lim_{n\to\infty}\left(\frac{1}{n}\int_{\pi}^{2\pi}\frac{sin(2jnx)dx}{x^3}\right) = 0$

Going to (1) we obtain: $I = \frac{1}{\pi^2} - \left(\frac{4}{\pi}\right)(0) = \frac{1}{\pi^2}$

$$\therefore \lim_{n\to\infty}\left(\int_{\pi}^{2\pi}\frac{|sin(nx)|dx}{x^2}\right) = \frac{1}{\pi^2}$$

3.95

$$\Omega = \sum_{n=1}^{\infty}\left[log\left(\frac{n^2+3n+2}{n^2+3n}\right) + \frac{1}{4}log\left(\frac{(n-1)^2+3(n-1)+2}{(n-1)^2+3(n-1)}\right) + \cdots + \frac{1}{n^2}log\frac{3}{2}\right]$$

$$= \frac{1}{1^2}log\left(\frac{3}{2}\right)$$

$$+\frac{1}{2^2}log\left(\frac{3}{2}\right) + log\left(\frac{6}{5}\right)$$

$$+\frac{1}{3^2}log\left(\frac{3}{2}\right) + \frac{1}{2^2}log\left(\frac{6}{5}\right) + log\left(\frac{10}{9}\right)$$

$$+\frac{1}{4^2}log\left(\frac{3}{2}\right) + \frac{1}{3^2}log\left(\frac{6}{5}\right) + \frac{1}{2^2}log\left(\frac{10}{9}\right) + log\left(\frac{15}{14}\right)$$

$$+\frac{1}{5^2}log\left(\frac{3}{2}\right)+\frac{1}{4^2}log\left(\frac{6}{5}\right)+\frac{1}{3^2}log\left(\frac{10}{9}\right)+\frac{1}{2^2}log\left(\frac{15}{14}\right)+log\left(\frac{21}{20}\right)$$

$$+\cdots$$

Adding columnise, we get

$$\Omega = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)\Omega_1 = \frac{\pi^2}{6}\Omega_1$$

where

$$\Omega_1 = log\left(\frac{3}{2}\right)+log\left(\frac{6}{5}\right)+log\left(\frac{10}{9}\right)+log\left(\frac{15}{14}\right)+log\left(\frac{21}{20}\right)+\cdots$$

$$= log\left(\frac{3}{2}\right)+log\left(\frac{3\times 4}{2\times 5}\right)+log\left(\frac{4\times 5}{3\times 6}\right)+log\left(\frac{5\times 6}{4\times 7}\right)+log\left(\frac{6\times 7}{5\times 8}\right)+\cdots$$

$$= log\left(\frac{3}{2}\right)+\sum_{k=1}^{\infty}log\left(\frac{(k+2)(k+3)}{(k+1)(k+4)}\right) = log\left(\frac{3}{2}\right)+log\,2 = log\,3$$

Thus, $\Omega = \frac{\pi^2}{6}log\,3$

3.96

$$\Omega_n(x) = \int_1^x t^0+t^1+\cdots+t^{n-1}dt = \Big|_1^x t+\frac{t^2}{2}+\frac{t^3}{3}+\cdots+\frac{t^n}{n}dt =$$

$$= x-1+\frac{x^2-1}{1}+\frac{x^3-1}{3}+\cdots+\frac{x^n-1}{n}$$

$$\Omega = \lim_{x\to 1} tan^{-1}(nx-n)\cdot\frac{1+\frac{1-x}{tan^{-1}(nx-n)}+\cdots+\frac{1-x^n}{n\,tan^{-1}(nx-n)}}{(x-1)^2} =$$

$$= \lim_{x\to 1}\frac{n+\frac{n(1-x)}{tan^{-1}(nx-n)}+\cdots+\frac{n(1-x^n)}{n\,tan^{-1}(nx-n)}}{x-1} =$$

$$\lim_{x\to 1}\frac{\sum_{k=1}^{n}\left(1+\frac{n(1-x^k)}{k\,tan^{-1}(nx-n)}\right)}{x-1}$$

$$\lim_{x\to1}\frac{1+\dfrac{n(1-x^k)}{k\tan^{-1}(nx-n)}}{x-1}=\lim_{x\to1}n\cdot\frac{1+\dfrac{n(1-x^k)}{k\tan^{-1}(nx-n)}}{\tan^{-1}(nx-n)}=$$

$$\overset{L'H}{\underset{\frac{0}{0}}{=}}\lim_{x\to1}n\cdot\frac{k\tan^{-1}(nx-n)+n(1-x^k)}{k(\tan^{-1}(nx-n))^2}\overset{L'H}{\underset{\frac{0}{0}}{=}}$$

$$=\lim_{x\to1}n\cdot\frac{k\dfrac{1}{(nx-n)^2+1}\cdot n-nkx^{k-1}}{2k\tan^{-1}(nx-n)\dfrac{1}{(nx-n)^2+1}\cdot n}$$

$$=\frac{n}{2}\lim_{x\to1}\frac{\dfrac{n}{(nx-n)^2+1}-nx^{k-1}}{nx-n}=\frac{1}{2}\lim_{x\to1}\frac{\dfrac{1}{(nx-n)^2+1}-x^{k-1}}{x-1}=$$

$$=\frac{1}{2}\lim_{x\to1}\frac{1-x^{k-1}((nx-2)^2+1)}{((nx-n)^2+1)(x-1)}=\lim_{x\to1}\frac{1-x^{k-1}((nx-n)^2+1)}{x-1}=$$

$$\lim_{x\to1}-\frac{1}{2}\cdot\left((k-1)x^{k-2}(nx-n)^2+2x^{k-1}(nx-n)+(k-1)x^{k-2}\right)=-\frac{1}{2}(k-1)$$

$$\Rightarrow\Omega=-\frac{1}{2}(0+1+2+\cdots+n-1)=-\frac{1}{2}\cdot\frac{(n-1)n}{2}=\frac{-(n-1)n}{4}$$

3.97

Ω can be written as: $\Omega=\lim_{n\to\infty}n\cdot\frac{\ln\left(\frac{B}{A}\right)^n-\ln A}{\ln B^n}$ *(1) where*

$B=1+\frac{10^{\frac{1}{n+1}}}{n+1},A=1+\frac{10^{\frac{1}{n}}}{n},\forall x>-1:\frac{x}{x+1}<\ln(1+x)<x$*, so*

$$\frac{\dfrac{10^{\frac{1}{n}}}{n}}{1+\dfrac{10^{\frac{1}{n}}}{n}}<\ln A<\frac{10^{\frac{1}{n}}}{n}\to\frac{10^{\frac{1}{n}}}{1+\dfrac{10^{\frac{1}{n}}}{n}}<\ln A^n<10^{\frac{1}{n}}\to\lim_{n\to\infty}\ln A^n=1$$

It holds that $\lim_{n\to\infty}10^{\frac{1}{n}}=\lim_{n\to\infty}10^{\frac{1}{n+1}}=1$*. Similarly* $\lim_{n\to\infty}\ln A^n=1$

(1) now becomes $\Omega=\lim_{n\to\infty}\left[\ln\left(\frac{B}{A}\right)^n-1\right]$*. We will show that:*

$$\frac{B}{A} = \frac{1+\frac{10^{\frac{1}{n+1}}}{n+1}}{1+\frac{10^{\frac{1}{n}}}{n}} < 1 - \frac{1}{n^2}$$ *(2) . Simplifications reduce (2) to:*

$$-\frac{B}{A} + 1 - \frac{1}{n^2} = \frac{10^{\frac{1}{n}}\cdot n^3 - 10^{\frac{1}{n+1}}\cdot n^3 + 10^{\frac{1}{n}}\cdot n^2 - n^2 - 10^{\frac{1}{n}}\cdot n - n - 10^{\frac{1}{n}}}{n^2(n+1)\left(n+10^{\frac{1}{n}}\right)} \quad (3)$$

Nominator of (3) is positive (*). Also, $\left(\frac{B}{A}\right)^{n^2} < \left(1 - \frac{1}{n^2}\right)^{n^2}$ *(4),* $\frac{B}{A} < 1$*. We can also show that:* $\left(\frac{B}{A}\right)^{n^2}$ *is increasing (*),* $\left(1 - \frac{1}{n^2}\right)^{n}$ ↑*, and* $\left(\frac{B}{A}\right)^{n^2} - \left(1 - \frac{1}{n^2}\right)^{n}$ *is also increasing (*). This together with (4) means* $lim_{n\to\infty}\left(\frac{B}{A}\right)^{n} = lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right)^{n^2} = \frac{1}{e}$ *therefore*

$$\Omega = ln \lim_{n\to\infty}\left(\frac{1}{e}\right) - 1 = -2$$

(*) It is quite a laboriuos task to show these through derivatives of the respective functions.

3.98

$$\Omega = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}\sum_{k=1}^{n}3^k\, tan^{-1}\left(\frac{3}{k^2-k-1}\right) = \frac{1}{3}(0) + \frac{1}{3^2}[3\, tan^{-1}(-3)] +$$

$$+\frac{1}{3^3}\left[3\, tan^{-1}(-3) + 3^2\, tan^{-1}\left(\frac{3}{3}\right)\right] +$$

$$+\frac{1}{3^4}\left[3\, tan^{-1}(-3) + 3^2\, tan^{-1}\left(\frac{3}{3}\right) + 3^3\, tan^{-1}\left(\frac{3}{5}\right)\right]$$

$$+\cdots$$

$$+\frac{1}{3^{n+1}}\left[3\, tan^{-1}(-3) + 3^2\, tan^{-1}\left(\frac{3}{3}\right) + \cdots + 3^n\, tan^{-1}\left(\frac{3}{n^2-n+1}\right)\right]$$

$$+\cdots$$

Adding columnwise

$$\Omega = \left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)tan^{-1}(-3) + \left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)tan^{-1}\left(\frac{2}{3}\right) + \left(\sum_{r=1}^{\infty}\frac{1}{3^r}\right)tan^{-1}\left(\frac{3}{5}\right) + \ldots$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} tan^{-1}\left(\frac{3}{n^2 - n - 1}\right) = \frac{1}{2}\sum_{n=1}^{\infty} tan^{-1}\left(\frac{(n+1)-(n-2)}{1+(n+1)(n-2)}\right) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}[tan^{-1}(n+1) - tan^{-1}(n-2)] = \frac{1}{2}[-tan^{-1}(-1) - tan^{-1}(0) - tan^{-1}(1)] = 0$$

3.99

$$\Omega(x) = -1 + 4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)}, x \in \mathbb{R}$$

$$\Rightarrow \left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < \Omega(a)\Omega(b) + 1, 0 < a < 1, b > 1$$

$$\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{n+2}\cdot\frac{1}{(n+1)(n+3)} = \frac{1}{n+2}\left(\frac{1}{2(n+1)} - \frac{1}{2(n+3)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$$

$$S_1 = \sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{2}\lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{(N+2)(N+3)}\right) = \frac{1}{4}$$

$$\frac{n}{(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$S_2 = \sum_{n=0}^{\infty}\frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=0}^{\infty}\frac{1}{(n+2)(n+3)} - S_1$$

$$= \lim_{N\to\infty}\sum_{n=0}^{N}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - S_1 = \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{N+3}\right) - \frac{1}{4} = \frac{1}{4}$$

$$\Omega(x) = -1 + 4\cdot\sum_{n=0}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)} = -1 + 4(S_2\cdot x + S_1) =$$

$$= -1 + 4\left(\frac{1}{4} + \frac{1}{4}x\right) = x$$

$$\left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < \Omega(a)\Omega(b) + 1$$

$$\Leftrightarrow a^b + b^a - ab - 1 < 0, 0 < a < 1, b > 1$$

$$Let\ f(b) = a^b + b^a - ab - 1, 0 < a < 1, b > 1$$

$$f'(b) = a^b \log(a) + ab^{a-1} - a = a^b \log(a) - a(1 - b^{a-1}) < 0\ \forall b > 1 \Rightarrow f \searrow (1, \infty)$$

$$For\ b > 1 \Leftrightarrow f(b) < f(1) = 0 \Leftrightarrow a^b + b^a < 1 + ab, 0 < a < 1, b > 1$$

3.100

$$For\ n \geq 2,\ let$$

$$a_n = x_n^2 - 2x_n x_{n-1} - x_{n-1}^2$$

$$a_2 = 3^2 - 2(3)(1) - 1^1 = 2$$

$$x_n = x_{n-2} + 2x_{n-1} = (x_n - x_{n-1})^2 = (x_{n-2} + x_{n-1})^2$$

$$= x_n^2 - 2x_n x_{n-1} + x_{n-1}^2 = x_{n-1}^2 + 2x_{n-2}x_{n-1} + x_{n-2}^2 \Rightarrow$$

$$\Rightarrow x_n^2 - 2x_n x_{n-1} - x_{n-1}^2 = -(x_{n-1}^2 - 2x_{n-2}x_{n-1} - x_{n-2}^2) \Rightarrow$$

$$\Rightarrow a_n = -a_{n-1}\ \ \forall n \geq 3 \Rightarrow a_n = (-1)^{n-2}a_2 = (-1)^n a_2 = (-1)^n(2)$$

$$\Omega = \lim_{n\to\infty} \frac{(a_n)(-1)^{n+1}}{n} = \lim_{n\to\infty} \frac{(-1)^{n+1}(-1)^n 2}{n} = 0$$

3.101

$$\cos^3 x = \frac{1}{4}(\cos(3x) + 3\cos x)$$

$$\cos^3(3^{n-1}x) = \frac{1}{4}\left((\cos 3^n x) + 3\cos(3^{n-1}x)\right)$$

$$\alpha(x) = \frac{1}{4} \times \frac{4}{3}\left(\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n (\cos 3^n x + 3\cos(3^{n-1}x))\right)$$

$$\alpha(x) = \frac{1}{3}\left(-\frac{1}{3}(\cos 3x + 3\cos x) + \frac{1}{9}(\cos 9x + 3\cos 3x)\right)$$

$$-\frac{1}{27}(\cos 27x + 3\cos(9x)) \ldots \alpha(x) = \frac{1}{3}\begin{Bmatrix} -\frac{\cos 3x}{3} - \cos x \\ \frac{\cos 9x}{9} + \frac{\cos 3x}{3} \\ -\frac{\cos 27x}{27} - \frac{\cos 9x}{9} \end{Bmatrix}$$

$$\alpha(x) = \frac{1}{3}\left(lim_{n\to\infty}\left(-\frac{1}{3}\right)^n \cos(3^n x) - \cos x\right),\ \alpha(x) = -\frac{\cos x}{3}; \alpha\left(\frac{\pi}{2} - x\right) = \frac{-\sin x}{3}$$

$$\beta(x) = -\frac{\sin x}{3},\quad \frac{\beta(x)\beta(3x)\ldots\beta(2n-1)x}{\beta(2x)\beta(4x)\ldots\beta(nx)} = M$$

$$lim_{x\to 0} M = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot(2n)} = N,\ \Omega = lim_{n\to\infty}(N)^{\frac{1}{n}}$$

$$\Omega = lim_{n\to\infty}\left(\frac{2n!}{\left(2\cdot 4\cdot 6\cdot 8\cdot\ldots\cdot(2n)\right)^2}\right)^{\frac{1}{n}},\ \Omega = lim_{n\to\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^{\frac{1}{n}}$$

$$\Omega = \frac{1}{4}\lim_{n\to\infty}(2C_n^n)^{\frac{1}{n}}$$

Using Stirling approximation $n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n,\ \Omega = \frac{1}{4} lim_{n\to\infty}\left(\frac{\sqrt{4\pi n}\cdot\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2n\pi}\right)\left(\frac{n}{e}\right)^{2n}}\right)^{\frac{1}{n}}$

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n\pi}}\right)^{\frac{1}{n}} = 1$$

3.102

$$\Omega(a) = \lim_{n\to\infty} n\left(\left(\frac{n+1}{n}\right)^{n+a} - e\right) = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n+a} - e}{1} \overset{Heine}{=}$$

$$\lim_{n\to\infty}\frac{\left(1+\frac{1}{x}\right)^{x+a} - e}{\frac{1}{x}} = \lim_{y\to 0}\frac{(1+y)^{\frac{1}{y}+a} - e}{y} \overset{L'H}{=}$$

$$\lim_{y\to 0}(1+y)^{\frac{1}{y}+a}\left[-\frac{1}{y^2}ln(1+y) + \left(\frac{1}{y}+a\right)\frac{1}{1+y}\right] = e\ \lim_{y\to 0}\left[-\frac{ln(1+y)}{y^2} + \frac{1+ay}{y(1+y)}\right] =$$

$$= e \lim_{y\to 0} \frac{-(1+y)\ln(1+y)+y+ay^2}{y^3+y^2} \overset{L'H}{=} e \lim_{y\to 0} \frac{-\ln(1+y)-1+1+2ay}{3y^2+2y} \overset{L'H}{=}$$

$$= e\, lim_{y\to 0} \frac{-1+2a}{6y+2} = \frac{e(2a-1)}{2} \quad (1)$$

From (1) we must show: $\frac{e^3}{8}(2a-1)(2b-1)(2c-1) \le \frac{1}{27} \Leftrightarrow$

$$\frac{e}{2}\sqrt[3]{(2a-1)(2b-1)(2c-1)} \le \frac{1}{3} \quad (2)$$

But $\sqrt[3]{(2a-1)(2b-1)(2c-1)} \le \frac{2(a+b+c)-3}{3}$ *(3)*

From (2)+(3) we must show:

$$\frac{e}{2}\left[\frac{2(a+b+c)}{3}-3\right] \le \frac{1}{3} \Leftrightarrow 2e(a+b+c)-3e \le 2 \Leftrightarrow$$

$\Leftrightarrow 2e(a+b+c) \le 3e+2$ *which its true.*

3.103

Consider a continuous function $f:(0,\infty)\to\mathbb{R}$ *such that:*

$$\lim_{x\to 0} f(x) = \alpha, \lim_{x\to\infty} f(x) = \beta$$

Then, for positive a, b *we can write:*

$$\int_{\frac{1}{n}}^{n} \frac{f(ax)-f(bx)}{x}dx = \int_{\frac{1}{n}}^{n} \frac{f(ax)}{x}dx - \int_{\frac{1}{n}}^{n} \frac{f(bx)}{x}dx = \int_{\frac{a}{n}}^{an} \frac{f(x)}{x}dx - \int_{\frac{b}{n}}^{bn} \frac{f(x)}{x}dx$$

$$= \int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)}{x}dx + \int_{bn}^{an} \frac{f(x)}{x}dx \quad (1)$$

$$= (\alpha-\beta)\ln\frac{b}{a} + \int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x}dx + \int_{bn}^{an} \frac{f(x)-\beta}{x}dx$$

But

$$\left|\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x} dx\right| \le sup\left\{|f(x)-\alpha|: 0 < x < \frac{max(a,b)}{n}\right\} \left|ln\frac{b}{a}\right|$$

and

$$\left|\int_{bn}^{\frac{a}{n}} \frac{f(x)-\beta}{x} dx\right| \le sup\{|f(x)-\beta|: x > n\,min(a,b)\} \left|ln\frac{a}{b}\right|$$

Thus

$$\lim_{n\to\infty}\int_{\frac{a}{n}}^{\frac{b}{n}} \frac{f(x)-\alpha}{x} dx = 0, \lim_{n\to\infty}\int_{bn}^{\frac{a}{n}} \frac{f(x)-\beta}{x} dx = 0$$

So, letting n tend to ∞ in (1) we get $lim_{n\to\infty} \int_{\frac{1}{n}}^{n} \frac{f(ax)-f(bx)}{x} = (\alpha-\beta)\,ln\frac{b}{a}$

In particular, if $a = 3, b = 2 f(x) = \frac{cos\,x}{1+x}$ we get:

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{n} \frac{1}{x}\left(\frac{cos\,3x}{1+3x} - \frac{cos\,2x}{1+2x}\right) dx = ln\frac{2}{3}$$

3.104

$$0 \le sin\left(\frac{k}{n}\right) \le \frac{k}{n}$$

$$n^{14} \le k + n^{14} \le n + n^{14} \Rightarrow n^2 \le \sqrt[7]{k+n^{14}} \le \sqrt[7]{n+n^{14}}$$

$$\Rightarrow \frac{0}{(n+n^{14})^{\frac{1}{7}}} \le \frac{sin\left(\frac{k}{n}\right)}{\sqrt[7]{k+n^{14}}} \le \frac{k}{n^3} \Rightarrow 0 \le \sum_{k=1}^{n} \frac{sin\left(\frac{k}{n}\right)}{(k+n^{14})^{\frac{1}{7}}} \le \frac{1}{n^3} \cdot \sum_{k=1}^{n} k$$

$$\Rightarrow 0 \le \sum_{k=1}^{n} \frac{sin\left(\frac{k}{n}\right)}{(k+n^{14})^{\frac{1}{7}}} \le \frac{n(n+1)}{2n^3}$$

$$As\ lim_{n\to\infty} \frac{n(n+1)}{2n^3} = lim_{n\to\infty} \frac{n+1}{2n^2} = 0$$

$$\therefore \text{ by the Sandwich theorem: } lim_{n\to\infty} \sum_{k=1}^{n} \frac{sin\left(\frac{k}{n}\right)}{[k+n^{14}]^{\frac{1}{7}}} = 0$$

3.105

$$\Omega(n) = \sum_{k=1}^{\infty} \frac{2k^2 + 2nk + k - 1}{(2k + 2n + 2)!!} = \sum_{k=1}^{\infty} \left(\frac{k}{(2k + 2n)!!} - \frac{k + 1}{(2k + 2n + 2)!!}\right) = \frac{1}{(2n + 2)!!}$$

Now,

$$\lim_{n\to\infty}\left(n! \cdot \Omega(n)\right) = \lim_{n\to\infty}\left(\frac{n!}{(2n + 2)!!}\right) = \lim_{n\to\infty}\left(\frac{1}{2^{n+1}(n + 1)}\right)$$

$$\therefore \lim_{n\to\infty}\left(n! \cdot \Omega(n)\right) = 0$$

3.106

For $n \geq 3$,

$$a_n + a_{n-2} = \int_0^{\frac{\pi}{4}} (tan\,x)^{n-2}\,(1 + tan^2\,x)dx = \int_0^{\frac{\pi}{4}} (tan\,x)^{n-2}\,sec^2\,x\,dx =$$

$$= \left[\frac{1}{n - 1}(tan\,x)^{n-1}\right]_0^{\frac{\pi}{4}} = \frac{1}{n - 1}$$

$$\therefore \lim_{n\to\infty}\left[\frac{1}{(n - 2)(a_n + a_{n-2})}\right]^{2n} = \lim_{n\to\infty}\left[\frac{1}{(n - 2)\left(\frac{1}{n - 1}\right)}\right]^{2n} = \lim_{n\to\infty}\left[\frac{n - 1}{n - 2}\right]^{2n}$$

$$= \lim_{n\to\infty}\left[\left(1 + \frac{1}{n - 2}\right)^{n-2}\right]^2 \left[\left(1 + \frac{1}{n - 2}\right)^n\right] = (e^2)(1) = e^2$$

3.107

$$\frac{k^2 + 1}{(k - 1)k(k + 1)!} = \frac{(k + 1)^2 - 2k}{(k - 1)k(k + 1)!} = \frac{k + 1}{(k - 1)kk!} - \frac{2}{(k - 1)(k + 1)!} =$$

$$= \frac{1}{(k-1)k!} + \frac{1}{(k-1)kk!} - \frac{2}{(k-1)(k+1)!} = \frac{1}{(k-1)k!} - \frac{2}{(k-1)(k+1)!} + \frac{1}{(k-1)kk!}$$

$$= \frac{k-1}{(k-1)(k+1)!} + \frac{1}{(k-1)kk!} = \frac{1}{(k+1)!} + \frac{1}{(k-1)kk!} = \frac{1}{(k+1)!} + \frac{1-k+k}{(k-1)kk!} =$$

$$= \frac{1}{(k+1)!} - \frac{k-1}{(k-1)kk!} + \frac{k}{(k-1)kk!} = \frac{1}{(k+1)!} - \frac{1}{kk!} + \frac{1}{(k-1)k!} =$$

$$= \frac{1}{(k+1)!} + \frac{1-k+k}{(k-1)k!} - \frac{1}{kk!} = \frac{1}{(k+1)!} - \frac{k-1}{(k-1)k!} + \frac{k}{(k-1)k!} - \frac{1}{kk!} \Rightarrow$$

$$\Rightarrow \frac{k^2+1}{(k-1)k(k+1)!} = \frac{1}{(k+1)!} - \frac{1}{k!} + \frac{1}{(k-1)(k-1)!} - \frac{1}{kk!} \Rightarrow$$

$$\Rightarrow \sum_{k=2}^{n} \frac{1+k^2}{(k-1)k(k+1)!} = \sum_{k=2}^{n} \left(\frac{1}{(k+1)!} - \frac{1}{k!} + \frac{1}{(k-1)(k-1)!} - \frac{1}{kk!} \right)$$

$$= \frac{1}{(n+1)!} - \frac{1}{2!} + 1 - \frac{1}{nn!} = \frac{1}{2} + \frac{1}{(n+1)!} - \frac{1}{nn!} \Rightarrow$$

$$\sum_{k=2}^{n} \frac{1+k^2}{(k-1)k(k+1)!} = \frac{1}{2} - \frac{1}{n(n+1)!} \ \textit{(1)}$$

From (1) we must calculate:

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}\left(\frac{1}{2} - \frac{1}{n(n+1)!}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n+1}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)!}\right)}{\frac{1}{n}\left(\frac{1}{2} - \frac{1}{n(n+1)!}\right)} =$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{(n+1)(n+2)!}}{\frac{1}{2} - \frac{1}{n(n+1)!}} = 1$$

3.108

$$\textit{Let } a_n = \prod_{k=1}^{n} \left(1 + \frac{(n+k)^2}{n^3}\right) \Rightarrow \ln a_n = \sum_{k=1}^{n} \ln\left(1 + \frac{(n+k)^2}{n^3}\right)$$

Now, using: $x - \frac{x^2}{2} \le \ln(1+x) \le x, \forall x \ge 0$

$$x = \frac{(n+k)^2}{n^3} \Rightarrow \frac{(n+k)^2}{n^3} - \frac{(n+1)^4}{2n^6} \le \ln\left(1 + \frac{(n+k)^2}{n^3}\right) \le \frac{(n+k)^2}{n^3} \Rightarrow$$

$$\Rightarrow \sum_{k=1}^{n}\frac{(n+k)^2}{n^3}-\sum_{k=1}^{n}\frac{(n+1)^4}{2n^6}\le \sum_{k=1}^{n} ln\left(1+\frac{(n+2)^2}{n^3}\right)\le \sum_{k=1}^{n}\frac{(n+k)^2}{n^3} \quad (1)$$

$$\text{But } lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(n+k)^2 = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^2 = \int_0^1(1+x)^2\,dx =$$

$$= \frac{(1+x)^3}{3}\bigg|_0^1 = \frac{7}{3} \quad (2)$$

$$\lim_{n\to\infty}\frac{1}{2n^6}\sum_{k=1}^{n}(n+k)^4 = \lim_{n\to\infty}\frac{1}{2n}\left[\frac{1}{n}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^4\right] =$$

$$= \lim_{n\to\infty}\frac{1}{2n}\left(\int_0^1(1+x)^4dx\right) = 0 \quad (3)$$

$$\text{From (1)+(2)+(3)} \Rightarrow lim_{n\to\infty} ln\, a_n = \frac{7}{3} \Rightarrow \Omega = lim_{n\to\infty} a_n = e^{\frac{7}{3}}$$

3.109

$$\Omega = \lim_{n\to\infty}\left(n^7\cdot\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx\right) = \lim_{n\to\infty}\left(\frac{\int_0^{\frac{1}{n^7}}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{\frac{1}{n^7}}\right)$$

$$\overset{t=\frac{1}{n^7}}{=} \lim_{t\to 0}\left(\frac{\int_0^{t}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{t}\right) \overset{DLH}{=} \lim_{t\to 0}\left(\frac{\frac{d}{dt}\int_0^{t}\frac{x\sin x+\cos x}{2\sin x+3\cos x+6}dx}{\frac{d}{dt}(t)}\right)$$

$$= \lim_{t\to 0}\frac{t\sin t+\cos t}{2\sin t+3\cos t+6} = \frac{0+1}{0+3+6} = \frac{1}{9}$$

3.110

$$lim_{n\to\infty}\Omega_n = \infty \text{ because } lim_{n\to\infty}(n-6)\binom{7}{7} = \infty$$

$$\Omega = \lim_{n\to\infty}(\Omega_n)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{ln(\Omega_n)}{n}} = \lim_{n\to\infty} e^{ln\frac{\Omega_{n+1}}{\Omega_n}} = \lim_{n\to\infty}\frac{\Omega_{n+1}}{\Omega_n} =$$

$$= \lim_{n\to\infty}\frac{\binom{n+1}{7}+2\binom{n}{7}+\dots+(n-5)\binom{7}{7}}{\binom{n}{7}+2\binom{n-1}{7}+\dots+(n-6)\binom{7}{7}} \underset{Stolz-Cesaro}{\overset{\frac{\infty}{\infty}}{=}}$$

$$\lim_{n\to\infty}\frac{\binom{n+2}{7}+\cdots+\binom{7}{7}}{\binom{n+1}{7}+\binom{n}{7}+\cdots+\binom{7}{7}}=\lim_{n\to\infty}\frac{\binom{n+3}{7}}{\binom{n+2}{7}}=\lim_{n\to\infty}\frac{\frac{(n+3)!}{(n-4)!}}{\frac{(n+2)!}{(n-5)!}}=$$

$$=\lim_{n\to\infty}\frac{(n-3)(n-2)(n-1)n(n+1)(n+2)(n+3)}{(n-4)(n-3)(n-2)(n-1)n(n+1)(n+2)}=1$$

$$\Omega=1.$$

3.111

$$\Omega=\lim_{n\to\infty}\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}}=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt[n]{5}-1\right)+\left(\sqrt[n]{15}-1\right)+\left(\sqrt[n]{25}-1\right)+\cdots+\left(\sqrt[n]{10n-5}-1\right)}{\left(\sqrt[n]{10}-1\right)+\left(\sqrt[n]{20}-1\right)+\left(\sqrt[n]{30}-1\right)+\cdots+\left(\sqrt[n]{10n}-1\right)}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n}(ln\,5+ln\,15+ln\,25+\cdots+ln(10n-5))+O\left(\frac{1}{n^2}\right)}{\frac{1}{n}(ln\,10+ln\,20+ln\,30+\cdots+ln(10n))+O\left(\frac{1}{n^2}\right)}=\lim_{n\to\infty}\frac{\sum_{r=1}^{n}ln(10n-5)}{\sum_{r=1}^{n}ln(10n)}$$

$$\overset{Stolz-Cesaro\ th^m}{=}\lim_{n\to\infty}\frac{ln(10n-5)}{ln(10n)}$$

$$\therefore\ \Omega=\frac{n-\sqrt[n]{5}-\sqrt[n]{15}-\sqrt[n]{25}-\cdots-\sqrt[n]{10n-5}}{n-\sqrt[n]{10}-\sqrt[n]{20}-\sqrt[n]{30}-\cdots-\sqrt[n]{10n}}=1$$

3.112

$$Let\ a_n=\frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right)=\prod_{k=1}^{n}\left(1-\frac{3}{6k-2}\right)=\prod_{k=1}^{n}\frac{6k-5}{6k-2}\quad (1)$$

$$\left.\begin{aligned}\sqrt{1\cdot 7}&<\frac{1+7}{2}=4\\ \sqrt{7\cdot 13}&<\frac{7+13}{2}=10\\ \sqrt{13\cdot 19}&<\frac{13+19}{2}=16\\ &\vdots\\ \sqrt{(6n-11)(6n-5)}&<6n-8\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow 1\cdot 7\cdot 13\cdot\ldots\cdot(6n-11)\sqrt{6n-5}<4\cdot 10\cdot\ldots\cdot(6n-8)\Rightarrow$$

$$\Rightarrow \frac{1\cdot 7\cdot 13\cdot \ldots \cdot (6n-11)\sqrt{6n-5}}{4\cdot 10\cdot \ldots \cdot (6n-8)} < 1 \Rightarrow$$

$$\Rightarrow \frac{1\cdot 7\cdot 13\cdot \ldots \cdot (6n-5)}{4\cdot 10\cdot \ldots \cdot (6n-2)} < \frac{\sqrt{6n-5}}{6n-2} \quad (2)$$

From (1)+(2) $\Rightarrow 0 < a_n < \frac{\sqrt{6n-5}}{6n-2} \Rightarrow lim_{n\to\infty} a_n = 0$

3.113

$$\Omega = \lim_{n\to\infty} n^2\left(7^{\frac{1}{n+5}} - 7^{\frac{1}{n+8}}\right) = \lim_{n\to\infty} n^2\, 7^{\frac{1}{n+8}}\left(7^{\frac{1}{n+5}-\frac{1}{n+8}} - 1\right) =$$

$$= \lim_{n\to\infty} n^2 \cdot 7^{\frac{1}{n+8}} \cdot \left(7^{\frac{3}{(n+5)(n+8)}} - 1\right) = \lim_{n\to\infty} 7^{\frac{1}{n+8}} \frac{\left(7^{\frac{3}{(n+5)(n+8)}} - 1\right)}{\frac{3}{(n+5)(n+8)}} \cdot \frac{3n^2}{(n+5)(n+8)} = 3\ln 7$$

3.114

$$x\cos x \le \sin x \le x, \forall x \ge 0 \Rightarrow \frac{1}{n+k}\cos\frac{1}{2n} \le \frac{1}{n+k}\cos\frac{1}{n+k} \le \sin\frac{1}{n+k} \le \frac{1}{n+k} \Rightarrow$$

$$\Rightarrow \cos\frac{1}{2n}\sum_{k=1}^{n}\frac{1}{n+k} \le \sum_{k=1}^{n}\sin\frac{1}{n+k} \le \sum_{k=1}^{n}\frac{1}{n+k} \quad (1)$$

But $lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x}dx = \ln 2 \quad (2)$

From (1)+(2) $\Rightarrow lim_{n\to\infty}\omega_n = \ln 2 \quad (3)$

$$\Omega = \lim_{n\to\infty}\left(\sqrt[n+1]{n+1}\cdot\omega_{n+1} - \sqrt[n]{n}\omega_n\right) =$$

$$= \lim_{n\to\infty}\left(\sqrt[n+1]{n+1}\cdot\omega_{n+1} - \sqrt[n]{n}\cdot\omega_{n+1} + \sqrt[n]{n}\omega_{n+1} - \sqrt[n]{n}\omega_n\right)$$

$$= lim_{n\to\infty}\omega_{n+1}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) + lim_{n\to\infty}\sqrt[n]{n}\left(\omega_{n+1} - \omega_n\right) \quad (4)$$

$$lim_{n\to\infty}\sqrt[n]{n}\left(\omega_{n+1} - \omega_n\right) = 1\cdot(\ln 2 - \ln 2) = 0 \quad (5)$$

Let $f: [n, n+1] \to \mathbb{R}; f(x) = x^{\frac{1}{x}}$

From Lagrange's theorem $\exists c \in (n, n+1)$*, so that:*

$$\frac{f(n+1)-f(n)}{n+1-n} = f'(c) \Rightarrow (n+1)^{\frac{1}{n+1}} - n^{\frac{1}{n}} = f'(c) \quad (6)$$

$$f'(x) = x^{\frac{1}{x}}\left(\frac{1-\ln x}{x^2}\right) \quad (7)$$

From (6)+(7)⇒ $\sqrt[n+1]{n+1} - \sqrt[n]{n} = c^{\frac{1}{c}}\left(\frac{1-\ln c}{c^2}\right) \Rightarrow$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) = \lim_{n\to\infty} c^{\frac{1}{c}}\left(\frac{1-\ln c}{c^2}\right) \quad (8)$$

Because $c \in (n, n+1)$, $\lim_{n\to\infty} x^{\frac{1}{x}} \cdot \left(\frac{1-\ln x}{x^2}\right) = 0 \quad (9)$

$$\left.\begin{array}{r} \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^0 = 1 \\ \lim_{x\to\infty} \frac{1-\ln x}{x^2} = \lim_{x\to\infty} \frac{-\frac{1}{x}}{2x} = 0 \end{array}\right\} \Rightarrow (9)$$

From (9) $\Rightarrow \lim_{n\to\infty} \omega_{n+1}\left(\sqrt[n+1]{n+1} - \sqrt[n]{n}\right) = \ln 2 \cdot 0 = 0$ *(10)*

From (4)+(5)+(10)⇒ $\Omega = 0$.

3.115

$$\Omega = \lim_{n\to\infty} \frac{\ln(1+7^{T_n}) \cdot \ln(1+2^{T_n})}{\ln(1+5^{T_n}) \cdot \ln(1+3^{T_n})} = \lim_{n\to\infty} \frac{\frac{\ln(1+7^{T_n})}{T_n} \cdot \frac{\ln(1+2^{T_n})}{T_n}}{\frac{\ln(1+5^{T_n})}{T_n} \cdot \frac{\ln(1+3^{T_n})}{T_n}} \quad (1)$$

$\lim_{n\to\infty} \frac{H_n}{G_n} = \infty$. *Let* $a > 1$.

$$\lim_{n\to\infty} \frac{\ln(1+a^{T_n})}{T_n} \overset{C.S.}{=} \lim_{n\to\infty} \frac{\ln(1+a^{T_{n+1}}) - \ln(1+a^{T_n})}{T_{n+1} - T_n} = \lim_{n\to\infty} \frac{\ln\left(\frac{1+a^{T_{n+1}}}{1+a^{T_n}}\right)}{T_{n+1} - T_n} =$$

$$= \lim_{n\to\infty} \frac{\ln\left(1 + \frac{1+a^{T_{n+1}}}{1+a^{T_n}} - 1\right)}{T_{n+1} - T_n} = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{a^{T_{n+1}} - a^{T_n}}{1+a^{T_n}}\right)}{\frac{a^{T_{n+1}} - a^{T_n}}{1+a^{T_n}}} \cdot \frac{a^{T_{n+1}} - a^{T_n}}{(1+a^{T_n})(T_{n+1} - T_n)} =$$

$$= \lim_{n\to\infty} \frac{a^{T_n}(a^{T_{n+1}-T_n} - 1)}{(1+a^{T_n})(T_{n+1} - T_n)} = \lim_{n\to\infty} \frac{a^{T_{n+1}-T_n} - 1}{\left(\frac{1}{a^{T_n}}+1\right)(T_{n+1} - T_n)} = \ln a \quad (1)$$

From (1) $\Rightarrow \Omega = \frac{\ln 7 \cdot \ln 2}{\ln 5 \cdot \ln 3}$

Observation: $\lim_{n\to\infty} T_{n+1} - T_n = 0$*, because.*

$$\lim_{n\to\infty}\left(\frac{H_{n+1}}{G_{n+1}}-\frac{H_n}{G_n}\right)=\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n+1}}{1+\frac{1}{2^2}+\cdots+\frac{1}{(n^2+1)^2}}-\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}}=$$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\right)}{G_n\cdot G_{n+1}}$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n+1}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\frac{1}{(n+1)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}{G_n\cdot G_{n+1}}=$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n+1}\left[\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)-\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n+1}\right]}{G_n\cdot G_{n+1}}=0$$

because $\lim_{n\to\infty} G_n=\frac{\pi^2}{6}$, *and* $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n+1}\overset{(1)}{=}\lim_{n\to\infty}\frac{1}{n+1}=0$

3.116

$$\Omega=\lim_{n\to\infty}\left(1-\ln 2+\sum_{k=1}^{n}\sin\frac{1}{k+n}\right)^n \quad (1)$$

Lemma: $\forall x\in[0,\Gamma]: x\left(1-\frac{x}{\Gamma}\right)\le \sin x\le x$. *LHS is easily proven by considering the function*

$f(x)=x\left(1-\frac{x}{\Gamma}\right)-\sin x$ *over* $[0,\Gamma]$.

$x=\frac{1}{k+n}\to\frac{1}{n+k}\left[1-\frac{1}{\Gamma(n+k)}\right]\le\sin\frac{1}{n+k}\le\frac{1}{n+k}$ *(2). We know that*

$\sum_1^\infty\frac{1}{n}=\infty$ *and that* $\sum_1^n\frac{1}{k}=\ln 2-\gamma+\varepsilon_n$ *where* $\gamma=$ *constant and* $\varepsilon\simeq\frac{1}{2}\to 0, k\to\infty$. *Also*

$\sum_1^{2n}\frac{1}{k}=\ln(2n)-\gamma+\varepsilon_{2n},\varepsilon_{2n}\simeq\frac{1}{4n}$. *This means that*

$\sum_{k=1}^n\frac{1}{n+k}=\left(\sum_1^{2n}\frac{1}{k}\right)-\left(\sum_1^n\frac{1}{k}\right)=\ln 2-\frac{1}{4n}$. *Now,* $\sum_1^\infty\frac{1}{n^2}=\frac{\Gamma^2}{6},\sum_1^\infty\frac{1}{(2n)^2}=\frac{\Gamma^2}{6}$ *therefore*

$\sum_{k=1}^n\frac{1}{(n+k)^2}\to 0$ *when* $n\to\infty$. *Taking sums to infity in (2)* $\to$

$lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k+n}-\sum_{k=1}^{n}\frac{1}{\Gamma(k+n)^2}\right) = ln\,2-0$ *and in the RHS* $lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k} = ln\,2$.

We can write also $ln\,2-\frac{1}{4n} < \sum_{k=1}^{n}\frac{1}{k+n} < ln\,2-\frac{1}{4n}$ *and in turn:*

$$\left(1-\frac{1}{4n}\right)^n < \left(1-ln\,2+\sum_{k=1}^{n}sin\frac{1}{k+n}\right)^n < \left(1-\frac{1}{4n}\right)^n$$

Taking limits $e^{-\frac{1}{4}} < \Omega < e^{-\frac{1}{4}}$ *hence* $\Omega = e^{-\frac{1}{4}}$. *Done!*

3.117

$$\Omega_n = \lim_{n\to\infty}\frac{\left(1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}\right)^{q+1}}{(1^q+3^q+\cdots+(2n-1)^q)^{p+1}}$$

$= lim_{n\to\infty}(x_n\cdot y_n)$ *where* $x_n = \frac{\left(1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}\right)^{q+1}}{n}$ *and*

$y_n = \frac{n}{(1+3^q+\cdots+(2n-1)^q)^{p+1}}$ *for all* $n\in\mathbb{N}, p, q \geq 1$

$$\lim_{n\to\infty}\sqrt[q+1]{x_n} = \lim_{n\to\infty}\frac{1+2^p\sqrt{2}+3^p\sqrt[3]{3}+\cdots+n^p\sqrt[n]{n}}{\sqrt[q+1]{n}} \overset{CAESARO\ STOLZ}{=} \lim_{n\to\infty}\frac{(n+1)^p\sqrt[n+1]{n+1}}{\sqrt[q+1]{n+1}-\sqrt[q+1]{n}}$$

$$= lim_{n\to\infty}\left(n^{\frac{q}{q+1}}\frac{(n+1)^p\sqrt[n+1]{n+1}}{\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{q+1}}-1}{\frac{1}{n}}}\right) = (q+1)\,lim_{n\to\infty}\,n^{\frac{q}{1+q}}(n+1)^p \text{ where } lim_{n\to\infty}\sqrt[n]{n} = 1$$

$$\lim_{n\to\infty}x_n = (q+1)^{q+1}\lim_{n\to\infty}n^q(n+1)^{p(q+1)}$$

$$\lim_{n\to\infty}\sqrt[p+1]{y_n} = \lim_{n\to\infty}\frac{\sqrt[p+1]{n}}{1+3^q+\cdots+(2n-1)^q} = \lim_{n\to\infty}\frac{\sqrt[p+1]{n+1}-\sqrt[p+1]{n}}{(2n+1)^q}$$

$$= \lim_{n\to\infty}\left(n^{-\frac{p}{1+p}}\frac{\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{p+1}}-1}{\frac{1}{n}}}{(2n+1)^q}\right) = \frac{1}{p+1}\lim_{n\to\infty}\frac{1}{n^{\frac{p}{1+p}}(2n+1)^q}$$

$$\lim_{n\to\infty} y_n = \frac{1}{(p+1)^{p+1}} \lim_{n\to\infty} \frac{1}{n^p(2n+1)^{q(p+1)}}$$

$$\therefore \lim_{n\to\infty} \Omega = \lim_{n\to\infty}(x_n y_n) = \frac{(q+1)^{q+1}}{(p+1)^{p+1}} \lim_{n\to\infty} \frac{n^q(n+1)^{p(q+1)}}{n^p(2n+1)^{q(p+1)}} = \frac{(q+1)^{q+1}}{2^{pq+q}(p+1)^{p+1}}$$ *(Answer)*

3.118

$$L = \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{(-1)^k (k)!\,(2n-k)!}{(2n)!}, \qquad L = \lim_{n\to\infty} \frac{1}{(2n)!} \sum_{k=0}^{2n} (2n-k)!\,(k)!\,(-1)^k$$

$$= \lim_{n\to\infty} \frac{1}{(2n)!}\left((2n)! - (2n-1)! + (2n-2)!\,(2)! - (2n-3)!\,(3)! \ldots\right) + (2n-1)! + (2n)!$$

$$= \lim_{n\to\infty} \frac{2\left((2n)! - (2n-1)! + (2n-2)!\,(2)!\right)}{(2n)!} + \frac{(n!)^2(-1)^n}{(2n)!} = 2 + \lim_{n\to\infty} \frac{(-1)^n (n!)^2}{(2n)!}$$

$$= 2 + \lim_{n\to\infty} \frac{(-1)^n 2n\pi\left(\frac{n}{e}\right)^{2n}}{\sqrt{2\pi}\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}}. \quad L = 2 + L_1$$

$$L_1 = \lim_{n\to\infty} \frac{(-1)^n\sqrt{n\pi}}{(2)^{2n}} \Rightarrow L_1 = 0, L = 2$$ *(Answer)*

3.119

$\Omega = e^{\frac{\pi}{4}}$ *Proof.*

$f = \prod_{k=1}^{n} \frac{k^2+n^2+n}{k^2+n^2}$ *Since* $\frac{x-1}{x} < \log(x) < x-1$,

$$\sum_{k=1}^{n} \frac{n}{k^2+n+n^2} < \log(f) = \sum_{k=1}^{n}\left(\log\left(\frac{k^2+n^2+n}{k^2+n^2}\right)\right) < \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$

Note that by removing the logs we have something resembling Dirichlet series.

We have the upper bound taken continuously $\int_0^n \frac{n}{k^2+n^2}\,dk = \frac{\pi}{4}$

and the lower bound $\int_0^n \frac{n}{k^2+n^2+n}\,dk = \frac{\sqrt{n}\tan^{-1}\left(\sqrt{\frac{n}{n+1}}\right)}{\sqrt{n+1}}$

$\lim_{n\to\infty} \frac{\sqrt{n}\tan^{-1}\left(\sqrt{\frac{n}{n+1}}\right)}{\sqrt{n+1}} = \frac{\pi}{4}$, *which completes the proof.*

3.120

$$\Omega = \lim_{n\to\infty}\int_0^{\frac{1}{2}} \frac{arcsec(nx)\log(1-x)}{2x^2-2x+1}dx$$

$$= \int_0^{\frac{1}{2}} \frac{\lim_{n\to\infty} arcsec(nx)\log(1-x)}{2\left\{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right\}}dx = \frac{\pi}{4}\int_0^{\frac{1}{2}} \frac{\log(1-x)}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx$$

$$= -\frac{\pi}{4}\int_{\frac{1}{2}}^{0} \frac{\log\left(z+\frac{1}{2}\right)}{z^2+\frac{1}{4}}dz \left[\begin{matrix} where \frac{1}{2}-x = z \Rightarrow dx - dz \\ when\ x = 0, z = \frac{1}{2}; when\ x = \frac{1}{2}, z = 0\end{matrix}\right]$$

$$= \frac{\pi}{4}\int_0^{\frac{1}{2}} \frac{\log\left(z+\frac{1}{2}\right)}{z^2+\frac{1}{4}}dz = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\tan\theta+1}{2}\right)d\theta \left[\begin{matrix} where\ z = \frac{1}{2}\tan\theta \Rightarrow dz = \frac{1}{2}\sec^2\theta\, d\theta \\ when\ z = 0, \theta = 0; when\ z = \frac{1}{2}, \theta = \frac{\pi}{4}\end{matrix}\right]$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left\{\frac{\tan\left(\frac{\pi}{4}-\theta\right)+1}{2}\right\}d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{4}} \log\left(\frac{\frac{1-\tan\theta}{1+\tan\theta}+1}{2}\right)d\theta$$

$$= \frac{\pi^2}{8}\log 2 - \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan\theta+1}{2}\right)d\theta \Rightarrow 2\Omega = \frac{\pi^2\log 2}{8} \Rightarrow \Omega = \frac{\pi^2\log 2}{16}$$

3.121

We know, $\lim_{u\to 0}\frac{(1+u)^r-1}{u} = r$

$x_{n+1} = \sum_{i=1}^{n}\frac{1}{x_i^p} = x_n + \frac{1}{x_n^p}, p \in \mathbb{N}^*$ *and* $x_0 > 0$

Now, $\{x_n\}_{n=1}^{\infty}$ *is an increasing function hence let* $\lim_{n\to\infty} x_n = l$

$\therefore l = l + \frac{1}{l^p} \Rightarrow l \to \infty$*, which is a contradiction,* $\therefore \lim_{n\to\infty} x_n = \infty$

$$\therefore \Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{\sum_{1\le i<j\le n} x_i x_j}\right) \Rightarrow \sqrt{2}\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{2\sum_{1\le i<j\le n} x_i x_j}\right)$$

$$\Rightarrow \sqrt{2}\Omega = \lim_{n\to\infty}\left(\frac{1}{\sqrt[p+1]{n^{p+2}}}\cdot\sqrt{\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2}\right) = \lim_{n\to\infty}\frac{x_1 + x_2 + \cdots + x_n}{\sqrt[p+1]{n^{p+2}}}$$

$$\left[\because \lim_{n\to\infty}\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n^{\frac{2p+4}{p+1}}} = 0\right]$$

$$\overset{CAESARO\ STOLZ}{=} \lim_{n\to\infty}\frac{x_{n+1}}{(n+1)^{\frac{p+2}{p+1}} - n^{\frac{p+2}{p+1}}} = \lim_{n\to\infty}\frac{\frac{x_{n+1}}{\sqrt[p+1]{n}}}{\frac{\left(1+\frac{1}{n}\right)^{\frac{p+2}{p+1}} - 1}{\frac{1}{n}}} = \frac{p+1}{p+2}\lim_{n\to\infty}\frac{x_{n+1}}{\sqrt[p+1]{n}}$$

$$= \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\frac{x_{n+1}^{p+1}}{n}} \overset{CAESARO\ STOLZ}{=} \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\left(x_{n+2}^{p+1} - x_{n+2}^{p+1}\right)}$$

$$= \frac{p+1}{p+2}\sqrt[p+1]{\lim_{n\to\infty}\left\{\left(x_{n+1} + \frac{1}{x_{n+1}^{p}}\right)^{p+1} - x_{n+1}^{p+1}\right\}} = \frac{p+1}{p+2}\sqrt[p+1]{\lim_{x_{n+1}\to\infty}\frac{\left(1+\frac{1}{x_{n+1}^{p+1}}\right)^{p+1} - 1}{\frac{1}{x_{n+1}^{p+1}}}}$$

$$= \frac{(p+1)\sqrt[p+1]{p+1}}{p+2} = \frac{\sqrt[p+1]{(p+1)^{p+2}}}{p+2} \Rightarrow \Omega = \frac{\sqrt[p+1]{(p+1)^{p+2}}}{\sqrt{2}(p+2)}$$ *(Answer)*

3.122

We know, $(\arcsin x)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{\left(n^2\binom{2n}{n}\right)}$ *where* $x \in [-1,1]$

$$f^{(2n)}(x) = \frac{2^{2n-1}(n!)^2}{n^2} + \frac{1}{2}\sum_{m=n+1}^{\infty}\frac{2^{2m}x\cdot(2m)!}{\left(m^2\binom{2m}{m}\right)} \Rightarrow f^{(2n)}(0) = \frac{2^{2n-1}(n!)^2}{n^2}$$

$$\Omega = \lim_{n\to\infty}\frac{\sqrt[2n]{f^{(2n)}(0)}}{n} = \lim_{n\to\infty}\sqrt{\frac{\sqrt[n]{\frac{2^{2n-1}(n!)^2}{n^2}}}{n^2}} = \sqrt{\lim_{n\to\infty}\sqrt[n]{\frac{2^{2n-1}\cdot(n!)^2}{n^{2(n+1)}}}}$$

$$\overset{CAUCHY-D'ALEMBERT}{=} \sqrt{\lim_{n\to\infty}\left(\frac{2^{2n+1}\cdot\{(n+1)!\}^2}{(n+1)^{2(n+2)}}\cdot\frac{n^{2(n+1)}}{2^{2n-1}\cdot(n!)^2}\right)} = \sqrt{\lim_{n\to\infty}\frac{4}{\left(1+\frac{1}{n}\right)^{2n}}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^4}} = \frac{2}{e}$$

3.123

Let $a_n = \frac{1}{2^n}\prod_{k=1}^{n}\left(2-\frac{3}{3k-1}\right) = \prod_{k=1}^{n}\left(1-\frac{3}{6k-2}\right) = \prod_{k=1}^{n}\frac{6k-5}{6k-2}$ *(1)*

$$\left.\begin{aligned}\sqrt{1\cdot 7} &< \frac{1+7}{2} = 4\\ \sqrt{7\cdot 13} &< \frac{7+13}{2} = 10\\ \sqrt{13\cdot 19} &< \frac{13+19}{2} = 16\\ &\vdots\\ \sqrt{(6n-11)(6n-5)} &< 6n-8\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow 1\cdot 7\cdot 13\cdot\ldots\cdot(6n-11)\sqrt{6n-5} < 4\cdot 10\cdot\ldots\cdot(6n-8) \Rightarrow$$

$$\Rightarrow \frac{1\cdot 7\cdot 13\cdot\ldots\cdot(6n-11)\sqrt{6n-5}}{4\cdot 10\cdot\ldots\cdot(6n-8)} < 1 \Rightarrow$$

$$\Rightarrow \frac{1\cdot 7\cdot 13\cdot\ldots\cdot(6n-5)}{4\cdot 10\cdot\ldots\cdot(6n-2)} < \frac{\sqrt{6n-5}}{6n-2} \quad (2)$$

From (1)+(2) $\Rightarrow 0 < a_n < \frac{\sqrt{6n-5}}{6n-2} \Rightarrow \lim_{n\to\infty} a_n = 0$

3.124

$$\lim_{n\to\infty}\frac{x_n^{p+1}}{n^p} = \left\{\lim_{n\to\infty}\frac{x_n^{\frac{p+1}{p}}}{n}\right\}^p = \left\{\lim_{n\to\infty}\left(x_{n+1}^{\frac{p+1}{p}} - x_n^{\frac{p+1}{p}}\right)\right\}^p \;[Stolz-Cesaro\ th^m] =$$

$$= \lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\left(\frac{x_{n+1}}{x_n}\right)^{\frac{p+1}{p}} - 1\right)\right\}^p$$

$$= \lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\left(1+\frac{1}{x_n+x_n^{\frac{p+1}{p}}}\right)^{\frac{p+1}{p}}-1\right)\right\}^p = \lim_{n\to\infty}\left\{x_n^{\frac{p+1}{p}}\left(\frac{\frac{p+1}{p}}{x_n+x_n^{\frac{p+1}{p}}}\right)\right\}^p =$$

$$= \lim_{n\to\infty}\left\{\left(\frac{\frac{p+1}{p}}{x_n^{-\frac{1}{p}}+1}\right)\right\}^p \therefore \lim_{n\to\infty}\frac{x_n^{p+1}}{n^p} = \left(\frac{p+1}{p}\right)^p$$

$$\left[Here \lim_{n\to\infty} x_n = \infty \therefore \lim_{n\to\infty}\frac{1}{x_n+x_n^{\frac{p+1}{p}}} = 0 \therefore \lim_{n\to\infty}\left(1+\frac{1}{x_n+x_n^{\frac{p+1}{p}}}\right)^{\frac{p+1}{p}} - 1\right.$$

$$\left. = \lim_{n\to\infty}\left(\frac{\frac{p+1}{p}}{x_n+x_n^{\frac{p+1}{p}}}\right) \& \lim_{n\to\infty} x_n^{-\frac{1}{p}} = 0\right]$$

3.125

We know, $lim_{u\to 0}\frac{(1+u)^r-1}{u} = r$ *and* $lim_{u\to 0}\frac{(1-u)^{-r}-1}{u} = r$

$x_{n+1} = x_n\sqrt[p]{1-x_n}$ *where* $x_0 \in (0,1)$*,*

then we have $1 > x_0 \geq x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq \cdots > 0$

$\therefore \{x_n\}_{n=1}^{\infty}$ *is a decreasing function similarly,* $\{y_n\}_{n=1}^{\infty}$ *defined by*

$y_{n+1} = y_n + \frac{1}{y_n^{p-1}}$ *is an increasing function.*

Let $lim_{n\to\infty} x_n = l$ *and* $lim_{n\to\infty} y_n = m$ *then* $l = l\sqrt[p]{1-l} \Rightarrow l = 0$ *and*

$m = m + \frac{1}{m^{p-1}} \Rightarrow m = \infty \therefore lim_{n\to\infty} x_n = 0$ *and* $lim_{n\to\infty} y_n = \infty$

$$\Omega = \lim_{n\to\infty}\left(y_n\sqrt[p]{x_n}\right) \Rightarrow \Omega^p = \lim_{n\to\infty}\left(\frac{y_n^p}{n}\cdot\frac{n}{\frac{1}{x_n}}\right)$$

$$\overset{CESARO\ STOLZ}{=} \lim_{n\to\infty} \frac{y_{n+1}^p - y_n^p}{n+1-n} \cdot \frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \left(\left(y_n + \frac{1}{y_n^{p-1}} \right)^p - y_n^p \right) \cdot \lim_{n\to\infty} \frac{1}{\frac{1}{x_n \sqrt[p]{1-x_n}} - \frac{1}{x_n}}$$

$$= \lim_{y_n\to\infty} \frac{\left(1 + \frac{1}{y_n^p}\right)^p - 1}{\frac{1}{y_n^p}} \cdot \lim_{x_n\to 0} \frac{1}{\frac{(1-x_n)^{-\frac{1}{p}} - 1}{x_n}} = p^2 \Rightarrow \Omega = \sqrt[p]{p^2}$$

3.126

$$\textit{For } p > 2. \textit{ Let } a_i = \left(\frac{n}{i}\right)^{\frac{1}{p}} = \frac{1}{\left(\frac{i}{n}\right)^{\frac{1}{p}}}$$

$$2 \sum_{1\le i<j\le n} a_i a_j = \left(\sum_{i=1}^n a_i \right)^2 - \sum_{i=1}^n a_i^2$$

$$\Rightarrow 2 \frac{1}{n^2} \sum_{1\le i<j\le n} a_i a_j = \left(\frac{1}{n} \sum_{i=1}^n a_i \right)^2 - \frac{1}{n} \cdot \frac{1}{n} \sum_{i=1}^n a_i^2$$

$$2 \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} a_i a_j = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n a_i \right)^2 - \lim_{n\to\infty} \frac{1}{n} \left(\frac{1}{n} \sum_i^n a_i^2 \right)$$

$$= \left(\int_0^1 \frac{1}{x^{\frac{1}{p}}} dx \right)^2 - \left(\lim_{n\to\infty} \frac{1}{n} \right) \int_0^1 \frac{1}{x^{\frac{2}{p}}} dx = \left(\left[\frac{x^{-\frac{1}{p}+1}}{1-\frac{1}{p}} \right]_0^1 \right)^2 - (0) \left[\frac{x^{1-\frac{2}{p}}}{1-\frac{2}{p}} \right]_0^1 = \frac{p^2}{(p-1)^2} - 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} a_i a_j = \frac{p^2}{2(p-1)^2}$$

3.127 $$I = \int \frac{x^2+1}{x^4+x^2+1} dx = \int \frac{\frac{x^2+1}{x^2}}{\frac{x^4+x^2+1}{x^2}} dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2 + \left(\sqrt{3}\right)^2} dx =$$

$$= \left\{ \begin{matrix} x - \frac{1}{x} = t \\ \left(1 + \frac{1}{x^2}\right) dx = dt \end{matrix} \right\} = \int \frac{dt}{t^2 + \left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}} arctan \frac{t}{\sqrt{3}} = \frac{1}{\sqrt{3}} arctan \frac{x^2-1}{\sqrt{3}x}$$

$$I = \int_0^{\frac{1}{n^5}} \frac{x^2+1}{x^4+x^2+1}dx = \frac{1}{\sqrt{3}} arctan \frac{x^2-1}{\sqrt{3}x}\Bigg|_0^{\frac{1}{n^5}} = \frac{1}{\sqrt{3}} arctan \frac{n^5\sqrt{3}}{n^{10}-1}$$

$$L = \lim_{n\to\infty} n^8 \cdot \frac{1}{\sqrt{3}} \cdot arctan \frac{n^5\sqrt{3}}{n^{10}-1} = \frac{1}{\sqrt{3}} \lim_{n\to\infty} \frac{arctan \frac{n^5\sqrt{3}}{n^{10}-1}}{\frac{1}{(n)^8}} = \frac{0}{0}$$

$$\lim_{n\to\infty} \frac{\frac{1}{1+\left(\frac{n^5\sqrt{3}}{n^{10}-1}\right)^2} \cdot \frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^5n^9}{(n^{10}-1)^2}}{-\frac{8n^7}{n^{16}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^{14}}{(n^{10}-1)^2+3n^{10}}}{-\frac{8}{n^9}} = -\lim_{n\to\infty} \frac{n^9[5\sqrt{3}n^{14}-5\sqrt{3}n^4-10\sqrt{3}n^{14}]}{8[n^{20}-2n^{10}+1+3n^{10}]} =$$

$$= \lim_{n\to\infty} \frac{n^9 5\sqrt{3}[n^{14}+n^4]}{8[n^{20}+n^{10}+1]} = \frac{5\sqrt{3}}{8} \lim_{n\to\infty} \frac{n^9n^{14}\left[1+\frac{1}{n^{10}}\right]}{n^{20}\left[1+\frac{1}{n^{10}}+\frac{1}{n^{20}}\right]} = \frac{5\sqrt{3}}{8} \lim_{n\to\infty} n^3 = +\infty$$

$$L = \frac{1}{\sqrt{3}} \cdot (+\infty) = +\infty$$

3.128 *We shall use the following results:*

1. $\sum_{k=0}^{n} k\binom{n}{k}\frac{1}{2^n} = \frac{n}{2}$

2. $\sum_{k=0}^{n} k(k-1)\binom{n}{k}\frac{1}{2^n} = \frac{n(n-1)}{4}$

3. (1), (2) $\Rightarrow \sum_{k=0}^{n} k^2\binom{n}{k}\frac{1}{2^n} = \frac{n(n+1)}{4}$

4. $\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{n}{2}-k\right)^2 = \frac{1}{4}n^2 - \frac{n^2}{2} + \frac{n(n+1)}{4} = \frac{n}{4}$

Main question

Let $f(x) = sin^{-1} x, 0 \le x \le 1$

As $f(x)$ *is continuous at* $x = \frac{1}{2}$*, given* $\varepsilon > 0\ \exists \delta > 0$ *such that:*

$$\left|f(x) - \frac{\pi}{6}\right| = \left|f(x) - f\left(\frac{1}{2}\right)\right| < \frac{\varepsilon}{2} \text{ whenever, } \left|x - \frac{1}{2}\right| < \delta, 0 \le x \le 1.$$

Now, $\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k} sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[sin^{-1}\left(\frac{k}{n}\right) - sin^{-1}\left(\frac{1}{2}\right)\right]$

$$\left[\because \sum_{k=0}^{n}\binom{n}{k} = 2^n\right]$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right]$$

$$\Rightarrow \left|\sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}\left\{sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6}\right\}\right| \le \sum_{k=0}^{n}\frac{1}{2^n}\binom{n}{k}\left|sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6}\right| \quad (1)$$

We split the set $\{0,1,2,\dots,n\}$ *into two subsets* A *and* B.

$0 \le k \le n, k \in A$ *if* $\left|\frac{k}{n} - \frac{1}{2}\right| < \delta$ *and* $k \in B$ *if* $\left|\frac{k}{n} - \frac{1}{2}\right| \ge \delta$

Now, $\sum_{k\in A}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right| \le \frac{\varepsilon}{2}\sum_{k\in A}\frac{1}{2^n}\binom{n}{k} \le \frac{\varepsilon}{2}(1) = \frac{\varepsilon}{2}$ *(2)*

If $k \in B$*, then* $\left|\frac{k}{n} - \frac{1}{2}\right| \ge \delta \Rightarrow \left(k - \frac{n}{2}\right)^2 \ge n^2\delta^2$

Now, $\sum_{k\in B}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right| \le \left(\frac{\pi}{2} + \frac{\pi}{2}\right)\frac{1}{n^2\delta^2}\sum_{k\in B}\left(k - \frac{n}{2}\right)^2\binom{n}{k}\frac{1}{2^n}$

$\le \frac{\pi}{n^2\delta^2}\cdot\frac{n}{4}$ *[using (4)]*

$$\Rightarrow \sum_{k\in B}\frac{1}{2^n}\binom{n}{k}\left|f\left(\frac{k}{n}\right) - f\left(\frac{1}{2}\right)\right| \le \frac{\pi}{4n\delta^2} \quad (3)$$

We choose n *sufficiently large, so that* $\frac{\pi}{4n\delta^2} < \frac{\varepsilon}{2}$ *(4)*

[This is possible as $\frac{1}{n} \to 0$ *as* $n \to \infty$*]Using (1), (2), (3), (4) we get:*

$\left|\frac{1}{2^n}\sum_{k=1}^{n}\binom{n}{k} sin^{-1}\left(\frac{k}{n}\right) - \frac{\pi}{6}\right| < \varepsilon$ *for sufficiently large values of* n.

$$\Rightarrow \lim_{n\to\infty}\frac{1}{2^n}\sum_{k}^{n}\binom{n}{k} sin^{-1}\left(\frac{k}{n}\right) = \frac{\pi}{6}$$

CHAPTER 9

INTEGRALS-SOLUTIONS

4.1

$$\int \left(x^{10} + \sqrt{1 + x^{20}}\right)^{\frac{21}{10}} = \int \left(\frac{(1 + x^{20}) - x^{20}}{\sqrt{1 + x^{20}} - x^{10}}\right)^{\frac{21}{10}} dx =$$

$$= \int \left(\sqrt{1 + x^{20}} - x^{10}\right)^{-\frac{21}{10}} dx = \int \left(\sqrt{1 + x^{-20}} - 1\right)^{-\frac{21}{10}} \cdot x^{-21} dx$$

$$\left(u = \sqrt{1 + x^{-20}} - 1 \Rightarrow x^{-20} = u^2 + 2u, -20x^{-21} dx = (2u + 2) du\right)$$

$$= \int u^{-\frac{21}{20}} \cdot \left[-\frac{1}{10}(u + 1)\right] du = -\frac{1}{10} \int \left(u^{-\frac{1}{20}} + u^{-\frac{21}{20}}\right) du = -\frac{1}{10}\left(\frac{20}{19} u^{\frac{19}{20}} - 20u^{-\frac{1}{20}}\right) + C =$$

$$= -\frac{2}{19}\left(1 - \sqrt{1 + x^{-20}}\right)^{\frac{19}{20}} + 2\left(1 - \sqrt{2 + x^{-20}}\right)^{-\frac{1}{20}} + C$$

4.2

$$L(2) = \lim_{x \to 0} \frac{(1 + x)^{2 + \frac{1}{x}} - e}{x}$$

$$L(a) = \lim_{x \to 0} \frac{(1 + x)^{\frac{1}{x}} (1 + x)^a - e}{x} = \lim_{x \to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24} x^2\right)(1 + ax + \cdots) - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left(1 + ax - \frac{x}{2} + \frac{ax^2}{2} + \cdots - 1\right)}{x} = \lim_{x \to 0} \frac{e(2a - 1)}{2} + 0(x)$$

$$L(a) = \frac{e(2a - 1)}{2}$$

$$L(a) + L(b) + L(c) \geq 3\big(L(a)L(b)L(c)\big)^{-\frac{1}{3}}$$

$$L(a)L(b)L(c) \leq \frac{\big(L(a) + L(b) + L(c)\big)^3}{27}$$

$$L(a) + L(b) + L(c) = \frac{e}{2}(2(a + b + c) - 3)$$

$$\text{and } a + b + c = \tfrac{3e+2}{2e}$$

$$2(a + b + c) = \frac{3e + 2}{e}$$

$$L(a) + L(b) + L(c) = \frac{e}{2}\left(3 + \frac{2}{e} - 3\right) = 1$$

$$\text{Hence } \prod L(a) \le \tfrac{1}{27} \quad \text{(proved)}$$

4.3

$$I_n = \int_n^{n+1} \frac{dx}{\sqrt{(x-n)(n+1-x)}} = \lim_{\substack{\varphi_1 \to 0^+ \\ \varphi_2 \to 0^+}} \int_{n+\varphi_1}^{n+1-\varphi_2} \frac{dx}{\sqrt{(x-n)(n+1-x)}} =$$

$$= \lim_{\substack{\varphi_1 \to 0^+ \\ \varphi_2 \to 0^+}} \int_{n+\varphi_1}^{n+1-\varphi_2} \left(\sqrt{\frac{n+1-x}{x-n}} + \sqrt{\frac{x-n}{n+1-x}}\right) dx$$

$$I = \int \left(\sqrt{\frac{n+1-x}{x-n}} + \sqrt{\frac{x-n}{n+1-x}}\right) dx$$

$$\frac{n+1-x}{x-n} = t^2 \Rightarrow x = n + \frac{1}{1+t^2} \Rightarrow dx = -\frac{2t}{(1+t^2)^2} dt$$

$$I = -\int \left(t + \frac{1}{t}\right)\left(\frac{2t}{(1+t^2)^2} dt\right) = -2 \arctan t = -2 \arctan \sqrt{\frac{n+1-x}{x-n}}$$

$$I_n = \lim_{\substack{\varphi_1 \to 0^+ \\ \varphi_2 \to 0^+}} \left(-2 \arctan \sqrt{\frac{n+1-x}{x-n}}\right)\Bigg|_{n+\varphi_1}^{n+1-\varphi_2} =$$

$$-2 \lim_{\substack{\varphi_1 \to 0^+ \\ \varphi_2 \to 0^+}} \left(\arctan \sqrt{\frac{\varphi_2}{1-\varphi_2}} - \arctan \sqrt{\frac{1-\varphi_1}{\varphi_1}}\right) = -2 \cdot \left(-\frac{\pi}{2}\right) = \pi$$

$$\Omega = \lim_{n\to\infty} \sin\left(\frac{1}{n} I_n\right) = \lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = 0$$

True or false

$$-1 < \Omega < 1$$

4.4

$$I_m = \int_0^1 \frac{\sqrt{e^x}(x+3)}{(x+1)\sqrt{e^x} + m} dx$$

Put $t = (x+1)\sqrt{e^x} + m$, *When* $x = 0, t = m+1$, $x = 1, t = m + 2\sqrt{e}$

$$\frac{dt}{dx} = (x+1)\frac{1}{2}\sqrt{e^x} + \sqrt{e^x} = \frac{1}{2}(x+3)\sqrt{e^x}$$

Thus,

$$I_m = 2\int_{m+1}^{m+2\sqrt{e}} \frac{dt}{t} = 2\ln\left(\frac{m+2\sqrt{e}}{m+1}\right), \qquad I_m < 1$$

$$\Leftrightarrow \ln\left(\frac{m+2\sqrt{e}}{m+1}\right) < \frac{1}{2} \Leftrightarrow m + 2\sqrt{e} < (m+1)\sqrt{e} \Leftrightarrow \sqrt{e} < \left(\sqrt{e} - 1\right)m \Leftrightarrow m < (m-1)\sqrt{e}$$

Not true for $m = 1,2$. *True, for* $m \geq 3$

4.5

Let us first prove that, $\forall x \in \left(0, \frac{\pi}{2}\right)$, $\sin x > x\sqrt[3]{\cos x}$

$$\sin x > x\sqrt[3]{\cos x} \Leftrightarrow \sin^3 x > x^3 \cos x \Leftrightarrow \sin^2 x \tan x > x^3 \quad (1)$$

Let $f(x) = \sin^3 x \tan x - x^3$, $f(0) = 0$

$$f'(x) = \sin^2 x \sec^2 x + \tan x\,(2\sin x \cos x) - 3x^2$$

$$= \tan^2 x + 2\sin^2 x - 3x^2 = g(x) \text{ (say)};\ g(0) = 0$$

$$g'(x) = 2\tan x \sec^2 x + 4\sin x \cos x - 6x$$

Let $\alpha(x) = \tan x \sec^2 x + 2\sin x \cos x - 3x$; $\alpha(0) = 0$

$$\alpha'(x) = (sec^2 x)(sec^2 x) + tan\, x\,(2\, sec\, x)(sec\, x\, tan\, x) + 2(cos^2 x - sin^2 x) - 3$$

$$= (1 + tan^2 x)^2 + 2\, tan^2 x\,(1 + tan^2 x) + 2(2\, cos^2 x - 1) - 3$$

$$= (1 + z)^2 + 2z(1 + z) + \frac{4}{1+z} - 5 \textit{ (taking } z = tan^2 x \textit{ and } sec^2 x = 1 + z\textit{)}$$

$$= \frac{(1 + z)^3 + 2z(1 + z)^2 + 4 - 5(1 + z)}{1 + z} = \frac{1 + z^3 + 3z + 3z^2 + 2z + 2z^3 + 4z^2 - 1 - 5z}{1 + z}$$

$$= \frac{3z^3+7z^2}{1+z} > 0 \textit{ (} z = tan^2 x > 0\textit{)}$$

$\alpha'(x) > 0$ *and* $\alpha(0) = 0$, $\forall x \in \left(0, \frac{\pi}{2}\right), \alpha(x) > \alpha(0) \Rightarrow \alpha(x) > 0 \Rightarrow g'(x) = 2\alpha(x) > 0$ *and* $g(0) = 0$,

$$\forall x \in \left(0, \frac{\pi}{2}\right), g(x) > g(0) = 0 \Rightarrow f'(x) > 0$$

$$f'(x) > 0 \textit{ and } f(0) = 0 \Rightarrow f(x) > f(0) = 0, \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$\forall x \in \left(0, \frac{\pi}{2}\right), sin^2 x\, tan\, x - x^3 > 0 \Rightarrow sin\, x > x\sqrt[3]{cos\, x} \textit{ from (1)}$$

For $x = \frac{\pi}{2}$, $sin\, x > x\sqrt[3]{cos\, x}$ *and for* $x \in \left(\frac{\pi}{2}, \pi\right)$, $\sqrt[3]{cos\, x} < 0$, $sin\, x > x\sqrt[3]{cos\, x}$

$$\forall x \in (0, \pi), sin\, x > x\sqrt[3]{cos\, x}$$

4.6

$$\int \frac{(x^2 + 1)^2}{\left(x^{\frac{113}{25}} + \frac{11}{3} x^{\frac{63}{25}} + 11\, x^{\frac{13}{25}}\right)^5}\, dx = \int \frac{x^4 + 2x^2 + 1}{x^{\frac{3}{5}}\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5} dx$$

$$= \int \frac{1}{\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5} \cdot \left(x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}\right) dx$$

(let $= x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}} \Rightarrow du = \frac{22}{5}\left(x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}\right) dx$ *)*

$$= \int \frac{1}{u^5} \cdot \frac{5}{22} du = -\frac{5}{22} \cdot \frac{1}{4} u^{-4} + Q = -\frac{5}{88}\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^{-4} + Q$$

$$\therefore \vartheta = -\frac{5}{(-4)}\left(\frac{22}{5}+\frac{12}{5}+\frac{2}{5}\right) = 9$$

4.7

$$\Omega = \lim_{t\to\infty}\int_0^t \frac{x^4}{(1+x^3)^2}dx; I = \int \frac{x^4}{(1+x^3)^2}dx = \begin{cases} u = x^2 \\ du = 2xdx \end{cases}$$

$$dv = \frac{x^2}{(1+x^3)^2}dx \Rightarrow v = \int \frac{x^2}{(1+x^3)^2}dx = \begin{Bmatrix} 1+x^3 = t \\ 3x^2dx = dt \end{Bmatrix} = \frac{1}{3}\int \frac{dt}{t} =$$

$$= -\frac{1}{3t} = -\frac{1}{3(1+x^3)}\Big\} = u\cdot\vartheta - \int \vartheta\cdot du = -\frac{x^2}{8(1+x^3)} + \frac{2}{3}\underbrace{\int \frac{x}{1+x^3}}_{I_1}dx$$

$$I_1 = \int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)} = \int \frac{A}{x+1}dx + \int \frac{Bx+c}{x^2-x+1}dx$$

$$\frac{x}{x^3+1} = \frac{A}{x+1} + \frac{Bx+c}{x^2-x+1}\Big| \cdot (x+1)(x^2-x+1) \Rightarrow$$

$$\begin{matrix} \Leftrightarrow x = A(x^2-x+1) + (Bx+c)(x+1) \\ \Leftrightarrow x = Ax^2 - Ax + A + Bx^2 + Bx + Cx + c \\ \Leftrightarrow x = (A+B)x^2 + (-A+B+C)x + A + C \end{matrix} \Rightarrow \begin{cases} A + D = 0 \\ -A + B + C = 1 \\ A + C \end{cases} \Leftrightarrow \begin{Bmatrix} A = -\frac{1}{3} \\ B = C = \frac{1}{3} \end{Bmatrix}$$

$$I_2 = \int \frac{A}{x+1}dx = -\frac{1}{3}ln(x+1); I_3 = \frac{1}{3}\int \frac{x+1}{x^2-x+1}dx = \frac{1}{6}\int \frac{2x+2}{x^2-x+1}dx =$$

$$= \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2+x+1} = \frac{1}{6}\cdot I_4 + \frac{1}{2}\cdot I_5$$

$$I_4 = \int \frac{2x-1}{x^2-x+1}dx = \begin{Bmatrix} x^{-x}+x+1 \\ (2x-1)dx = dt \end{Bmatrix} = \int \frac{dt}{t} = ln\, t = ln(x^2-x+1)$$

$$I_5 = \int \frac{dx}{x^2-x+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \begin{Bmatrix} x - \frac{1}{2} = t \\ dx = dt \end{Bmatrix} = \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{2}{\sqrt{3}}\cdot arctan\frac{2x-1}{\sqrt{3}}$$

$$I_3 = \frac{1}{6}\cdot ln(x^2-x+1) + \frac{1}{\sqrt{3}}arctan\frac{2x-1}{\sqrt{3}}; I_1 = I_2 + I_3$$

$$I = -\frac{x^2}{3(1+x^3)} + \frac{2}{3} \cdot I_1 = \frac{1}{9} \cdot \left[ln\left(\frac{x^2-x+1}{(x+1)^2} - \frac{3x^2}{x^3+1} + 2\sqrt{3}\,arctan\frac{2x-1}{\sqrt{3}}\right)\right]\Big|_0^t$$

$$\Omega = \frac{1}{9}\lim_{t\to\infty}\left[ln\frac{t^2 - t + 1}{(t + 1)^2} - \frac{3t^2}{t^3 + 1} + e\sqrt{3}\frac{2t - 1}{\sqrt{3}} + \frac{2\sqrt{3}\pi}{6}\right] =$$

$$= \frac{1}{9} \cdot \left[2\sqrt{3}\frac{\pi}{2} + 2\sqrt{3}\frac{\pi}{6}\right] = \frac{1}{9} \cdot 2\sqrt{3} \cdot \frac{4\pi}{6} = \frac{4\sqrt{3}\pi}{27}$$

4.8

$$\int_{-\pi}^{\pi} \frac{cos^2(-x)}{-x + a + \sqrt{x^2 + a^2}}dx$$

$$2I(a) = 2\int_0^{\pi}\left(\frac{cos^2 x}{\sqrt{x^2 + a^2} + a + x} + \frac{cos^2 x}{\sqrt{x^2 + a^2} + a - x}\right)dx$$

$$I(a) = \frac{1}{a}\int_0^{\pi} cos^2(x)\, dx \Rightarrow \frac{1}{a}\int_{-\pi}^{\pi}\frac{1+cos(2x)}{2}dx \Rightarrow \frac{1}{a}\left[\frac{\pi}{2} - 0\right] \text{ (OR) } I(a) = \frac{\pi}{2a}$$

$$\sum_{cyc}\left(I(a)\right) \overset{BERGSTROM}{\geq} \frac{9\pi}{2(a + b + c)}$$

4.9

$$Let\ x = tan\, b$$

$$I = \int_b^{\frac{\pi}{2}-b} ln\, tan\, b\, db = \int_b^{\frac{\pi}{2}-b} ln\, cot\, b\, db$$

$$I = \frac{1}{2}\int_b^{\frac{\pi}{2}-b}[ln\, tan\, b + ln\, cot\, b]\, db = 0$$

4.10

$$x = -t \Rightarrow \Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{f(-t)}{cos^3 t}dt$$

$$2\Omega = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{f(x) + f(-x)}{\cos^3 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a \cos x}{\cos^3 x} = a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}$$

$$\Omega = \frac{a}{2} \tan x|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a}{2}\left(\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right)$$

$$\Omega = \frac{a}{2}(1 + 1) = a, \Omega = a$$

4.11

Let us denote by $\varphi(x) = x \sin x\,(x + \cos x)$

$$\Rightarrow \varphi'(x) = x^2 \cos x + 2x \sin x + \sin x \cos x + x \cos x$$

then

$$\int \frac{x^2 \cos x + x + \sin x \cos x}{x \sin x\,(x + \cos x)} dx = \int \frac{\varphi'(x)}{\varphi(x)} dx + \int \frac{x - 2x \sin x - x \cos x}{\varphi(x)} dx$$

$$= \ln|\varphi(x)| - 2\int \frac{x \sin x\,(1 - \sin x)}{x \sin x\,(x + \cos x)} dx = \ln|\varphi(x)| - 2\int \frac{(x + \cos x)'}{x + \cos x} dx$$

$$= \ln|\varphi(x)| - 2\ln|x + \cos x| + \lambda = \ln\left|\frac{\varphi(x)}{(x+\cos x)^2}\right| + \lambda, \text{ whith } \lambda \in \mathbb{R}$$

Finally we get

$$\int \frac{x^2 \cos x + x + \sin x \cos x}{x \sin x\,(x + \cos x)} dx = \ln\left|\frac{x \sin x}{x + \cos x}\right| + \lambda$$

4.12

$$\int \frac{\cot 2x \cdot \cot x}{(\cot^2 x - \tan^2 x) \sin^3 2x} dx = \int \frac{\cos 2x \cdot \cos x}{\sin 2x \cdot \sin x} \cdot \frac{1}{\frac{\cos^2 x \cdot \sin^2 x}{\cos^2 x \cdot \sin^2 x}} \cdot \frac{1}{8 \sin^2 x \cos^3 x} dx$$

$$= \frac{1}{8}\int \frac{\cos x}{\sin 2x \cdot \sin^2 x \cdot \cos x} dx = \frac{1}{16}\int \frac{1}{\sin^3 x \cos x} dx$$

$$= \frac{1}{16}\int \frac{1}{\sin^3 x \cdot \cos^2 x} \cdot \cos dx = \frac{1}{16}\int \frac{1}{y^3 \cdot (1-y^2)} dx = (*), y = \sin x, dy = \cos x\, dx$$

$$\frac{1}{y^3(1 - y^2)} = \frac{1 - y^2 + y^2}{y^3(1 - y^2)} = \frac{1}{y^3} + \frac{1}{y(1 - y^2)} = \frac{1}{y^3} + \frac{1}{y} + \frac{y}{1 - y^2}$$

$$(*) = \frac{1}{16}\left(\int \frac{1}{y^3} dy + \int \frac{1}{y} dy + \int \frac{y}{1-y^2} dy\right)$$

$$= \frac{1}{16}\left(\frac{y^{-2}}{-2} + ln\, y - \frac{1}{2} ln\, \underbrace{(1-y^2)}_{+}\right) + C =$$

$$= \frac{1}{16}\left(-\frac{+1}{2y^2} + ln \frac{y}{\sqrt{1-y^2}}\right) + C = \frac{1}{16}\left(\frac{-1}{2\, sin^2 x} + ln \frac{sin\, x}{cos\, x}\right) + C$$

$$= \frac{1}{16}\left(\frac{-1}{2\, sin^2 x} + ln(tan\, x)\right) + C$$

4.13

$$cos\, 2x = cos^2 x - sin^2 x = cos^4 x - sin^4 x = sin^2 x\, cos^2 x\, (cot^2 x - tan^2 x)$$

$$* \, sin^3 2x = 8\, sin^3 x\, cos^3 x$$

$$\Rightarrow \Omega = \int \frac{sin^2 x\, cos^2 x\, (cot^2 x - tan^2 x) \frac{cos\, x}{sin\, x}}{(cot^2 x - tan^2 x) \cdot 8\, sin^3\, cos^3 x} dx = \frac{1}{8} \int \frac{1}{sin^2 x} dx = -\frac{1}{8} cot\, x + C$$

4.14

$$\Omega = \int_a^b tan(arccos(sin(arctan(x))))\, dx$$

$$= \int_a^b tan\left(\frac{\pi}{2} - arcsin(sin(arctan(x)))\right) dx \because arctan\, x \in \left]0; \frac{\pi}{2}\right[$$

$$= \int_a^b tan\left(\frac{\pi}{2} - arctan(x)\right) dx = \int_a^b tan\left(arctan\left(\frac{1}{x}\right)\right) dx = \int_a^b \frac{1}{x} dx = ln\left(\frac{b}{a}\right)$$

4.15

$$I = \int \frac{sin\, 3x\, cos^2 x}{cos\, 3x\, cos\, 2x} dx = \int \frac{(3\, sin\, x - 4\, sin^3 x)\, cos^2 x}{(4\, cos^3 x - 3\, cos\, x)(2\, cos^2 x - 1)} dx$$

$$= \int \frac{(3 - 4\, sin^2 x)\, sin\, x\, cos\, x}{(4\, cos^2 x - 3)(2\, cos^2 x - 1)} dx$$

$$I = \int \frac{\left(3 - 4(1 - cos^2 x)\right) cos\, x\, sin\, x}{(4\, cos^2 x - 3)(2\, cos^2 x - 1)} dx$$

$$\text{Put } cos^2 x = t$$

$$-2\, sin\, x\, cos\, x\, dx = dt$$

$$I = -\frac{1}{2}\int \frac{(4t - 1)}{(4t - 3)(2t - 1)} dt = -\int \frac{4t - 1}{(4t - 3)(4t - 2)} dt$$

$$= -\int \left[\frac{2}{4t - 3} - \frac{1}{4t - 2}\right] dt = \int \left[\frac{1}{4t - 2} - \frac{2}{4t - 3}\right] dt$$

$$= \frac{1}{4} ln|4t - 2| - \frac{1}{2} ln|4t - 3| + C = \frac{1}{4} ln|2\, cos^2 x - 1| - \frac{1}{2} ln|4\, cos^2 x - 3| + c$$

4.16

$$I = \int \frac{[x^2(1 - e^{-x}) - 1]dx}{x^4 + 2x^3 e^{-x} + (3 + e^{-2x})x^2 + 2xe^{-x} + 1} dx$$

Dividing numerator and denominator by x^2

$$I = \int \frac{\left(1 - e^{-x} - \frac{1}{x^2}\right) dx}{x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) e^{-x} + 3 + e^{-2x}} = \int \frac{\left(1 - e^{-x} - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) e^{-x} + e^{-2x} + 1}$$

$$I = \int \frac{\left(1 - e^{-x} - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x} + e^{-x}\right)^2 + 1} \quad \text{Put } x + \frac{1}{x} + e^{-x} = t$$

$$\left(1 - \frac{1}{x^2} - e^{-x}\right) dx = dt$$

$$I = \int \frac{dt}{t^2 + 1} = tan^{-1} t + c = tan^{-1}\left(x + \frac{1}{x} + e^{-x}\right) + c$$

4.17

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x - j} = \frac{(x - 1) + (x - 2) + \cdots + (x - n)}{(x - 1)(x - 2) \ldots (x - n)}$$

$$= \frac{nx - \frac{1}{2}n(n+1)}{(x-1)(x-2)\dots(x-n)} = \sum_{k=1}^{n} \left(n(k) - \frac{1}{2}n(n+1)\right)\frac{a_k}{x-k}$$

where

$$a_k = \frac{1}{\underbrace{(k-1)(k-2)\dots(1)}_{(k-1)\ factors}\underbrace{(-1)(-2)\dots(k-n)}_{(n-k)\ factors}}$$

$$= \frac{1}{(k-1)!\,(-1)^{n-k}(n-k)!} = \frac{(-1)^{n-k}}{(n-1)!}\binom{n-1}{k-1}$$

Thus,

$$\int_{n+1}^{n+2} \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{x-j} \right)$$

$$= \sum_{k=1}^{n} \frac{nk - \frac{1}{2}n(n+1)}{(n-1)!}(-1)^k \binom{n-1}{k-1} ln\left(\frac{n+2-k}{n+1-k}\right)$$

4.18

$$\int_a^b \left(\sum l\,n(x+b)^{\frac{1}{x+a}}\right)dx = \int_a^b \left(\sum \frac{1}{x+a} l\,n(x+b)\right)dx =$$

$$= \int_a^b \left(\prod ln(x+a)\right)' dx = ln(b+a)\,ln(b+b)\,ln(b+c) - ln(a+a)\,ln(a+b)\,ln(a+c) =$$

$$= ln(a+b)\,ln(a+c)\,(ln(2b) - ln(2a)) = ln\frac{b}{a}ln(a+b)\,ln(a+c)$$

4.19

$$(x+1)^2(ab)^x + (x+1)(a^x+b^x) + 1 = a^x b^x \left(x+1+\tfrac{1}{a^x}\right)\left(x+1+\tfrac{1}{b^x}\right) \quad (1)$$

(easy if you consider $y^2(ab)^x + y(a^x+b^x) + 1$ *)*

$$\frac{(b^x - a^x) + (x+1)(b^x \ln a - a^x \ln b) + \ln a - \ln b}{(x+1)^2(ab)^x + (x+1)(a^x + b^x) + 1} =$$

$$\overset{(1)}{=} \frac{\left(\frac{1}{a^x} - \frac{1}{b^x}\right) + (x+1)\left(\frac{\ln a}{a^x} - \frac{\ln b}{b^x}\right) + \frac{\ln a - \ln b}{a^x b^x}}{\left(x+1+\frac{1}{a^x}\right)\left(x+1+\frac{1}{b^x}\right)} =$$

$$= \frac{\left[x+1+\frac{1}{a^x} - \left(x+1+\frac{1}{b^x}\right)\right] + \frac{\ln a}{a^x}\left(x+1+\frac{1}{b^x}\right) - \frac{\ln b}{b^x}\left(x+1+\frac{1}{a^x}\right)}{\left(x+1+\frac{1}{a^x}\right)\left(x+1+\frac{1}{b^x}\right)} =$$

$$= \frac{x+1+\frac{1}{a^x} - \left(x+1+\frac{1}{b^x}\right)}{\left(x+1+\frac{1}{a^x}\right)\left(x+1+\frac{1}{b^x}\right)} + \frac{\frac{\ln a}{a^x}}{x+1+\frac{1}{a^x}} - \frac{\frac{\ln b}{b^x}}{x+1+\frac{1}{b^x}} =$$

$$= \frac{1}{x+1+\frac{1}{b^x}} - \frac{1}{x+1+\frac{1}{a^x}} + \frac{\frac{\ln a}{a^x}}{x+1+\frac{1}{a^x}} - \frac{\frac{\ln b}{b^x}}{x+1+\frac{1}{b^x}} =$$

$$= \frac{1 - \frac{\ln b}{b^x}}{x+1+\frac{1}{b^x}} - \frac{1 - \frac{\ln a}{a^x}}{x+1+\frac{1}{a^x}} = \frac{\left(x+1+\frac{1}{b^x}\right)'}{x+1+\frac{1}{b^x}} - \frac{\left(x+1+\frac{1}{a^x}\right)'}{x+1+\frac{1}{a^x}}$$

$$So \int f(x)dx = \ln\left(\frac{x+1+\frac{1}{b^x}}{x+1+\frac{1}{a^x}}\right) + c, c \in \mathbb{R}$$

4.20

$$\int_0^1 \int_0^2 \dots \int_0^{2018} \left\{\sum_{k=1}^{2018} x_{2018}\right\} dx_1 dx_2 \dots dx_{2018} = \int_0^1 \int_0^2 \dots \int_0^{2018} \left\{\sum_{k=1}^{2018} x_{2018}\right\} dx_{2018} dx_{2017} \dots dx_1$$

$$= \int_0^1 \int_0^2 \dots \int_0^{2017} \left[\int_{\sum_{k=1}^{2017} x_{2017}}^{2018+\sum_{k=1}^{2017} x_{2017}} \{t\}\, dt\right] dx_{2017} \dots dx_1 = \int_0^1 \int_0^2 \dots \int_0^{2017} \left[\frac{2018}{2}\right] dx_{2017} \dots dx_2 dx_1$$

$$= \frac{2018}{2} * 2017! = \frac{2018!}{2}$$

4.21

$$\Omega = \int_1^6 f^2(x) + 18xf^{-1}(3x)dx = \int_1^6 f^2(x)dx + 18\int_1^6 xf^{-1}(3x)dx$$

for second integral $3x = t \Rightarrow x = \frac{t}{3}|' \Rightarrow dx = \frac{1}{3}dt$

$$x = 1 \Rightarrow t = 3; x = 6 \Rightarrow t = 18$$

$$\Omega = \int_1^6 f^2(x)\,dx + 18\int_3^{18} \frac{t}{3}\cdot f^{-1}(t)\cdot\frac{1}{3}dt = \int_1^6 f^2(x)dx + 2\int_3^{18} t\cdot f^{-1}(t)dt$$

for second integral: $f^{-1}(t) = y \Rightarrow t = f(y)$

$$t = 3 \Rightarrow y = 1; t = 18 \Rightarrow t = 6; dt = f'(y)dy$$

$$\Omega = \int_1^6 f^2(x)\,dx + 2\int_1^6 f(y)\cdot y\cdot f'(y)dy = \int_1^6 f^2(x)dx + \int_1^6 y\cdot(f(y)^2)'dy =$$

$$= \int_1^6 f^2(x)\,dx + y\big(f(y)\big)^2|_1^6 - \int_1^6 \big(f(y)\big)^2\,dy = 6f(6)^2 - f^2(1) = 6\cdot 18^2 - 9 = 1935$$

4.22

$$\int_0^{\frac{\pi}{2}} n(\sin x)^{n-1}\cdot(\cos x)^{m+1}\,dx = \int_0^{\frac{\pi}{2}} m(\sin x)^{n+1}\cdot(\cos x)^{m-1}\,dx$$

$$Let\ I = \int_0^{\frac{\pi}{2}} n(\sin x)^{n-1}\cdot(\cos x)^{m+1}dx =$$

$$= \int_0^{\frac{\pi}{2}} n(\sin x)^{n-1}\cdot\cos(\cos x)^m\,dx = \int_0^{\frac{\pi}{2}} ((\sin x)^n)'(\cos x)^m\,dx$$

$$= (\sin x)^n(\cos x)^m\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (\sin x)^n\cdot m(\cos x)^{m-1}\cdot(-\sin x)dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin x)^n \cdot m(\cos x)^{m-1} \cdot \sin x \, dx = \int_0^{\frac{\pi}{2}} m(\sin x)^{n+1} (\cos x)^{m-1} dx$$

4.23

$$I = \int \frac{(\ln \ln \ln \ln x + 100)^{99}}{x(\ln x)(\ln \ln x)(\ln \ln \ln x)} dx = \left\{ \begin{matrix} t = \ln \ln \ln \ln x \\ dt = \frac{1}{\ln \ln \ln x} \cdot \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} dx \end{matrix} \right\}$$

$$= \int (t + 100)^{99} \, dt = \left\{ \begin{matrix} t + 100 = u \\ dt = du \end{matrix} \right\} = \int u^{99} \, du =$$

$$= \frac{u^{100}}{100} = \frac{(t + 100)^{100}}{1000} = \frac{(\ln \ln \ln \ln x + 100)^{100}}{100} \Bigg|_{e^{e^{e^{e^2}}}}^{e^{e^{e^{e^3}}}} = \frac{(103)^{100} - (102)^{100}}{100}$$

4.24

For $0 < \alpha < x \le \frac{\pi}{2}$

$$\frac{\sin 7x}{\sin x} = \frac{\sin 7x - \sin 5x + \sin 5x - \sin 3x + \sin 3x - \sin x + \sin x}{\sin x}$$

$$= \frac{2\cos 6x \sin x + 2\cos 4x \sin x + 2\cos 2x \sin x + \sin x}{\sin x}$$

$$= 2\cos 6x + 2\cos 4x + 2\cos 2x + 1$$

$$\Rightarrow \left(\frac{\sin 7x}{\sin x}\right)^2 = 4\cos^2 6x + 4\cos^2 4x + 4\cos^2 2x + 1 + 8\cos 6x \cos 4x +$$

$$+8\cos 6x \cos 2x + 4\cos 6x + 8\cos 4x \cos 2x + 4\cos 4x + 4\cos 2x$$

$$= 2(1 - \cos 12x) + 2(1 + \cos 8x) + 2(1 + \cos 4x) + 1 + 4(\cos 10x + \cos 2x) +$$

$$+4(\cos 8x + \cos 4x) + 4(\cos 6x + \cos 2x) + 4\cos 6x + 4\cos 4x + 4\cos 2x$$

$$= 7 + 12\cos 2x + 10\cos 4x + 8\cos 6x + 6\cos 8x + 4\cos 10x + 2\cos 12x$$

$$\int_\alpha^{\frac{\pi}{2}} \left(\frac{\sin 7x}{\sin x}\right)^2 dx = \int_\alpha^{\frac{\pi}{2}} \left(\begin{matrix} 7 + 12\cos 2x + 10\cos 4x + 8\cos 6x + \\ +6\cos 8x + 4\cos 10x + 2\cos 12x \end{matrix} \right) dx$$

$$= \left[7x - 6\sin 2x - \frac{5}{2}\sin 4x - \frac{4}{3}\sin 6x - \frac{3}{4}\sin 8x - \frac{2}{5}\sin 10x - \frac{1}{6}\sin 12x\right]_{\alpha}^{\frac{\pi}{2}}$$

$$= 7\left(\frac{\pi}{2}\right) + 6\sin 2\alpha + \frac{5}{2}\sin 4\alpha + \frac{4}{3}\sin 6\alpha + \frac{3}{4}\sin 8\alpha + \frac{2}{5}\sin 10\,\alpha + \frac{1}{6}\sin 12\alpha - 7\alpha$$

4.25

We know $\tan x = \cot x - 2\cot 2x$ *(easy to prove)* $\Rightarrow$

$$\left.\begin{aligned} \tan x &= \cot 2x - 2\cot 2x \\ 2\tan 2x &= 2\cot 2x - 4\cot 4x \\ 4\tan 4x &= 4\cot 4x - 8\cot 8x \end{aligned}\right\} \Rightarrow$$

$$\tan x + 2\tan 2x + 4\tan 4x = \cot 2x - 8\cot 8x \Rightarrow \Omega = \int x\,dx = \frac{x^2}{2} + C$$

4.26

Because for any $x \in \mathbb{R}$ *we have:* $\tan x = \cot x - 2\cot 2x \Rightarrow$

$$\left.\begin{aligned} \tan x &= \cot x - 2\cot 2x \\ 2\tan 2x &= 2\cot 2x - 4\cot 4x \\ 4\tan 4x &= 4\cot 4x - 8\cot 8x \end{aligned}\right\} \Rightarrow \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x \Rightarrow$$

$$\Rightarrow \Omega = \int \tan^2 x \cdot \cot x\,dx = \int \tan x\,dx = -\ln|\cos x| + C$$

4.27

$$\Omega = \int \frac{\tanh^2 x + \tanh^2 x\,(1 + \tanh^2 x)^2}{(1 + \tanh^2 x)^2}\,dx$$

$\Omega = \int \frac{t^2 + t^2(1+t^2)^2}{(1+t^2)^2} \cdot \frac{dt}{1-t^2}$ *where* $t = \tanh x$*. To split into partial fractions, put* $t^2 = y$

$$\frac{y + y(1+y)^2}{(1+y)^2(1-y)} = -1 + \frac{A}{1+y} + \frac{B}{(1+y)^2} + \frac{C}{1-y} \Rightarrow y + y(1+y)^2 =$$

$$= -(1-y)(1+y)^2 + A(1+y)(1-y) + B(1-y) + C(1+y)^2$$

Put $y = 1; 1 + 2^2 = C(1+1)^2 \Rightarrow C = \frac{5}{2}$*. Put* $y = -1; -1 = -2B \Rightarrow B = \frac{1}{2}$*. Put* $y = 0$

$$0 = -1 + A + B + C \Rightarrow A = 1 - B - C = 1 - 3 = -2$$

Thus

$$, \Omega = \int \left[-1 - \frac{2}{1+t^2} + \frac{1}{2}\cdot\frac{1}{(1+t^2)^2} + \frac{5}{2}\cdot\frac{1}{1-t^2}\right] dt = -t - 2\tan^{-1} t + \frac{1}{2} I_1 + \frac{5}{2}\cdot\frac{1}{2} \ln\left|\frac{1+t}{1-t}\right| + C$$

where $I_1 = \int \frac{1}{(1+t^2)^2} dt$. *Let* $I_2 = \int \frac{dt}{1+t^2} = \frac{t}{1+t^2} + \int \frac{t\cdot 2t}{(1+t^2)^2} = \frac{t}{1+t^2} + 2\int \frac{t^2+1-1}{(1+t^2)^2} dt =$

$$= \frac{t}{1+t^2} + 2I_2 - 2I_1 \Rightarrow -2I_1 = -I_2 - \frac{t}{1+t^2} = -\tan^{-1} t - \frac{t}{1+t^2}$$

$$\therefore \Omega = -t - 2\tan^{-1} t - \frac{1}{4}\tan^{-1} t - \frac{1}{4}\cdot\frac{t}{1+t^2} + \frac{5}{4}\ln\left|\frac{1+t}{1-t}\right| + C = -t - \frac{9}{4}\tan^{-1} t -$$

$$-\frac{1}{4}\cdot\frac{t}{1+t^2} + \frac{5}{4}\ln\left|\frac{1+t}{1-t}\right| + C \text{ where } t = \tanh x.$$

4.28

Note that $\frac{\tan^4 x(\tan^2 x - 2)}{(1-\tan^2 x)^2} = \frac{\tan^2 x\left((\tan^2 x - 1)^2 - 1\right)}{(1-\tan^2 x)^2} = \tan^2 x - \frac{\tan^2 x}{(1-\tan^2 x)^2} = \tan^2 x - \frac{1}{4}\tan^2 2x$

$$= (1 + \tan^2 x) - \frac{1}{4}(\tan^2 2x + 1) - \frac{3}{4} = \left(\tan x - \frac{1}{8}\tan 2x - \frac{3x}{4}\right)'$$

$$\text{So, } \Omega = \left[\tan x - \frac{1}{8}\tan 2x - \frac{3x}{4}\right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{8} - \frac{\pi}{8} = \frac{5\sqrt{3}}{24} - \frac{\pi}{8}$$

4.29

$$\Omega = \int \frac{\tan^{2\cdot\frac{n(n+1)}{2}}}{-2\cos^2 x\sqrt{1-\tan^{n^2+n+1} x}} dx = \int \frac{(\tan x)'\tan^{n^2+n} x}{-2\sqrt{1-\tan^{n^2+n+1} x}} dx =$$

$$= \frac{1}{n^2+n+1}\int \frac{\left(1-\tan^{n^2+n+1} x\right)'}{2\sqrt{1-\tan^{n^2+n+1} x}} dx = \frac{1}{n^2+n+1}\sqrt{1-\tan^{n^2+n+1} x} + C$$

4.30

$$\therefore I = \int \frac{2x^2 + x}{\sqrt{x^2+2x+2}} dx + \int \frac{4x+12}{(x^2+2x+2)^{\frac{3}{2}}} = I_1 + I_2$$

Where

$$I_1 = \int \frac{2x^2 + 4x + 4 - 3x - 4}{\sqrt{x^2 + 2x + 2}} dx = 2\int \sqrt{x^2 + 2x + 2}\, dx - \int \frac{3x + 4}{\sqrt{x^2 + 2x + 2}} dx$$

Put $x + 1 = t$

$$I_1 = 2\int \sqrt{t^2 + 1}\, dt - \int \frac{3(t-1) + 4}{\sqrt{t^2 + 1}} dt = 2\left[\frac{1}{2}t\sqrt{t^2 + 1} + \frac{1}{2}ln\left(t + \sqrt{t^2 + 1}\right)\right] -$$

$$-3\int \frac{t}{\sqrt{t^2 + 1}} - \int \frac{dt}{\sqrt{t^2 + 1}}$$

$$= t\sqrt{t^2 + 1} + ln\left(t + \sqrt{t^2 + 1}\right) - 3\sqrt{t^2 + 1} - ln\left(t + \sqrt{t^2 + 1}\right) =$$

$$= (t - 3)\sqrt{t^2 + 1} = (x - 2)\sqrt{x^2 + 2x + 2}$$

$$I_2 = \int \frac{2x + 2}{(x^2 + 2x + 2)^{\frac{3}{2}}} dx + 4\int \frac{dx}{(x^2 + 2x + 2)^{\frac{3}{2}}} = -\frac{2}{\sqrt{x^2 + 2x + 2}} + 4I_3$$

where $I_3 = \int \frac{dx}{[(x+1)^2+1]^{\frac{3}{2}}}$. *Put* $x + 1 = tan\,\theta$. $I_3 = \int \frac{sec^2\,\theta d\theta}{(tan^2\,\theta+1)^{\frac{3}{2}}} = \int cos\,\theta\, d\theta = sin\,\theta =$

$$= \frac{tan\,\theta}{sec\,\theta} = \frac{tan\,\theta}{\sqrt{tan^2\,\theta + 1}} = \frac{x + 1}{\sqrt{x^2 + 2x + 1}}$$

Thus, $I = (x - 2)\sqrt{x^2 + 2x + 2} + \frac{4+8x}{\sqrt{x^2+2x+2}} + C$

4.31

$$\Omega = \int \frac{tanh(x)}{1 + e^{3x}} dx = \int \frac{e^{2x} - 1}{(e^{2x} + 1)(e^{3x} + 1)} dx$$

Put $e^x = t, e^x dx = dt$

$$\Omega = \int \frac{t^2 - 1}{(t^2 + 1)(t^3 + 1)} dt = \int \frac{t - 1}{(t^2 + 1)(t^2 - t + 1)t} dt$$

$$\frac{t - 1}{(t^2 + 1)(t^2 - t + 1)} \equiv \frac{A}{t} + \frac{Bt + C}{t^2 + 1} + \frac{Dt + E}{t^2 - t + 1} \Rightarrow t - 1 = A(t^2 + 1)(t^2 - t + 1) +$$

$$+(Bt + c)t(t^2 - t + 1) + D(Dt + E)t(t^2 + 1)$$

Put $t = 0$; $-1 = A \Rightarrow A = -1$. *Put* $t = i; i - 1 = (Bi + C)i(-i) = Bi + C \Rightarrow$

$\Rightarrow B = 1, C = -1$. *Compare coefficient of* t^n; $D = A + B + D \Rightarrow D = 0$. *Put* $t = -\omega$,

($\omega \neq 1$ *is cube root of unity)*

$$-\omega - 1 = E(-\omega)(\omega^2 + 1) = E\omega^2 \Rightarrow \omega^2 = E\omega^2 \Rightarrow E = 1$$

Thus, $\Omega = \int \left[-\frac{1}{t} + \frac{t-1}{t^2+1} + \frac{1}{t^2-t+1}\right] dt = -\ln t + \frac{1}{2}\ln(t^2+1) - \tan^{-1}(t) +$

$$+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C = -x + \frac{1}{2}\ln(e^{2x}+1) - \tan^{-1}(e^x) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2e^x-1}{\sqrt{3}}\right) + C$$

4.32

$$\Omega = \int_1^{16} \arctan \sqrt[4]{x-1}\, dx = \int_0^{\sqrt[4]{15}} 4t^3 \arctan t\, dt = \int_0^{\sqrt[4]{15}} (t^4)' \arctan t\, dt$$

$$x - 1 = t^4, dx = 4t^3 dt$$

$$\Omega = t^4 \arctan t \Big|_0^{\sqrt[4]{15}} - \int_0^{\sqrt[4]{15}} \frac{t^4}{t^2+1} dt = 15 \arctan \sqrt[4]{15} - 0 - \int_0^{\sqrt[4]{15}} \frac{t^4 - 1 + 1}{t^2+1} dt$$

$$\Omega = 15 \arctan \sqrt[4]{15} - \int_0^{\sqrt[4]{15}} \left(t^2 - 1 + \frac{1}{t^2+1}\right) dt$$

$$\Omega = 15 \arctan \sqrt[4]{15} - \left(\frac{t^3}{3} - t + \arctan t\right)\Bigg|_0^{\sqrt[4]{15}}$$

$$\Omega = 15 \arctan \sqrt[4]{15} - \frac{\sqrt[4]{15^3}}{3} + \sqrt[4]{15} \cdot \arctan \sqrt[4]{15}$$

$$\Omega = 14 \arctan \sqrt[4]{15} - \frac{\sqrt[4]{15^3}}{3} + \sqrt[4]{15}$$

4.33

Put $\ln x = t \Rightarrow x = e^t$

$$\therefore I = \int e^t \cdot \frac{\sqrt{1+t^2}(1+t) + t^2}{t\sqrt{1+t^2} + (t^2+1)} dt = \int e^t \frac{\sqrt{1+t^2} + t\left(\sqrt{1+t^2} + t\right)}{\sqrt{t^2+1}\left(t + \sqrt{t^2+1}\right)} dt$$

$$= \int e^t \left[\frac{1}{t + \sqrt{t^2 + 1}} + \frac{t}{\sqrt{1 + t^2}}\right] dt = \int e^t \left[\sqrt{t^2 + 1} - t + \frac{t}{\sqrt{t^2 + 1}}\right] dt$$

$$= \int e^t \sqrt{t^2 + 1} dt + \int e^t \frac{t}{\sqrt{t^2 + 1}} dt - \int t e^t \, dt$$

$$= e^t \sqrt{t^2 + 1} - \int e^t \cdot \frac{t}{\sqrt{t^2 + 1}} + \int e^t \frac{t}{\sqrt{t^2 + 1}} - \{t e^t - e^t dt\}$$

[Integrating by parts]

$$= e^t \sqrt{t^2 + 1} - t e^t + e^t + C = e^t \left[\sqrt{t^2 + 1} - t + 1\right] + C = x \left[1 - \ln x + \sqrt{1 + (\ln x)^2}\right] + C$$

4.34

$$\int \frac{x^4 e^x dx}{(x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x)^2} =$$

$$= \int \frac{x^4 e^{-x} dx}{(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)^2} =$$

$$= -\int \frac{d(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)}{(e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72)^2}$$

$$\therefore \int \frac{x^4 e^x dx}{(x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x)^2} =$$

$$= \frac{1}{e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + 72} + C =$$

$$= \frac{e^x}{x^4 + 4x^3 + 12x^2 + 24x + 24 + 72e^x} + C$$

4.35

$$\Omega = \int \frac{\tanh(x)}{1 + e^{3x}} dx = \int \frac{e^{2x} - 1}{(e^{2x} + 1)(e^{3x} + 1)} dx$$

Put $e^x = t, e^x dx = dt$

$$\Omega = \int \frac{t^2 - 1}{(t^2 + 1)(t^3 + 1)} dt = \int \frac{t - 1}{(t^2 + 1)(t^2 - t + 1)t} dt$$

$$\frac{t-1}{(t^2+1)(t^2-t+1)} \equiv \frac{A}{t} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{t^2-t+1} \Rightarrow t-1 = A(t^2+1)(t^2-t+1) +$$

$$+(Bt+c)t(t^2-t+1) + D(Dt+E)t(t^2+1)$$

Put $t = 0;\ -1 = A \Rightarrow A = -1$. *Put* $t = i; i-1 = (Bi+C)i(-i) = Bi + C \Rightarrow$

$\Rightarrow B = 1, C = -1$. *Compare coefficient of* $t^n; D = A + B + D \Rightarrow D = 0$. *Put* $t = -\omega$,

($\omega \neq 1$ *is cube root of unity)*

$$-\omega - 1 = E(-\omega)(\omega^2+1) = E\omega^2 \Rightarrow \omega^2 = E\omega^2 \Rightarrow E = 1$$

Thus, $\Omega = \int \left[-\frac{1}{t} + \frac{t-1}{t^2+1} + \frac{1}{t^2-t+1}\right] dt = -\ln t + \frac{1}{2}\ln(t^2+1) - \tan^{-1}(t) +$

$$+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C = -x + \frac{1}{2}\ln(e^{2x}+1) - \tan^{-1}(e^x) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2e^x-1}{\sqrt{3}}\right) + C$$

4.36

Let $u = \sqrt{x + \sqrt{x^2+1}} \Rightarrow u^2 - x = \sqrt{x^2+1} \Rightarrow x = \frac{1-u^4}{2u^2}$

$$\Omega = \int \tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right)dx = \int \tan^{-1} u \cdot \left(\frac{1-u^2}{2u^2}\right) = \frac{1-u^4}{2u^2} \cdot \tan^{-1} u -$$

$$-\int \frac{1}{1+u^2} \cdot \frac{1-u^4}{2u^2} du = \frac{1-u^4}{2u^2} \cdot \tan^{-1} u - \int \frac{1-u^2}{2u^2} du = \frac{1-u^4}{2u^2} \cdot \tan^{-1} u -$$

$$-\frac{1}{2}\left(-\frac{1}{u} - u\right) + C = \frac{1-u^4}{2u^2} \cdot \tan^{-1} u + \frac{1}{2} \cdot \frac{1+u^2}{u} + C =$$

$$= \frac{1-\left(\sqrt{x+\sqrt{x^2+1}}\right)^4}{2\left(\sqrt{x+\sqrt{x^2+1}}\right)^2} \cdot \tan^{-1}\left(\sqrt{x+\sqrt{x^2+1}}\right) + \frac{1}{2} \cdot \frac{1+\left(\sqrt{x+\sqrt{x^2+1}}\right)^2}{\sqrt{x+\sqrt{x^2+1}}} + C$$

4.37

Put $x + 2 = t$, *then:*

$$\Omega = \int \frac{242t^5 - \{(t-1)^5 + (t+1)^5\}}{26t^3 - \{(t-1)^3 + (t+1)^3\}} dx$$

$$= \int \frac{242t^5 - 2(t^5 + 10t^3 + 5t)}{26t^3 - 2(t^3 + 3t)} dx = \int \frac{240t^5 - 20t^3 - 10t}{24t^3 - 6t} dt$$

$$= \int \frac{10t(24t^4 - 2t^2 - 1)}{6t(4t^2 - 1)} dt = \frac{5}{3} \int \frac{24t^4 - 2t^2 - 1}{4t^2 - 1} dt$$

$$= \frac{5}{3} \int \frac{(4t^2 - 1)(6t^2 + 1)}{4t^2 - 1} dt = \frac{5}{3} \int (6t^2 + 1) dt = \frac{5}{3} \times \left(\frac{6}{3} t^3 + t\right) + C$$

$$= \frac{10}{3} t^3 + \frac{5}{3} t + C = \frac{10}{3} (x + 2)^3 + \frac{5}{3} (x + 2) + C = \frac{10}{3} x^3 + 20x^2 + \frac{125}{3} x + C$$

4.38

For $x \geq 0, 0 \leq x^2 n^2 e^{-nx} < \frac{4!}{x^2 n^2}$, *since,* $e^{nx} > \frac{(nx)^4}{4!}$

Similarly, for $x \geq 0, 0 \leq n^2 e^{-nx} \tan^{-1} x < \frac{4! \tan^{-1} x}{x^4 n^2}$

$$0 \leq \lim_{n \to \infty} n^2 \int_0^1 \frac{x^2 + \tan^{-1} x}{e^{nx}} dx < \lim_{n \to \infty} \int_0^1 \left(\frac{4!}{x^2 n^2} + \frac{4! \tan^{-1} x}{x^4 n^2} \right) dx$$

$= \int_0^1 \lim_{n \to \infty} \left(\frac{4!}{x^2 n^2} + \frac{4! \tan^{-1} x}{x^4 n^2} \right) dx = 0$*so, by sandwich theorem* $\lim_{n \to \infty} \int_0^1 \frac{x^2 + \tan^{-1} x}{e^{nx}} dx = 0$

4.39

$$\int_0^{\frac{\pi}{2}} \frac{10 \tan^3 x - 19 \tan^{\frac{8}{3}} x - 36 \tan^2 x + 10 \tan x - 19 \tan^{\frac{2}{3}} x - 36}{2 \tan^{\frac{8}{3}} x + 35 \tan^{\frac{5}{3}} x + 108 \tan^{\frac{2}{3}} x} dx =$$

$$= \int_0^{\frac{\pi}{2}} \frac{10 \tan x - 19 \tan^{\frac{2}{3}} x - 36}{\tan^{\frac{2}{3}} x \, (2 \tan x + 27)(\tan x + 4)} \sec^2 x \, dx =$$

$$= 18 \int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{\tan^{\frac{2}{3}} x \, (2 \tan x + 27)} [8 \tan x \to 27 \tan x] - 4 \int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{\tan^{\frac{2}{3}} x \, (\tan x + 4)} -$$

$$- \int_0^{\frac{\pi}{2}} \left(\frac{1}{\tan x + 4} - \frac{2}{2 \tan x + 27} \right) \sec^2 x \, dx$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{sec^2 x\, dx}{tan^{\frac{2}{3}} x\,(tan\, x + 4)} - 4\int_0^{\frac{\pi}{2}} \frac{sec^2 x\, dx}{tan^{\frac{2}{3}} x\,(tan\, x + 4)} - \left[ln\frac{tan\, x + 4}{2\, tan\, x + 27}\right]_0^{\frac{\pi}{2}} = ln\, 2 + ln\left(\frac{4}{27}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{10\, tan^3 x - 19\, tan^{\frac{8}{3}} x - 36\, tan^2 x + 10\, tan\, x - 19\, tan^{\frac{2}{3}} x - 36}{2\, tan^{\frac{8}{3}} x + 35\, tan^{\frac{5}{3}} x + 108\, tan^{\frac{2}{3}} x} dx = 3\, ln\left(\frac{2}{3}\right)$$

4.40

$$\Omega \overset{\frac{0}{0}\, L.H,}{=} \lim_{x\to 0} \frac{\sqrt[3]{1 + 2x}\cdot\sqrt[5]{1 + 3x} - 1}{x} \cdot \frac{x}{\sqrt[3]{1 + 3x}\sqrt[5]{1 + 2x} - 1} =$$

$$\Omega \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\sqrt[3]{1 + 2x}\cdot\sqrt[5]{1 + 3x} - 1}{\sqrt[3]{1 + 3x}\cdot\sqrt[5]{1 + 2x} - 1}$$

$$\Omega \overset{L.H.}{=} \lim_{x\to 0} \frac{\frac{1}{3}(1 + 2x)^{-\frac{2}{3}}\cdot 2\cdot\sqrt[5]{1 + 3x} + \frac{1}{5}(1 + 3x)^{-\frac{4}{5}}\cdot 3\sqrt[3]{1 + 2x}}{\frac{1}{3}(1 + 3x)^{-\frac{2}{3}}\cdot 3\cdot\sqrt[5]{1 + 2x} + \frac{1}{5}(1 + 2x)^{-\frac{4}{5}}\cdot 2\cdot\sqrt[3]{1 + 3x}}$$

$$\Omega = \frac{\frac{2}{3} + \frac{3}{5}}{1 + \frac{2}{5}} = \frac{19}{15}\cdot\frac{5}{7} = \frac{19}{21}$$

4.41

For $a > 1$, *let* $I(a) = \int_0^a \frac{(2x+3)dx}{x(x+1)(x+2)(x+3)+a} = \int_0^a \frac{(2x+3)dx}{(x^2+3x)(x^2+3x+2)+a}$

Put $x^2 + 3x = y$.

$$I(a) = \int_0^{a^2+3a} \frac{dy}{y(y + 2) + a} = \int_0^{a^2+3a} \frac{dy}{(y + 1)^2 + a - 1}$$

$$= \frac{1}{\sqrt{a - 1}} tan^{-1}\left(\frac{y + 1}{\sqrt{a - 1}}\right)\Bigg]_0^{a^2+3a}$$

$$= \frac{1}{\sqrt{a - 1}}\left[tan^{-1}\left(\frac{a^2 + 3a + 1}{\sqrt{a - 1}}\right) - tan^{-1}\left(\frac{1}{\sqrt{a - 1}}\right)\right]$$

$$\lim_{a\to\infty} I(a) = 0\left[\because for\ b > 0, 0 < tan^{-1}(b) < \frac{\pi}{2}\right]$$

4.42

$$I_n = \int_0^{\frac{\pi}{2}} \frac{sin^n x}{sin^n x + cos^n x} dx,\ \ J_n = \int_0^{\frac{\pi}{2}} \frac{cos^n x}{sin^n x + cos^n x} dx,\ \ I_n + J_n = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

$$I_n = -\int_{\frac{\pi}{2}}^{0} \frac{sin^n\left(\frac{\pi}{2} - y\right)}{sin^n\left(\frac{\pi}{2} - y\right) + cos^n\left(\frac{\pi}{2} - y\right)} dy = \int_0^{\frac{\pi}{2}} \frac{cos^n y}{sin^n y + cos^n y} dy = J_n$$

$$2I_n = \frac{\pi}{2} \to I_n = \frac{\pi}{4}, \qquad \Omega_n = (n+1)\cdot I_n^n = (n+1)\cdot\left(\frac{\pi}{4}\right)^n$$

$$\lim_{n\to\infty} \frac{\Omega_{n+1}}{\Omega_n} = \frac{\pi}{4} < 1 \to \lim_{n\to\infty} \Omega_n = 0$$

4.43

$\int_a^b f(x)dx = lim_{|x_r - x_{r-1}|\to 0} \sum_{r=1}^n f(\xi_r)\,(x_r - x_{r-1})$ *where* $\xi_r \in (x_{r-1}, x_r)$

Let $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, *considering a partition*

$P = \left(\frac{S_1}{S_n}, \frac{S_2}{S_n}, \ldots, 1\right)$ *now* $\left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right| = \frac{2r}{n(n+1)} \to 0$ *as* $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n(n+1)} \sum_{k=1}^{n} k\, sin\left(\frac{k^2 + k}{n^2 + n}\right)$$

$$= \frac{1}{2} \lim_{\left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right|\to 0} \sum_{r=1}^{n} \left(\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right) sin\left(\frac{S_r}{S_n}\right) = \frac{1}{2}\int_0^1 sin\,x\,dx = \frac{1 - cos\,1}{2}$$

4.44

We know $\int_a^b f(x)\,dx = lim_{|x_r - x_{r-1}|\to 0} \sum_{r=1}^n f(\xi_r)\,(x_r - x_{r-1})$ *where* $\xi_r \in (x_{r-1}, x_r)$

Let $S_n = \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$, *considering a partition*

$P = \left(\frac{S_1}{S_n}, \frac{S_2}{S_n}, \dots, 1\right)$ *now* $\left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right| = \frac{6r^2}{n(n+1)(2n+1)} \to 0$ *as* $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n(n+1)(2n+1)} \sum_{k=1}^{n} tan^{-1}\left(\frac{k(k+1)(2k+1)}{n(n+1)(2n+1)}\right)$$

$$= \frac{1}{6} \lim_{\left|\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right| \to 0} \sum_{r=1}^{n} \left(\frac{S_r}{S_n} - \frac{S_{r-1}}{S_n}\right) tan^{-1}\left(\frac{S_r}{S_n}\right)$$

$$= \frac{1}{6}\int_0^1 tan^{-1} x\, dx = \frac{1}{6}[x\, tan^{-1} x]_{x=0}^{x=1} - \frac{1}{6}\int_0^1 \frac{xdx}{1+x^2} = \frac{\pi}{24} - \frac{ln\, 2}{12}$$

4.45

$$\left.\begin{matrix} \Omega(n) = \int_{-1}^{1} x\, ln(1+n^{3x})\, dx \\ x = -t|' \Rightarrow dx = -dt\; x = -1 \Rightarrow t = 1\; x = 1 \Rightarrow t = -1 \end{matrix}\right\} \Rightarrow$$

$$\Omega(n) = \Omega(n) = \int_{1}^{-1} -t\, ln(1+n^{-3t})\,(-dt) = -\int_{1}^{-1} t\, ln\left(1 + \frac{1}{n^{3t}}\right) dt =$$

$$= -\int_{-1}^{1} t\, ln\left(\frac{n^{3t}+1}{n^{3t}}\right) dt = -\int_{-1}^{1} t\, ln(1+n^{3t})\, dt + \int_{-1}^{1} t\, ln\, n^{3t} dt \Rightarrow$$

$$2\Omega(n) = \int_{-1}^{1} 3t^2\, ln\, n\, dt \Rightarrow 2\Omega(n) = t^3\, ln\, n \Big|_{-1}^{1} \Rightarrow \Omega(n) = ln\, n \Rightarrow$$

we must show this: $9\left(1+\sqrt{2}+\dots+\sqrt{n}\right)^2 > 4n^2(n+1) \Leftrightarrow$

$$1+\sqrt{2}+\dots+\sqrt{n} > \frac{2n\sqrt{n+1}}{3}, \forall n \geq 1$$

$P(1): 1 > \frac{2\sqrt{2}}{3} \Leftrightarrow 3 > 2\sqrt{2}$ *true.*

Now: $P(k): 1+\sqrt{2}+\dots+\sqrt{k} > \frac{2k\sqrt{k+1}}{3}$

$$P(k+1): 1+\sqrt{2}+\dots+\sqrt{k+1} > \frac{2(k+1)\sqrt{k+2}}{3}$$

From $P(k) \Rightarrow 1+\sqrt{2}+\dots+\sqrt{k}+\sqrt{k+1} > \frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}$

We must show this: $\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1} > \frac{2(k+1)\sqrt{k+2}}{3} \Leftrightarrow$

$2k+3 > 2\sqrt{(k+1)(k+2)} \Leftrightarrow 4k^2+12k+9 > 4k^2+12k+8 \Leftrightarrow 9 > 8$ *true.*

4.46

$$\omega(n) = \int_{-1}^{1} \frac{(1+2x+x^2)^n(1-2x+x^2)^n}{(1-x^2)(1+2x^2+x^4)^n}dx = \int_{-1}^{1} \frac{(1+x)^{2n}(1-x)^{2n}}{(1-x^2)(1+x^2)^{2n}}dx =$$

$$= \int_{-1}^{1} \frac{(1-x^2)^{2n-1}}{(1+x^2)^{2n}}dx = 2\int_{-1}^{1} \left(\frac{1-x^2}{1+x^2}\right)^{2n-1} \frac{dx}{1+x^2}$$

Put $x = tan\,\theta$, $dx = sec^2\,\theta$

$$\Omega = 2\int_{0}^{\frac{\pi}{4}} cos^{2n-1}(2\theta)\,d\theta$$

Put $2\theta = \varphi$,

$$\omega(n) = \int_{0}^{\frac{\pi}{2}} cos^{2n-1}\varphi\,d\varphi = \int_{0}^{\frac{\pi}{2}} cos\,\varphi\,cos^{2n-2}\varphi\,d\varphi =$$

$$= [sin\,\varphi\,cos^{2n-2}\varphi]_0^{\frac{\pi}{2}} + (2n-2)\int_{0}^{\frac{\pi}{2}} cos^{2n-3}\varphi\,sin^2\varphi\,d\varphi$$

$$= (2n-2)\int_{0}^{\frac{\pi}{2}} cos^{2n-3}(1-cos^2\varphi)\,d\varphi \Rightarrow (2(n-1)+1)\varphi(n) = (2n-2)\omega(n-1)$$

$\omega(n) = \frac{2n-2}{2n-1}\omega(n-1)$. *Now,* $\omega(n+2) = \frac{2n+2}{2n+3}\omega(n+1)$ *and* $\omega(n+1) = \frac{2n}{2n+1}\omega(n)$

$$\therefore \Omega = \lim_{n\to\infty}\frac{\omega(n+2)-\omega(n+1)}{\omega(n+1)-\omega(n)} = \lim_{n\to\infty}\frac{\left(\frac{2n+2}{2n+3}-1\right)\omega(n+1)}{\left(\frac{2n}{2n+1}-1\right)\omega(n)} = \lim_{n\to\infty}\frac{2n+1}{2n+3}\cdot\frac{2n}{2n+1}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{3}{2n}} = 1$$

4.47

$$\Omega = lim_{n\to\infty}\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx} \quad (*)$$

$$\int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\int_0^1 x^n\cdot\frac{2x}{(1+x^2)^{n+1}}dx \quad \textit{(Integrate by parts)}$$

$$= \frac{1}{2}\left(\frac{x^n(1+x^2)^{-n}}{-n}\Big|_0^1 + \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}\right) = \frac{1}{2}\left(-\frac{1}{n(2)^n} + \int_0^1\frac{x^{n-1}dx}{(1+x^2)^n}\right)$$

$$\therefore \int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\left(\frac{-1}{n(2)^n} + \int_0^1\frac{x^{n-1}dx}{(1+x^2)^n}\right) \quad (1)$$

Then

$$\int_0^1 \frac{x^n dx}{(1+x^2)^n} = \frac{1}{2}\left(\frac{-1}{(n-1)(2)^{n-1}} + \int_0^1\frac{x^{n-2}dx}{(1+x^2)^{n-1}}\right) \quad (2)$$

Let $I_n \coloneqq \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n}$, *let* $x = \frac{1}{y}, dx = \frac{-dy}{yz}$

$$\therefore I_n = \int_1^\infty \frac{x^{1-n-2}}{(1+x^2)^n}x^{2n}\cdot dx = \int_1^\infty \frac{x^{n-1}dx}{(1+x^2)^n}$$

$$\therefore 2I_n = \int_1^\infty \frac{x^{n-1}dx}{(1+x^2)^n}, \textit{ let } x^2 = y \Rightarrow x = y^{\frac{1}{2}}, dx = \frac{1}{2}y^{-\frac{1}{2}}dy$$

$$\therefore 2I_n = \frac{1}{2}\int_1^\infty \frac{x^{\frac{n}{2}-1}dx}{(1+x^2)^n} = \frac{1}{2}\beta\left(\frac{n}{2},\frac{n}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n)}$$

$$\therefore I_n = \int_0^1 \frac{x^{n-1}dx}{(1+x^2)^n} = \frac{1}{4}\cdot\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n)} \quad (3)$$

$$\textit{Then } I_{n-1} = \int_0^1 \frac{x^{n-2}dx}{(1+x^2)^{n-1}} = \frac{1}{4}\cdot\frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \quad (4)$$

$$\therefore \int_0^1 \frac{x^{n+1}dx}{(1+x^2)^{n+1}} = \frac{1}{2}\left(\frac{-1}{n(2)^n} + \frac{\Gamma^2\left(\frac{n}{2}\right)}{4\Gamma(n)}\right) \quad (5)$$

$$\text{and} \int_0^1 \frac{x^n dx}{(1+x^2)^n} = \frac{1}{2}\left(\frac{-1}{(n-1)(2)^{n-1}} + \frac{\Gamma^2\left(\frac{n-1}{2}\right)}{4\Gamma(n-1)}\right) \quad (6)$$

$$\text{But } \Omega \coloneqq 2 \cdot lim_{n\to\infty}\left(\frac{\frac{x^{n+1}dx}{(1+x^2)^{n+1}}}{\int_0^1 \frac{x^n dx}{(1+x^2)^n}}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{-\frac{1}{n(2)^n} + \frac{\Gamma^2\left(\frac{n}{2}\right)}{4\Gamma(n)}}{-\frac{1}{(n-1)(2)^{n-1}} + \frac{\Gamma^2\left(\frac{n-1}{2}\right)}{4\Gamma(n-1)}}\right) \therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{4\Gamma\left(\frac{n}{2}\right)}{4\Gamma(n)} \cdot \frac{\Gamma(n-1)}{\Gamma^2\left(\frac{n-1}{2}\right)}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{\Gamma(n-1)}{\Gamma(n)} \cdot \frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n-1}{2}\right)}\right) = \lim_{n\to\infty}\left(\frac{\Gamma(2n-1)\Gamma^2(n)}{\Gamma(2n)\Gamma^2\left(n-\frac{1}{2}\right)}\right)$$

$$\therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \left(\frac{\Gamma(n)}{\Gamma\left(n-\frac{1}{2}\right)}\right)^2\right) \therefore \frac{\Omega}{2} = \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \left(\frac{(n-1)!}{\frac{\sqrt{\pi}(2n)!}{4^n \cdot n!\left(n-\frac{1}{2}\right)}}\right)^2\right)$$

$$\frac{\Omega}{2} = \lim_{n\to\infty} \frac{1}{(2n-1)} \cdot \left(\frac{(n-1)! \cdot 4^n \cdot n! \cdot \left(n-\frac{1}{2}\right)}{\sqrt{\pi} \cdot (2n)!}\right)^2$$

$$= \lim_{n\to\infty}\left(\frac{1}{(2n-1)} \cdot \frac{\left(n-\frac{1}{2}\right)^2 \cdot 4^{2n}(n!)^2\left((n-1)!\right)^2}{\pi\left((2n)!\right)^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\left(n-\frac{1}{2}\right)^2}{\pi n^2} \cdot \frac{(4)^{2n} \cdot (n!)^4}{(2n-1)(2n!)^2}\right) = \frac{1}{\pi} \lim_{n\to\infty}\left(\frac{(4)^{2n}(n!)^4}{(2n)(2n!)^2}\right) =$$

$$= \frac{1}{2\pi} \lim_{n\to\infty}\left(\frac{2^{4n}(n!)^4}{n \cdot \left((2n)!\right)^2}\right) = \left(\frac{1}{2\pi}\right)\left(\lim_{n\to\infty}\left(\frac{2^{2n}(n!)^2}{\sqrt{n} \cdot (2n)!}\right)\right)^2$$

$$\text{Using Stirling's formula } \left(n! \sim \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\right)$$

$$\text{Then } \Omega = \frac{1}{\pi}\left(lim_{n\to\infty}\frac{(2\pi n)(n)^{2n}\cdot e^{-2n}\cdot 2^{2n}}{\sqrt{n}\cdot\sqrt{n}\cdot 2\sqrt{\pi}e^{-2n}2^{2n}(n)^{2n}}\right)^2$$

$$\Omega = \frac{1}{\pi}\lim_{n\to\infty}\left(\sqrt{\pi}\right)^2 = \frac{1}{\pi}\cdot\pi = 1$$

$$\therefore \Omega \coloneqq \lim_{n\to\infty}\left(\frac{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx}{\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx}\right) = 1$$

4.48

$$m = min\, f(x) \le f(x) \le max\, f(x) = M$$

$$\Rightarrow (f(x)-m)(f(x)-M) \le 0 \Rightarrow f^2(x) + Mm \le (M+m)f(x)$$

$$\Rightarrow f(x) + \frac{Mm}{f(x)} \le m + M, \text{ since, } f(x) \ne 0, f \text{ is continuous hence } f \text{ is } R - \text{Integrable}$$

$$\Rightarrow \int_a^b f(x)\,dx + Mm\int_a^b \frac{dx}{f(x)} \le (M+m)\int_a^b dx = (M+m)(b-a)$$

$$\Rightarrow (b-a)(M+m) \overset{AM\ge GM}{\ge} 2\sqrt{Mm\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{dx}{f(x)}\right)}$$

$$\therefore \left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{dx}{f(x)}\right) \le \frac{\left((b-a)(M+m)\right)^2}{4Mm}$$

4.49

$$\int_0^{\frac{\pi}{2}} \frac{ln\left(\frac{1-sin\,x}{1+sin\,x}\right)\sqrt{cos\,x}}{(1+sin\,x)\sqrt{1-sin\,x}} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{cos\,x}\,[ln(1-sin\,x)-ln(1+sin\,x)]}{(1+sin\,x)\sqrt{1-sin\,x}}dx$$

$$\overset{integral\ part}{=} \left\{-\frac{2\sqrt{cos\,x}}{\sqrt{1-sin\,x}}\cdot\frac{cos\left(\frac{x}{2}\right)-sin\left(\frac{x}{2}\right)}{sin\left(\frac{x}{2}\right)+cos\left(\frac{x}{2}\right)}\left[ln\left(\frac{1-sin\,x}{1+sin\,x}\right)-4\right]\right\}\Bigg|_0^{\frac{\pi}{2}} = 0-(4\cdot 2) = -8 \text{ (Proved)}$$

4.50

$$\cos a - \cos b > \sqrt{b^2+1} - \sqrt{a^2+1}$$

$$\Leftrightarrow \cos a + \sqrt{a^2+1} > \cos b + \sqrt{b^2+1}, \forall\, a, b \in \left(0, \frac{\pi}{2}\right) \quad \textit{(1)}$$

Let $f(x) = \cos x + \sqrt{x^2+1}, \forall x \in \left(0, \frac{\pi}{2}\right), f'(x) = -\sin x + \frac{x}{\sqrt{x^2+1}}$

Now, $\forall x \in \left(0, \frac{\pi}{2}\right), x < \tan x \Rightarrow x^2 + 1 < 1 + \tan^2 x$

$$\Rightarrow x^2 + 1 < \sec^2 x \Rightarrow \frac{1}{x^2+1} > \cos^2 x \Rightarrow \frac{1}{x^2+1} > 1 - \sin^2 x$$

$$\Rightarrow \sin^2 x > 1 - \frac{1}{x^2+1} \Rightarrow \sin^2 x > \frac{x^2}{x^2+1} \Rightarrow \frac{x}{\sqrt{x^2+1}} - \sin x < 0$$

$\Rightarrow f'(x) < 0 \Rightarrow f(x) = \cos x + \sqrt{x^2+1}$ *is decreasing on* $\left(0, \frac{\pi}{2}\right)$

$b > a; f(b) < f(a) \Rightarrow \cos a + \sqrt{a^2+1} > \cos b + \sqrt{b^2+1} \Rightarrow$ *(1) is proved.*

4.51

We have for all $t, z > 0$*;*

$$\frac{t\sqrt{z} + z\sqrt{t}}{t + z - \sqrt{zt}} = \sqrt{zt} \cdot \frac{\sqrt{z} + \sqrt{t}}{\underbrace{t + z - \sqrt{zt}}_{\geq \sqrt{zt}}} \leq \sqrt{z} + \sqrt{t} \leq \sqrt{2(z+t)}$$

(because $x + y \leq \sqrt{2}\sqrt{x^2 + y^2}$*)*

So put $z = f(x)$ *and* $t = g(x)$ *to find:*

$$\frac{f(x)\sqrt{g(x)} + g(x)\sqrt{f(x)}}{f(x) - \sqrt{f(x)g(x)} + g(x)} \leq \sqrt{2}\sqrt{\left(f(x) + g(x)\right)} \leq 4 \Rightarrow$$

$$\Rightarrow \int_a^b \frac{f(x)\sqrt{g(x)} + g(x)\sqrt{f(x)}}{f(x) - \sqrt{f(x)g(x)} + g(x)}\, dx \leq 4(b - a)$$

4.52

Let

$$I = \int_0^1 f(x)\,dx = [xf(x)]_{x=0}^{x=1} - \int_0^1 xf'(x)dx = f(1) - \int_0^1 xf'(1-x)\,dx$$

$$= f(1) + \int_0^1 x\,f'(1-x)d(1-x)$$

$$= f(1) + \int_0^1 f'\,(1-x)d(1-x) - \int_0^1 (1-x)f'(1-x)d(1-x)$$

Let $z = 1 - x$ *then*, $dx = -dz$ *and when* $x = 0$, *then* $z = 1$: $x = 1, z = 0$

$$= f(1) + \int_1^0 f'(z)\,dz - \int_1^0 zf'(z)dz = f(1) + f(0) + \int_1^0 \left[\frac{d}{dz}(z)\int f'(z)dz\right]dz$$

$$= f(1) + f(0) - \int_0^1 f(z)\,dz = f(1) + f(0) - I.$$

So, $I = \frac{f(1)+f(0)}{2} \geq \sqrt{f(0)\cdot f(1)}$

4.53

We know,

$$min\left(\int_i^{i+1} f(x)\,dx\right) \leq \int_i^{i+1} f(x)\,dx \leq max\left(\int_i^{i+1} f(x)\,dx\right)$$

$$\alpha \leq min\left(\int_i^{i+1} f(x)\,dx\right) \leq \int_i^{i+1} f(x)\,dx \leq max\left(\int_i^{i+1} f(x)\,dx\right) \leq \beta$$

$$\left(\int_i^{i+1} f(x)\,dx - \alpha\right)\left(\int_i^{i+1} f(x)dx - \beta\right) \leq 0$$

$$\Rightarrow \alpha\beta + \left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha + \beta)\int_i^{i+1} f(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{n-1} \alpha\beta + \sum_{i=1}^{n-1}\left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha + \beta)\sum_{i=1}^{n-1}\int_i^{i+1} f(x)\,dx$$

$$(n-1)\alpha\beta + \sum_{i=1}^{n-1}\left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha + \beta)\Omega$$

4.54

We have for all $x \in [a, b]$; $0 < m \leq f(x) \leq M \Rightarrow \begin{cases} m^k \leq f^k(x) \leq M^k \\ \frac{1}{M^k} \leq \frac{1}{f^k(x)} \leq \frac{1}{m^k} \end{cases}$

$$\Rightarrow \begin{cases} m^k(b-a) \leq \int_a^b f^k(x) \leq M^k(b-a) \\ \frac{1}{M^k}(b-a) \leq \int_a^b \frac{1}{f^k(x)} \leq \frac{1}{m^k}(b-a) \end{cases} \Rightarrow$$

$$\left(\frac{m}{M}\right)^k (b-a)^2 \leq \left(\int_a^b f^k(x)\right)\left(\int_a^b \frac{1}{f^k(x)}\right) \leq \left(\frac{M}{m}\right)^k (b-a)^2$$

$$\Rightarrow \prod_{k=1}^{n}\left(\left(\frac{m}{M}\right)(b-a)^2\right) \leq \prod_{k=1}^{n}\left(\int_a^b f^k(x)\right)\left(\int_a^b \frac{1}{f^k(x)}\right) \leq \prod_{k=1}^{n}\left(\left(\frac{M}{m}\right)^k (b-a)^2\right)$$

$$\Rightarrow \left(\left(\frac{m}{M}\right)^{\frac{n(n+1)}{2}} (b-a)^{2n}\right) \leq \prod_{k=1}^{n}\left(\int_a^b f^k(x)\right)\left(\int_a^b \frac{1}{f^k(x)}\right) \leq \left(\left(\frac{M}{m}\right)^{\frac{n(n+1)}{2}} (b-a)^{2n}\right)$$

since

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

4.55

Let

$$f(t) = (t-a)\int_a^b \frac{x}{\log x}dx + (b-a)\int_t^a \frac{x}{\log x} \quad \textit{for all } t \in [e, e^2]$$

$$f'(t) = \int_a^b \frac{x}{\log x}dx - (b-a)\frac{t}{\log t}, \qquad \int_a^b \frac{x}{\log x}dx = \left[\frac{x^2}{\log x}\right]_a^b + \int_a^b \frac{x(1-\log x)}{(\log x)^2}dx$$

where: $\int_a^b \frac{x(1-\log x)}{(\log x)^x}dx \geq 0$

Let $\varphi(t) = \frac{t^2}{\log t}$ *for all* $t \in [e, e^2]$,

$\varphi'(t) = \frac{t(2\log t - 1)}{(\log t)^2} \geq 0$ *for all* $t \in [e, e^2]$ *so,* $\varphi(t)$ *is continuous on* $[e, e^2], \varphi'(t) \geq 0$ *for all* $t \in [e, e^2]$

so, for $a \leq t \leq b, \varphi(a) \leq \varphi(b), \frac{b^2}{\log b} - \frac{a^2}{\log a} \geq 0$ *where* $a, b \in [e, e^2]$ *so,*

$\int_a^b \frac{x}{\log x}dx \geq (b-a)\frac{t}{\log t}$ *where* $t \in [a, b]$

$f'(t) \geq 0$ *for all* $t \in [a, b] \subset [e, e^2]$

so, $f(t)$ *is increasing and for* $c \in [a, b] \subset [e, e^2], f(c) \geq 0$

$$(c-a)\int_a^b \frac{x}{\log x}dx \geq (b-a)\int_a^c \frac{x}{\log x}dx$$

4.56

Let $0 < a, b < 1$. *For* $a \leq x \leq b$,

$$\frac{1}{\sin^{-1} x} + \frac{1}{\cos^{-1} x} \geq \frac{4}{\sin^{-1} x + \cos^{-1} x} = \frac{8}{\pi}$$

and $\sqrt{\sin^{-1} x \cos^{-1} x} \leq \frac{1}{2}(\sin^{-1} x + \cos^{-1} x) = \frac{\pi}{4} \Rightarrow \frac{1}{\sin^{-1} x \cos^{-1} x} \geq \frac{16}{\pi^2}$

$$\left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)\geq 1+\frac{8}{\pi}+\frac{16}{\pi^2}$$

$$\Rightarrow \frac{1}{b-a}\int_a^b\left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)dx\geq\frac{b-a}{b-a}\left(1+\frac{8}{\pi}+\frac{16}{\pi^2}\right)=\left(1+\frac{4}{\pi}\right)^2$$

4.57

For $a\geq 2$, let

$$\Omega(a)=\int_0^1\frac{1-x^2}{1+ax^2+x^4}dx$$

$\Omega'(a)=\int_0^1\frac{(1-x^2)(-1)x^2}{(1+ax^2+x^4)^2}dx<0\Rightarrow\Omega(a)$ *is strictly decreasing on* $[2,\infty)$. *Also,*

$$\Omega(2)=\int_0^1\frac{1-x^2}{1+2x^2+x^4}dx=\int_0^1\frac{-1-x^2+2}{(1+x^2)^2}dx=\int_0^1\left[\frac{2}{(1+x^2)^2}-\frac{1}{1+x^2}\right]dx$$

$$\frac{\pi}{4}=\int_0^1\frac{dx}{1+x^2}=\frac{x}{1+x^2}\Big|_0^1+\int_0^1\frac{x(2x)}{(1+x^2)^2}dx=\frac{1}{2}+\int_0^1\frac{2(x^2+1)-2}{(1+x^2)^2}dx=$$

$$=\frac{1}{2}+\frac{\pi}{4}-\Omega(2)\Rightarrow\Omega(2)=\frac{1}{2}$$

Thus, $0<\Omega(a)<\frac{1}{2}\ \ \forall a>2$. *Now,*

$$2bc\Omega(a)+2ca\Omega(b)+2ab\Omega(c)<$$

$$<bc+ca+ab\leq\frac{1}{2}(b^2+c^2)+\frac{1}{2}(c^2+a^2)+\frac{1}{2}(a^2+b^2)=a^2+b^2+c^2$$

4.58

Let be $f:[0,1]\to\mathbb{R};f(x)=log(1+e^x)$.

$$f'(x)=\frac{e^x}{1+e^x};f''(x)=\frac{(e^x)'(e^x+1)-e^x(e^x+1)'}{(e^x+1)^2}=\frac{e^x}{(e^x+1)^2}>0$$

$$M=max\,f'(x)=f'(1)=\frac{e}{(e+1)^2}$$

Let be $g, h: [0,1] \to \mathbb{R}$*;*

$$g(x) = log(1 + e^x) - \frac{e}{(e+1)^2} \cdot x$$

$$h(x) = log(1 + e^x) + \frac{e}{(e+1)^2} \cdot x$$

$$g'(x) = f'(x) - \frac{e}{(e+1)^2} \le 0 \Rightarrow g \text{ decreasing}$$

$$h'(x) = f'(x) + \frac{e}{(e+1)^2} \ge 0 \Rightarrow h \text{ increasing}$$

By Cebyshev – integral form:

$$\int_0^1 \left(g(x) \cdot h(x)\right) dx \le \left(\int_0^1 g(x) dx\right)\left(\int_0^1 h(x) dx\right)$$

$$\int_0^1 \left(log(1 + e^x) - \frac{e}{(e+1)^2} \cdot x\right)\left(log(1 + e^x) + \frac{e}{(e+1)^2} \cdot x\right) dx \le$$

$$\le \int_0^1 \left(log(1 + e^x) - \frac{e}{(e+1)^2} \cdot x\right) dx \cdot \int_0^1 \left(log(1 + e^x) + \frac{e}{(e+1)^2} \cdot x\right) dx$$

$$\int_0^1 log^2(1 + e^x)\, dx - \frac{e^2}{(e+1)^4} \cdot \frac{x^3}{3}\Big|_0^1 \le \left(\int_0^1 log(1 + e^x)\, dx\right)^2 - \frac{1}{4} \cdot \frac{e^2}{(e+1)^4}$$

$$\int_0^1 log^2(1 + e^x)\, dx - \frac{e^2}{3(e+1)^4} \le \left(\int_0^1 log(1 + e^x)\, dx\right)^2 - \frac{e^2}{4(e+1)^2}$$

$$\int_0^1 log^2(1 + e^x)\, dx \le \left(\int_0^1 log(1 + e^x)\, dx\right)^2 + \frac{e^2}{3(e+1)^4} - \frac{e^2}{4(e+1)^4}$$

$$\int_0^1 log^2(1 + e^x)\, dx \le \left(\int_0^1 log(1 + e^x)\, dx\right)^2 + \frac{e^2}{12(e+1)^4} < \left(\int_0^1 log(1 + e^x)\, dx\right)^2 + \frac{1}{12}$$

4.59

Let $f(x) = tan^{-1} x$ *for all* $x \in [0,1]$

$$f'(x) = \frac{1}{1+x^2}, f''(x) = -\frac{2x}{(1+x^2)^2} \le 0 \text{ for all } x \in [0,1]$$

Applying HERMITE – HADAMARD Inequality

$$\frac{f(1) + f(0)}{2} \le \frac{1}{1-0}\int_0^1 f(x)\, dx \le f\left(\frac{1+0}{2}\right)$$

$$\Rightarrow \frac{tan^{-1}(1) + tan^{-1}(0)}{2} \le \int_0^1 tan^{-1} x\, dx \le tan^{-1}\left(\frac{1}{2}\right)$$

$$\frac{\pi}{8} \le \int_0^1 tan^{-1} x\, dx \le tan^{-1}\left(\frac{1}{2}\right)$$

4.60

$$\int \frac{\sqrt{3 + cos\, 2x}}{cos\, x} dx = \sqrt{2}\int \frac{1 + cos^2 x}{cos\, x\sqrt{1 + cos^2 x}} dx =$$

$$= \sqrt{2}\int \frac{sec^2 x}{\sqrt{tan^2 x + 2}} dx + \sqrt{2}\int \frac{cos\, x}{\sqrt{2 - sin^2 x}} dx$$

$$\therefore \int \frac{\sqrt{3 + cos\, 2x}}{cos\, x} dx = \sqrt{2}\left(sinh^{-1}\left(\frac{tan\, x}{\sqrt{2}}\right) + sin^{-1}\left(\frac{sin\, x}{\sqrt{2}}\right)\right) + C$$

4.61

Let

$$A = \int_0^1 e^{-x^2}\, dx, B = \int_0^1 e^{x^2}\, dx$$

Consider

$$f(t) = \left(1 + \frac{1}{t}\right)^c A^{c+1} + (1+t)^c B^{c+1}, t > 0$$

$$f'(t) = c\left(1+\frac{1}{t}\right)^{c-1}\left(-\frac{1}{t^2}\right)A^{c+1} + c(1+t)^{c-1}B^{c+1} == c\left(1+\frac{1}{t}\right)^{c-1}\frac{B^{c+1}}{t^2}\left[t^{c+1} - \left(\frac{A}{B}\right)^{c+1}\right]$$

Note that

$f'(t) < 0$ *for* $0 < t < \frac{A}{B}$, $f'(0) > 0$ *for* $t > \frac{A}{B}$

$f(t)$ *is least when* $t = \frac{A}{B}$.

Thus,

$$f(t) \geq f\left(\frac{A}{B}\right)\ \forall\, t > 0 \Rightarrow f\left(\frac{a}{b}\right) \geq f\left(\frac{A}{B}\right)$$

$$\Rightarrow \left(1+\frac{b}{a}\right)^c\left(\int_0^1 e^{-x^2}\,dx\right)^{c+1} + \left(1+\frac{a}{b}\right)^c\left(\int_0^1 e^{x^2}\,dx\right)^{c+1} \geq$$

$$\geq \left(1+\frac{A}{B}\right)^c B^{c+1} + \left(1+\frac{B}{A}\right)A^{c+1} = (A+B)^c B + (A+B)^c A =$$

$$= (A+B)^{c+1} = \left(\int_0^1 \left(e^{-x^2} + e^{x^2}\right)dx\right)^{c+1}$$

4.62

$$g(x) = \frac{e^{f(x)}}{1+e^{f(x)}}$$

$e^{\frac{1}{1-0}\int_0^1 log(g(x))dx} \leq \frac{1}{1-0}\int_0^1 g(x)\,dx$ *– (AM-GM – integral form)*

$$e^{\int_0^1 (f(x) - log(1+e^{f(x)}))dx} \leq \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx$$

$$\frac{e^{\int_0^1 f(x)dx}}{e^{\int_0^1 (log(1+e^{f(x)}))dx}} \leq \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx \quad (1)$$

$$\varphi(x) = log(1+e^x),\ \varphi'(x) = \frac{e^x}{1+e^x},\ \varphi'(x) = \frac{e^x}{(1+e^x)^2} > 0$$

$$log\left(1+e^{\int_0^1 f(x)dx}\right) \overset{Jensen}{\leq} \int_0^1 \left(log\left(1+e^{f(x)}\right)\right) dx$$

$$\frac{1}{1+e^{\int_0^1 f(x)dx}} \leq \frac{e^{\int_0^1 f(x)dx}}{e^{\int_0^1 \left(log\left(1+e^{f(x)}\right)\right)dx}} \quad (2)$$

By (1), (2): $\frac{1}{1+e^{\int_0^1 f(x)dx}} \leq \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}} dx$

4.63

Let $f(x) = \sqrt{e^x}$ for all $x \in [a, b]$ where $a < b$ [WLOG let us assume $a < b$]

$f''(x) = \frac{\sqrt{e^x}}{4} > 0$, hence $f(x)$ is strictly convex. Applying HERMITE – HADAMARD Inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}$$

$$\Rightarrow e^{\frac{a+b}{4}} \leq \frac{1}{b-a}\int_a^b \sqrt{e^x}\,dx \leq \frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\Rightarrow e^{\frac{a+b}{4}} \leq \frac{2}{b-a}\left(\sqrt{e^b}-\sqrt{e^a}\right) \leq \frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\Rightarrow e^{\frac{a+b}{2}} \leq \frac{4}{(b-a)^2}\left(\sqrt{e^b}-\sqrt{e^a}\right)^2 \leq \frac{\left(\sqrt{e^a}+\sqrt{e^b}\right)^2}{4} \leq \frac{e^b+e^a}{2}$$

$$\Rightarrow 2(b-a)^2\sqrt{e^{a+b}} \leq 8\left(\sqrt{e^b}-\sqrt{e^a}\right)^2 \leq (e^b+e^a)(b-a)^2$$

4.64

$$sin\,x\,tan\,x - \left(2\,tan\frac{x}{2}\right)^2 = \frac{sin^2 x}{cos\,x} - \frac{4\,sin^2\frac{x}{2}}{cos^2\frac{x}{2}} = \frac{4\,sin^2\frac{x}{2}}{cos\,x\,cos^2\frac{x}{2}}\left(cos^2\frac{x}{2}-1\right)^2 > 0$$

$$\sqrt{sin\,x\,tan\,x} > 2\,tan\frac{x}{2} > 2\cdot\frac{x}{2} = x, \quad (1)$$

$$f:\left(0,\frac{\pi}{2}\right) \to \mathbb{R}, f(x) = sin\,x + tan\,x - 2x, f'(x) = cos\,x + tan^2 x - 1$$

$$f''(x) = \frac{\sin x\,(2-\cos^3 x)}{\cos^3 x} > 0, \inf f'(x) = \inf f(x) = 0 \to f(x) > 0$$

$$\sin x + \tan x > 2x \to \sin\frac{x}{2} + \tan\frac{x}{2} > x, \quad (2)$$

$$\tan x > x, \quad (3)$$

By multiplying (1), (2), (3) $\to x^3 \tan x \left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x} > x^6$

$$\int_a^b x^3 \tan x \left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x}\, dx > \frac{1}{7}(b^7 - a^7)$$

4.65

$$x = a^y \to \frac{1}{\log a}\int_1^a f(x^4)\, dx = \int_0^1 a^y f(a^{4y})\, dy$$

$$\frac{1}{\log a}\int_1^a f(x^4)\, dx + \frac{1}{\log b}\int_1^b f(x^4)\, dx = \int_0^1 \left(a^x f(a^{4x}) + b^x f(b^{4x})\right) dx \overset{JENSEN}{\geq}$$

$$\geq \int_0^1 (a^x + b^x) f\left(\frac{a^{5x} + b^{5x}}{a^x + b^x}\right) dx \overset{AM-GM}{\geq}$$

$$\geq 2\int_0^1 \sqrt{a^x b^x}\, f(a^{4x} - a^{3x}b^x + a^{2x}b^{2x} - a^x b^{3x} + b^{4x}) dx$$

4.66

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x} dx \overset{AM-GM}{\leq} \frac{1}{5}\int_a^b \frac{1}{x^{\frac{4}{5}}}\left(\frac{\cos x}{\sin x}\right)^{\frac{1}{5}} dx$$

and by Holder inequality

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x} dx \leq \frac{1}{5}\left(\int_a^b \frac{dx}{x}\right)^{\frac{4}{5}}\left(\int_a^b \frac{\cos x}{\sin x} dx\right)^{\frac{1}{5}} = \frac{1}{5}\left(\ln\frac{b}{a}\right)^{\frac{4}{5}}\left(\ln\frac{\sin b}{\sin a}\right)^{\frac{1}{5}}$$

$$< \left(ln\frac{b}{a}\right)^{\frac{4}{5}}\left(ln\frac{sin\,b}{sin\,a}\right)^{\frac{1}{5}}$$

4.67

$$(g\circ f)(x)\in[0,c]\Rightarrow(g\circ f)(x)\le c;(\forall)x\in[0,a]$$

$$\frac{1}{c}\int_0^a(g\circ f)^2\,(x)dx\le\frac{1}{c}\int_0^a c\cdot(g\circ f)(x)\,dx=\int_0^a(g\circ f)(x)\,dx$$

$$(f^{-1}\circ g^{-1})(x)\in[0,a]\Rightarrow(f^{-1}\circ g^{-1})(x)\le a(\forall)x\in[0,c]$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2(x)\,dx\le\frac{1}{a}\int_0^c a(f^{-1}\circ g^{-1})\,(x)dx=\int_0^c(f^{-1}\circ g^{-1})\,(x)dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2\,(x)dx\le\int_0^a(g\circ f)\,(x)dx$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2\,(x)dx\le\int_0^c(f^{-1}\circ g^{-1})(x)\,dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2\,(x)dx+\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2\,(x)dx\le\int_0^a(g\circ f)(x)\,dx+\int_0^c(f^{-1}\circ g^{-1})\,(x)dx=ac$$

4.68

Let $f_n(x)=\left|\left[nx+\frac{1}{2}\right]-nx\right|, x\in\mathbb{R}$*. Note:*

$$f_n\left(x+\frac{1}{n}\right)=\left|\left[n\left(n+\frac{1}{n}\right)+\frac{1}{2}\right]-n\left(x+\frac{1}{n}\right)\right|=\left|\left[nx+\frac{1}{2}+1\right]-nx-1\right|$$

$=\left|\left[nx+\frac{1}{2}\right]+1-nx-1\right|=f_n(x)\therefore f_n(x)$ *is periodic with period* $\frac{1}{n}$*.*

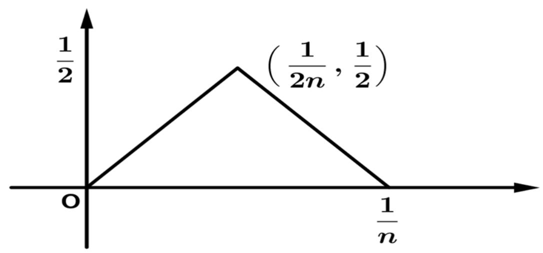
$$\Rightarrow\int_0^{2018}f_n(x)\,dx=2018n\int_0^{\frac{1}{n}}f_n(x)\,dx$$

For $0\le x<\frac{1}{2n}, 0\le nx<\frac{1}{2}\Rightarrow\frac{1}{2}\le nx+\frac{1}{2}<1\Rightarrow\left[nx+\frac{1}{2}\right]=0$ *and for* $\frac{1}{2n}\le x<\frac{1}{n}$

$$\frac{1}{2} \le nx < 1 \Rightarrow 1 \le nx + \frac{1}{2} < \frac{3}{2} \Rightarrow \left[nx + \frac{1}{2}\right] = 1$$

Thus, $f_n(x) = |0 - nx| = nx$ *for* $0 \le x < \frac{1}{2n} = |1 - nx| = 1 - nx$ *for* $\frac{1}{2n} \le n < \frac{1}{n}$

$$\therefore \int_0^{\frac{1}{n}} f_n(x)\, dx = \frac{1}{2}\left(\frac{1}{n}\right)\left(\frac{1}{2}\right) = \frac{1}{4n}$$



Thus,

$$\int_0^{2018} f_n(x) dx = (2018n)\left(\frac{1}{4n}\right) = \frac{1009}{2} \Rightarrow \lim_{n\to\infty} \int_0^{2018} f_n(x)\, dx = \frac{1009}{2}$$

4.69

$$Put\ x = \tan\theta, 0 < \theta < \frac{\pi}{2}$$

$$\frac{\sqrt{1+x^2}-1}{x} = \frac{\sec\theta - 1}{\tan\theta} = \frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \Rightarrow \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{\theta}{2}$$

Let $I = \int_\varepsilon^{\frac{\pi}{2}} \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) dx = \int_a^b \frac{\theta}{2} \cdot \sec^2\theta\, d\theta$ *where*

$$b = \tan^{-1}\left(\frac{\pi}{2}\right)$$

$$a = \tan^{-1}(\varepsilon)$$

$$I = \frac{1}{2}\theta\tan\theta\Big]_a^b - \int_a^b \tan\theta\, d\theta = \frac{1}{2}b\tan b - \frac{1}{2}a\tan a - \log(\sec\theta)\Big]_a^b =$$

$$= \frac{1}{2}b\tan b - \frac{1}{2}a\tan a - \frac{1}{2}\log(1+\tan^2 b) + \frac{1}{2}\log(1+\tan^2 a)$$

$$As\ \varepsilon \to 0_+, a \to 0$$

$$\therefore \lim_{\varepsilon \to 0_+} I = \frac{\pi}{4} \tan^{-1}\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2} \log\left(1 + \frac{\pi^2}{4}\right)$$

4.70

f is increasing convex function on $\left[a, \frac{1}{a}\right]$ then $f\left(\frac{1}{x}\right)$ is convex too,

then by H-H inequality we have

$$2f\left(\frac{1+a^2}{2a}\right) \leq \frac{a}{1-a^2}\left(\int_a^{\frac{1}{a}} f(x)\, dx + \int_a^{\frac{1}{a}} f\left(\frac{1}{x}\right) dx\right) \leq \left(f(a) + f\left(\frac{1}{a}\right)\right)$$

in the second integral we use the changment $y = \frac{1}{x}$ we get the inequality

4.71

Let $f: [0, c] \to \mathbb{R}^+$ defined by

$$f(x) = \tan x\,(4 \tan y \tan z - 2 \tan y - 2 \tan z + 1) + \tan y + \tan z - 2 \tan y \tan z$$

for all $x \in [0, c]$. Now,

$$f'(x) = \sec^2 x\,(4 \tan y \tan z - 2 \tan y - 2 \tan z + 1) \geq 0 \text{ since}$$

$$x \in (0, c) \subseteq \left(0, \frac{\pi}{4}\right)$$

and $y, z \in \left(0, \frac{\pi}{4}\right)$. So, f is continuous on $[0, c]$ and $f'(x) \geq 0$ hence

f is increasing on $[0, c]$. So, $f\left(\frac{\pi}{4}\right) \geq f(c) \geq f(x) \geq f(0)$

$$\Rightarrow 4 \tan y \tan z - 2 \tan y - 2 \tan z + 1 \geq f(x) \geq \tan y + \tan z - 2 \tan y \tan z$$

$$\Rightarrow (2 \tan y - 1)(2 \tan z - 1) \geq f(x) \geq \frac{1}{2} - \frac{1}{2}(2 \tan y - 1)(2 \tan z - 1)$$

$$\Rightarrow 1 \geq f(x) \geq 0 \text{ for all } y, z \in \left(0, \frac{\pi}{4}\right)$$

$$\therefore 0 \leq \int_0^a \left(\int_0^b \left(\int_0^c \left(\sum (\tan x + 2\tan y \tan z) + 4 \prod \tan x \right) dx \right) dy \right) dz \leq abc$$

4.72

For $x > 0$

$$\frac{x^4+1}{x^6+1} \leq \frac{1}{x} \Leftrightarrow x^5 + x \leq x^6 + 1$$

$$\Leftrightarrow x^6 - x^5 + 1 - x \geq 0 \Leftrightarrow x^5(x-1) - (x-1) \geq 0$$

$$\Leftrightarrow (x-1)(x^5-1) \geq 0 \Leftrightarrow (x-1)^2(x^4+x^3+x^2+x+1) \geq 0$$

$$\therefore \int_b^c \frac{x^4+1}{x^6+1} dx \leq \int_b^c \frac{1}{x} dx = \ln\left(\frac{c}{b}\right) \Rightarrow e^{a\int_b^c \frac{x^4+1}{x^6+1} dx} \leq e^{a \ln\left(\frac{c}{b}\right)} = \left(\frac{c}{b}\right)^a$$

Similarly for other expressions.Thus

$$e^{\sum a \int_b^c \frac{x^4+1}{x^6+1} dx} \leq \left(\frac{c}{b}\right)^a \left(\frac{a}{c}\right)^b \left(\frac{b}{a}\right)^c$$

4.73

Let $f(x) = \arctan(x) \arctan\left(\frac{1}{x}\right)$ *for all* $x \in (0, \infty)$ *we have*

$$f'(x) = \frac{1}{x^2+1}\left(\arctan\left(\frac{1}{x}\right) - \arctan(x)\right)$$

then f *increasing on* $(0,1)$ *and decreasing on* $(0, +\infty)$ *it follow that for all*

$$x \in (0, +\infty), f(x) \leq f(1) = \frac{\pi^2}{16}.$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \arctan(x) \arctan\left(\frac{1}{x}\right) dx \leq \frac{\pi}{6} f(1) = \frac{\pi^3}{96}$$

4.74

Let $M = max\{f'(x), x \in [a,b]\}$ *and* $g(x) = f(x) - Mx$*, clearly* g *is decreasing function then by Chebyshev inequality we have*

$$\int_a^b xf(x)\,dx \le \frac{1}{b-a}\left(\int_a^b g(x)\,dx\right)\left(\int_a^b x\,dx\right)$$

then

$$\int_a^b xf(x)\,dx \le \frac{1}{b-a}\left(\int_a^b f(x) - Mxdx\right)\left(\int_a^b x\,dx\right) + M\int_a^b x^2\,dx$$

$$= M\left(\int_a^b x^2\,dx - \frac{1}{b-a}\left(\int_a^b x\,dx\right)^2\right) = M\frac{(b-a)^3}{12}.$$

4.75

$$\frac{tan^{-1}x - tan^{-1}0}{x-0} = \frac{1}{1+x^2} \le 1$$

$$\frac{tan^{-1}x}{x} \le 1$$

$$\int_0^b \frac{arctan\,x}{x}dx < \int_0^b dx = b$$

$$a^2\int_0^b \frac{arctan\,x}{x}dx + b^2\int_0^c \frac{arctan\,x}{x}dx + c^2\int_0^a \frac{arctan\,x}{x}dx < a^2b + b^2c + c^2a$$

$$\begin{cases} a^3 + a^3 + b^3 \ge 3a^2b \\ b^3 + b^3 + c^3 \ge 3b^2c \\ c^3 + c^3 + a^3 \ge 3c^2a \end{cases} \Rightarrow a^3 + b^3 + c^3 \ge a^2b + b^3c + c^2a$$

$$a^2\int_0^b \frac{arctan\,x}{x}dx + b^2\int_0^c \frac{arctan\,x}{x}dx + c^2\int_0^a \frac{arctan\,x}{x}dx < a^3 + b^3 + c^3$$

4.76

Let $f(x) = sin\,x + tan\,x - 2x, 0 \le x \le 1$

$$f'(x) = cos\,x + sec^2 x - 2 \ge 2\sqrt{cos\,x\,sec^2 x} - 2 = 2\left(\sqrt{sec\,x} - 1\right) > 0$$

for $0 < x < 1$

Thus, $f(x) > f(0) = 0$, *for* $0 < x < 1$

$$\Rightarrow sin(cos\,x) + tan(cos\,x) > 2\,cos\,x\,, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{2}\int_a^b [sin(cos\,x) + tan(cos\,x)]\,dx > \int_a^b cos\,x\,dx = sin\,b - sin\,a$$

$$\left[0 < a < b < \frac{\pi}{2}\right]$$

4.77

Let

$$a = \int_3^4 e^{-x^2}\,dx, b = \int_2^3 e^{-x^2}\,dx\,, \qquad a + b = \int_2^4 e^{-x^2}\,dx$$

(a) $2a^2 + 2b^2 = (a + b)^2 + (a - b)^2 \geq (a + b)^2$

(b) $(a + b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2 \quad [\because ab > 0]$

4.78

$$\Omega(a) = \int_0^a \frac{\sqrt{x(x^2 + x + 1)}}{\sqrt{x + 1}\sqrt{x^4 + x^2 + 1}}$$

$$x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1)$$

$$\therefore \Omega(a) = \int_0^a \frac{\sqrt{x}}{\sqrt{(x + 1)(x^2 - x + 1)}}\,dx = \int_0^a \frac{\sqrt{x}}{\sqrt{\left(x^{\frac{3}{2}}\right)^2 + 1}}\,dx$$

Put $x^{\frac{3}{2}} = t, \frac{3}{2}\sqrt{x}dx = dx$

$$\therefore \Omega(a) = \frac{2}{3}\int_0^{a^{\frac{3}{2}}} \frac{dt}{\sqrt{t^2 + 1}} = \frac{2}{3}ln\left(t + \sqrt{t^2 + 1}\right)\Bigg|_0^{a^{\frac{3}{2}}} = \frac{2}{3}ln\left(\sqrt{a^3 + 1} + \sqrt{a^3}\right)$$

Now,

$$\Omega(a) + \Omega(b) + \Omega(c) \geq 3\left[\left(\Omega(a)\right)\left(\Omega(b)\right)\left(\Omega(c)\right)\right]^{\frac{1}{3}}$$

$$\Rightarrow [\Omega(a) + \Omega(b) + \Omega(c)]^3 \geq 27\Omega(a)\Omega(b)\Omega(c)$$

$$= 8\prod\left[log\sqrt{a^3 + 1} + \sqrt{a^3}\right] \geq \prod\left[log\sqrt{(a^3 + 1)} + \sqrt{a^3}\right]$$

4.79

$$Let\ f(x) = sin\,x - \left(x - \frac{x^3}{6}\right), 0 \leq x \leq 1$$

$$f'(x) = cos\,x - \left(1 - \frac{1}{2}x^2\right)$$

$$f''(x) = -sin\,x + x > 0\ for\ 0 < x < 1$$

$$\Rightarrow f'(x) \uparrow on\ [0,1] \Rightarrow f'(x) > f'(0) = 0\ for\ 0 < x \leq 1$$

$$\therefore f(x) \uparrow on\ [0,1] \Rightarrow f(x) > 0\ for\ 0 < x \leq 1$$

$$\therefore sin\,x > x - \frac{1}{6}x^3, 0 < x \leq 1$$

$$\Rightarrow \int_0^1 \sqrt{x}\,sin\,x\,dx > \int_0^1 \left(x^{\frac{3}{2}} - \frac{1}{6}x^{\frac{7}{2}}\right)dx = \frac{2}{5} - \frac{2}{6 \times 9} = \frac{49}{135}$$

4.80

$$Let\ f(x) = \frac{1}{cos\,a}\ for\ all\ x \in \left(0, \frac{\pi}{2}\right)\ and$$

$$g(x) = 1\ for\ all\ x \in \left(0, \frac{\pi}{2}\right)$$

1. $f(x)$ *and* $g(x)$ *are integrable on* $\left(0, \frac{\pi}{2}\right)$

2. $g(x)$ *keeps the same sign on* $\left(0, \frac{\pi}{2}\right)$

$$\therefore \int_a^b tan\,x\,dx \leq \int_a^b \frac{dx}{cos\,x} = \frac{1}{cos\,x}(b - a)$$

where $\alpha \in (a, b)$

Similarly,

$$\int_a^b \cot x\, dx \le \int_a^b \frac{dx}{\sin x} = \frac{b-a}{\sin\beta}$$

where $\beta \in (a, b)$

$$\therefore \left(\int_a^b \tan x\, dx\right)\left(\int_a^b \cot x\, dx\right) \le \frac{(b-a)^2}{\sin\beta\cos\alpha}$$

4.81

Let $F(x) = \int_0^x f(t)\, dt$

As f *is continuous on* $\mathbb{R}$, F *is differentiable on* $\mathbb{R}$*, and* $F'(x) = f(x)$

Let $G(t) = \int_{p\sin t}^{q\cos t} f(x)\, dx, t \in \left(0, \frac{\pi}{2}\right) = F(q\cos t) - F(p\sin t)$

we have G *is differentiable on* $\left(0, \frac{\pi}{2}\right)$ *and* $G(t) \le G\left(\frac{\pi}{4}\right)\ \forall t \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow G$ *attains maximum value at* $t = \frac{\pi}{4}$

$$\therefore G'\left(\frac{\pi}{4}\right) = 0 \Rightarrow -q\sin\left(\frac{\pi}{4}\right)F'\left(\frac{q}{\sqrt{2}}\right) - b\cos\frac{\pi}{4}F'\left(\frac{p}{\sqrt{2}}\right) = 0$$

$$\Rightarrow qf\left(\frac{q}{\sqrt{2}} + bf\left(\frac{p}{\sqrt{2}}\right)\right) = 0.$$

4.82

Let $\cos^{-1} x = \theta, \theta \in [0, \pi]$

$$\int_{-1}^{1} \sqrt{1-x^2}\cos^{-1} x\, dx \Rightarrow x = \cos\theta \quad (\because in\ [0, \pi], \sin\theta \ge 0)$$

$$\Rightarrow \sqrt{1-x^2} = \sqrt{\sin^2\theta} = \sin\theta$$

$$dx = -\sin\theta d\theta$$

$$= \int_{\pi}^{0} \theta \sin\theta\,(-\sin\theta)\,d\theta$$

$$= -\tfrac{1}{2}\int_0^{\pi} \theta(-2\sin\theta)d\theta = -\tfrac{1}{2}\int_0^{\pi} \theta(\cos 2\theta - 1)d\theta = -\tfrac{1}{2}\left(\int_0^{\pi} \theta\cos\theta\,d\theta - \int_0^{\pi} \theta d\theta\right) \to \text{ (1)}$$

$$\int \theta\cos 2\theta\,d\theta = \theta\int \cos 2\theta\,d\theta - \frac{1}{2}\int \sin 2\theta\,d\theta = \frac{\theta}{2}(\sin 2\theta) + \frac{1}{4}\cos 2\theta + c$$

$$\therefore \int_0^{\pi} \theta\cos 2\theta\,d\theta = \left[\frac{\theta\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right]_0^{\pi}$$

$$= \left(\frac{\pi}{2}\sin 2\pi + \frac{\cos 2\pi}{4}\right) - \left(\frac{0\cdot\sin 0}{2} + \frac{\cos 0}{4}\right) = 1 - 1 = 0$$

$$\therefore \text{(1)} \Rightarrow \int_{-1}^{1}\sqrt{1-x^2}\cos^{-1}x\,dx = -\tfrac{1}{2}\left(0 - \tfrac{1}{2}[\theta^2]_0^{\pi}\right)$$

$$= -\frac{1}{2}\left(-\frac{\pi^2}{2}\right) = \frac{\pi^2}{4} \because \pi > e \therefore \frac{\pi^2}{4} > \frac{e^2}{4}$$

4.83

For $a > 0, k > 0$, *let*

$$I = \int_{-k}^{k}\frac{e^{x^2}+e^{-x^2}}{a^x+1} \text{ (1)}$$

Put $x = -t$, *so that*

$$I = \int_{k}^{-k}\frac{e^{t^2}+e^{-t^2}}{a^{-t}+1}(-1)dt = \int_{-k}^{k}\frac{a^x\left(e^{x^2}+e^{-x^2}\right)}{a^x+1}dx \quad \text{(2)}$$

Adding (1) and (2), we get

$$2I = \int_{-k}^{k}\left(e^{x^2} + e^{-x^2}\right)dx \Rightarrow I = \int_0^k\left(e^{x^2} + e^{-x^2}\right)dx \geq 2\int_0^k = 2k$$

$$\therefore \int_{-a}^{a}\frac{e^{x^2} + e^{-x^2}}{1+2^x}dx + \int_{-b}^{b}\frac{e^{x^2} + e^{-x^2}}{1+3^x}dx + \int_{-c}^{c}\frac{e^{x^2} + e^{-x^2}}{1+5^x}dx \geq 2(a+b+c) \geq 6(abc)^{\frac{1}{3}}$$

4.84

$$f(a) = f'(a) = 0$$

$$\left(\int_0^a f(x)dx\right)^2 = \left([xf(x)]_{x=0}^{x=a} - \int_0^a xf'(x)dx\right)^2 \le \left(\int_0^a x^2 dx\right)\left(\int_0^a \left(f'(x)\right)^2 dx\right)$$

$$= \frac{a^3}{3}\left(\int_0^a \left(f'(x)\right)^2 dx\right)$$

again,

$$\left(\int_0^a f(x)dx\right)^2 = \left(\int_0^a xf'(x)dx\right)^2 = \left(\left[\frac{x^2}{2}f'(x)\right]_{x=0}^{x=a} - \frac{1}{2}\int_0^a x^2 f''(x)dx\right)^2$$

$$= \left(\frac{1}{2}\int_0^a x^2 f''(x)dx\right)^2 \le \frac{1}{4}\left(\int_0^a x^4 dx\right)\left(\int_0^a \left(f''(x)\right)^2 dx\right) = \frac{a^5}{20}\left(\int_0^a \left(f''(x)\right)^2 dx\right)$$

$$\therefore \left(\int_0^a f(x)dx\right)^4 \le \frac{a^8}{60}\left(\int_0^a \left(f'(x)\right)^2 dx\right)\left(\int_0^a \left(f''(x)\right)^2 dx\right)$$

4.85

$$\sum_{u=1}^{n} \int_{-1}^{1} \frac{e^{x^2}}{a_u^x + 1} dx < ne$$

We have

$$\sum_{u=1}^{n} \int_{-1}^{1} \frac{e^{x^2}}{a_u^x + 1} dx = \sum_{u=1}^{n} \int_{0}^{1} e^{x^2} dx < \sum_{u=1}^{n} e \int_{0}^{1} dx = ne$$

We have

$$\sum_{u=1}^{n} \int_{-1}^{1} \frac{e^{x^2}}{a_u^x + 1} dx < ne$$

4.87

We have known that for any $x \geq 0$, $tan(x) \leq x$ that is:

$$\sum \Omega(a) \leq \sum \int_a^{2a} \frac{arctan(x+1)}{x} dx \leq \sum \int_a^{2a} \frac{x+1}{x} dx =$$

$$= \sum a + ln(2) < \pi(1 + log(2))$$

4.88

$$\int_0^1 \sqrt[3]{f(x)}\,dx \overset{HOLDER'S\ INEQUALITY}{\leq} \sqrt[3]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\int_0^1 \sqrt[5]{f(x)}\,dx \overset{HOLDER'S\ INEQUALITY}{\leq} \sqrt[5]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\int_0^1 \sqrt[7]{f(x)}\,dx \overset{HOLDER}{\leq} \sqrt[7]{\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 dx\right)\left(\int_0^1 f(x)dx\right)} = 1$$

$$\left(\int_0^1 \sqrt[3]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[5]{f(x)}\,dx\right)\left(\int_0^1 \sqrt[7]{f(x)}\,dx\right) \leq 1$$

4.89

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\left(\int_0^1 f^3(x)dx\right) =$$

$$= \int_0^1 \left(f^2(x)\sqrt{f(x)}\right)^2 dx \cdot \int \left(f^3(x)\sqrt{f(x)}\right)^2 dx \cdot \int_0^1 \left(f^4(x)\sqrt{f(x)}\right)^2 dx$$

$$\cdot \left(\int_0^1 f(x)\sqrt{f(x)}\right)^2 dx \overset{CBS}{\geq}$$

$$\geq \left(\int_0^1 f^6(x)dx\right)^2 \cdot \left(\int_0^1 f^6(x)dx\right)^2 dx =$$

$$= \left(\left(\int_0^1 f^6(x)dx\right)\left(\int_0^1 1^2 dx\right)\right)^4 \overset{CBS}{\geq} \left(\int_0^1 f^3(x)dx\right)^8 = \sqrt[7]{2^8}$$

$$\sqrt[7]{2}\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right) \geq \sqrt[7]{2^8}$$

$$\left(\int_0^1 f^5(x)\,dx\right)\left(\int_0^1 f^7(x)\,dx\right)\left(\int_0^1 f^9(x)\,dx\right) \geq 2$$

4.90

$$e^{x^2} \geq x^2 + 1 \to e^{-x^2} \leq \frac{1}{x^2 + 1} \to$$

$$\int_0^{\frac{h_a^2}{h_ah_b+h_bh_c+h_ah_c}} e^{-t^2}\,dt \leq tan^{-1}\left(\frac{h_a^2}{h_ah_b + h_bh_c + h_ch_a}\right)$$

$$\sum_{cyc} \int_0^{\frac{h_a^2}{h_ah_b+h_bh_c+h_ah_c}} e^{-t^2}\,dt \leq \sum tan^{-1}\left(\frac{h_a^2}{h_ah_b + h_bh_c + h_ch_a}\right) \overset{JENSEN}{\overbrace{\leq}}$$

$$\leq 3\,tan^{-1}\left(\frac{1}{3}\sum \frac{h_a^2}{h_ah_b + h_bh_c + h_ch_a}\right) \overset{LEMMA}{\overbrace{\leq}} 3\,tan^{-1}\frac{R}{6r}$$

LEMMA:

$$\frac{h_a^2 + h_b^2 + h_c^2}{h_ah_b + h_bh_c + h_ah_c} \leq \frac{R}{2r}$$

By Adil Abdullayev

We have, $h_a = \frac{2\Delta}{a}, h_b = \frac{2\Delta}{b}, h_c = \frac{2\Delta}{c}, a + b + c = 2p$ *and*

$$ab + bc + ca = p^2 + r^2 + 4Rr$$

$$\frac{h_a^2 + h_b^2 + h_c^2}{h_ah_b + h_bh_c + h_ah_c} \leq \frac{R}{2r} \Leftrightarrow \frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}} \leq \frac{R}{2r}$$

$$\Leftrightarrow \frac{a^2b^2+b^2c^2+a^2c^2}{abc(a+b+c)} \le \frac{R}{2r} \Leftrightarrow \frac{(p^2+r^2+4Rr)^2}{abc(a+b+c)} \le \frac{R+4r}{2r}$$

$$\Leftrightarrow \frac{p^4+r^4+16r^2r^2+2p^2r^2+8Rr^3+8Rrp^2}{8Rrp^2} \le \frac{R+4r}{2r}$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3+8Rrp^2 \le 4R^2p^2+16Rrp^2$$

$$\Leftrightarrow p^4+r^4+16R^2r^2+2p^2r^2+8Rr^3 \le 4R^2p^2+8Rrp^2$$

We know, $p^2 \le 4R^2+4Rr+3r^2$*, then we need to prove,*

$$p^2(4R^2+4Rr+3r^2)+(r^2+4Rr)^2+2p^2r^2 \le 4R^2p^2+8Rrp^2$$

$$\Leftrightarrow p^2(5r^2-4Rr)+(r^2+4Rr)^2 \le 0 \Leftrightarrow p^2 \ge \frac{(r^2+4Rr)^2}{4Rr-5r^2}$$

Again, we know, $p^2 \ge 16Rr-5r^2$*, we will show,*

$$16Rr-5r^2 \ge \frac{(r^2+4Rr)^2}{4Rr-5r^2}$$

$\Leftrightarrow 4R^2-9Rr+2r^2 \ge (R-2r)(4R-r) \ge 0$*, which is true.*

4.91 *If* x *in positive real number we have:*

$$\Omega^2(x)=\frac{x(x+3)}{(x+2)(x+1)}$$

Now we must prove

$$\sum\left[\frac{x(x+3)}{(x+2)(x+1)}-\frac{2}{x+2}\right] < 3 \Leftrightarrow \sum\frac{x-1}{x+1} < 3 \Leftrightarrow$$

$$\Leftrightarrow \sum\left(1-\frac{2}{x+1}\right) < 3 \Leftrightarrow 3-2\sum\frac{1}{x+1} < 3$$

4.92

$$\int_0^{\frac{\pi}{2}} sin^p\, x\, cos^q\, x\, dx = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, -1 < p,q < 1$$

$$\Gamma(p)\Gamma(1-p) = \pi \csc \pi p$$

$$\left(\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{m}} x \cos^{-\frac{1}{m}} x\, dx\right)\left(\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{n}} x \cos^{-\frac{1}{n}} x\, dx\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{m+1}{2m}\right)\Gamma\left(\frac{m-1}{2m}\right)\cdot\frac{1}{2}\Gamma\left(\frac{n+1}{2n}\right)\Gamma\left(\frac{n-1}{2n}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{m+1}{2m}\right)\Gamma\left\{1-\frac{m+1}{2m}\right\}\cdot\frac{1}{2}\Gamma\left(\frac{n+1}{2n}\right)\Gamma\left\{1-\frac{n+1}{2n}\right\}$$

$$= \frac{\pi^2}{4}\csc\frac{\pi(m+1)}{2m}\csc\frac{\pi(n+1)}{2n} = \frac{\pi^2}{4\cos\frac{\pi}{2m}\cos\frac{\pi}{2n}} \overset{AM\geq GM}{\geq} \frac{\pi^2}{\left(\cos\frac{\pi}{2m}+\cos\frac{\pi}{2n}\right)}$$

4.93

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1} x$$

$$J = \int_1^{\sqrt{3}} (\pi - 2\tan^{-1} x)(\tan^{-1} x)^2 dx$$

$$(\pi - 2\tan^{-1}x)(\tan^{-1} x)^2 \overset{AM-GM}{<} \left(\frac{\pi}{3}\right)^3$$

$$\max_{[1,\sqrt{3}]}(\pi - 2\tan^{-1} x)(\tan^{-1} x)^2 = \frac{\pi^3}{27}$$

$$(\pi - 2\tan^{-1} x)(\tan^{-1} x)^2 < \frac{\pi^3}{27}$$

$$\int_1^{\sqrt{3}} (\pi - 2\tan^{-1} x)(\tan^{-1} x)^2 dx < \int_1^{\sqrt{3}} \frac{\pi^3}{27} dx$$

$$J < \frac{\pi^3}{27}(\sqrt{3}-1)$$

4.94

Lemma: Let f be a convex function defined on $I \subseteq \mathbb{R}$ then for any

$x \le y \le z$ in I we have, $f(x-y+z) \le f(x) - f(y) + f(z)$

Now, $\left\{e^{m^2}\right\}'' = 2e^{m^2} + 4m^2e^{m^2} > 0$ *for all* $m \in \mathbb{R}$

Let $z = n + 12$ *and* $y = n + 8$ *then from*

$$f(x-y+z) \le f(z) - f(y) + f(x) \Rightarrow$$

$$\Rightarrow f(n+4) + f(n+8) \le f(n) + f(n+12)$$

where $x \in [a, a+3]$ *then*

$$\int_a^{a+3} f(n+4)\,dn + \int_a^{a+3} f(n+8)\,dn \le \int_a^{a+3} f(n)\,dn + \int_a^{a+3} f(n+12)\,dn$$

$$\Rightarrow \int_{a+4}^{a+7} f(x)\,dx + \int_{a+8}^{a+11} f(x)\,dx \le \int_a^{a+3} f(x)\,dx + \int_{a+12}^{a+15} f(x)\,dx$$

$$\therefore \int_{a+8}^{a+11} e^{x^2}\,dx + \int_{a+4}^{a+7} e^{x^2}dx \le \int_a^{a+3} e^{x^2}dx + \int_{a+12}^{a+15} e^{x^2}dx$$

4.95

$$e^x \ge 1 + x, x \in \mathbb{R},$$

$$log(1+x) \le x, x > -1 \rightarrow log\left(1+\sqrt{sin\,x}\right) \le \sqrt{sin\,x} \rightarrow$$

$$log^2\left(1+\sqrt{sin\,x}\right) \le sin\,x \le x$$

$$\int_0^1 log^2\left(1+\sqrt{sin\,x}\right) < \int_0^1 x\,dx = \frac{1}{2}$$

4.96

$$\frac{a\,sin\,x}{b + a\,cos\,x} = \frac{a\left(2\,tan\frac{x}{2}\right)}{b\left(1+tan^2\frac{x}{2}\right) + a\left(1-tan^2\frac{x}{2}\right)}$$

$$= \frac{2a\,tan\left(\frac{x}{2}\right)}{(a+b)+(b-a)\,tan^2\left(\frac{x}{2}\right)} = \frac{\frac{2a}{b+a}tan\left(\frac{x}{2}\right)}{1+\frac{b-a}{b+a}tan^2\frac{x}{2}} = \frac{tan\frac{x}{2}-\frac{b-a}{b+a}tan\frac{x}{2}}{1+\frac{b-a}{b+a}tan^2\left(\frac{x}{2}\right)}$$

Put $\frac{b-a}{b+a}tan\frac{x}{2} = tan\,\theta$

$$\therefore \frac{a\sin x}{b+a\cos x} = \frac{tan\frac{x}{2}-tan\,\theta}{1+tan\frac{x}{2}tan\,\theta} = tan\left(\frac{x}{2}+\theta\right)$$

$$\Rightarrow arctan\left(\frac{a\sin x}{b+a\cos x}\right) = \frac{x}{2}+\theta = \frac{x}{2}+tan^{-1}\left(\frac{b-a}{b+a}tan\frac{x}{2}\right)$$

Similarly,

$$arctan\left(\frac{b\sin x}{a+b\cos x}\right)$$

$$= \frac{x}{2}+arctan\left(\frac{a-b}{a+b}tan\frac{x}{2}\right) = \frac{x}{2}-arctan\left(\frac{b-a}{b+a}tan\frac{x}{2}\right)$$

$$\therefore I(a,b) = \int_a^b \left(\frac{x}{2}+\frac{x}{2}\right)dx = \frac{1}{2}(b^2-a^2) \Rightarrow \frac{2}{b-a}I(a,b) = b+a$$

Thus, $\frac{2}{b-a}I(a,b)+\frac{2}{c-b}I(b,c)+\frac{2}{c-a}I(c,a) = 2(a+b+c)$

Now,

$$\frac{a+b}{2} \geq \sqrt{ab} \quad [AM \geq GM]$$

and $\frac{a+b}{2} \geq \sqrt{\frac{a^2+b^2}{2}}$

$$\Leftrightarrow (a+b)^2-2(a^2+b^2) \geq 0 \Leftrightarrow (a-b)^2 \geq 0$$

$$\therefore a+b \geq \sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}} \Rightarrow \sum(a+b) \geq \sum\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)$$

4.97

Let $t = a\,tan\,\theta\,,dt = a\,sec^2\,\theta\,d\theta$

when $t = 0, \theta = 0$, *when* $t = x, \theta = tan^{-1}x$

$$\Omega(a) = \lim_{x\to\infty}\int_0^x \frac{\log t}{t^2 + a^2}dt = \frac{1}{a}\lim_{x\to\infty}\int_a^{tan^{-1}x} \log(a\tan\theta)\,d\theta$$

$$= \frac{1}{a}\lim_{x\to\infty}\int_0^{tan^{-1}x} \log(a\tan(tan^{-1}x - \theta))\,d\theta = \frac{1}{a}\lim_{x\to\infty}\int_0^{tan^{-1}x} \log\left(a\cdot\frac{x - \tan\theta}{1 + x\tan\theta}\right)d\theta$$

$$= \frac{1}{a}\lim_{x\to\infty}\int_0^{tan^{-1}x} \log\left(a\cdot\frac{1 - \frac{\tan\theta}{x}}{\frac{1}{x} + \tan\theta}\right)d\theta = \frac{1}{a}\int_0^{\frac{\pi}{2}} \log\left(\frac{a^2}{a\tan\theta}\right) = \pi\log a^{\frac{1}{a}} - \Omega(a)$$

$\Rightarrow 2\Omega(a) = \pi\log a^{\frac{1}{a}} \Rightarrow \Omega(a) = \frac{\pi}{2}\log a^{\frac{1}{a}}$. *So,* $\sum_{cyc}\Omega^2(a) = \frac{\pi^2}{4}\log^2\left(a^{\frac{1}{a}}\right)$

$$\geq \frac{\pi^2}{12}\left(\sum_{cyc}\log\left(a^{\frac{1}{a}}\right)\right)^2 = \frac{\pi^2}{12}\log^2\left(a^{\frac{1}{a}}\cdot b^{\frac{1}{b}}\cdot c^{\frac{1}{c}}\right)$$

4.98

If $0 < a < b$ *then* $\frac{2}{\pi}\ln\left(\frac{b}{a}\right) + b - a < \frac{\pi}{2}\int_a^b \frac{dx}{\arctan x} < \frac{\pi}{2}\ln\left(\frac{b}{a}\right) + b - a$

We need to prove that $\frac{2}{\pi x} + 1 < \frac{1}{2\arctan x} < \frac{\pi}{2x} + 1$ *(1)* $\forall x > 0$

Put $\arctan x = t \Rightarrow 0 < t < \frac{\pi}{2}$. *We have (1)* $\Rightarrow \frac{2}{\pi\tan t} + 1 < \frac{\pi}{2t} < \frac{\pi}{2\tan t} + 1$

$$* f(t) = \frac{\pi}{2t} - \frac{2}{\pi\cdot\tan t} - 1$$

We have $f'(t) = \frac{2}{\pi\cdot\sin^2 t} - \frac{\pi}{2t^2} = \frac{4t^2 - \pi^2\cdot\sin^2 t}{2t^2\cdot\pi\cdot\sin^2 t}$

On the other hand, by Jordan's inequality, we have

$$\sin t > \frac{2t}{\pi} \Rightarrow \sin^2 t > \frac{4t^2}{\pi^2} \Rightarrow 4t^2 - \pi^2\cdot\sin^2 t < 4t^2 - \pi^2\cdot\frac{4t^2}{\pi^2} = 0 \Rightarrow f'(t) < 0$$

$\Rightarrow f(t)$ *is a decreasing function* $\Rightarrow$

$\Rightarrow f(t) > \lim_{t\to\frac{\pi}{2}}\left(\frac{\pi}{2t} - \frac{2}{\pi\cdot\tan t} - 1\right) \Rightarrow f(t) > 0 \Rightarrow \frac{2}{\pi\cdot\tan t} + 1 < \frac{\pi}{2t}$ *(2)*

$$*\, g(t) = \frac{\pi}{2\tan t} + 1 - \frac{\pi}{2t}$$

We have $g'(t) = \frac{\pi}{2t^2} - \frac{\pi}{2\sin^2 t} = \frac{2\pi\cdot\sin^2 t - 2\pi\cdot t^2}{4t^2\cdot\sin^2 t} = \frac{2\pi(\sin t - t)(\sin t + t)}{4t^2\cdot\sin^2 t}$

On the other hand, by Jordan's inequality, we have

$\sin t \le t \Rightarrow g'(t) \le 0 \Rightarrow g(t)$ *is a decreasing functioin*

$\Rightarrow f(t) > \lim_{t\to\frac{\pi}{2}}\left(\frac{\pi}{2\tan t} + 1 - \frac{\pi}{2t}\right) \Rightarrow f(t) > 0 \Rightarrow \frac{\pi}{2t} < \frac{\pi}{2\cdot\tan t} + 1$ *(3)*

(2) and (3) $\Rightarrow \frac{2}{\pi\cdot\tan t} + 1 < \frac{\pi}{2t} < \frac{\pi}{2\cdot\tan t} + 1 \Rightarrow$ *(1) True* $\Rightarrow$

$$\Rightarrow \int_a^b \left(\frac{2}{\pi\cdot x} + 1\right)dx < \int_a^b \frac{\pi}{2\arctan x}dx < \int_a^b \left(\frac{\pi}{2x} + 1\right)dx$$

$$\Rightarrow \frac{2}{\pi}\ln\left(\frac{b}{a}\right) + b - a < \frac{\pi}{2}\int_a^b \frac{dx}{\arctan x} < \frac{\pi}{2}\ln\left(\frac{b}{a}\right) + b - a$$

4.99

Set $f(x) = \frac{1}{x^5}, x > 0$ *and* $g(x) = x^5\cdot e^{x^2}, x > 0$

It's $f'(x) = -\frac{5}{x^6} < 0, x > 0$ *and* $g'(x) = x^4 e^{x^2}(2x^2 + 5) > 0, \forall x > 0.$

So f *strictly decreasing when* $x > 0$ *and* g *strictly increasing*

Using the Chebyshev's integral inequality, we have that:

$$\int_a^b \frac{1}{x^5}dx \cdot \int_a^b x^5\cdot e^{x^2}dx > \int_a^b \frac{1}{x^5}\cdot x^5 e^{x^2}dx\cdot(b-a)$$

$$\Rightarrow \left[-\frac{1}{4x^4}\right]_a^b \cdot \int_a^b x^5 e^{x^2}dx > \int_a^b e^{x^2}dx\cdot(b-a)$$

$$\Rightarrow \frac{1}{4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\int_a^b x^5 e^{x^2}dx > \int_a^b e^{x^2}dx\,(b-a)$$

$$\Rightarrow \frac{1}{4}(b-a)\cdot\frac{(b+a)}{a^2b^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\int_a^b x^5e^{x^2}dx > \int_a^b e^{x^2}dx\,(b-a)$$

$$\Rightarrow \frac{1}{4}\left(\frac{1}{a^4b}+\frac{1}{a^2b^3}+\frac{1}{a^3b^2}+\frac{1}{ab^4}\right)\int_a^b x^5e^{x^2}dx > \int_a^b e^{x^2}dx$$

4.100

$$\Omega(a)=\sqrt[n]{\int_0^a \sqrt[m]{e^{(m+n)x^2}}\,dx}\cdot\sqrt[m]{\int_0^a \sqrt[n]{e^{-(m+n)x^2}}\,dx}\overset{HOLDER'S}{\geq}\int_0^a\left|e^{\frac{(m+n)x^2}{mn}}\cdot\frac{1}{e^{\frac{(m+n)x^2}{mn}}}\right|dx=a$$

Similarly, $\Omega(b)\geq b,\Omega(c)\geq c$

$$\therefore \sum_{cyc}\Omega^2(a)=\sum_{cyc}a^2\geq\sum_{cyc}ab$$

4.101

Using Hölder inequality for integrals, I have that.

$$\left(\int_0^a 1^{\frac{3}{2}}\,dx\right)^{\frac{2}{3}}\cdot\left(\int_0^a e^{3x^2}dx\right)^{\frac{1}{3}}\geq\int_0^a e^{x^2}\,dx\Rightarrow$$

$$a^{\frac{2}{3}}\left(\int_0^a e^{3x^2}\,dx\right)^{\frac{1}{3}}\geq\int_0^a e^{x^2}\,dx\Rightarrow\int_0^a e^{3x^2}\,dx\geq\frac{1}{a^2}\left(\int_0^a e^{x^2}dx\right)^3\ \ (1)$$

Just the same:

$$\left(\int_0^a 1^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\cdot\left(\int_0^a e^{-3x^2}dx\right)^{\frac{1}{3}}\geq\int_0^a e^{-x^2}dx\Rightarrow\Rightarrow\cdots\int_0^a e^{-3x^2}\,dx\geq\frac{1}{a^2}\left(\int_0^a e^{-x^2}dx\right)^3\ \ (2)$$

(1) × (2) (everything is positive) we have that

$$\int_0^a e^{3x^2}\,dx\cdot\int_0^a e^{-3x^2}dx\geq\frac{1}{a^4}\left(\int_0^a e^{x^2}dx\int_0^a e^{-x^2}dx\right)^3$$

4.102

$$J=\int_0^{\frac{\pi}{4}} ln(1+tan\,x)\,dx \overset{x=\frac{\pi}{4}-t}{=} \int_0^{\frac{\pi}{4}} ln\left(1+tan\left(\frac{\pi}{4}-t\right)\right)dt$$

$$=\int_0^{\frac{\pi}{4}} ln\left(1+\frac{1-tan\,t}{1+tan\,t}\right)dt=\int_0^{\frac{\pi}{4}} ln\left(\frac{2}{1+tan\,t}\right)dt$$

$$=\int_0^{\frac{\pi}{4}} ln(2)\,dt-\int_0^{\frac{\pi}{4}} ln(1+tan\,t)\,dt=\frac{\pi}{4}ln\,2-J\to J=\frac{\pi}{8}ln\,2$$

$$I=\int_0^{\frac{\pi}{4}}\left(\frac{1-sin^2x}{1+sin^2x}+\frac{1-cos^2x}{1+cos^2x}\right)ln(1+tan\,x)\,dx$$

$$=\int_0^{\frac{\pi}{4}}\left(\frac{2}{1+sin^2x}+\frac{2}{1+cos^2x}-2\right)ln(1+tan\,x)\,dx$$

$$=2\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+sin^2x}+\frac{1}{2-sin^2x}-1\right)ln(1+tan\,x)\,dx$$

$\therefore$ *let* $f(x)=\frac{1}{1+x^2}+\frac{1}{2-x^2}\quad \forall x\in\left[0;\frac{1}{\sqrt{2}}\right]$

$$f'(x)=-\frac{2x}{(1+x^2)^2}+\frac{2x}{(2-x^2)^2}=2x\left(\frac{1}{(x^2-2)^2}-\frac{1}{(x^2+1)^2}\right)$$

$$=\frac{2x((x^2+1)^2-(x^2-2)^2)}{(x^2-2)^2(x^2+1)^2}=\frac{6x(2x^2-1)}{(x^2-2)^2(x^2+1)^2}\le 0\quad\forall x\in\left[0;\frac{1}{\sqrt{2}}\right]$$

$$0\le x\le\frac{\pi}{4}\Rightarrow 0\le sin\,x\le\frac{1}{\sqrt{2}}\Rightarrow f(sin\,x)\ge f\left(\frac{1}{\sqrt{2}}\right)=\frac{4}{3}$$

$$\Rightarrow\frac{1}{1+sin^2x}+\frac{1}{2-sin^2x}-1\ge\frac{1}{3}\Rightarrow I\ge 2\int_0^{\frac{\pi}{4}}\frac{1}{3}ln(1+tan\,x)\,dx\Leftrightarrow I\ge\frac{2}{3}J\Leftrightarrow I\ge\frac{\pi}{12}ln\,2$$

4.103

If we consider the functions

$f(x) = \frac{1}{x^3}, x \in [a, 2a]$ *(Strictly decreasing on* $[a, 2a]$*)*

$g(x) = e^x, x \in [a, 2a]$ *(Strictly increasing on* $[a, 2a]$*)*

Using Chebyshev integral inequality we have:

$$a \cdot \int_a^{2a} \frac{e^x}{x^3} dx < \int_a^{2a} \frac{1}{x^3} dx \cdot \int_a^{2a} e^x dx = \left[-\frac{1}{2x^2}\right]_a^2 (e^{2a} - e^a)$$

$$\Rightarrow a \int_a^{2a} \frac{e^x}{x^3} dx < \frac{3}{8a^2}(e^{2a} - e^a) \Rightarrow \int_a^{2a} \frac{e^x}{x^3} dx < \frac{3}{8a^3} e^a(e^a - 1)$$

4.104

$$\forall x > 1: f(x) = \int_1^x log^2 t\, dt$$

$$\therefore f'(x) = log^2 x \;\&\; f''(x) = 2\frac{log\, x}{x} \geq 0 \quad \forall x > 1$$

So by Jensen's inequality: $\begin{cases} f\left(\frac{a+3b}{4}\right) \leq \frac{f(a)+3f(b)}{4} \\ f\left(\frac{3a+b}{4}\right) \leq \frac{3f(a)+f(b)}{4} \end{cases} \Rightarrow f(a) + f(b) \geq f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)$

$$\Leftrightarrow \int_1^a log^2 t\, dt + \int_1^b log^2\, dt \geq \int_1^{\frac{a+3b}{4}} log^2 t\, dt + \int_1^{\frac{3a+b}{4}} log^2 t\, dt$$

4.105

$c \leq f(x) \leq d$ *and* $c \leq g(x) \leq d; \frac{c}{d} \leq \frac{f}{g} \leq \frac{d}{c}$ *for all* $x \in [a, b]$

$$\Rightarrow \left(\frac{f}{g} - \frac{c}{d}\right)\left(\frac{f}{g} - \frac{d}{c}\right) \leq 0 \Rightarrow \frac{f}{g} + \frac{g}{f} \leq \frac{c}{d} + \frac{d}{c} \Rightarrow \int_a^b \frac{f(x)}{g(x)} dx + \int_a^b \frac{g(x)}{f(x)} dx \leq \left(\frac{c}{d} + \frac{d}{c}\right)(b - a)$$

$$\Rightarrow cd\left(\int_a^b \frac{f(x)}{g(x)}dx + \int_a^b \frac{g(x)}{f(x)}dx\right) < (c^2 + d^2)(b-a)$$

4.106

f is increasing function then for all $s \in [a, \sqrt{ab}]$ and $t \in [\sqrt{ab}, b]$ we have $f(s) \leq f(t)$ then

$$\left(b - \sqrt{ab}\right)\int_a^{\sqrt{ab}} f(s)\,dx = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right)\int_a^{\sqrt{ab}} f(s)\,ds \leq$$

$$\leq \sqrt{a}\left(\sqrt{b} - \sqrt{a}\right)\int_{\sqrt{ab}}^{b} f(t)dt = \left(\sqrt{ab} - a\right)\int_{\sqrt{ab}}^{b} f(t)\,dt$$

it follow that

$$\sqrt{b}\int_a^{\sqrt{ab}} f(x)\,dx \leq \sqrt{a}\int_{\sqrt{ab}}^{b} f(x)dx = \sqrt{a}\left(\int_a^b f(x)\,dx - \int_a^{\sqrt{ab}} f(x)dx\right)$$

then

$$\left(\sqrt{b} + \sqrt{a}\right)\int_a^{\sqrt{ab}} f(x)\,dx \leq \sqrt{a}\int_a^b f(x)\,dx$$

4.107

$$sin\left(\frac{\pi}{2} - x\right) \overset{JORDAN}{\geq} \frac{2}{\pi}\left(\frac{\pi}{2} - x\right) \rightarrow cos\,x \geq 1 - \frac{2}{\pi}x \rightarrow$$

$$cos\,x + \frac{2}{\pi}x + x \geq 1 + x$$

$$\zeta(A) = \int_0^A \frac{1}{\sqrt{cos\,x + \frac{2}{\pi}x + x}}dx \leq \int_0^A \frac{1}{\sqrt{1+x}}dx = 2\sqrt{1+A} - 2$$

$$\sum \zeta(A) \leq 2\sum\sqrt{1+A} - 6 \overset{JENSEN}{\leq} 2\cdot 3\sqrt{1 + \frac{A+B+C}{3}} - 6 = 2\sqrt{3(1+\pi)} - 6$$

4.108

$$1 \le x_n^k \le y_n^k \le z_n^k \le t_n^k \le 13$$

$$x_n^k = 1 + \frac{2k}{n}, y_n^k = 5 + \frac{2k}{n}, z_n^k = 7 + \frac{2k}{n}, t_n^k = 11 + \frac{2k}{n}$$

$$f - convexe \to \frac{f(y_n^k) - f(x_n^k)}{y_n^k - x_n^k} \le \frac{f(t_n^k) - f(z_n^k)}{t_n^k - z_n^k} \to$$

$$\lim_{n\to\infty} \frac{2}{n}\sum_{n=1}^{n} f(y_n^k) + \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f(z_n^k) \ge \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n} f(t_n^k) + \lim_{n\to\infty} \frac{2}{n}\sum_{\substack{k=1\\ n\to\infty}}^{n} f(x_n^k)$$

$$\int_5^7 f(x)\,dx + \int_7^9 f(x)dx \le \int_{11}^{13} f(x)dx + \int_1^3 f(x)dx$$

$$\int_5^9 f(x)dx \le \int_{11}^{13} f(x)dx + \int_1^3 f(x)dx$$

4.109

Applying Cauchy – Schwarz,

$$\left(\int_a^b \frac{x^2}{x^3+1}dx\right)^2 \le \left(\int_a^b \frac{dx}{(x^3+1)^2}\right)\left(\int_a^b x^4 dx\right) = \frac{(b^5-a^5)}{5}\left(\int_a^b \frac{dx}{(x^3+1)^2}\right)$$

$$\Rightarrow \left(\frac{1}{3}[ln(x^3+1)]_{x=a}^{x=b}\right)^2 \le \frac{b^5-a^5}{5}\left(\int_a^b \frac{dx}{(x^3+1)^2}\right)$$

$$\therefore ln^2\left(\frac{b^3+1}{a^3+1}\right)\cdot\frac{5}{9(b^5-a^5)} \le \int_a^b \frac{dx}{(x^3+1)^2}$$

4.110

$$Let\ I_n = \int_{\frac{1}{2}}^{\frac{3}{2}} \left(1 + \frac{1}{x^{\frac{1}{n}}}\right)\left(1 + \frac{1}{(2-x)^{\frac{1}{n}}}\right)dx$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} \left[1 + \frac{1}{x^{\frac{1}{n}}} + \frac{1}{(2-x)^{\frac{1}{n}}} + \frac{1}{[x(2-x)]^{\frac{1}{n}}}\right] dx$$

Also $\frac{1}{2^{\frac{1}{n}}} + \frac{1}{(2-x)^{\frac{1}{n}}} \geq \frac{2}{[x(2-x)]^{\frac{1}{2n}}}$ *and* $x(2-x) = 1 - (1-x)^2 \leq 1$

$\therefore \frac{1}{x^{\frac{1}{n}}} + \frac{1}{(2-x)^{\frac{1}{n}}} \geq 2$ *and* $\frac{1}{[x(2-x)]^{\frac{1}{n}}} \geq 1$. *Thus,*

$$I_n \geq \int_{\frac{1}{2}}^{\frac{3}{2}} u\, dx = 4 \Rightarrow \Omega(n) \geq 4^{n-1}u = 4^n$$

Now, $\Omega(n)\Omega(m)\Omega(p) \geq 4^{m+n+p} \geq 4^{3(mnp)^{\frac{1}{3}}} = 64^{(mnp)^{\frac{1}{3}}}$

4.111

Let $\varphi(x) = f(x) - 3x^2$ *for all* $x \in [a, a+2], \varphi''(x) = f''(x) - 6 \geq 0$

hence φ *is convex. By Hermite Hadamard inequality*

$$\varphi(a+1) \leq \frac{1}{2}\int_a^{a+2} \varphi(x)\, dx \Rightarrow f(a+1) - 3(a+1)^2 \leq \frac{1}{2}\int_a^{a+2} f(x)\, dx - \frac{3}{2}\int_a^{a+2} x^2\, dx$$

$\Rightarrow 1 + f(a+1) \leq \frac{1}{2}\int_a^{a+2} f(x)\, dx$. *Let* $G(x) = 6x^2 - f(x)$ *for all*

$x \in [a, a+2]$. $G''(x) = 12 - f''(x) \geq 0$ *for all* $x \in [a, a+2]$ *hence* G *is convex. By Hermite Hadamard inequality*

$$\frac{1}{2}\int_a^{a+2} G(x)\, dx \geq G(a+1) \Rightarrow \frac{6}{2}\int_a^{a+2} x^2 dx - \frac{1}{2}\int_a^{a+2} f(x)\, dx \geq 6(a+1)^2 - f(a+1)$$

$$\Rightarrow 2 + f(a+1) \geq \frac{1}{2}\int_a^{a+2} f(x)\, dx \therefore 2 + f(a+1) \geq \frac{1}{2}\int_a^{a+2} f(x)\, dx \geq 1 + f(a+1)$$

4.112

$\forall x > 0: f(x) = e^x \ln(x+1)$ & $F(x)$ *his primitive function.*

$$So,\ F''(x) = f'(x) = \left(\frac{1}{x+1} + ln(x+1)\right)e^x > 0.$$

$$So,\ F \text{ is a convex function} \Rightarrow F\left(\frac{a+b}{2}\right) \le \frac{F(a)+F(b)}{2}$$

$$I = 2\int_0^{\sqrt{ab}} f(x)\,dx \overset{AM-GM}{\le} \int_0^{\frac{a+b}{2}} f(x)\,dx \quad \because f(x) \ge 0$$

$$= \int_0^a f(x)\,dx + \int_a^{\frac{a+b}{2}} f(x)\,dx + \int_0^b f(x)\,dx + \int_b^{\frac{a+b}{2}} f(x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_0^b f(x)\,dx + 2F\left(\frac{a+b}{2}\right) - F(a) - F(b) \le \int_0^a f(x)\,dx + \int_0^b f(x)\,dx$$

$$\rightarrow 2\int_0^{\sqrt{ab}} e^x\, ln(x+1) \le \int_0^a e^x\, ln(x+1)\,dx + \int_0^b e^x\, ln(x+1)\,dx$$

4.113

$$\Omega(a) = \int_0^a \frac{(x^4-3x^3+5x^2-3x+4)+(10x^3-30x^2+40x)}{x^4-3x^3+5x^2-3x+4}\,dx =$$

$$= \int_0^a \left(1 + \frac{10x(x^2-3x+4)}{(x^2+1)(x^2-3x+4)}\right)dx = a + \int_0^a \frac{10x}{x^2+1}\,dx = a + 5ln(a^2+1)$$

$$\sum \Omega(a) = \sum a + 5\sum ln(a^2+1) \overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{abc} + 5ln\prod(1+a^2) = 3 + 5ln\prod(1+a^2)$$

4.114

$$Let\ f(x) = \sqrt{(x+a)(x+b)},\ x \in [0,1],\ a,b > 0$$

$$f'(x) = \frac{a+b+2x}{2\sqrt{(x+a)(x+b)}} > 0 \rightarrow f\ increasing,\ f(x) \ge f(0) = \sqrt{ab}$$

$$\sqrt{(x+a)(x+b)} \geq \sqrt{ab} \to \Omega(a,b) = \int_0^1 \sqrt{(x+a)(x+b)}\,dx > \sqrt{ab}$$

$$\sum (a+b)\Omega(a,b) > \sum (a+b)\sqrt{ab} \overset{AM-GM}{\geq} \sum 2\sqrt{ab}\cdot\sqrt{ab} =$$

$$= 2\sum ab \overset{AM-GM}{\geq} 2\cdot 3\sqrt[3]{(abc)^2} = 6$$

4.115

$$(sin^2\theta)^{cos^2\theta}(cos^2\theta)^{sin^2\theta} + (sin^2\theta)^{sin^2\theta}(cos^2\theta)^{cos^2\theta} > (sin^2\theta)^{sin^2\theta}(cos^2\theta)^{cos^2\theta} =$$

$$= (sin^2\theta)^{\frac{sin^2\theta}{sin^2\theta+cos^2\theta}}(cos^2\theta)^{\frac{sin^2\theta}{sin^2\theta+cos^2\theta}} \overset{GM-HM}{\geq} \frac{sin^2\theta + cos^2\theta}{\frac{sin^2\theta}{sin^2\theta} + \frac{cos^2\theta}{cos^2\theta}} = \frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} \left((sin^2\theta)^{cos^2\theta}(cos^2\theta)^{sin^2\theta} + (sin^2\theta)^{sin^2\theta}(cos^2\theta)^{cos^2\theta}\right)d\theta > \int_0^{\frac{\pi}{2}} \frac{1}{2}d\theta = \frac{\pi}{4}$$

4.116

$$\Omega = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\pi^2 - 4\pi x + 4x^2)\sqrt{cosx}dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} (\pi - 2x)^2\sqrt{cosx}dx =$$

$$= \frac{1}{8}\int_0^{\pi} y^2\sqrt{cos\left(\frac{\pi}{2} - \frac{y}{2}\right)}\,dy = \frac{1}{8}\int_0^{\pi} y^2\sqrt{sin\frac{y}{2}}\,dy \overset{JORDAN}{>}$$

$$> \frac{1}{8}\int_0^{\pi} y^2\sqrt{\frac{y}{\pi}}\,dy = \frac{1}{8\sqrt{\pi}}\int_0^{\pi} y^{\frac{5}{2}}\,dy = \frac{\pi^3\sqrt{\pi}}{8\sqrt{\pi}}\cdot\frac{2}{7} = \frac{\pi^3}{28}$$

4.117

$$I = \int_0^1 e^{2\sqrt{x}}\,dx \overset{y=\sqrt{x}}{=} 2\cdot\int_0^1 ye^{2y}\,dy = \int_0^1 y\,(e^{2y})'dy = \left[ye^{2y} - \int e^{2y}\right]_0^1 =$$

$$= \left[ye^{2y} - \frac{1}{2}e^{2y}\right]_0^1 = \frac{e^2 + 1}{2}$$

$$e^t \geq t + 1 \ \ \forall\, t \in \mathbb{R} \overset{t=x^2}{\Rightarrow} e^{x^2} \geq x^2 + 1 \Rightarrow \int_0^1 e^{x^2}\,dx \geq \int_0^1 (x^2 + 1)\,dx$$

$$\Rightarrow \int_0^1 e^{x^2}\,dx \geq \left[\frac{x^3}{3} + 1\right]_0^1 = \frac{4}{3} \Rightarrow \left(\int_0^1 e^{x^2}\,dx\right)^2 \geq \frac{16}{9} \Rightarrow -\left(\int_0^1 e^{x^2}\,dx\right)^2 \leq -\frac{16}{9} \quad : (1)$$

$$x^2 \leq \sqrt{x} \ \ \forall x \in [0{,}1] \Rightarrow 2x^2 \leq 2\sqrt{x} \ \ \forall x \in [0{,}1] \overset{e^x \uparrow [0,1]}{\Rightarrow} e^{2x^2} \leq e^{2\sqrt{x}} \ \ \forall x \in [0{,}1]$$

$$\overset{e^x \uparrow [0,1]}{\Rightarrow} \int_0^1 e^{2x^2}\,dx \leq \int_0^1 e^{2\sqrt{x}}\,dx \Rightarrow \int_0^1 e^{2x^2} dx \leq I = \frac{e^2+1}{2} \quad : (2)$$

$$\begin{Bmatrix}(1)\\(2)\end{Bmatrix} \Rightarrow \int_0^1 e^{2x^2}\,dx - \left(\int_0^1 e^{x^2}\,dx\right)^2 \leq \frac{e^2+1}{2} - \frac{16}{9} \quad : (3)$$

$$\frac{e^2+1}{2} - \frac{16}{9} - \frac{e^2}{3} = \frac{3e^2-23}{18} < 0 \quad : (4)$$

$$\begin{Bmatrix}(3)\\(4)\end{Bmatrix} \Rightarrow \int_0^1 e^{2x^2}\,dx - \left(\int_0^1 e^{x^2}\,dx\right)^2 < \frac{e^2}{3}$$

4.118

$$lnx \leq x - 1 \rightarrow ln\frac{b}{a} \leq \frac{b}{a} - 1 \rightarrow ln(a+b)\, ln\frac{b}{a} \leq \frac{b-a}{a} ln(a+b) \rightarrow$$

$$ln(a+b)\, ln\frac{2b}{2a} \leq \frac{1}{a} ln(a+b)^{b-a} \rightarrow$$

$$ln(b+a)\, ln2b - ln(a+b)\, ln2a \leq \frac{1}{a} ln(a+b)^{b-a} \rightarrow$$

$$\int_a^b (ln(x+a)\ln(x+b))'dx \leq \frac{1}{a} ln(a+b)^{b-a} \rightarrow$$

$$\int_a^b \left(\frac{ln(x+a)}{x+b} + \frac{ln(x+b)}{x+a}\right) dx \leq \frac{1}{a} ln(a+b)^{b-a}$$

4.119

$$\int_a^b e^{-\sqrt{x}} sin\left(\frac{\sqrt{x}+100}{100}\right)dx \le \int_a^b e^{-\sqrt{x}}dx = \int_a^b \frac{1}{e^{\sqrt{x}}}dx \le$$

$$\le \int_a^b \frac{1}{1+\sqrt{x}}dx \overset{AM-GM}{\le} \int_a^b \frac{1}{2\sqrt[4]{x}}dx = \frac{1}{2}\int_a^b x^{-\frac{1}{4}}dx = \frac{1}{2}\cdot\frac{4}{3}\left(b^{\frac{3}{4}} - a^{\frac{3}{4}}\right) = \frac{2}{3}\left(b^{\frac{3}{4}} - a^{\frac{3}{4}}\right)$$

4.120

$$(sin^2 x)^{cos^2 x}\cdot(cos^2 x)^{sin^2 x} \overset{Mg\le Mg}{\le}$$

$$\le \left(\frac{cos^2 x\cdot sin^2 x + sin^2 x\cdot cos^2 x}{cos^2 x + sin^2 x}\right)^{cos^2 x + sin^2 x} =$$

$$= 2\,sin^2 x\cdot cos^2 x \overset{Mg\le Ma}{\le} 2\cdot\left(\frac{cos^2 x + sin^2 x}{2}\right) = \frac{1}{2}$$

$$2\cdot\sum\int_0^{\frac{a+b}{a^2+b^2}}\left((sin\,x)^{2\,cos^2 x}\cdot(cos\,x)^{2\,sin^2 x}\right)dx \overset{Mg\le Ma}{\le} 2\cdot\frac{1}{2} = 1$$

$$2\cdot\sum\int\left((sin^2 x)^{cos^2 x}\cdot(cos^2 x)^{sin^2 x}\right)dx \le \sum\int_0^{\frac{a+b}{a^2+b^2}} 1\,dx =$$

$$= \sum\frac{a+b}{a^2+b^2} \overset{Mg\le Ma}{\le} \sum\frac{a+b}{2ab} = \frac{1}{2}\left(2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right) = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

4.121

Let $x \le 1$ but $\ne 0$ and m be a rational number which does not lie between 0 and 1 then $(1-x)^m \ge 1 - mx$

$$11^{-a}\int_1^{10}\left(x+\frac{10}{x}\right)^a dx = 11^p\int_1^{10}\frac{dx}{\left(x+\frac{10}{x}\right)^p}dx \text{ [let } -a = p > 0\text{]}$$

$$\ge 11^p\int_1^{10}\frac{dx}{(x+10)^p} \text{ [since, } 1 \le x \le 10 \Rightarrow 1 + x \le x + \frac{10}{x} \le 10 + x\text{]}$$

$$= \frac{20\left(\frac{11}{20}\right)^p - 11}{1-p} \text{ we need to prove, } \frac{20\left(\frac{11}{20}\right)^p - 11}{1-p} \geq 9$$

$$\Leftrightarrow \left(1 - \frac{9}{20}\right)^p \geq 1 - \frac{9p}{20}, \text{ which is true. Hence proved.}$$

4.122

$$\int_0^{\frac{\pi}{2}} e^{sinx} dx \overset{JORDAN}{\geq} \int_0^{\frac{\pi}{2}} e^{\frac{2x}{\pi}} dx = \frac{\pi}{2}\left(e^{\frac{2}{\pi}\cdot\frac{\pi}{2}} - e^{\frac{2}{\pi}\cdot 0}\right) = \frac{\pi}{2}(e-1)$$

$$\int_0^{\frac{\pi}{2}} e^{sinx} dx \leq \int_0^{\frac{\pi}{2}} e^{x} dx = e^{\frac{\pi}{2}} - 1$$

4.123

Function f is strictly increasing in $[a, b]$ so does $g(x) = x^2, x \in [a, b]$.

It suffices to prove that

$$(b-a)\int_a^b x^2 f(x)\, dx \geq (b-a)ab \int_a^b f(x)\, dx$$

$$\text{or } \frac{(b-a)}{ab}\int_a^b x^2 f(x)\, dx \geq (b-a)\int_a^b f(x)\, dx$$

$$\text{or } \left(\frac{1}{a} - \frac{1}{b}\right)\int_a^b x^2\, f(x) dx \geq (b-a)\int_a^b f(x)\, dx$$

$$\text{or } \int_a^b \frac{1}{x} dx \cdot \int_a^b x^2 f(x)\, dx \geq (b-a)\int_a^b f(x)\, dx$$

That is true due to Chebyshev's inequality, because $\frac{1}{x^2}$ is strictly decreasing in $[a, b], a > 0$

(easy to check) and $x^2, f(x)$ are strictly increasing in $[a, b]$.

4.124

For $0 \leq x \leq 1, k \in \mathbb{N}$, let $f_k(x) = e^{x-[x]} + e^{2x-[2x]} + \cdots + e^{kx-[kx]}$

For $0 \leq x < \frac{1}{k}, [x] = [2x] = \cdots = [kx] = 0$

$$\Rightarrow f_k(x) = e^x + e^{2x} + \cdots + e^{kx} \text{ for } 0 \le x < \frac{1}{k}; \quad > \underbrace{e^x + e^x + \cdots + e^x}_{k\ times} = ke^x > k$$

for $0 \le x < \frac{1}{k}$. *We now show that* $a_n = \int_0^1 \frac{f_n(x)}{nx} dx = \infty$ *for each given* n.

For a given $n \in \mathbb{N}$, *write* $a_n = b_n + c_n$ *where* $b_n = \int_0^{\frac{1}{n}} \frac{f_n(x)}{nx} dx, c_n = \int_{\frac{1}{n}}^1 \frac{f_n(x)}{nx} dx$

Note $c_n > 0$, *therefore* $a_n > b_n \ge \int_0^{\frac{1}{n}} \frac{n}{nx} dx = \int_0^{\frac{1}{n}} \frac{1}{x} dx = \infty$

$$\therefore a_n \to \infty \Rightarrow lim_{n\to\infty} a_n = \infty$$

4.125

Applying Chebyshev's Integral Inequality for $n = 2$.

$$\int_a^b x^2 f(x)dx \ge \frac{1}{b-a}\left(\int_a^b x^2\, dx\right)\left(\int_a^b f(x)dx\right) = \frac{a^2 + ab + b^2}{3}\int_a^b f(x)\, dx \ge ab\int_a^b f(x)\, dx$$

4.126

$\forall x \in [0,1]$ *it's* $\sqrt{x} \ge x$ *"="* *for* $x = 0, x = 1 \Leftrightarrow \sqrt{x} + x \ge 2x$

$$\Leftrightarrow \sqrt{\sqrt{x} + x} \ge \sqrt{2} \cdot \sqrt{x} \ge \sqrt{2} \cdot x \Leftrightarrow \sqrt{\sqrt{x} + x} + x \ge \left(\sqrt{2} + 1\right) \cdot x$$

$$\Leftrightarrow \sqrt{\sqrt{x + x} + x} \ge \sqrt{\left(\sqrt{2} + 1\right) \cdot \sqrt{x}} \text{ so,}$$

$$\int_0^1 \sqrt{\sqrt{x + \sqrt{x}} + x}\, dx \ge \sqrt{\sqrt{2} + 1} \cdot \int_0^1 \sqrt{x}\, dx = \sqrt{\sqrt{2} + 1} \cdot \left[\frac{2}{3} \cdot x^{\frac{3}{2}}\right]_0^1 = \sqrt{\sqrt{2} + 1} \cdot \frac{2}{3}$$

4.127

$$e^{nx} + e^{(n-1)x} + \cdots + e^{2x} + e^x + 1 \overset{AM-GM}{\ge} (n+1)\sqrt[n+1]{e^{nx} \cdot e^{(n-1)x} \ldots e^{2x} \cdot e^x \cdot 1}$$

$$= (n+1)\sqrt[n+1]{e^{nx+(n-1)x+\cdots+2x+x}} = (n+1) \cdot \sqrt[n+1]{e^{x(1+2+\cdots+n)}}$$

$$= (n+1) \cdot \sqrt[n+1]{e^{\frac{x \cdot n \cdot (n+1)}{2}}} = (n+1) \cdot \left(e^{\frac{xn \cdot (n+1)}{2}}\right)^{\frac{1}{n+1}}$$

$$= (n+1)\cdot e^{\frac{xn}{2}} \text{ so,}$$

$$\int_{\ln a}^{\ln b} \frac{\sqrt{e^{nx}}\cdot dx}{e^{nx}+\cdots+e^{2x}+e^{x}+1} \le \int_{\ln a}^{\ln b} \frac{e^{\frac{nx}{2}}}{(n+1)\cdot e^{\frac{nx}{2}}}dx$$

$$= \int_{\ln a}^{\ln b} \frac{dx}{n+1} = \frac{1}{n+1}\cdot[\ln b - \ln x] = \frac{1}{n+1}\cdot \ln\left(\frac{b}{a}\right) = \ln \sqrt[n+1]{\frac{b}{a}}$$

4.128

I will use the well-known inequality $e^x \ge x+1,\ \forall x \in \mathbb{R}$

$$2^{\sin x} = e^{\ln 2 \sin x} \ge \ln 2\cdot \sin x + 1$$

$$2^{\cos x} = e^{\ln 2 \cos x} \ge \ln 2 \cos x + 1$$

Adding those two inequalities, we have that

$$2^{\sin x} + 2^{\cos x} \ge \ln 2\,(\sin x + \cos x) + 2 \Rightarrow$$

$$\int_0^{\frac{\pi}{2}} \left(2^{\sin x} + 2^{\cos x}\right)dx > \ln 2 \int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx + 2\int_0^{\frac{\pi}{2}} dx \Rightarrow$$

$$\int_0^{\frac{\pi}{2}} \left(2^{\sin x} + 2^{\cos x}\right)dx > \ln 2\cdot[\sin x - \cos x]_0^{\frac{\pi}{2}} + \pi \Rightarrow \int_0^{\frac{\pi}{2}} \left(2^{\sin x} + 2^{\cos x}\right)dx > 2\ln 2 + \pi$$

4.129

Using Cauchy – Schwarz inequality for integrals, we have that:

$$\int_0^1 f^2(x)dx \ge \left(\int_0^1 f(x)dx\right)^2 \Rightarrow \left(\int_0^1 f^2(x)dx\right)^3 \ge \left(\int_0^1 f(x)\,dx\right)^6 \Rightarrow$$

$1 + \left(\int_0^1 f^2(x)dx\right)^3 \ge 1 + \left(\int_0^1 f(x)dx\right)^6$. *So, it suffices to prove that:*

$$1 + \left(\int_0^1 f(x)\,dx\right)^6 > \left(\int_0^1 f(x)dx\right)^3 \quad (1)$$

Setting $\int_0^1 f(x)\,dx = a$*, its easy to see that (1) holds, because:*

$1 + a^6 > a^3 \Leftrightarrow a^6 - a^3 + 1 > 0$ *for every* $a \in \mathbb{R} \Leftrightarrow (a^3)^2 - a^3 + 1 > 0$ *or* $y^2 - y + 1 > 0$

for every $y \in \mathbb{R}.$ *;* $\Delta = -3 < 0$ *so, this is true!*

4.130

Put $f(x) = \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 + \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3$ *when* $\frac{\pi}{6} \le x < \frac{\pi}{2}$

We have $\sin 2x \le 1 \Rightarrow 2\sin x \cos x \le 1 \Rightarrow \sin x \cos x \le \frac{1}{2}$

Lemma: $a^3 + b^3 \ge \frac{(a+b)^3}{4}$ *when* $a > 0, b > 0$

Applying the lemma, we have

$$f(x) \ge \frac{\left(\sin^2 x + \frac{1}{\sin^2 x} + \cos^2 x + \frac{1}{\cos^2 x}\right)^3}{4} \Rightarrow f(x) \ge \frac{\left(1 + \frac{1}{\sin^2 x \cos^2 x}\right)^3}{4}$$

Since $\sin x \cos x \le \frac{1}{2} \Rightarrow \frac{1}{\sin^2 x \cos^2 x} \ge 4 \Rightarrow 1 + \frac{1}{\sin^2 x \cos^2 x} \ge 5$

So, $f(x) \ge \frac{(1+4)^3}{4} \Rightarrow f(x) \ge \frac{125}{4} \Rightarrow$

$$\Rightarrow \int_{\frac{\pi}{6}}^{a} \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 dx + \int_{\frac{\pi}{6}}^{a} \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3 dx \ge \int_{\frac{\pi}{6}}^{a} \frac{125}{4} dx$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{a} \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 dx + \int_{\frac{\pi}{6}}^{a} \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3 dx \ge \frac{125}{4}a - \frac{125\pi}{24}$$

$$\Rightarrow \frac{125\pi}{24} + \int_{\frac{\pi}{6}}^{a} \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 dx + \int_{\frac{\pi}{6}}^{a} \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3 dx \ge \frac{125}{4}a \Rightarrow$$

$$\Rightarrow \frac{125\pi}{24} + \int_{\frac{\pi}{6}}^{a} \left(\sin^2 x + \frac{1}{\sin^2 x}\right)^3 + \int_{\frac{\pi}{6}}^{a} \left(\cos^2 x + \frac{1}{\cos^2 x}\right)^3 > \frac{125}{6}a$$

(QED). The equality doesn't occur.

4.131

Using the generalized Hölder's inequality for three functions we have that: $(f(x) \geq 0)$

$$\int_0^1 f(x)\,dx \leq \left(\int_0^1 f^3(x)dx\right)^{\frac{1}{3}} \left(\int_0^1 1^3 dx\right)^{\frac{1}{3}} \left(\int_0^1 1^3 dx\right)^{\frac{1}{3}}$$

$$\Rightarrow \int_0^1 f^3(x)dx \geq \left(\int_0^1 f(x)dx\right)^3 \quad (1)$$

So, it suffices to prove that

$$\int_0^1 f^3(x)\,dx - 10\left(\int_0^1 f(x)dx\right)^2 + 25\int_0^1 f(x)dx \geq 0$$

But

$$\int_0^1 f^3(x)dx - 10\left(\int_0^1 f(x)dx\right)^2 + 25\int_0^1 f(x)dx \overset{(1)}{\geq}$$

$$\left(\int_0^1 f(x)dx\right)^3 - 10\left(\int_0^1 f(x)dx\right)^2 + 25\int_0^1 f(x)dx \geq 0$$

$$\int_0^1 f(x)dx\left(\left(\int_0^1 f(x)dx\right)^2 - 10\int_0^1 f(x)dx + 25\right) \geq 0 \Rightarrow$$

$$\int_0^1 f(x)\,dx \cdot \left(\int_0^1 f(x)\,dx - 5\right)^2 \geq 0$$

which holds! $(f(x) \geq 0)$

4.132

Using Cauchy-Schwarz inequality we get:

$$\left(\int_a^b xf(x)dx\right)^2 \leq \int_a^b x^2 dx \int_a^b f^2(x)dx = \frac{b^3-a^3}{3}\int_a^b f^2(x)dx$$

It is given that function in increasing in the interval [a , b] ,which implies f(x) ≤ f(b)

for all x ≤ b. Squaring we get $f^2(x) \le f^2(b)$ for all x ≤ b

We can say that $\int_a^b f^2(x)dx \le \int_a^b f^2(b)dx = (b-a)\ f^2(b)$

$$\left(\int_a^b xf(x)dx\right)^2 \le \int_a^b x^2dx \int_a^b f^2(x)dx =$$

$$= \frac{b^3-a^3}{3}(b-a)\ f^2(b) \le (a^2+b^2+ab)b^2f^2(b)$$

4.133

Lemma: $\sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}} \ge \frac{8}{9\cdot\sqrt[3]{x+1}}\ \forall x \in \left(0;\frac{\pi}{2}\right)$

Surely, we have 1) $x > 0 \Rightarrow \frac{8}{9\cdot\sqrt[3]{x+1}} < \frac{8}{9}$ *2)* $x > 0, \sin^4 x \le 1$ *and* $\cos^4 x \le 1 \Rightarrow \sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}} > \sqrt{0+\frac{1}{4}}+\sqrt{1+\frac{1}{4}} > \frac{8}{9}$.

So, $\sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}} \ge \frac{8}{9\cdot\sqrt[3]{x+1}}$

Applying the lemma, we have

$$\int_a^b\left(\sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}}\right)dx \ge \int_a^b\left(\frac{8}{9\sqrt[3]{x+1}}\right)dx$$

$$\Rightarrow \int_a^b\left(\sqrt{x+\frac{1}{4\sin^4 x}}+\sqrt{1+\frac{1}{4\cos^4 x}}\right)dx \ge \frac{4}{3}\left(\sqrt[3]{(b+1)^2}-\sqrt[3]{(a+1)^2}\right)$$

(Q.E.D). The equality occurs when $a = b$.

4.134

We have that:

$$s(x)+\sqrt[3]{p(x)q(x)r(x)}+\sqrt[3]{p(x)q(x)r(x)}+\sqrt[3]{p(x)q(x)r(x)} \overset{AM-GM}{\ge}$$

$$4\sqrt[4]{s(x)\left(\sqrt[3]{p(x)}\right)^3\left(\sqrt[3]{q(x)}\right)^3\left(\sqrt[3]{r(x)}\right)^3} = 4\sqrt[4]{p(x)q(x)r(x)s(x)}$$

So $s(x) + 3\sqrt[3]{p(x)q(x)r(x)} \geq 4\sqrt[4]{p(x)q(x)r(x)s(x)}$

Integrate from 0 *to* a*, we have what we want.* $(a > 0)$

(if $a = 0$*, it's obvious)*

4.135

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\frac{1}{3}\sum\int_0^1 sin^{-1}(x^a(1-x)^b)\,dx \geq \frac{1}{3}\sum\int_0^1 x^a(1-x)^b dx\ [\because x \geq sinx, \forall x \geq 0] =$$

$$= \frac{1}{3}\sum\beta(a+1,b+1) = \frac{1}{3}\sum\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right] \overset{AM-GM}{\geq}$$

$$\geq \sqrt[3]{\prod\frac{(\Gamma(a+1))^2}{\Gamma(a+b+1)}} = \sqrt[3]{\prod\frac{(a!)^2}{(b+a+1)!}}$$

4.136

Let $f(x) = x - ln(1-x)$ *for all* $x \in (0,1)$ *then* $f'(x) = 1 + \frac{1}{1-x} > 0$

hence f *is increasing on* $(0,1)$ *then* $f(x) > f(0) = 0 \Rightarrow x > ln(1-x)$

$$\frac{1}{b-a}\int_a^b \frac{dx}{ln(1-x)} + \frac{2(ln\,b - ln\,a)}{b-a} > \frac{1}{b-a}\int_a^b \frac{dx}{x} + \frac{2}{b-a}\int_a^b \frac{dx}{x} = \frac{3}{b-a}\int_a^b \frac{dx}{x}$$

$$> \frac{6}{a+b}\left[\begin{matrix}\because \frac{1}{x}\ is\ convex\ then\ applying \\ Hermite\ Hadamard\ Inequality\end{matrix}\right]$$

We need to prove, $\frac{6}{a+b} > \frac{2}{a+b} + \frac{1}{2} \Leftrightarrow \frac{4}{a+b} > \frac{1}{2} \Leftrightarrow 2 > a + b$

Which is true since $1 > a > b$*. Hence true.*

4.137

$$Let\ y = tanh\,x \Rightarrow x = tanh^{-1}\,y\,, dx = \frac{1}{1-y^2}\,dy$$

$$\Omega = \int \frac{tanh^6 x + 2\,tanh^4 x}{(1 + tanh^2 x)^2}\,dx = \int \frac{y^6 + 2y^4}{(1 + y^2)^2} \cdot \frac{1}{1 - y^2}\,dy$$

$$= \int \left(-1 + \frac{y^4 + y^2 + 1}{(1 + y^2)^2(1 - y^2)}\right) dy = -y + \frac{3}{4}\int \frac{1}{1 - y^2}\,dy + \frac{1}{4}\int \frac{1 - y^2}{(1 + y^2)^2}\,dy$$

$$= -y + \frac{3}{4}tanh^{-1}\,y + \frac{1}{4}\int \frac{1}{\left(\frac{1}{y} + y\right)^2} \cdot (-1)d\left(\frac{1}{y} + y\right) = -y + \frac{3}{4}tanh^{-1}\,y + \frac{1}{4} \cdot \frac{1}{\frac{1}{y} + y} + C$$

$$= -y + \frac{3}{4}tanh^{-1}\,y + \frac{1}{4} \cdot \frac{y}{1 + y^2} + C = \frac{3}{4}x - tanh\,x + \frac{1}{4} \cdot \frac{tanh\,x}{1 + tanh^2 x} + C$$

4.138

As $tan^{-1}\,x$ is an increasing function,

$$tan^{-1}(2n) \le tan^{-1}\,x \le tan^{-1}(3n) \quad \forall x \in [2n, 3n]$$

$$\Rightarrow \frac{tan^{-1}(2n)}{x} \le \frac{tan^{-1}\,x}{x} \le \frac{tan^{-1}(3n)}{x} \quad \forall x \in [2n, 3n]$$

$$\Rightarrow tan^{-1}(2n) \int_{2n}^{3n} \frac{1}{x}\,dx \le \int_{2n}^{3n} \frac{tan^{-1}(x)}{x}\,dx \le tan^{-1}(3n) \int_{2n}^{3n} \frac{1}{x}\,dx$$

$$\Rightarrow (tan^{-1}(2n))\,ln\left(\frac{3}{2}\right) \le \int_{2n}^{3n} \frac{tan^{-1}(x)}{x}\,dx \le tan^{-1}(3n)\,ln\left(\frac{3}{2}\right)$$

Taking limit as $n \to \infty$, we get

$$\frac{\pi}{2}ln\left(\frac{3}{2}\right) \le \lim_{n\to\infty} \int_{2n}^{3n} \frac{tan^{-1}(x)}{x}\,dx \le \frac{\pi}{2}ln\left(\frac{3}{2}\right)$$

$$\therefore \lim_{n\to\infty} \int_{2n}^{3n} \frac{tan^{-1}\,x}{x}\,dx = \frac{\pi}{2}ln\left(\frac{3}{2}\right)$$

4.139

$$\Omega = \int_{-1}^{0} \sqrt{\frac{1 + e^x}{1 - e^x}}\, dx$$

$$\frac{1 + e^x}{1 - e^x} = t \Rightarrow e^x = \frac{t - 1}{1 + t} \Rightarrow x = \ln\left(\frac{t - 1}{1 + t}\right) \Rightarrow dx = \frac{2}{t^2 - 1} dt$$

$$\Omega = 2 \int_{\frac{e+1}{e-1}}^{\infty} \frac{\sqrt{t}}{t^2 - 1} dt$$

$$t = z^2 \Rightarrow dt = 2z\, dz$$

$$\Omega = 4 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{z^2}{z^4 - 1} dz = 4 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{z^2 - 1}{z^4 - 1} dz + 4 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^4 - 1} dz$$

$$= 4 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^2 + 1} dz + 2 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^2 - 1} dz - 2 \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^2 + 1} dz$$

$$= 2\left[\int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^2 + 1} dz + \int_{\sqrt{\frac{e+1}{e-1}}}^{\infty} \frac{1}{z^2 - 1} dz\right] = 2\left[\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{e + 1}{e - 1}} + \frac{1}{2}\ln\left|\frac{y - 1}{y + 1}\right|\Bigg|_{\sqrt{\frac{e + 1}{e - 1}}}^{\infty}\right]$$

$$= 2\left[\tan^{-1}\sqrt{\frac{e - 1}{e + 1}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{e + 1}{e - 1}} - 1}{\sqrt{\frac{e + 1}{e - 1}} + 1}\right|\right] = 2\tan^{-1}\sqrt{\frac{e - 1}{e + 1}} - \ln\left|\frac{\sqrt{e + 1} - \sqrt{e - 1}}{\sqrt{e + 1} + \sqrt{e - 1}}\right|$$

4.140

We know that: $\ln x \le x - 1, \forall x > 0$

Setting $x \to \frac{1}{x} > 0$*, we have that,* $\ln x \ge \frac{x-1}{x}, \forall x > 0$ *(1)*

So, setting $x \to 1 - x > 0$*, we have that*

$$0 > ln(1-x) \geq -\frac{x}{1-x} \Rightarrow \frac{1}{ln(1-x)} \leq \frac{x-1}{x} \Rightarrow \frac{1}{ln(1-x)} \leq 1 - \frac{1}{x} \Rightarrow$$

$$\Rightarrow \int_a^b \frac{1}{ln(1-x)} dx \leq b - a - (ln\, b - ln\, a)$$

$$\Rightarrow 2\frac{ln\, b - ln\, a}{b-a} + \frac{1}{b-a}\int_a^b \frac{1}{ln(1-x)} dx \leq \frac{ln\, b - ln\, a}{b-a} + 1$$

So, it suffices to prove that $\frac{ln\, b - ln\, a}{b-a} + 1 < 1 + \frac{1}{\sqrt{ab}}$ *or* $\sqrt{ab} < \frac{b-a}{ln\, b - ln\, a}$ *which holds as a fundamental property of the Lograrithmic Mean!*

4.141

For $0 < a \leq x \leq b \Rightarrow xe^{x^2} \leq be^{b^2}$*. Now,*

$$2\int_a^b \frac{be^{b^2}}{1+e^{2x^2}} dx \geq 2\int_a^b \frac{xe^{x^2}}{1+e^{2x^2}} dx = tan^{-1}\left(e^{x^2}\right)\Big]_a^b = tan^{-1}\left(e^{b^2}\right) - tan^{-1}\left(e^{a^2}\right)$$

$$\Rightarrow 2\int_a^b \frac{dx}{1+e^{2x^2}} \geq \frac{tan^{-1}\left(e^{b^2}\right) - tan^{-1}\left(e^{a^2}\right)}{be^{b^2}}$$

4.142 *Let* $f(x) = (15x+2)x - (36x^3+1) = -(4x+1)(9x^2-6x+1)$

$= -(4x+1)(3x-1)^2 \leq 0$ *for all* $x > 0 \therefore \frac{15x+2}{36x^3+1} \leq \frac{1}{x}$ *for* $x > 0$

$$\Rightarrow \int_\alpha^{2\alpha} \frac{15x+2}{36x^3+1} dx \leq ln\, x\Big]_\alpha^{2\alpha} = ln\, 2, \forall \alpha > 0$$

Thus $b\int_a^{2a} \frac{15x+2}{36x^3+1} dx + c\int_a^{2b} \frac{15x+2}{36x^3+1} dx + a\int_a^{2c} \frac{15x+2}{36x^3+1} dx$

$\leq (b+c+a)\, ln\, 2 = (a+b+c)\, ln\, 2$

4.143

Let $f(x) = e^{x^2} - [1 + 2a(x-a)]e^{a^2}, a \leq x \leq b, a > 0$

$$f'(x) = 2xe^{x^2} - 2ae^{a^2}$$

If $x > 0, e^{x^2} > e^{a^2} \Rightarrow 2xe^{x^2} > 2ae^{a^2} \therefore f'(x) > 0$ *for* $0 < x < b \Rightarrow$

$\Rightarrow f(x)$ *increases on* $[a,b] \Rightarrow f(x) > f(a) = 0$ *for* $a < x \leq b \Rightarrow$

$$\Rightarrow \int_a^b \left[e^{x^a} - [1 + 2a(x-a)]e^{a^2}\right] dx > 0 \Rightarrow$$

$$\Rightarrow \int_a^b e^{x^2} dx > e^{a^2}(b-a) + 2ae^{a^2}(b-a)^2$$

$$\Rightarrow \frac{1}{b-a}\int_a^b e^{x^2} dx > [1 + a(b-a)]e^{a^2} \quad (1)$$

Let $g(x) = [1 + 2b(b-x)]e^{b^2} - e^{x^2}, a \leq x \leq b$

$$g'(x) = -2be^{b^2} - 2xe^{x^2} < 0 \ [\because 0 < a \leq x < b]$$

$\Rightarrow g(x)$ *is strictly decreasing on* $[a,b]$. *As* $g(b) = e^{b^2} - e^{b^2} = 0$, *we get*

$g(x) > g(b) = 0$ *for* $a \leq x < b \Rightarrow [1 + 2b(b-x)]eb^2 > e^{x^2}$

$$\Rightarrow \int_a^b [1 + 2b(b-x)]\, e^{b^2} dx > \int_a^b e^{x^2}\, dx$$

$$\Rightarrow \int_a^b e^{x^2} dx < [x - b(b-x)^2]e^{b^2}]_a^b = [(b-a) + b(a-b)^2]e^{b^2} =$$

$$= (b-a)[1 + ab - b^2]e^{b^2}$$

$$\Rightarrow \frac{1}{b-a}\int_a^b e^{x^2} dx < (1 + ab - b^2)e^{b^2} \quad (2)$$

From (1), (2) we get the desired inequality.

4.144 $\because a \leq x \leq \frac{\pi}{2}, \therefore 0 < x \leq \frac{\pi}{2}$ $(\because a > 0)$. *By Jordan's inequality,* $\forall x \in \left(0, \frac{\pi}{2}\right]$,

$$\frac{\sin x}{x} \geq \frac{2}{\pi} \Rightarrow \pi \int_a^{\frac{\pi}{2}} \frac{\sin x}{x} dx \geq \pi \int_a^{\frac{\pi}{2}} \frac{2}{\pi} dx = 2\left(\frac{\pi}{2} - a\right) = \pi - 2a \Rightarrow LHS \geq a\pi + \pi - 2a$$

$$\overset{?}{>} \pi + a\pi - \pi \sin a - 2 - 2a + 2\sin a \Leftrightarrow \pi \sin a + 2 \underset{(1)}{\overset{?}{\geq}} 2\sin a$$

$\because 2 > 2\sin a, \therefore \pi \sin a + 2 > 2\sin a \quad (\because \sin a > 0) \Rightarrow$ *(1) is true (proved)*

4.145

$$1 \geq x \geq 0 \Rightarrow \frac{1}{\pi^a} \geq \frac{1}{(x+\pi)^a} \geq \frac{1}{(1+\pi)^a}, \frac{1}{\pi^b} \geq \frac{1}{(x+\pi)^b} \geq \frac{1}{(1+\pi)^b}$$

$$\frac{1}{\pi^c} \geq \frac{1}{(x+\pi)^c} \geq \frac{1}{(1+\pi)^c}, \Omega(a) = \lim_{n\to\infty} n \int_0^1 \frac{x^n}{(x+\pi)^a} dx \geq \frac{1}{(1+\pi)^a} \lim_{n\to\infty} n \int_0^1 x^n\, dx$$

$$= \frac{1}{(1+\pi)^a} \lim_{n\to\infty} \frac{n}{n+1} [x^{n+1}]_{x=0}^{x=1} = \frac{1}{(1+\pi)^a}$$

$$\sum_{cyc} (1+\pi)^b \Omega(a) \geq \sum_{cyc} \frac{(1+\pi)^b}{(1+\pi)^a} \overset{AM \geq GM}{\geq} 3$$

4.146

$$0 < a \leq b < \frac{\pi}{2} \text{ now } \int_a^b \frac{\cos x \cdot \sin^2(\sin x)}{\sin^2 x} dx = \int_{\sin a}^{\sin b} \frac{\sin^2 y}{y^2} dy$$

Where $1 \geq y \geq 0$, *we need to prove,* $\sin^2 y \geq \frac{y^2}{1+y^2}$ *or* $\sin^2 y + \frac{1}{1+y^2} - 1 \geq 0$

Let $f(y) = \sin^2 y + \frac{1}{1+y^2} - 1$ *for all* $1 \geq y \geq 0$, $f'(y) = \sin 2y - \frac{2y}{(1+y^2)^2}$

Now we know, $\sin 2y \geq 2y - \frac{4y^3}{3}$. *So, we will prove,* $2y - \frac{4y^2}{3} \geq \frac{2y}{(1+y^2)^2}$

$\Leftrightarrow (1+y^2)^2(3-2y^2) \geq 0 \Leftrightarrow y^2(4 - y^2 - 2y^4) \geq 0$, *which is true*

[since, $1 \geq y \geq 0 \Rightarrow 4 > 3 \geq y^2 + 2y^4$*]*

$$\therefore \sin 2y - \frac{2y}{(1+y^2)^2} \geq 0 \Rightarrow f'(y) \geq 0 \Rightarrow f(y) \geq f(0) = 0 \Rightarrow \frac{\sin^2 y}{y^2} \geq \frac{1}{1+y^2}$$

$$\Rightarrow \int_{\sin a}^{\sin b} \frac{\sin^2 y}{y^2} dy \geq \int_{\sin a}^{\sin b} \frac{dy}{1+y^2} = \tan^{-1}(\sin b) - \tan^{-1}(\sin a) = \tan^{-1}\left(\frac{\sin b - \sin a}{1 + \sin a \cdot \sin b}\right)$$

4.147

Let $f(x) = \sqrt[3]{x}$ *for all* $x \geq 0, f'(x) = \frac{x^{-\frac{2}{3}}}{3}, f''(x) = -\frac{2x^{-\frac{5}{3}}}{9} \leq 0, f$ *is a concave function,*

Applying Hermite – Hadamard,

$\frac{1}{b-a}\int_a^b \sqrt[3]{x} dx \leq \sqrt[3]{\left(\frac{a+b}{2}\right)} \leq \sqrt[3]{b}$, *since* $0 \leq a < b, \int_a^b \sqrt[3]{x}\, dx \leq (b-a)\sqrt[3]{b}$

$$\int_a^b \sin\left(\sqrt[3]{x}\right) dx \leq \int_a^b \sqrt[3]{x}\, dx \leq (b-a)\sqrt[3]{b}$$

4.148

$$F(a) = \int_0^a \frac{\cos^7 x}{\cos 6x + 6\cos 4x + 15\cos 2x + 10} dx =$$

$$= \int_0^a \frac{\cos^7 x}{4\cos^3 2x - 3\cos 2x + 6(2\cos^2 2x - 1) + 15\cos 2x + 10} dx =$$

$$= \int_0^a \frac{\cos^7 x}{4\cos^3 2x + 12\cos^2 2x + 12\cos 2x + 4} dx = \frac{1}{4}\int_0^a \frac{\cos^7 x}{\cos^3 2x + 3\cos^2 x + 3\cos 2x + 1}$$

$$= \frac{1}{4}\int_0^a \frac{\cos^7 x}{(1+\cos^2 x)^3} dx = \frac{1}{4}\int_0^a \frac{\cos^7 x}{(2\cos^2 x)^3} dx =$$

$\frac{1}{2^4}\int_0^a \cos x\, dx = \frac{1}{2^5}\sin x \Big|_0^a = \frac{1}{2^5}\sin a$ *(1)*

But $\sin a \leq a, \forall a \in \left[0, \frac{\pi}{2}\right]$ *(2). From (1)+(2)⇒* $F(a) \leq \frac{1}{2^5}a$ *(3).*

From (3) we must show:

$$\frac{1}{2^{15}} abc\left(\frac{a+b+c}{2^5}\right) \leq \frac{1}{2^{20}}(a^4 + b^4 + c^4) \Leftrightarrow$$

$$\Leftrightarrow abc(a+b+c) \leq a^4+b^4+c^4 \quad (4)$$

But $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+a^2c^2 \geq abc(a+b+c) \Rightarrow$ *(4) its true.*

4.149

Let $f(x) = \sqrt{\sin x}$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$, $f'(x) = \frac{\cos x}{\sqrt{\sin x}}; f''(x) = -\sqrt{\sin x} - \frac{\cos^2 x}{2(\sin x)^{\frac{3}{2}}} =$

$= \frac{\sin^2 x+1}{2(\sin x)^{\frac{3}{2}}} \leq 0$ *for all* $x \in \left[0, \frac{\pi}{2}\right]$, *hence* f *is concave, by Hermite – Hadamard Inequality*

$$\sqrt{\sin\left(\frac{\frac{\pi}{2}+0}{2}\right)} \cdot \frac{\pi}{2} \geq \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx \geq \frac{\pi}{2} \cdot \frac{\sqrt{\sin\frac{\pi}{2}} + \sqrt{\sin 0}}{2} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cdot x^{\frac{3}{2}} dx \overset{\substack{CAUCHY \\ SCHWARZ}}{\leq} \sqrt{\left(\int_0^{\frac{\pi}{2}} \sin x\, dx\right)\left(\int_0^{\frac{\pi}{2}} x^3\, dx\right)} = \sqrt{\left[\frac{x^4}{4}\right]_{x=0}^{x=\frac{\pi}{2}}} = \frac{\pi^2}{8}$$

$$\int_0^{\frac{\pi}{2}} x^{\frac{3}{2}} \sqrt{\sin x}\, dx \overset{\substack{CHEBYSHEV'S \\ INEQUALITY}}{\geq} \left(\frac{\pi}{2} - 0\right) \int_0^{\frac{\pi}{2}} x^{\frac{3}{2}} dx \cdot \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$$

$\geq \frac{\pi}{2} \cdot \frac{\pi^{\frac{5}{2}}}{10\sqrt{2}} \cdot \frac{\pi}{4}$, *we need to prove,* $\frac{\pi^2}{8} \cdot \frac{\pi^{\frac{5}{2}}}{10\sqrt{2}} \geq \frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \Leftrightarrow \pi^2 \geq \frac{20}{3}$, *which is true*

$$\therefore \frac{\pi^{\frac{5}{2}}}{12\sqrt{2}} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \cdot x^{\frac{3}{2}}\, dx \leq \frac{\pi^2}{8}$$

4.150

Applying Berström:

$$\int_0^{\frac{\pi}{2}} \left(\frac{9^{\sin x}}{4^{\sin x} + 5^{\sin x}} + \frac{16^{\sin x}}{3^{\sin x} + 5^{\sin x}} + \frac{25^{\sin x}}{3^{\sin x} + 4^{\sin x}}\right)^3 dx \geq$$

$$\geq \int_0^{\frac{\pi}{2}} \frac{\left(3^{\sin x}+4^{\sin x}+5^{\sin x}\right)^3}{8} dx \overset{AM\geq GM}{\geq} \int_0^{\frac{\pi}{2}} \frac{27}{8}\left(60^{\sin x}\right) dx \geq \int_0^{\frac{\pi}{2}} \frac{27}{8}(60)^{\sin x} \cos x \, dx$$

$$\geq \frac{27}{8}\left(\frac{(60)^{\sin x}}{\ln 60}\right)\Bigg|_0^{\frac{\pi}{2}} \geq \frac{27\times 60}{8\ln 60} \geq \frac{27\times 51}{8\ln 60} \geq \frac{405}{2\ln 60} \geq \frac{1593}{8\ln 60} \text{ (proved)}$$

4.151

$$\int_0^1 \left(e^x + e^{x^2} + \dots + e^{x^n}\right) dx \overset{Jensen}{\geq} \int_0^1 n \cdot e^{\frac{x+x^3+\dots+x^n}{n}} dx \overset{e^x\geq x+1}{\geq}$$

$$\geq \int_0^1 n \cdot \left(\frac{x+x^2+\dots+x^n}{n}+1\right) dx = \int_0^1 n \cdot \frac{x+\dots+x^n+n}{n} dx =$$

$$= \int_0^1 (x+x^2+\dots+x^n+n)\, dx = \left[\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^{n+1}}{n+1}+nx\right]_0^1 =$$

$$= \frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n+1}+n \geq n\cdot\sqrt[n]{\frac{1}{2}\cdot\frac{1}{3}\cdot\ldots\cdot\frac{1}{n+1}}+n = n\cdot\frac{1}{[(n+1)!]^{\frac{1}{n}}}+n$$

4.152

Let $f(a) = \frac{\ln(1+a\cos x)}{\cos x}$ *is a continuous function in* $a \Rightarrow \Omega'(a) = \int_0^{\pi} \frac{1}{1+a\cos x} dx$

$$\left.\begin{array}{c} Let\ \tan\frac{x}{2} = t \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}dt \\ x=0\Rightarrow t=0; x=\pi \Rightarrow t=\infty \end{array}\right\} \Rightarrow \Omega'(a) = \int_0^{\infty} \frac{1}{1+a\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt$$

$$= 2\int_0^{\infty} \frac{1}{1+t^2+a-at^2} dt = 2\int_0^{\infty} \frac{1}{(1-a)t^2+1+a} dt = \frac{2}{1-a}\int_0^{\infty} \frac{1}{t^2+\left(\sqrt{\frac{1+a}{1-a}}\right)^2} dt =$$

$$= \frac{2}{1-a}\cdot\frac{1}{\sqrt{\frac{1+a}{1-a}}}\arctan\frac{t}{\sqrt{\frac{1+a}{1-a}}}\Bigg|_0^{\infty} = \frac{\pi}{\sqrt{1-a^2}} \Rightarrow$$

$$\left.\begin{array}{r}\Omega(a)=\pi\int\frac{1}{\sqrt{1-a^2}}da=\pi\arcsin a+c\\ But\ \Omega(a)=0\Rightarrow c=0\end{array}\right\}\Rightarrow\Omega(a)=\pi\arcsin a\Rightarrow \textit{we must show:}$$

$\sum(\arcsin a)^2\geq\sum\arcsin a\cdot\arcsin b$, *which its true because* $\sum x^2\geq\sum xy$

4.153

$$\left.\begin{array}{r}\Omega(n)=\int\limits_{-1}^{1}x\ln(1+n^{3x})\,dx\\ x=-t|'\Rightarrow dx=-dt\ x=-1\Rightarrow t=1;\ x=1\Rightarrow t=-1\end{array}\right\}\Rightarrow$$

$$\Omega(n)=\int\limits_{1}^{-1}-t\ln(1+n^{-3t})\,(-dt)=-\int\limits_{-1}^{1}t\ln\left(1+\frac{1}{n^{3t}}\right)dt=-\int\limits_{-1}^{1}t\ln\left(\frac{n^{3t}+1}{n^{3t}}\right)dt=$$

$$=-\int\limits_{-1}^{1}t\ln(1+n^{3t})\,dt+\int\limits_{-1}^{1}t\ln n^{3t}\,dt\Rightarrow 2\Omega(n)=\int\limits_{-1}^{1}3t^2\ln dt\Rightarrow 2\Omega(n)=t^3\ln n\Big|_{-1}^{1}\Rightarrow$$

$\Rightarrow\Omega(n)=\ln n$. *We must show this:* $9\left(1+\sqrt{2}+\cdots+\sqrt{n}\right)^2>4n^2(n+1)\Leftrightarrow$

$$\Leftrightarrow 1+\sqrt{2}+\cdots+\sqrt{n}>\frac{2n\sqrt{n+1}}{3},\forall n\geq 1$$

$P(1): 1>\frac{2\sqrt{2}}{3}\Leftrightarrow 3>2\sqrt{2}$ *true.*

Now: $P(k): 1+\sqrt{2}+\cdots+\sqrt{k}>\frac{2k\sqrt{k+1}}{3}$

$$P(k+1): 1+\sqrt{2}+\cdots+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}$$

From P(k) $\Rightarrow 1+\sqrt{2}+\cdots+\sqrt{k}+\sqrt{k+1}>\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}$

We must show this: $\frac{2k\sqrt{k+1}}{3}+\sqrt{k+1}>\frac{2(k+1)\sqrt{k+2}}{3}$

$2k+3>2\sqrt{(k+1)(k+2)}\Leftrightarrow 4k^2+12k+9>4k^2+12k+8\Leftrightarrow 9>8$ *true.*

4.154

$$(1+ax-x^2)e^{x^2} \overset{(1)}{<} \frac{1}{a-x}\int_x^a e^{x^2}\,dx \overset{(2)}{<} (a^2-ax)e^{a^2}+e^{x^2}$$

Let $f(x)=e^{x^2}$*. Then* $f''(x)=(4x^2+2)e^{x^2}>0 \therefore f(x)$ *is convexe*

$\therefore$ *by Hermite – Hadamard inequality,*

$$\frac{1}{a-x}\int_x^a e^{x^2}\,dx \le \frac{e^{x^2}+e^{a^2}}{2} \overset{?}{<} (a^2-ax)e^{a^2}+e^{x^2} \Leftrightarrow e^{x^2}+e^{a^2} < 2(a^2-ax)e^{a^2}+2e^{x^2}$$

$$\Leftrightarrow e^{a^2}-e^{x^2} < 2ae^{a^2}(a-x) \Leftrightarrow \frac{e^{a^2}-e^{x^2}}{a-x} \overset{(2a)}{<} 2ae^{a^2}(\because a-x>0)$$

By Cauchy's MVT, there exists θ *satisfying* $a>\theta>x$*, such that* $\frac{e^{a^2}-e^{x^2}}{a-x}=\frac{d}{dx}\left(e^{x^2}\right)_{x=\theta}$

$\therefore \frac{e^{a^2}-e^{x^2}}{a-x}=2\theta e^{\theta^2}$*, where* $a>\theta>x<2ae^{a^2}(\because \theta<a) \Rightarrow$ *(2a) is true* $\Rightarrow$ *(2) is true.*

Also, by Hermite – Hadamard's inequality,

$$\frac{1}{a-x}\int_x^a e^{x^2}dx \ge e^{\left(\frac{a+x}{2}\right)^2} \overset{?}{>} (1+ax-x^2)e^{x^2} \Leftrightarrow \left(\frac{a+x}{2}\right)^2 \overset{(1a)}{>} ln(1+ax-x^2)+x^2$$

Now, $ln\{1+(ax-x^2)\} \overset{(i)}{\le} ax-x^2(\because ln(1+m)\le m)$

(i)$\Rightarrow$ *RHS of (1a)*$\le ax \overset{?}{<} \left(\frac{a+x}{2}\right)^2 \Leftrightarrow (a-x)^2 \overset{?}{>} 0 \rightarrow$ *true* $\therefore$ *(1a) is true* $\Rightarrow$*(1) is true (Done)*

4.155

Since f *is a convex function, we have:* $f(ta+(1-t)b)\le tf(a)+(1-t)f(b)$

For $a=0$ *and* $b=1$ *we have:* $f(1-t)\le(1-t)f(1)=3(1-t)$ *(1)*

By integrating the inequality a) side by side for t *from* $\frac{1}{2}$ *to* 1 *we have.*

$$\int_{\frac{1}{2}}^{1} f(1-t)\,dt \leq 3\int_{\frac{1}{2}}^{1}(1-t)\,dt$$

We substitute $1-t = x$ *and we have:* $\int_0^1 f(x)\,dx \leq 3\int_0^1 f(x)\,dx = \frac{3}{2}x^2\Big|_0^{\frac{1}{2}} = \frac{3}{8}$ *(2)*

By integrating the inequality (1) side by side for t *from 0 to* $\frac{1}{2}$ *we have.*

$$\int_0^1 f(1-t)\,dt \leq 3\int_0^{\frac{1}{2}}(1-t)\,dt$$

Also, by substituting $1-t = x$ *we have:* $\int_{\frac{1}{2}}^1 f(x)\,dx \leq 3\int_{\frac{1}{2}}^1 x\,dx = \frac{3}{2}x^2\Big|_{\frac{1}{2}}^{1} = \frac{9}{8}$ *(3)*

From (2) and (3) we have:

$$\int_{\frac{1}{2}}^{1} f(x)\,dx \leq \frac{9}{8} = 3\cdot\frac{3}{8} = 3\cdot\int_0^{\frac{1}{2}} f(x)\,dx \Rightarrow \int_{\frac{1}{2}}^{1} f(x)\,dx - 3\int_0^{\frac{1}{2}} f(x)dx \leq 0 \Rightarrow$$

$$\Rightarrow 3\int_{\frac{1}{3}}^1 f(x)\,dx - \int_0^1 f(x)\,dx \leq 2\int_{\frac{1}{2}}^1 f(x)\,dx + 2\int_0^{\frac{1}{2}} f(x)\,dx \quad (4)$$

Substituting (1) and (3) to (4) and we have:

$$3\int_{\frac{1}{2}}^{1} f(x)\,dx - \int_0^{\frac{1}{2}} f(x)\,dx \leq 2\int_{\frac{1}{2}}^{1} f(x) + 2\int_0^{\frac{1}{2}} f(x)\,dx \leq 2\frac{9}{8} + 2\frac{3}{8} = \frac{9}{4} + \frac{3}{4} = \frac{12}{4} = 3$$

Deductively,

$$3\int_{\frac{1}{2}}^{1} f(x)\,dx - \int_0^{\frac{1}{2}} f(x)\,dx \leq 3 \Rightarrow 3\int_{\frac{1}{2}}^{1} f(x)\,dx \leq 3 + \int_0^{\frac{1}{2}} f(x)\,dx \Rightarrow$$

$$\Rightarrow \int_{\frac{1}{2}}^{1} f(x)\,dx \le 1 + \frac{1}{3}\int_{0}^{\frac{1}{2}} f(x)\,dx$$

4.156

By Bergström inequality

$$x^4 + 2x^2 + 1 + (tan^{-1} x)^2 = (x^2 + 1)^2 + (tan^{-1} x)^2 \ge \frac{(x^2 + 1 + tan^{-1} x)^2}{2}$$

Therefore $\frac{x^2+1+tan^{-1} x}{x^4+2x^2+1+(tan^{-1} x)^2} \le 2\frac{x^2+1+tan^{-1} x}{(x^2+1+tan^{-1} x)^2} = \frac{2}{x^2+1+tan^{-1} x}$ *(1)*

Since $\frac{\pi}{4} \le x \le a \Rightarrow tan^{-1} x > -1$. *So,* $\frac{x^2+1+tan^{-1} x}{x^4+2x^2+1+\left(tan^{-1} x\right)^2} \le \frac{2}{x^2+1-1} = \frac{2}{x^2}$ *(2)*

From (2) we have: $\Omega(a) \le 2\int_{\frac{\pi}{4}}^{a} \frac{dx}{x^2} = -\frac{2}{x} = -\frac{2}{a} + \frac{2}{\frac{\pi}{4}} = -\frac{2}{a} + \frac{8}{\pi} < -\frac{2}{0} + 8$ *(3)*

From (3) we have: $\left(1 + 2\Omega(0)\right)b^2 \le \left(17 - \frac{4}{a}\right)b^2 = (17a - 4)\frac{b^2}{a}$ *(4)*

Since $a \ge \frac{\pi}{4} > 0$ *we have:* $(a + 1)^3 \ge 0 \Rightarrow$

$$a^3 + 3a^2 + 3a + 1 \ge a^3 - 2a + 3a + 1 \ge 0 \Rightarrow$$

$$\Rightarrow a^3 \ge 17a - 1 > 17a - 4 \quad (5)$$

By substituting (5) to (4) we have: $\left(1 + 2\Omega(a)\right)b^2 \le \frac{a^3}{a}b^2 = a^2b^2$. *Hence,*

$LHS \le a^2b^2 + b^2c^2 + a^2c^2$. *By Cauchy-Schwarz inequality we have:*

$LHS \le a^2b^2 + b^2c^2 + a^2c^2 \le a^4 + b^4 + c^4$ *Q.E.D.*

4.157

For $\theta \in \mathbb{R}$,

$$cos(11\theta) + cos(9\theta) = 2\,cos(10\theta)\,cos\,\theta \Rightarrow cos(11\,cos^{-1} x) + cos(9\,cos^{-1} x) =$$

$$= 2\,cos(10\,cos^{-1} x)\,cos(cos^{-1} x) = 2x\,cos(10\,cos^{-1} x)$$

$$\therefore \Omega(a) = \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)\,dx}{(x^2+a^2)[\cos(11\cos^{-1}x+\cos(9\cos^{-1}x))]} =$$

$$= \frac{8}{\pi}\int_0^a \frac{x\cos(10\cos^{-1}x)}{(x^2+a^2)2x\cos(10\cos^{-1}x)}dx = \frac{4}{\pi}\cdot\frac{1}{a}\tan^{-1}\frac{x}{a}\Big]_0^a = \frac{1}{a}\cdot\frac{4}{\pi}\cdot\frac{\pi}{4} = \frac{1}{a}$$

Now, $[\Omega(a)+\Omega(b)+\Omega(c)]\left[6+\frac{1}{\Omega^3(a)}+\frac{1}{\Omega^3(b)}+\frac{1}{\Omega^3(c)}\right] = \left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right][6+a^3+b^3+c^3] \geq$

$$\geq 3\left(\frac{1}{abc}\right)^{\frac{1}{3}} 9(1\cdot 1\cdot 1\cdot 1\cdot 1\cdot 1)^{\frac{1}{9}} = 27\left(\frac{1}{abc}\right)^{\frac{1}{3}}(abc)^{\frac{1}{3}} = 27$$

4.158

Integral form of $AM \geq GM$

Suppose $f:[a,b]\to(0,\infty)$ *be continuous function, then*

$$e^{\frac{1}{b-a}\int_a^b \ln f(x)dx} \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

Now, $e^{\frac{1}{1-0}\int_0^1 \ln f(x)\cdot g(x)\cdot h(x)dx} = e^{\int_0^1(\ln f(x)+\ln g(x)+\ln h(x))dx}$

$$= e^{\int_0^1 \ln f(x)dx+\int_0^1 \ln g(x)dx+\int_0^1 \ln h(x)dx} = \prod_{cyc}\left(\int_0^1 f(x)\,dx\right) \overset{AM\geq GM}{\leq} \frac{1}{27}\left(\sum_{cyc}\int_0^1 f(x)\,dx\right)^3$$

$$27e^{\int_0^1 \ln f(x)\cdot g(x)\cdot h(x)dx} \leq \left(\sum_{cyc}\int_0^1 f(x)\,dx\right)^3$$

4.159

According to Chebyshev's inequality

$$\left(\int_0^1 e^{1-x}\cdot\gamma^x\,dx\right)\left(\int_0^1 e^x\cdot\pi^{1-x}dx\right) < (1-0)^2\int_0^1 e^{1-x}\gamma^x e^x\pi^{1-x}\,dx = e\int_0^1 \gamma^x\pi^{1-x}\,dx$$

4.160

By Bergström's inequality:

$$\cos^4 x + \tan^2 x \geq \frac{(\cos^2 x + \tan x)^2}{2}$$

Therefore

$$\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2 \leq 4\frac{(\cos^2 x+\tan x)^2}{(\cos^2 x+\tan x)^4} = \frac{4}{(\cos^2 x+\tan x)^2} \quad (1)$$

By AM-GM inequality we have:

$$(\cos^2 x + \tan x)^2 \geq 4\cos^2 x \tan x \quad (2)$$

We substitute (2) to (1) and we get: $\left(\frac{\cos^2 x+\tan x}{\cos^4 x+\tan^2 x}\right)^2 \leq \frac{1}{\cos^2 x \tan x}$

$$\int_a^b \left(\frac{\cos^2 x + \tan x}{\cos^4 x + \tan^2 x}\right)^2 dx \leq \int_a^b \frac{dx}{\cos^2 x \tan x} = \int_a^b \frac{\frac{dx}{\cos^2 x}}{\tan x} = \int_a^b \frac{d(\tan x)}{\tan x} = \ln\left|\frac{\tan b}{\tan a}\right|$$

4.161

By Hölder's inequality we have for $g: [a, b] \to (0, \infty)$

$$\int_a^b g(x)\, dx \leq \left(\int_a^b dx\right)^{\frac{2}{7}} \left(\int_a^b \left(g(x)\right)^{\frac{7}{5}} dx\right)^{\frac{5}{7}}$$

Taking seventh power and applying this to $g(x) = \left(f(x)\right)^5$ *we get*

$$\left(\int_a^b \left(f(x)\right)^5 dx\right)^7 \leq (b-a)^2 \left(\int_a^b \left(f(x)\right)^7 dx\right)^5$$

With equality if f *is constant.*

The desired inequality follows, with no conditions on a *and* b.

4.162

Theorem: Let $f: [0,1] \to \mathbb{R}, f$ *continuos. Then:*

$$\lim_{n\to\infty} n\int_0^1 x^n f(x)dx = f(1)$$

In our case $f(x) = \frac{1}{(x+a)^2} \Rightarrow \Omega(a) = \frac{1}{(1+a)^2}$ *(1)*

From (1) we must show:

$$\frac{b^3}{(1+a)^2} + \frac{c^3}{(1+b)^2} + \frac{a^3}{(1+c)^2} \geq \frac{8}{25} \quad (2)$$

From Hölder's inequality we have:

$$\frac{b^3}{(1+a)^2} + \frac{c^3}{(1+b)^2} + \frac{a^3}{(1+c)^2} \geq \frac{(a+b+c)^3}{(a+b+c+3)^2} \quad (3)$$

We have $a + b + c = 2$ *(4)*

From (3) + (4) $\Rightarrow \frac{b^3}{(1+a)^2} + \frac{c^3}{(1+b)^2} + \frac{a^3}{(1+c)^2} \geq \frac{8}{25} \Rightarrow$ *(2) its true.*

4.163

$$\Omega(k) = \int_0^1 \frac{x^k - kx^{k-1} - x^{k-1} + 1}{x^{2k+1} + 2x^{k+1} + x^k + x + 1}dx = \int_0^1 \frac{x^k - (k+1)x^{k-1} + 1}{(x^k + 1)(x^{k+1} + x + 1)}dx$$

$$= \int_0^1 \left(\frac{(k+1)x^k + 1}{x^{k+1} + x + 1} - \frac{(k+1)x^{k-1}}{x^k + 1}\right)dx = \left(ln|x^{k+1} + x + 1| - \frac{k+1}{k}ln|x^k + 1|\right)\Big|_0^1$$

$$= ln\,3 - \frac{k+1}{k}ln\,2$$

$$\Omega = \lim_{n\to\infty}\left(ln\,2\,ln\,n - n\,ln\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\Omega(k)\right)$$

$$= \lim_{n\to\infty}\left(ln\,2\,ln\,n - n\,ln\left(\frac{3}{2}\right) + \sum_{k=1}^{n}\left(ln\,3 - \frac{k+1}{k}ln\,2\right)\right)$$

$$= ln\,2 \times \lim_{n\to\infty}\left(ln\,n + n - \sum_{k=1}^{n}\left(1 + \frac{1}{k}\right)\right) = ln\,2 \times (-\gamma) = -\gamma\,ln\,2$$

4.164

$$f(x)=\frac{x\sec x\,(1+\sin x)-(1+\cos x+\sin x)}{x+\sin x-\cos x-1}+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$=\frac{\sec x\left(x(1+\sin x)-(\cos x+\cos^2 x+\sin x\cos x)\right)}{x+\sin x-\cos x-1}+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$=\frac{\sec x\left(x(1+\sin x)-(\cos x+1-\sin^2 x+\sin x\cos x)\right)}{x+\sin x-\cos x-1}+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$=\frac{\sec x\left(x(1+\sin x)+(\sin x-\cos x-1)(1+\sin x)\right)}{x+\sin x-\cos x-1}+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$=\frac{\sec x\,(1+\sin x)(x+\sin x-\cos x-1)}{x+\sin x-\cos x-1}+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$=\sec x+\tan x+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}$$

$$\therefore I=\int\left(\sec x+\tan x+\frac{1+\cos x+\sin x}{x+\sin x-\cos x-1}\right)dx$$

$$=\ln|\tan x+\sec x|-\ln|\cos x|+\ln|x+\sin x-\cos x-1|+C$$

4.165

$f:\mathbb{R}\Rightarrow(0,\infty)$, *continuous, we have*

$$3\left(f^4(x)+f^4(y)+f^4(z)\right)^5\geq\left(f^4(x)+f(x)f^3(y)+f^5(x)f^3(z)\right)^5+$$

$$+\left(f(y)f^3(x)+f^4(y)+f(y)f^3(z)\right)^5+\left(f(z)f^3(x)+f(z)f^3(y)+f^4(z)\right)^5$$

$$=\left(f^5(x)+f^5(y)+f^5(z)\right)\left(f^3(x)+f^3(y)+f^3(z)\right)^5$$

Hence $\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^{3(z)}}\right)^5\geq\frac{f^5(x)+f^5(y)+f^5(z)}{3}$

Hence $\int_0^a\int_0^a\int_0^a\left(\frac{f^4(x)+f^4(y)+f^4(z)}{f^3(x)+f^3(y)+f^3(z)}\right)^5 dx\,dy\,dz\geq\frac{1}{3}\int_0^a\int_0^a\int_0^a\left(f^5(x)+f^5(y)+f^5(z)\right)dx\,dy\,dz=$

$$=a^2\int_0^a f^5(x)\,dx$$

4.166

$$\int \frac{x^6 \ln x}{(3+x^7)^5} dx$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} + \frac{1}{28}\int \frac{1}{x(3+x^7)^6} dx \begin{pmatrix} u = \ln x & dv = \frac{x^6}{(3+x^7)^5} dx \\ du = \frac{1}{x} dx & v = -\frac{1}{28}\cdot\frac{1}{(3+x^7)^6} \end{pmatrix}$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} + \underbrace{\frac{1}{28}\int \frac{x^{-35}}{(3x^{-7}+1)^6}\cdot x^{-8} dx}_{let\ y=3x^{-7}+1 \Rightarrow x^{-7}=\frac{1}{3}(y-1)}$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} + \frac{1}{28}\int \frac{\frac{1}{3^5}(y-1)^5}{y^6}\cdot\left(-\frac{1}{21} dy\right)$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} - \frac{1}{142884}\int \left(\frac{1}{y} - \frac{5}{y^2} + \frac{10}{y^3} - \frac{10}{y^4} + \frac{5}{y^5} - \frac{1}{y^6}\right) dy$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} - \frac{1}{142884}\left(\ln|y| + \frac{5}{y} - \frac{5}{y^2} + \frac{10}{3y^3} - \frac{5}{4y^4} + \frac{1}{5y^5}\right) + C$$

$$= -\frac{1}{28}\cdot\frac{\ln x}{(3+x^7)^6} - \frac{1}{142884}\left(\ln|3x^{-7}+1| + \frac{5}{3x^{-7}+1} - \frac{5}{(3x^{-7}+1)^2} + \frac{10}{3(3x^{-7}+1)^3} - \frac{5}{4(3x^{-7}+1)^4} + \frac{1}{5(3x^{-7}+1)^5}\right) + C$$

4.167

$$e^{2x} + (\cos x - \sin x + 2)e^x + \cos x - \sin x \cos x - \sin x + 1$$

$$= e^{2x} + [(1+\cos x) + (1-\sin x)] + (1+\cos x)(1-\sin x)$$

$$= (e^x + 1 + \cos x)(e^x + 1 - \sin x)$$

Also,

$$(e^x + \cos x + 1)(e^x - \cos x) - (e^x - \sin x)(e^x - \sin x + 1)$$

$$= e^{2x} - \cos^2 x + e^x - \cos x - [e^{2x} - 2e^x \sin x + \sin^2 x + e^x - \sin x]$$

$$= 2e^x \sin x + \sin x - \cos x - 1 = (2e^x + 1)\sin x - \cos x - 1 = \text{Numerator}$$

Thus,

$$I = \int \left[\frac{e^x - \cos x}{e^x + 1 - \sin x} - \frac{e^x - \sin x}{e^x + 1 + \cos x}\right] dx = \ln\left(\frac{e^x + 1 - \sin x}{e^x + 1 + \cos x}\right) + c$$

4.168

Put $e^x = t$

$I = \int \frac{\ln(1+t) - t}{[(1+t)\ln(1+t)]^2} dt = I_1 - I_2$ *where:* $I_1 = \int \frac{\ln(1+t)+1}{[(1+t)\ln(1+t)]^2} dt$

Put $(1+t)\ln(1+t) = u \Rightarrow (1 + \ln(1+t))dt = du$

$$I_1 = \int \frac{du}{u^2} = -\frac{1}{u}$$

$$= -\frac{1}{(1+t)\ln(1+t)} = -\frac{1}{(1+e^x)\ln(1+e^x)}$$

$$I_2 = \int \frac{t+1}{(t+1)^2(\ln(t+1))^2} dt$$

Put $\ln(1+t) = v$

$$\frac{1}{1+t} dt = dv$$

$$I_2 = \int \frac{dv}{v^2} = -\frac{1}{v}$$

$$= -\frac{1}{\ln(1+t)} = -\frac{1}{\ln(1+e^x)}$$

$$I = \frac{1}{\ln(1+e^x)} - \frac{1}{(1+e^x)\ln(1+e^x)} + C = \frac{e^x}{[\ln(1+e^x)](1+e^x)} + C$$

4.169

Let $A = \begin{pmatrix} a & bc & 1 \\ 1 & cb & a \end{pmatrix}, A^T = \begin{pmatrix} a & 1 \\ b & c \\ c & b \\ 1 & a \end{pmatrix}$*, then*

$$A \cdot A^T = \begin{pmatrix} 1 + \sum_{cyc} a^2 & 2(a + bc) \\ 2(a + bc) & 1 + \sum_{cyc} a^2 \end{pmatrix}$$

Applying Cauchy – Binet $\det(A \cdot A^T) \geq 0 \Rightarrow 1 + \sum_{cyc} a^2 \geq 2(a + bc)$

putting $b = \sqrt{\cos x}$, $c = \frac{1}{x}$ *and* $a = e^{\frac{\pi}{12}}$, *then*

$\left(e^{\frac{\pi}{12}} - 1\right)^2 + \cos x + \frac{1}{x^2} \geq 2\frac{\sqrt{\cos x}}{x}$, *then integrating both sides*

$$\frac{\pi}{12}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos x\, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{x^2} \geq 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx \Rightarrow \frac{\pi}{24}\left(e^{\frac{\pi}{12}} - 1\right)^2 + \frac{\sqrt{2} - 1}{4} + \frac{152}{\pi^3}$$

$$\geq \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sqrt{\cos x}}{x} dx$$

4.170

For $0 < x < 1$,

$$1 - x^2 + x^{2015} - x^{2016} = (1 - x)[1 + x + x^{2015}] > 0$$

Also, $x^2 - x^{2015} + x^{2016} = x^2 + x^{2016} - x^{2015} \geq 2(x^{1009}) - x^{2015}$

$$= x^{1009} + x^{1009} - x^{2015} > 0 \Rightarrow 1 - x^2 + x^{2015} - x^{2016} < 1$$

$$0 < 1 - x^2 + x^{2015} - x^{2016} < 1 \quad (1)$$

Also, $x^{2015} - x^{2016} = x^{2015}(1 - x) > 0$ *for* $0 < x < 1$

$$\Rightarrow 1 - x^2 + x^{2015} - x^{2016} > 1 - x^2, \quad (2) \quad 0 < x < 1$$

From (1) and (2) for $0 < x < 1$, $1 - x^2 < 1 - x^2 + x^{2015} - x^{2016} < 1$

$$\Rightarrow 1 < \frac{1}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{1}{\sqrt{1 - x^2}}$$

$$1 < \int_0^1 \frac{dx}{\sqrt{1 - x^2 + x^{2015} - x^{2016}}} < \frac{\pi}{2}$$

CHAPTER 10

ADVANCED CALCULUS-SOLUTIONS

5.1.

WLOG we suppose that $x \geq y \geq z$*. Then*

$$\frac{1}{1+\sqrt{xy}} \leq \frac{1}{1+\sqrt{y^2}} = \frac{1}{1+y}, 0 < a \leq y \leq b$$

$$\frac{1}{1+\sqrt{yz}} \leq \frac{1}{1+\sqrt{z^2}} = \frac{1}{1+z}, 0 < a \leq z \leq b$$

$$\frac{1}{1+\sqrt{zx}} \leq \frac{1}{1+\sqrt{z^2}} = \frac{1}{1+z}, 0 < a \leq z < b$$

By summing the above inequalities we get:

$$\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}} \leq \frac{1}{1+y} + \frac{2}{1+z}$$

Hence,

$$\int_a^b\int_a^b\int_a^b \left(\frac{1}{1+\sqrt{xy}} + \frac{1}{1+\sqrt{yz}} + \frac{1}{1+\sqrt{zx}}\right) dx\,dy\,dz \leq \int_a^b\int_a^b\int_a^b \left(\frac{1}{1+y} + \frac{2}{1+z}\right) dx\,dy\,dz =$$

$$= \int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+y} + 2\int_a^b\int_a^b\int_a^b \frac{dx\,dy\,dz}{1+z} =$$

$$= (b-a)^2 \log(1+y)\Big|_0^b + 2(b-a)^2 \log(1+z)\Big|_0^b =$$

$$= (b-a)^2 \log\left(\frac{1+b}{1+a}\right) + 2(b-a)^2 \log\left(\frac{1+b}{1+a}\right) = 3(b-a)^2 \log\left(\frac{1+b}{1+a}\right)$$

q.e.d.

5.2

$$I = \int_0^\infty \frac{1-\cos x}{8-4x\sin x + x^2(1-\cos x)} dx = \int_0^\infty \frac{2\sin^2\left(\frac{x}{2}\right)}{8-4x\sin x + 2\left(x\sin\left(\frac{x}{2}\right)\right)^2} dx$$

Setting $\frac{x}{2} = t$

$$I = \frac{1}{2}\int_0^\infty \frac{\sin^2 t}{1 - t\sin(2t) + (t\sin t)^2} dt$$

noticing that $(\cos t - t\sin t)^2 = \cos^2 t - 2t\sin t\cos t + (t\sin t)^2$

$$I = \frac{1}{2}\int_0^{\infty} \frac{\sin^2 t}{\sin^2 t + (\cos t - t\sin t)^2}dt = \frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{1 + (x - \cot x)^2}$$

we have

$$\frac{\sin x}{x} = \lim_{n\to\infty}\prod_{k=1}^{n}\left(1 - \frac{x^2}{(k\pi)^2}\right) \to \ln\left(\frac{\sin x}{x}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(1 - \frac{x^2}{(k\pi)^2}\right)$$

Differentiating with respect to x *gives:*

$$\cot x - \frac{1}{x} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2x}{x^2 - (k\pi)^2} = \lim_{n\to\infty}\left(\frac{1}{x+\pi} + \frac{1}{x-\pi} + \cdots + \frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right)$$

$$x - \cot x = \lim_{n\to\infty}\left(x - \left(\frac{1}{x} + \frac{1}{x+\pi} + \frac{1}{x-\pi} + \cdots + \frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right)\right)$$

Since the above series converges as $n \to \infty$ *and by Glasser's Master Theorem (see: "A Remarkable Property of Definite Integrals" By M.L. Glasser) we have that:*

$$\int_{-\infty}^{\infty} f(x - \cot x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

It follows that:

$$I = \frac{1}{4}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \frac{\pi}{4}$$

5.3

$$\lim_{n\to\infty} 6\sum_{k=1}^{n}\frac{(n-k+1)^2(k)}{kn((n+1)(2n+1)} = \lim_{n\to\infty}\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n}\frac{(n-k+1)^2 f(k)}{k}$$

Now,

$a \le f(k) \le b$,

And,

$$\lim_{n\to\infty}\frac{6}{n(n+1)(2n+1)}\sum_{k=1}^{n}\frac{(n-k+1)^2 a}{k}$$

$$= \lim_{n\to\infty}\frac{6a}{n(n+1)(2n+1)}\cdot\sum_{k=1}^{n}\frac{n^2 + k^2 + 1 - 2k - 2nk + 2n}{k}$$

$$= \lim_{n\to\infty} \frac{6a}{n(n+1)(2n+1)} \cdot \left[(n+1)^2 H_n + \frac{n(n+1)}{2} - 2n(n+1)\right]$$

$$= \lim_{n\to\infty} \frac{6a}{n(n+1)(2n+1)} \left((n+1)^2 H_n - \frac{3n(n+1)}{2}\right)$$

$$= a \lim_{n\to\infty} 6\left(\frac{(n+1)H_n}{n(2n+1)} - \frac{3}{2(2n+1)}\right) = 6a\left(\frac{1}{2}\lim_{n\to\infty} \frac{H_n}{n}\right)$$

$$= 3a\, \lim_{n\to\infty} \frac{H_n}{n} = 0 \text{ (Cauchy first theorem)}$$

Similarly,

$$\lim_{n\to\infty} \frac{6b}{n(n+1)(2n+1)} \sum_{k=1}^{n} \frac{(n-k+1)^2}{k} = 0$$

Thus by squezze theorem, the given limit is 0

5.4.

$$\sum_{n=0}^{\infty} \frac{1}{16^n (2n+1)^3} \binom{2n}{n} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{16^n} \binom{2n}{n} \left(\int_0^1 x^{2n} \ln^2 x \, dx\right) =$$

$$= \frac{1}{2} \ln^2 x \left[\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \left(\frac{x^2}{4}\right)^n\right] dx = \frac{1}{2} \int_0^1 \frac{\ln^2 x}{\sqrt{1 - \frac{x^2}{4}}} dx \left[x \to 2\sin\frac{x}{2}\right] = \frac{1}{2} \int_0^{\frac{\pi}{3}} \ln^2\left(2\sin\frac{x}{2}\right) dx$$

$$\therefore \sum_{n=0}^{\infty} \frac{1}{16^n (2n+1)^3} \binom{2n}{n} = \frac{7\pi^3}{216}$$

5.5.

$$\Omega = \int_0^{\infty} \frac{x^2 + 2}{x^6 + 1} dx = \underbrace{\int_0^{\infty} \frac{x^2}{x^6 + 1} dx}_{I_1} + \underbrace{2\int_0^{\infty} \frac{1}{x^6 + 1} dx}_{I_2}$$

$$I_1 = \int_0^{\infty} \frac{x^2}{x^6 + 1} dx$$

$$x^6 = y \Rightarrow dx = \frac{1}{6} y^{-\frac{5}{6}} dy$$

$$I_1 = \int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{1}{6} \int_0^{\infty} \frac{y^{-\frac{1}{2}}}{y + 1} dy$$

$$\frac{1}{y + 1} = 1 - t \Rightarrow dy = \frac{1}{(1 - t)^2} dt$$

$$I_1 = \frac{1}{6}\int_0^{\infty} t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \frac{1}{6}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{1}{6}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$I_2 = 2\int_0^{\infty} \frac{1}{x^6+1}dx = 2\left[\frac{\pi}{6\,sin\left(\frac{\pi}{6}\right)}\right] = \frac{2\pi}{3}$$

$$\Omega = \int_0^{\infty} \frac{x^2+2}{x^6+1}dx = I_1 + I_2 = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$$

5.6

$$tan\,x + tan\left(x+\frac{\pi}{3}\right) + tan\left(x+\frac{2\pi}{3}\right) = tan\,x + tan\left(x+\frac{\pi}{3}\right) + tan\left\{\pi - \left(\frac{\pi}{3} - x\right)\right\}$$

$$= tan\,x + tan\left(x+\frac{\pi}{3}\right) - tan\left(\frac{\pi}{3} - x\right) = tan\,x + \frac{tan\,x+\sqrt{3}}{1-\sqrt{3}\,tan\,x} - \frac{\sqrt{3}-tan\,x}{1+\sqrt{3}\,tan\,x}$$

$$= tan\,x + \frac{\left(\sqrt{3}+tan\,x\right)\left(1+\sqrt{3}\,tan\,x\right) - \left(\sqrt{3}-tan\,x\right)\left(1-\sqrt{3}\,tan\,x\right)}{1-3\,tan^2 x}$$

$$= \frac{tan\,x - 3\,tan^3 x + 8\,tan\,x}{1-3\,tan^2 x} = 3\,tan\,3x$$

$$\therefore \Omega = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}}\int_{\frac{\pi}{18}}^{\frac{\pi}{12}} \frac{3\,tan\,3x}{tan\,3x\,tan\,3y}dx\,dy = \int_{\frac{\pi}{18}}^{\frac{\pi}{12}} 3\left(\frac{\pi}{36}\right)cot\,3y\,dy$$

$$= \frac{\pi}{36}log|sin(3y)|\Big]_{\frac{\pi}{18}}^{\frac{\pi}{12}} = \frac{\pi}{36}\left\{log\left(\frac{1}{\sqrt{2}}\right) - log\left(\frac{1}{2}\right)\right\} = \frac{\pi}{36}log\left(\sqrt{2}\right) = \frac{\pi}{72}log\,2$$

5.7

$$\Omega = \int_1^2\left(\int_1^2\left(\int_1^2\left(\frac{x^5+y^5+z^5-(x+y+z)^5}{x^3+y^3+z^3-(x+y+z)^3}\right)dx\right)dy\right)dz =$$

$$= \int_1^2\left(\int_1^2\left(\int_1^2\left(\frac{-5(y+z)(z+x)(x+y)(x^2+y^2+z^2+xy+yz+zx)}{-3(y+z)(z+x)(x+y)}\right)dx\right)dy\right)dz =$$

$$= \frac{5}{3}\int_1^2 \left(\int_1^2 \left(\int_1^2 (x^2 + y^2 + z^2 + xy + yz + zx)dx\right)dy\right)dz =$$

$$= \frac{5}{3}\int_1^2 \left(\int_1^2 \left(y^2 + z^2 + \frac{7}{3} + \frac{3y}{2} + \frac{3z}{2} + yz\right)dy\right)dz =$$

$$= \frac{5}{3}\int_1^2 \left(\frac{7}{3} + z^2 + \frac{7}{3} + \frac{9}{4} + \frac{3z}{2} + \frac{3z}{2}\right)dz = \frac{5}{3}\int_1^2 \left(z^2 + 3z + \frac{83}{12}\right)dz = \frac{275}{12}$$

5.8

$$\because \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = e^{-x}$$

$$\therefore \int_0^\infty e^{-x}x^{z-1}\,dx = \lim_{n\to\infty}\int_0^n \left(1 - \frac{x}{n}\right)^n x^{z-1}\,dx$$

$$Let\ x = ns \Rightarrow dx = nds$$

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{z-1}dx = \int_0^1 (1-s)^n\,(ns)^{z-1}ds = n^z\int_0^1 (1-s)^n\,s^{z-1}ds$$

$$= n^2\left[\frac{s^z(1-s)^n}{z}\right]_0^1 + \frac{n^z n(n-1)}{z(z+1)}\int_0^1 (1-s)^{n-z}\,s^{z+1}ds$$

$$= \frac{n^z n(n-1)\dots 3\cdot 2\cdot 1}{z(z+1)(z+2)\dots(z+n-1)}\int_0^1 s^{z+n-1}\,ds = \frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)}$$

$$\therefore \lim_{n\to\infty}\int_0^n \left(1 - \frac{x}{n}\right)^n x^{z-1} = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)}$$

$$\because \Gamma(z) = (z-1)!$$

$$\because (z-1)! = \frac{z!}{z} = \frac{1}{z}\cdot\frac{1\cdot 2\cdot 3\cdot \dots \cdot z(z+1)(z+2)\dots(z+n)}{(z+1)(z+2)\dots(z+n)}$$

$$= \frac{n!\,(n+1)(n+2)\dots(n+z)}{z(z+1)(z+2)\dots(z+2)} = \frac{n!\,n^z\cdot\frac{(n+1)}{n}\cdot\frac{(n+2)}{n}\dots\frac{(n+z)}{n}}{z(z+1)(z+2)\dots(z+n)}, n\to\infty$$

$$\therefore \lim_{n\to\infty}\frac{n!\,n^z\,\frac{(n+1)}{n}\cdot\frac{(n+2)}{n}\dots\frac{(n+z)}{n}}{z(z+1)(z+2)\dots(z+n)} = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)}$$

$$\therefore \Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)}$$

$$\therefore \int_0^\infty e^{-x}x^{z-1}\,dx = \lim_{n\to\infty}\int_0^n \left(1-\frac{x}{n}\right)x^{z-1}\,dx = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)(z+2)\dots(z+n)} = \Gamma(z)$$

$$\therefore \Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx\,, Re\,(z) > 0$$

5.9

$$(x+y)^3 - x^3 - y^3 = 3x^2y + 3xy^2 = 3xy(x+y)$$

$$(x+y)^5 - x^5 - y^5 = 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4$$

$$= 5xy(x^3 + 2x^2y + 2xy^2 + y^3) = 5xy[x^3 + y^3 + 2xy(x+y)]$$

$$= 5xy(x+y)(x^2 + xy + y^2)$$

$$(x+y)^7 - x^7 - y^7 = 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6$$

$$= 7xy[(x^5 + y^5) + 3xy(x^3 + y^3) + 5x^2y^2(x+y)]$$

$$= 7xy(x+y)[x^4 - x^3y + x^2y^2 - xy^3 + y^4 + 3xy(x^2 + y^2 - xy) + 5x^2y^2]$$

$$= 7xy(x+y)[x^4 + 2x^3y + 3x^2y^2 + 3xy^3 + y^4]$$

$$= 7xy(x+y)(x^2 + xy + y^2)^2$$

$$\Omega = \int_1^2 dx\left[\int_1^2 \frac{[-3xy(x+y)][-7xy(x+y)(x^2+xy+y^2)^2]}{25x^2y^2(x+y)^2(x^2+yx+y^2)^2}\right]dy = \int_1^2\left(\int_1^2 \frac{21}{25}dy\right)dx = \frac{21}{25}$$

5.10

Numerator

$$= x^2y^2(x^3+y^3) - xy(x^4+y^4) + x^5 + y^5 + xy(x+y) + 2(x+y) - (x+y)^2 - 2$$

$$= x^5(y^2 - y + 1) + y^5(x^2 - x + 1) + x(y^2 - y + 1) + y(x^2 - x + 1) -$$

$$-(x^2 - x + 1) - (y^2 - y + 1)$$

$$= (x^5 + x - 1)(y^2 - y + 1) + (y^5 + y - 1)(x^2 - x + 1)$$

$$= (x^3 + x^2 - 1)(x^2 - x + 1)(y^2 - y + 1) + (y^3 + y^2 - 1)(y^2 - y + 1)(x^2 - x + 1)$$

$$= (x^3 + y^3 + x^2 + y^2 - 2)(x^2 - x + 1)(y^2 - y + 1)$$

Thus,

$$\Omega = \int_0^1\int_0^1 (x^3 + y^3 + x^2 + y^2 - 2)\,dxdy = \int_0^1 \left[\frac{1}{4}x^4 + xy^3 + \frac{1}{3}x^3 + xy^2 - 2x\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{1}{4} + y^3 + \frac{1}{3} + y^2 - 2\right] dy = \left(\frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{17}{12}y\right)\Big]_0^1$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{17}{12} = \frac{3 + 4 - 17}{12} = -\frac{5}{6}$$

5.11

Let $I_n = \int_0^\infty \frac{(n-1)e^{nx} - ne^{(n-1)x} + 1}{xe^{nx}(e^x - 1)} dx$

For $n = 1, I_n = 0$. *Let* $n > 1$.

$$(n-1)e^{nx} - ne^{(n-1)x} + 1$$

$$= (e^x - 1)\left[(n-1)e^{(n-1)x} - e^{(n-2)x} - e^{(n-3)x} - \cdots - e^{2x} - e^x - 1\right]$$

Thus

$$I_n = \int_0^\infty \frac{(n-2)e^{(n-1)x} - \sum_{k=0}^{n-2} e^{kx}}{xe^{nx}} dx = \int_0^\infty \left[\sum_{k=0}^{(n-1)} \left(\frac{e^{(n-1)x} - e^{kx}}{xe^{nx}}\right)\right] dx =$$

$$= \sum_{k=0}^{n+1} \int_0^\infty \frac{e^{-x} - e^{-(n-k)x}}{x} dx \quad (1)$$

We now use, if f *is continuous for* $x \geq 0$ *and* $lim_{n\to\infty} f(x) = l$, *then for* $a, b > 0$,

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(0) - l)\, ln\left(\frac{b}{a}\right)$$

Let $f(x) = e^{-x}$, *then* $f(0) = 1, l = 0$,

$$\therefore \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx = ln\left(\frac{b}{a}\right) \quad (2)$$

Thus, from (1), (2)

$$I_n = \sum_{k=0}^{n-1} ln\left(\frac{n-k}{1}\right) = ln(n!)$$

5.12

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + cos\,x} dx = \int_0^{\frac{\pi}{2}} (1 + cos\,x)^{-1} dx = \int_0^{\frac{\pi}{2}} \left[\sum_{\gamma}^{\infty} (cos\,x)^{2\gamma} - \sum_{\gamma=0}^{\infty} (cos\,x)^{2\gamma+1}\right] dx$$

$$= \frac{\pi}{2} + \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

...

$$-\left(1+\frac{2}{3}+\frac{4}{5}\cdot\frac{2}{3}+\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot\ldots\right)$$

$\therefore$ *Given sum* $= 1$

5.13

$$\Omega = \prod_{n=0}^{\infty}\left[1+\left(\frac{1}{e}\right)^{3^n}+\left(\frac{1}{e}\right)^{2\cdot 3^n}\right]$$

Note, $1 + x_n + x_n^2 = \frac{1-x_n^3}{1-x_n}$

$$\therefore \Omega = \prod_{n=0}^{\infty}\frac{\left(1-e^{-3^{n+1}}\right)}{\left(1-e^{-3^n}\right)} = \frac{1-e^{-3}}{1-e^{-1}} \times \frac{1-e^{-9}}{1-e^{-3}} \quad (1)$$

$[\because e^{-\infty} = 0]$

$$= \frac{e}{e-1}$$

5.14

$$I_n = \int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)^n} = \int_0^{\infty}\frac{x^2dx}{\left(x^2+\frac{1}{4}\right)^{n+1}} + \int_0^{\infty}\frac{\frac{1}{4}dx}{\left(x^2+\frac{1}{4}\right)^{n+1}}$$

$$= \left[-\frac{x}{2n\left(x^2+\frac{1}{4}\right)^n}\right] + \int_0^{\infty}\frac{\frac{1}{4}dx}{2n\left(x^2+\frac{1}{4}\right)^n} + I_{n+1} = \frac{1}{2n}I_n + \frac{I_{n+1}}{4}$$

$$\to \frac{I_{n+1}}{I_n} = 2\cdot\frac{2n-1}{n} \Rightarrow \prod_{k=1}^{n}\frac{I_{k+1}}{I_k} = \prod_{k=1}^{n}\frac{2k-1}{k}\times\frac{2k}{k} \Leftrightarrow \frac{I_{n+1}}{I_1} = \frac{(2n)!}{(n!)^2}$$

$$I_1 = \int_0^{\infty}\frac{dx}{\left(x^2+\frac{1}{4}\right)} = [2a\tan 2x]_0^{\infty} = \pi \to I_{n+1} = \pi\frac{(2n)!}{(n!)^2}$$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{I_{n+1}} = \lim_{n\to\infty}\sqrt[n]{\prod\frac{(2n)!}{2^{2n}(n!)^2}} = \lim_{n\to\infty}\frac{\pi\frac{(2n+2)!}{\left((n+1)!\right)^2}}{\pi\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2} = 4$$

5.15

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$, *when* $x = 0, \theta = 0; x = 1, \theta = \frac{\pi}{2}$

$$\Omega = \lim_{n\to\infty}\sqrt[n]{\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx} = \lim_{n\to\infty}\sqrt[n]{\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\cos^{2n+2}\theta\, d\theta}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\beta(n,n+1)} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\cdot\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}} \overset{CAUCHY-D'ALEMBERT}{=}$$

$$= \lim_{n\to\infty}\left(\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+3)}\cdot\frac{\Gamma(2n+1)}{\Gamma(n)\Gamma(n+1)}\right) = \lim_{n\to\infty}\left(\frac{n(n+1)}{(2n+1)(2n+2)}\right) = \frac{1}{4}$$

5.16

$$\Rightarrow \int_0^\infty \frac{xe^{-x}\,ln^2(x)}{1-e^{-x}}\,dx \Rightarrow \sum_{n=0}^{\infty}\int_0^\infty x\,ln^2(x)\,e^{-(n+1)x}\,dx$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{\partial^2}{\partial p^2}\left[\int_0^\infty x^p e^{-(n+1)x}\,dx\right]_{p=1} \Rightarrow \sum_{n=0}^{\infty}\frac{\partial^2}{\partial p^2}\left[\frac{\Gamma(p+1)}{(n+1)^{p+1}}\right]_{p=1} \Rightarrow \sum_{n=1}^{\infty}\frac{\partial^2}{\partial p^2}\left[\frac{\Gamma(p+1)}{n^{p+1}}\right]_{p=1}$$

$$\Rightarrow \frac{\partial^2}{\partial p^2}[\Gamma(p+1)\tau(p+1)]_{p=1} \Rightarrow \frac{\partial}{\partial p}[\Gamma(p+1)\tau'(p+1) + \tau(p+1)\Gamma'(p+1)]_{p=1}$$

$$\Rightarrow [\Gamma(p+1)\tau''(p+1) + \tau'(p+1)\Gamma'(p+1) + \Gamma'(p+1)\tau'(p+1) + \tau(p+1)\Gamma''(p+1)]_{p=1}$$

$$\Rightarrow \Gamma(2)\tau''(2) + \tau'(2)\Gamma'(2) + \Gamma'(2)\tau'(2) + \tau(2)\Gamma''(2)$$

$$\Rightarrow \tau''(2) + \frac{\pi^2}{3}(1-\gamma)[\gamma + ln(2\pi) - 12\,ln(A)] + \frac{\pi^2}{6}\left[(1-\gamma)^2 + \frac{\pi^2}{6} - 1\right]$$

$$\Rightarrow \Gamma''(2) + \left(\frac{\pi^2}{3} - \frac{\pi^2}{3}\gamma\right)[\gamma + ln(2\pi) - 12\,ln(A)] + \frac{\pi^2}{6}\left[1 + \gamma^2 + \gamma + \frac{\pi^2}{6} - 1\right]$$

$$\Rightarrow \tau''(2) + \frac{\pi^2}{3}\gamma + \frac{\pi^2}{3}ln(2\pi) - 4\pi^2\,ln(A) - \frac{\pi^2}{3}\gamma^2 - \frac{\pi^2}{3}\gamma\,ln(2\pi) + 4\pi^2\gamma\,ln(A) +$$

$$+ \frac{\pi^2}{6}\gamma^2 - \frac{\pi^2}{3}\gamma + \frac{\pi^4}{36}$$

(OR)

$$I = \tau''(2) + \frac{\pi^2}{3}ln(2\pi)\,[1-\gamma] + \frac{\pi^4}{36} - \frac{\pi^2}{6}\gamma^2 - 4\pi^2\,ln(A)\,[1-\gamma]$$

5.17

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} ln(sin(x+y))\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}+x} ln(sin(u))\,du\,dx$$

$$= \left[x\int_{x}^{\frac{\pi}{2}+x} ln(sin(u))\,du\right]_{x}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} x\left[ln\left(sin\left(\frac{\pi}{2}+x\right)\right) - ln(sin(x))\right]dx$$

$$= \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi} ln(sin(u))\,du - \int_{0}^{\frac{\pi}{2}} x[ln(cot(x))]\,dx$$

$$I_1 = \int_{\frac{\pi}{2}}^{\pi} ln(sin(u))\,du\Bigg]_{\pi-u=z} = \int_{0}^{\frac{\pi}{2}} ln(sin(z))\,dz = -\frac{\pi}{2}ln\,2$$

$$I_2 = \int_{0}^{\frac{\pi}{2}} x[ln(cot(x))]\,dx = \int_{0}^{\frac{\pi}{2}} \left(\frac{\pi}{2}-x\right)\int_{0}^{\frac{\pi}{2}} x[ln(tan(x))]\,dx$$

$$= \frac{\pi}{2}\int_{0}^{\frac{\pi}{2}} ln(tan(x))\,dx - \int_{0}^{\frac{\pi}{2}} x[ln(tan(x))]\,dx = \int_{0}^{\frac{\pi}{2}} x[ln(tan(x))]\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} x\left[\sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k}\right)cos(2kx)\right]dx = \sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k}\right)\int_{0}^{\frac{\pi}{2}} x\,cos(2kx)\,dx$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k}\right)\left[\left[\frac{x\,sin(2kx)}{2k}\right]_{0}^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} \frac{sin(2kx)}{2k}\,dx\right]$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k}\right)\left[-\frac{1}{2k}\int_{0}^{\frac{\pi}{2}} sin(2kx)\,dx\right]$$

$$= \sum_{k=1}^{\infty}\left(\frac{(-1)^k-1}{k}\right)\left[\frac{cos(\pi k)-1}{4k^2}\right] = \sum_{k=1}^{\infty}\frac{((-1)^k-1)^2}{4k^3}$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}\frac{2-2(-1)^k}{k^3} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k^3} = \frac{1}{2}\left[\sum_{k=1}^{\infty}\frac{1}{k^3} + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}\right]$$

$$= \frac{1}{2}\left[\zeta(3) + \left(1-\frac{1}{4}\right)\zeta(3)\right] = \frac{7}{8}\zeta(3)$$

$$I = \frac{\pi}{2}\left(-\frac{\pi}{2}\ln 2\right) - \frac{7}{8}\zeta(3) = -\frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3)$$

5.18

$$I = \int_0^1\int_0^1 \left\{\frac{x}{y}\right\}^n dx\,dy = \int_0^1\int_x^\infty \frac{x\{u\}^n}{u^2}du\,dx = \left[\frac{x^2}{2}\int_x^\infty \frac{\{u\}^n}{u^2}du\right]_0^1 + \frac{1}{2}\int_0^1 \{x\}^n dx =$$

$$= \frac{1}{2}\int_1^\infty \frac{\{u\}^n}{u^2}du + \frac{1}{2}\int_0^1 x^n\,dx = \frac{1}{2}\sum_{l\ge1}^{l+1}\frac{(u-1)^n}{u^2}du + \frac{1}{2(n+1)}$$

$$I_1 = \sum_{l\ge1}\int_l^{l+1}\frac{(u-l)^n}{u^2}du = \sum_{l\ge1}\int_0^1\frac{u^n}{(u+l)^2}du = \int_0^1 u^n\sum_{l\ge1}\frac{1}{(u+l)^2}dx =$$

$$= \int_0^1 u^n\left[\sum_{l\ge1}\int_0^\infty xe^{-(1+u)x}dx\right]du = \int_0^1 u^n\left[\int_0^\infty xe^{-ux}\sum_{l\ge1}e^{-lx}dx\right]du =$$

$$= \int_0^1 u^n\left[\int_0^\infty\frac{xe^{-ux}}{e^x-1}\right]du = \int_0^\infty\frac{x}{e^x-1}\left[\int_0^1 u^n e^{-ux}\right]dx =$$

$$= \int_0^\infty\frac{x}{e^x-1}\left[n!\,e^{-x}\sum_{k\ge1}\frac{x^{k-1}}{(n+k)!}\right]dx = \sum_{k\ge1}\frac{n!}{(n+k)!}\int_0^\infty\frac{x^k e^{-x}}{e^x-1}dx$$

$$= \sum_{k\ge1}\frac{n!}{(n+k)!}\int_0^\infty\left[x^k e^{-x}\sum_{i\ge1}e^{-ix}\right]dx = \sum_{k\ge1}\frac{k!}{(n+k)!}\sum_{i\ge1}\int_0^\infty x^k e^{-(i+1)x}dx$$

$$= \sum_{k\ge1}\frac{n!}{(n+k)!}\sum_{i\ge1}\frac{k!}{(i+1)^{k+1}} = \sum_{k\ge1}\frac{n!\,k!}{(n+k)!}\sum_{i\ge1}\frac{1}{(i+1)^{k+1}} = \sum_{k\ge1}\frac{n!\,k!}{(n+k)!}[\zeta(k+1)-1]$$

$$= \sum_{n\ge1}\frac{\zeta(k+1)-1}{\binom{n+k}{k}}$$

$$I = \frac{1}{2}I_1 + \frac{1}{2(n+1)} = \frac{1}{2}\sum_{n\ge1}\frac{\zeta(k+1)-1}{\binom{n+k}{k}} + \frac{1}{2(n+1)}$$

Note 1. $\int_0^1 u^n e^{-ux}\,du = n!\,e^{-x}\sum_{k\ge1}\frac{x^{k-1}}{(n+k)!}$

Note 2. $\int_0^\infty x^k e^{-(i+1)x}\,dx = \frac{\Gamma(k+1)}{(i+1)^{k+1}} = \frac{k!}{(i+1)^{k+1}}$

5.19

$$\int_0^1\int_0^1 ln\,\Gamma\,(x+y+1)\,dx\,dy = \int_0^1\int_{x+1}^{x+2} ln\,\Gamma\,(u)\,du\,dx$$

$$= \left[x\int_{x+1}^{x+2} ln\,\Gamma(u)\,du\right]_0^1 - \int_0^1 x\,ln\frac{\Gamma(x+2)}{\Gamma(x+1)}dx = \int_0^2 ln\,\Gamma(u)\,du - \int_0^1 x\,ln(x+1)\,dx$$

$$I_1 = \int_1^2 ln\,\Gamma(u)\,du$$

$$I_1(a) = \int_a^{a+1} ln\,\Gamma(u)\,du$$

$$I_1'(a) = ln\,\Gamma(a+1) - ln\,\Gamma(a) = ln\,a\,,\qquad I_1(a) = a\,ln\,a - a + C$$

$$I_1(0) = \int_0^1 ln\,\Gamma\,(u)\,du = ln\left(\sqrt{2\pi}\right) = C\,,\qquad I_1(a) = a\,ln\,a - a + ln\left(\sqrt{2\pi}\right)$$

$$I_1 = -1 + ln\left(\sqrt{2\pi}\right)\,,\qquad I_2 = \int_0^1 x\,ln(x+1)\,dx = \frac{1}{4}$$

$$I = \frac{3}{4} + ln\left(\sqrt{2\pi}\right)$$

5.20

$$\int_0^\infty \frac{1}{(1+y^k)(1+y^2)}dy = \int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy + \underbrace{\int_1^\infty \frac{1}{(1+y^k)(1+y^2)}dy}_{let\ y=\frac{1}{x}\Rightarrow dy=-\frac{1}{x^2}dx}$$

$$= \int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy + \int_1^0 \frac{1}{\left(1+\frac{1}{x^k}\right)\left(1+\frac{1}{x^2}\right)}\cdot\left(-\frac{1}{x^2}dx\right)$$

$$= \int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy + \int_0^1 \frac{x^k}{(x^k+1)(x^2+1)}dx$$

$$= \int_0^1 \frac{1}{(1+y^k)(1+y^2)}dy + \int_0^1 \frac{y^k}{(y^k+1)(y^2+1)}dx = \int_0^1 \frac{1+y^k}{(1+y^k)(1+y^2)}dy$$

$$= \int_0^1 \frac{1}{1+y^2}dy = (tan^1 y)|_0^1 = \frac{\pi}{4}$$

So

$$\int_0^\infty \frac{1}{(1+y^{(20n)!!})(1+y^2)}dy = \frac{\pi}{4}$$

5.21

$$\Omega = \int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{(a+bx)^{p+q}}dx, let\ t = \frac{bx}{a(1-x)} \Rightarrow x = \frac{at}{at+b}$$

$dx = \frac{ab}{(at+b)^2}dt$; *when* $x = 0, t = 0$; *when* $x = 1, t \to \infty$

$$\Omega = \int_0^\infty \frac{\left(\frac{at}{at+b}\right)^{p-1}\left(1-\frac{at}{at+b}\right)^{q-1}}{\left(a+\frac{abt}{at+b}\right)^{p+q}} \cdot \frac{ab}{(at+b)^2}dt =$$

$$= \frac{ab}{a^{p+q}}\int_0^\infty \frac{\left(\frac{at}{at+b}\right)^{p-1}\left(\frac{b}{at+b}\right)^{q-1}}{\left(1+\frac{bt}{at+b}\right)^{p+q}} \cdot \frac{dt}{(at+b)^2}$$

$$= \frac{b^q}{a^q}\int_0^\infty \frac{t^{p-1}}{\left(b+t(a+b)\right)^{p+q}}dt = \frac{1}{a^q b^p}\int_0^\infty \frac{t^{p-1}}{\left(1+\frac{a+b}{b}\cdot t\right)^{p+q}}dt$$

$$= \frac{1}{a^q b^p}\cdot\frac{b}{a+b}\cdot\left(\frac{b}{a+b}\right)^{p-1}\int_0^\infty \frac{z^{p-1}}{(1+z)^{p+q}}dz \left[we\ put\ z = \frac{a+b}{b}t\right] = \frac{\beta(p,q)}{a^q(a+b)^{p+q}}$$

5.22

$$H_{m-\frac{1}{2}} + \frac{d}{dm}\int_0^1 \left(\frac{x^{2m}-x}{ln\,x}\cdot\frac{dx}{1+x}\right) = H_{m-\frac{1}{2}} + \frac{d}{dm}\int_0^1 \frac{1}{1+x}\left(\int_1^{2m} x^y\,dy\right)dx$$

$$= H_{m-\frac{1}{2}} + \frac{d}{dm}\int_1^{2m}\left(\frac{x^y}{1+x}dx\right)dy = H_{m-\frac{1}{2}} + 2\left(\int_0^1 \frac{x^{2m}}{1+x}dx\right)$$

$$= \int_0^1 \frac{1-x^{m-\frac{1}{2}}}{1-x}dx + 2\int_0^1 \frac{x^{2m}}{1+x}dn = \int_0^1 \frac{1-x^{2m-1}}{1-x^2}\cdot 2xdx + 2\int_0^1 \frac{x^{2m}}{1+x}dn$$

$$= 2\int_0^1 \frac{x - x^{2m}}{(1+x)(1-x)} dn + 2\int_0^1 \frac{x^{2m}}{1+x} = 2\int_0^1 \frac{x - x^{2m} + x^{2m}(1-x)}{1-x^2} dx = 2\int_0^1 \frac{x - x^{2m-1}}{1-x^2} dn$$

$$= \int_0^1 \frac{1-x^{2m}}{1-x^2} d(x^2) = \int_0^1 \frac{1-x^m}{1-x} dx = H_m$$

5.23

As we know, the series representation of $tan(x)$.

$$tan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1} \quad \textit{(1)}$$

$\Rightarrow \int \frac{tan(x)}{x} dx + \int \frac{ln(x)\,ln(2-x)}{x} dx$*. Using (1), we get,*

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} \int x^{2n-2}\, dx + \int \sum_{n=1}^{\infty} \frac{-x^{n-1}}{n} ln(x)\, dx \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} \left[\frac{x^{2n-1}}{2n-1}\right] - \sum_{n=1}^{\infty} \frac{1}{n} \int x^{n-1}\, ln(x)\, dx \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{\partial}{\partial n}\left(\int x^{n-1} \partial x\right)\right]$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{x^n\, ln(x)}{n} - \frac{x^n}{n^2}\right] + c \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n} - 1}{(2n)!\,(2n-1)} + \sum_{n=1}^{\infty} \left(\frac{x^n}{n^3}\right) + c$$

(OR)

$$I = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}x^{2n-1}}{(2n)!\,(2n-1)}\right) + Li_3(x) + c$$

5.24

$$\int_0^1 \left(\int_0^t \frac{ln\, x}{1-x^5} dx\right) dt = \left[t \int_0^t \frac{ln\, x}{1-x^5} dx\right]_0^1 - \int_0^1 \frac{t\, ln\, t}{1-t^5} dt = \int_0^1 \frac{(1-t)\, ln\, t}{1-t^5} dt \left[t \to t^{\frac{1}{5}}\right] =$$

$$= \frac{1}{25}\left(\int_0^1 \frac{t^{\frac{1}{5}-1}\, ln\, t}{1-t} dt - \int_0^1 \frac{t^{\frac{2}{5}}\, ln\, t}{1-t} dt\right)$$

$$\therefore \int_0^1 \left(\int_0^t \frac{\ln x}{1-x^5}dx\right)dt = \frac{-\Psi_1\left(\frac{1}{5}\right)+\Psi_1\left(\frac{2}{5}\right)}{25} = \frac{A\Psi_1\left(\frac{2}{5}\right)+B\Psi_1\left(\frac{1}{5}\right)}{C}$$

$$\therefore 25(A-B)+2C = -25\times 2 + 2\times 50 = 0$$

5.25

$$\frac{1-a^2}{1-2a\cos x + a^2} = 2\sum_{k=0}^{\alpha} a^k \cos kx + I$$

$$\therefore \int_0^{\alpha} \frac{dx}{(1+x^2)(3-\cos x)} = \frac{I}{3}\int_0^{\alpha} \frac{dx}{(1+x^2)\left(1-\frac{\cos x}{3}\right)} = \frac{I}{3}\int_0^{\alpha} \frac{dx}{(1+x^2)\left(1-\frac{2a\cos x}{1+a^2}\right)}$$

$$\frac{2a}{1+a^2} = \frac{1}{3} \text{ or } a^2-6a+1=0$$

$$a = \frac{6-\sqrt{36-4}}{2} = 3-2\sqrt{2}$$

$$= \frac{(1+a^2)}{3}\int_0^{\alpha} \frac{dx}{(1+x^2)(1+a^2-2a\cos x)} = \frac{1+a^2}{3}\cdot\frac{1}{1-a^2}\int_0^{\alpha}\left[2\sum_{k=0}^{\alpha} a^k\cos kx + I\right]\frac{dx}{1+x^2}$$

$$= \frac{2(1+a^2)}{3(1-a^2)}\sum_{k=0}^{\alpha} a^k \int_0^{\alpha}\frac{\cos kx}{1+x^2}dx + \frac{1+a^2}{3(1-a^2)}\int_0^{\alpha}\frac{1}{1+x^2}dx =$$

$$= \frac{2(1+a^2)}{3(1-a^2)}\sum_{k=0}^{\alpha}\frac{\pi}{2}a^k e^{-k} + \frac{1+a^2}{3(1-a^2)}\frac{\pi}{2} = \frac{2(1+a^2)}{3(1-a^2)}\cdot\frac{\pi}{2\left(1-\frac{a}{e}\right)} + \frac{\pi}{6}\left(\frac{1+a^2}{1-a^2}\right) =$$

$$= \frac{(1+a^2)}{(1-a^2)}\cdot\frac{\pi}{6}\left[\frac{2}{1-\frac{a}{e}} + I\right] = \frac{\pi}{6}\cdot\frac{(1+a^2)}{(1-a^2)}\left[\frac{2+1-\frac{a}{e}}{1-\frac{a}{e}}\right] = \frac{\pi}{6}\cdot\frac{(1+a^2)}{(1-a^2)}\left[\frac{3e-a}{e-a}\right] =$$

$$= \frac{\pi}{6}\left[\frac{1+\left(3-2\sqrt{2}\right)^2}{1-\left(3-2\sqrt{2}\right)^2}\right]\left[\frac{3e-3+2\sqrt{2}}{e-3+2\sqrt{2}}\right] = \frac{\pi}{6}\left[\frac{18-12\sqrt{2}}{12\sqrt{2}-16}\right]\left(\frac{3e+2\sqrt{2}-2}{2\sqrt{2}+e-3}\right) =$$

$$= \frac{\pi}{12}\cdot\frac{\left(9-6\sqrt{2}\right)}{\left(3\sqrt{2}-4\right)}\left(\frac{3e+2\sqrt{2}-2}{2\sqrt{2}+e-3}\right)$$

5.26

$$\int\left(\sum_{n=1}^{\infty} 3^n \sinh^3\left(\frac{x}{3^n}\right)\right)dx = \frac{1}{4}\int\left(\sum_{n=1}^{\infty} 3^n \sinh\left(\frac{x}{3^{n-1}}\right) - 3^{n+1}\sinh\left(\frac{x}{3^n}\right)\right)dx =$$

$$= \frac{1}{4}\int (3\sinh(x) - 3x)\,dx$$

$$\therefore \int \left(\sum_{n=1}^{\infty} 3^n \sinh^3\left(\frac{x}{3^n}\right)\right) dx = \frac{3}{4}\left(\cosh(x) - \frac{x^2}{2}\right) + C$$

5.27

$$\Psi(m) := \int_0^1 x^2 \ln(x) \ln(m + x)\, dx$$

Find $\int_0^1 \frac{\Psi(m)dm}{1+m^2}$

Let $I := \int_0^\infty \frac{\Psi(m)dm}{1+m^2} = \int_0^1 \left(x^2 \ln(x) \int_0^1 \frac{\ln(m+x)}{1+m^2} dm\right) dx = I$ *(1)*

Let $F(\beta) := \int_0^\infty \frac{\ln(\beta+x)dx}{1+x^2}, \beta \geq 0 \Rightarrow F'(\beta) = \int_0^\infty \frac{dx}{(1+x^2)(\beta+x)}$

$$F'(\beta) = \left(\frac{1}{1+\beta^2}\right)\int_0^\infty \left(\frac{\beta}{1+x^2} + \frac{1}{\beta+x} - \frac{x}{1+x^2}\right) dx =$$

$$= \left(\frac{1}{1+\beta^2}\right)\left[\beta \tan^{-1}(x)\Big|_0^\infty + \frac{1}{2}\ln\left(\frac{\beta^2 + x^2 + 2\beta x}{1+x^2}\right)\Big|_0^\infty\right] =$$

$= \left(\frac{1}{1+\beta^2}\right)\left(\frac{\pi\beta}{2} - \ln(\beta)\right)$, *But* $f(0) = 0$

$$\therefore F(\beta) = \left(\frac{\pi}{2}\right)\int_0^\beta \frac{ydy}{1+y^2} - \int_0^\beta \frac{\ln(y)\,dy}{1+y^2} = \frac{\pi}{4}\ln(1+\beta^2) - \int_0^\beta \frac{\ln(y)\,dy}{1+y^2}$$

$$\therefore I = \int_0^1 \left(x^2 \ln(x)\left(\frac{\pi}{4}\ln(1+x^2) - \int_0^x \frac{\ln(y)\,dy}{1+y^2}\right)\right) dx =$$

$$= \frac{\pi}{4}\int_0^1 x^2 \ln(x)\ln(1+x^2) dx - \int_0^1\int_0^x \frac{\ln(y)\cdot x^2\ln(x)}{1+y^2} dydx$$

$$= \frac{\pi}{4}\int_0^1 x^2 \ln(x)\ln(1+x^2) dx - \int_0^1 \left(\frac{\ln(y)}{1+y^2}\int_y^1 x^2\ln(x)\,dx\right) dy$$

$$= \frac{\pi}{4}\int_0^1 x^2 \ln(x)\ln(1+x^2) dx + \int_0^1 \frac{\ln(x)}{1+x^2}\left(\frac{1}{9} + \frac{x^3}{3}\ln(x) - \frac{x^3}{9}\right) dx$$

$\therefore I = \frac{\pi}{4}\int_0^1 x^2 \ln(x)\ln(1+x^2)\, dx + \frac{1}{9}\int_0^1 \frac{\ln(x)dx}{1+x^2} - \frac{1}{9}\cdot\int_0^1 \frac{x^3\ln(x)dx}{1+x^2} + \frac{1}{3}\int_0^1 \frac{x^3\ln^2(x)dx}{1+x^2}$ *(1)*

Let $I = \int_0^1 x^2 \ln(x)\ln(1+x^2)\, dx$, *integrating by parts*

$$I_1 = \left(\frac{x^3}{3}ln(x) - \frac{x^3}{9}\right)ln(1+x^2)\Big|_0^1 - \int_0^1 \left(\frac{x^3}{3}ln(x) - \frac{x^3}{9}\right)\frac{2x}{1+x^2}dx$$

$$\Rightarrow I_1 = -\frac{1}{9}ln(2) + \left(\frac{2}{9}\right)\int_0^1 \frac{x^4 dx}{1+x^2} - \frac{2}{3}\int_0^1 \frac{x^4\, ln(x)\, dx}{1+x^2}$$

$$I_1 = -\frac{1}{9}ln(2) + \left(\frac{2}{9}\right)\int_0^1 \left(x^2 - 1 + \frac{1}{1+x^2}\right)dx - \left(\frac{2}{3}\right)\int_0^1 \left((x^2-1)\,ln(x) + \frac{ln(x)}{1+x^2}\right)dx$$

$$I_1 = -\frac{1}{9}ln(2) + \left(\frac{2}{9}\right)\left(\frac{1}{3} - 1 + \frac{\pi}{4}\right) - \left(\frac{2}{3}\right)\left(\frac{8}{9}\right) - \left(\frac{2}{3}\right)\int_0^1 \frac{ln(x)\,dx}{1+x^2}$$

$$\therefore I_1 = -\frac{1}{9}ln(2) - \frac{4}{27} + \frac{\pi}{18} - \frac{16}{27} + \frac{2}{3}G$$

$$\therefore I_1 = -\frac{20}{27} + \frac{\pi}{18} - \frac{1}{9}ln(2) + \frac{2}{3}G$$

$$I_2 \coloneqq \int_0^1 \frac{ln(x)\,dx}{1+x^2} = -G \Rightarrow I_2 = -G$$

$$I_3 \coloneqq \int_0^1 \frac{x^3\,ln(x)\,dx}{1+x^2} = \int_0^1 \left(x - \frac{x}{1+x^2}\right)ln(x)\,dx = \int_0^1 x\,ln(x)\,dx - \int_0^1 \frac{x\,ln(x)dx}{1+x^2}$$

$$= \frac{x^2}{2}ln(x) - \frac{x^2}{4}\Big|_0^1 - \left(\frac{1}{2}ln(x)\,ln(1+x^2)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{ln(1+x^2)}{x}dx\right) =$$

$$= -\tfrac{1}{4} + \tfrac{1}{2}\int_0^1 \tfrac{ln(1+x^2)dx}{x} \rightarrow \textit{let } x^2 = y \Rightarrow \tfrac{dx}{x} = \tfrac{dy}{2y}$$

$$\therefore I_3 = -\frac{1}{4} + \frac{1}{4}\int_0^1 \frac{ln(1+x)\,dx}{x} = -\frac{1}{4} + \frac{1}{4}\left(\frac{\pi^2}{12}\right)$$

$$\therefore I_3 = -\frac{1}{4} + \frac{\pi^2}{48}$$

$$\textit{Let } I_4 \coloneqq \int_0^1 \tfrac{x^3\,ln^2(x)dx}{1+x^2} = \int_0^1 x\,ln^2(x)\,dx - \int_0^1 \tfrac{x\,ln^2(x)dx}{1+x^2} =$$

$$= \frac{1}{4} - \left(\frac{1}{2}ln^2(x)\,ln(1+x^2)\Big|_0^1 - 2\int_0^1 \frac{1}{2}\frac{ln(1+x^2)\,ln(x)}{x}dx\right) =$$

$$= \tfrac{1}{4} + \int_0^1 \tfrac{ln(1+x^2)\,ln(x)}{x}dx \rightarrow \textit{let } x^2 = y \Rightarrow x = y^{\frac{1}{2}}, \tfrac{dx}{x} = \tfrac{dy}{2y}$$

$$= \frac{1}{4} + \frac{1}{4}\int_0^1 \frac{ln(1+x)\,ln(x)}{x}dx = \frac{1}{4} + \left(\frac{1}{4}\right)\left[-ln(x)\,Li_2(-x)\Big|_0^1 + \int_0^1 +Li_2\frac{(-x)}{x}dx\right]$$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)\int_0^1 Li_2\frac{(-x)}{x}dx = \frac{1}{4} + \frac{1}{4}Li_3(-1) = \frac{1}{4} - \frac{3}{16}\zeta(3)$$

$$\therefore I_4 = \frac{1}{4} - \frac{3}{16}\zeta(3)$$

$$\therefore I = \left(\frac{\pi}{4}\right)\left(-\frac{20}{27} + \frac{\pi}{18} - \frac{1}{9}ln(2) + \frac{2}{3}G\right) - \frac{1}{9}G$$

$$-\frac{1}{9}\left(-\frac{1}{4} + \frac{\pi^2}{48}\right) + \frac{1}{3}\left(\frac{1}{4} - \frac{3}{16}\zeta(3)\right)$$

$$\therefore I = -\frac{5\pi}{27} + \frac{\pi^2}{72} - \frac{\pi\,ln(2)}{36} + \frac{\pi G}{6} - \frac{1}{9}G$$

$$+\frac{1}{36} - \frac{\pi^2}{(48)(9)} + \frac{1}{12} - \frac{\zeta(3)}{16}$$

$$I = \frac{1}{9} + \frac{\pi G}{6} - \frac{G}{9} - \frac{\zeta(3)}{16} - \frac{5\pi}{27} + \frac{5\pi^2}{432} - \frac{\pi\,ln(2)}{36}$$

5.28

$$\Omega = \int_{-\beta}^{\beta} \frac{dx}{(\beta+x)^{\frac{2}{3}}(\beta+x)^{\frac{1}{3}}}$$

Put $x = \beta\cos(2\theta)$

$$dx = -2\beta\sin(2\theta)\,d\theta$$

$$\Omega = \int_{\frac{\pi}{2}}^{0} \frac{-2\beta\sin 2\theta\, d\theta}{\beta(1-\cos 2\theta)^{\frac{2}{3}}(1+\cos 2\theta)^{\frac{1}{3}}}$$

$$\Omega = \int_0^{\frac{\pi}{2}} \frac{4\sin\theta\cos\theta d\theta}{(2)\sin^{\frac{4}{3}}\theta\cos^{\frac{2}{3}}\theta},\ \Omega = \int_0^{\frac{\pi}{2}} 2\sin^{-\frac{1}{3}}\theta\cos^{\frac{1}{3}}\theta\,d\theta$$

$$\Omega = \beta\left(\frac{1}{3},\frac{2}{3}\right) = \Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right)$$

Using reflection formula

$$\Gamma(m)\Gamma(1-m) = \pi\csc(m\pi) \Rightarrow \Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right) = \pi\csc\left(\frac{2\pi}{3}\right) = \frac{2\pi}{\sqrt{3}}$$

5.29

$$\Omega = \int_0^\infty \frac{1}{x} tan^{-1}\left(\frac{3x^2}{4x^4 + 5x^2 + 2}\right) dx = \int_0^\infty \frac{1}{x} tan^{-1}\left[\frac{(4x^2 + 1) - (x^2 + 1)}{1 + (4x^2 + 1)(x^2 + 1)}\right] dx =$$

$$= \int_0^\infty \frac{1}{x}[tan^{-1}(4x^2 + 1) - tan^{-1}(x^2 + 1)]\, dx$$

Let $\phi(x) = tan^{-1}(x^2 + 1)$. *Note* φ *is continuous on* $[0, \infty)$

$$\therefore \Omega = \int_0^\infty \frac{\varphi(2x) - \varphi(x)}{x} dx \text{ [Frullani's Integral]}$$

$$= [\phi(0) - \varphi(\infty)]\, ln\left(\frac{1}{2}\right) = \left(\frac{\pi}{4} - \frac{\pi}{2}\right)(-ln\, 2) = \frac{\pi}{4} ln(2)$$

5.30

We have

$$e^{\frac{k}{n^2}} = 1 + \frac{k}{n^2} + \frac{\left(\frac{k}{n^2}\right)^2}{2} + O\left(\frac{k^3}{n^6}\right) = 1 + \frac{k}{n^2} + \frac{k^2}{2n^4} + O\left(\frac{k^3}{n^6}\right)$$

Thus

$$\sum_{k=1}^{n} e^{\frac{k}{n^2}} = n + \frac{1}{n^2}\sum_{k=1}^{n} k + \frac{1}{2n^4}\sum_{k=1}^{n} k^2 + O\left(\frac{1}{n^6}\sum_{k=1}^{n} k^3\right)$$

$$= n + \frac{n(n + 1)}{2n^2} + \frac{n(n + 1)(2n + 1)}{12n^4} + O\left(\frac{\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}}{n^6}\right)$$

$$= n + \frac{1}{2} + \frac{1}{2n} + \frac{2n^2 + 3n + 1}{12n^3} + O\left(\frac{1}{n^2}\right)$$

$$\sum_{k=1}^{n} e^{\frac{k}{n^2}} - n - \frac{1}{2} = \frac{1}{2n} + \frac{1}{6n} + O\left(\frac{1}{n^2}\right),$$

so that

$$n\left(\sum_{k=1}^{n} e^{\frac{k}{n^2}} - n - \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{6} + O\left(\frac{1}{n}\right) \to \frac{2}{3}$$

as $n \to \infty$.

5.31

$$Let\ S_k = \sum_{\substack{n=1\\n\neq k}}^{\infty}\frac{1}{n^2-k^2} = \sum_{n=1}^{k-1}\frac{1}{n^2-k^2} + \sum_{n=k+1}^{\infty}\frac{1}{n^2-k^2} = \frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{n-k}-\frac{1}{n+k}\right) +$$

$$+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right) = -\frac{1}{2k}\sum_{n=1}^{k-1}\left(\frac{1}{k-n}+\frac{1}{k+n}\right)+\frac{1}{2k}\sum_{n=k+1}^{\infty}\left(\frac{1}{n-k}-\frac{1}{n+k}\right)$$

$$= -\frac{1}{2k}\left(\frac{1}{k-1}+\frac{1}{k+1}+\frac{1}{k-2}+\frac{1}{k+2}+\cdots+\frac{1}{1}+\frac{1}{2k-1}\right)+$$

$$+\frac{1}{2k}\left[\left(1-\frac{1}{2k+1}\right)+\left(\frac{1}{2}-\frac{1}{2k+2}\right)+\left(\frac{1}{3}-\frac{1}{2k+3}\right)+\cdots\right] =$$

$$= -\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}+\frac{1}{k+1}+\cdots+\frac{1}{2k-1}\right)+\frac{1}{2k}\left(1+\frac{1}{2}+\cdots+\frac{1}{2k}\right) =$$

$$= -\frac{1}{2k}\left(-\frac{1}{k}\right)+\frac{1}{4k^2} = \frac{3}{4k^2}$$

$$Thus\ \Omega = \sum_{k=1}^{\infty}\left(\sum_{\substack{n=1\\n\neq k}}^{\infty}\frac{1}{n^2-k^2}\right) = \sum_{k=1}^{\infty}S_k = \frac{3}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{3}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$$

5.32

$$\lim_{n\to\infty}\left((n+1)\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1}dx - n\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx\right)[x\to \tan x] =$$

$$= \lim_{n\to\infty}\left((n+1)\left(\int_0^{\frac{\pi}{4}}2^{n+1}\sin^{n+1}x\cdot\cos^{n-1}x\,dx\right)-n\left(\int_0^{\frac{\pi}{4}}2^{n}\sin^{n}x\cdot\cos^{n-2}x\,dx\right)\right)$$

$$= \lim_{n\to\infty}\left((n+1)\left(-\frac{1}{n}+\frac{\sqrt{2\pi n}}{n+1}\right)-n\left(-\frac{1}{n-1}+\frac{\sqrt{2\pi(n-1)}}{n}\right)\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{n(n-1)}+\sqrt{2\pi}\left(\sqrt{n}-\sqrt{n-1}\right)\right)$$

$$\therefore \lim_{n\to\infty}\left((n+1)\int_0^1\left(\frac{2x}{1+x^2}\right)^{n+1} - n\int_0^1\left(\frac{2x}{1+x^2}\right)^{n}dx\right) = 0$$

5.33

$$\omega(x) = \frac{1}{x}+\sum_{n=1}^{\infty}\frac{1}{2^n}\tanh\left(\frac{x}{2^n}\right) = \frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}}\coth\left(\frac{x}{2^{n-1}}\right)-\frac{1}{2^n}\coth\left(\frac{x}{2^n}\right)\right) = \coth(x)$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(\frac{x^n}{coth(x)\cdot coth(2x)\cdot\ldots\cdot coth(nx)}\right)} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to 0 \\ x>0}} \Big((x\, tanh(x))\cdot(x\, tanh(2x))\ldots(x\, tanh(nx))\Big)} = 0$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(\frac{x^n}{\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)}\right)} = 0$$

OR

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(x^n\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)\right)} =$$

$$= \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} (x^n\, coth(x)\cdot coth(2x)\cdot\ldots\cdot coth(nx))}$$

$$= \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(\left(\frac{x}{tanh(x)}\right)\cdot\left(\frac{x}{tanh(2x)}\right)\ldots\left(\frac{x}{tanh(nx)}\right)\right)} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n!}\right)} =$$

$$= \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty} \left(\frac{1}{n+1}\right)$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{\lim_{\substack{x\to\infty \\ x>0}} \left(x^n\omega(x)\cdot\omega(2x)\cdot\omega(3x)\cdot\ldots\cdot\omega(nx)\right)} = 0$$

5.34

$$\Omega = \int_0^\infty \frac{x\, tan^{-1} x}{x^4+x^2+1}\, dx \quad (1)$$

Put $x = \frac{1}{t}$

$$dx = -\frac{dt}{t^2}$$

$$\Omega = \int_0^\infty \frac{x\, tan^{-1}\left(\frac{1}{x}\right)dx}{x^4+x^2+1} \quad (2)$$

(1)+(2)

$$\Omega = \frac{\pi}{4}\int_0^\infty \frac{x\, dx}{(x^4+x^2+1)} = \frac{\pi}{8}\int_0^\infty \frac{dx}{x^2+x+1}$$

$$\Omega = \frac{\pi}{8}\int_0^\infty \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\pi}{8}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\frac{2}{\sqrt{3}} = \frac{\pi^2}{12\sqrt{3}}$$

5.35

$$\int_0^1\int_0^1 x\,ln\,x\,ln(x+m)\,dx\,dm = \int_0^1\int_x^{x+1} x\,ln\,x\,ln\,t\,dt\,dx$$

$$= \left[\left(\frac{x^2}{2}ln\,x-\frac{x^2}{4}\right)\int_x^{x+1} ln\,t\,dt\right]_0^1 - \int_0^1\left(\frac{x^2}{2}ln\,x-\frac{x^2}{4}\right)(ln(x+1)-ln\,x)dx$$

$$= -\frac{1}{4}\int_1^2 ln\,t\,dt - \frac{1}{2}\int_0^1 x^2\,ln\,x\,ln(x+1)\,dx + \frac{1}{2}\int_0^1 x^2\,ln^2\,x\,dx + \frac{1}{4}\int_0^1 x^2\,ln(x+1)\,dx$$

$$-\frac{1}{4}\int_0^1 x^2\,ln\,x\,dx$$

$$= -\frac{1}{4}(2\,ln\,2-1) - \frac{1}{2}\int_0^1 x^2\,ln\,x\,ln(x+1)\,dx + \frac{1}{27} + \frac{1}{4}\left(\frac{2\,ln\,2}{3}-\frac{5}{18}\right) + \frac{1}{36}$$

$$= -\frac{1}{3}ln\,2 + \frac{53}{216} - \frac{1}{2}\int_0^1 x^2\,ln\,x\,ln(x+1)\,dx$$

$$\int_0^1 x^2\,ln\,x\,ln(x+1)\,dx = \int_0^1 x^2\,ln\,x\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{k+2}\,ln\,x\,dx$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left[\frac{x^{k+3}}{k+3}ln\,x-\frac{x^{k+2}}{(k+3)^2}\right]_0^1 = -\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k(k+3)^2}$$

$$= -\frac{1}{9}\sum_{k=1}^{\infty}(-1)^{k-1}\left[\frac{1}{k}-\frac{1}{k+3}-\frac{3}{(k+3)^2}\right]$$

$$= -\frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} + \frac{1}{9}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+3} + \frac{1}{3}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(k+3)^2}$$

$$= -\frac{1}{9}ln\,2 + \frac{1}{9}\left(\frac{5}{9}-ln\,2\right) + \frac{1}{3}\left(\frac{31}{36}-\frac{\pi^2}{12}\right)$$

$$= -\frac{2}{9}ln\,2 + \frac{41}{108} - \frac{\pi^2}{36}$$

$$\int_0^1\int_0^1 x\ln x\ln(x+m)\,dx\,dm = -\frac{1}{3}\ln 2 + \frac{53}{216} - \frac{1}{2}\int_0^1 x^2\ln x\ln(x+1)\,dx$$

$$= -\frac{1}{3}\ln 2 + \frac{53}{216} - \frac{1}{2}\left(-\frac{2}{9}\ln 2 + \frac{41}{108} - \frac{\pi^2}{36}\right) = \frac{\pi^2 - 16\ln 2 + 4}{72}$$

5.36

Substitute $x^2 = t$

$$\Omega = 2\int_0^\infty \frac{\pi\sqrt{t} - \ln t}{(\pi^2 + \ln^2 t)(1+t)^2}\frac{dt}{\sqrt{t}}$$

$$= 2\pi\int_0^\infty \frac{1}{(\pi^2 + \ln^2 t)(1+t)^2}dt - 2\int_0^\infty \frac{\ln t}{(\pi^2 + \ln^2 t)(1+t)^2}\frac{dt}{\sqrt{t}}$$

$$\Omega = 2\pi I_1 - 2I_2 = \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$$

Proofs: I_1 *is a Schroder integral, which generally is defined as:*

$$\int_0^\infty \frac{1}{(\pi^2 + \ln^2 x)(1+x)^n}dx = (-1)^{n-1}G_n$$

and G_n *are Gregory coefficients that respects the following recurrence relation:*

$$\frac{G_1}{n} - \frac{G_2}{n-1} + \cdots + (-1)^{n-1}\frac{G_n}{1} = \frac{1}{n+1}$$

$$I_1 = \int_0^\infty \frac{1}{(\pi^2 + \ln^2 t)(1+t)^2}dt = (-1)^{2-1}G_2 = \frac{1}{12}$$

$$I_2 = \int_0^\infty \frac{\ln x}{(\pi^2 + \ln^2 x)(1+x)^2}\frac{\sqrt{x}}{x}dx$$

Substitute $\ln x = t$*, then let* $t = -y$*, add both results and simplify:*

$$I_2 = \int_0^\infty \frac{x}{\pi^2 + x^2}\cdot\frac{e^{\frac{x}{2}}}{(1+e^x)^2}dx = \int_0^\infty \frac{-x}{\pi^2 + x^2}\cdot\frac{e^{-\frac{\pi}{2}}}{(1+e^{-x})^2}dx$$

$$2I_2 = \int_{-\infty}^\infty \frac{x}{\pi^2 + x^2}\left(\frac{e^{\frac{x}{2}}}{(1+e^x)^2} - \frac{e^{-\frac{x}{2}}}{(1+e^{-x})^2}\right)dx$$

$$I_2 = -\frac{1}{4}\int_{-\infty}^{\infty} \frac{x}{\pi^2 + x^2} \cdot \frac{sinh\left(\frac{x}{2}\right)}{cosh^2\left(\frac{x}{2}\right)} dx$$

Now, integrate by parts, using that:

$$-\frac{1}{2}\int \frac{sinh\left(\frac{x}{2}\right)}{cosh^2\left(\frac{x}{2}\right)} = \frac{1}{cosh\left(\frac{x}{2}\right)} + C$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{x^2 - \pi^2}{(x^2 + \pi^2)^2}\right)\left(\frac{1}{cosh\left(\frac{x}{2}\right)}\right) dx$$

Using the following property of the fourier transform:

$$\int_{-\infty}^{+\infty} f(x)g(x)\, dx = \int_{-\infty}^{+\infty} (F^{-1}g)(s)(Ff)\,(s) ds$$

The integral simplifies to:

$$I_2 = \int_0^{\infty} \left(\sqrt{\frac{x}{2}}x(-e^{-\pi x})\right)\left(\sqrt{2\pi}\frac{1}{cosh(\pi x)}\right) dx = -\frac{1}{\pi}\int_0^{\infty} \frac{x}{cosh(x)} e^{-x} dx$$

The latter integral is equal to the Laplace transform in $s = 1$ *of*

$$f(t) = \frac{t}{cosh(t)} \to F(s) = \frac{1}{8}\left(\psi_1\left(\frac{s+1}{4}\right) - \psi_1\left(\frac{s+3}{4}\right)\right)$$

$$I_2 = F(s = 1) = \frac{1}{8}\left(\psi_1\left(\frac{1}{2}\right) - \psi_1(1)\right) = \frac{1}{8}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{24}$$

$$\to I_2 = \int_0^{\infty} \frac{ln\, x}{(\pi^2 + ln^2 x)(1 + x)^2}\frac{\sqrt{x}}{x} dx = -\frac{\pi}{24}$$

5.37

$$\Omega_1 + \Omega_2 = \int_0^c \int_0^c \left(\sqrt{x^2 + y^2 - 2ax + a^2} + \sqrt{x^2 + y^2 - 2by + b^2}\right) dx\, dy$$

Now, $\sqrt{x^2 + y^2 - 2ax + a^2} + \sqrt{x^2 + y^2 - 2by + b^2} =$

$$= \sqrt{(a - x)^2 + y^2} + \sqrt{(b - y)^2 + x^2}$$

Again, $\sqrt{(a - x)^2 + y^2} \leq a - x + y$

$$[0 \leq x \leq a]$$

and $\sqrt{(b-y)^2+x^2} \le b-y+x$

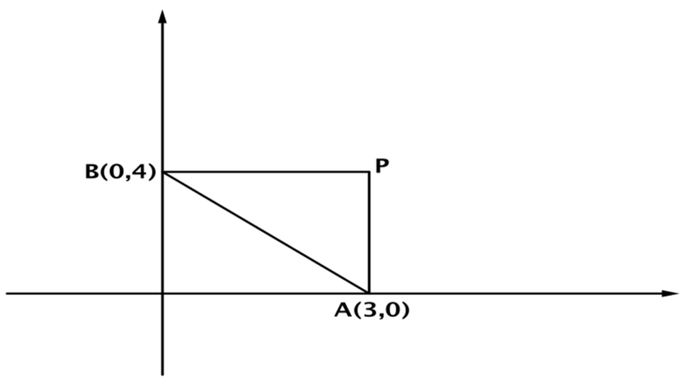
Hence, $\sqrt{(a-x)^2+y^2} + \sqrt{(b-y)^2+x^2} \le a-x+y+b-y+x = a+b$

$$\Omega_1 + \Omega_2 \le \int_0^c \int_0^c (a+b)\,dx\,dy = (a+b)c^2$$

5.38

$$PA + PB = \sqrt{x^2+y^2-6x+9} + \sqrt{x^2+y^2-8y+16}$$

$$= \sqrt{(x-3)^2+y^2} + \sqrt{(x^2)+(y-4)^2} \ge \sqrt{3^2+y^2} = 5$$



$$\Omega_1 + \Omega_2 = \int_0^a \left(\int_0^a \left(\sqrt{x^2+y^2-6x+9} + \sqrt{x^2+y^2-8y+16} \right) dx \right) dy \ge \int_0^a \left(\int_0^a 5dx \right) dy$$
$$= 5a^2$$

5.39

$$x^2 + 34y^2 - 10xy - 6y + 2 =$$

$$= (x^2 + 25y^2 - 10xy) + (9y^2 - 6y + 1) + 1 = (x-5y)^2 + (3y-1)^2 + 1 \ge 1$$

$$\int_0^1 \int_0^1 (x^2 + 34y^2 - 10xy - 6y + 2)^2 dx\,dy \ge 1$$

5.40

$$\Rightarrow 2 \int_0^1 \frac{[ln(1-x) + ln(1+x)]\,ln(1-x)}{x} dx$$

$$\Rightarrow 2\int_0^1 \frac{ln^2(1-x)}{x}dx + 2\int_0^1 \frac{ln(1-x)\,ln(1+x)}{x}dx$$

Let,

$$A = \int_0^1 \frac{ln^2(1-x)}{x}dx$$

$$\Rightarrow \int_0^1 \frac{ln^2(x)}{1-x}dx \Rightarrow \sum_{n=0}^{\infty}\int_0^1 x^n\,ln^2(x)\,dx \Rightarrow \sum_{n=0}^{\infty}\frac{\partial^2}{\partial n^2}\left[\int_0^1 x^n dx\right]$$

$$\Rightarrow \sum_{n=0}^{\infty}\frac{\partial^2}{\partial n^2}\left[\frac{x^{n-1}}{n+1}\right]_0^1 \Rightarrow \sum_{n=0}^{\infty}\left[\frac{x^{n-1}\,ln^2(x)}{n+1} - 2\frac{x^{n+1}\,ln(x)}{(n+1)^2} + 2\frac{x^{n-1}}{(n+1)^3}\right]_0^1$$

$\Rightarrow 2\sum_{n=0}^{\infty}\left(\frac{1}{n^3}\right)$ *(OR)* $A = 2\zeta(3)$

Let

$$B = \int_0^1 \frac{ln(1-x)\,ln(1+x)}{x}dx$$

$$\Rightarrow \frac{1}{4}\left[\frac{1}{2}\int_0^1 \frac{ln^2(1-x)}{x}dx - 2\int_0^1 \frac{ln^2(x)}{(1-x)(1+x)}dx\right] \Rightarrow \frac{1}{4}\left[-\frac{1}{2}\int_0^1 \frac{ln^2(x)}{1-x}dx - \int_0^1 \frac{ln^2(x)}{1+x}dx\right]$$

Now, applying I.B.P., we get,

$$\Rightarrow \frac{1}{4}\left[-\int_0^1 \frac{ln(x)\,ln(1-x)}{x}dx + 2\int_0^1 \frac{ln(x)\,ln(1+x)}{x}dx\right]$$

Now, again applying I.B.P., we get

$$\Rightarrow \frac{1}{4}\left[-\int_0^1 \frac{Li_2(x)}{x}dx + 2\int_0^1 \frac{Li_2(-x)}{x}dx\right]$$

Let,

$x = -u$, *in second integral, we get*

$$dx = -du$$

$$\Rightarrow \frac{1}{4}\left([-Li_3(x)]_0^1 + 2\int_0^1 \frac{Li_2(u)}{u}du\right) \Rightarrow \frac{1}{4}\left(Li_3(1) + 2[Li_3(x)]_0^1\right)$$

$$\Rightarrow \frac{1}{4}\left[-\frac{5}{2}\left(Li_3(1)\right)\right] \Rightarrow \frac{1}{4}\left[-\frac{5}{2}\zeta(3)\right] (OR)\ B = -\frac{5}{8}\zeta(3)$$

Combining all, we get, $I = 2A + 2B \Rightarrow 2(2\zeta(3)) + 2\left(-\frac{5}{8}\zeta(3)\right)$ *(OR)*

$$I = \frac{11}{4}\zeta(3) > \frac{5}{2}\zeta(3)$$

5.41

We have, by $R.M \geq A.M \geq G.M;$

$$8(x^4 + y^4) \geq (x + y)^4 \geq 16x^2y^2$$

$$8\left(\int_0^1 dy\right)\left(\int_0^1 x^4\, dx\right) + 8\left(\int_0^1 dx\right)\left(\int_0^1 y^4\, dy\right) \geq \int_0^1\int_0^1 (x + y)^4 dxdy \geq$$

$$\geq 16\left(\int_0^1 x^2 dx\right)\left(\int_0^1 y^2 dy\right)$$

$$\frac{16}{5} \geq \int_0^1\int_0^1 (x + y)^4\, dxdy \geq \frac{16}{9} > 1 \ \text{(Proved)}$$

We have, $0 \leq t \leq 1 \Rightarrow 0 \leq xt \leq x$, *similarly,* $0 \leq 1 - t \leq 1$

$\Rightarrow 0 \leq y(1 - t) \leq y$. *Adding we have,* $0 \leq xt + y(1 - t) \leq x + y$

$$\int_a^b\int_a^b\int_0^1 \left(xt + y(1 - t)\right)^2 dxdydt \leq \int_a^b\int_a^b\int_0^1 (x + y)^4 dxdydt = \int_a^b\int_a^b (x + y)^4\, dxdy$$

5.42

We have, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p, q > 0$ *now,* $e^{\frac{x}{p}}$ *and* $e^{\frac{y}{p}}$ *are convex functions, hence by Hermite – Hadamard Inequality*

$$\int_a^b e^{\frac{x}{p}}\, dx \geq (b - a)e^{\frac{a+b}{2p}}, \int_a^b e^{\frac{y}{q}}\, dy \geq (b - a)e^{\frac{a+b}{2q}}$$

$$\therefore \int_a^b\int_a^b e^{\frac{qx+py}{p+q}}\, dx\, dy = \left(\int_a^b e^{\frac{x}{p}}\, dx\right)\left(\int_a^b e^{\frac{y}{q}}\, dy\right) \geq (b - a)^2 e^{\frac{a+b}{2}\left(\frac{1}{p}+\frac{1}{q}\right)}$$

$$= (b - a)^2\sqrt{e^{a+b}}$$

now, $e^{\frac{qx+py}{p+q}} \leq \frac{e^x}{p} + \frac{e^y}{q}$ $\left[\begin{matrix}\because e^m \text{ is a convex function} \\ \text{and } \frac{1}{p} + \frac{1}{q} = 1\end{matrix}\right]$

$$\Rightarrow \int_a^b \int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy \le \frac{1}{p}\int_a^b \int_a^b e^{x}\,dx\,dy + \frac{1}{q}\int_a^b \int_a^b e^{y}\,dx\,dy$$

$$= (b-a)(e^b - e^a)\left(\frac{1}{p} + \frac{1}{q}\right) = (b-a)(e^b - e^a)$$

$$\therefore (b-a)^2\sqrt{e^{a+b}} \le \int_a^b \int_a^b e^{\frac{qx+py}{p+q}}\,dx\,dy \le (b-a)(e^b - e^a)$$

5.43

$$\frac{2x_1}{\pi} < \sin x_1 < x_1 \quad \text{(Jordan)}$$

$$\frac{2}{\pi}\int_0^{a_1} x_1 dx \le \int_0^{a_1} \sin x_1\,dx_1 \le \int_0^{a_1} x_1 dx_1$$

$$\frac{1}{\pi}\cdot a_1^2 \le \int_0^{a_1} \sin x_1\,dx_1 \le \frac{1}{2}\cdot a_1^2,$$

$$\frac{1}{\pi}\cdot a_2^2 \le \int_0^{a_2} \sin x_2\,dx_1 \le \frac{1}{2}\cdot a_2^2$$

....

$$\frac{1}{\pi}\cdot a_n^2 \le \int_0^{a_n} \sin x_n\,dx_1 \le \frac{1}{2}\cdot a_n^2,$$

$$\frac{1}{\pi^n}\cdot \prod_{i=1}^{n} a_i^2 \le \prod_{i=1}^{n}\int_0^{a_1} \sin x_i\,dx_i \le \frac{1}{2^n}\prod_{i=1}^{n} a_i^2$$

$$\frac{1}{\pi^n}\cdot \prod_{i=1}^{n} a_i^2 \le \int_0^{a_n} \dots \int_0^{a_1} \left(\prod_{i=1}^{n} \sin x_i\right) dx_1 \dots dx_n \le \frac{1}{2^n}\prod_{i=1}^{n} a_i^2$$

5.44

$$\left(\frac{px + qy}{p + q}\right)^2 = \left(\frac{x}{q} + \frac{y}{p}\right)^2 = \frac{x^2}{q^2} + \frac{2xy}{pq} + \frac{y^2}{p^2}$$

$$\int_a^b \int_a^b \left(\frac{px + qy}{p + q}\right)^2 dx\,dy = \frac{1}{q^2}\int_a^b \int_a^b x^2\,dx\,dy + \frac{2}{pq}\left(\int_a^b x\,dx\right)\left(\int_a^b y\,dy\right) + \frac{1}{p^2}\int_a^b \int_a^b y^2 dy\,dx$$

$$= \frac{(b-a)(b^3-a^3)}{3q^2} + \frac{(b-a)(b^3-a^3)}{3p^2} + \frac{(b^2-a^2)^2}{2pq}$$

$$\frac{1}{(b-a)^2}\int_a^b \int_a^b \left(\frac{px+qy}{p+q}\right)^2 dx\, dy = \frac{a^2+ab+b^2}{3}\left(\frac{1}{p^2}+\frac{1}{q^2}\right) + \frac{(b+a)^2}{2pq}$$

$$\geq \frac{(a+b)^2}{4}\left(\frac{1}{p^2}+\frac{1}{q^2}\right) + \frac{(a+b)^2}{2pq}\left[\because a^2+ab+b^2 \geq \frac{3(a+b)^2}{4}\right]$$

$$= \frac{(a+b)^2}{4}\left(\frac{1}{p}+\frac{1}{q}\right)^2 = \frac{a^2+2ab+b^2}{4}$$

Similarly, $\frac{1}{(b-a)^2}\int_a^b \int_a^b \left(\frac{px+qy}{p+q}\right)^2 dx\, dy$

$$\leq \frac{a^2+ab+b^2}{3}\left(\frac{1}{p^2}+\frac{1}{q^2}\right) + \frac{2(a^2+ab+b^2)}{3pq}$$

$$= \frac{a^2+ab+b^2}{3}\left(\frac{1}{p}+\frac{1}{q}\right)^2 = \frac{a^2+ab+b^2}{3}$$

5.45

Using the inequality

$$(x+y)^2 \geq 4xy, x, y > 0$$

we have that:

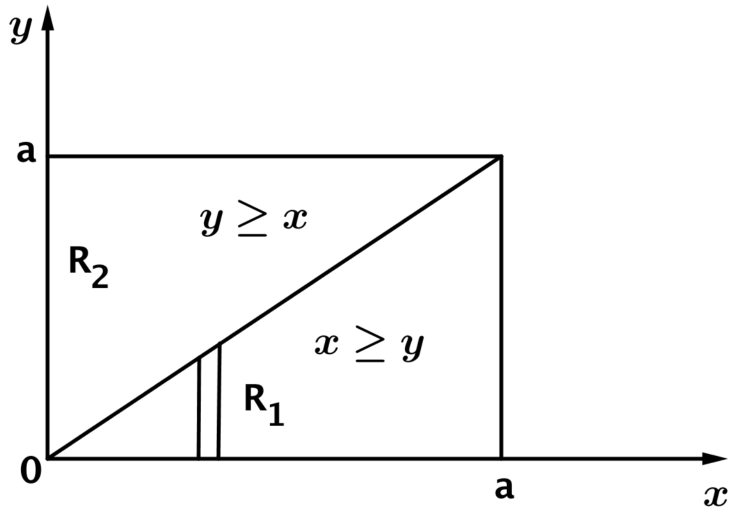
$$x+y \geq \frac{4xy}{x+y} \Leftrightarrow \frac{1}{x+y} \leq \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right) \Rightarrow$$

$$\int_a^b \int_a^b \frac{1}{x+y} dxdy \leq \frac{1}{4}\int_a^b \int_a^b \left(\frac{1}{x}+\frac{1}{y}\right) dx\, dy \Leftrightarrow \int_a^b \int_a^b \frac{1}{xy} dx\, dy \leq \frac{1}{4}\cdot 2(b-a)\, ln\left(\frac{b}{a}\right)$$

$\Leftrightarrow \frac{1}{b-a}\int_a^b \int_a^b \frac{1}{x+y} dx\, dy \leq \frac{1}{2} ln\left(\frac{b}{a}\right)$ *so, it suffices to prove that*

$\frac{1}{2} ln\left(\frac{b}{a}\right) < \frac{13}{25} ln\left(\frac{b}{a}\right)$ *or* $25 < 26$ *which holds!*

5.46



Let

$$f(x,y) = \sqrt{x^2 + 2xy} + \sqrt{y^2 + 2xy}, \qquad x, y \geq 0$$

$$\Omega(a) = \int_0^a \int_0^a f(x,y)\,dxdy = \int\int_{R_1} f(x,y)\,dxdy + \int\int_{R_2} f(x,y)dxdy$$

$$\int\int_{R_1} f(x,y)dxdy = \int_0^a \int_{y=0}^{y=x} f(x,y)dxdy$$

$$\geq \int_0^a \int_{y=0}^{y=x} \left(\sqrt{y^2 + 2yy} + \sqrt{y^2 + 2yy}\right)dydx$$

$$= \int_0^a \int_{y=0}^{y=x} 2\sqrt{3}ydydx = \int_0^a \sqrt{3}[y^2]_0^x\,dx = \int_0^a \sqrt{3}\,x^2dx = \frac{1}{\sqrt{3}}a^3$$

Similarly,

$$\int\int_{R_2} f(x,y)dxdy \geq \frac{1}{\sqrt{3}}a^3$$

$$\therefore \Omega(a) \geq \frac{2}{\sqrt{3}}a^3$$

Now

$$\frac{\Omega(a)}{b^3} + \frac{\Omega(b)}{c^3} + \frac{\Omega(c)}{a^3} \geq \frac{2}{\sqrt{3}}\left(\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}\right) \geq 2\sqrt{3}$$

5.47

We have that

$$\int_0^1\int_0^1 \dots \int_0^1 (1 + x_1^2)\, dx_1 dx_2 \dots dx_2 =$$

$$\int_0^1 (1 + x_1^2)\, dx_i \int_0^1 (1 + x_2^2)\, dx_x \dots . \int_0^1 (1 + x_n^2)\, dx_n = \left(\frac{4}{3}\right)^2$$

Doing the same

$$\int_0^1\int_0^1 \dots \int_0^1 (1 - x_i^2) dx_1 dx_2 \dots dx_n = \left(\frac{2}{3}\right)^n$$

So it suffices to prove that

$$\left(\frac{4}{3}\right)^n + \left(\frac{2}{3}\right)^n \le 2^n \text{ or } 2^n + 1 \le 3^n \text{ or}$$

$1 \le 3^n - 2^n$ *(*) which clearly holds for every* $n \in \mathbb{N}^*$

(*) $3^n - 2^n = 3^{n-1} + 3^{n-2} \cdot 2 + \dots + 2^{n-2} \cdot 3 + 2^{n-1} > 1$

when $n > 1$

5.48

By Cauchy – Schwarz inequality we have that:

$$x\sqrt{x^2 + z^2} + y\sqrt{y^2 + z^2} \le \sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2 + z^2} \Leftrightarrow$$

$$\Leftrightarrow x\sqrt{x^2 + z^2} + y\sqrt{y^2 + z^2} \le x^2 + y^2 + z^2 \Rightarrow$$

$$\int_0^1\int_0^1\int_0^1 \left(x\sqrt{x^2 + z^2} + y\sqrt{y + z^2}\right) dxdydz \le \int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\, dxdydz$$

But $\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\, dxdydz = 1$

cause $\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\, dxdydz = \int_0^1\int_0^1\int_0^1 x^2 dxdydz +$

$$\int_0^1\int_0^1\int_0^1 y^2 dxdydz + \int_0^1\int_0^1\int_0^1 z^2 dxdydz = \frac{x^3}{3}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1 + \frac{z^3}{3}\Big|_0^1 = 1$$

5.49

From the well – known inequality $\ln a \le a - 1, \forall a > 0$
we have that:

$$ln\frac{x}{y} \le \frac{x}{y} - 1 \Rightarrow 2y\, ln\frac{x}{y} \le 2x - 2y = 0$$

$$2y\int_1^a ln\frac{x}{y}dx \le \int_1^a 2x\,dx - \int_1^a 2y\,dx$$

$$\Rightarrow 2\int_1^b \left(y\int_1^a ln\frac{x}{y}dx\right)dy \le (a^2-1)(b-1) - (a-1)(b^2-1)$$

$$\Rightarrow 2\int_1^b \left(y\int_1^a ln\frac{x}{y}dx\right)dy \le (a-1)(b-1)(a+1-b-1)$$

$$\Rightarrow 2\int_1^b \left(y\int_1^a ln\frac{x}{y}dx\right)dy \le (a-1)(b-1)(a-b)$$

5.50

We have that

$$(x\,sin^2 a + y\,cos^2 a)(x\,cos^2 a + y\,sin^2 a) =$$

$$\left[\left(\sqrt{x}\,sin\,a\right)^2 + \left(\sqrt{y}\,cos\,a\right)^2\right]\cdot\left[\left(\sqrt{x}\,cos\,a\right)^2 + \left(\sqrt{y}\,sin\,a\right)^2\right]$$

$$\overset{B-C-S}{\ge} \left(\sqrt{xy}\,sin^2 a + \sqrt{xy}\,cos^2 a\right)^2 = xy$$

so $\Omega(a,b) = \int_0^a\int_0^b xy\,dx\,dy = \int_0^a xdx\cdot\int_0^b y\,dy = \frac{(ab)^2}{4}$

Doing exactly the same work, we have that $\Omega(b,c) \ge \frac{(bc)^2}{4}, \Omega(c,a) \ge \frac{(ca)^2}{4}$

So $4\Omega(a,b) + 4\Omega(b,c) + 4\Omega(c,a) \ge 4\frac{(ab)^2}{4} + 4\cdot\frac{(bc)^2}{4} + 4\frac{(ca)^2}{4} =$

$$(ab)^2 + (bc)^2 + (ca)^2 \ge ab^2c + a^2cb + abc^2 = abc(a+b+c)$$

5.51

$$\Omega(x) = \int_0^x \frac{ln(1+ax)}{1+a^2}da;$$

$$= \int_0^x\int_0^x \frac{a\,da\,dt}{(1+at)(1+a^2)} = \int_0^x\int_0^x \frac{a(1+t^2)dtda}{(1+t^2)(1+at)(1+a^2)}$$

$$= \int_0^x \left[\frac{1}{1+t^2}\int_0^x \frac{a\,da}{1+a^2}\right]dt + \int_0^x \left[\frac{t}{1+t^2}\int_0^x \frac{da}{1+a^2}\right]dt -$$

$$-\int_0^x \left[\frac{t}{1+t^2}\int_0^x \frac{da}{1+at}\right]dt$$

$$= \left(\int_0^x \frac{dt}{1+t^2}\right)\left(\int_0^x \frac{a\,da}{1+a^2}\right) + \left(\int_0^x \frac{t\,dt}{1+t^2}\right)\left(\int_0^x \frac{da}{1+a^2}\right) - \int_0^x \frac{ln(xt+1)}{1+t^2}dt$$

$$\therefore 2\Omega(x) = \frac{tan^{-1}x}{2} ln(1+x^2) + \frac{ln(1+x^2)}{2} tan^{-1}x = tan^{-1}x\, ln(1+x^2)$$

Hence, $2\big(\Omega(x)+\Omega(y)+\Omega(z)\big)$

$$= tan^{-1}x\, ln(1+x^2) + tan^{-1}y\, ln(1+y^2) + tan^{-1}z\, ln(1+z^2)$$

Now, $x \in (0,1)$. *By* $AM \geq GM$ $ln(1+x^2) \geq ln(2x)$

$tan^{-1}x \geq 1$ *for* $x \in (0,1)$

$$\therefore LHS \geq ln(2x) + ln(2y) + ln(2z) = 3\ln 2 + ln(xyz)$$

5.52

We have that $(x+y)^2 \geq 4xy$

$$\frac{x+y}{4xy} \geq \frac{1}{x+y} \Rightarrow \frac{1}{x+y} \leq \frac{1}{9}\left(\frac{1}{x}+\frac{1}{9}\right) \quad (1)$$

So, using (1) (integrating (1)), we have:

$$\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)dx\,dy\,dz \leq \frac{1}{2}\int_a^{2a}\int_b^{2b}\int_c^{2c}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dx\,dy\,dz$$

$$= \frac{1}{2}(bc\ln 2 + ca\ln 2 + ab\ln 2) = \frac{1}{2}(\ln 2^{bc+ca+ab}) = \ln 2^{\frac{ab+bc+ca}{2}} = \ln\sqrt{2^{ab+bc+ca}}$$

5.53

$$\Omega = lim_{n\to\infty}(\pi+H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} =$$

$$lim_{n\to\infty}(\pi+H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{H_n}} + lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} \quad (1)$$

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \lim_{n\to\infty} H_n = \infty$$

$$\Omega_1 = \lim_{n\to\infty}(\pi + H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{H_n}} = \lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}}\left(\left(\frac{\pi + H_n}{H_n}\right)^{1+\frac{1}{H_n}} - 1\right) =$$

$$= lim_{n\to\infty} H_n^{\frac{1}{H_n}} \cdot H_n\left(e^{\left(1+\frac{1}{H_n}\right)ln\left(1+\frac{\pi}{H_n}\right)} - 1\right) \quad (2)$$

$$\lim_{n\to\infty} H_n^{\frac{1}{H_n}} = e^{\lim_{n\to\infty}\frac{ln\,H_n}{H_n}} \overset{C.S.}{=} e^{\lim_{n\to\infty}\frac{ln\,H_{n+1} - ln\,H_n}{H_{n+1} - H_n}} = e \cdot \frac{\lim_{n\to\infty} ln\frac{H_n + 1}{H_n}}{\frac{1}{n+1}}$$

$$= e^{\lim_{n\to\infty}(n+1)} \cdot \frac{ln\left(1 + \frac{H_{n+1} - H_n}{H_n}\right)}{\frac{H_{n+1} - H_n}{H_n}} \cdot \frac{H_{n+1} - H_n}{H_n}$$

$$= e^{lim_{n\to\infty}(n+1)} \cdot \frac{1}{(n+1)H_n} = e^{lim_{n\to\infty}\frac{1}{H_n}} = e^0 = 1 \quad (3)$$

$$\lim_{n\to\infty} H_n \cdot \frac{\left(e^{\left(1+\frac{1}{H_n}\right)ln\left(1+\frac{\pi}{H_{n-1}}\right)}\right)}{\left(1 + \frac{1}{H_n}\right)ln\left(1 + \frac{\pi}{H_n}\right)} \cdot \left(1 + \frac{1}{H_n}\right)ln\left(1 + \frac{\pi}{H_n}\right) =$$

$$= \lim_{n\to\infty}(1 + H_n)\frac{ln\left(1 + \frac{\pi}{H_n}\right)}{\frac{\pi}{H_n}} \cdot \frac{\pi}{H_n} = \lim_{n\to\infty}(1 + H_n) \cdot \frac{\pi}{H_n} =$$

$$= lim_{n\to\infty}\left(\frac{\pi}{1+n} + \pi\right) = \pi \quad (4)$$

From (2)+(3)+(4) $\Rightarrow \Omega_1 = \pi$ *(5)*

$$\Omega_2 = \lim_{n\to\infty}(H_n)^{1+\frac{1}{H_n}} - (H_n)^{1+\frac{1}{\pi+H_n}} = \lim_{n\to\infty}(H_n)^{1+\frac{1}{\pi+H_n}}\left(H_n^{\frac{\pi}{H_n(1+H_n)}} - 1\right) =$$

$$= lim_{n\to\infty} H_n^{\frac{1}{\pi+H_n}} \cdot H_n\left(H_n^{\frac{\pi}{H_n(1+H_n)}} - 1\right) \quad (6)$$

$$lim_{n\to\infty} H_n^{\frac{1}{\pi+H_n}} = 1 \quad \textit{(from 3)}$$

$$\lim_{n\to\infty}\frac{H_n\left(e^{\frac{\pi\,ln\,H_n}{H_n(1+H_n)}} - 1\right)}{\frac{\pi\,ln\,H_n}{H_n(1+H_n)}} \cdot \frac{\pi\,ln\,H_n}{H_n(1 + H_n)} = \pi\lim_{n\to\infty}\frac{ln\,H_n}{1 + H_n} \overset{C.S}{=} \pi\lim_{n\to\infty}\frac{ln\,H_{n+1} - ln\,H_n}{H_{n+1} - H_n} =$$

$$\pi\,lim_{n\to\infty}\frac{ln\frac{H_{n+1}}{H_n}}{\frac{1}{n+1}} = 0 \quad \textit{(from (3)) (7)}$$

$$\text{From (6)+(7)} \Rightarrow \Omega_2 = 0 \quad (8)$$

$$\text{From (5)+(8)} \Rightarrow \Omega = \pi$$

5.54

$$I \coloneqq \sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(n+1)-1\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{n=1}^{\infty}\frac{1}{j^{n+1}}-1\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\left(\sum_{j=2}^{\infty}\frac{1}{j^{n+1}}\right) = \sum_{j=2}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{j^{n+1}}$$

$$= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{j}\right)^{n+1} = \sum_{j=2}^{\infty}\frac{1}{j^2}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{j}\right)^{n-1}$$

$$= \sum_{j=2}^{\infty}\frac{1}{j^2}\left(\frac{1}{1+\frac{1}{j}}\right) = \sum_{j=2}^{\infty}\frac{1}{j^2+j} = \sum_{j=2}^{\infty}\left(\frac{1}{j}-\frac{1}{j+1}\right)$$

$$= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\cdots = \frac{1}{2}$$

$$\therefore I \coloneqq \sum_{n=1}^{\infty}(-1)^{n-1}\left(\zeta(n+1)-1\right) = \frac{1}{2}$$

5.55

$$\sqrt[3]{xyz}+\sqrt[3]{yzt}+\sqrt[3]{ztx}+\sqrt[3]{txy} \le \frac{x+y+z+y+z+t+z+t+x+t+x+y}{3}$$

$$= \frac{3(x+y+z+t)}{3} = x+y+z+t$$

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 (x+y+z+t)\,dxdydzdt = \int_0^1\int_0^1\int_0^1\left(\frac{x^2}{2}+yx+zx+tx\right)_0^1 dydzdt =$$

$$= \int_0^1\int_0^1\int_0^1\left(\frac{1}{2}+y+z+t\right)dydzdt$$

$$= \int_0^1\int_0^1\left(\frac{1}{2}y+\frac{y^2}{2}+zy+ty\right)_0^1 dzdt = \int_0^1\int_0^1\left(\frac{1}{2}+\frac{1}{2}+z+t\right)dzdt = \int_0^1\left(z+\frac{z^2}{2}+tz\right)_0^1 dt$$

$$= \int_0^1\left(1+\frac{1}{2}+t\right)dt = \left(t+\frac{t}{2}+\frac{t^2}{2}\right)_0^1 = 1+\frac{1}{2}+\frac{1}{2} = 2 \Rightarrow I \le 2$$

5.56

$$Let\ f(t)=ln(t+1)-tln2, f'(t)=\frac{1}{t+1}-ln2, f''(t)=-\frac{1}{(t+1)^2}<0, t\geq 1$$

$$f'-decreasing, f'(t)\leq f'(1)=\frac{1}{2}-ln2<0\to f-decreasing$$

$$\underbrace{max}_{t\in[0,\infty)} f(t)=f(1)=0\to ln(t+1)-tln2\leq 0\to ln(t+1)\leq tln2$$

Equality holds for $t=1$

$$[ln(x+y)-lnx]\cdot[ln(x+y)-lny]=ln\left(\frac{x+y}{x}\right)\cdot ln\left(\frac{x+y}{y}\right)=$$

$$=ln\left(1+\frac{y}{x}\right)ln\left(1+\frac{x}{y}\right)\leq\frac{x}{y}\cdot ln2\cdot\frac{y}{x}\cdot ln2=ln^2 2$$

$$\Omega(a,b)=\int_a^{2a}\left(\int_b^{2b}\big((ln(x+y)-lnx)(ln(x+y)-lny)\big)dy\right)dx\leq$$

$$\leq\int_a^{2a}\left(\int_b^{2b}(ln^2 2)dy\right)dx=ln^2 2\cdot ab$$

$$\frac{1}{ln2}(\Omega(a,b)+\Omega(b,c)+\Omega(c,b))<ln2\cdot(ab+bc+ca)\leq$$

$$\leq(a^2+b^2+c^2)ln2=ln2^{a^2+b^2+c^2}$$

5.57

$$\Omega(a,b)=\int_0^1\left(\int_0^1\sqrt{(x^2+a^2+b^2)(y^2+a^2+b^2)}dy\right)dx\overset{CBS}{\geq}$$

$$\geq\int_0^1\left(\int_0^1(ax+ab+by)dy\right)dx=\frac{a+b}{2}+ab$$

$$\sum\Omega(a,b)\geq\sum\frac{a+b}{2}+\sum ab=$$

$$=a+b+c+ab+bc+ca\overset{AM-GM}{\geq}6\sqrt[6]{(abc)^3}=6\sqrt{abc}\geq 6abc$$

5.58

$$x^2+xy+y^2\geq\frac{3}{4}(x+y)^2\to(x^2+xy+y^2)^2\geq\frac{9}{16}(x+y)^4, (1)$$

$$(2x+y)(x+2y)\overset{AM-GM}{\leq}\frac{9(x+y)^2}{4}, (2)$$

$$By\ (1),(2)\to\frac{(x^2+xy+y^2)^2}{(2x+y)(x+2y)}\geq\frac{\frac{9}{16}(x+y)^4}{\frac{9}{4}(x+y)^2}=\frac{1}{4}(x+y)^2\geq xy$$

$$\Omega(a,b) = \int_a^{2a}\left(\int_b^{2b} \frac{(x^2+xy+y^2)^2}{(2x+y)(x+2y)}dy\right)dx \geq \int_a^{2a}\int_b^{2b} xydydx = \frac{9}{4}a^2b^2$$

$$\frac{\Omega(a,b)}{a^2b^2} \geq \frac{9}{4} \to \frac{\Omega(a,b)}{a^2b^2} + \frac{\Omega(b,c)}{b^2c^2} + \frac{\Omega(c,a)}{c^2a^2} \geq \frac{27}{4}$$

5.59

$$\int_1^2\int_1^2\int_1^2 \frac{x}{y}dxdydz = \int_1^2 xdx \cdot \int_1^2 \frac{1}{y}dy \cdot \int_1^2 dz = \frac{3ln2}{2}$$

$$\Omega = \int_1^2\int_1^2\int_1^2 \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{x}{z}+\frac{y}{x}+\frac{z}{y}\right)dxdydz = 6\cdot\frac{3ln2}{2} = 9ln2$$

$$9ln2 < \frac{15}{2} \leftrightarrow ln2 < \frac{5}{6} \leftrightarrow ln2 < 0.833333..(true)$$

5.60

$$|sin(x-y)\,cos(x+y) - sin(x+y)| \leq$$

$$\leq |sin(x-y)|\cdot|cos(x+y)| + |sin(x+y)| \leq |cos(x+y)| + |sin(x+y)| \leq$$

$$\leq \sqrt{2}\cdot\sqrt{cos^2(x+y)+sin^2(x+y)} = \sqrt{2}$$

$$\Omega(a,b) \leq \sqrt{2}\int_a^{2a}\left(\int_b^{2b} dy\right)dx = \sqrt{2}ab$$

$$\Omega(a,b) + \Omega(b,c) + \Omega(c,a) \leq \sqrt{2}(ab+bc+ca) \leq \sqrt{2}(a^2+b^2+c^2)$$

5.61

$$\int_a^b\int_a^b \frac{x^4+y^4+x^2y^2}{x^2+y^2+xy}dxdy = \int_a^b\int_a^b \frac{(x^2+y^2+xy)(x^2+y^2-xy)}{x^2+y^2+xy}dxdy =$$

$$= \int_a^b\int_a^b (x^2+y^2-xy)dxdy = \frac{2(b^3-a^3)(b-a)}{3} - \frac{(b^2-a^2)^2}{4} \geq \frac{(b^2-a^2)^2}{4} \leftrightarrow$$

$$\leftrightarrow \frac{2(b^3-a^3)(b-a)}{3} \geq \frac{(b^2-a^2)^2}{2} \leftrightarrow 4(b^2+ab+a^2) \geq 3(a^2+2ab+b^2) \leftrightarrow$$

$$\leftrightarrow (b-a)^2 \geq 0$$

5.62

$a \geq 0$ then:

$$\int_0^a \int_0^a \int_0^a \sqrt{x(x+y)+z(x+y)}\,dxdydz = \int\int\int \sqrt{(x+y)(z+x)}\,dxdydz \overset{CBS}{\geq}$$

$$\geq \int\int\int \left(\sqrt{xz}+\sqrt{xy}\right)dx\,dy\,dz = \int_0^a \int_0^a \left(\sqrt{y}+\sqrt{z}\right)dy\,dz \cdot \int_0^a \sqrt{x}\,dx =$$

$$= \int_0^a \sqrt{x}\ dz \cdot \int_0^a \left(\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \sqrt{z}\cdot y\right)\Bigg|_0^a\right)dz = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^a \cdot \int_0^a \left(\frac{2}{3}\cdot a^{\frac{3}{2}} + a\cdot\sqrt{z}\right)dz =$$

$$= \frac{2}{3}\cdot a^{\frac{3}{2}}\cdot\left(\frac{2}{3}a^{\frac{2}{3}}\cdot z + \frac{a\cdot z^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_0^a = \frac{2}{3}\cdot a^{\frac{2}{3}}\cdot\left(\frac{2}{3}\cdot a^{\frac{3}{2}} + \frac{2}{3}\cdot a^{\frac{3}{2}}\right) = \frac{8}{9}\cdot a^4$$

5.63

By Schweitzer's inequality:

$$(x+y)\left(\frac{1}{x}+\frac{1}{y}\right) \leq \frac{(1+2)^2}{4\cdot 1\cdot 2}\cdot 2^2 \to 2+\frac{x}{y}+\frac{y}{x} \leq \frac{9}{2} \to \sqrt{\frac{x}{y}+\frac{y}{x}} \leq \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$$

$$\sqrt{\frac{x}{y}+\frac{y}{x}} \geq \sqrt{2} \to \sqrt{2} \leq \sqrt{\frac{x}{y}+\frac{y}{x}} \leq \frac{\sqrt{10}}{2} \to$$

$$\int_a^b \int_a^b \sqrt{2}\,dxdy \leq \int_a^b \int_a^b \sqrt{\frac{x}{y}+\frac{y}{x}}\,dxdy \leq \int_a^b \int_a^b \frac{\sqrt{10}}{2}\,dxdy$$

$$\sqrt{2}(b-a)^2 \leq \int_a^b \int_a^b \sqrt{\frac{x}{y}+\frac{y}{x}}\,dxdy \leq \frac{\sqrt{10}}{2}(b-a)^2$$

5.64

For $0 < x, y < 1$

$$\frac{(x^y+y^x)(x+y)^{x+y}}{(2x)^y(2y)^x} = \frac{(x^y+y^x)(x+y)^{x+y}}{2^{x+y}x^y y^x}$$

$$= (x^{-y}+y^{-x})\left(\frac{x+y}{2}\right)^{x+y} \geq 2\sqrt{x^{-y}y^{-x}}\left(\sqrt{xy}\right)^{x+y} = 2[x^{-y}y^{-x}x^{x+y}y^{x+y}]^{\frac{1}{2}}$$

$$= 2[x^x y^x]^{\frac{1}{2}}. \text{ But } x^x \geq \left(\frac{1}{e}\right)^{\frac{1}{e}} \quad \forall x > 0$$

$$[f(x) = x^x, x > 0, f'(x) = x^x(1 + \log x), f'(x) < 0 \text{ if } 0 < x < \frac{1}{e}$$

$$> 0 \text{ if } x > \frac{1}{e}, = 0 \text{ if } x = \frac{1}{e}] \text{ Thus,}$$

$$E = \frac{(x^y + y^x)(x + y)^{x+y}}{(2x)^y(2y)^x} \geq 2\left(e^{\frac{1}{e}}e^{\frac{1}{e}}\right)^{\frac{1}{2}} \Rightarrow E \geq 2e^{\frac{1}{e}}$$

$$\text{Now, } 2e^{\frac{1}{e}} > 1 \Leftrightarrow 2^e(e) > 1 \;\therefore E > 1$$

$$\Rightarrow \int_a^b \int_a^b E dx\, dy > \int_a^b (b-a)dy = (b-a)^2 \Rightarrow \frac{1}{(b-a)^2}\int_a^b \int_a^b E\, dx\, dy > 1$$

5.65

$$\sum xy = \sum \sqrt{((x-1)+1)\cdot((y-1)+1)} \overset{CBS}{\geq} 2\cdot\left(\sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}\right)$$

$$\frac{1}{2abc}\cdot\int\left(\int\left(\int \frac{\sum\sqrt{xy}}{\sum\sqrt{x-1}}dx\right)dy\right)dz \overset{CBS}{\geq} \frac{1}{2abc}\cdot\int_c^{2c}\left(\int_b^{2b}\left(\int_a^{2a}\frac{2\cdot\sum\sqrt{x-1}}{\sum\sqrt{x-1}}dx\right)dy\right)dz =$$

$$= \frac{1}{2abc}\cdot 2(2a-a)(2b-b)(2c-c) = 1$$

5.66

$$\text{For } 0 \leq x, y, z \leq \frac{\pi}{4}$$

$$(\tan x - 1)(\tan y - 1)(\tan z - 1) \leq 0$$

$$\Rightarrow \tan x \tan y \tan z - \sum \tan x \tan y + \sum \tan x \leq 1$$

$$\text{Also, } \tan x \tan y \tan z < \tan x \tan y, \tan y \tan z, \tan z \tan x$$

$$\therefore 4 \tan x \tan y \tan z - 2\sum \tan x \tan y + \sum \tan x \leq 1$$

$$\Rightarrow 4\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\prod \tan x\right)dxdydz - 2\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\left(\sum \tan x \tan y \tan z\right)dxdydz$$

$$+ \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\sum \tan x\, dxdydz \leq \int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} dxdydz$$

$$\Rightarrow \Omega_1 - 2\Omega_2 + 4\Omega_4 \leq \left(\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right)\left(\frac{\pi}{4}\right) = \frac{\pi^3}{64}$$

5.67

$$2\int_1^a\int_1^b\int_1^c\left(\sum_{cyc}\frac{x}{x^2+yz}\right)dxdydz \overset{AM\geq GM}{\leq} 2\int_1^a\int_1^b\int_1^c\left(\sum_{cyc}\frac{x}{2\sqrt{x^2yz}}\right)dxdydz$$

$$=\int_1^a\int_1^b\int_1^c\left(\sum_{cyc}\frac{1}{\sqrt{xy}}\right)dxdydz \leq \int_1^a\int_1^b\int_1^c\left(\sum_{cyc}\frac{1}{x}\right)dxdydz \left[\because \sum_{cyc}\frac{1}{x}\geq\sum_{cyc}\frac{1}{\sqrt{xy}}\right]$$

$$=\sum_{cyc}\int_1^c\int_1^b\left(\int_1^a\frac{dx}{x}\right)dydz=\sum_{cyc}(b-1)(c-1)\ln a=\sum_{cyc}\ln a^{(b-1)(c-1)}$$

5.68

$$\frac{x+y+\sqrt{xy}}{\sqrt{x}+\sqrt{y}+\sqrt[4]{xy}}=\frac{\sqrt{x^3}-\sqrt{y^3}}{\sqrt{x}-\sqrt{y}}\cdot\frac{\sqrt[4]{x}-\sqrt[4]{y}}{\sqrt[4]{x^3}-\sqrt[4]{y^3}}=$$

$$=\frac{\sqrt[4]{x^3}+\sqrt[4]{y^3}}{\sqrt[4]{x}+\sqrt[4]{y}}=\sqrt{x}+\sqrt{y}-\sqrt[4]{xy}\geq\sqrt[4]{xy}$$

$$\int_a^b\int_a^b\sqrt{\frac{x+y+\sqrt{xy}}{\sqrt{x}+\sqrt{y}+\sqrt[4]{xy}}}dxdy\geq\int_a^b\int_a^b\sqrt[8]{xy}=\int_a^b\sqrt[8]{x}dx\int_a^b\sqrt[8]{y}dy=\frac{64}{81}\left(b^{\frac{9}{8}}-a^{\frac{9}{8}}\right)^2$$

5.69

$$\begin{cases}e^x\geq x+1\\ e^y\geq y+1\end{cases}\to e^x+e^y\geq x+y+2 \overset{AM-GM}{\geq} 2\sqrt{xy}+2\to$$

$$e^x+e^y-2\geq 2\sqrt{xy}\to\frac{e^x+e^y-2}{\sqrt{xy}}>2\to\left(\frac{e^x+e^y-2}{\sqrt{xy}}\right)^{10}>2^{10}\to$$

$$\int_a^b\int_a^b\left(\frac{e^x+e^y-2}{\sqrt{xy}}\right)^{10}\geq\int_a^b\int_a^b 2^{10}dxdy=2^{10}(b-a)^2$$

5.70

$$1\leq a\leq b\leq 2, \frac{x^2+xy}{y^2}+\frac{y^2+xy}{x^2}=\frac{(x^2+y^2)^2+xy(x-y)^2}{x^2y^2}\geq\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

$$\int_a^b\int_a^b\sqrt{\frac{x^2+xy}{y^2}+\frac{y^2+xy}{x^2}}dxdy\geq\int_a^b\int_a^b\left(\frac{x}{y}+\frac{y}{x}\right)dxdy\geq 2\int_a^b\int_a^b dxdy=2(b-a)^2$$

$$\frac{x^2+xy}{y^2}+\frac{y^2+xy}{x^2}=\left(\frac{x}{y}+\frac{y}{x}+2\right)\left(\frac{x}{y}+\frac{y}{x}-1\right)\le\frac{1}{4}\left(\frac{2x}{y}+\frac{2y}{x}+1\right)^2$$

$$\int_a^b\int_a^b\sqrt{\frac{x^2+xy}{y^2}+\frac{y^2+xy}{x^2}}\,dxdy\le\int_a^b\int_a^b\left(\frac{x}{y}+\frac{y}{x}+\frac{1}{2}\right)dxdy$$

$$=2(b-a)\ln\frac{b}{a}+\frac{(b-a)^2}{2},\ \textit{need to prove,}$$

$$2(b-a)\ln\frac{b}{a}+\frac{(b-a)^2}{2}\le\frac{3\sqrt{3}}{2}(b-a)^2\Leftrightarrow\frac{\ln b-\ln a}{b-a}\le\frac{3\sqrt{3}-1}{4}\approx 1.04$$

applying MVT $\frac{\ln b-\ln a}{b-a}=\frac{1}{c}\le 1$ *since* $1\le a\le c\le b\le 2$ *hence* $\frac{1}{c}\le\frac{3\sqrt{3}-1}{4}$, *which is true.*

5.71

$$x,y,z,t\in[0;1]$$

$$\to\begin{cases}(1-\sqrt{x})(1-\sqrt{y})\ge 0\\(1-\sqrt{z})(1-\sqrt{t})\ge 0\\(1-\sqrt{xy})(1-\sqrt{2t})\ge 0\end{cases}\to\begin{cases}\sqrt{x}+\sqrt{y}\le 1+\sqrt{xy}\\\sqrt{z}+\sqrt{t}\le 1+\sqrt{2t}\\\sqrt{xy}+\sqrt{zt}\le 1+\sqrt{xyzt}\end{cases}$$

$$\to(\sqrt{x}+\sqrt{y})(\sqrt{z}+\sqrt{t})\le(1+\sqrt{xy})(1+\sqrt{2t})=1+\sqrt{xy}+\sqrt{zt}+\sqrt{xyzt}\le 2(1+\sqrt{xyzt})$$

$$\to\frac{(\sqrt{x}+\sqrt{y})(\sqrt{z}+\sqrt{t})}{1+\sqrt{xyzt}}\le 2$$

$$\textit{So: } LHS\le\int_0^a\int_0^b\int_0^c\int_0^d 2\,dxdydzdt=2abcd$$

5.72

$$\int_0^1\int_0^1\int_0^1\sin^{10}(x^4+y^4+z^4)\,dxdydz\le\int_0^1\int_0^1\int_0^1\sin(x^4+y^4+z^4)\,dxdydz\le$$

$$\le\int_0^1\int_0^1\int_0^1(x^4+y^4+z^4)\,dxdydz$$

$$=\int_0^1\int_0^1\int_0^1 x^4dxdydz+\int_0^1\int_0^1\int_0^1 y^4dxdydz+\int_0^1\int_0^1\int_0^1 z^4dxdydz$$

$$=\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$$

5.73

$$x^5+y^5=(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4)$$

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4$$

$$(x+y)^4-\frac{x^5+y^5}{x+y}=x^4+4x^3y+6x^2y^2+4xy^3+y^4-x^4+x^3y-x^2y^2+xy^3-y^4$$

$$=5x^3y+5x^2y^2+5xy^3=5xy(x^2+xy+y^2)\geq 5xy\cdot\frac{3}{4}(x+y)^2$$

$$=\frac{25xy}{4}(x+y)^2\geq\frac{15xy}{4}\cdot 4xy=15x^2y^2$$

$$LHS\geq\int_a^b\int_a^b 15x^2y^2\,dxdy=\int_a^b 15x^2\left(\frac{y^3}{3}\right)_a^b dx=\int_a^b 5x^2(b^3-a^3)\,dx$$

$$=\left(\frac{5x^3}{3}\right)_a^b(b^3-a^3)=\frac{5}{3}(b^3-a^3)^2$$

5.74

$$\sqrt{\frac{x+a}{(x+b)(y+b)}}+\sqrt{\frac{x+b}{(x+a)(y+a)}}\geq 2\sqrt[4]{\frac{x+a}{(x+b)(y+b)}\cdot\frac{x+b}{(x+a)(y+a)}}=$$

$$=\frac{2}{\sqrt[4]{(y+a)(y+b)}}\geq\frac{2}{\sqrt[4]{(b+a)(b+b)}}=\frac{\sqrt[4]{8}}{\sqrt[4]{b(a+b)}},$$

$$\int_a^b\int_a^b\left(\sqrt{\frac{x+a}{(x+b)(y+b)}}+\sqrt{\frac{x+b}{(x+a)(y+a)}}\right)dxdy\geq\int_a^b\int_a^b\left(\frac{\sqrt[4]{8}}{\sqrt[4]{b(a+b)}}\right)dxdy=$$

$$=\frac{\sqrt[4]{8}(b-a)^2}{\sqrt[4]{b(a+b)}},\ (M_a\geq M_g)$$

5.75

We have that (By A.M-GM)

$$\frac{1+x+1+1+1}{5}\geq\sqrt[5]{x\cdot 1\cdot 1\cdot 1\cdot 1}\Rightarrow 4+x\geq 5\sqrt[5]{x}$$

Doing the same:
$$4+y\geq 5\sqrt[5]{y}$$
$$4+z\geq 5\sqrt[5]{z}$$
$$4+t\geq 5\sqrt[5]{t}$$

Adding these inequalities, we have that,

$$16+x+y+z+t\geq 5\left(\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{t}\right)\Leftrightarrow\frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{t}}{16+x+y+z+t}\leq\frac{1}{5}$$

$$\int_0^1\int_0^1\int_0^1\int_0^1\frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{t}}{16+x+y+z+t}dxdydzdt\leq\frac{1}{5}$$

5.76

$$x^2+y^2+z^2\geq xy+yz+zx|\cdot 2$$

$$\sum(x^4+1)\overset{M_a\geq M_g}{\geq}2\cdot\sum x^2\geq 2\cdot\sum xy\Leftrightarrow$$

$$\Leftrightarrow\sum x^4+3\geq 2\cdot\sum xy\,|(+2)$$

$$\sum x^4+5\geq 2\cdot\sum xy+2\Leftrightarrow$$

$$\frac{\sum x^4+5}{\sum xy+1}\geq 2\quad (*)$$

$$\frac{1}{16}\int\int\int\left(\frac{\sum x^4+5}{\sum xy+1}\right)^4dxdydz\geq\frac{1}{16}\int_0^1\int_0^1\int_0^1 16\,dxdydz$$

$$=\int_0^1 1\,dx\cdot\int_0^1 1\,dy\cdot\int_0^1 1\,dz=1$$

5.77

$$\left(\frac{x+y+z+t}{4}\right)^5\leq\frac{x^5+y^5+z^5+t^5}{4}\Rightarrow\frac{(x+y+z+t)^5}{x^5+y^5+z^5+t^5}\leq 256$$

$$\Rightarrow\frac{x+y+z+t}{(x^5+y^5+z^5+t^5)^{\frac{1}{5}}}\leq(256)^{\frac{1}{5}}\Rightarrow\frac{1}{a^4}\int_a^{2a}\int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{x+y+z+t}{(x^5+y^5+z^5+t^5)^{\frac{1}{5}}}dxdydzdt$$

$$\leq\frac{1}{a^4}(256)^{\frac{1}{5}}\int_a^{2a}\int_a^{2a}\int_a^{2a}\int_a^{2a}dxdydzdt=(256)^{\frac{1}{5}}$$

5.78

$$\frac{x^3}{x^2+y^2+xy}\geq\frac{x^3}{x^2+y^2+\frac{x^2+y^2}{2}}=\frac{2x^3}{3(x^2+y^2)}\text{ so,}$$

$$\sum_{cyc}\frac{x^3}{x^2+y^2+xy}\geq\frac{2}{3}\cdot\left[\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}\right]$$

$$\overset{AM-GM}{\geq} \frac{2}{3} \cdot 3 \cdot \sqrt[3]{\frac{(xyz)^3}{\prod(x^2+y^2)}} = 2 \cdot \frac{xyz}{\sqrt[3]{\prod(x^2+y^2)}}$$

$$\left.\begin{matrix} 0 < a \leq x \leq 2a \\ 0 < a \leq y \leq 2a \\ 0 < a \leq z \leq 2a \end{matrix}\right\} \begin{matrix} \Rightarrow a^3 \leq xyz \leq 8a^3 \\ \Rightarrow 2a^2 \leq x^2 + y^2 \leq 4a^2 \Rightarrow 8a^6 \leq \prod(x^2+y^2) \leq 4^3 \cdot a^6 \end{matrix}$$

$$\Rightarrow 2a^2 \leq \sqrt[3]{\prod(x^2+y^2)} \leq 4a^2 \Rightarrow \frac{1}{\sqrt[3]{\prod(x^2+y^2)}} \geq \frac{1}{4a^2}$$

$$xyz \geq a^3$$

$$\frac{xyz}{\sqrt[3]{\prod(x^2+y^2)}} \geq \frac{a^3}{4a^2} = \frac{a}{4} \Rightarrow \frac{2xyz}{\sqrt[3]{\prod(x^2+y^2)}} \geq \frac{a}{2}$$

$$\textit{thus } \frac{1}{a^4}\Big|_a^{2a}\Big|_a^{2a}\Big|_a^{2a}\left(\sum_{cyc}\frac{x^3}{x^2+y^2+xy}\right)dxdydz$$

$$\geq \frac{1}{a^4}\Big|_a^{2a}\Big|_a^{2a}\Big|_a^{2a} 3\left(\frac{a}{2}\right)dxdydz = \frac{1}{a^4} \cdot \frac{3a}{2} \cdot a \cdot a \cdot a = \frac{3}{2}$$

5.79

$$(x^2+y^2+z^2)(x+y+z) \overset{AM-GM}{\geq} 3\sqrt[3]{x^2y^2z^2} \cdot 3\sqrt[3]{xyz} = 9xyz$$

$$\frac{1}{(x^2+y^2+z^2)(x+y+z)} \leq \frac{1}{9xyz} \to \frac{xyz}{(x^2+y^2+z^2)(x+y+z)} \leq \frac{1}{9}$$

$$\int_1^2\int_1^2\int_1^2 \frac{xyz}{(x^2+y^2+z^2)(x+y+z)}dxdydz \leq \int_1^2\int_1^2\int_1^2 \frac{1}{9}dxdydz = \frac{1}{9}$$

5.80

The function $f(x) = \frac{1}{x}$ *is convex on* $(0, \infty)$ *then*

$$\frac{xy}{x\cos^2 z + y\sin^2 z} = \frac{1}{\frac{\cos^2 z}{y} + \frac{\sin^2 z}{x}} \leq (x\sin^2 z + y\cos^2 z)$$

we have $\frac{xy}{x\cos^2 z+y\sin^2 z} \leq (y\sin^2 z + x\cos^2 z) \leq a$, $(x, y \in [0, a])$ *and* $\frac{xy}{a} \leq \frac{xy}{x\cos^2 z+y\sin^2 z}$

then

$$\frac{1}{4} = \frac{1}{a^4}\int_0^a\int_0^a xy\,dxdy \leq \frac{1}{a^3}\int_0^a\int_0^a \frac{xy}{x\cos^2 z + y\sin^2 z} \leq \frac{1}{a^2}\int_0^a\int_0^a dx\,dy = 1$$

It follow that

$$\frac{1}{4} \le \frac{1}{a^3} \int_0^a \int_0^a \frac{xy\, dxdy}{x \sin^2 z + y \cos^2 z} \le 1$$

5.81

Using AM-GM we have that

$$1 + f(x)f(y) \ge 2\sqrt{f(x)f(y)} \quad (1) \text{ So, by GM-HM we have}$$

$$\sqrt{f(x)f(y)} \ge \frac{2}{\frac{1}{f(x)} + \frac{1}{f(y)}} \Rightarrow 2\sqrt{f(x)f(y)} \ge \frac{4}{\frac{1}{f(x)} + \frac{1}{f(y)}} \quad (2)$$

So, using (1)+(2) we have that

$$1 + f(x)f(y) \ge \frac{4}{\frac{1}{f(x)} + \frac{1}{f(y)}} \overset{f(x)>0}{\underset{f(y)>0}{\Leftrightarrow}} \frac{1}{1 + f(x)f(y)} \le \frac{\frac{1}{f(x)} + \frac{1}{f(y)}}{4} \Rightarrow$$

$$\int_0^1 \int_0^1 \frac{dxdy}{1 + f(x)f(y)} \le \int_0^1 \int_0^1 \frac{\frac{1}{f(x)} + \frac{1}{f(y)}}{4} dxdy \Rightarrow$$

$$\int_0^1 \int_0^1 \frac{dxdy}{1 + f(x)f(y)} \le \frac{\int_0^1 \frac{1}{f(x)} dx + \int_0^1 f(y) dy}{4} = \frac{\int_0^1 \frac{1}{f(x)} dx}{2}$$

5.82

$$\int_1^b \int_1^a \frac{x + y}{x^2 + y^2} dxdy \le 2 \int_1^b \int_1^a \frac{dxdy}{x + y} \overset{AM \ge GM}{\le} \int_1^b \int_1^a \frac{dxdy}{\sqrt{xy}}$$

$$\overset{Cauchy-Schwarz}{\le} \sqrt{\left(\int_1^a \frac{dx}{x}\right)\left(\int_1^a dx\right)} \cdot \sqrt{\left(\int_1^b \frac{dy}{y}\right) \cdot \left(\int_1^b dy\right)} = \sqrt{(a-1) \ln b \cdot (b-1) \ln a}$$

$$\overset{AM \ge GM}{\le} \frac{(a-1) \ln b + (b-1) \ln a}{2} = \frac{\ln(a^b \cdot b^a) - \ln(ab)}{2} = \ln \sqrt{\frac{a^b b^a}{ab}}$$

5.83

$$2\sqrt{x} + 3 = \sqrt{x} + \sqrt{x} + 1 + 1 + 1 \overset{AM-GM}{\ge} 5\sqrt[5]{\sqrt{x} \cdot \sqrt{x} \cdot 1 \cdot 1 \cdot 1} = 5\sqrt[5]{x}$$

$$\sum (2\sqrt{x} + 3) \ge \sum 5\sqrt[5]{x} \to 2\sum \sqrt{x} + 12 \ge 5\sum \sqrt[5]{x} \to$$

$$\rightarrow \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{t}+6}{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{z}} \geq \frac{5}{2} \rightarrow$$

$$\int_1^2\int_1^2\int_1^2\int_1^2 \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{t}+6}{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}+\sqrt[5]{z}} dxdydzdt \geq \int_1^2\int_1^2\int_1^2\int_1^2 \frac{5}{2} dxdydzdt = \frac{5}{2}$$

5.84

$$\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x} \leq \frac{x+y}{4}+\frac{y+z}{4}+\frac{z+x}{4}=\frac{x+y+z}{2}$$

so, $\left[\sum_{cyc}\frac{xy}{x+y}\right]^2 \leq \frac{(x+y+z)^2}{4} \leq \frac{3}{4}(x^2+y^2+z^2)$ *thus,*

$$\int_1^2\int_1^2\int_1^2 \left(\frac{3}{4}(x^2+y^2+z^2)\right) dxdydz = \int_1^2\int_1^2 \frac{3}{4}\cdot\left[\frac{x^3}{3}+y^2x+z^2x\right]_1^2 dydz$$

$$=\frac{3}{4}\int_1^2\int_1^2 \left(\frac{7}{3}+y^2+z^2\right) dydz = \frac{3}{4}\cdot\int_1^2 \left[\frac{7}{3}z+y^2z+\frac{z^3}{3}\right]_1^2 dy$$

$$=\frac{3}{4}\int_1^2 \left(\frac{7}{3}+y^2+\frac{7}{3}\right) dy = \frac{3}{4}\cdot\left[\frac{14}{3}y+\frac{y^3}{3}\right]_1^2 = \frac{3}{4}\cdot\frac{21}{3}=\frac{21}{4}$$

$$\int_1^2\int_1^2\int_1^2 \left(\sum_{cyc}\frac{xy}{x+y}\right) dxdydz \leq \frac{21}{4}$$

5.85

$$e^x+e^y-4 \geq x+1+y+1-4 \overset{AM-GM}{\geq} 2\sqrt{xy}-2$$

$$\left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5 \geq 2^5 \rightarrow \int_2^3\int_2^3 \left(\frac{e^x+e^y-4}{\sqrt{xy}-1}\right)^5 dxdy \geq \int_2^3\int_2^3 32dxdy = 32$$

5.86

$$I = \int_2^3 \log x \log(x^2-1)\, dx$$

$$I = \left[\left(\log(x^2-1)\,(x\log x - x)\right)\right]_2^3 - \int_2^3 \frac{(2x)(x\log x - x)}{x^2-1} dx$$

$$I = \left[\left(log(8)\,(3\,log\,3 - 3)\right) - \left(log(3)\,(2\,log\,2 - 2)\right)\right] - 2\int_{2}^{3} \frac{(x^2)(log\,x - 1)}{x^2 - 1}\,dx$$

$$I = [2\,log(3) + log(2)\,(7\,log(3) - 9)] - 2\int_{2}^{3} \frac{(x^2\,log\,x) - x^2}{x^2 - 1}\,dx$$

$$I = [2\,log(3) + log(2)\,(7\,log(3) - 9)] - 2\int_{2}^{3} \frac{(x^2\,log\,x)}{x^2 - 1}\,dx + 2\int_{2}^{3} \frac{x^2}{x^2 - 1}\,dx$$

$$I = [2\,log(3) + log(2)\,(7\,log(3) - 9)] - 2\int_{2}^{3} log\,x\,dx - 2\int_{2}^{3} \frac{log\,x}{x^2 - 1}\,dx + 2\int_{2}^{3} dx$$

$$+ 2\int_{2}^{3} \frac{dx}{x^2 - 1}$$

$$I = [2\,log(3) + log(2)\,(7\,log(3) - 9)] - 2[x\,log\,x - x]_2^3 - \int_{2}^{3} \frac{log\,x}{x - 1}\,dx + \int_{2}^{3} \frac{log\,x}{x + 1}\,dx$$

$$+ \left[log\left|\frac{x - 1}{x + 1}\right|\right]_2^3$$

$$I = [2\,log(3) + log(2)\,(7\,log(3) - 9)] - 2[x\,log\,x - 2]_2^3 - \int_{2}^{3} \frac{log\,x}{x - 1}\,dx + \int_{2}^{3} \frac{log\,x}{x + 1}$$

$$+ \left[log\left|\frac{1}{2}\right| - log\left|\frac{1}{3}\right|\right]$$

Let

$$A = \int_{2}^{3} \frac{log\,x}{x - 1}\,dx$$

$$A = [-Li_2(1 - x) + c]_2^3$$

$$A = -Li_2(-2) + Li_2(-1)$$

$$A = -Li_2(-2) - \frac{\pi^2}{12} \cong 0.614279 \quad (1)$$

$$B = \int_{2}^{3} \frac{log\,x}{x + 1}\,dx = \sum_{k=0}^{\infty} (-1)^k \int_{2}^{3} x^k\,log\,x\,dx$$

$$B = \sum_{k=0}^{\infty}(-1)^k\left[\frac{x^{k+1}\log x}{(x+1)} - \frac{x^{k+1}}{(k+1)^2}\right]_2^3$$

$$B = \sum_{k=0}^{\infty}(-1)^k\left[\frac{3(3^k\log 3)}{(k+1)} - \frac{3(3^k)}{(k+1)^2} - \frac{2(2^k\log 2)}{(k+1)} + \frac{2(2^k)}{(k+1)^2}\right]$$

$$B = -\sum_{k=1}^{\infty}(-1)^k\left[\frac{(3^k\log 3)}{(k)} - \frac{(3^k)}{(k)^2} - \frac{(2^k\log 2)}{(k)} + \frac{(2^k)}{(k)^2}\right]$$

$$B = -[\log 3\, Li_1(-3) - Li_2(-3) - \log 2\, Li_1(-2) + Li_2(-2)]$$

$$B = [Li_2(-3) - Li_2(-2) + \log(2)\log(3)] \cong 0.258871 \quad (2)$$

$$\Rightarrow I = [2\log(3) + \log(2)(7\log(3) - 9)] - 2[x\log x - x]_2^3 - \left(-Li_2(-2) - \frac{\pi^2}{12}\right) +$$

$$+([Li_2(-3) - Li_2(-2) + \log(2)\log(3)]) + \left[\log\left|\frac{1}{2}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)(7\log(3) - 9)] - 2[3\log 3 - 3 - 2\log 2 + 2] -$$

$$-\left(-Li_2(-2) - \frac{\pi^2}{12}\right) + ([Li_2(-3) - Li_2(-2) + \log(2) + \log(2)\log(3)]) +$$

$$+\left[\log\left|\frac{1}{2}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)(7\log(3) - 9)] - 2[3\log 3 - 3 - 2\log 2 + 2] -$$

$$-\left(-Li_2(-2) - \frac{\pi^2}{12}\right) + ([Li_2(-3) - Li_2(-2) + \log(2)\log(3)]) + \left[\log\left|\frac{3}{2}\right|\right]$$

$$\Rightarrow I = [2\log(3) + \log(2)(7\log(3) - 9)] - 2\left[\log\left(\frac{27}{4}\right) - 1\right] - (0.614279) + (0.258871)$$

$$+\left[\log\left|\frac{3}{2}\right|\right]$$

$$\Rightarrow I = \log\left(\frac{9}{512}\right) + 7(\log 2\log 3) - 2\log\left(\frac{27}{4}\right) + \log\left(\frac{3}{2}\right) + 1.644592 < \frac{35}{8} + \log\left(\frac{3}{2}\right)$$

5.87

Because for all $x, y, z > 0$

$$\frac{3}{4}(x+y) + \frac{1}{(x+y)} \geq \sqrt{3} \Rightarrow \frac{3}{4}(x+y)^2 + 1 \geq \sqrt{3}(x+y) \Rightarrow$$

$$\Rightarrow x^2 + xy + y^2 + 1 \geq \sqrt{3}(x+y) \Rightarrow x^2 + 2xy + y^2 + 1 \geq xy + \sqrt{3}(x+y) \Rightarrow$$

$$\Rightarrow \frac{(x+y)^2+1}{xy+\sqrt{3}(x+y)} \ge 1$$

$$\textit{Hence } \Omega(a) = \int_a^{2a}\int_a^{2a} \frac{(x+y)^2+1}{xy+(x+y)\sqrt{3}}\,dx\,dy \ge \int_a^{2a}\int_a^{2a} 1\,dx\,dy = xy|_a^{2a}|_a^{2a} =$$

$$= (2a-a)(2a-a) = a^2$$

$$\textit{Similarly } \Omega(b) \ge b^2 \textit{ and } \Omega(c) \ge c^2$$

$$\textit{Hence } \Omega(a)+\Omega(b)+\Omega(c) \ge a^2+b^2+c^2 \ge ab+bc+ca$$

5.88

Using Cauchy – Schwarz inequality, we have that:

$$\sqrt{f^2(x)+f^2(y)} + \sqrt{2f(x)f(y)} \le \sqrt{2}\cdot\sqrt{\left(\sqrt{f^2(x)+f^2(y)}\right)^2 + \left(\sqrt{2f(x)+2f(y)}\right)^2}$$

$$= \sqrt{2}\cdot\sqrt{f^2(x)+f^2(y)+2f(x)f(y)} = \sqrt{2}\cdot\sqrt{\big(f(x)+f(y)\big)^2} =$$

$$= \sqrt{2}\big(f(x)+f(y)\big) \underset{in\ [0,a]\times[0,a)}{\overset{integrate}{\Rightarrow}}$$

$$\int_0^a\int_0^a \sqrt{f^2(x)+f^2(y)}\,dx\,dy + \int_0^a\int_0^a \sqrt{2f(x)f(y)} \le \sqrt{2}\int_0^a\int_0^a \big(f(x)+f(y)\big)$$

$$= \sqrt{2}\cdot 2a\int_0^a f(x)\,dx$$

5.89

$$\frac{x}{x^2+yz}+\frac{y}{y^2+zx}+\frac{z}{z^2+xy} \overset{A-G}{\le} \frac{x}{2\sqrt{x^2yz}}+\frac{y}{2\sqrt{y^2zx}}+\frac{z}{2\sqrt{z^2xy}}$$

$$= \frac{1}{2}\sum\frac{1}{\sqrt{xy}} \underset{(1)}{\overset{C-B-S}{\le}} \frac{1}{2}\sqrt{\sum\frac{1}{x}}\sqrt{\sum\frac{1}{y}} = \frac{1}{2}\left(\sum\frac{1}{x}\right)$$

$$\textit{(1)} \Rightarrow \textit{LHS} \le \frac{1}{2}\int_1^a\int_1^b\int_1^c\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dxdydz = \frac{1}{2}\int_1^a\int_1^b\left(\int_1^c\frac{dx}{x}+\left(\frac{1}{y}+\frac{1}{z}\right)\right)dy\,dz =$$

$$= \frac{1}{2}\int_1^a \int_1^b \left(\ln c + \left(\frac{1}{y} + \frac{1}{z}\right)(c-1)\right) dy dz$$

$$= \frac{1}{2}\int_1^a \left[(\ln c)\int_1^b dy + \frac{1}{z}(c-1)\int_1^b dy + (c-1)\int_1^b \frac{dy}{y}\right] dz =$$

$$= \frac{1}{2}\int_1^a \left[(b-1)\ln c + \frac{(b-1)(c-1)}{z} + (c-1)\ln b\right] dz =$$

$$= \frac{1}{2}\left[(b-1)\ln c \int_1^a dz + (b-1)(c-1)\int_1^a \frac{dz}{z} + (c-1)\ln b \int_1^a dz\right]$$

$$= \frac{1}{2}[(a-1)(b-1)\ln c + (b-1)(c-1)\ln a + (c-1)(a-1)\ln b]$$

$$= \frac{1}{2}\left[\ln c^{(a-1)(b-1)} + \ln a^{(b-1)(c-1)} + \ln b^{(c-1)(a-1)}\right] = \frac{1}{2}\left[\ln \prod a^{(b-1)(c-1)}\right] =$$

$$\ln \sqrt{\prod a^{(b-1)(c-1)}}$$

5.90

$$\Omega_1 = \left[\int_a^b (1+x^2)dx\right]^n = \left[b - a + \frac{b^3 - a^3}{3}\right]^n$$

and

$$\Omega_2 = \left[\int_a^b (1-x^2)dx\right]^n = \left[b - a - \frac{b^3 - a^3}{3}\right]^n$$

$$\Omega_2 + \Omega_2 < \left[2(b-a) + \frac{b^3 - a^3}{3} - \frac{b^3 - a^3}{3}\right]^n = 2^n(b-a)^n$$

[Because of $x^n + y^n < (x+y)^n$ for positive x, y]

$$\text{So, } \Omega_1 + \Omega_2 < [2(b-a)]^n = \left[\underbrace{1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1}_{\text{"}n-1\text{" times}} \cdot 2(b-a)\right]^n \leq$$

$$\underset{AM}{\overset{GM}{\leq}} \left[\left[\frac{1+1+1+\cdots+1+[2(b-a)]}{n}\right]^n\right]^n = \left[\frac{n-1+2(b-a)}{n}\right]^{n^2}$$

$$\Rightarrow \sqrt[n^2]{\Omega_1 + \Omega_2} < \frac{n-1}{n} + \frac{2(b-a)}{n} \Rightarrow \sqrt[n^2]{\Omega_1 + \Omega_2} < 1 - \frac{1}{n} + \frac{2(b-a)}{n}$$

$$\Rightarrow \sqrt[n^2]{\Omega_1 + \Omega_2} + \frac{1}{n} < 1 + \frac{2(b-a)}{n}$$

5.91

$$RHS = \int_a^{2a}\int_a^{2a}\int_0^1 [tx + (1-t)y]^4\, dz\, dy\, dx = \frac{1}{5}\int_a^{2a}\int_a^{2a} \left.\frac{[tx + (1-t)y]^5}{x-y}\right]_0^1 dy\, dx$$

$$= \frac{1}{5}\int_a^{2a}\int_a^{2a} \frac{x^5 - y^5}{x-y}\, dy\, dx = \frac{1}{5}\int_a^{2a}\int_a^{2a} [x^4 + x^3y + x^2y^2 + xy^3 + y^4]\, dy\, dx$$

$$= \frac{1}{5}\left[\left[\int_a^{2a}\int_a^{2a} x^4\, dy\, dx + \int_a^{2a}\int_a^{2a} x^3y\, dy\, dx + \int_a^{2a}\int_a^{2a} x^2y^2\, dy\, dx + \right.\right. \\ \left. + \int_a^{2a}\int_a^{2a} xy^3\, dy\, dx + \int_a^{2a}\int_a^{2a} y^4\, dy\, dx \right] =$$

$$= \frac{1}{5}\left[\frac{x}{5}((2a)^5 - a^5)a + \frac{7}{4 \times 2}((2a)^4 - a^4)(3a^2) + \frac{1}{3 \times 3}((2a)^3 - a^3)^2\right]$$

$$= \frac{1}{5}\left[\frac{62}{5} + \frac{45}{4} + \frac{4a}{a}\right]a^6 = \frac{5237}{5 \times 180}a^5 \qquad (1)$$

$$LHS = \frac{1}{16}\int_0^a\int_0^a (x+y)^4\, dy\, dx = \frac{1}{16}\int_0^a \left[\frac{1}{5}(x+y)^5\right]_0^a dx =$$

$$= \frac{1}{16} \times \frac{1}{5}\int_0^a [(x+a)^5 - x^5]\, dx = \frac{1}{16 \times 5 \times 6}[(x+a)^6 - x^6]_0^a =$$

$$= \frac{1}{96 \times 5}[(2a)^6 - a^6 - a^6] = \frac{31}{240}a^6 \quad (2)$$

We wish to show: $\frac{31}{240}a^6 < \frac{5237}{5 \times 180}a^6 \Leftrightarrow \frac{31}{240} < \frac{5237}{900}$

It is clearly true as $LHS < 1$ *and* $RHS > 1$.

5.92

For a complex z *such that* $|z| \leq 1, z \neq -1$ *we have*

$$log(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^k}{k}$$

Hence for $\theta \in (-\pi, \pi)$ *we have*

$$log(1 + e^{i\theta}) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} e^{ik\theta}}{k}$$

Taking real parts we get

$$log\left|1+e^{i\theta}\right| = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\pi)}{k}$$

Finally, for $\theta \in (-\pi,\pi)$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\theta)}{k} = \frac{1}{2}log(2+2\cos(\theta))$$

Or, for $a \in (-1,1]$*, we have*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cos(k\arccos a)}{k} = log\sqrt{2+2a} \le log\sqrt{2e^a}$$

Because $1+a \le e^a$

5.93

We use a fundamental inequality: $\sqrt{x^2+y^2}+\sqrt{a^2+b^2} \ge \sqrt{(x+a)^2+(y+b)^2}$ *(1)*

Let's prove it. On Squaring both sides, we get:

$$x^2+y^2+a^2+b^2+2\sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge (x+a)^2+(y+b)^2 \Rightarrow$$

$$\Rightarrow 2\sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge 2ax+2by \Rightarrow \sqrt{x^2+y^2}\sqrt{a^2+b^2} \ge ax+by$$

Again, squaring both sides, we get: $(x^2+y^2)(a^2+b^2) \ge (ax+by)^2 \Rightarrow$

$\Rightarrow x^2a^2+b^2x^2+a^2y^2+b^2y^2 \ge a^2x^2+b^2y^2+2axby \Rightarrow (ax-by)^2 \ge 0$ *which is true and the inequality in equation (1) is true. Using the inequality in equation (1)*

$$\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2} \ge \sqrt{(2a)^2+(2x+2y)^2} \ge 2\sqrt{a^2+(x+y)^2} \Rightarrow$$

$$\Rightarrow \frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{\sqrt{a^2+(x+y)^2}} \ge 2$$

$$\int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}}dx\,dy\,dz \ge \int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{2}{z}dx\,dy\,dz \ge \int_a^{2a}\int_a^{2a}[2\ln z]_a^{2a}\,dx\,dy$$

$$\ge \int_a^{2a}\int_a^{2a}(2\log 2)\,dx\,dy \ge (2\log 2)\int_a^{2a}dx\int_a^{2a}dy \ge 2\log 2\,(a\cdot a) \ge 2a^2\log 2$$

$$\therefore \int_a^{2a}\int_a^{2a}\int_a^{2a}\frac{\sqrt{a^2+4x^2}+\sqrt{a^2+4y^2}}{z\sqrt{a^2+(x+y)^2}} \ge 2a^2\log 2$$

5.94

Since e^x is a convex function we apply the Hermite – Hadamard inequality for double integral:

$$\frac{1}{a^2}\int_a^{2a}\int_a^{2a} \sqrt[16]{c^{(x+3y)^2}}\,dx\,dy \ge e^{\left(\frac{\frac{3a}{2}+\frac{9a}{2}}{4}\right)^2} = e^{\left(\frac{3a}{2}\right)^2} = \sqrt[4]{e^{9a^2}} \Rightarrow$$

$$\Rightarrow \int_a^{2a}\int_a^{2a} \sqrt[16]{e^{(x+3y)^2}}\,dx\,dy \ge a^2\sqrt[4]{e^{9a^2}}$$

Also, since e^x is convex we have: $e^{\left(\frac{x+3y}{4}\right)^2} \le \frac{1}{4}e^{x^2} + \frac{3}{4}e^{y^2}$. Consequently:

$$\int_a^{2a}\int_a^{2a} \sqrt[16]{e^{(x+3y)^2}}\,dx\,dy \le \frac{1}{4}\int_a^{2a}\int_a^{2a} e^{x^2}\,dx + \frac{3}{4}\int_a^{2a}\int_a^{2a} e^{y^2}\,dy =$$

$$= \frac{a}{4}\int_a^{2a} e^{x^2}\,dx + \frac{3a}{4}\int_a^{2a} e^{y^2}\,dy = \frac{a}{4}\int_a^{2a} e^{x^2}\,dx + \frac{3a}{4}\int_a^{2a} e^{x^2}\,dx = a\int_a^{2a} e^{x^2}\,dx$$

5.95

Let $P(x) = 5(x^4 - x^3) - x^5 + 1$. Clearly we have $P(1) = P'(1) = 0$, so $P(x)$ is divisible by $(x-1)^2$. An easy calculation shows that:

$P(x) = (x-1)^2(x^2(3-x) + 2x + 1)$. Thus, for

$x \in [0,3]$ we have $P(x) \ge 0$. Consider, positive numbers t, u, v and define

$$x = \frac{\sqrt[5]{3}t}{\sqrt[5]{t^5 + u^5 + v^5}}, y = \frac{\sqrt[5]{3}u}{\sqrt[5]{t^5 + u^5 + v^5}}, z = \frac{\sqrt[5]{3}v}{\sqrt[5]{t^5 + u^5 + v^5}}$$

These numbers belong to $[0,3]$. From $P(x) + P(y) + P(z) \ge 0$ we conclude that

$3^{\frac{4}{5}} \cdot \frac{t^4+u^4+v^4}{\left(\sqrt[5]{t^5+u^5+v^5}\right)^4} \ge 3^{\frac{3}{5}} \cdot \frac{t^3+u^3+v^3}{\left(\sqrt[5]{t^5+u^5+v^5}\right)^3}$*. Equivalently* $\left(\frac{t^4+u^4+v^4}{t^3+u^3+v^3}\right)^5 \ge \frac{1}{3}(t^5 + u^5 + v^5)$

It follows that for $f: [0, a] \to (0, +\infty)$ we have

$$\int_0^a\int_0^a\int_0^a \left(\frac{f^4(z) + f^4(y) + f^4(z)}{f^3(x) + f^3(y) + f^3(z)}\right)^5 dx\,dy\,dz \ge \int_0^a\int_0^a\int_0^a \frac{1}{3}\left(f^5(x) + f^5(y) + f^5(x)\right) dx\,dy\,dz$$

$$= a^2\int_0^a f^5(x)\,dx$$

5.96

$$2\int_a^{2c}\int_a^{2b}\int_a^{2a}\frac{(2x+y)(2y+z)(2z+x)}{(x+y+z)^2}dx \overset{MG\le MA}{\le} 2\int\int\int\frac{\left(\frac{3(x+y+z)}{3}\right)^3}{(x+y+z)^2}dx\,dy\,dz =$$

$$= 2\int\int\int(x+y+z)\,dx\,dy\,dz = 2\cdot\int\int\left(\frac{x^2}{2}+(y+z)x\right)\Bigg|_a^{2a} dy\,dz =$$

$$= 2\cdot\int\int_b^{2b}\left(\frac{3a^2}{2}+a(y+z)\right)dy\,dz = 2\int\int_b^{2b}\left(ay+\frac{3a^2}{2}+az\right)dy\,dz =$$

$$= 2\cdot\int_c^{2c}\left(\frac{ay^2}{2}+\left(\frac{3a^2}{2}+az\right)y\right)\Bigg|_b^{2b} dz = 2\cdot\int_c^{2c}\left(\frac{3ab^2}{2}+\left(\frac{3a^2}{2}+az\right)\cdot b\right)dz =$$

$$= 2\cdot\int_c^{2c}\left(abz+\frac{3ab^2}{2}+\frac{3a^2b}{2}\right)dz = 2\left(\frac{abz^2}{2}+\left(\frac{3a^2b}{2}+\frac{3ab^2}{2}\right)z\right)\Bigg|_c^{2c} =$$

$$= 2\left(\frac{3abc^2}{2}+\left(\frac{3a^2b}{2}+\frac{3ab^2}{2}\right)c\right) = 3abc(a+b+c)$$

5.97

Let $f(x) = \cot x$ *for all* $x \in \left[0,\frac{\pi}{4}\right]$ *then* $f'(x) = -\csc^2 x\,;\, f''(x) = 2\csc^2 x\cot x > 0$

for all $x \in \left[0,\frac{\pi}{4}\right]$*, hence* f *is convex*

$$\cot\left(\frac{4x+3y}{7}\right) \le \frac{4}{7}\cot x + \frac{3}{7}\cot y \Rightarrow \int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy \le$$

$$\le \frac{4}{7}\int_a^{2a}\int_a^{2a}\cot x\,dx\,dy + \frac{3}{7}\int_a^{2a}\int_a^{2a}\cot y\,dx\,dy = \frac{4a}{7}\int_a^{2a}\cot x\,dx + \frac{3a}{7}\int_a^{2a}\cot y\,dy =$$

$$= \frac{4a}{7}|\log(\sin x)|_{x=a}^{x=2a} + \frac{3a}{7}|\log(\sin y)|_{y=a}^{y=2a} = a\log\left(\frac{\sin 2a}{\sin a}\right) = a\log(2\cos x) =$$

$= \log(2\cos x)^a$*. Applying Hermite – Hadamard for Double Integral;*

$$\frac{1}{(2a-a)(2a-a)}\int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy \ge \cot\left(\frac{4\cdot\left(\frac{2a+a}{2}\right)+3\cdot\left(\frac{2a+a}{2}\right)}{7}\right) \Rightarrow$$

$$\Rightarrow \int_a^{2a}\int_a^{2a}\cot\left(\frac{4x+3y}{7}\right)dx\,dy \ge a^2\cot\left(\frac{3a}{2}\right)$$

$$\therefore a^2 \cot\left(\frac{3a}{2}\right) \le \int_a^{2a}\int_a^{2a} \cot\left(\frac{4x+3y}{7}\right) dx\, dy \le \log(2\cos a)^a$$

5.98

Since

$$(xy+yz+zx)\left(\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\right) = (x+y+z) + \left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right) + \left(\frac{x^2}{z}+\frac{z^2}{y}+\frac{y^2}{x}\right) \ge$$

$$\ge 3(x+y+z).$$

Hence $\frac{3(x+y+z)}{\frac{x}{y^2}+\frac{y}{zx}+\frac{z}{xy}} \le xy+yz+zx \Rightarrow \frac{3(xyz)(x+y+z)}{x^2+y^2+z^2} \le xy+yz+zx \Rightarrow$

$$\Rightarrow \frac{3(x+y+z)}{x^2+y^2+z^2} \le \frac{xy+yz+zx}{xyz} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

Hence

$$3\int_a^{2a}\int_b^{2b}\int_c^{2c} \frac{x+y+z}{x^2+y^2+z^2} dx\,dy\,dz = \int_a^{2a}\int_b^{2b}\int_c^{2c} \frac{3(x+y+z)}{(x^2+y^2+z^2)} dx\,dy\,dz \le$$

$$\le \int_a^{2a}\int_b^{2b}\int_c^{2c} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) dx\,dy\,dz = xy\ln z + yz\ln x + zx\ln y \Big|_a^{2a}\Big|_b^{2b}\Big|_c^{2c} =$$

$$= [ab\ln(2c) + bc\ln(2a) + ca\ln(2b)] - [ab\ln c + bc\ln a + ca\ln b] =$$

$$= [ab\ln 2 + bc\ln 2 + ca\ln 2 + ab\ln c + bc\ln a + ca\ln b] - [ab\ln c + bc\ln a + ca\ln b]$$

$$= ab\ln 2 + bc\ln 2 + ca\ln 2 = (ab+bc+ca)\ln 2$$

5.99

Using Hölder's inequality, for $p = 10, q = \frac{10}{9}$*, we have that:*

$$1 = \int_0^1 f(x)\,dx \le \left(\int_0^1 f^{10}(x)dx\right)^{\frac{1}{10}} \cdot \left(\int_0^1 1^{\frac{10}{9}} dx\right)^{\frac{9}{10}} = \left(\int_0^1 f^{10}(x)dx\right)^{\frac{1}{10}} \Rightarrow$$

$$\Rightarrow \int_0^1 f^{10}(x)\,dx \ge 1 \quad (1)$$

So,

$$\int_0^1\int_0^1 f(x)f(y)dx\,dy = \int_0^1 f(x)\,dx \int_0^1 f(y)\,dy = 1 \overset{(1)}{\le} \int_0^1 f^{10}(x)\,dx$$

5.100

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_{\varepsilon}^{x} \frac{\sin t}{t}\,dt$$

then $\Omega'(x) = \frac{\sin x}{x}, \Omega''(x) = \frac{\cos x}{x^2}(x - \tan x) \le 0$

since, $\sin x \le x \le \tan x$ *for all* $x \ge 0$*, hence* Ω *is concave so, from the definition of concave function*

$$\sum_{cyc} \frac{a}{a+b+c}\Omega(a) \le \Omega\left(\sum_{cyc}\left(\frac{a}{a+b+c}\right)\cdot a\right)$$

$$\sum_{cyc} a\Omega(a) \le (a+b+c)\Omega\left(\frac{a^2+b^2+c^2}{a+b+c}\right)$$

5.101

$$u = x + y, v = y + z, w = z + x$$

$$\sum_{cyc(x,y,z)} \frac{(x+y)^2}{(y+z)\sin(z+x)} = \sum_{cyc(u,v,w)} \frac{u^2}{v\cdot \sin w} \overset{\sin w<w}{\overbrace{>}}$$

$$> \sum_{cyc(u,v,w)} \frac{u^2}{vw} \overset{BERGSTROM}{\overbrace{\ge}} \frac{(u+v+w)^2}{uv+vw+wu} \ge 3$$

$$(u+v+w)^2 \ge 3(uv+vw+wu)$$

$$\int_a^b\int_a^b\int_a^b \sum_{cyc(x,y,z)} \frac{(x+y)^2}{(y+z)\sin(z+x)} \ge \int_a^b\int_a^b\int_a^b 3dxdydz = 3(b-a)^3$$

5.102

$$\because a,b,c,d,e,f,x,y,z \le 1 \therefore 1 \ge a^4,b^4,c^4,d^4,e^4,f^4,x^4,y^4,z^4$$

$$\therefore a^2+b^2+x^2+3 = a^2+b^2+x^2+1+1+1 \ge a^2+b^2+x^2+a^4+b^4+x^4 \ge$$

$$\overset{A-G}{\ge} 6abx \Rightarrow a^2+b^2+x^2+3 \overset{(1)}{\ge} 6abx$$

Again

$$, c^2+d^2+y^2+3 = c^2+d^2+y^2+1+1+1 \ge c^2+d^2+y^2+c^4+d^4+y^4 \overset{A-G}{\ge}$$

$$\geq 6cdy \Rightarrow c^2 + d^2 + y^2 + 3 \overset{(2)}{\geq} 6cdy$$

Also,

$$e^2 + f^2 + z^2 + 3 = e^2 + f^2 + z^2 + 1 + 1 + 1 \geq e^2 + f^2 + z^2 + e^4 + f^4 + z^4 \overset{A-G}{\geq}$$

$$\geq 6efz \Rightarrow e^2 + f^2 + z^2 + 3 \overset{(3)}{\geq} 6efz$$

(1).(2).(3) $\Rightarrow (a^2 + b^2 + x^2 + 3)(c^2 + d^2 + y^2 + 3)(e^2 + f^2 + z^2 + 3) \overset{(a)}{\geq} 216abcdefxyz$

Case 1: $3abcdefxyz - 1 \leq 0$

Then, $LHS \leq \int_0^1 \int_0^1 \int_0^1 0\, dx\, dy\, dz = 0 < 1 \Rightarrow$ *given inequality is true*

Case 2: $3abcdefxyz - 1 > 0$

(a) $\Rightarrow LHS \overset{(i)}{\leq} \frac{1}{2}\int_0^1 \int_0^1 \int_0^1 \frac{3m-1}{m} dx\, dy\, dz$, *where* $m = abcdefxyz$

$\because m \leq 1 \therefore \frac{3m-1}{m} = 3 - \frac{1}{m} \overset{(ii)}{\leq} 2 \therefore$ *(i), (ii)* $\Rightarrow LHS \leq \int_0^1 \int_0^1 \int_0^1 dx\, dy\, dz = 1 \Rightarrow$ *given inequality is true (Hence proved)*

5.103

$$\left.\begin{array}{l} 0 < a \leq x \leq b < \dfrac{\pi}{4} \\ 0 < a \leq y \leq b < \dfrac{\pi}{4} \end{array}\right. \Rightarrow 0 < x + y < \frac{\pi}{2}$$

So, $0 < \sin(x + y) < 1 < \frac{\pi}{2} \Rightarrow$

So, $\sin(\sin(x + y)) > 0$ *and final* $\sin(\sin(\sin(x + y))) > 0$

So, we know, that for positive x *holds that* $\sin x < x$. *This means that:*

$$\sin(\sin(\sin(x + y))) < \sin(\sin(x + y)) < \sin(x + y) < x + y.$$

Taking integrals we have:

$$\int_a^b \int_a^b \sin(\sin(\sin(x + y)))\, dxdy < \int_a^b \int_a^b (x + y)\, dx\, dy =$$

$$= \int_a^b \left[\frac{x^2}{2} + yx\right]_a^b dy = \int_a^b \left(\frac{b^2}{2} - \frac{a^2}{2}\right) + y(b - a) dy$$

$\left[\left(\frac{b^2-a^2}{2}\right)y + \frac{y^2}{2}(b - a)\right]_a^b = \frac{(b-a)(b+a)}{2}b - a + \frac{b^2-a^2}{2}(b - a) = \frac{(b-a)^2(b+a)}{2} \cdot 2$. *So,*

$$\frac{1}{a+b}\int_a^b\int_a^b sin(sin(sin(x+y)))\,dxdy < (b-a)^2$$

5.104

$$si\,(x) = -\int_x^\infty \frac{sin\,\theta}{\theta}d\theta$$

$$si'(x) = -\left[\lim_{\theta\to\infty}\frac{sin\,\theta}{\theta} - \frac{sin\,x}{x}\right] = \frac{sin\,x}{x}$$

Now,

$$\Omega_1 = \int_\gamma^e \frac{1}{x}\big(si\,(e^2x) - si\,(\pi x)\big)dx = \int_\gamma^e\left(\int_\pi^{e^2} si'(tx)dt\right)dx =$$

$$= \int_\pi^{e^2}\left(\int_\gamma^e si'(tx)dx\right)dt$$

[Interchange order of integration]

$$= \int_\pi^{e^2}\frac{si\,(tx)}{t}\Bigg]_\gamma^e dt = \int_\pi^{e^2}\frac{si\,(te) - si\,(\gamma t)}{t}dt = \int_\pi^{e^2}\frac{si(ex) - sin(\gamma x)}{x}dx = \Omega_2$$

5.105

We consider the function $f(x) = x\,ln\,x$*, for* $x > 0$

$f'(x) = ln\,x + 1 > 0 \Rightarrow f -$ *convex*

Therefore, we apply the Hermite – Hadamard inequality for double integral:

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b log\left(\frac{x+y}{2}\right)^{x+y} \geq log\left(\frac{\frac{a+b}{2}+\frac{a+b}{2}}{2}\right)^{\left(\frac{a+b}{2}+\frac{a+b}{2}\right)} = log\left(\frac{a+b}{2}\right)^{(a+b)} \Rightarrow$$

$$\Rightarrow \int_a^b\int_a^b log\left(\frac{x+y}{2}\right)^{x+y} \geq (b-a)^2\,log\left(\frac{a+b}{2}\right)^{(a+b)}$$

5.106

$$\textit{For } a,b>0, \left(\frac{a^2+b^2}{a+b}\right)^3 \ge \frac{1}{2}(a^3+b^3) \Leftrightarrow 2(a^2+b^2)^3 \ge (a^3+b^3)(a+b)^3 \Leftrightarrow$$

$$\Leftrightarrow 2(a^6+3a^4b^2+3a^2b^4+b^6) \ge (a^3+b^3)(a^3+3a^2b+3ab^2+b^3) \Leftrightarrow$$

$$\Leftrightarrow a^6+b^6-3a^5b-2a^3b^3-3ab^5+3a^2b^4+3a^4b^2 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (a^3-b^3)^2-3ab(a^3-b^3)(a-b) \ge 0 \Leftrightarrow (a^3-b^3)[a^3-b^3-3ab(a-b)] \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (a^3-b^3)(a-b)^3 \ge 0 \Leftrightarrow (a-b)^4(a^2+ab+b^2) \ge 0 \textit{ which is true}$$

Thus,

$$\Omega(x,y,z) = \left(\frac{f(x)^2+f(y)^2}{f(x)+f(y)}\right)^3 + \left(\frac{f(y)^2+f(z)^2}{f(y)+f(z)}\right)^3 + \left(\frac{f(z)^2+f(x)^2}{f(z)+f(x)}\right)^3 \ge$$

$$\ge \frac{1}{2}(f(x)^3+f(y)^3)+\frac{1}{2}(f(y)^3+f(z)^3)+\frac{1}{2}(f(z)^3+f(x)^3) = f(x)^3+f(y)^3+f(z)^3$$

$$\therefore \int_a^b\int_a^b\int_a^b \Omega(x,y,z)\,dx\,dy\,dz \ge \int_a^b\int_a^b\int_a^b [f(x)^3+f(y)^3+f(z)^3]\,dxdydz =$$

$$= 3(b-a)^2\int_a^b f(x)^3\,dx$$

5.107

$$x^4+x^2y^2+y^4 = (x^4+y^4+2x^2y^2)-x^2y^2 = (x^2+y^2)^2-(xy)^2 \overset{(1)}{=}$$

$$= (x^2+y^2+xy)(x^2+y^2-xy)$$

$$\textit{Now, } (x-y)^2 \ge 0 \Rightarrow x^2+y^2-2xy \ge 0 \Rightarrow 2x^2+2y^2-2xy \ge x^2+y^2 \Rightarrow$$

$$\Rightarrow x^2+y^2 \overset{(2)}{\le} 2(x^2-xy+y^2)$$

$$\textit{(1), (2)} \Rightarrow \textit{LHS} \le \frac{3}{2}\cdot 2\int_a^b\int_a^b \frac{(x^2-xy+y^2)dx\,dy}{(x^2+xy+y^2)(x^2-xy+y^2)} = 3\int_a^b\int_a^b \frac{dx\,dy}{x^2+xy+y^2} \overset{A-G}{\le} 3\int_a^b\int_a^b \frac{dx\,dy}{3xy}$$

$$= \int_a^b\int_a^b \frac{dx\,dy}{xy} = \int_a^b\left[\int_a^b \frac{dx}{x}\right]\frac{dy}{y} = \left(ln\frac{b}{a}\right)\int_a^b \frac{dy}{y} = \left(ln\frac{b}{a}\right)^2$$

5.108

$$I = \int_0^1 \frac{ln(1+x)}{x(1+x^2)}dx = \int_0^1 ln(1+x)\left[\frac{A}{x}+\frac{Bx+C}{1+x^2}\right]dx$$

$$A=1; B=-1; C=0$$

$$I=\int_0^1 ln(1+x)\left[\frac{1}{x}-\frac{x}{1+x^2}\right]dx=\int_0^1\frac{ln(1+x)}{x}dx-\int_0^1\frac{x\,ln(1+x)}{1+x^2}dx$$

$$N=\int_0^1\frac{ln(1+x)}{x}dx;M=\int_0^1\frac{x\,ln(1+x)}{1+x^2}dx$$

$$N=\int_0^1\frac{ln(1+x)}{x}dx=ln(x)\,ln(1+x)\Big|_0^1-\int_0^1\frac{ln\,x}{1+x}dx=-\int_0^1\frac{ln\,x}{1+x}dx=-Li_2(-1)=\frac{\pi^2}{12}$$

$$M=\int_0^1\frac{x\,ln(1+x)}{1+x^2}dx;M(a)=\int_0^1\frac{x\,ln(1+ax)}{1+x^2}dx$$

$$M'(a)=\int_0^1\frac{x^2}{(1+x^2)(1+ax)}dx=\int_0^1\left[\frac{Ax+B}{1+x^2}+\frac{C}{1+ax}\right]dx$$

$$A=\frac{a}{1+a^2};B=-\frac{1}{1+a^2};C=\frac{1}{1+a^2}$$

$$M'(a)=\int_0^1\left[\frac{\frac{a}{1+a^2}x-\frac{1}{1+a^2}}{1+x^2}+\frac{\frac{1}{1+a^2}}{1+ax}\right]dx=\frac{1}{1+a^2}\int_0^1\frac{ax-1}{1+x^2}dx+\frac{1}{1+a^2}\int_0^1\frac{dx}{1+ax}$$

$$=\frac{a}{1+a^2}\int_0^1\frac{x}{1+x^2}dx-\frac{1}{1+a^2}\int_0^1\frac{1}{1+x^2}dx+\frac{1}{1+a^2}\int_0^1\frac{dx}{1+ax}$$

$$=\frac{1}{2}ln\frac{a}{1+a^2}-\frac{\pi}{4}\cdot\frac{1}{1+a^2}+\frac{ln(1+a)}{a(1+a^2)}$$

$$M(a)=M(1)-M(0)=M(1)$$

$$M=\frac{1}{2}ln\,2\int_0^1\frac{a}{1+a^2}da-\frac{\pi}{4}\int_0^1\frac{1}{1+a^2}da+\int_0^1\frac{ln(1+a)}{a(1+a^2)}da=\frac{1}{4}ln^2 2-\frac{\pi^2}{16}+I$$

$$I=N-M=\frac{\pi^2}{12}-\left(\frac{1}{4}ln^2 2-\frac{\pi^2}{16}+I\right)$$

$$I=\frac{7\pi^2}{96}-\frac{1}{8}ln^2 2$$

5.109

The inequality can be written as:

$$\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}-\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\cdot\ldots\cdot(1+H_k)}{H_1\cdot H_2\cdot\ldots H_{k+1}}>2$$

$$\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}+1=\frac{\sum_{k=1}^{n+1}\frac{1}{H_k}}{n+1}+\frac{n+1}{n+1}=\frac{\sum_{k=1}^{n+1}\left(\frac{1}{H_k}+1\right)}{n+1}>\sqrt[n+1]{\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)}\succ$$

$$\succ\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}>\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)$$

We need to prove that

$$\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\left(1+\frac{1}{H_2}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}\geq 2$$

$$\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}=\Omega(n+1)=$$

$$=\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_{n+1}}\right)-\left(\frac{1}{H_1}+1\right)\cdot\ldots\cdot\left(1+\frac{1}{H_n}\right)\cdot\frac{1}{H_{n+1s}}-\sum_{k=1}^{n-1}\left(1+\frac{1}{H_1}\right)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}$$

$$=\prod_{k=1}^{n}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n-1}(1+H_1)\cdot\ldots\cdot\left(1+\frac{1}{H_k}\right)\cdot\frac{1}{H_{k+1}}=\Omega(n)$$

$$\Rightarrow\Omega(n+1)=\Omega(n)=\cdots=\Omega(2)=\left(1+\frac{1}{H_1}\right)\left(1+\frac{1}{H_2}\right)-\left(1+\frac{1}{H_1}\right)\frac{1}{H_2}=$$

$$=1+\frac{1}{H_1}=2\Rightarrow\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\left(1+\frac{1}{H_1}\right)\ldots\left(1+\frac{1}{H_k}\right)\frac{1}{H_{k+1}}\geq 2\Leftrightarrow$$

$$\Leftrightarrow\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}-\sum_{k=1}^{n}\frac{(1+H_1)\cdot\ldots\cdot(1+H_k)}{H_1\cdot\ldots\cdot H_{k+1}}$$

$$>\prod_{k=1}^{n+1}\left(1+\frac{1}{H_k}\right)-\sum_{k=1}^{n}\frac{(1+H_1)\cdot\ldots\cdot(1+H_k)}{H_1\cdot\ldots\cdot H_k\cdot H_{k+1}}\geq 2$$

$$\Leftrightarrow 2+\sum_{k=1}^{n}\frac{(1+H_1)(1+H_2)\ldots(1+H_k)}{H_1H_2\ldots H_{k+1}}<\left(1+\frac{1}{n+1}\sum_{k=1}^{n+1}\frac{1}{H_k}\right)^{n+1}$$

(Q.E.D)

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