ICE MATH

CONTESTS PROBLEMS

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MARIAN URSĂRESCU

Daniel Sitaru, born on 9 August 1963 in Craiova, Romania, is teacher at National Economic College "Theodor Costescu" in Drobeta Turnu - Severin. He published 36 mathematical books, last seven of these "Math Phenomenon", Algebraic Phenomenon", Analytical Phenomenon", "The Olympic Mathematical Marathon" and "699 Olympic Mathematical Challenges",”Olympic Mathematical Energy”,”Calculus Marathon” (the last one with Marian Ursărescu), were very appreciated world wide. He is the founding editor of "Romanian Mathematical Magazine", an Interactive Mathematical Journal with 5,000.000 visitors in the last three years (www.ssmrmh.ro).Many problems from his books were published in famous journals such as "American Mathematical Monthly", "Crux Mathematicorum", "Math Problems Journal", "The Pentagon Journal", "La Gaceta de la RSME", "SSMA Magazine".He also published an impressive number of original problems in all mathematical journals from Romania (GMB, Cardinal, Elipsa, Argument, Recreații Matematice). His articles from "Crux Mathematicorum" and "The Pentagon Journal" were also very appreciated.

Marian Ursărescu, was born on 1st of June 1965, in Focșani. He graduated from A.I. Cuza University, Faculty of Mathematics, in 1988. He is a teacher of mathematics from 1988 at "Roman Vodă" National College in Roman. Starting from 1990 until now, he had 47 pupils that participated on the Mathematical National Olympiad, which from 28 had obtained prizes and Olympic mentions. He published over 100 problems and articles in Mathematical National Gazette. Also, he published several problems and articles in mathematical magazines such as "Mathematical Recreations", "Romanian Mathematical Magazine", "Let's understand math." A lot of his proposed problems had been selected in various mathematical contests, olympiads and mathematical books. He co-authored "Functional Equations" together with M. O. Drâmbe.

FROM AUTHORS

In July 2016 was founded "Romanian Mathematical Magazine" (RMM) (www.ssmrmh.ro) as an Interactive Mathematical Journal.

Same date was founded "Romanian Mathematical Magazine"-Online Mathematical Journal (ISSN-2501-0099) and "Romanian Mathematical Magazine"-Paper Variant (ISSN-1584-4897).

It three years the RMM website was visited by over 5,000,000 people from all over the world.

With over 7,000 proposed problems posted, over 11,000 solutions and many math articles and math notes, RMM represents a big opportunity for young mathematicians around the world to be recognized as great proposers and solvers.

This book is a small part of RMM-Interactive Journal.

Many thanks to RMM-Team for proposed problems and solutions.

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CYCLIC INEQUALITIES-PROBLEMS

PROBLEM 1.01

If $a, b, c \geq 0$ then:

$$12 + \sum (a^8 + 1)\left(\frac{1}{b^4 + 1} + \frac{1}{c^4 + 1}\right) \geq 12\sqrt{2}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 1.02

If $a, b, c \in (0; +\infty), abc = 1$ then:

$$\frac{a^2 + b^2}{c} + \frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} \leq 2(a^4 + b^4 + c^4).$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.03

If $a, b, c > 0, a + b + c = 3$ then:

$$5(a^4 + b^4 + c^4) \geq 12 + a^5 + b^5 + c^5$$

Proposed by Marian Ursărescu-Romania

PROBLEM 1.04

If $a, b, c > 0$ then:

$$(7 + a^3 + b^3 + c^3)\left(7 + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) \geq 100$$

Proposed by Daniel Sitaru-Romania

PROBLEM 1.05

For $a, b, c > 0 \wedge ab + bc + ca = 3abc$. Prove:

$$\frac{\sqrt{ab}}{\left(\sqrt{a} + \sqrt{b}\right)^4} + \frac{\sqrt{bc}}{\left(\sqrt{b} + \sqrt{c}\right)^3} + \frac{\sqrt{ca}}{\left(\sqrt{c} + \sqrt{a}\right)^4} \leq \frac{3}{16}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.06

If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{ab^2}{\sqrt{b^2 + bc + c^2}} + \frac{bc^2}{\sqrt{c^2 + ca + a^2}} + \frac{ca^2}{\sqrt{a^2 + ab + b^2}} + \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) \geq \frac{7\sqrt{3}}{4}$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

PROBLEM 1.07

If $a, b, c > 0, abc = 1$ then:

$$\sqrt{\frac{a^5 + b^5}{a^2 + b^2}} + \sqrt{\frac{b^5 + c^5}{b^2 + c^2}} + \sqrt{\frac{c^5 + a^5}{c^2 + a^2}} \geq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 1.08

If $a, b, c > 0, a + b + c = 3$ then:

$$3 + \sum\left(\frac{b}{12a + 1} + \frac{c}{6b + 1}\right) > \sum\left(\frac{c}{10b + 1} + \frac{b}{2a + 1}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.09

If $x, y, z > 0, n \in \mathbb{N}, n \geq 2$, $x^3 + y^3 + z^3 = 3$ ***then:***

$$\sum \frac{x}{y^4 + z^4 + y^2 z^2} \leq \frac{1}{(xyz)^n}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaidian

PROBLEM 1.10

If $a, b, c \geq 0$ ***then:***

$$3\left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}}\right) \geq 2\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right) + \sqrt[4]{ab} + \sqrt[4]{bc} + \sqrt[4]{ca}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.11

If $a, b, c \geq 0$ ***then:***

$$3\sqrt{3}(a+b)(b+c)(c+a) \leq 8\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.12

If $a, b, c > 0$ ***then:***

$$\frac{4}{a+b+c}\left(\sum a^2\right)\left(\sum a^4\right) \leq 3\sum a^5 + \frac{1}{(a+b+c)^3}\left(\sum a^2\right)^4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.13

If $x, y, z > 0, x + y + z = 3$ ***then:***

$$\frac{1}{\sqrt{x+y^2+z^2}} + \frac{1}{\sqrt{x^2+y+z^2}} + \frac{1}{\sqrt{x^2+y^2+z}} \leq \sqrt{3}$$

Proposed by Marian Ursărescu-Romania

PROBLEM 1.14

For $a, b, c > 0$***. Prove:***

$$\frac{(a+b)a^3}{a^2+ab+b^2} + \frac{(b+c)b^3}{b^2+bc+c^2} + \frac{(c+a)c^3}{c^2+ca+a^2} \geq \frac{2(a+b+c)^2}{9}$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

PROBLEM 1.15

For $x, y, z > 0 \wedge xyz = 1$***. Prove:***

$$\frac{x}{x^{12}+2y^4+1} + \frac{y}{y^{12}+2z^4+1} + \frac{z}{z^{12}+2x^4+1} \leq \frac{x^8+y^8+z^8}{4}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.16

Prove that for all non-negative numbers x, y, z ***satisfying*** $x + y + z = 1$

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}$$

Germany NMO

PROBLEM 1.17

If x, y and z are positive real numbers such that $xyz \geq 7 + 5\sqrt{2}$, then:

$$x^2 + y^2 + z^2 - 2(x + y + z) \geq 3.$$

Proposed by Neculai Stanciu-Romania

PROBLEM 1.18

If a, b and c are positive real numbers, then prove that

$$\frac{a(b-c)}{c(a+b)} + \frac{b(c-a)}{a(b+c)} + \frac{c(a-b)}{b(c+a)} \geq 0$$

Proposed by Neculai Stanciu – Romania

PROBLEM 1.19

If x, y and z are positive real numbers, then prove that

$$\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)} \geq \frac{8}{9}$$

Proposed by Neculai Stanciu-Romania

PROBLEM 1.20

If $0 < a, b, c \leq 1$ then:

$$\frac{1}{a+a^a} + \frac{1}{b+b^b} + \frac{1}{c+c^c} \geq \frac{9}{3+a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.21

If $a, b, c > 0, ab + bc + ca = 6abc$ then:

$$\frac{1}{\sqrt{ab(a+b)}} + \frac{1}{\sqrt{bc(b+c)}} + \frac{1}{\sqrt{ca(c+a)}} \leq 3 + \frac{a+b+c}{4abc}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.22

If $x, y > 0$ then:

$$4\left(x + \frac{x+1}{y}\right)\left(y + \frac{y+1}{x}\right) \leq \left(2 + x + y + \frac{1}{x} + \frac{1}{y}\right)^2$$

Proposed by Mihalcea Andrei Stefan-Romania

PROBLEM 1.23

If $a, b, c > 1, ab + bc + ca = abc$ then:

$$abc^c + bca^a + cab^b \geq a^2b^2c^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.24

If $a, b, c, d, e > 0, c + d + e = 1$ then:

$$\left(a + \frac{b}{c}\right)^4 + \left(a + \frac{b}{d}\right)^4 + \left(a + \frac{b}{e}\right)^4 \geq 3(a+3b)^4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.25

If $x, y, z, t > 0$ then:

$$\sum \frac{yzt}{\left(\sqrt[3]{ztx} + \sqrt[3]{txy} + \sqrt[3]{xyz}\right)^3} \geq \frac{4}{27}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.26

If $a, b, c > 1$ then:

$$\frac{1}{2\sqrt{2}} \sin\frac{\pi}{3a} \sin\frac{\pi}{3b} \sin\frac{\pi}{3c} > \frac{1}{\sqrt{(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.27

For $a, b, c \in (0, +\infty) \wedge x, y \in [1; +\infty)$. Prove:

$$\frac{a^x}{(b+c)^y} + \frac{b^x}{(a+c)^y} + \frac{c^x}{(a+b)^y} \geq \frac{(a+b+c)^{x-y}}{2^y 3^{x-y-1}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.28

If $a, b, c \in (0, 1), a^2 + b^2 + c^2 = \sqrt{3}$ then:

$$(1-a^2)^{\frac{1}{a}} \cdot (1-b^2)^{\frac{1}{b}} \cdot (1-c^2)^{\frac{1}{c}} < \frac{1}{e^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.29

If $a, b, c \geq 0$ then:

$$3^e \cdot (a^e + b^e + c^e)^\pi \leq 3^\pi \cdot (a^\pi + b^\pi + c^\pi)^e$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.30

If $a, b, c \geq 0$ then:

$$a^2\sqrt{b^2+c^2} + b^2\sqrt{c^2+a^2} + c^2\sqrt{a^2+b^2} \geq \sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.31

If $x, y, z > 0, x + y + z = 3$ then:

$$\sum \sqrt{(x+y+1)(y+z+1)} \leq 6 + \sum \frac{x^3+z^3}{x^2+z^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.32

If $x, y, z > 0$ then:

$$\frac{(x^4+y^4)^2 + (y^4+z^4)^2 + (z^4+x^4)^2}{\sqrt{x^4+y^4+z^4}} \geq 4\sqrt{3}x^2y^2z^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.33

For $a,b,c,d\geq 1$. Prove:

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}\geq\frac{4}{1+abcd}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.34

If $x,y,z,t\geq 1$ then:

$$x^x\cdot y^y\cdot z^z\cdot t^t\geq x^{\sqrt[3]{yzt}}\cdot y^{\sqrt[3]{ztx}}\cdot z^{\sqrt[3]{txy}}\cdot t^{\sqrt[3]{xyz}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.35

If $x,y,z>0$ then:

$$\frac{1}{x^2+y^2+2z^2}+\frac{1}{y^2+z^2+2x^2}+\frac{1}{z^2+x^2+2y^2}\geq\frac{2xyz\sqrt{3(x^2+y^2+z^2)}}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.36

If $x,y,z>0$ then:

$$8(x+y+z)^9\sum\left(\frac{yz}{xy+xz}\right)^3\geq 3^{10}x^3y^3z^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.37

If $0<x,y,z\leq 2,\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=3$ then:

$$\sqrt{2}<\frac{3+\sqrt{y(2-x)}+\sqrt{z(2-y)}+\sqrt{x(2-z)}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\leq 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.38

For $a,b,c\in(0;1]\wedge m\in\mathbb{N}^*$. Prove:

$$\frac{1}{\sqrt{1+a^m}}+\frac{1}{\sqrt{1+b^m}}+\frac{1}{\sqrt{1+c^m}}\leq\frac{3\sqrt{2}}{1+(abc)^{\frac{m}{6}}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.39

Let $a,b,c\in(0;+\infty)\wedge ab+bc+ca=3$. Prove:

$$\frac{1}{\sqrt[6]{a^2+3}}+\frac{1}{\sqrt[6]{b^2+3}}+\frac{1}{\sqrt[6]{c^2+3}}\leq\frac{\sqrt[3]{36}}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{1}{3}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.40

If $a,b,c\in\mathbb{R}$, then:

$$(a^3+b^3+c^3-3abc)^2\leq(a^2+b^2+c^2)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.41

$$\textit{If } a,b,c>0, a+b+c=3 \textit{ then: } a^4+b^4+c^4\ge 3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.42

If $a,b,c>0, abc=1$ ***then:***

$$\frac{1}{a^4+b^4+c}+\frac{1}{b^4+c^4+a}+\frac{1}{c^4+a^4+b}\le 1$$

Proposed by Marian Ursărescu – Romania

PROBLEM 1.43

If $a,b,c>0, abc=1$ ***then:***

$$\sum\frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)}\ge\frac{1}{a^{21}}+\frac{1}{b^{21}}+\frac{1}{c^{21}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.44

If $a,b,c>0$, $a+b+c=3$ ***then:***

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge\frac{3}{2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.45

If $a,b,c,d>0$ **then:**

$$\frac{a^7}{a^3+bcd}+\frac{b^7}{b^3+cda}+\frac{c^7}{c^3+dab}+\frac{d^7}{d^3+abc}\ge 2abcd$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.46

If $x,y,z,t>0$ **then:**

$$\frac{1}{\sqrt[3]{xyzt}}\left(\frac{x^2}{x+1}+\frac{y^2}{y+1}+\frac{z^2}{z+1}+\frac{t^2}{t+1}\right)\ge\frac{\sqrt[3]{x^2}}{x+1}+\frac{\sqrt[3]{y^2}}{y+1}+\frac{\sqrt[3]{z^2}}{z+1}+\frac{\sqrt[3]{t^2}}{t+1}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.47

If $0\le a,b,c\le 3$ ***then:***

$$1\le\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}+2^{abc}\le 8^9+\frac{9}{10}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.48

If $x,y,z>0$ ***then:***

$$\frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)}\ge\frac{(xy+1)(yz+1)(zx+1)}{(x^4+1)(y^4+1)(z^4+1)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.49

If $a, b, c \in \mathbb{R}$ then:

$$(a-b)^2(b-c)^2(c-a)^2 \leq 3(a^2+b^2+c^2)(a^4+b^4+c^4)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.50

If $x, y, z > 0$ then:

$$\frac{x}{3}\cdot\left(\frac{8}{3y+5z}\right)^7+\frac{y}{3}\left(\frac{8}{3z+5x}\right)^7+\frac{z}{3}\cdot\left(\frac{8}{3x+5y}\right)^7 \geq \left(\frac{3}{x+y+z}\right)^6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.51

If $a, b, c > 0, a+b+c=3$ then:

$$\sum(a+b-c)^3\cdot\sum(a+b-c)^5 \geq 9abc$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.52

If $1 \leq a \leq x, 1 \leq b \leq y, 1 \leq c \leq z$ then:

$$\frac{\sqrt{2}(a+x)(b+y)(c+z)}{(a+1)(b+1)(c+1)} \leq \sqrt{(abc)^2+(xyz)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.53

If $a, b, c, d > 0, a+b+c+d=1$ then:

$$2^{16}abcd(1-a)(1-b)(1-c)(1-d) \leq 81$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.54

Let $x, y, z \in (0; +\infty) \wedge xyz = 1$ and $\theta \geq 1$. Prove:

$$\frac{1}{\left(2\sqrt{x}+xy\right)^\theta}+\frac{1}{\left(2\sqrt{y}+yz\right)^\theta}+\frac{1}{\left(2\sqrt{z}+zx\right)^\theta} \geq 3^{1-\theta}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.55

If $a, b, c \in \mathbb{N}, a+b+c=8$ then:

$$\frac{81}{(a+1)(b+1)(c+1)} > \frac{1}{\sqrt[4]{27}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.56

If $a, b, c \in \mathbb{R}, a^2+b^2+c^2=3$ then:

$$2(a^4+b^4+c^4)+12 \geq 3(a^3+b^3+c^3+a+b+c)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.57

If $a, b, c \geq e$ then:

$$(ln(ae))(ln(be))(ln(ce))+4 \geq 4\,ln(abc)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.58

If $a, b, c > 0$ then:

$$2(a^2+b^2+c^2+a^3+b^3+c^3) \le \sqrt{2} \cdot \sum_{cyc(a,b,c)} \sqrt{a^6+b^6} + \sqrt[3]{4} \cdot \sum_{cyc(a,b,c)} \sqrt[3]{a^6+b^6}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.59

If $a, b, c > 0$ then:

$$\left(e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\right)^2 \le \left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.60

Let $\Delta\ ABC \wedge abc = 1$ and $x \in (0;1)$. Prove:

$$\frac{1}{(a^2+2ab+3)^x}+\frac{1}{(b^2+2bc+3)^x}+\frac{1}{(c^2+2ca+3)^x} \le \frac{3}{6^x}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.61

If $x, y, z > 0, x^4+y^4+z^4 = x^2y^2z^2$ then:

$$\left(\frac{zx^2+zy^2}{x^4+y^4}\right)^2+\left(\frac{xy^2+xz^2}{y^4+z^4}\right)^2+\left(\frac{yz^2+zx^2}{z^4+x^4}\right)^2 \le 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.62

If $x, y, z > 1, xyz = 2\sqrt{2}$ then:

$$x^y+y^z+z^x+y^x+z^y+x^z > 9$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.63

If $x, y, z > 0, \frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3} = 1$ then:

$$\sum_{cyc(x,y,z)} \frac{y^3+z^3+3}{x^3} \ge 3xyz$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.64

Let $a, b, c \in [0;+\infty) \wedge a+b+c = 3$. Prove:

$$a^2+b^4+c^4 \le 81+abc$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.65

If $a, b, c > 0$ then:

$$\sum_{cyc(a,b,c)} \left(\frac{1}{a^2b^2}-\frac{1}{ab}\right)+2\sum_{cyc(a,b,c)} \frac{bc^2(ab+1)}{a(b^2c^2+1)} \ge 6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.66

If $x,y,z>0,\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=3\sqrt{xyz}$ ***then:***

$$\frac{(x^2+1)(y^2+1)}{(x^3+1)(y^3+1)}+\frac{(y^2+1)(z^2+1)}{(y^3+1)(z^3+1)}+\frac{(z^2+1)(x^2+1)}{(z^3+1)(x^3+1)}\le 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.67

If $x,y,z\ge 0$ ***then:***

$$\left(\sqrt{x^2+y^2}+\sqrt{y^2+z^2}+\sqrt{z^2+x^2}\right)^2\ge 2\sqrt{3(x^2y^2+y^2z^2+z^2x^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.68

If $a,b,c>0,a^3+b^3+c^3=3$ **then:**

$$\left(\frac{a^2+1}{a+1}\right)^3+\left(\frac{b^2+1}{b+1}\right)^3+\left(\frac{c^2+1}{c+1}\right)^3\ge 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.69

Let $x,y,z>0\wedge x+y+z\le 1$**. Prove:**

$$(x+y+z)^3+\frac{1}{xyz}\ge 28$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.70

If $a,b,c\in\mathbb{N},a,b,c\ge 4$ **then:**

$$a^{\frac{1}{a+1}}\cdot b^{\frac{1}{b+1}}\cdot c^{\frac{1}{c+1}}>(a+1)^{\frac{1}{a+2}}\cdot(b+1)^{\frac{1}{b+2}}\cdot(c+1)^{\frac{1}{c+2}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.71

If $x,y,z,t\in[-5,5],x+y+z+t=0$ ***then:***

$$\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\le 20$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.72

If $a,b,c>0$ ***then:***

$$\sum\frac{1}{a+1}-3\sum\frac{1}{a+2}+3\sum\frac{1}{a+3}-\sum\frac{1}{a+4}<\frac{\sqrt{6}}{8}\left(\frac{\sqrt{a}}{a^2}+\frac{\sqrt{b}}{b^2}+\frac{\sqrt{c}}{c^2}\right)$$

PROBLEM 1.73

If $x,y,z\in(0,1),x^6+y^6+z^6=\frac{1}{9}$ **then:**

$$\left(\frac{2}{1-x^2}\right)^6+\left(\frac{2}{1-y^2}\right)^6+\left(\frac{2}{1-z^2}\right)^6\ge 3^7$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.74

If $x, y, z > 0$ then:

$$\frac{(x+y)^4+1}{(x+y)^6+1}+\frac{(y+z)^4+1}{(y+z)^6+1}+\frac{(z+x)^4+1}{(z+x)^6+1}\le\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.75

If $a, b, c > 0, a+b+c=3$ then:

$$a^6+b^6+c^6+\frac{1}{32}\left((3-a)^6+(3-b)^6+(3-c)^6\right)\ge 9$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.76

If $a, b > 0$ then:

$$\left(\frac{2ab}{a+b}-\sqrt{ab}+\frac{a+b}{2}\right)^2+ab\le\left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.77

Let $a, b, c > 0$. Prove:

$$\frac{\left((a+b)(b+c)(c+a)\right)^2}{\sqrt{(a^2+b^2)(b^2+c^2)(c^2+a^2)}}\ge 16\sqrt{2}abc$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.78

Let $x, y, z > 0$ and $\prod x = 1$. Prove:

$$8\sum x\sqrt{\sum x}\le 3\sqrt{3}\prod(x+y)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.79

Let $x, y, z > 0$ and $xyz = 1$. Prove:

$$8\sum x\sqrt{\sum x}\le 3\sqrt{3}\prod(x+y)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.80

If $x, y, z > 0$ then:

$$\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2\cdot\sum\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}\right)^2\cdot\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2\ge\left(\sum\left(\frac{x}{y}+\frac{y}{x}\right)\right)^3$$

Proposed by Daniel Sitaru-Romania

PROBLEM 1.81

If $x, y, z > 0, xyz = 9$ then:

$$\sqrt{z}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)^2+\sqrt{x}\cdot\left(\frac{y+z}{\sqrt{y}+\sqrt{z}}\right)^2+\sqrt{y}\cdot\left(\frac{z+x}{\sqrt{z}+\sqrt{x}}\right)^2\ge 9$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.82

If $a, b, c, d > 0, a + b + c + d = 4$ then:

$$\frac{(a+1)(b+1)(c+1)(d+1)}{abcd} \geq 16$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.83

Let $a, b, c \in (0; +\infty) \wedge a + b + c = 3$. Prove:

$$\sqrt{a^8 + \frac{1}{a^2} + \frac{1}{a}} + \sqrt{b^8 + \frac{1}{b^2} + \frac{1}{b}} + \sqrt{c^8 + \frac{1}{c^2} + \frac{1}{c}} \geq 3\sqrt{3}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.84

If $a, b, c \geq 0$ then:

$$e^{2\sqrt{3}(a+b+c)} \geq \left((a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)\right)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.85

Let $a, b, c \in (0, \infty)$ then $9(a^2 + b^2 + c^2)^2 \geq 8(a + b + c)(a^3 + b^3 + c^3)$

Proposed by Richdad Phuc – Hanoi – Vietnam

PROBLEM 1.86

1) If a, b, c, k are nonnegative real numbers such that $a + b + c > 0$, then

$$\frac{ab}{b + 2kc + k^2a} + \frac{bc}{c + 2ka + k^2b} + \frac{ca}{a + 2kb + k^2c} \leq \frac{a + b + c}{(1 + k)^2}.$$

2) If x, y, z are nonnegative real numbers and a, b, c are positive real numbers such that $4ab \geq c^2$, then

$$\frac{xy}{ax + by + cz} + \frac{yz}{ay + bz + cx} + \frac{zx}{az + bx + cy} \leq \frac{x + y + z}{a + b + c}.$$

Proposed by Le Khansy Sy-Long An-Vietnam

PROBLEM 1.87

If $a, b, c > 0$ then:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c + a}{c + b} + \frac{a + b}{a + c} + \frac{b + c}{b + a}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

PROBLEM 1.88

Prove that if $a, b, c \geq 0$ then:

$$\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right)^6 \leq 27(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.89

If $a, b, c > 0, a + b + c = 3$ then:

$$\frac{a^4}{b^4\sqrt{2c(a^3 + 1)}} + \frac{b^4}{c^4\sqrt{2a(b^3 + 1)}} + \frac{c^4}{a^4\sqrt{2b(c^3 + 1)}} \geq \frac{a^2 + b^2 + c^2}{2}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 1.90

If $a,b,c>0, a^4+b^4+c^4=1$ ***then:***

$$\frac{a^7+b^7}{ab(a+b)}+\frac{b^7+c^7}{bc(b+c)}+\frac{c^7+a^7}{ca(c+a)}\geq 3(a^2b^2+b^2c^2+c^2a^2)-2$$

Proposed by Marin Chirciu – Romania

PROBLEM 1.91

Let be: $a,b,c>0$***. Prove that the following relationship holds:***

$$\frac{abc}{7\sqrt{7}}\leq\frac{(a^2-ab+b^2)(b^2-bc+b^2)(c^2-ca+a^2)}{\sqrt{(a^2+5ab+b^2)(b^2+5bc+c^2)(c^2+5ca+a^2)}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.92

If $a,b,c>0, a^6+b^6+c^6=9$ ***then:***

$$2\left(\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}+\frac{b+c}{\left(b^3\sqrt{c}+c^3\sqrt{b}\right)^2}+\frac{c+a}{\left(c^3\sqrt{a}+a^3\sqrt{c}\right)^2}\right)\geq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.93

If $a,b,c>0, a+b+c=1$ ***then:***

$$5\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\leq\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\leq 5$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.94

If $a,b,c\in(0;+\infty)$ ***and*** $k\in\mathbb{R}$***, prove that.***

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{(k-1)^2}{2}\geq\frac{(k^2+2k+13)(a^2+b^2+c^2)}{2(a+b+c)^2}$$

Proposed by Le Khanh Sy-Long An-Vietnam

PROBLEM 1.95

If $a,b,c>0$ ***then:***

$$\sum\frac{a^5}{(2a+3b)^3}+\sum\frac{a^5}{(2a+3c)^3}\geq\frac{2(a^2+b^2+c^2)}{125}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.96

If $a,b,c>0, a\neq b\neq c\neq a$ ***then:***

$$\frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}>\frac{81}{4(a^2+b^2+c^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.97

If $a,b,c\in\mathbb{R}, a\neq b\neq c\neq a$

$\omega=min(|a+b|,|b+c|,|c+a|), \Omega=max(|a|,|b|,|c|)$ ***then:***

$$\omega<\frac{1}{3}\left(\frac{a|a|-b|b|}{a-b}+\frac{b|b|-c|c|}{b-c}+\frac{c|c|-a|a|}{c-a}\right)<2\Omega$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.98

If $a,b,c>0, a+b+c=1$ *then:*

$$a^3+b^3+c^3+6abc\geq a^{a^2+2bc}\cdot b^{b^2+2ac}\cdot c^{c^2+2ab}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.99

If $a,b,c\in\mathbb{R}$ *then:*

$$\sum(a^2+b^2-c^2)^2+8\sum a^2b^2\geq 27abc\sqrt[3]{abc}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.100

If $x,y\geq 0$ *then:*

$$(x^3+y^3)^3(x^2-xy+y^2)\geq x^2y^2\sqrt{xy}(x^2+y^2)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.101

If $a,b,c\in(0,\infty)$ *then:*

$$a^8b^8+b^8c^8+c^8a^8\geq a^5b^5c^5\sqrt[4]{27(a^4+b^4+c^4)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.102

If $a,b,c>0, a^2+b^2+c^2=26(a+b+c)$ *then:*

$$\frac{1}{\sqrt{a+b^2}}+\frac{1}{\sqrt{b+c^2}}+\frac{1}{\sqrt{c+a^2}}\geq\frac{1}{\sqrt{a+b+c}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.103

If $a,b,c\in(0,\infty)$ *then:*

$$\sum\left(a^2b+ab\sqrt{ab}+ab^2\right)\leq\frac{3\sqrt{2}}{2}\sum\sqrt{a^6+b^6}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.104

If $a,b,c,d>1, abcd=e^4$ *then:*

$$\frac{lnd}{log_d(ab^2c^3)}+\frac{lnc}{log_c(da^2b^3)}+\frac{lnb}{log_b(cd^2a^3)}+\frac{lna}{log_a(bc^2d^3)}\geq\frac{2}{3}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

PROBLEM 1.105

If $x,y,z>0$ *then:*

$$x^2+y^2+z^2+xy+yz+zx\geq 2\sqrt{3xyz(x+y+z)}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.106

If $a,b,c>0, a+b+c=3$ *then:*

$$\frac{\sqrt{2}}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+3\right)\geq\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.107

If $x, y, z > 0, x + y + z = 3$ ***then:***

$$\frac{1}{(x+y)^3}+\frac{1}{(y+z)^3}+\frac{1}{(z+x)^3}+\frac{3}{8}\geq 16\left(\frac{1}{(2x+y+z)^3}+\frac{1}{(x+2y+z)^3}+\frac{1}{(x+y+2z)^3}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 1.108

If $a, b, c, d > 0, a + b + c + d = 4$ ***then:***

$$\frac{(a+1)(b+1)(c+1)(d+1)}{abcd}\geq 16$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.109

If $a, b, c > 0, a + b + c \leq 1$ ***then:***

$$\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.110

If $a, b \geq 1$ ***then:***

$$\left|\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right|\leq (a-b)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.111

If $a, b, c > 0$ ***then:***

$$(a+b+c)\left(\frac{a}{b^{10}}+\frac{b}{c^{10}}+\frac{c}{a^{10}}\right)\geq\left(\frac{a}{b^5}+\frac{b}{c^5}+\frac{c}{a^5}\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.112

If $0 < a, b, c < \frac{\pi}{2}$ ***then:***

$$\left(\frac{a+b+c}{ab+bc+ca}\sin\left(\frac{ab+bc+ca}{a+b+c}\right)\right)^{a+b+c}\geq\left(\frac{\sin b}{b}\right)^a\left(\frac{\sin c}{c}\right)^b\left(\frac{\sin a}{a}\right)^c$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.113

If $a, b, c \geq 0$ ***then:***

$$3abc\leq\sqrt{a^2+b^2+c^2}\cdot\sqrt[3]{a^3+b^3+c^3}\cdot\sqrt[5]{a^5+b^5+c^5}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.114

If $x, y, z > 0, \frac{x+y}{2x+y}+\frac{y+z}{2y+z}+\frac{z+x}{2z+x}=2$ ***then:***

$$\frac{3x^2+xy+2y^2}{2x^2+y^2}+\frac{3y^2+yz+2z^2}{2y^2+z^2}+\frac{3z^2+zx+2x^2}{2z^2+x^2}\leq 6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.115

If $a, b, c > 0$ ***then:***

$$a^a \cdot b^b \cdot c^c \cdot (4a + 4b + 4c)^{a+b+c} \geq 3^{a+b+c} \cdot (a+b)^{a+b} \cdot (b+c)^{b+c} \cdot (c+a)^{c+a}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 1.116

If $x, y, z \geq 0, x + y + z = 2$ ***then:***

$$\frac{2}{5} \leq \frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \leq \frac{18}{13}$$

Proposed by Vasile Mircea Popa-Romania

PROBLEM 1.117

If $a, b > 0$ ***then:***

$$\frac{\left((ab)^6 + \left(\frac{a+b}{2}\right)^{12}\right)\left(ab + \left(\frac{a+b}{2}\right)^2\right)}{\left((ab)^3\sqrt{ab} + \left(\frac{a+b}{2}\right)^7\right)^2} \leq \frac{\left(a^5 + b^5\right)^2}{4(ab)^5}$$

Proposed by Daniel Sitaru – Romania

ACYCLIC,ASYMMETRICAL INEQUALITIES

PROBLEMS

PROBLEM 2.01

If $a,b\geq 0, a+b+c+d=0$ then:

$$4\sum a^3\geq 3(a+b)(ac+ad+bc+bd+4cd)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.02

If $a,b,c,d>0$ ***then:***

$$\left(2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\right)\left(\sum\frac{1}{a}\right)^4\geq\frac{4672}{3}$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

PROBLEM 2.03

If $a,b,c,d>0, x,y\in\mathbb{R}$ then:

$$\frac{sin^2x}{a}+\frac{cos^2x}{b}+\frac{sin^2y}{c}+\frac{cos^2y}{d}>\frac{2}{a+b+c+d}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.04

If $a,b,c,d,e,f>0$ then:

$$\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{\left(c^5+d^5\right)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.05

If $0<a\leq b$ then:

$$\left(1+\frac{a+3b}{4}\right)^{3a+b}\leq\left(1+\frac{3a+b}{4}\right)^{a+3b}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.06

If $0\leq x\leq y\leq z$ then:

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)}\geq\frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.07

If $a,b,c,d\in\mathbb{N}^*, 1\leq a\leq b\leq c\leq d$ then:

$$4\log_{a+1}a\leq\log_{a+1}a+\log_{b+1}b+\log_{c+1}c+\log_{d+1}d\leq 4\log_{d+1}d$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.08

If $a, b, c, d > 0$ then:

$$\frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b}+\frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c}+\frac{\left(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}\right)^4}{a+b+c+d}\le 75$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.09

If $a, b, c \ge 0$ then:

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3}\le\sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}$$

Proposed by Kiran Kedlaya-Berkeley-California-USA

PROBLEM 2.10

Prove that if $x, y, z > 0$ then:

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\le\sqrt{6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.11

Prove that if $a, b, c > 0$ then:

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\le\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.12

If $0 \le a \le b \le c$ then:

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}\le 0$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.13

If a, b, c be positive real number such that $a \le b \le c$ then

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\ge(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

PROBLEM 2.14

If $a, b, c, d, e > 0, 2b = a + c, 2c = b + d$ then:

$$a^2+b^2+c^2+d^2\ge 4\sqrt[8]{e}\left(a+d-\sqrt[8]{e}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.15

Prove that if x, y and z are in $[-5, 3]$ then

$$\sqrt{3x-5y-xy+15}+\sqrt{3y-5z-yz+15}+\sqrt{3z-5x-xz+15}\le 12$$

When does equality occur?

Hungary NMO

PROBLEM 2.16

If $a,b,c,d>0, abcd=1$ then:

$$a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)+c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\le\frac{1}{2}(ab+bc+cd+da)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.17

In $\Delta ABC, a\neq b$ the following relationship holds:

$$\frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)}>1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.18

Let a,b,c be positive real numbers such that:

$$\begin{cases} ab>6 \\ \frac{a}{8}+3b+\frac{2c}{3}=\frac{abc}{9}+\frac{67}{4a} \end{cases}$$

Find the minimum value of the expression:

$$P=3a+2b+c$$

Proposed by Do Quoc Chinh-Vietnam

PROBLEM 2.19

If $a,b,c\ge 1$ then:

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)}\le\frac{15}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.20

If $0<a\le b$ and $c,d,e\ge 0$ then:

$$a^3\le\frac{\left(a+c\sqrt{ab}+b\right)\left(a+d\sqrt{ab}+b\right)\left(a+e\sqrt{ab}+b\right)}{(c+2)(d+2)(e+2)}\le b^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.21

If $0<a\le b\le c$ then:

$$3a^2b\le\prod\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le 3bc^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.22

If $a,b,c\ge 0$ then:

$$2a^2+6ab+7b^2\ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.23

If $0\le x,y,z\le a$ then:

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}\le a\left(1+\sqrt{2}+\sqrt{3}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.24

If $x, y, z, t \geq 1$ then:

$$\frac{xy + 2yz + 2zt + 2xz + ty + tx + 9}{2x + 2y + 3z + 2t} \geq 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.25

If $x, y, z, t > 0$ then:

$$\frac{(x^6 + y^6)^2(x^4 + y^4 + z^4)^3(x^3 + y^3 + z^3 + t^3)^4}{(x^{12} + y^{12})(x^{12} + y^{12} + z^{12})(x^{12} + y^{12} + z^{12} + t^{12})} \leq 1152$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.26

If $0 < a \leq b \leq c$ then:

$$\frac{1}{1 + e^{a-b+c}} + \frac{1}{1 + e^b} \leq \frac{1}{1 + e^a} + \frac{1}{1 + e^c}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.27

If $a, b \in \mathbb{R}, |3a + 4b + 2| = 5$ then:

$$a^2 + b^2 + 4b + 7 \geq 4a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.28

If $a, b, c > 0, a^2 + b^2 = 1, b^2 + c^2 = 1$ then:

$$a + 2b + c + \frac{a + c}{abc} \geq 4 + 2\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.29

If $a, b, c > 0$ then:

$$\left(\frac{a^4}{4} + \frac{b^8}{8} + \frac{5\sqrt[5]{c^8}}{8}\right)\left(\frac{5\sqrt[5]{a^8}}{8} + \frac{b^8}{8} + \frac{c^4}{4}\right) \geq \frac{27(abc)^4}{(ab + bc + ca)^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.30

If $a \geq b \geq c$ then:

$$\sqrt{a^2 - b^2} + \sqrt{b^2 - c^2} + \sqrt{a^2 - c^2} + \sqrt{2}(a + b + c) \geq \sqrt{a^2 + b^2} + \sqrt{b^2 + c^2} + \sqrt{a^2 + c^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.31

If $0 < a \leq b \leq c \leq d \leq e$ then:

$$2\sqrt{ab} + 3\sqrt[3]{abc} + 4\sqrt[4]{abcd} \leq 9\sqrt[5]{abcde}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.32

Let be $x = \frac{(a+b+c+d)^4}{256\,abcd}, y = \frac{(a+b+c)^3}{27abc}, z = \frac{(a+b)^2}{4ab}, a, b, c, d \ge 1$

Prove that:

$$ab(1 + c + cd)(x + y + z) \le 3(abcdx + abcy + abz)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.33

Let be: $a, b, c, d \ge 0, p \ge q \ge r \ge 0$

$$x = \frac{a+b+c+d}{4} - \sqrt[4]{abcd}, y = \frac{a+b+c}{3} - \sqrt[3]{abc}, z = \frac{a+b}{2} - \sqrt{ab}$$

Prove that:

$$3(px + 3qy + 2rz) \ge (4x + 3y + 2z)(p + q + r)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.34

If $0 < z < y < x < \frac{\pi}{2}$ *then:*

$$\frac{\sin x}{\sin y} + \frac{\sin x + \sin y}{\sin z} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{y}\right)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.35

If $a, b > 0, a + b + c = 0$ *then:*

$$6\left(a^5 + b^5 + c^5\right) \ge 5(2ab + c^2)\left(2ab\sqrt{ab} + c^3\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.36

Let x, y, z *be positive real numbers such that* $xyz = x + 27y + 125z$.

Prove that: $x + y + z \ge 27$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 2.37

If $a, b, c, d \in \mathbb{R}, a \le b \le c \le d$ *then:*

$$e^a - e^c + e^b - e^d \ge 2\left(\sqrt{e^{a+b}} - \sqrt{e^{c+d}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.38

If $a,b,c\in\mathbb{R}, a+b\ge 0, a\le c\le b$ ***then:***

$$2\sinh\left(\frac{a+b}{2}\right)\le \sinh(c)+\sinh(a+b-c)\le \sinh(a)+\sinh(b)$$

Proposed by Abdallah El Farissi – Bechar – Algerie

PROBLEM 2.39

If $a,b,c,d\in\mathbb{R}$ ***then:***

$$2(ad-bc)^4+2(ac+bd)^4\ge (a^2+b^2)^2(c^2+d^2)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.40

If $0<x\le y; a,b,c>0$

$$A=\frac{a+b+c}{3}, B=\frac{ab+bc+ca}{a+b+c}, C=\frac{3abc}{ab+bc+ca} \text{ then:}$$

$$A+B+C\ge\frac{3\left(Ax+Cy+B\sqrt{xy}\right)}{x+y+\sqrt{xy}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.41

If $0<a\le b\le c, x,y\ge 0$ ***then:***

$$(9a+12b+18c)(x^2+y^2)^2+(18a+12b)xy\ge$$
$$\ge (a+b+c)(13x^2+10xy+13y^2)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.42

If $0\le a,b\le 1$ ***then:***

$$\frac{a+1}{2^b}+\frac{b+3}{3^a}+(1-a)e^{1-b}\le e+4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.43

If $a,b,c>0$ ***then:***

$$(a+2b+c)(a^2+b^2+c^2+ab+bc-ac)+4\ge 6b+3a+3c$$

Proposed by Rustem Zeynalov-Baku-Azerbaidian

PROBLEM 2.44

If $2<a<b<c$ ***then:***

$$\left(\sqrt{b}+\sqrt{c}\right)\left(a^2+\frac{1}{a^3}\right)+\left(\sqrt{a}-\sqrt{c}\right)\left(b^2+\frac{1}{b^3}\right)<\left(\sqrt{a}+\sqrt{b}\right)\left(c^2+\frac{1}{c^3}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.45

If $0\le a<b<c<d$ ***then:***

$$3\sqrt{a}+3\sqrt[4]{b}+2\sqrt[6]{c}+\sqrt[8]{d}>\sqrt[6]{ab}+\sqrt[12]{abc}+\sqrt[20]{abcd}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.46

If $a,b,c>0, a^2+b^2+c^2=4$ *then:*

$$\frac{6}{a}-\frac{1}{b}+\frac{1}{c}<\frac{12}{abc}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.47

If $0\le a<3, 0\le b<5, 0\le c<7$ *then:*

$$\sqrt[3]{a+1}+\sqrt[5]{b+1}+\sqrt[7]{c+1}<6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.48

If $0<a\le b$ *then:*

$$3a\le\frac{a^3+b^3}{a^2+b^2}+\frac{a^5+b^5}{a^4+b^4}+\frac{a^7+b^7}{a^6+b^6}\le 3b$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.49

If $a,b,c\ge 0$ *then:*

$$a+b+c\ge\left(\sqrt[4]{a}+\sqrt[4]{b}\right)\sqrt[4]{abc}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.50

If $1\le a<b<c$ *then:*

$$\frac{b^3-a^3}{3}+\frac{c^5-b^5}{5}+\frac{c^7-a^7}{7}<\frac{b^4-a^4}{4}+\frac{c^6-b^6}{6}+\frac{c^8-a^8}{8}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.51

If $0<a\le b\le c$ *then:*

$$\frac{9(a+b)\sqrt{ab}+6(a+b+c)\sqrt[3]{abc}+18c^2}{(5a+5b+8c)\left(c+\sqrt{ab}+\sqrt[3]{abc}\right)}\ge 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.52

If $a,b,c>0, abc=1$ *then:*

$$\frac{a}{1+a}+\frac{b}{(1+a)(1+b)}+\frac{c}{(1+a)(1+b)(1+c)}\ge\frac{7}{8}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.53

If $a,b,c,d\ge 1$ *then:*

$$\frac{(1+a)^3(1+b)^2(1+c)}{(1+ab)(1+abc)(1+abcd)}\le\frac{64}{(1+b)(1+c)(1+d)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.54

If $0<x\le y\le z<1$ *then:*

$$(y-z)\tan^{-1}x+(z-x)\tan^{-1}y+(z-y)\tan^{-1}z<\frac{\pi}{2}-\log 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.55

If $a, b, c, d \geq 1$ ***then:***

$$3\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\right) > \frac{2}{1+\sqrt{ab}}+\frac{3}{1+\sqrt[3]{abc}}+\frac{4}{1+\sqrt[4]{abcd}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.56

If $2 < a \leq b \leq c$ ***then:***

$$ln(a-1)\cdot ln(bc)+ln(b-1)\cdot ln\, c \leq ln(c-1)\cdot ln(ab)+ln(b-1)\cdot ln\, a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.57

If $a, b > 0, a+b+c=0$ ***then:***

$$6\left(a^5+b^5+c^5\right) \geq 5(2ab+c^2)\left(2ab\sqrt{ab}+c^3\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.58

If $0 < a \leq b \leq c$ ***then:***

$$(b-a)(c-a)(c-b)e^{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}} \leq \left(e^b-e^a\right)(e^c-e^a)\left(e^c-e^b\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.59

If $0 < a < b < c < d$ ***then:***

$$\left(\frac{a}{c}\right)^{\frac{1}{a-c}} > \left(\frac{b}{d}\right)^{\frac{1}{b-d}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.60

If $0 \leq x, y, z < 1$ ***then:***

$$\sqrt[3]{\frac{(1+x^3)(1+y^6)(1+z^9)}{(1-x^3)(1-y^6)(1-z^9)}} \geq \frac{1+xy^2z^3}{1-xy^2z^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.61

If $0 < a < b < c$ ***then:***

$$\frac{1}{a-c}arctan\left(\frac{a-c}{1+ac}\right) > \frac{1}{b-d}arctan\left(\frac{b-d}{1+bd}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.62

If $a, b, c, d \geq 0$ ***then:***

$$\sqrt{a^2+b^2-ab\sqrt{2}}+\sqrt{b^2+c^2-bc\sqrt{3}}+\sqrt{c^2+d^2-\frac{cd\left(\sqrt{6}+\sqrt{2}\right)}{2}} \geq \sqrt{a^2+d^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.63

If $a,b,c\in(0,\infty)$, $a^2+b^2+c^2=10$ then:

$$27\left(\frac{1}{c}+\frac{5}{b}-\frac{1}{a}\right)\le 25\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.64

Let a,b,c be positive real numbers. Prove that

$$\frac{2a^2+bc}{b+c}+\frac{2b^2+ca}{c+a}+\frac{2c^2+ab}{a+b}\ge\frac{3}{2}(a+b+c)+\frac{16abc(a-c)^2}{(a+b+c)(a+b)(b+c)(c+a)}$$

Solution by Nguyen Ngoc Tu – Ha Giang – Vietnam

PROBLEM 2.65

If $a,b,c,\eta\in\mathbb{R}$ then:

$$|(a-b)(b-c)(c-a)|\le\sum|(a-b)(c+b+\eta)(c+a+\eta)|$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.66

If $a,b,c\in\mathbb{R}^*$; $a\ne b\ne c\ne a$ then:

$$\frac{\sqrt[6]{a^2+b^2}}{\sqrt[3]{|a|}+\sqrt[3]{|b|}}+\frac{\sqrt[10]{b^2+c^2}}{\sqrt[5]{|b|}+\sqrt[5]{|c|}}+\frac{\sqrt[14]{c^2+a^2}}{\sqrt[7]{|c|}+\sqrt[7]{|a|}}<3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.67

Prove that for any real numbers x,y,z:

$$(x^2+2y^2+2z^2+3xy+5yz+3zx)^2\ge 8(x+y)(y+z)(z+x)(x+y+z)$$

Proposed by Nguyen Viet Hung –Hanoi- Vietnam

PROBLEM 2.68

If $a,b,c,d>0$ then:

$$\frac{ac+bd+|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}+\frac{(a^2+b^2)(c^2+d^2)}{(ac+bd)|ad-bc|}\ge 2+\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.69

If $0<x\le y\le z\le t$ then:

$$\left(4\sqrt[4]{xyzt}-3\sqrt[3]{xyz}\right)\left(3\sqrt[3]{xyz}-2\sqrt{xy}\right)\le zt$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.70

If $x,y,z\in[0,\infty)$ then:

$$\sqrt{x^2-xy\sqrt{3}+y^2}+\sqrt{y^2-yz\sqrt{2}+z^2}\ge\sqrt{x^2-xz+z^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.71

If $x, y, z \in (0, \infty), xyz = 1$ ***then:***

$$x\left(x - 3(y+z)\right)^2 + \left(3x - (y+z)\right)^2 (y+z) \geq 27$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.72

If $0 < a \leq b \leq c, a+b+c = 3$ ***then:***

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a}{c} + \frac{b}{a} + \frac{c}{b} + (b-a)(a-c)(b-c)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.73

If $a, b \geq 0, a+b+c+d = 0$ ***then:***

$$4\sum a^3 \geq 3(a+b)(ac+ad+bc+bd+4cd)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.74

If $a, b, c, d \in \mathbb{R}$ ***then:***

$$(2a+3b+4c+5d)^2 \geq 8(3ab+5ad+6bc+10cd)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 2.75

If $a, b, c, d > 0, x, y \in \mathbb{R}$ ***then:***

$$\frac{\sin^2 x}{a} + \frac{\cos^2 x}{b} + \frac{\sin^2 y}{c} + \frac{\cos^2 y}{d} > \frac{2}{a+b+c+d}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.76

If $a, b, c, d, e, f > 0$ ***then:***

$$\frac{(a^3+b^3)^4}{(c^6+d^6)^5} \cdot \frac{\left(c^5+d^5\right)^6}{(e^8+f^8)^7} \cdot \frac{(e^7+f^7)^8}{(a^4+b^4)^3} > 1$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.77

GENERALIZATION FOR HUNG NGUYEN VIET'S INEQUALITY

If $a, b, c, x, y, z > 0, a^3x + b^3y + c^3z = xyz$ ***then:***

$$x + y + z \geq (a+b+c)\sqrt{a+b+c}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.78

If $0 \leq x \leq y \leq z$ ***then:***

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)} \geq \frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 2.79

If $a,b,c,d\in\mathbb{N}^*, 1\le a\le b\le c\le d$ *then:*

$$4\log_{a+1}a\le \log_{a+1}a+\log_{b+1}b+\log_{c+1}c+\log_{d+1}d\le 4\log_{d+1}d$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.80

If $x,y\in\mathbb{R}$ *then:*

$$(x^3+2y^3-3xy^2)^2\le (x^2+2y^2)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.81

Prove that if $x,y,z>0$ *then:*

$$\sqrt{\frac{x}{y}}+2\sqrt{\frac{y}{z}}+3\sqrt{\frac{z}{x}}\le\sqrt{6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.82

Prove that if $a,b,c>0$ *then:*

$$\sqrt{\frac{a}{b+c}}+2\sqrt{\frac{b}{c+a}}+4\sqrt{\frac{c}{a+b}}\le\sqrt{7\left(\frac{a}{b+c}+\frac{2b}{c+a}+\frac{4c}{a+b}\right)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.83

If $0\le a\le b\le c$ *then:*

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}\le 0$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.84

If a,b,c *be positive real number such that* $a\le b\le c$ *then*

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Proposed by Pham Quoc Sang-Ho Chi Minh-Vietnam

PROBLEM 2.85

If $a,b,c,d,e>0, 2b=a+c, 2c=b+d$ *then:*

$$a^2+b^2+c^2+d^2\ge 4\sqrt[8]{e}\left(a+d-\sqrt[8]{e}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.86

Prove that if x,y *and* z *are in* $[-5,3]$ *then*

$$\sqrt{3x-5y-xy+15}+\sqrt{3y-5z-yz+15}+\sqrt{3z-5x-xz+15}\le 12$$

When does equality occur?

Hungary NMO

PROBLEM 2.87

If $a,b,c,d>0, abcd=1$ *then:*

$$a\left(\frac{b}{b+a}+\frac{d}{d+a}\right)+c\left(\frac{b}{b+c}+\frac{d}{d+c}\right)\le\frac{1}{2}(ab+bc+cd+da)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.88

In $\Delta ABC, a \neq b$ the following relationship holds:

$$\frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)} > 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.89

Let a, b, c be positive real numbers such that:

$$\begin{cases} ab > 6 \\ \frac{a}{8}+3b+\frac{2c}{3}=\frac{abc}{9}+\frac{67}{4a} \end{cases}$$

Find the minimum value of the expression:

$$P = 3a+2b+c$$

Proposed by Do Quoc Chinh-Vietnam

PROBLEM 2.90

If $a, b, c \geq 1$ then:

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^3+c^4)}{(1+a^2)(1+b^3)(1+c^4)} \leq \frac{15}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.91

If $0 < a \leq b$ and $c, d, e \geq 0$ then:

$$a^3 \leq \frac{\left(a+c\sqrt{ab}+b\right)\left(a+d\sqrt{ab}+b\right)\left(a+e\sqrt{ab}+b\right)}{(c+2)(d+2)(e+2)} \leq b^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.92

If $0 < a \leq b \leq c$ then:

$$3a^2b \leq \prod \sqrt[3]{a^3+ab\sqrt{ab}+b^3} \leq 3bc^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.93

If $a, b, c \geq 0$ then:

$$2a^2+6ab+7b^2 \geq 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.94

If $0 \leq x, y, z \leq a$ then:

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2} \leq a\left(1+\sqrt{2}+\sqrt{3}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.95

If $x, y, z, t \geq 1$ then:

$$\frac{xy+2yz+2zt+2xz+ty+tx+9}{2x+2y+3z+2t} \geq 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.96

If $x,y,z>0, xyz(3x+2y+36z)=6$ ***then:***

$$\left(\frac{x^2y^2}{36}+1\right)(4y^2z^2+1)(9z^2x^2+1)\ge 64x^4y^4z^4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.97

If $0<a\le b\le c$ ***then:***

$$\frac{1}{1+e^{a-b+c}}+\frac{1}{1+e^{b}}\le\frac{1}{1+e^{a}}+\frac{1}{1+e^{c}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.98

If $a,b\in\mathbb{R}, |3a+4b+2|=5$ ***then:***

$$a^2+b^2+4b+7\ge 4a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.99

If $0<a\le b\le c\le d\le e$ ***then:***

$$2\sqrt{ab}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}\le 9\sqrt[5]{abcde}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.100

If $a,b,c\in\mathbb{R}, a^2+b^2+c^2=3$ ***then:***

$$|a+(a+c)b+c|\le 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.101

If $0\le x\le\frac{\sqrt{6}}{3}$ ***then:***

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}\ge 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 2.102

If $a,b,c,d,e,f\ge 1$ ***then:***

$$a+b+2c+2d+e+f\le abc^2d^2ef+7$$

Proposed by Daniel Sitaru – Romania

ELEGANT INEQUALITIES AND IDENTITIES

PROBLEMS

PROBLEM 3.01

If $a,b,c>0, a+b+c=3, 0\le x\le 1$ ***then:***

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\le 6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.02

If $x,y\in\left(0,\frac{\pi}{2}\right)$ ***then:***

$$\frac{(\sin^2 x+\sin^2 y)^{\sin^2 x+\sin^2 y}\cdot(\cos^2 x+\cos^2 y)^{\cos^2 x+\cos^2 y}}{(\sin x)^{2\sin^2 x}\cdot(\sin y)^{2\sin^2 y}\cdot(\cos x)^{2\cos^2 x}\cdot(\cos y)^{2\cos^2 y}}\le 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.03

Let $a,b,c>0$ ***and*** $a+b+c=3$***. Prove that:***

$$a\cdot\arcsin\left(\frac{b}{b+1}\right)+b\cdot\arcsin\left(\frac{c}{c+1}\right)+c\cdot\arcsin\left(\frac{a}{a+1}\right)\le\frac{\pi}{2}$$

Proposed by Dimitris Kastriotis-Athens-Greece

PROBLEM 3.04

Let $n\in\mathbb{N}^*\wedge n\ge 2$ ***and*** $x_1,x_2,\dots,x_n\in(0;+\infty)$***. Prove:***

$$e^n x_1^{\frac{1}{x_1}}x_2^{\frac{1}{x_2}}\dots x_n^{\frac{1}{x_n}}\le e^{x_1+x_2+\dots+x_n}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.05

If $x,y\ge 0$ ***then:***

$$(e^x+1)\sqrt{e^y}+(e^y+1)\sqrt{e^x}\le(e^x+1)(e^y+1)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.06

If $a,b,c>0, abc=1$ ***then:***

$$e^{a^{3a^{3}}}+e^{b^{3b^{3}}}+e^{c^{3c^{3}}}\ge 3e$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

PROBLEM 3.07

$$A=(a_{ij})_{\substack{1\le i\le n\\1\le j\le n}}, a_{ij}=10i+j, n\ge 2, n\in\mathbb{N}^*.$$

Find $X,Y\in M_n(\mathbb{R})$ ***such that:***

$$\det X<0, \det Y<0, A+Y=X$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.08

If $n \in \mathbb{N}, n \geq 2$ then:

$$log(n!) + 1 - n < \sum_{k=2}^{n} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k}\right) < log(n!)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.09

If $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1$ then:

$$\sqrt[3]{abc} \cdot tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right) \leq \sqrt{\frac{ab+bc+ca}{3}} \cdot tan^{-1}\left(\sqrt[3]{abc}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.10

If $a \geq 4, b, c \geq 0, a + c \leq 2b, x, y, z \in \mathbb{R}$ then:

$$(a-3)(c - x^2 - y^2 - z^2) \leq (b - x - y - z)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.11

Let $x, y \in (0; +\infty) \wedge x + y = 1$ and $n \in \mathbb{N}^*$.

Prove: $(xy)^n \geq \frac{16^n+1}{4^n} - \frac{1}{x^n y^n}$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.12

If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{(tan\,x + cot\,x)(tan\,y + cot\,y)(tan\,z + cot\,z)}{(tan\,x + cot\,y)(tan\,y + cot\,z)(tan\,z + cot\,x)} \geq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.13

If in ΔABC, $a \leq b \leq c$ then:

$$h_a^{20} - h_b^{20} + h_c^{20} \geq (h_a - h_b + h_c)^{20}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.14

If $x > 0$ then:

$$\left(e^{x^2} + e^{(x+3)^2}\right)\left(\frac{1}{1+e^x} + \frac{1}{1+e^{x+3}}\right) > \left(e^{(x+1)^2} + e^{(x+2)^2}\right)\left(\frac{1}{1+e^{x+1}} + \frac{1}{1+e^{x+2}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.15

If $a, b, c, d, e, f > 0, a + d = b + e = c + f = 5$ then:

$$(a+b+c)\left(\frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right) \leq 3\left(\frac{a}{d} + \frac{b}{e} + \frac{c}{f}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.16

Let x, y, z be positive real numbers such that: $x^2+y^2+z^2=3$.

Find the minimum of value:

$$P=\frac{x}{\sqrt{y}+\sqrt{z}}+\frac{y}{\sqrt{z}+\sqrt{x}}+\frac{z}{\sqrt{x}+\sqrt{y}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 3.17

If $x, y \in \mathbb{R}$ then:

$$\frac{5\sin^2 x}{1+\cos^2 x}+\frac{5\cos^2 x\cdot\sin^2 y}{1+\sin^2 x+\cos^2 x\cdot\cos^2 y}+\frac{5\cos^2 x\cdot\cos^2 y}{1+\sin^2 x+\cos^2 x\cdot\sin^2 y}\geq 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.18

If $a, b, c > 0$ then:

$$\frac{9+4a+4a^2}{1+a}+\frac{9+4b+4b^2}{1+b}+\frac{9+4c+4c^2}{1+c}\geq 24$$

Proposed by Eliezer Okeke-Nigeria

PROBLEM 3.19

If $a, b, c, d \in \mathbb{N}-\{0\}, a > b > c > d$ then:

$$bd(2^a-1)(2^c-1) > ac\left(2^b-1\right)\left(2^d-1\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.20

If $x, y > 0, x+2y \leq 5, 3x+y \geq 7, (x+2y)(3x+y) \geq 20$ then:

$$4x+3y \geq 9$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.21

If $0 < x < \frac{\pi}{2}$ then:

$$\pi \cdot e^{\sum_{k=1}^{n} \log\left(\cos\left(\frac{x}{2^k}\right)\right)} > 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.22

Let x, y, z be positive real numbers such that $x+y+z=3$. Find the minimum of value:

$$P=\frac{x}{\sqrt{\frac{y^4+z^4}{2}+2yz}}+\frac{y}{\sqrt{\frac{z^4+x^4}{2}+2zx}}+\frac{z}{\sqrt{\frac{x^4+y^4}{2}+2xy}}+\frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

PROBLEM 3.23

If $x, y, z \in \left(0, \frac{\pi}{2}\right), \sin x+\sin y+\sin z=1$ then:

$$\cos^2 x\cdot\cos^2 y\cdot\cos^2 z \geq 512\sin^2 x\cdot\sin^2 y\cdot\sin^2 z$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.24

If $x, y, z \in \left(0, \frac{\pi}{2}\right), x + y + z = \pi$ then:

$$\frac{xy(\tan x + \sin x)}{x^2 + \sin x \cdot \tan x} + \frac{yz(\tan y + \sin y)}{y^2 + \sin y \cdot \tan y} + \frac{zx(\tan z + \sin z)}{z^2 + \sin z \cdot \cos z} > \pi$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.25

If $x_1, x_2, \dots x_n > 0, n \in \mathbb{N}, n \geq 2, x_1 x_2 \cdot \dots \cdot x_n = 1$ *then:*

$$\frac{x_1 e^{x_1} + x_2 e^{x_2} + \dots + x_n e^{x_n}}{x_1 + x_2 + \dots + x_n} \geq e$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.26

If $x, y, z, t \in \left(0, \frac{\pi}{2}\right)$ *then:*

$$64 \cdot \cos x \cdot \cos z \cdot \sin y \cdot \sin t \cdot \sin(x - y) \cdot \sin(z - t) \leq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.27

If $x, y, z \in (0, 1)$ *then:*

$$\sum_{cyc(x,y,z)} \frac{y(sin^{-1} x + tan^{-1} x)}{x^2 + tan^{-1} x \cdot sin^{-1} x} > 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.28

If $x \geq 0$ *then:*

$$sin\, x\,(16\, sin^4 x + 5) \leq 5x(4x^2 + 1)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.29

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, $\cos x \cdot \cos y \cdot \cos z = \frac{\sqrt{2}}{2}$ then:

$$15(\cos 2x + \cos 2y + \cos 2z) + 6(\cos 4x + \cos 4y + \cos 4z) +$$
$$+ \cos 6x + \cos 6y + \cos 6z \geq 18$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.30

Find $x, y, z \in \left(0, \frac{\pi}{2}\right]$ *such that:*

$$\frac{sin^2 x}{1 + sin^2 x} + \frac{sin^2 y}{(1 + sin^2 x)(1 + sin^2 y)} + \frac{sin^2 z}{(1 + sin^2 x)(1 + sin^2 y)(1 + sin^2 z)}$$
$$+ \frac{1}{8\, sin\, x\, sin\, y\, sin\, z} \leq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.31

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$, $sin\, x + sin\, y + sin\, z = 1$ *then:*

$$cos^2 x \cdot cos^2 y \cdot cos^2 z \geq 512\, sin^2 x \cdot sin^2 y \cdot sin^2 z$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.32

If $a,b,c>0$ ***then:***

$$tan^{-1}\left(\frac{(2a+b)(b+2a)}{9ab}\right)+tan^{-1}\left(\frac{(2b+c)(c+2b)}{9bc}\right)+tan^{-1}\left(\frac{(2c+a)(2a+c)}{9ca}\right)\geq\frac{3\pi}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.33

If $0<x,y,z,t<\frac{\pi}{2}$ ***then:***

$$\sum_{cyc(x,y,z,t)}(sin^2x+csc^2x)^3+\sum_{cyc(x,y,z,t)}(cos^2x+sec^2x)^3\geq 125$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.34

If $x\in\left(0,\frac{\pi}{2}\right)$ ***then:***

$$2\cdot(sin\,x)^{1-sin\,x}\cdot(1-sin\,x)^{sin\,x}\leq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.35

If $x\in\left[0,\frac{\pi}{14}\right)$ ***then:***

$$(cos\,3x)^{21}\cdot(cos\,5x)^7\cdot(cos\,7x)\leq(cos\,x)^{413}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.36

If $x\in\left(0,\frac{\pi}{2}\right)$ ***then:***

$$\pi\left(\frac{\sin x}{x}+\frac{\cos x}{\frac{\pi}{2}-x}\right)>4+(\pi-2)(\sin x+\cos x)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.37

If $2\,sin^2x+2\,sin^2y=1,x,y\in\left(0,\frac{\pi}{2}\right)$ ***then:***

$$2\,tan\,x\,tan\,y+2\,tan\,x+2\,tan\,y<3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.38

If $x,y>0,xy\geq\frac{1}{8}$ ***then:***

$$\frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}}>\frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.39

If $0<a\leq b<\frac{\pi}{2}$ ***then:***

$$(a+b)\left(\sin(\sqrt{ab})-\cos(\sqrt{ab})\right)\leq 2\sqrt{ab}\left(\sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.40

If $x,y,z>0$ then:

$$e^{x^2+y^2+z^2}\geq 2exyz\sqrt{2e}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

PROBLEM 3.41

If $m,n\in\mathbb{N},m,n\geq 1$ then:

$$3(m+n)+\log(m!\cdot n!)^{10}\geq 6\sqrt{m\cdot n\cdot H_m\cdot H_n}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.42

If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\sqrt{2}(1+\tan x)(1+\tan y)(1+\tan z)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.43

If $a,b,c\geq 1$ then:

$$\frac{a}{c\cdot\log(eb-\log b)}+\frac{b}{a\cdot\log(ec-\log c)}+\frac{c}{b\cdot\log(ea-\log a)}\geq\frac{9}{a+b+c}$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

PROBLEM 3.44

If $a,b,c>0,\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=6,0\leq x\leq 1$ then:

$$a\left(\frac{b}{a}\right)^x+b\left(\frac{c}{b}\right)^x+c\left(\frac{a}{c}\right)^x+b\left(\frac{a}{b}\right)^x+c\left(\frac{b}{c}\right)^x+a\left(\frac{c}{a}\right)^x\geq 12$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.45

If $a,b,c,x,y,z>0,a+b+c=x+y+z=1$ then:

$$\frac{(a+x)^{a+x}\cdot(b+y)^{b+y}\cdot(c+z)^{c+z}}{a^a\cdot b^b\cdot c^c\cdot x^x\cdot y^y\cdot z^z}\leq 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.46

If $x,y,z>0$ then:

$$(x+y+z)\left(\frac{\sqrt{3}}{3}+\tan 20^\circ\right)>4\sum_{cyc}\left(\frac{xy}{x\cot 50^\circ+y\cot 10^\circ}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.47

If $0<a<b$ then:

$$4\left(\frac{3^a}{4^a}-\frac{3^b}{4^b}\right)<5\left(\frac{4^a}{5^a}-\frac{4^b}{5^b}\right)<6\left(\frac{5^a}{6^a}-\frac{5^b}{6^b}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.48

If $\boldsymbol{a, b, c} \in \mathbb{N}^*$

$$\Omega(a,b) = \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots + \frac{b(b-1)\cdot \ldots \cdot 2\cdot 1}{(a+b-1)(a+b-2)\cdot \ldots \cdot a}$$

then:

$$b\cdot\Omega(a,b) + c\cdot\Omega(b,c) + a\cdot\Omega(c,a) \geq a+b+c$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.49

If $\boldsymbol{a, b, c} > 0, a+b+c = 64$ ***then:***

$$\frac{csc^4\left(\frac{\pi}{7}\right)}{\sqrt{ab}} + \frac{csc^4\left(\frac{2\pi}{7}\right)}{\sqrt{bc}} + \frac{csc^4\left(\frac{3\pi}{7}\right)}{\sqrt{ca}} > 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.50

If $\boldsymbol{x, y, z} \in \left(0, \frac{\pi}{2}\right)$ ***then:***

$$\tan x + \tan y + \tan z > \tan x \cdot \tan y \cdot \tan z - \frac{1}{\cos x \cdot \cos y \cdot \cos z}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.51

If $\boldsymbol{a, b, c} > 0$ ***then:***

$$\tan^{-1}\left(\frac{(2a+b)(b+2a)}{9ab}\right) + \tan^{-1}\left(\frac{(2b+c)(c+2b)}{9bc}\right) + \tan^{-1}\left(\frac{(2c+a)(2a+c)}{9ca}\right) \geq \frac{3\pi}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.52

If $\boldsymbol{a, b, c, d, e, f > 0, a+d = b+e = c+f = 5}$ ***then:***

$$(a+b+c)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right) \leq 3\left(\frac{a}{d}+\frac{b}{e}+\frac{c}{f}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.53

If $\boldsymbol{a, b, c} > 1$ ***then:***

$$\frac{\sin\left(\frac{2}{a+b}\right)\sin\left(\frac{2}{b+c}\right)\sin\left(\frac{2}{c+a}\right)}{\sin\left(\frac{1}{\sqrt{ab}}\right)\sin\left(\frac{1}{\sqrt{bc}}\right)\sin\left(\frac{1}{\sqrt{ca}}\right)} \geq \left(\frac{8abc}{(a+b)(b+c)(c+a)}\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.54

Prove that if $\boldsymbol{0} < a \leq b$ ***then:***

$$\left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \leq \frac{(a+b)^2}{ab}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.55

If $x, y, z, t > 0$ then:

$$4\left(\left(x-\sqrt{xy}+y\right)\left(z-\sqrt{zt}+t\right)\right)^2 \geq (x^2+y^2)(z^2+t^2)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.56

If $a, b, c, d > 0, a+b+c+d=1$ then:

$$\frac{ab}{1+c+d}+\frac{ac}{1+b+d}+\frac{ad}{1+b+c}+\frac{bc}{1+a+d}+\frac{bd}{1+a+c}+\frac{cd}{1+a+b} \leq \frac{1}{4}$$

Proposed by Vasile Mircea Popa – Romania

PROBLEM 3.57

If $a, b, c \in \mathbb{N}, a, b, c \geq 4$ then:

$$\sqrt[a+1]{b}+\sqrt[a+1]{c}+\sqrt[b+1]{a}+\sqrt[b+1]{c}+\sqrt[c+1]{a}+\sqrt[c+1]{b} \leq 6\sqrt[4]{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.58

If $0 \leq x \leq \frac{\pi}{4}$ then:

$$\sin x+\cos x+\sin x \cdot \tan x+x^2 \geq 1+x \cdot \sin x+x \cdot \tan x$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 3.59

If $a, b, c > 0, a+b+c=3, x=\sqrt{\frac{3-ab-bc-ca}{3}}$ then:

$$a^3+b^3+c^3+3abc \geq 6+8x^2-10x^3+log(1+x^2-2x^3)$$

Proposed by Andrei Bâră – Romania

PROBLEM 3.60

Let $x, y, z \geq 0$.Prove that:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}+k$$

$$k=\frac{3\left|\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{y}-\sqrt[3]{z}\right)\left(\sqrt[3]{z}-\sqrt[3]{x}\right)\right|}{4}$$

Proposed by Adil Abdullayev – Baku – Azerbaidjian

PROBLEM 3.61

Prove that if $a, b \in (0,1)$ then:

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{a+2ab+b}} \leq \left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}} \leq \left(\frac{a+b}{2}\right)^{\frac{2}{a+b+2}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.62

Let a, b, c be real number such that $a, b, c \geq \frac{1}{2}$ and $a+b+c=6$.

Prove that: $ab+bc+ca \geq 3\sqrt{abc+ab+bc+ca-4}$

Proposed at Hai Phong-Contest-Vietnam

PROBLEM 3.63

Let $a, b \in (0, \infty)$ and $a + b = 2$. Prove that:

$$(a+1)^a(b+1)^b + 2ab \geq 6$$

Proposed by Richdad Phuc-Hanoi -Vietnam

PROBLEM 3.64

Let $a, b, c > 0$ and $ab + bc + ca + abc = 4$ then

$$a^3 + b^3 + c^3 + abc \geq 4$$

Proposed by Richdad Phuc-Hanoi-Vietnam

PROBLEM 3.65

If $0 < a_1 \leq 1 \leq a_2 \leq 2 \leq \cdots \leq 2015 \leq a_{2016} \leq 2016$ then:

$$\sum_{k=1}^{2016}\left(a_k + \frac{k^2}{a_k}\right) > 2016\left(2016 + \frac{1}{\sqrt[2016]{a_1 a_2 \cdot \ldots \cdot a_{2016}}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.66

1. If $a, b > 0$ then: $b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b$

2. If $a > 0, 0 < b \leq 1$ then: $b^b \cdot e^{1+\frac{1}{a}} \geq 2b \cdot e^b$

Proposed by Abdallah El Farissi-Bechar-Algerie

PROBLEM 3.67

Prove that, for positive a, b:

$$\frac{a}{b\sqrt{2}} + \frac{b\sqrt{2}}{a} + 2\left(\frac{\sqrt{a^2+b^2}}{b} + \frac{b}{a^2+b^2}\right) \geq \frac{9\sqrt{2}}{2}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 3.68

If $x \in \mathbb{R}$ then:

$$\left(\sqrt{x^2-x+1} - \sqrt{x^2+x+1}\right)^2 + \left(\sqrt{x^2-x+1} - \sqrt{4x^2+3}\right)^2 + \left(\sqrt{x^2+x+1} - \sqrt{4x^2+3}\right)^2 < 6x^2 + 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.69

If $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n > 0, n \in \mathbb{N}^*$ then:

$$exp\left(\sum_{i=1}^{n}(x_i - y_i)\right) \geq \left(\frac{x_1}{y_1}\right)^{y_1} \cdot \left(\frac{x_2}{y_2}\right)^{y_2} \cdot \ldots \cdot \left(\frac{x_n}{y_n}\right)^{y_n}$$

Proposed by Abdallah El Farissi – Bechar – Algerie

PROBLEM 3.70

If $a, b, c > 0, m \geq 0$ then:

$$\frac{a}{(b+c)^{m+1}} + \frac{b}{(c+a)^{m+1}} + \frac{c}{(a+b)^{m+1}} \geq \frac{3^{m+1}}{2^{m+1}(a+b+c)^m}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

PROBLEM 3.71

If $a, b, c > 0$ ***then:***

$$\frac{(\sqrt{a}+\sqrt{b})^2}{4}+\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{9}+\frac{(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{16}<4(a+b+c+d)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.72

If $x, y, z, t \in \mathbb{R}, x+y+z+t=0$ ***then:***

$$2^x+2^y+2^z+2^t+8 \geq 3\left(\frac{1}{\sqrt[3]{2^x}}+\frac{1}{\sqrt[3]{2^y}}+\frac{1}{\sqrt[3]{2^z}}+\frac{1}{\sqrt[3]{2^t}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.73

If $n \in \mathbb{N}, n \geq 3, x, y \geq 0$ ***then:***

$$\sqrt[3]{x^3+y^3}+\sqrt[4]{x^4+y^4}+\cdots+\sqrt[n]{x^n+y^n} \leq (n-2)\sqrt{x^2+y^2}$$

Proposed by Daniel Sitaru - Romania

PROBLEM 3.74

If $a, b \in (1, \infty)$ ***then:***

$$\left(\frac{1+\ln a}{2}\right)^{2016}+\left(\frac{\ln a \ln b+1}{2 \ln a \ln b}\right)^{2016}+\left(\frac{1+\ln b}{2}\right)^{2016} \geq 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.75

If $a, b, c \in (0, \infty)$ ***then:***

$$a+b+c \geq 3\sqrt[9]{\frac{abc(a+b)^2(b+c)^2(c+a)^2}{64}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.76

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ ***then:***

$$\frac{\tan x}{\sin y+\sin z}+\frac{\tan y}{\sin z+\sin x}+\frac{\tan z}{\sin x+\sin y}>\frac{3}{2}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

PROBLEM 3.77

If $a, b, c > 0, a+b+c=\sqrt{3}$ ***then:***

$$\sum\left(2\arctan\left(\frac{a+b}{2}\right)+\arctan c\right) \leq \frac{3\pi}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.78

Prove that if $a, b, c, d, e, f \in (0, \infty)$ ***and*** $a+b+c=2; d+e+f=3$ ***then:***

$$\left(\frac{d}{a}\right)^a \cdot \left(\frac{c}{b}\right)^b \cdot \left(\frac{f}{c}\right)^c \leq \frac{9}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.79

If $a, b, c \in \mathbb{N} - \{0, 1\}, a + b + c = 100$ *then:*

$$\binom{a+b}{a} + \binom{b+c}{b} + \binom{c+a}{c} > 200$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.80

If $\forall x \in \mathbb{R}, \begin{cases} a^x + b^x \geq a + b \\ a^x + b^x + c^x \geq a + b + c \\ a^x + b^x + c^x + d^x \geq a + b + c + d \end{cases}, a, b, c, d > 0$ *then:*

$$a^{3a} \cdot b^{3b} \cdot c^{2c} \cdot d^d = 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.81

If $, b \in \mathbb{R}^+$ *,* $a < b$ *then:*

$$\frac{2\left(\sqrt{b^2+1} - \sqrt{a^2+1}\right)^2}{b^2 - a^2} < ln\frac{b + \sqrt{b^2+1}}{a + \sqrt{a^2+1}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.82

If $x, y > 0; z \in \mathbb{R}$ *then:*

$$\frac{(x+y)^2}{(x\sin^2 z + y\cos^2 z)(x\cos^2 z + y\sin^2 z)} + \frac{x}{y} + \frac{y}{x} \geq 6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.83

If $a, b, c, d > 0$ *then:*

$$(a+c)^c(b+d)^d(c+d)^{c+d} \leq c^c \cdot d^d \cdot (a+b+c+d)^{c+d}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.84

If $x \in (0, \pi);\ y > 0$ *then:*

$$\frac{2}{y + \sin^2 x} + \frac{2}{y^2 + \sin x} \leq \frac{1}{\sin x\sqrt{\sin x}} + \frac{1}{y\sqrt{y}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.85

Prove that if $a, b \in (0, \infty)$ *then:*

$$e^{(b-a)(6-a^2-b^2-ab)} \leq \left(\frac{b + \sqrt{b^2+1}}{a + \sqrt{a^2+1}}\right)^6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.86

If $a, b, c, x \in \mathbb{R}$ *then:*

$$a^2 + b^2 + c^2 + (\sin x + \cos x + \sin x\cos x)(ab + bc + ca) \geq 0$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.87

$$\text{If } a,b,c\in\mathbb{R}^*;\ x,y,z\in\mathbb{R}$$
$$\Omega_1 = x^2\sum\frac{a^2}{b^2}+y^2\sum\frac{1}{a^2}+z^2\sum a^4$$
$$\Omega_2 = xy\sum\frac{1}{a}+xz\sum ab+yz\sum\frac{b^2}{a}$$
then:
$$\Omega_1+2\Omega_2\ge 0$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.88

If $a,b\in\left(0,\frac{\pi}{2}\right)$ *then:*

$$\frac{\cos a}{1+\cos^4 a}+\frac{\sin a\cos b}{1+\sin^4 a\cos^4 b}+\frac{\sin a\sin b}{1+\sin^4 a\sin^4 b}<\frac{9\sqrt{3}}{10}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.89

Prove that if $a,b,c\in(0,\infty)$; $abc=1$ *then:*

$$be^a+ce^b+ae^c\ge\frac{15}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.90

If $x,y,z\ge 1$ *then:*

$$x^2y^4z^6-1\ge 6yz^2(x^2-1)(y^2-1)(z^2-1)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.91

If $x,y,z\ge 0, x^2+y^2+z^2=\frac{\pi}{2}$ *then:*

$$\arctan(x^2)+\arctan(y^2)+\arctan(z^2)\ge\frac{(x+y+z)^2}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.92

If $a,b>0, n\in\mathbb{N}, n\ge 2$ *then:*

$$\left(1+\frac{2\sqrt{ab}}{a+b}\right)^{\frac{1}{n}}+\left(1-\frac{2\sqrt{ab}}{a+b}\right)^{\frac{1}{n}}<2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.93

If $a,b,c\ge 0, a^2+b^2+c^2=2$ *then:*

$$\sqrt{ab}\csc\frac{\pi}{7}+\sqrt{bc}\csc\frac{2\pi}{7}+\sqrt{ca}\csc\frac{3\pi}{7}\le 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.94

If $x, y \in \mathbb{R}, x \neq y$ then:

$$\frac{(2^x + 4^x + 8^x) - (2^y + 4^y + 8^y)}{x - y} > \log 64 \cdot \sqrt[6]{128^{x+y}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.95

If $x, y \in \left(0, \frac{\pi}{2}\right)$ then:

$$\sin(x + y) < \sin x \left(\frac{\sin y}{y}\right)^3 + \sin y \left(\frac{\sin x}{x}\right)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.96

If $m \in \mathbb{N}, m \geq 2$ then:

$$m + \tan^2 \frac{\pi}{36} + \tan^2 \frac{11\pi}{36} + \tan^2 \frac{13\pi}{36} + \tan^2 \frac{21\pi}{36} > 2 + \left(\frac{3}{16}\right)^{\frac{m-2}{m}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.97

If $a_1, a_2, \ldots, a_n > 0, n \in \mathbb{N}^*, f: \mathbb{R} \to \mathbb{R}, f(x) = x + 2x^2 + \cdots + nx^n$ then:

$$f^2\left(\frac{a_1}{a_2}\right) + f^2\left(\frac{a_2}{a_3}\right) + \cdots + f^2\left(\frac{a_{n-1}}{a_n}\right) + f^2\left(\frac{a_n}{a_1}\right) \geq \frac{n^3(n+1)^2}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.98

If $A = \tan\beta \tan\gamma + 5, B = \tan\gamma \tan\alpha + 5, C = \tan\alpha \tan\beta + 5$,
$\alpha, \beta, \gamma = \frac{\pi}{2}$, then: $\sqrt{A} + \sqrt{B} + \sqrt{C} \leq 4\sqrt{3}$

Proposed by Boris Colakovic-Belgrade-Serbia

PROBLEM 3.99

If in $\Delta\, ABC, a \geq b \geq c$ then:

$$m_a \cos^2 \frac{\pi}{7} + m_b \cos^2 \frac{2\pi}{7} + m_c \cos^2 \frac{3\pi}{7} < \frac{5s}{6}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.100

If $a, b, c \in (4, \infty), abc = 2^{11}$ then:

$$\prod \left(a^2 \sin\frac{2\pi}{a} + (a+1)^2 \sin\frac{2\pi}{a+1}\right) > 2^{16}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.101

If $n \in \mathbb{N}^*, n \geq 2, a, b, c > 1, a + b + c = 3^{n+1}$ then:

$$\sum \left(\sqrt[n]{a + \sqrt[n]{a}} + \sqrt[n]{a - \sqrt[n]{a}}\right) < 18$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.102

If $a, b, c, d > 0, a + b + c + d = 1$ *then:*

$$a^3 + b^3 + c^3 + d^3 + 3(ab + ac + ad + bc + bd + cd) \geq$$
$$\geq 1 + 6\left(ab\sqrt{cd} + cd\sqrt{ab}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.103

If $n \in \mathbb{N}^*, a > 0$ *then:*

$$\frac{n^n}{n!}\sum_{k=1}^{n+1}\frac{(-1)^{k-1}\binom{n}{k-1}}{a+k} < \left(\sum_{k=1}^{n}\frac{1}{a+k}\right)^n$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.104

If $x, y \geq 0, n \in \mathbb{N}^*$ *then:*

$$\sum_{k=1}^{n}\binom{n}{k} \cdot x^{2n-2k} \cdot y^{2k} \geq (2^n - 2)x^n y^n$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.105

If $a, b, c > 0, a + b + c + d = 0$ *then:*

$$3|bcd + cda + dab + abc| \geq \left|d^3 + 3abc\right|$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.106

If $a, b, c \geq -2$ *then:*

$$\sum b\left(e^b - a^a\right) \geq \sum ae^a(b - a)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.107

If $1 < x < y$ *then:*

$$x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.108

Prove that $2^{\cos x} + 2^{\sin x} \geq 2^{\frac{\sqrt{2}-1}{\sqrt{2}}}, \forall x \in \mathbb{R}.$

Proposed by Ibrahim Abdulazeez-Zaria-Nigeria

PROBLEM 3.109

Prove that:

$$\sin 2^\circ + \sin 3^\circ + \sin 4^\circ + \cdots + \sin 10^\circ < 54 \cdot \sin 1^\circ$$

Proposed by Ilkin Guliyev-Azerbaidian

PROBLEM 3.110

If $x, y, z \in \left[0, \frac{\pi}{2}\right)$ *then:*

$$xyz(\sin x + \sin y + \sin z) \leq y^2 z \sin x + z^2 x \sin y + x^2 y \sin z$$

Proposed by Daniel Sitaru-Romania

PROBLEM 3.111

If $0 \le x, y, z < 1$ then:

$$\sqrt[3]{\frac{(1+x^3)(1+y^6)(1+z^9)}{(1-x^3)(1-y^6)(1-z^9)}} \ge \frac{1+xy^2z^3}{1-xy^2z^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.112

If $x, y, z > 0$ then:

$$\sum \frac{\sqrt{x}}{3\sqrt{y}+5\sqrt{z}} + \frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{8(x+y+z)} \ge \frac{1}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.113

If $a, b, c, d > 0$ then:

$$\sum a \tan^{-1} a \ge \sum \sqrt[3]{bcd} \tan^{-1} a \ge 4\sqrt[4]{abcd \prod \tan^{-1} a}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.114

If $x, y, z \in \mathbb{R}, a > 0, |x| \le a, |y| \le a, |z| \le a$ then:

$$\sqrt{7(a^2-x^2)} + \sqrt{7(a^2-y^2)} + \sqrt{7(a^2-z^2)} + 9\sqrt[3]{xyz} \le 12a$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.115

If $a, b, c, x, y, z > 0, a+b+c \ge 3$ then:

$$(ax+by+cz)(ay+bz+cx)(az+bx+cy) \ge \frac{729}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.116

If $a, b, c, d > 0$ then:

$$(a+b)^2(a+c)^2(a+d)^2(b+c)^2(b+d)^2(c+d)^2 \ge$$
$$\ge \left(a+\sqrt[3]{bcd}\right)^3\left(b+\sqrt[3]{cda}\right)^3\left(c+\sqrt[3]{dab}\right)^3\left(d+\sqrt[3]{abc}\right)^3$$

Proposed by Mihály Bencze-Romania

PROBLEM 3.117

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\prod ln(1+\tan^2 x) \cdot \prod ln(1+\cot^2 y) \le \prod ln^2\left(\frac{2}{\sin 2z}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.118

If $x, y \ge 0, n \ge 1, n \in \mathbb{Q}, AM = \frac{x+y}{2}, GM = \sqrt{xy}$ then :

$$\left(\frac{x^n+y^n}{\sqrt{2}}\right)^2 \ge AM^{2n} + GM^{2n}$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

PROBLEM 3.119

If $\mathbf{0} < a < b$ ***then:***

$$e^{\frac{1}{b}} < \left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}} < e^{\frac{1}{a}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.120

If $P \in \mathbb{R}[x]$ ***with distinct roots*** $x_1, x_2, \dots, x_n \in \mathbb{R}, n \in \mathbb{N}^*$ ***then:***

$$\frac{P''(x)}{P(x)} < \left(\frac{P'(x)}{P(x)}\right)^2 + \sum_{k=1}^{n} \frac{P''(x_k)}{P'(x_k)}, \forall x \in \mathbb{R} - \{x_1, x_2, \dots, x_n\}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.121

If $a, b, c > 0, a+b+c = 3, x \in \mathbb{R}$ ***then:***

$$\left(\sqrt[3]{a\sin^2 x} + \sqrt[3]{b\cos^2 x}\right)\left(\sqrt[3]{b\sin^2 x} + \sqrt[3]{c\cos^2 c}\right)\left(\sqrt[3]{c\sin^2 x} + \sqrt[3]{a\cos^2 x}\right) \le 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.122

If $a, b, c, d \in \mathbb{R}$ ***then:***

$$a+b+c+d \le \frac{1}{2} + (a+b)(c+d) + a^2 + b^2 + c^2 + d^2$$

Proposed by Uche Eliezer Okeke-Anambra-Nigeria

PROBLEM 3.123

If $0 < a \le b < \frac{\pi}{2}$ ***then:***

$$\frac{tanb}{tana} \ge e^{2(b-a)}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

PROBLEM 3.124

Prove that:

$$2^x + 3^x + 4^x \ge x\ln 24 + 3, \forall x \in \mathbb{R}$$

Proposed by Nho Nguyen Van-Nghe An-Vietnam

PROBLEM 3.125

For $\mathbf{0} < a < b \wedge x_1, x_2, \dots, x_n \in [a; b] \wedge \alpha > 0$***. Prove:***

$$\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}} \le a^\alpha + b^\alpha$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.126

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 \in \mathbb{R}[X], n \ge 2$$

If $a_0, a_1, \dots, a_n > 0$ ***then:*** $P\left(1 + \frac{1}{n}\right) \ge P(1) + \frac{1}{n}P'(1)$

Proposed by Marian Ursărescu-Romania

PROBLEM 3.127

If $a,b,c>0$ then:

$$a^a\cdot b^b\cdot c^c\ge\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}\left(\frac{b+c}{2}\right)^{\frac{b+c}{2}}\left(\frac{c+a}{2}\right)^{\frac{c+a}{2}}\ge(abc)^{\frac{a+b+c}{3}}$$

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PROBLEM 3.128

If $0<x_1\le x_2\le x_3\le\cdots\le x_n$ is an arithmetical progression with common difference d then:

$$\tan^{-1}\frac{d}{1+x_1x_2}+\tan^{-1}\frac{d}{1+x_2x_3}+\cdots+\tan^{-1}\frac{d}{1+x_{n-1}x_n}\le\ln\sqrt{\frac{x_n}{x_1}}$$

Proposed by Mihaly Bencze-Romania

PROBLEM 3.129

For $a,b\in(0;+\infty)\wedge 0\le\theta\le\pi$. Prove:

$$\frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})}\le 1+\sin\theta$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.130

If $x,y,z>0$ then:

$$x+y+z\ge\ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)+\ln\left(\frac{y+2}{(z-1)^2-2z+5}\right)+\ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3$$

Proposed by Lazaros Zachariadis-Thessaloniki-Greece

PROBLEM 3.131

For $0<a<b$. Prove:

$$\frac{e^{b^2}-e^{a^2}}{b-a}\ge(a+b)(ab+1).$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.132

For $a\ge 1\wedge b\ge 1$. Prove:

$$\frac{\sum_{k=0}^{8}b^{8-k}a^k}{\sum_{k=0}^{7}a^{7-k}b^k}\ge\frac{9}{8}.$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.133

For $a,b,c\in(0;+\infty)$. Prove:

$$\frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}}\ge e^6$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.134

If $a,b,c>0$ then:

$$\frac{e^a+e^b+e^c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}>2$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.135

For $\Delta\ ABC$ have $\widehat{BAC}=\frac{\pi}{2}$, put $\widehat{ABC}=\alpha,\widehat{ACB}=\beta$ and $\theta\geq 2$

$$\textit{Prove: } \frac{2}{(\sqrt{2})^\theta}\leq \sin^\theta\alpha+\sin^\theta\beta\leq 1$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.136

For $0<a<b<1$. Prove:

$$\frac{b\sqrt[3]{b}-a\sqrt[3]{a}}{b\sqrt{b}-a\sqrt{a}}\geq\frac{8}{9}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.137

If $a,b,c>0,x,y,z>1$ then:

$$\log_{y^bz^c}x^a+\log_{z^bx^c}y^a+\log_{x^by^c}z^a\geq\frac{3a}{b+c}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

PROBLEM 3.138

Prove without computer:

$$e^e(1-e^{\tan e})>e^\pi-\pi^\pi$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaidian

PROBLEM 3.139

If $a,b,c\geq 0$ then:

$$3(\sinh a+\sinh b+\sinh c)\geq(a+b+c)\left(\sqrt[3]{\cosh a}+\sqrt[3]{\cosh b}+\sqrt[3]{\cosh c}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.140

If $a,b,c\in\mathbb{N}^*$

$$\Omega(a,b)=\frac{b}{a+b-1}+\frac{b(b-1)}{(a+b-1)(a+b-2)}+\cdots+\frac{b(b-1)\cdot\ldots\cdot 2\cdot 1}{(a+b-1)(a+b-2)\cdot\ldots\cdot a}$$

then:

$$b\cdot\Omega(a,b)+c\cdot\Omega(b,c)+a\cdot\Omega(c,a)\geq a+b+c$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.141

If $a,b,c\in(0,1),2(a+b+c)=3$ then:

$$\sum(3+(\log_a c)^4)\left(3+\frac{1}{(a+b)^4}\right)\geq 48$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.142

If $x,y,z,t\geq 1$ then:

$$(\ln xy)\left(\ln^2 x+\ln^2 y-\ln x\ln y-\ln z\ln t\right)\geq(\ln zt)\left(\ln x\ln y+\ln z\ln t-\ln^2 z-\ln^2 t\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.143

If $1\leq x<y$ then:

$$\frac{(y^5-x^5)(y^7-x^7)(y^9-x^9)}{(y^6-x^6)(y^8-x^8)(y^{10}-x^{10})}<\frac{21}{32}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.144

If $0 \leq a, b, c, d \leq 2$ ***then:***

$$\frac{9a}{1+bcd}+\frac{9b}{1+cda}+\frac{9c}{1+dab}+\frac{9d}{1+abc}+9e^{abcd} \leq 8+9e^{16}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.145

For $b > a \geq 1 \wedge n \in \mathbb{N} \wedge n \geq 2$. ***Prove:***

$$\prod_{k=1}^{n}\frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}} \geq \frac{(2n+1)!}{4^n(n!)^2}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.146

In acute ΔABC ***the following relationship holds:***

$$\frac{1}{\pi}(A\tan^{\alpha}A+B\tan^{\alpha}B+C\tan^{\alpha}C) \geq \sqrt{3^{\alpha}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.147

If $a, b, c \in (0,1], x, y > 0$ ***then:***

$$\frac{3}{2}\log(x^2+y^2) > (a+b+c)\log x+(3-a-b-c)\log y$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.148

For $a, b \in [1;+\infty) \wedge m, n \in \mathbb{N}^* \wedge m \geq n \geq 2$. ***Prove:***

$$\frac{\sum_{k=0}^{m} a^{m-k}b^k}{\sum_{l=0}^{n} a^{n-l}b^l} \geq \frac{m+1}{n+1}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.149

If $a, b, c, d, e, f > 0$ ***then:***

$$\frac{a+b+c}{\sqrt[3]{abc}\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)} \leq \frac{\sqrt[3]{def}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{d+e+f}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.150

In ΔABC ***the following relationship holds:***

$$\frac{\left((a+1)(b+1)(c+1)\right)^{\frac{1}{2}}}{e^{a+b+c}} < 1$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.151

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right)$$

Find an increasing order for:

$$\Omega_1 = \gamma^{\sqrt{\pi e}}, \Omega_2 = \pi^{\sqrt{e\gamma}}, \Omega_3 = e^{\sqrt{\gamma\pi}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.152

If $a, b, c > 1, n \in \mathbb{N}, n \geq 2$ then:

$$\sum \frac{\sqrt[n]{a^n+1}}{a^n+1} + \sum \frac{\sqrt[n]{a^n-1}}{a^n-1} > \frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.153

If $x, y, z \in \mathbb{R}$ then:

$$\frac{1}{e^{\sin^2 x}} + \frac{1}{e^{\sin^2 y}} + \frac{1}{e^{\sin^2 z}} + \frac{1}{e^{\cos^2 x}} + \frac{1}{e^{\cos^2 y}} + \frac{1}{e^{\cos^2 z}} > 3\left(\frac{1}{2} + \frac{\sqrt{e}}{e}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.154

If $x, y, z, t \in \left(0, \frac{\pi}{2}\right)$ then:

$$64 \cdot \cos x \cdot \cos z \cdot \sin y \cdot \sin t \cdot \sin(x-y) \cdot \sin(z-t) \leq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.155

Let $n \in \mathbb{N} \wedge n \geq 2$ and $\theta \geq 1$. Prove:

$$\sum_{k=0}^{n} \left(C_n^k\right)^{\theta} > (n+1)\left(\frac{2^n}{n+1}\right)^{\theta}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.156

If $x, y, z \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{x(\cos x + \cos z) + y(\cos y + \cos x) + z(\cos z + \cos y)}{x(\cos x + \cos y) + y(\cos y + \cos z) + z(\cos z + \cos x)} \geq 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.157

For $0 < a < b < 1 \wedge m, n \in \mathbb{N} \wedge m \geq n \geq 2$. Prove:

$$\frac{b\sqrt[m]{b} - a\sqrt[m]{b}}{b\sqrt[n]{b} - a\sqrt[n]{a}} \geq \frac{mn+n}{mn+m}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.158

If $x \in \left(0, \frac{\pi}{2}\right), n \in \mathbb{N}, n \geq 3$ then:

$$\prod_{k=3}^{n} \sqrt[k]{\sin^k x + \cos^k x} \geq 2^{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.159

If $a, b \in \mathbb{N}, a, b \geq 2$ then:

$$(2a-1)(3a-1)\cdot \ldots \cdot (a^2-1) + (2b-1)\cdot(3b-1)\cdot \ldots \cdot (b^2-1) > 2\sqrt{\frac{a!\cdot b!\cdot a^2\cdot b^b}{ab\cdot \sqrt[ab]{a^b\cdot b^a}}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.160

If $m, n \in \mathbb{N}, a, b, c > 0, u \geq 0$ ***– fixed then:***

$$\sum (m + a^{m+1})\left(n + \frac{1}{(b+c+u)^{m+1}}\right) \geq \frac{3(m+1)(n+1)(a+b+c)}{2(a+b+c)+3u}$$

Proposed by D.M.Batinetu-Giurgiu, Neculai Stanciu-Romania

PROBLEM 3.161

If $a, b, c > 1$ ***then:***

$$\frac{1}{\log_a c + 2\log_a b} + \frac{1}{\log_b a + 2\log_b c} + \frac{1}{\log_c b + 2\log_c a} \geq 1$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.162

If $x, y, z \in \mathbb{R}, x + y + z = 0$ ***then:***

$$\frac{|2x+3| + |2y+3| + |2z+3| + 9}{2} \geq |x-3| + |y-3| + |z-3|$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.163

If $x \in \left(0, \frac{\pi}{2}\right)$ ***then:***

$$\left|\frac{2}{\sin x + \cos x} + \frac{(\sin x - \cos x)(1 - \tan x)}{1 + \tan x}\right| \leq \sqrt{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.164

If $x, y, z > 0$ ***then:***

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} + \frac{3\sqrt{3}}{\sqrt{2(x+y+z)}} \geq 2\sqrt{2}\left(\frac{1}{\sqrt{x+2y+z}} + \frac{1}{\sqrt{y+2x+z}} + \frac{1}{\sqrt{x+2z+y}}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.165

If $a < b < c < d < e < f < g < h, a, b, c, d, e, f, g, h \in \mathbb{R}$ ***then:***

$$(a+b+c+d+e+f+g+h)^2 \geq 16(ah + bg + cf + de)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 3.166

If $b > a \geq e$ ***then:***

$$\frac{\pi^b - \pi^a}{e \cdot \log \frac{b}{a}} > \pi^e$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 3.167

If $x, y, z \geq 0, x + y + z = \frac{\pi}{4}$ ***then:***

$$\sum \tan x\,(1 + \tan y) \geq 2\sqrt{\tan x \cdot \tan y \cdot \tan z}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 3.168

If $x, y, z \in \mathbb{R}, x + y + z = 0$ ***then:***

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)} \geq \left(1+\frac{1}{\sqrt{e^x}}\right)\left(1+\frac{1}{\sqrt{e^y}}\right)\left(1+\frac{1}{\sqrt{e^z}}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.169

If $x, y, z > 0$ ***then:***

$$\frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.170

If $x, y, z \in \mathbb{R}, x + y + z = 0$ ***then:***

$$4^x + 4^y + 4^z \geq 2(2^{x+y} + 2^{y+z} + 2^{z+x}) - 3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 3.171

If $a, b, c, x, y, z > 0, a + b + c = x + y + z = 1$ ***then:***

$$\frac{(a+x)^{a+x} \cdot (b+y)^{b+y} \cdot (c+z)^{c+z}}{a^a \cdot b^b \cdot c^c \cdot x^x \cdot y^y \cdot z^z} \leq 4$$

Proposed by Daniel Sitaru – Romania

GEOMETRICAL INEQUALITIES AND IDENTITIES-PROBLEMS

PROBLEM 4.01

In $ABCD$ convexe quadrilateral: $AB = a, BC = b, CD = c, DA = d$. Prove that:

$$\sum \sqrt{a^2 + b^2 + c^2} > 2\sqrt{3 \cdot AC \cdot BD}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.02

In $ABCD$ cyclic quadrilater, $AB = a, BC = b, CD = c, DA = d, s -$ semiperimeter:

$$\sin A \sin B \le \left(1 - \frac{s}{a}\right)\left(1 - \frac{s}{b}\right)\left(1 - \frac{s}{c}\right)\left(1 - \frac{s}{d}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.03

In $ABCD$ cyclic quadrilater, $AB = a, BC = b, CD = c, DA = d$,

$S - area\ [ABCD]$

$$\sin A + \sin B + \sin C + \sin D \le \frac{4S}{\sqrt{abcd}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.04

In $ABCD$ convexe quadrilater:

$$\frac{\sin(\widehat{ABD})}{\sin(\widehat{DBC})} + \frac{\sin(\widehat{BDC})}{\sin(\widehat{ADB})} + \frac{\sin(\widehat{ACB})}{\sin(\widehat{ACD})} + \frac{\sin(\widehat{DAC})}{\sin(\widehat{CAB})} \ge 4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.05

In $ABCD$ cyclic quadrilateral, a, b, c, d – sides, R - circumradius:

$$\frac{\sqrt[4]{a^3b^3c^3d^3}}{(a+b+c+d)^2} \le \frac{R}{8\sqrt{2}}$$

Proposed by Adil Abdullayev-Baku-Azerbaidian

PROBLEM 4.06

Let $A_1A_2A_3A_4$ be a tetrahedron and let M be its interior point. Denote respectively by S_i and d_i the are and distance from M to face opposite to vertex A_i. Prove that

$$\sum_{1 \le i < j \le 4} S_iS_jd_id_j \le \frac{27}{8}V^2$$

where V is the volume of the tetrahedron.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 4.07

In $ABCD$ tetrahedron, $h_A, h_B, h_C, h_D -$ altitudes, $R -$ circumradius, $r -$ inradius:

$$R(h_A + h_B + h_C + h_D) \ge 48r^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.08

In acute $\Delta\ ABC, \Psi = \begin{vmatrix} \sin A & \sin B & \sin C \\ \sin 2A & \sin 2B & \sin 2C \\ \sin 3A & \sin 3B & \sin 3C \end{vmatrix}$ *, O – circumcentre,*

I – incentre, H – orthocentre. Prove that:

$$S[OIH] = \frac{R^6}{2abcs} \cdot |\Psi|$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.09

In ΔABC *the following relationship holds:*

$$\frac{r + r_a}{h_a - r} + \frac{r + r_b}{h_b - r} + \frac{r + r_c}{h_c - r} = 2\left(\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c}\right)$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.10

In ΔABC *the following relationship holds:*

$$\frac{1}{2}\left(\frac{h_a}{AI} + \frac{h_b}{BI} + \frac{h_c}{CI}\right) = \cos\frac{A}{2}\cos\frac{B}{2} + \cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{A}{2}\ , I - incenter$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.11

In $\Delta ABC, K - Lemoine's\ point, KD \perp BC, KE \perp CA, KF \perp AB,$

$KD = x, KE = y, KF = z$ *. Prove that:*

$$\frac{m_a^2}{x \cdot h_a} + \frac{m_b^2}{y \cdot h_b} + \frac{m_c^2}{z \cdot h_c} = \frac{3}{16S^2} \cdot (a^2 + b^2 + c^2)^2$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.12

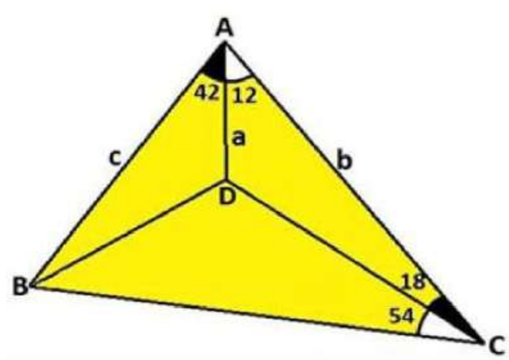
In ΔABC *the following relationship holds:*

$$\frac{h_a + h_b}{r_a + r_b} + \frac{h_b + h_c}{r_b + r_c} + \frac{h_c + h_a}{r_c + r_a} = \frac{2(R + r)}{R}$$

Proposed by Bogdan Fustei-Romania

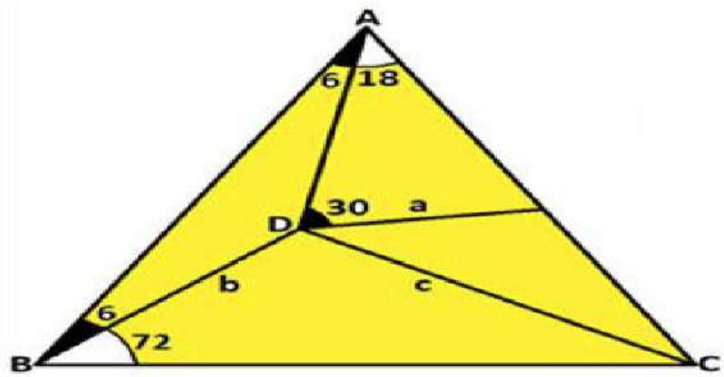
PROBLEM 4.13

Prove that: $a^2 + b^2 = c^2$



Proposed by Mohamed Ozcelic-Turkey

PROBLEM 4.14



Prove that: $c^2 = a^2 + b^2$

Proposed by Mohamed Ozcelik-Turkey

PROBLEM 4.15

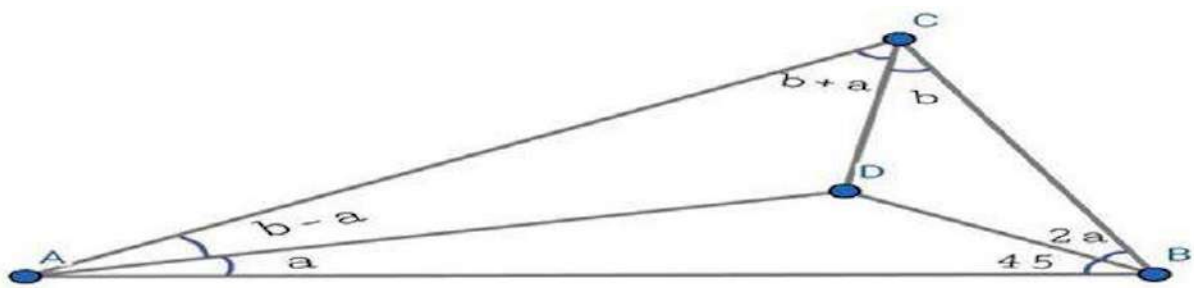
In $\Delta ABC, K - Lemoine's\ point, KD \perp BC, KE \perp CA, KF \perp AB,$

$KD = x, KE = y, KF = z$ *. Prove that:*

$$\frac{xh_a}{r_b r_c} + \frac{yh_b}{r_c r_a} + \frac{zh_c}{r_a r_b} = \frac{x}{r_a} + \frac{y}{r_b} + \frac{z}{r_c}$$

Proposed by Mehmet Sahin-Ankara-Turkey

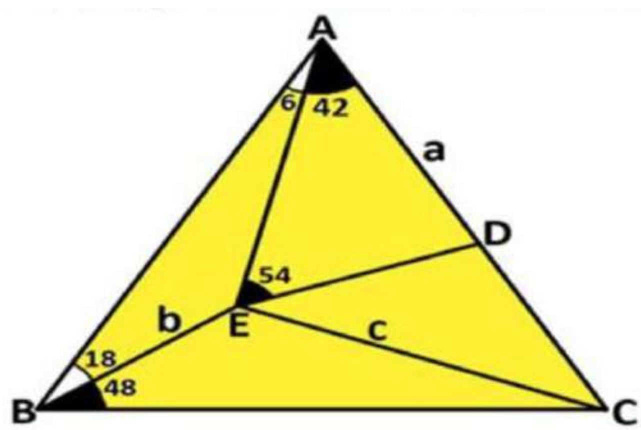
PROBLEM 4.16



$a = ?$

Proposed by Murat Oz-Turkey

PROBLEM 4.17



$c^2 = a^2 + b^2$

Proposed by Mohamed Ozcelic-Turkey

PROBLEM 4.18

In ΔABC , $K - Lemoine's\ point$, $KD \perp BC, KE \perp CA, KF \perp AB$,

$KD = x, KE = y, KF = z$. Prove that:

$$\frac{xr_a + yr_b + zr_c}{x + y + z} = \frac{ar_a + br_b + cr_c}{a + b + c} = 2R - r$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.19

In ΔABC the following relationship holds:

$$\sum \sqrt{2s - 2\sqrt{a(2s - a)}} \geq (\sqrt{2} - 1)(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.20

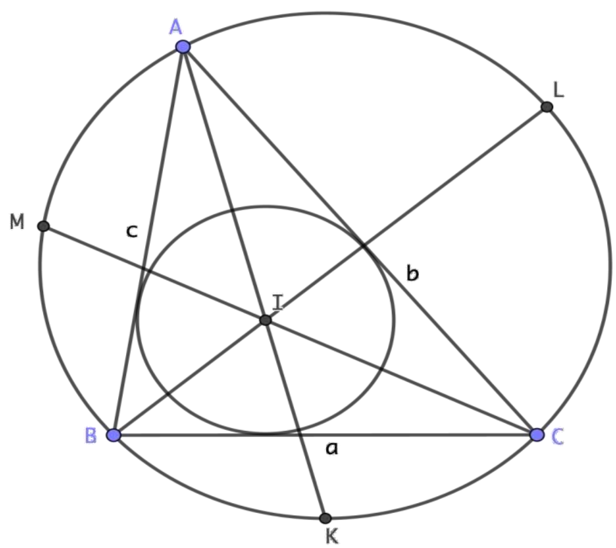
In $\Delta ABC, R_a, R_b, R_c$-circumradii of ΔVI_bI_c, ΔVI_cI_a, ΔVI_aI_b

I_a, I_b, I_c-excenters, V –Bevan's point. Prove that:

$$\sum_{cyc} \frac{1}{R_a^2} = \frac{2R - r}{2R^3}, \qquad \prod_{cyc} R_a = \frac{4R^4}{r}, \qquad \frac{a}{R_a^2} + \frac{b}{R_b^2} + \frac{c}{R_c^2} = \frac{[I_aI_bI_c] - 2[ABC]}{2R^3}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.21



Prove:

$$(IK)(IL)(IM) = \frac{2R^2S}{s}$$

Proposed by Thanasis Gakopoulos-Greece

PROBLEM 4.22

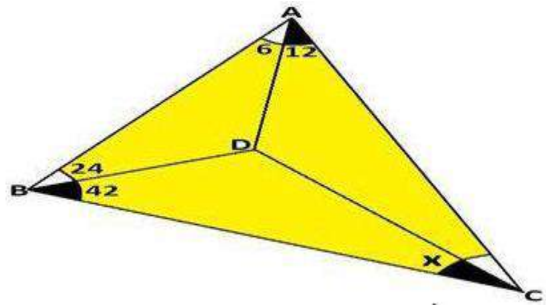
In $\Delta ABC, I$ – incenter, AA', BB', CC' - internal bisectors,

$$\frac{IA}{IA'}, \frac{IB}{IB'}, \frac{IC}{IC'} \in \mathbb{N}^*$$

Find: $\Omega = \frac{w_aw_bw_c}{m_am_bm_c} + \frac{h_ah_bh_c}{w_aw_bw_c} + \frac{m_am_bm_c}{h_ah_bh_c}$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.23



Find "x".

Proposed by Mohamed Ozcelik-Turkey

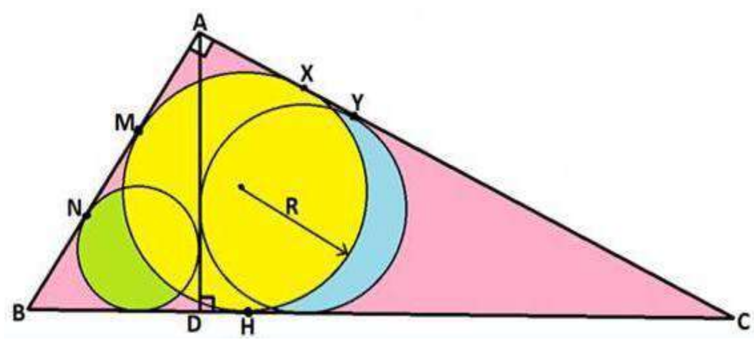
PROBLEM 4.24

In ΔABC the following relationship holds:

$$\sum \frac{1}{b\cos B + c\cos C - a\cos A} = \frac{R}{2S\cos A\cos B\cos C} + \frac{1}{a\cos A + b\cos B + c\cos C}$$

Proposed by Daniel Sitaru – Romania

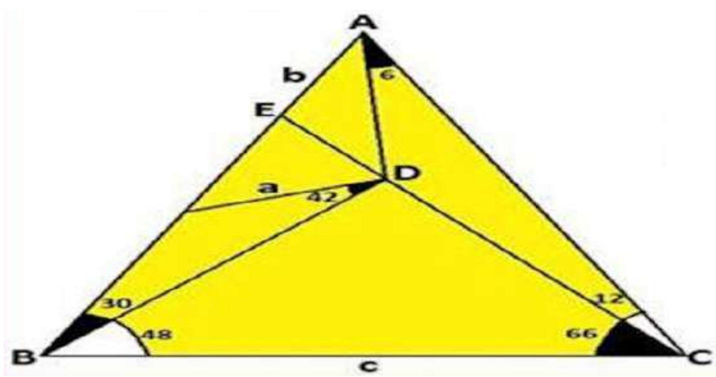
PROBLEM 4.25



Prove that: $AD - MN - XY = R$

Proposed by Muhammad Ozcelik-Turkey

PROBLEM 4.26



E, D, C linear

Prove that: $a^2 + b^2 = c^2$

Proposed by Mohamed Ozcelik-Turkey

PROBLEM 4.27

ΔDEF pedal triangle of I – incenter in $\Delta ABC, R_a, R_b, R_c$ – circumradii of $\Delta AEF, \Delta BFD, \Delta CDE, \varphi_a, \varphi_b, \varphi_c$ – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$\frac{R_a \cdot R_b \cdot R_c}{\varphi_a \cdot \varphi_b \cdot \varphi_c} = \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.28

In ΔABC the following relationship holds:

$$\sqrt[5]{\frac{2(s-a)}{c}} + \sqrt[5]{\frac{2(s-b)}{a}} + \sqrt[5]{\frac{2(s-c)}{b}} \le 3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.29

If in $ABCD$ - tetrahedron $AD = BC = a, BD = AC = b, CD = AB = c$, R - radii of circumsphere then:

$$8(4R^2 - a^2)(4R^2 - b^2)(4R^2 - c^2) \le a^2 b^2 c^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.30

In any scalene acute – angled $\Delta\ ABC$:

$$\sqrt{\sum (\sin A)^{2\cos A}} + \sqrt{\sum (\cos A)^{2\sin A}} > \frac{\sqrt{3}}{3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.31

In $\Delta\ ABC$ the following relationship holds:

$$\frac{\sin^2 A}{\sin^{-1}\frac{4}{5}} + \frac{\sin^2 B}{\sin^{-1}\frac{5}{13}} + \frac{\sin^2 C}{\sin^{-1}\frac{16}{65}} \ge \frac{2s^2}{\pi R^2}$$

$$\frac{\sin^2 A}{\tan^{-1}\frac{1}{2}} + \frac{\sin^2 B}{\tan^{-1}\frac{1}{5}} + \frac{\sin^2 C}{\tan^{-1}\frac{1}{8}} \ge \frac{4s^2}{\pi R^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.32

In $\Delta\ ABC$ the following relationship holds:

$$\sum \frac{h_a}{\sin\frac{A}{2}} \ge \frac{2}{3}\sum m_a + \frac{4}{3}\sum w_a$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.33

Let $\Delta\ A'B'C'$ be the pedal triangle of I – incentre in $\Delta\ ABC$. Prove that:

$$m_a^2 \cdot \frac{AI}{A'I} + m_b^2 \cdot \frac{BI}{B'I} + m_c^2 \cdot \frac{CI}{C'I} \ge 2(m_a m_b + m_b m_c + m_c m_a)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.34

In $\Delta\ ABC$ the following relationship holds:

$$3a^2 + 2b^2 - c^2 > 4S$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.35

In $\Delta\ ABC$ the following relationship holds:

$$\frac{bc}{aw_a} + \frac{ca}{bw_b} + \frac{ab}{cw_c} \geq \frac{18r}{s}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.36

In ΔABC the following relationship holds:

$$\frac{a}{bc + r^2} + \frac{b}{ca + r^2} + \frac{c}{ab + r^2} \geq \frac{12\sqrt{3}}{13R}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.37

In $\Delta\ ABC$ the following relationship holds:

$$\frac{(b + c)w_a}{a} + \frac{(c + a)w_b}{b} + \frac{(a + b)w_c}{c} \geq (a + b + c)\sqrt{3}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.38

In $\Delta\ ABC$ the following relationship holds:

$$h_b h_c \cos\frac{A}{2} + h_c h_a \cos\frac{B}{2} + h_a h_b \cos\frac{C}{2} \leq \frac{\sqrt{3}}{2}\left(h_a^2 + h_b^2 + h_c^2\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.39

In ΔABC the following relationship holds:

$$\left(\frac{1}{r_a} + \frac{1}{h_a}\right)m_a^2 + \left(\frac{1}{r_b} + \frac{1}{h_b}\right)m_b^2 + \left(\frac{1}{r_c} + \frac{1}{h_c}\right)m_c^2 \geq 12R - 6r$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.40

In $\Delta\ ABC$ the following relationship holds:

$$\frac{16r^4}{R} \leq \frac{(h_a + h_b)(h_b + h_c)(h_c + h_a)}{27} \leq R^3$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.34

In $\Delta\ ABC$ the following relationship holds:

$$12r^2 \tan^2 75^\circ \leq (a + r + r_a)^2 + (b + r + r_b)^2 + (c + r + r_c)^2 \leq 3R^2 \tan^2 75^\circ$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.42

In $\Delta\ ABC$ the following relationship holds:

$$(s - a)\sin\frac{A}{2} + (s - b)\sin\frac{B}{2} + (s - c)\sin\frac{C}{2} \leq \frac{S\left(r_a^2 + r_b^2 + r_c^2\right)}{2r_a r_b r_c}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.43

In ΔABC the following relationship holds:

$$\sqrt{2}\left(\sqrt{\frac{s-a}{a}}+\sqrt{\frac{s-b}{b}}+\sqrt{\frac{s-c}{c}}\right)\le\sqrt{6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.44

In ΔABC the following relationship holds:

$$\left(\frac{1}{h_a}+\frac{1}{r_a}\right)bc+\left(\frac{1}{h_b}+\frac{1}{r_b}\right)ca+\left(\frac{1}{h_c}+\frac{1}{r_c}\right)ab\ge 28r-2R$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.45

In $\Delta\ ABC$ the following relationship holds:

$$\sum a^2\,(b\cos B+c\cos C)\le 9\sqrt{3}R^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.46

In ΔABC the following relationship holds:

$$\frac{a+b+c}{2}\ge\frac{9S}{h_a+h_b+h_c}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.47

In acute $\Delta\ ABC$ the following relationship holds:

$$2\sum a^2\cos^2 A\,(b\cos B+c\cos C)^2\le\left(\sum a\cos A\right)\prod(b\cos B+c\cos C)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.48

Let $\Delta A'B'C'$ be the pedal triangle of I – incenter in ΔABC. Prove that:

$$m_a^2\cdot\frac{AI}{A'I}+m_b^2\cdot\frac{BI}{B'I}+m_c^2\cdot\frac{CI}{C'I}\ge 2(m_am_b+m_bm_c+m_cm_a)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.49

In ΔABC the following relationship holds:

$$\frac{\sqrt{b^2+c^2}}{h_a}+\frac{\sqrt{c^2+a^2}}{h_b}+\frac{\sqrt{a^2+b^2}}{h_c}\ge\frac{18\sqrt{2}r^2}{S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.50

In acute $\Delta\ ABC$ the following relationship holds:

$$\frac{\cos A}{bc}+\frac{\cos B}{ca}+\frac{\cos C}{ab}\ge\frac{1}{2R^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.51

In acute $\Delta ABC, K - Lemoine's\ point, KD \perp BC, KE \perp CA, KF \perp AB,$
$KD = x, KE = y, KF = z, H - orthocenter$ *. Prove that:*

$$\frac{x}{AH} + \frac{y}{BH} + \frac{z}{CH} \geq \frac{8S^2}{9R^4}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.52

In ΔABC *the following relationship holds:*

$$\frac{(R - r_a)^2}{h_a} + \frac{(R - r_b)^2}{h_b} + \frac{(R - r_c)^2}{h_c} \geq \frac{13r^2 - 3R^2}{r}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.53

In $\Delta\ ABC$ *the following relationship holds:*

$$3\sum_{cyc} \frac{h_a}{\sin\frac{B}{2}\sin\frac{C}{2}} \geq 4\sum_{cyc} m_a + 8\sum_{cyc} w_a$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.54

In $\Delta\ ABC$ *the following relationship holds:*

$$\frac{24r^2}{R} \leq \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq \frac{4R^2 - 2Rr}{r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 4.55

In acute ΔABC *the following relationship holds:*

$$\frac{2R - a}{2R + a} + \frac{2R - b}{2R + b} + \frac{2R - c}{2R + c} \geq 3\tan^2 15^\circ$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.56

In acute $\Delta\ ABC$ *the following relationship holds:*

$$r_a^2 \tan A + r_b^2 \tan B + r_c^2 \tan C \geq \sqrt{3}s^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.57

In $\Delta ABC, \Omega - first\ Brocard\ point, \omega - Brocard\ angle, I_aI_bI_c - excentral\ triangle.$
Prove that:

$$\frac{1}{[A\Omega B]} + \frac{1}{[B\Omega C]} + \frac{1}{[C\Omega A]} \geq \frac{9}{[I_aI_bI_c] \cdot \sin^2\omega}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.58

In ΔABC *the following relationship holds:*

$$\frac{h_a}{r_b + r_c} + \frac{h_b}{r_c + r_a} + \frac{h_c}{r_a + r_b} \geq 15 - \frac{s^2}{2r^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.59

In ΔABC the following relationship holds:

$$\frac{a^3}{h_b+h_c}+\frac{b^3}{h_c+h_a}+\frac{c^3}{h_a+h_b}\geq 2sR$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 4.60

In ΔABC the following relationship holds:

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\geq\frac{\sqrt{3}}{3R^3}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.61

In ΔABC the following relationship holds:

$$\frac{a^2}{h_b+h_c}+\frac{b^2}{h_c+h_a}+\frac{c^2}{h_a+h_b}\geq\frac{4s^2}{9R}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.62

In ΔABC , $K-Lemoine's\ point$, $KD\perp BC, KE\perp CA, KF\perp AB$,
$KD=x, KE=y, KF=z$. Prove that:

$$6\sqrt{3}\leq\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\leq\frac{27R^2}{2S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.63

In ΔABC the following relationship holds:

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\geq\frac{16s^2}{27R^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.64

In acute ΔABC the following relationship holds:

$$\sum a^3\cos^3 A+\frac{3abc(a^2+b^2+c^2-8R^2)}{8R^2}\geq 2\sum ba^2\cos B\cos^2 A$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.65

In ΔABC the folowing relationship holds:

$$\sin A+\sin B+\sin C+\frac{3\sqrt{3}}{2}\leq 2\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.66

If $x,y>0, x+y\leq 1$ then in ΔABC the following relationship holds:

$$\frac{h_a}{bx+cy}+\frac{h_b}{cx+ay}+\frac{h_c}{ax+by}\geq\frac{2S}{R^2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.67

In ΔABC the following relationship holds:

$$m_a+m_b+m_c\ge\frac{h_ah_c}{h_b}+\frac{h_bh_a}{h_c}+\frac{h_ch_b}{h_a}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.68

In ΔABC the following relationship holds:

$$12\left(\frac{r}{R}\right)^2\le\frac{h_a^2}{r_br_c}+\frac{h_b^2}{r_cr_a}+\frac{h_c^2}{r_ar_b}\le 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 4.69

In Δ ABC the following relationship holds:

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\le\frac{3R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.70

In Δ ABC the following relationship holds:

$$s_a^2\cot\frac{A}{2}+s_b^2\cot\frac{B}{2}+s_c^2\cot\frac{C}{2}\ge\sqrt{3}(s_as_b+s_bs_c+s_cs_a)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.71

In ΔABC the following relationship holds:

$$\frac{a}{h_b+h_c}+\frac{b}{h_c+h_a}+\frac{c}{h_a+h_b}\ge\sqrt{3}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.72

If in $\Delta ABC, \mu(A)\ge\mu(B)\ge\mu(C)$ then:

$$\frac{c}{\sin A}+\frac{a}{\sin B}+\frac{b}{\sin C}\le\frac{a^2b+b^2c+c^2a}{2S}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.73

In ΔABC the following relationship holds:

$$\frac{\left(\frac{2}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}{\frac{1}{ab}+\frac{2}{bc}+\frac{1}{ca}}+\frac{\left(\frac{2}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)^2}{\frac{1}{bc}+\frac{2}{ca}+\frac{1}{ab}}+\frac{\left(\frac{2}{ca}+\frac{1}{ab}+\frac{1}{bc}\right)^2}{\frac{1}{ca}+\frac{2}{ab}+\frac{1}{bc}}\ge\frac{4}{R^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.74

In ΔABC the following relationship holds:

$$\frac{s}{ab+bc+ca}+\frac{8Rr}{(2s-a)(2s-b)(2s-c)}\ge\frac{2\sqrt{3}}{9R}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.75

In $\Delta\ ABC$ the following relationship holds:

$$\frac{2(m_a+m_b+m_c)}{\sqrt{3(a^2+b^2+c^2)}}+\frac{3\sqrt{3(a^2+b^2+c^2)^3}}{8m_am_bm_c}\geq 4\sqrt{3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.76

In ΔABC the following relationship holds:

$$\frac{h_a}{h_bh_c}+\frac{h_b}{h_ch_a}+\frac{h_c}{h_ah_b}\leq\frac{R}{2r^2}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.77

In ΔABC the following relationship holds:

$$\frac{h_b+h_c}{h_a}+\frac{h_c+h_a}{h_b}+\frac{h_a+h_b}{h_c}\leq\frac{3R}{r}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.78

In ΔABC the following relationship holds:

$$\sum_{cyc}\left(\frac{\left(\frac{1}{ab\sin\frac{A}{2}\sin\frac{B}{2}}\right)^7}{\left(\frac{1}{bc\sin\frac{B}{2}\sin\frac{C}{2}}\right)^6+\left(\frac{1}{ca\sin\frac{C}{2}\sin\frac{A}{2}}\right)^6}\right)\geq\frac{1}{2r^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.79

If ΔDEF is pedal triangle of I – incentre of ΔABC then:

$$\frac{S[ABC]}{S[DEF]}\leq\frac{R}{r}+\frac{r}{R}+\frac{3}{2}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.80

In ΔABC the following relationship holds:

$$\frac{m_aw_b}{h_c}+\frac{m_bw_c}{h_a}+\frac{m_cw_a}{h_b}\geq\frac{2\sqrt{3}S}{R}$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 4.81

In ΔABC the following relationship holds:

$$a^2\cos^2 A+b^2\cos^2 B+c^2\cos^2 C\geq 8\sqrt{3}S\cdot\cos A\cos B\cos C$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.82

In acute $\Delta\ ABC$ the following relationship holds:

$$AI+BI+CI\leq\sqrt{6R(h_a+h_b+h_c-6r)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.83

If in $\Delta ABC, N$ – Nagel's point then:

$$\sum_{\substack{cyc\,(a,b,c)\\ cyc(A,B,C)}} \frac{a^2 \cdot AN^2}{5(b^2 \cdot BN^2 + c^2 \cdot CN^2) - a^2 \cdot AN^2} \ge \frac{1}{3}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.84

In ΔABC the following relationship holds:

$$\frac{32}{27R^2 r} \le \left(\frac{1}{r_a} + \frac{1}{r_b}\right)\left(\frac{1}{r_b} + \frac{1}{r_c}\right)\left(\frac{1}{r_c} + \frac{1}{r_a}\right) \le \frac{4R}{27r^4}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.85

In ΔABC the following relationship holds:

$$(a^2 \cos 7^\circ + b^2 \cos 65^\circ + c^2 \cos 79^\circ)\left(\frac{1}{\cos 29^\circ} + \frac{5}{\cos 35^\circ} + \frac{1}{\cos 43^\circ}\right) > 108r^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.86

In ΔABC the following relationship holds:

$$\frac{\sqrt{(r_b - r)(r_c - r)}}{a} + \frac{\sqrt{(r_c - r)(r_a - r)}}{b} + \frac{\sqrt{(r_a - r)(r_b - r)}}{c} \ge \sqrt{3}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.87

ADIL ABDULLAYEV'S REFINEMENT FOR TERESHIN'S INEQUALITY

In ΔABC the following relationship holds:

$$m_a^2 \ge \left(\frac{b^2 + c^2}{4R}\right)^2 + \frac{(b - c)^2(a^2 - b^2 - c^2)^2}{16b^2c^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.88

In ΔABC the following relationship holds:

$$\sqrt{\frac{h_a}{r} - 2} + \sqrt{\frac{h_b}{r} - 2} + \sqrt{\frac{h_c}{r} - 2} \le \sqrt{r + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}}$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.89

In ΔABC the following relationship holds:

$$\frac{m_a^2}{a} + \frac{m_b^2}{b} + \frac{m_c^2}{c} \ge 6s\left(\frac{r}{R}\right)^2$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 4.90

In acute ΔABC the following relationship holds:

$$\sum_{cyc} a^4(b^2 + c^2 - a^2) \ge 32RS^2\sqrt{2(a^2 + b^2 + c^2)\cos A \cos B \cos C}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.91

In ΔABC the following relationship holds:

$$4\sum m_b m_c - 4R\sum \frac{h_b h_c}{h_a} \le s^2 + r(4R+r)$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.92

In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{\sin A}{\sin B}} + \sqrt[3]{\frac{\sin B}{\sin C}} + \sqrt[3]{\frac{\sin C}{\sin A}} - \sqrt[3]{\frac{\sin A}{\sin C}} - \sqrt[3]{\frac{\sin B}{\sin A}} - \sqrt[3]{\frac{\sin C}{\sin B}} < 1$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.93

In $\Delta ABC, I$ – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$. Prove that:

$$2R^2 - 2Rr - r^2 \le \frac{1}{4R^2}\left(R_a^4 + R_b^4 + R_c^4\right) \le 4R^2 - 8Rr + 3r^2$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.94

In ΔABC the following relationship holds:

$$\frac{((r_a - r_b)^2 + (r_b - r_c)^2 + (r_c - r_a)^2)r}{3s^2} \le R - 2r$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.95

In ΔABC the following relationship holds:

$$2R(m_a w_a h_a + m_b w_b h_b + m_c w_c h_c) \ge 9r^2(s^2 + r^2 + 4Rr)$$

Proposed by Seyran Ibrahimov-Maasilli-Azerbaijan

PROBLEM 4.96

In ΔABC the following relationship holds:

$$(m_a + m_b + m_c)^2 \ge 9\sqrt{3}S$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.97

In ΔABC the following relationship holds:

$$\sqrt[4]{108} \le \sqrt{\frac{a}{r_a}} + \sqrt{\frac{b}{r_b}} + \sqrt{\frac{c}{r_c}} \le \sqrt{3\sqrt{3}\cdot\frac{R}{r}}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.98

In ΔABC , R_a, R_b, R_c –circumradii of $\Delta VI_bI_c, \Delta VI_cI_a, \Delta VI_aI_b$, I_a, I_b, I_c-excenters, V-Bevan's point. Prove that:

$$\frac{w_a}{R_a} + \frac{w_b}{R_b} + \frac{w_c}{R_c} \ge \frac{9r}{2R}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.99

In acute ΔABC the following relationship holds:

$$\sum a^3 \cos^3 A + \frac{3abc(a^2+b^2+c^2-8R^2)}{8R^2} \ge 2\sum ba^2 \cos B \cos^2 A$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.100

If in $\Delta ABC, \mu(A) = \frac{\pi}{3}$ then the following relationship holds:

$$3\sqrt{3}R + a \ge \frac{4bc}{a}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.101

In $\Delta ABC, I_a, I_b, I_c$ –excenters, $V = X(40) - Bevan's\ point$

R_a, R_b, R_c-circumradii in ΔI_bVI_c, $\Delta I_cVI_a, \Delta I_aVI_b$. Prove that:

$$\frac{h_a}{R_a^2} + \frac{h_b}{R_b^2} + \frac{h_c}{R_c^2} = \frac{r}{2R^3}(r_a + r_b + r_c)$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.102

In $\Delta ABC, a \ge b \ge c, a + b \ge 3c.$

Prove that: $4R - 9r \ge 0.$

Proposed by Nguyen Van Canh-Vietnam

PROBLEM 4.103

In ΔABC the following relationship holds:

$$\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \le 4\left(\frac{R}{r} - 1\right)$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.104

In ΔABC the following relationship holds:

$$\frac{1}{a\cos B\cos C} + \frac{1}{b\cos C\cos A} + \frac{1}{c\cos A\cos B} \ge \frac{18}{s}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.105

In ΔABC the following relationship holds:

$$3 \le \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \le \left(\frac{R}{r}\right)^2 - \frac{R}{2r}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 4.106

In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$(a+a')(b+b')(c+c') \ge 64rr'\sqrt{ss'} + 4\left(\sqrt{Rrs} - \sqrt{R'r's'}\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.107

In ΔABC the following relationship holds:

$$a\cos A + b\cos B + c\cos C \ge 12\sqrt{3}R\cos A\cos B\cos C$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.108

In ΔABC the following relationship holds:

$$m_a m_b m_c(m_a+m_b+m_c) \ge 9S^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.109

In ΔABC the following relationship holds:

$$\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right) \le \frac{9m_a m_b m_c}{h_a h_b h_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.110

In ΔABC the following relationship holds:

$$\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}+\frac{2m_a}{w_a}+\frac{2m_b}{w_b}+\frac{2m_c}{w_c} \le \frac{w_a+w_b+w_c}{r}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.111

If $\Delta A'B'C'$ - circumcevian triangle of I – incentre in ΔABC then:

$$IA'+IB'+IC' \ge 48\sqrt{3}r^3\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.112

In ΔABC the following relationship holds:

$$2\sqrt[3]{abc} \le \sqrt{3}(3R-2r)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.113

URSĂRESCU's REFINEMENT OF EULER'S INEQUALITY

In ΔABC the following relationship holds:

$$R \ge \frac{1}{6}\left(\frac{a(b+c-a)}{h_a}+\frac{b(c+a-b)}{h_b}+\frac{c(a+b-c)}{h_c}\right) \ge 2r$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.114

In ΔABC the following relationship holds:

$$\left(\frac{2m_a+2m_b}{m_c}\right)^7+\left(\frac{2m_b+2m_c}{m_a}\right)^7+\left(\frac{2m_c+2m_a}{m_b}\right)^7 > \left(\frac{3a}{m_a}\right)^7+\left(\frac{3b}{m_b}\right)^7+\left(\frac{3c}{m_c}\right)^7$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.115

In ΔABC the following relationship holds:

$$\sqrt{3\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)} \le \frac{m_a m_b m_c}{S^2}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.116

In ΔABC the following relationship holds:

$$2\left(\sqrt{\cos\frac{A}{2}}+\sqrt{\cos\frac{B}{2}}+\sqrt{\cos\frac{C}{2}}\right)-\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\geq\frac{3^{\frac{5}{4}}}{2^{\frac{1}{2}}}$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.117

In acute ΔABC , I – incenter the following relationship holds:

$$\frac{m_a}{AI^2}+\frac{m_b}{BI^2}+\frac{m_c}{CI^2}\leq\frac{4R+r}{4r^2}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.118

In ΔABC the following relationship holds:

$$\sum\sqrt{r_a(r_b+r_c)}\leq(m_a+m_b+m_c)\sqrt{\frac{R}{r}}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.119

If $x,y,z>0$ then in ΔABC the following relationship holds:

$$\frac{x}{y+z}\cdot r_a^2+\frac{y}{z+x}\cdot r_b^2+\frac{z}{x+y}\cdot r_c^2\geq\frac{91r^2-16R^2}{2}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.120

In $\Delta ABC, I$ – incentre, $AI=x, BI=y, CI=z$
the following relationship holds:

$$\frac{2r^3}{27}(x+y+z)^3+r^2(x^4+y^4+z^4)\geq x^2y^2z^2$$

Proposed by Mustafa Tarek-Cairo-Egypt

PROBLEM 4.121

In ΔABC the following relationship holds:

$$\frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2}\geq\left(\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.122

If $M\in Int(\Delta ABC)$ then:

$$27\cdot[MAB]\cdot[MBC]\cdot[MCA]\leq[ABC]^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.123

In ΔABC the following relationship holds:

$$4(m_a+m_b+m_c)\leq\frac{r_a}{\sin^2\frac{A}{2}}+\frac{r_b}{\sin^2\frac{B}{2}}+\frac{r_c}{\sin^2\frac{C}{2}}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.124

If in $\Delta ABC, AD, BE, CF$ – internal bisectors then:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\geq 18r^2$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.125

In acute ΔABC the following relationship holds:

$$\frac{2\sqrt{3}}{R}\leq\frac{1}{a\cos A}+\frac{1}{b\cos B}+\frac{1}{c\cos C}\leq\frac{\sqrt{3}}{4R\cos A\cos B\cos C}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.126

In acute ΔABC the following relationship holds:

$$\cos A\sin(\sin A)+\cos B\sin(\sin B)+\cos C\sin(\sin C)\leq\frac{3}{2}\sin\left(\frac{\sqrt{3}R}{4r}\right)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.127

If in $\Delta\ ABC, I$ – incenter then:

$$\left(\frac{AI+BI}{CI}\right)^5+\left(\frac{BI+CI}{AI}\right)^5+\left(\frac{CI+AI}{BI}\right)^5>\left(\frac{BC}{AI}\right)^5+\left(\frac{CA}{BI}\right)^5+\left(\frac{AB}{CI}\right)^5$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.128

In ΔABC the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{r_a}{m_a}}+\sum_{cyc}\frac{h_b+h_c}{w_a}\geq 6\sum_{cyc}\sin\frac{A}{2}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.129

If a, b and c are the lengths of the sides of a triangle, then:

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}-2\left[\left(\frac{a-b}{a+b}\right)^2+\left(\frac{b-c}{b+c}\right)^2+\left(\frac{c-a}{c+a}\right)^2\right]\geq 3$$

Proposed by Titu Zvonaru, Neculai Stanciu-Romania

PROBLEM 4.130

In ΔABC the following relationship holds:

$$\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_b}+\frac{r_c}{r_a+r_b}+\frac{3}{2}\leq\frac{12}{6-\frac{R}{r}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.131

If in $\Delta ABC, \omega$ – Brocard angle then:

$$\sin\omega\leq\frac{(a+b)^2+(b+c)^2+(c+a)^2}{16(a^2+b^2+c^2)}+\frac{S}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.132

In ΔABC the following relationship holds:

$$16\left(\sum ab\sin^2 A\right)\left(\sum ab\cos^2 A\right) \le 729R^4$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.133

If in $\Delta ABC, I$ – incentre, $\Delta A'B'C'$ - pedal triangle of incentre then:

$$\frac{IA\cdot IA'}{w_a}+\frac{IB\cdot IB'}{w_b}+\frac{IC\cdot IC'}{w_c} \le \frac{3\sqrt{3}}{4S}\cdot IA\cdot IB\cdot IC$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.134

Let ΔABC, put $P = e^{(\sin A+2\sin B)(\sin B+2\sin C)(\sin C+2\sin A)}$

Find: $\max P$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.135

In acute ΔABC the following relationship holds:

$$a\cos A + b\cos B + c\cos C \le \frac{3\sqrt{3}R}{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.136

In ΔABC the following relationship holds:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c} \ge \frac{1}{2}\left(\frac{h_b+h_c}{h_a}+\frac{h_c+h_a}{h_b}+\frac{h_a+h_b}{h_c}\right)$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.137

In $\Delta ABC, I$ – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$.

Prove that:

$$2R^2 - 2Rr - r^2 \le \frac{1}{4R^2}\left(R_a^4+R_b^4+R_c^4\right) \le 4R^2 - 8Rr + 3r^2$$

Proposed by Marian Ursărescu – Romania

PROBLEM 4.138

In ΔABC the following relationship holds:

$$a^3\cos B\cos C + b^3\cos C\cos A + c^3\cos A\cos B \ge \frac{27abc}{\left(\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}\right)^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.139

In ΔABC the following relationship holds:

$$\left(\frac{h_bh_c}{h_a}\right)^2+\left(\frac{h_ch_a}{h_b}\right)^2+\left(\frac{h_ah_b}{h_c}\right)^2 \ge \left(\frac{2S}{R}\right)^2$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.140

In acute ΔABC the following relationship holds:

$$\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}>A^2+B^2+C^2+\cos A+\cos B+\cos C$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.141

In ΔABC the following relationship holds:

$$12R\le\frac{b^2+c^2}{h_a}+\frac{c^2+a^2}{h_b}+\frac{a^2+b^2}{h_c}\le\frac{9\sqrt{3}R^2}{S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.142

In ΔABC the following relationship holds:

$$\frac{a\cdot m_a}{\sin\frac{A}{2}}+\frac{b\cdot m_b}{\sin\frac{B}{2}}+\frac{c\cdot m_c}{\sin\frac{C}{2}}\ge 6sR$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.143

In ΔABC the following relationship holds:

$$\frac{\sqrt{b+c}}{r_a}+\frac{\sqrt{c+a}}{r_b}+\frac{\sqrt{a+b}}{r_c}\le\frac{4R-2r}{r\cdot\sqrt[4]{27r^2}}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.144

In ΔABC the following relationship holds:

$$(m_a+m_b+m_c)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)+\frac{9S^2}{m_am_bm_c(m_a+m_b+m_c)}\ge 10$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.145

In acute ΔABC the following relationship holds:

$$\prod\left(\frac{a}{c}\cos A+\frac{b}{c}\cos B-\cos C\right)\le\cos A\cos B\cos C$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.146

If in ΔABC: $a\le b\le c$ then:

$$\frac{bm_c}{cm_b}+\frac{am_b}{bm_a}+\frac{cm_a}{am_c}\ge\frac{cm_b}{bm_c}+\frac{bm_a}{am_b}+\frac{am_c}{cm_a}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.147

If $x,y,z\in\left(0,\frac{\pi}{2}\right)$ then:

$$\sum_{cyc}(\tan x+2\sin x)>3(x+y+z)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.148

Prove that:

$$\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}=\frac{\sqrt{13}}{6}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{5}{2\sqrt{13}}\right)\right)+\frac{1}{12}$$

Proposed by Vasile Mircea Popa-Romania

PROBLEM 4.149

Prove that if $A,B,C,D>0, A+B+C+D=\frac{\pi}{4}$

$$\Omega_1=\sum\tan A+\sum\tan A\tan B-\sum\tan A\tan B\tan C$$

$$\Omega_2=\frac{\sin^2(A+B)\sin^2(C+D)}{\cos^2 A\cos^2 B\cos^2 C\cos^2 D}$$

then:

$$16(\Omega_1-1)\le\Omega_2$$

Proposed by Daniel Sitaru-Romania

PROBLEM 4.150

In ΔABC the following relationship holds:

$$4\left(\sum_{cyc}m_a(h_b-h_c)\right)^2<9\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}h_a^2\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.151

Prove that:

$$\frac{1}{\sin x}+\frac{2\sqrt{2}}{\cos x}\ge 3\sqrt{3}\quad\left(0<x<\frac{\pi}{2}\right)$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 4.152

$$\frac{p^3}{\cos\theta}+\frac{q^3}{\sin\theta}\ge(p^2+q^2)^{\frac{3}{2}}\quad\left(0<\theta<\frac{\pi}{2}\right)$$

p,q are positive constants

Proposed by Kunihiko Chikaya-Tokyo-Japan

PROBLEM 4.153

In acute ΔABC with sides different in pairs, AA_1,BB_1,CC_1 – altitudes, AA_2,BB_2,CC_2 – medians, AA_3,BB_3,CC_3 – symedians. Prove that:

$$\frac{A_2A_3}{A_2A_1}+\frac{B_2B_3}{B_2B_1}+\frac{C_2C_3}{C_2C_1}>\frac{108r^2}{a^2+b^2+c^2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.154

In ΔABC the following relationship holds:

$$\frac{2m_am_bm_c}{h_ah_bh_c} \geq 1 + \frac{r_a^2 + r_b^2 + r_c^2}{r_ar_b + r_br_c + r_cr_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.155

In ΔABC the following relationship holds:

$$\frac{am_a^5 + bm_b^5 + cm_c^5}{(am_a + bm_b + cm_c)^5} \geq \frac{1}{729R^4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.156

In ΔABC the following relationship holds:

$$\frac{1}{m_a}\sin\frac{A}{2} + \frac{1}{m_b}\sin\frac{B}{2} + \frac{1}{m_c}\sin\frac{C}{2} \leq \frac{m_a^2 + m_b^2 + m_c^2}{2m_am_bm_c}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.157

If $M \in Int(\Delta ABC), AM = x, BM = y, CM = z$ then:

$$\frac{ax}{ax + by + 98cz} + \frac{by}{by + cz + 98ax} + \frac{cz}{cz + ax + 98by} \geq \frac{3}{100}$$

Proposed by Daniel Sitaru-Romania

PROBLEM 4.158

In ΔABC the following relationship holds:

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \geq \frac{1}{2}\left(\frac{h_b + h_c}{h_a} + \frac{h_c + h_a}{h_b} + \frac{h_a + h_b}{h_c}\right)$$

Proposed by Bogdan Fustei-Romania

PROBLEM 4.159

In ΔABC the following relationship holds:

$$\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} \leq 8 + \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2}\right)^2$$

(I – incentre, R_a, R_b, R_c – circumradii of $\Delta BIC, \Delta CIA, \Delta AIB$)

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.160

In acute ΔABC the following relationship holds:

$$(am_a + bm_b + cm_c)(s_am_a + s_bm_b + s_cm_c) \leq \frac{243\sqrt{3}R^4}{8}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.161

In ΔABC the following relationship holds:

$$\sqrt{h_a + h_b} + \sqrt{h_b + h_c} + \sqrt{h_c + h_a} \leq \frac{a + b + c}{\sqrt{R}}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.162

In ΔABC the following relationship holds:

$$a(2s-a)\cos\frac{A}{2}+b(2s-b)\cos\frac{B}{2}+c(2s-c)\cos\frac{C}{2}\geq 36\sqrt{3}r^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.163

Let ΔABC. Prove:

$$\left(1+\frac{1}{\sin A}+\frac{1}{\sin B+\sin C}\right)\left(1+\frac{1}{\sin B}+\frac{1}{\sin A+\sin C}\right)\left(1+\frac{1}{\sin C}+\frac{1}{\sin A+\sin B}\right)\geq\left(1+\sqrt{3}\right)^3$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM 4.164

If in ΔABC: $ab=12R^2\sin^2\frac{C}{2}$ then: $r\leq\frac{c\sqrt{3}}{6}$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.165

In ΔABC the following relationship holds:

$$27a^2b^2c^2\leq(8R-10r)^6$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.166

In ΔABC the following relationship holds:

$$\frac{aw_a^2}{h_a}+\frac{bw_b^2}{h_b}+\frac{cw_c^2}{h_c}\geq 2r^2\sqrt{\frac{486r}{R}}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.167

If in ΔABC, I – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$ then:

$$\sqrt{6}\leq\sqrt{\frac{R_a}{h_a}}+\sqrt{\frac{R_b}{h_b}}+\sqrt{\frac{R_c}{h_c}}\leq\sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

PROBLEM 4.168

Find $\Omega\in\mathbb{R}$ such that in acute ΔABC holds:

$$\Omega=\left(\frac{b\cos B}{c\cos c}+\frac{c\cos c}{b\cos B}\right)\cos 2A+\left(\frac{c\cos C}{a\cos A}+\frac{a\cos A}{c\cos C}\right)\cos 2B+\left(\frac{a\cos B}{b\cos B}+\frac{b\cos B}{a\cos A}\right)\cos 2C$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.169

In ΔABC the following relationship holds:

$$\frac{\sqrt{b^2+c^2}}{h_a}+\frac{\sqrt{c^2+a^2}}{h_b}+\frac{\sqrt{a^2+b^2}}{h_c}\leq\frac{9R^2}{\sqrt{2}\cdot S}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.170

In Δ ABC the following relationship holds:

$$\frac{am_a}{h_a}+\frac{bm_b}{h_b}+\frac{cm_c}{h_c}\geq 2\sqrt{3\sqrt{3}S}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.171

In ΔABC the following relationship holds:

$$\sqrt{h_a - 2r} + \sqrt{h_b - 2r} + \sqrt{h_c - 2r} \le \sqrt{h_a + h_b + h_c}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.172

In ΔABC, ΔA'B'C' the following relationship holds:

$$(a + a')(b + b')(c + c') \ge 32\sqrt{RR'SS'} + 4\left(\sqrt{RS} - \sqrt{R'S'}\right)^2$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.173

In ΔABC the following relationship holds:

$$\sqrt{\frac{r_b r_c}{a}} + \sqrt{\frac{r_c r_a}{b}} + \sqrt{\frac{r_a r_b}{c}} \le \sqrt{\frac{s(h_a + h_b + h_c)}{2r}}$$

Proposed by Bogdan Fustei – Romania

PROBLEM 4.174

In ΔABC the following relationship holds:

$$\frac{a(s-a)}{b+c} + \frac{b(s-b)}{c+a} + \frac{c(s-c)}{a+b} \le \frac{3\sqrt{3}R}{4}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.175

In ΔABC the following relationship holds:

$$\left(\frac{h_a}{aw_a^2}\right)^2 + \left(\frac{h_b}{bw_b^2}\right)^2 + \left(\frac{h_c}{cw_c^2}\right)^2 \ge \frac{1}{R^2(2R^2 + r^2)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 4.176

In ΔABC the following relationship holds:

$$4\sqrt{3} \le \frac{b^2 + c^2}{ar_a} + \frac{c^2 + a^2}{br_b} + \frac{a^2 + b^2}{cr_c} \le \frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3 - 8\sqrt{3}$$

Proposed by Mehmet Sahin-Ankara-Turkey

PROBLEM 4.177

In ΔABC the following relationship holds:

$$\frac{r_a h_a}{a} + \frac{r_b h_b}{b} + \frac{r_c h_c}{c} \le \frac{3(a + b + c)}{4}$$

Proposed by Bodgan Fustei – Romania

ANALYTICAL INEQUALITIES AND IDENTITIES-PROBLEMS

PROBLEM 5.01

$$\Omega(x)=-\frac{1}{2}+4\sum_{n=1}^{\infty}\frac{n+x}{(n+1)(n+2)(n+3)},x\in\mathbb{R}$$

If $a\in(0,1), b>1$ then:

$$(\Omega(a))^{\Omega(b)}+(\Omega(b))^{\Omega(a)}<1+\Omega(a)\cdot\Omega(b)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.02

Find the limit:

$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\left(\sum_{n=1}^{\infty}\frac{n^{10}}{10^n\cdot n!}\right)\right)^{k^4}$$

Proposed by Naren Bhandari-Bajura-Nepal

PROBLEM 5.03

Find:

$$\Omega=\lim_{n\to\infty}n^8\int_0^{\frac{1}{n^5}}\frac{x^2+1}{x^4+x^2+1}dx$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.04

$$\Omega=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n+2)!!}$$

Find:

$$\Omega=\lim_{n\to\infty}\left((\pi+n\omega)^{1+\frac{1}{n\omega}}-(n\omega)^{1+\frac{1}{\pi+n\omega}}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.05

Find:

$$\Omega=\lim_{n\to\infty}\left(\lim_{x\to 0}\left(\frac{\underbrace{(\sin x)^{(\sin x)^{(\sin x)^{\cdots}}}}_{for\ "n"\ times}}{\underbrace{x^{x^{x^{\cdots}}}}_{for\ "n"\ times}}\right)\right)$$

Proposed by Naren Bhandari-Bajura-Nepal

PROBLEM 5.06

Find:

$$\Omega_1 = \lim_{n\to\infty}\left(1-\frac{\pi^4}{90}+\sum_{k=1}^{n}\frac{1}{k^4}\right)^n$$

$$\Omega_2 = \lim_{n\to\infty}\left(4-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^n$$

$$\Omega_3 = \lim_{n\to\infty}\left(5-4\log 2-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)^n$$

$$\Omega_4 = \lim_{n\to\infty}\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)^n$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.07

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2}{n\left(\sqrt[n+1]{1-\sin x}+\sqrt[n+1]{1+\sin x}-2\right)}\right), x\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.08

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\left[\sqrt[6]{n(n+1)(n+2)}\right]}{n\left[\sqrt{n}\right]}\right), [*] \text{ - great integer function}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.09

$$f:\mathbb{R}\to\mathbb{R}, f(x)=\frac{1}{x^2+2x+3}, f^{(0)}(x)=f(x), f^{(n)}(x) - \text{n}^{\text{th}} \text{ derivative}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{f^{(n)}(0)}{n!}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.10

$$x_1, y_1 > 0, a\in\mathbb{R}, a>1, n\in\mathbb{N}, n\geq 1,$$

$$x_{n+1} = a^{-(x_1+x_2+\cdots+x_n)}, y_{n+1} = a^{\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_n}}$$

Find:

$$\Omega = \lim_{n\to\infty}(x_n\cdot y_n)$$

Proposed by Marian Ursărescu – Romania

PROBLEM 5.11

$$H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}, n\geq 1$$

Find:

$$\Omega=\lim_{n\to\infty}\left(H_n^2\left(\left(\frac{1+H_n}{H_n}\right)^{H_n}-\log\left(\frac{1+H_n}{H_n}\right)^{eH_n}\right)\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.12

Find:

$$\Omega=\lim_{n\to\infty}\left(n\left(\left(\sum_{k=1}^{n}\frac{1}{k^2}\right)^{\frac{\pi^2}{6}}-\left(\frac{\pi^2}{6}\right)^{\sum_{k=1}^{n}\frac{1}{k^2}}\right)\right)$$

Proposed by Marian Ursărescu-Romania

PROBLEM 5.13

Let $n\in\mathbb{N}\geq 0$. Find:

$$\Phi=\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n)!}\left[\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1}dx\right]$$

Proposed by Naren Bhandari-Bajura-Nepal

PROBLEM 5.14

$\omega(n)=\sum_{i=1}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]$, $[*]$ **- great integer function**

Find:

$$\Omega=\lim_{n\to\infty}\left(\log(3n+1)-\sum_{k=1}^{n}\frac{1}{\omega(k)}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.15

Find:

$$\Omega=\lim_{n\to\infty}\left(\log(2n+1)-\sum_{k=1}^{n}\left(\frac{1}{k[\sqrt{k}]}\cdot\left[\frac{\sqrt{k}+\sqrt{k+1}+\sqrt{k+2}}{3}\right]\right)\right), [*] \text{ - GIF function}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 5.16

$$x_0>0, x_{n+1}=\sqrt[3]{x_n^2+\frac{1}{3}x_n+\frac{1}{27}}-\frac{1}{3}$$

Find:

$$\Omega=\lim_{n\to\infty}(n\cdot x_n)$$

Proposed by Marian Ursărescu-Romania

PROBLEM 5.17

$$\Omega(n)=\int_{0}^{2\pi}\log(n^2-2n\cos t+1)\,dt, n\geq 1$$

Find:

$$\Omega = \lim_{n\to\infty}\left(1+\frac{\Omega(n)}{4\pi}\right)^{\log(n+1)}$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.18

GENERALIZATION OF MARIAN URSĂRESCU's SEQUENCE

$$x_n > 0, n \ge 1, \lim_{n\to\infty}\big(n(n+1)(x_{n+1}-x_n)\big) = a, \lim_{n\to\infty} x_n = b, a, b \in \mathbb{R}$$

Find in terms of a, b:

$$\Omega = \lim_{n\to\infty}\left(n\left(x_n^b - b^{x_n}\right)\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.19

$$x_n = \sum_{i=1}^{n}\left[\frac{\sqrt{i}-i}{\sqrt{i+\sqrt{i}}-i}\right], [*] - great\ integer\ function$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1+x_n^2\log\left(\frac{1+x_n}{x_n}\right)}{x_n}\right)^{x_n}$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.20

Find:

$$\Omega = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}\left(\int_{\varepsilon}^{\frac{\pi^2}{4}-\varepsilon}\left(\frac{1}{1+\tan(\sqrt{x})+\cot(\sqrt{x})}\right)dx\right)$$

Proposed by Vasile Mircea Popa-Romania

PROBLEM.5.21

If $0 < a \le b < \frac{\pi}{2}$ then:

$$\left(\int_0^{\sqrt{ab}}\left(\sqrt[3]{x}\cdot\sin x\right)dx\right)\left(\int_0^{\frac{a+b}{2}}\left(\sqrt[3]{x}\cdot\cos x\right)dx\right) \le \left(\int_0^{\sqrt{ab}}\left(\sqrt[3]{x}\cdot\cos x\right)dx\right)\left(\int_0^{\frac{a+b}{2}}\left(\sqrt[3]{x}\cdot\sin x\right)dx\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.22

If $a, b, c > 1, a+b+c = 6$ then:

$$\frac{\Gamma'(a)}{\Gamma(a)}+\frac{\Gamma'(b)}{\Gamma(b)}+\frac{\Gamma'(c)}{\Gamma(c)}+\frac{ab+bc+ca}{2abc} < 3\log 2$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.23

Prove that:

$$\ln\left(\int_0^{\frac{\pi}{2}}\left(\frac{8^{\sin x}}{3^{\sin x}+4^{\sin x}}+\frac{27^{\sin x}}{2^{\sin x}+4^{\sin x}}+\frac{64^{\sin x}}{2^{\sin x}3^{\sin x}}\right)dx\right)>\ln\left(\frac{9\left((4!)^{\frac{2}{3}}-1\right)}{4\ln(4!)}\right)$$

Proposed by Nguyen Van Nho-Nghe An-Vietnam

PROBLEM.5.24

Let f, g, h be continuously differentiable functions on $(0,1)$ so that:

i) $\forall x\in[0,1], 0<g(x)\le f(x)\le h(x)$

ii) g **and** h **both have fixed points on** $[0,1]$**.**

iii) $f(0)=0$

iv) $\forall x\in(0,1), x<f(x)<1$

Prove that there are n distinct numbers $\alpha_i\in(0,1)$ with $i=1,2,\dots,n$ such that

$$\sum_{i=1}^{n}\left(f'(\alpha_i)-\sqrt{f'(\alpha_i)}\right)>0.$$

Proposed by by Anas Adlany-El Jadida-Morocco

PROBLEM.5.25

Prove that if $a\in(0,\infty), n\in N, k=1,2,\dots,n$, then the following inequalities hold:

i) $2\prod_{i=1}^{n}\left(e^k-1\right)\ge e^{\frac{n(n-1)}{2}}$ **and**

ii) $\prod_{i=1}^{n}\left(e^k+a^k-2\right)\ge 2^{n-1}\cdot a^{\frac{11n^2-14n-1}{24}}$

Proposed by Anas Adlany - El Jadida – Morroco

PROBLEM.5.26

If $\alpha\ge 2$ then $\sum_{k=1}^{\infty}(\xi(\alpha k)-1)\le\frac{3}{4}$ where ξ denote the Riemann function.

Proposed by Mihály Bencze – Romania

PROBLEM.5.27

$$-1<a,b,c<1, \Omega(a)=\int_0^{\pi}\frac{\log(1+a\cos x)}{\cos x}dx$$

Prove that:

$$\frac{1}{\pi^2}\left(\Omega^2(a)+\Omega^2(b)+\Omega^2(c)\right)\ge\sum\left(\sin^{-1}a\cdot\sin^{-1}b\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.28

If

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+(\pi n)^2+(n+\pi^2)+\pi^2}=\frac{\cosh(p)}{\cos(q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}$$

then show that:

$$p = q\sqrt{2\pi^2 + 3}$$

Proposed by Srinivasa Raghava-AIRMC-India

PROBLEM.5.29

$$\tan\left[\frac{\pi}{64}\right]^2 \tan\left[\frac{\pi}{32}\right]^2 \tan\left[\frac{\pi}{16}\right]^2 \tan\left[\frac{\pi}{8}\right]^2 =$$

$$= \frac{(2-\sqrt{2})\left(2-\sqrt{2+\sqrt{2}}\right)\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}{(2+\sqrt{2})\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}$$

Proposed by John Horton Conway-Grenoble-France

PROBLEM.5.30

$$\frac{4}{\pi} + \int_0^1 \left(\frac{\pi}{1!} + x1^7\frac{\pi^3}{3!} + x^2 2^7\frac{\pi^5}{5!} - x^3 3^7\frac{\pi^7}{7!} + x^4 4^7\frac{\pi^9}{9!} - x^5 5^7\frac{\pi^{11}}{11!} + \cdots\right) dx = \frac{\pi}{64}(63 - 7\pi^4)$$

Proposed by Srinivasa Raghava-AIRMC-India

PROBLEM.5.31

π, e, γ with Riemann Zeta function.

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n+1} \cdot \frac{4^{-n}}{n} = \ln\left(\frac{\pi}{e}\right)$$

$$\sum_{n=1}^{\infty} \left(\frac{\zeta(2n)}{n} - \frac{\zeta(2n+1)}{4^n}\right)\frac{1}{2n+1} = \gamma + \ln\left(\frac{\pi}{e}\right)$$

γ – Euler's Gamma Constant

Proposed by Srinivasa Raghava-AIRMC-India

PROBLEM.5.32

Let $\Omega(x) = 2\sum_{k=0}^{\infty} \frac{1}{2k+1} \tanh^{2k+1}\left(\frac{1}{2\Gamma(x)}\right)$

Prove that:

$$\int_{0.5}^{1.5} e^{-\Gamma(x)} \psi(x)\{1 + \Omega(x)\}\, dx = \frac{e^{-\sqrt{\pi}} - 2e^{-\frac{\sqrt{\pi}}{2}}}{\sqrt{\pi}}$$

Proposed by Obidah Al Sharafy-Sana'a-Yemen

PROBLEM.5.33

Find:

$$\Omega = \int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx, x \in \left(0, \frac{\pi}{2}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.34

Find:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt[m]{1 + \frac{k^{m-1}}{n^m}} - 1 \right)$$

$$m \in \mathbb{N}^*$$

Proposed by Pierre Mounir-Cairo-Egypt

PROBLEM.5.35

$f: (0, \infty) \to (1, \infty)$, f **– continuous. Prove that if** $0 < a \le b$ **then:**

$$4(b-a)^3 + 6(b-a)^2 \int_a^b \log(f(x))\, dx \le 3(b-a)^2 \int_a^b f(x) dx + \left(\int_a^b f(x) dx \right)^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.36

If $a, b \in \mathbb{R}$ **then:**

$$b^3 + 6\int_a^b (\tan^{-1} x)\, dx \ge 3 \log\left(\frac{1+b^2}{1+a^2}\right) + a^3$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.37

Find:

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{H_n} \sum_{k=1}^{n} \left(\frac{1 \cdot \sqrt{2!} \cdot \sqrt[3]{3!} \cdot \ldots \cdot \sqrt[k]{k!}}{(k+1)!} \right) \right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.38

$$\lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k+1} \binom{n}{k} \right\}^{\sqrt{n}}$$

Proposed by Pierre Mounir-Cairo-Egypt

PROBLEM.5.39

In two different ways, find:

$$\frac{(a+1)}{(b+1)(b+2)} + \frac{(a+1)(a+2)}{(b+1)(b+2)(b+3)} + \cdots$$

$$b > a + 1 > 0$$

Proposed by Pierre Mounir-Cairo-Egypt

PROBLEM.5.40

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{\sin^2\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi i}{2n}\right)+\cos\left(\frac{\pi i}{2n}\right)+1}\cdot\frac{i}{n}\int_{i}^{i+1}\sqrt{(i+1-x)(x-i)}\,dx\right)\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM.5.41

Find:

$$\Omega=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\frac{\sin^2\left(\frac{\pi}{2n}\right)+\cos\left(\frac{\pi}{2n}\right)}{\sin\left(\frac{\pi i}{2n}\right)+\cos\left(\frac{\pi i}{2n}\right)+1}\cdot\frac{1}{n}\int_{i}^{i+1}\sqrt{(i+1-x)(x-i)}\,dx\right)\right)$$

Proposed by Daniel Sitaru – Romania

FAMOUS INEQUALITIES-PROBLEMS

PROBLEM 6.01- ALBU'S INEQUALITY

In $\Delta\ ABC$ acuteangled the following relationship holds:

$$\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\le 3\sqrt[3]{\frac{3}{4}}$$

PROBLEM 6.02- ANDERSON'S INEQUALITY

In $\Delta\ ABC$ the following relatonsip holds:

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}\le\frac{abc}{r}$$

PROBLEM 6.03-*ARKADY'S INEQUALITY – 1*

In ΔABC the following relationship holds:

$$\left(\sum(2ab-c^2)\right)^{\frac{3}{2}}\le 4abc+8\left(3\sqrt{3}-4\right)\prod(s-a)$$

PROBLEM 6.04- *ARSLANGIC – MILOSEVIC'S INEQUALITY*

In right triangle $\Delta\ ABC, a-$ hypothenuse:

$$h_a\le b+c-\left(\sqrt{a}-\frac{1}{2}\right)a$$

PROBLEM 6.05-*BAGER'S INEQUALITY – 1*

In $\Delta\ ABC$ the following relationship holds:

$$\frac{m_a^2}{bc}+\frac{m_b^2}{ca}+\frac{m_c^2}{ab}\ge\frac{9}{4}$$

PROBLEM 6.06- *BAGER'S INEQUALITY – 2*

In $\Delta\ ABC$ the following relationship holds:

$$\frac{a^2}{m_bm_c}+\frac{b^2}{m_cm_a}+\frac{c^2}{m_am_b}\ge 4$$

PROBLEM 6.07 - *BAGER'S INEQUALITY – 3*

In ΔABC the following relationship holds:

$$\sum h_a^2\le\sum w_a^2\le s^2\le\sum m_a^2\le\sum r_a^2$$

PROBLEM 6.08 - ***BAITAN'S INEQUALITY***

In acute angled $\Delta\ ABC$ the following relationship holds:

$$\prod\left(\frac{1-\cos A}{\cos A}\right)\geq\frac{8(\sum\tan A)^3}{27\prod(\tan A+\tan B)}\geq 1$$

PROBLEM 6.09 - ***BANICA'S INEQUALITY***

In acute $\Delta\ ABC$ the following relationship holds:

$$\frac{r}{R}\geq\frac{11-2k}{2(k-1)},k=\frac{1}{2\cos A\cos B\cos C}$$

PROBLEM 6.10 - ***BANKHOFF'S INEQUALITY – 2***

In ΔABC the following relationship holds:

$$h_a+h_b+h_c\leq 2R+5r$$

PROBLEM 6.11 - ***BARRERO'S INEQUALITY – 1***

In $\Delta\ ABC$ non – obtuse the following relationship holds:

$$\sqrt[4]{\sin A\cos^2 B}+\sqrt[4]{\sin B\cos^2 C}+\sqrt[4]{\sin C\cos^2 A}\leq\sqrt[8]{\frac{3}{64}}$$

PROBLEM 6.12 - ***BARROW – JANIC'S INEQUALITY***

If $x,y,z\in\mathbb{R},xyz>0$ then in $\Delta\ ABC$ the following relationship holds:

$$x\cos A+y\cos B+z\cos C\leq\frac{1}{2}\left(\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\right)$$

PROBLEM 6.13 - ***BEATTY'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$a^2+b^2+c^2=2H,ab+bc+ca=K$$

$$\frac{(K-H)(3K-5H)}{12}\leq S^2\leq\frac{(K-H)^2}{12},S=area\ [ABC]$$

PROBLEM 6.14 - ***BENCZE'S REFINEMENT FOR IONESCU – WEITZENBOCK'S INEQUALITY-1***

In ΔABC the following relationship holds:

$$a^2+b^2+c^2\geq 4\sqrt{3}\cdot S\cdot max\left(\frac{m_a}{h_a},\frac{m_b}{h_b},\frac{m_c}{h_c}\right)\geq 4\sqrt{3}\cdot S$$

PROBLEM 6.15 - ***BENCZE'S REFINEMENT OF HADWIGER – FINSLER'S INEQUALITY***

In ΔABC the following relationship holds:

$$2\sum ab-\sum a^2\geq 4\left(2\sum\tan\frac{A+B}{2}-\sqrt{3}\right)S\geq 4\sqrt{3}S$$

PROBLEM 6.16 - ***BENCZE'S REFINEMENT OF IONESCU – WEITZENBOCK'S INEQUALITY-2***

In ΔABC the following relationship holds:

$$a^2+b^2+c^2\geq\sqrt{48S^2+8r(4R+r)\sum(a-b)^2+\left(\sum(a-b)^2\right)^2}\geq 4\sqrt{3}S$$

PROBLEM 6.17 - ***BLUNDON-GERRETSEN'S INEQUALITY***

In ΔABC the following relationship holds:

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

PROBLEM 6.18 - ***BLUNDON'S INEQUALITY***

In ΔABC the following relationship holds:

$$abc\leq 8R^2r+\left(12\sqrt{3}-16\right)Rr^2$$

PROBLEM 6.19 - ***BODAN'S INEQUALITY***

In acute-angled $\Delta\ ABC$ the following relationship holds:

$$\tan A+\tan B+\tan C\geq\frac{9R}{\sqrt[3]{4SR}}$$

PROBLEM 6.20 - ***BOTTEMA'S INEQUALITY***

In ΔABC the following relationship holds:

$$64s^3(s-a)(s-b)(s-c)\leq 27a^2b^2c^2$$

PROBLEM 6.21 - ***BRETSCHNEIDER'S INEQUALITY***

In $ABCD$ convexe quadrilateral the following relationship holds:

$$AC\cdot BD\geq|AB\cdot CD-AD\cdot BC|$$

PROBLEM 6.22 - ***CARLITZ INEQUALITY***

In ΔABC the following relationship holds:

$$(abc)^2 \geq \left(\frac{4S}{\sqrt{3}}\right)^3$$

PROBLEM 6.23 - ***CERIN'S INEQUALITY***

In ΔABC the following relationship holds:

$$b+c-a > \frac{ab+ac-bc}{4R}$$

PROBLEM 6.24 - ***CHILD'S INEQUALITY GENERALIZED***

In ΔABC the following relationship holds:

$$\frac{1}{\sin^n \frac{A}{2}} + \frac{1}{\sin^n \frac{B}{2}} + \frac{1}{\sin^n \frac{C}{2}} \geq 3 \cdot 2^n, n \in \mathbb{N}$$

PROBLEM 6.25 - ***CHILD'S INEQUALITY***

In ΔABC the following relationship holds:

$$\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}} + \sqrt{\sin\frac{B}{2}\sin\frac{C}{2}} + \sqrt{\sin\frac{C}{2}\sin\frac{A}{2}} \leq \frac{3}{2}$$

PROBLEM 6.26 - ***CHILD'S INEQUALITY – 2***

In ΔABC the following relationship holds:

$$\frac{1}{\sin\frac{A}{2}} + \frac{1}{\sin\frac{B}{2}} + \frac{1}{\sin\frac{C}{2}} \geq 6$$

PROBLEM 6.27 - ***CHILD'S INEQUALITY – 3***

In ΔABC the following relationship holds:

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}} \geq 12$$

PROBLEM 6.28 - ***CHUNG'S INEQUALITY***

If $a_1 \geq a_2 \geq a_3 \geq 0, b_1, b_2, b_3 \in \mathbb{R}$,
$a_1 \leq b_1, a_1 + a_2 \leq b_1 + b_2, a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$ then
$a_1^2 + a_2^2 + a_3^2 \leq b_1^2 + b_2^2 + b_3^2$

PROBLEM 6.29 - ***CIOPLEA'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$\left(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}\right)\left(\frac{1}{\sqrt{\sin A}}+\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)\le\frac{9R}{2r}$$

PROBLEM 6.30 - ***CÎRTOAJE'S INEQUALITY***

In $\Delta\ ABC, a\neq b\neq c\neq a$, the following relationship holds:

$$\left|\frac{a}{a-b}+\frac{b}{b-c}+\frac{c}{c-a}\right|>\sqrt{6}-1$$

PROBLEM 6.31 - ***CURRY'S INEQUALITY***

In ΔABC the following relationship holds:

$$4S\sqrt{3}\le\frac{9abc}{a+b+c}$$

PROBLEM 6.32 - ***DINCA'S REFINEMENT FOR NESBITT'S INEQUALITY***

If $a,b,c>0$ then:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{(a+b+c)^2}{2(ab+bc+ca)}\ge\frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)}\ge\frac{3}{2}$$

PROBLEM 6.33 - ***DORDEVIC'S INEQUALITY***

In ΔABC the following relationship holds:

$$a\cos A+b\cos B+c\cos C\le s$$

PROBLEM 6.34 - ***DOUCET'S INEQUALITY***

In ΔABC the following relationship holds:

$$9r(4R+r)\le 3s^2\le(4R+r)^2$$

PROBLEM 6.35 - ***EMMERICH'S INEQUALITY***

In right angle $\Delta\ ABC$ the following relationship holds:

$$\frac{R}{r}\ge 1+\sqrt{2}$$

PROBLEM 6.36 - ***ERDOS INEQUALITY***

In acuteangled $\Delta\ ABC$ the following relationship holds:

$$R+r\le\max(h_a,h_b,h_c)$$

PROBLEM 6.37 - ***FINTA'S INEQUALITY***

In ΔABC the following relationship holds:

$$\frac{2s}{3} \leq \frac{a\sin\frac{A}{2} + b\sin\frac{A}{2} + c\sin\frac{C}{2}}{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}} < s$$

PROBLEM 6.38 - ***GERASIMOV'S INEQUALITY***

In ΔABC the following relationship holds:

$$r^2 + r_a^2 + r_b^2 + r_c^2 \geq 7R^2$$

PROBLEM 6.39 - ***GOLDNER'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq 16S^2, S = [ABC] - \text{area}$$

PROBLEM 6.40 - ***GOLDSTONE'S INEQUALITY***

In ΔABC the following relationship holds:

$$16r^2s^2 \leq \sum a^2b^2 \leq 4R^2s^2$$

PROBLEM 6.41 - ***GOTMAN'S INEQUALITY***

In ΔABC the following relationship holds:

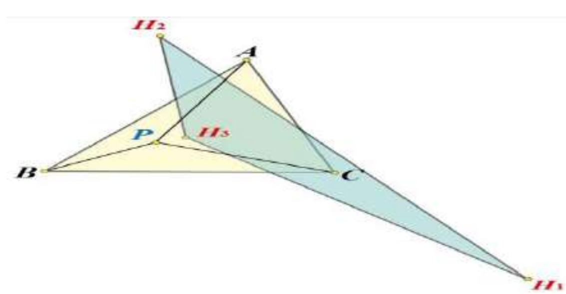
$$9r \leq h_a + h_b + h_c \leq m_a + m_b + m_c \leq 4R + r \leq \frac{9R}{2}$$

PROBLEM 6.42 - ***GROENMAN'S INEQUALITY***

In ΔABC the following relationship holds:

$$p^2 \geq \frac{r(4R + r)^2}{R + r}$$

PROBLEM 6.43



P, any point in the plane of ABC does not belong to side lines AB, BC, CA
H_1, H_2, H_3 orthocenters of PAB, PBC, PCA. Prove: $[ABC] = [H_1H_2H_3]$
Reference: J.T. Groenman – D.J. Smeenk, CRUX 717

PROBLEM 6.44 - ***IONESCU – LEUENBERGER'S GENERALIZED INEQUALITY – 1***

In ΔABC the following relationship holds:

$$\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m}\geq\frac{\left(\sqrt{3}\right)^{2-m}}{R^m}, m\geq 0$$

Proposed by D.M. Bătinețu – Giurgiu; Daniel Sitaru – Romania

PROBLEM 6.45 - ***IONESCU – LEUENBERGER'S GENERALIZED INEQUALITY – 2***

In ΔABC the following relationship holds:

$$\frac{1}{(ax+by)^m}+\frac{1}{(bx+cy)^m}+\frac{1}{(cx+ay)^m}\geq\frac{\left(\sqrt{3}\right)^{2-m}}{(x+y)^m R^m}, m\geq 0, x,y>0$$

Proposed by D.M. Bătinețu – Giurgiu; Daniel Sitaru – Romania

PROBLEM 6.46 - ***JANIC'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$m_a m_b m_c(h_a+h_b+h_c)\geq h_a h_b h_c(m_a+m_b+m_c)$$

PROBLEM 6.47 - ***KARAMATA'S INEQUALITY***

If $a,b>0, a\neq b$ then:

$$\frac{a-b}{\log a-\log b}>\frac{a\sqrt[3]{b}+b\sqrt[3]{a}}{\sqrt[3]{b}+\sqrt[3]{a}}$$

PROBLEM 6.48 - ***KATSUURA'S INEQUALITY***

If $\Omega=\left(0,\frac{\pi}{2}\right)$ then:

$$\sin\Omega<2\sin\frac{\Omega}{2}<\Omega<\sin\frac{\Omega}{2}+\tan\frac{\Omega}{2}<2\tan\frac{\Omega}{2}<$$

$$<\sqrt{\sin\Omega\tan\Omega}<\frac{\sin\Omega+\tan\Omega}{2}<\tan\Omega$$

PROBLEM 6.49

$$\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\right),$$

$$R_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log\left(n+\frac{1}{2}\right)$$

Prove that:

$$\frac{1}{23(n+1)^2}<R_n-\gamma<\frac{1}{24n^2}, n\in\mathbb{N}^*$$

Proposed by D.W.de Temple-AMM

PROBLEM 6.50 - ***KLAMKIN'S INEQUALITY***

In ΔABC the following relationship holds:
$$s^2 + 5r^2 \geq 16Rr$$

PROBLEM 6.51 - ***KLAMKIN'S INEQUALITY***

$$x, y, z \in \mathbb{R}, n \in \mathbb{Z}. \text{ In } \Delta\ ABC:$$
$$x^2 + y^2 + z^2 \geq (-1)^{n+1}(2yz\cos(nA) + 2zx\cos(nB) + 2xy\cos(nC))$$

PROBLEM 6.52 - ***KLAMKIN'S INEQUALITY – 4***

In ΔABC the following relationship holds:
$$s \leq 2R + \left(3\sqrt{3} - 4\right)r$$

PROBLEM 6.53- ***LAZAREVIC'S INEQUALITY***
$$\cosh x < \left(\frac{\sinh x}{x}\right)^3, x \in \mathbb{R}^*$$

PROBLEM 6.54 - ***LESSEL-PELLING'S INEQUALITY***

In ΔABC the following relationship holds:
$$w_a + w_b + m_c \leq p\sqrt{3}$$
$$p - semiperimeter$$

PROBLEM 6.55 - ***LEUENBERGER-CARLITZ'S INEQUALITY***

In ΔABC the following relationship holds:
$$6r(4R + r) \leq 2p^2 \leq 2(2R + r)^2 + R^2$$

PROBLEM 6.56 - ***LEUENBERGER'S INEQUALITY***

In ΔABC the following relationship holds:
$$\frac{9r}{2S} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9R}{4S}$$

PROBLEM 6.57 - ***LIU'S INEQUALITY***
In acute – angled $\Delta\ ABC$:
$$\cos(B - C) \leq \frac{h_a}{m_a}$$

PROBLEM 6.58 - ***MAFTEI'S INEQUALITY***

In acute ΔABC the following relationship holds:

$$\frac{18r}{s} \le \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \le \frac{9R^2}{2S}$$

PROBLEM 6.59 - ***MAKOWSKI'S INEQUALITY – 2***

In ΔABC the following relationship holds:

$$\frac{2}{R} \le \sqrt[4]{\frac{27}{S^2}} \le \frac{1}{r}$$

PROBLEM 6.60

Refinement of Dorin Marghidanu's lemmas

Proposed by Marian Dincă-Romania

Lemmas and it's refinement:

Let $a \in (0,1), b \in (0,1)$

$$a^b \ge \frac{a}{a+b-ab} > \frac{a}{a+b}$$

PROBLEM 6.61 - ***MATIC'S INEQUALITY***

In ΔABC the following relationship holds:

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \ge \left(\frac{2}{3}\right)^{n-2} \cdot s^{n-1}, n \in \mathbb{N}^*$$

PROBLEM 6.62 - ***MAZUR'S INEQUALITY***

Let V be the volume of a tetrahedron $ABCD$ and let $a = AB \cdot CD$,
$b = AC \cdot BD, c = AD \cdot BC$. Then:

$$(a+b-c)(b+c-a)(c+a-b) \ge 72V^2$$

PROBLEM 6.63 - ***MILNE'S – INEQUALITY***

$$(a_1+b_1+a_2+b_2)\left(\frac{a_1b_1}{a_1+b_1} + \frac{a_2b_2}{a_2+b_2}\right) \le (a_1+a_2)(b_2+b_2)$$

$$(a_1+b_1+a_2+b_2+a_3+b_3)\left(\frac{a_1b_1}{a_1+b_1} + \frac{a_2b_2}{a_2+b_2} + \frac{a_3b_3}{a_3+b_3}\right) \le$$
$$\le (a_1+a_2+a_3) \cdot (b_1+b_2+b_3)$$

$$\left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right) \le \left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right), i \in \overline{1,n}; n \ge 2, a_i > 0; b_i > 0$$

PROBLEM 6.64 - ***MILOSLAVLJEVIC'S INEQUALITY***

In ΔABC the following relationship holds:

$$9r \leq a\cos\frac{A}{2} + b\cos\frac{B}{2} + c\cos\frac{C}{2} \leq \frac{9R}{2}$$

PROBLEM 6.65 - ***MITRINOVIC – ADAMOVIC'S INEQUALITY***

$$\cos x < \left(\frac{\sin x}{x}\right)^3, x \in \left(0, \frac{\pi}{2}\right)$$

PROBLEM 6.66 - ***MITRINOVIC'S GENERALIZED INEQUALITY***

$A_1A_2 \dots A_n, n \geq 3$ *polygon circumscribed to a circle of radius* r

$$a_k = A_kA_{k+1}, A_{n+1} = A_1, k \in \overline{1, n}$$

$$s \geq nr\tan\frac{\pi}{n}, s = \text{semiperimeter}$$

PROBLEM 6.67 - ***MONGOLIAN INEQUALITY***

If $x, y, z \in (0, \infty)$ then:

$$\left(\frac{x+y+z}{3}\right)^3 \geq \frac{(x+y)(y+z)(z+x)}{8}$$

PROBLEM 6.68 - ***MOSER'S INEQUALITY***

In ΔABC the following relationship holds:

$$a^2 + b^2 + c^2 \geq \frac{36}{35}\left(\frac{abc}{s} + s^2\right)$$

$s -$ *semiperimeter*

PROBLEM 6.69 - ***NABIEV'S INEQUALITY***

In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{R}{2S^2}} \leq \frac{1}{3}\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right)$$

PROBLEM 6.70 - ***NAKAJIMA'S INEQUALITY***

In ΔABC the following relationship holds:

$$a^2 + b^2 + c^2 \leq 8R^2 + \frac{4}{3\sqrt{3}}S$$

PROBLEM 6.71 - ***OTHOV'S INEQUALITY***

In ΔABC the following relationship holds:

$$a^4 + b^4 + c^4 \geq 16S^2$$

PROBLEM 6.72 - ***PEDOE'S INEQUALITY***

In $\Delta\ ABC: a, b, c$ – sides, $f = S[ABC]$.
In $\Delta\ MNP: A, B, C$ - sides, $F = S[MNP]$

$$A^2(b^2 + c^2 - a^2) + B^2(c^2 + a^2 - b^2) + C^2(a^2 + b^2 - c^2) \geq 16fF$$

PROBLEM 6.73 - ***REFINEMENT OF GERRETSEN'S INEQUALITY***

In acute-angled $\Delta\ ABC$ the following relationship holds:

$$\prod \left(\frac{1 - \cos A}{\cos A}\right) \geq \frac{8(\sum \tan A)^3}{27 \prod(\tan A + \tan B)} \geq 1$$

PROBLEM 6.74 - ***REFINEMENTS OF EULER'S INEQUALITY***

In acute – angled $\Delta\ ABC$ the following relationship holds:

$$2r \leq \frac{1}{3}(HA + HB + HC) \leq R$$

In $\Delta\ ABC$ the following relationship holds:

$$2r \leq \frac{r}{2} + \frac{1}{4}(IA + IB + IC) \leq R$$

H – orthocentre, I – incentre

PROBLEM 6.75 - ***RIGBY'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$\sum \frac{2\sqrt{s(s-a)}}{a} \geq 3\sqrt{3}$$

PROBLEM 6.76 - ***RIGBY'S INEQUALITY – 2***

In ΔABC the following relationship holds:

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq a^3(b + c) + b^3(c + a) + c^3(a + b)$$

PROBLEM 6.77 - ***RUSSIAN INEQUALITY***

$$\frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} > \frac{3}{2}, \qquad x, y, z > 0$$

PROBLEM 6.78 - ***RUSSIAN INEQUALITY – 2***

If $a, b, c \in (0, \infty)$ ***then:***

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{c+a}} + \sqrt{\frac{c}{a+b}} > 2$$

PROBLEM 6.79 - ***SANDOVICI'S INEQUALITY***

In acute – angled $\Delta\ ABC$ ***the following relationship holds:***

$$\tan A + \tan B + \tan C \geq 3\sqrt[3]{\frac{4S}{R^2}}$$

PROBLEM 6.80 - ***SCHAUMBERGER'S INEQUALITY***

$A_1A_2 \dots A_n, n \geq 3$ ***polygon circumscribed to a circle of radius*** r

$$a_k = A_kA_{k+1}, A_{n+1} = A_1, k \in \overline{1, n}$$

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq 4F \tan\frac{\pi}{n}, F = area\ [A_1A_2 \dots A_n]$$

PROBLEM 6.81 - ***SECLAMAN'S INEQUALITY***

In $\Delta\ ABC, S = area\ [ABC] = \frac{1}{2}$

$$\min(a, b, c) \leq \frac{a^2 + b^2 + c^2}{abc(\sin A + \sin B + \sin C)} \leq \max(a, b, c)$$

PROBLEM 6.82 - ***SECLAMAN'S INEQUALITY – 2***

If $a > 1$ ***then:***

$$\frac{\sqrt{a+1} - \sqrt{a}}{(a+1)^{\sqrt{a+1}} - a^{\sqrt{a}}} < \frac{1}{2a^{\sqrt{a}}}$$

PROBLEM 6.83 - ***SECLAMAN'S INEQUALITY – 3:***

If $a, b, c \geq 0, a + b + c = 3$ ***then:*** $(1-a)(1-b)(1-c) + 2 \geq 2abc$

If $a, b, c \geq 0, a + b + c = 3$ ***find:***

$$\max(2(a^3 + b^3 + c^3) + 15(ab + bc + ca) + 6abc)$$

Proposed by Dan Radu Seclaman-Romania

PROBLEM 6.84 - ***SECLAMAN'S INEQUALITY – 4***

If $A, B \in \mathcal{M}_n(R), n \geq 2, (A-B)^2 = O_n, AB = BA, a \in R, |a| < 2$ ***then:***

$$\det(I_n - a(A+B) + a^2AB) \geq 0$$

Proposed by Dan Radu Seclaman-Romania

PROBLEM 6.85 - ***SECLAMAN'S INEQUALITY – 5***

In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2+m_c^2}{r_a+r_b+r_c}\le 2R-r$$

PROBLEM 6.86

If $a,b>0, a\neq b$ then:

$$\sqrt{2}<\frac{\sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}<2$$

Proposed by Shan He Wu-China

PROBLEM 6.87 - ***SONDAT'S IDENTITY***

If in acute $\Delta\ ABC, H$ – orthocentre, I - incentre, O – circumcentre then:

$$(Area[OIH])^2=\frac{(a-b)^2(b-c)^2(c-a)^2}{64r^2}$$

PROBLEM 6.88 - ***STANCIU'S GENERALIZATION OF NESBITT'S INEQUALITY***

If $a,b,c,x,y>0$ then:

$$\frac{a}{xb+yc}+\frac{b}{xc+ya}+\frac{c}{xa+yb}\ge\frac{(a+b+c)^2}{(x+y)(ab+bc+ca)}\ge$$
$$\ge\frac{3\sqrt{3(a^2+b^2+c^2)}}{(x+y)(a+b+c)}\ge\frac{3}{x+y}$$

PROBLEM 6.89 - ***STANCIU'S INEQUALITY***

In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{\sin B}+\frac{1}{\sin C}\ge\frac{2}{\cos\frac{A}{2}}$$

PROBLEM 6.90 - ***SZOLLOSY'S INEQUALITY***

In ΔABC the following relationship holds:

$$\sqrt{bc(s-a)}+\sqrt{ca(s-b)}+\sqrt{ab(s-c)}\le 3R\sqrt{s}$$

PROBLEM 6.91 - ***TEPPER'S IDENTITY***

$$n!=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(a-k)^n, a\in\mathbb{R}$$

PROBLEM 6.92 - ***THEBAULT'S INEQUALITY***

In ΔABC the following relationship holds:

$$r_a^2 + r_b^2 + r_c^2 \geq \frac{27R^2}{4}$$

PROBLEM 6.93 - ***TRUCHT'S INEQUALITY – 1***

In ΔABC the following relationship holds:

$$4R + r \geq s\sqrt{3}$$

PROBLEM 6.94 - ***TRUCHT'S – INEQUALITY – 2***

In ΔABC the following relationship holds:

$$s^2 + r^2 \geq 14Rr$$

PROBLEM 6.95 - ***TSINTSIFAS INEQUALITY***

In ΔABC the following relationship holds:

$$\frac{m}{n+p}a^2 + \frac{n}{p+m}b^2 + \frac{p}{m+n}c^2 \geq 2\sqrt{3}S$$

$$m, n, p \in (0, \infty), S - \text{area}$$

PROBLEM 6.96 - ***TSINTSIFAS – MURTY'S INEQUALITY***

In ΔABC the following relationship holds:

$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}$$

PROBLEM 6.97 – ***URSĂRESCU'S INEQUALITY***

In acute $\Delta\ ABC$ the following relationship holds:

$$\frac{\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}}{\sqrt{\cot\frac{A}{2}} + \sqrt{\cot\frac{B}{2}} + \sqrt{\cot\frac{C}{2}}} \geq \sqrt{\frac{r}{R}}$$

PROBLEM 6.98 - ***VASIC'S INEQUALITY***

If $x, y, z \in \mathbb{R}, xyz > 0$ then in $\Delta\ ABC$:

$$x\sin A + y\sin B + z\sin C \leq \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)$$

PROBLEM 6.99 - *WALKER'S INEQUALITY-1*

In ΔABC the following relationship holds:

$$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

PROBLEM 6.100 - *WALKER'S INEQUALITY – 2*

In acute – angled ABC triangle the following relationship holds:

$$a^2+b^2+c^2\geq 4(R+r)^2$$

PROBLEM 6.101 - *WEISSTEN'S INEQUALITY:*

In Δ ABC the following relationship holds:

$$\frac{\sin A+\sin B+\sin C}{\cot A+\cot B+\cot C}\leq\frac{3}{2}$$

PROBLEM 6.102 - *WILKER – ANGLESIO'S INEQUALITY*

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2+\frac{16}{\pi^4}\cdot x^3\tan x\,,x\in\left(0,\frac{\pi}{2}\right)$$

PROBLEM 6.103 - *WILLIAMS-HARDY'S INEQUALITY*

If $x>1$ then:

$$\frac{\log x}{x^3-1}<\frac{1}{3}\cdot\frac{x+1}{x^3+x}$$

PROBLEM 6.104 - *WU'S INEQUALITY – 1*

If $x,y,z\in(0,\infty)$ then:

$$(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)\geq(xy+yz+zx)^3$$

PROBLEM 6.105 - *YANG'S INEQUALITY*

In Δ ABC the following relationship holds:

$$16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\leq s^2\leq 4R^2+4Rr+3r^2-\frac{r^2(R-2r)}{R-r}$$

Proposed by Yang Xue Zhi – China

PROBLEM 6.106

If $a, b > 0$ then:

$$\sqrt{\frac{a^2+b^2}{2}}+\frac{2}{\frac{1}{a}+\frac{1}{b}}\ge\frac{a+b}{2}+\sqrt{ab}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

PROBLEM 6.107 - ***ROMANIAN INEQUALITY***

In acute – angled $\Delta\ ABC, \omega -$ the Brocard angle:

$$\frac{R}{r}\ge\max\left\{\frac{1}{\sin\omega},\frac{(a+b)(b+c)(c+a)}{16RS},\frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right\}$$

PROBLEM 6.108

If $a, b > 0, a\neq b$ then:

$$0<\frac{\frac{a-b}{\ln a-\ln b}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}<\frac{1}{3}$$

Proposed by B.G.Carlson-USA

MISCELANEOUS PROBLEMS

PROBLEM 7.01

Prove that:

$$\sqrt[3]{2\left(\cos\frac{\pi}{13}+\cos\frac{5\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{2\pi}{13}-\cos\frac{3\pi}{13}\right)}-\sqrt[3]{2\left(\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}\right)}=\sqrt[3]{7-3\sqrt[3]{13}}$$

Proposed by Vasile Mircea Popa-Romania

PROBLEM 7.02

Find all $n \in \mathbb{N}$ such that $\Omega(n) \in \mathbb{N}$:

$$\Omega(n)=\sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2}+\log_n\left(\sqrt{\binom{2n}{3}}\right)$$

Proposed by Ajao Yinka-Nigeria

PROBLEM 7.03

Let $f(x)=\log\left(\frac{1}{x+\sqrt{x^2+m^2}}\right)$, then prove that:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\lambda|(n+1)}{n^5}+\sum_{k=1}^{\infty}\frac{(m+1+k)}{(k+m)^3m^2}=\frac{\pi^2}{6}\left(\frac{\pi^2}{6}+\zeta(3)\right)$$

where

$\lambda=\lim_{x\to 0}\frac{f^{n+2}(x)}{f^n(x)}$, f^n **is n^th^ derivatives**

Proposed by Naren Bhandari-Bajura-Nepal

PROBLEM 7.04

If $n,k \in \mathbb{N}$ then:

$$\sum_{j=0}^{n}\frac{(-1)^j}{1+j}\cdot\binom{k}{j}\binom{k-1-j}{n-j}=\frac{1}{k+1}\left((-1)^n+\binom{k}{n+1}\right)$$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 7.05

If $z \in \mathbb{C}, |z^2-2|=|4z+i|$ then:

$$|z|<2\sqrt{5}$$

Proposed by Marian Ursărescu – Romania

PROBLEM 7.06

Solve for natural numbers:

$$(x+y)^{x^n+y^n}=(x+1)^{x^n}\cdot(y+1)^{y^n}, n\in\mathbb{N}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

PROBLEM 7.07

Solve for real numbers:

$$(1+\sin x)\cdot(\sin x)^{\cos x}+(1+\cos x)\cdot(\cos x)^{\sin x}=1+\sin x+\cos x$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

PROBLEM 7.08

Find all function $f:\mathbb{R}\to\mathbb{R}$ satisfying:

$$f(x+ny^2)\geq(y+1)^n f(x),\forall x,y\in\mathbb{R},1\leq n\in\mathbb{N}$$

Proposed by Nguyen Van Canh-Vietnam

PROBLEM 7.09

Find all functions $f:\mathbb{R}\to\mathbb{R}$ continuous in $x=0$ such that:

$$f(2018x)=f(2019)x+x^2$$

Proposed by Nguyen Van Canh-Romania

PROBLEM 7.10

Find all ROLLE functions $f:[0,1]\to\mathbb{R}$ such that:

$$\begin{cases} f(0)=f(1)=\dfrac{2019}{2018} \\ 2017f'(x)+2018f(x)\leq 2019,\forall x\in(0,1)\end{cases}$$

Proposed by Nguyen Van Canh-Vietnam

PROBLEM 7.11

If $a,b\in(0,\infty)$ then:

$$\frac{2\sqrt{ab}}{a+b}+\frac{4ab}{(a+b)^2}+\frac{(a+b)^2}{4ab}+\frac{a+b}{2\sqrt{ab}}\leq 2\left(\frac{a}{b}+\frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.12

Prove that if: $z_1,z_2,\dots,z_n\in\mathbb{C}^*,n\in\mathbb{N}$ then:

$$\frac{\left|\sum_{i=1}^n(Re\,z_i+Im\,z_i)\right|}{\sum_{i=1}^n|z_i|}\leq\sqrt{2}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.13

Solve for real numbers:

$$\begin{cases}\dfrac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}\\ \dfrac{\tan^{-1}y}{\cot^{-1}y}=e^{\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)}\end{cases}$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

PROBLEM 7.14

If $x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n\in(0,\infty),n\in\mathbb{N}^*$ then:

$$\left(\sum_{i=1}^n\frac{x_i^2+y_i^2}{x_iy_i}\right)\left(\sum_{i=1}^n\frac{x_iy_i}{x_i^2+y_i^2}\right)\leq\left(\sum_{i=1}^n\frac{x_i}{y_i}\right)\left(\sum_{i=1}^n\frac{y_i}{x_i}\right)$$

Proposed by Daniel Sitaru-Romania

PROBLEM 7.15

If $a_i > 0, i \in \overline{1,n}, a_1 + a_2 + \cdots + a_n = 1$ then:

$$\frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{\frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}} \ge n+1$$

Proposed by Regragui El Khammal-Morocco

PROBLEM 7.16

If $a, r \in (0, \infty)$ then:

$$\sum_{k=1}^{n} \frac{k}{\left(\sum_{i=1}^{k}\left(\frac{1}{a+(i-1)r}\right)\right)} < (2a + (n-1)r)n, n \in \mathbb{N}^*$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.17

If $a_i, b_i \in (0, \infty), i \in \overline{1,n}, n \in \mathbb{N}^*$ then:

$$\frac{(2n)^n}{\prod_{i=1}^{n}(a_i + b_i)} \le \frac{1}{2}\left[\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)^n + \left(\sum_{i=1}^{n}\frac{1}{b_i}\right)^n\right]$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.18

Let $n \in \mathbb{N}^*, a_1, \ldots, a_n \in \left(0, \frac{\pi}{2}\right)$ such that

$$\sum_{1\le k\le n} a_k \le n.$$

Prove that

$$\left(\sum_{1\le k\le n}\frac{1}{a_k}\right)\sin\left(\frac{n}{\sum_{1\le k\le n}\frac{1}{a_k}}\right) + \frac{n}{\pi} > n$$

Proposed by Mihalcea Andrei Stefan-Romania

PROBLEM 7.19

If $a_k \in (0, \infty)$ where $k = 1, 2, 3, \ldots, n$ and $a_1 + a_2 + \cdots + a_n = 1$ then:

$$\sum_{k=1}^{n}\frac{1}{a_k} \ge (n+1)\left(\sum_{k=1}^{n}\frac{1}{a_k+1}\right)$$

Proposed by Marin Chirciu – Romania

PROBLEM 7.20

If $a, b, c \in (0, \infty)$ then:

$$(a^4 + b^4 + c^4)\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) \ge 2\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.21

Let $a, b, c > 0$ such that: $a^2 + b^2 + c^2 = 3abc$. Find the maximum of expression:

$$P = \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} + \frac{bc}{2b^6 - b^5 + c^4 + b^2 + 1} + \frac{ca}{2c^6 - c^5 + a^4 + c^2 + 1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 7.22

$$x_1, x_2, \dots, x_n > 0, n \in \mathbb{N}^*, \sum_{i=1}^{n} x_i = n^2$$

Find:

$$\Omega = max \prod_{i=1}^{n} (x_i)^i$$

Proposed by Madan Beniwal-Varanasi-India

PROBLEM 7.23

If a, b, c are maximum positives values such that:

$$\sum_{k=1}^{2017} \psi(n) = aH_b - c\gamma - d$$

find: $a + b + c + d$

Proposed by Shivam Sharma-New Delhi-India

PROBLEM 7.24

Let x, y, z be positive real numbers. Find the minimum possible value of

$$\frac{x}{y+z} + \frac{y}{z+x} + 2\sqrt{\frac{1}{2} + \frac{z}{x+y}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 7.25

$$a, b, c \in (1, \infty)$$

$$\Omega(a, b, c) = \sum \frac{\log_a^2 b + \log_a b \cdot \log_b c + \log_b^2 c}{\log_a b + \log_b c}$$

Find $\min \Omega(a, b, c)$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.26

Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum of expression:

$$P = \frac{x^3}{y\sqrt{x^3+8}} + \frac{y^3}{z\sqrt{y^3+8}} + \frac{z^3}{x\sqrt{z^3+8}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

PROBLEM 7.27

Let x, y, z be non-negative real numbers such that $x + y + z = 1$. Find the maximum and minimum possible values of

$$(y+z)\sqrt{1+x} + (z+x)\sqrt{1+y} + (x+y)\sqrt{1+z}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

PROBLEM 7.28

Let x, y, z be positive real numbers such that: $x+y=z=3$. Find the minimum of expression:$Q=\frac{1}{x(2y^2-yz+2z^2)}+\frac{1}{y(2z^2-zx+2x^2)}+\frac{1}{z(2x^2-xy+2y^2)}+2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)$

Proposed by Hoang Le Nhat Tung-Hanoi-Vietnam

PROBLEM 7.29

In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}}{\frac{w_a+w_b}{h_c}+\frac{w_b+w_c}{h_a}+\frac{w_c+w_a}{h_b}}=\frac{r}{4R}$$

Proposed by Mustafa Tarek-Cairo-Egypt

PROBLEM 7.30

Solve for real numbers:

$$\begin{cases}\left(\frac{xy}{z}\right)^4+\left(\frac{yz}{x}\right)^4+\left(\frac{zx}{y}\right)^4=xyz\sqrt[4]{27\sum x^4}\\ x^4-4y^3+6z^2-4x+1=0\end{cases}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.31

$$A\in M_3(\mathbb{R}),\det(A^2+2A+2I_3)=\det(A+I_3)=0$$

Find:

$$\Omega=\det A$$

Proposed by Marian Ursărescu-Romania

PROBLEM 7.32

$$\alpha=\begin{vmatrix}\frac{1}{x+a} & \frac{1}{x+b} & \frac{1}{x+c}\\ \frac{1}{y+a} & \frac{1}{y+b} & \frac{1}{y+c}\\ \frac{1}{z+a} & \frac{1}{z+b} & \frac{1}{z+c}\end{vmatrix},\beta=\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix},\gamma=\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$$

If $a,b,c,x,y,z>0$ **then:**

$$3^9|\alpha|\geq\frac{|\beta\gamma|}{(a+b+c+x+y+z)^9}$$

Proposed by Daniel Sitaru – Romania

PROBLEM 7.33

Solve for real numbers:

$$2^x\cdot 3^{\frac{1}{x}}+3^x\cdot 2^{\frac{1}{x}}=\sqrt{6}(\sqrt{2}+\sqrt{3})(5-\sqrt{6})$$

Proposed by Daniel Sitaru – Romania

CYCLIC INEQUALITIES-SOLUTIONS

SOLUTION 1.01

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality} \Leftrightarrow \sum(a^8+1)\left(\frac{1}{b^4+1}+\frac{1}{c^4+1}\right) \overset{(1)}{\geq} 12(\sqrt{2}-1)$$

$$\textit{LHS of (1)} = \sum_{cyc}\frac{a^8+1}{b^4+1}+\sum_{cyc}\frac{b^8+1}{a^4+1} = \sum_{cyc}\left(\frac{a^8+1}{b^4+1}+\frac{b^8+1}{a^4+1}\right) \underset{(2)}{\overset{A-G}{\geq}} 2\sum_{cyc}\sqrt{\frac{a^8+1}{a^4+1}\cdot\frac{b^8+1}{b^4+1}}$$

$$\textit{Let } f(x) = \frac{x^8+1}{x^4+1}\ \forall x \geq 0.\ \textit{We have } f'(x) = \frac{4x^3(x^8+2x^4-1)}{(x^4+1)^2}$$

$$\textit{and } f''(x) = \frac{4x^2(3x^{12}+9x^8+19x^4-3)}{(x^4+1)^3},\ f'(x) = 0 \textit{ iff } x = 0 \textit{ or } x = \sqrt[4]{\sqrt{2}-1}$$

$$f''(0) = 0 \textit{ with } f(0) = 1 \textit{ and } f''\left(\sqrt[4]{\sqrt{2}-1}\right) > 0 \textit{ with } f\left(\sqrt[4]{\sqrt{2}-1}\right) = 2(\sqrt{2}-1)$$

$$\therefore f(x)\forall x \geq 0 \textit{ attains its minimum at } x = \sqrt[4]{\sqrt{2}-1} \textit{ and } f_{\min} \overset{(3)}{=} 2(\sqrt{2}-1)$$

$$\textit{(2), (3)} \Rightarrow LHS \geq 6\sqrt{\left(2(\sqrt{2}-1)\right)^2} = 12(\sqrt{2}-1) \Rightarrow \textit{(1) is true (proved)}$$

SOLUTION 1.02

Solution by Marian Ursărescu-Romania

$$abc = 1 \textit{ we show this:}$$

$$(a^2+b^2)ab+(b^2+c^2)bc+(c^2+a^2)ca \leq 2(a^4+b^4+c^4)$$

$$\textit{But } a^4+b^4 \geq ab(a^2+b^2) \quad \textit{(1)} \Leftrightarrow$$

$$a^4-a^3b+b^4-ab^3 \geq 0 \Leftrightarrow$$

$$a^3(a-b)+b^3(b-a) \geq 0 \Leftrightarrow (a-b)(a^3-b^3) \geq 0 \Leftrightarrow$$

$$(a-b)^2(a^2+ab+b^2) \geq 0 \textit{ true}$$

$$\textit{From (1)} \Rightarrow \left.\begin{matrix} a^4+b^4 \geq ab(a^2+b^2) \\ b^2+c^4 \geq bc(b^2+c^2) \\ c^4+a^4 \geq ac(a^2+c^2) \end{matrix}\right\} \Rightarrow$$

$$2(a^4+b^4+c^4) \geq ab(a^2+b^2)+bc(b^2+c^2)+ac(a^2+c^2)$$

SOLUTION 1.03

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\textit{We have } 5a^4-a^5 \geq 15a-11\ \forall 0 < a < 3 \quad \textit{(1)}$$

It is true since (1) $\Rightarrow (a-1)^2(a^3-3a^2-7a-11)\le 0 \Rightarrow$

$$\Rightarrow (a-1)^2[a^2(a-3)-7a-11]\le 0$$

(True since $(a-1)^2\ge 0$ *and* $a^2(a-3)-7a-11<-11<0)$

Similalry, we have $5b^4-b^5\ge 15b-11\ \forall 0<b<3$ *(2) and*

$5c^4-c^5\ge 15c-11\ \forall 0<c<3$ *(3)*

(1), (2) and (3) $\Rightarrow 5(a^4+b^4+c^4)-\left(a^5+b^5+c^5\right)\ge 15(a+b+c)-33\Rightarrow$

$$\Rightarrow 5(a^4+b^4+c^4)-\left(a^5+b^5+c^5\right)\ge 12\Rightarrow 5(a^4+b^4+c^4)\ge 12+a^5+b^5+c^5$$

The equality occurs when $a=b=c=1$*.*

SOLUTION 1.04

Solution by Abdallah El Farissi-Bechar-Algerie

$$(7+a^3+b^3+c^3)\left(7+\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=$$

$$=49+7\left(a^3+\frac{1}{a^3}+b^3+\frac{1}{b^3}+c^3+\frac{1}{c^3}\right)+(a^3+b^3+c^3)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

$$\ge 49+42+9=100$$

SOLUTION 1.05

Solution by Marian Ursărescu-Romania

$$ab+bc+ca=3abc\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$$

Let $\frac{1}{\sqrt{a}}=x,\frac{1}{\sqrt{b}}=y,\frac{1}{\sqrt{c}}=z$ *with* $x,y,z>0\wedge x^2+y^2+z^2=3$

With this notation the inequality becomes:

$$\sum\frac{\frac{1}{xy}}{\left(\frac{1}{x}+\frac{1}{y}\right)^4}\le\frac{3}{16}\Leftrightarrow\sum\frac{x^3y^3}{(x+y)^4}\le\frac{3}{16}\quad (1)$$

But $x+y\ge 2\sqrt{xy}\Rightarrow (x+y)^4\ge 16x^2y^2\Rightarrow \frac{1}{(x+y)^4}\le\frac{1}{16x^2y^2}\Rightarrow\frac{x^3y^3}{(x+y)^4}\le\frac{1}{16}xy$ *(2)*

From (1) + (2) the inequality becomes: $\frac{1}{16}\sum xy\le\frac{3}{16}\Leftrightarrow\sum xy\le 3$ *(3)*

But $\sum xy\le\sum x^2=3$ *(4) (from hypothesis) .From (3) + (4)* $=1$ *the inequality is true.*

SOLUTION 1.06

Solution by Christos Eythimiou-Greece

$$a,b,c>0\wedge a+b+c=3$$

$$\Rightarrow \frac{ab^2}{\sqrt{b^2+bc+c^2}}+\frac{bc^2}{\sqrt{c^2+ca+a^2}}+\frac{ca^2}{\sqrt{a^2+ab+b^2}}+\frac{\sqrt{3}}{4}(a^2+b^2+c^2)=$$

$$\frac{a^2b^2}{a\sqrt{b^2+bc+c^2}}+\frac{b^2c^2}{b\sqrt{c^2+ca+a^2}}+\frac{c^2a^2}{c\sqrt{a^2+ab+b^2}}$$

$$+\frac{\sqrt{3}}{4}\left((a+b+c)^2-2(ab+bc+ca)\right)\geq$$

$$\frac{(ab+bc+ca)^2}{\sqrt{a}\sqrt{ab^2+abc+ac^2}+\sqrt{b}\sqrt{bc^2+bca+ba^2}+\sqrt{c}\sqrt{ca^2+cab+cb^2}}$$

$$+\frac{\sqrt{3}}{4}\left(3^2-2(ab+bc+ca)\right)\geq$$

$$\frac{(ab+bc+ca)^2}{\sqrt{a+b+c}\sqrt{ab^2+abc+ac^2+bc^2+bca+ba^2+ca^2+cab+cb^2}}$$

$$+\frac{\sqrt{3}}{4}\left(9-2(ab+bc+ca)\right)=$$

$$\frac{(ab+bc+ca)^2}{\sqrt{3}\sqrt{(a+b+c)(ab+bc+ca)}}+\frac{9\sqrt{3}}{4}-\frac{\sqrt{3}}{2}(ab+bc+ca)=$$

$$\frac{\left(\sqrt{ab+bc+ca}\right)^4}{\sqrt{3}\sqrt{3(ab+bc+ca)}}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{9\sqrt{3}}{4}=$$

$$\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}+\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}+\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{7\sqrt{3}}{4}\geq$$

$$3\sqrt[3]{\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}\cdot\frac{\left(\sqrt{ab+bc+ca}\right)^3}{6}\cdot\frac{\sqrt{3}}{2}}-\frac{\sqrt{3}}{2}(ab+bc+ca)+\frac{7\sqrt{3}}{4}=\frac{7\sqrt{3}}{4}$$

SOLUTION 1.07

Solution by Marian Ursărescu-Romania

We use breaking method: we show the following inequality:

$$\sqrt{\frac{a^2+b^5}{a^2+b^2}}\geq\sqrt[4]{a^3b^3}\quad \textit{(1)}$$

Proof: $\frac{a^2+b^5}{a^2+b^2}\geq\sqrt{a^3b^3}\Rightarrow a^5+b^5\geq(a^2+b^2)ab\sqrt{ab}\Rightarrow$

$$a^5 - a^3b\sqrt{ab} + b^5 - b^3a\sqrt{ab} \geq 0 \Rightarrow$$

$$a^3\sqrt{a}\left(a\sqrt{a} - b\sqrt{b}\right) + b^3\sqrt{4}\left(b\sqrt{b} - a\sqrt{a}\right) \geq 0 \Rightarrow$$

$$\left(a\sqrt{a} - b\sqrt{b}\right)\left(a^3\sqrt{a} - b^3\sqrt{b}\right) \geq 0 \Leftrightarrow \left(\sqrt{a^3} - \sqrt{b^3}\right)\left(\sqrt{a^5} - \sqrt{b^5}\right) \geq 0$$

obvious, because if $a \geq b \Rightarrow \sqrt{a^3} - \sqrt{b^3} \geq 0$ *and* $\sqrt{a^5} - \sqrt{b^5} \geq 0$

$a \leq b \Rightarrow \sqrt{a^3} - \sqrt{b^3} \leq 0$ *and* $\sqrt{a^5} - \sqrt{b^5} \leq 0$

Using relation (1) $\Rightarrow \sum \sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq \sum \sqrt[4]{a^3b^3} \geq 3\sqrt[3]{\sqrt[4]{a^6b^6c^6}} \Rightarrow \sum \sqrt{\frac{a^5+b^5}{a^2+b^2}} \geq 3$

SOLUTION 1.08

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a, b, c > 0, a + b + c = 3$ *then* $3 + \sum\left(\frac{b}{12a} + \frac{c}{6b+1}\right) > \sum\left(\frac{c}{10b+1} + \frac{b}{2a+1}\right)$ *(1)*

We have (1) $\Rightarrow a + b + c + \frac{b}{12a+1} + \frac{c}{6b+1} + \frac{c}{12b+1} + \frac{a}{6c+1} + \frac{a}{12c+1} + \frac{b}{6a+1} >$

$$> \frac{c}{10b+1} + \frac{b}{2a+1} + \frac{a}{10c+1} + \frac{c}{2b+1} + \frac{b}{10a+1} + \frac{a}{2c+1}$$

$$\Rightarrow a\left(1 + \frac{1}{12c+1} + \frac{1}{6c+1} - \frac{1}{2c+1} - \frac{1}{10c+1}\right) +$$

$$+b\left(1 + \frac{1}{12a+1} + \frac{1}{6a+1} - \frac{1}{2a+1} - \frac{1}{10a+1}\right) +$$

$$+c\left(1 + \frac{1}{12b+1} + \frac{1}{6b+1} - \frac{1}{2b+1} - \frac{1}{10b+1}\right) > 0$$

$$\Rightarrow a \cdot \frac{1440c^4 + 720c^3 + 204c^2 + 24c + 1}{(2c+1)(6c+1)(10c+1)(12c+1)} + b \cdot \frac{1440a^4 + 720a^3 + 204a^2 + 24a + 1}{(2a+1)(6a+1)(10a+1)(12a+1)} +$$

$+c \cdot \frac{1440b^4+720b^3+240b^2+24b+1}{(2b+1)(6b+1)(10b+1)(12b+1)} > 0$ *(True)*$\Rightarrow$ *Q.E.D.*

SOLUTION 1.09

Solution by Daniel Sitaru-Romania

$$3 = x^3 + y^3 + z^3 \overset{AM-GM}{\geq} 3xyz \rightarrow xyz \leq 1 \rightarrow (xyz)^2 \geq (xyz)^n, n \geq 2$$

$$\sum \frac{x}{y^4 + z^4 + y^2z^2} \overset{AM-GM}{\leq} \sum \frac{x}{3\sqrt[3]{(yz)^6}} = \frac{1}{3}\sum \frac{x}{y^2z^2} =$$

$$= \frac{1}{3}\sum \frac{x^3}{(xyz)^2} = \frac{1}{3(xyz)^2}\sum x^3 \leq \frac{1}{3(xyz)^2} \cdot 3 \leq \frac{1}{(xyz)^n}$$

SOLUTION 1.10

Solution by Ravi Prakash-New Delhi-India

For $x, y \geq 0$, we first show

$$9(x^4+y^4) \geq 2(x^2+xy+y^2)^2$$

$$\Leftrightarrow 9(x^4+y^4) \geq 2(x^4+y^4+x^2y^2+2x^3y+2xy^3+2x^2y^2)$$

$$\Leftrightarrow 7(x^4+y^4)-6x^2y^2-4x^3y-4xy^3 \geq 0 \Leftrightarrow 3(x^2-y^2)^2+4(x^3-y^3)(x-y) \geq 0$$

$$\Leftrightarrow 3(x^2-y^2)^2+4(x-y)^2(x^2+xy+y^2) \geq 0$$

Which is true.Putting $x = a^{\frac{1}{4}}, y = b^{\frac{1}{4}}$, we get

$$9(a+b) \geq 2\left(\sqrt{a}+(ab)^{\frac{1}{4}}+\sqrt{b}\right)^2 \Rightarrow \frac{3}{\sqrt{2}}\sqrt{a+b} \geq \sqrt{a}+\sqrt{b}+(ab)^{\frac{1}{4}} \quad (1)$$

Similarly, $3\sqrt{\frac{b+c}{2}} \geq \sqrt{b}+\sqrt{c}+(bc)^{\frac{1}{4}}$ *(2)* $\quad 3\sqrt{\frac{c+a}{2}} \geq \sqrt{c}+\sqrt{a}+(ca)^{\frac{1}{4}}$ *(3)*

Adding (1), (2), (3) we get

$$3\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\right) \geq 2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)+(ab)^{\frac{1}{4}}+(bc)^{\frac{1}{4}}+(ca)^{\frac{1}{4}}$$

SOLUTION 1.11

Solution by Andrew Okukura-Romania

$$a^2+ab+b^2 \geq \frac{3}{4}(a+b)^2 \ \left(a^2+b^2 \geq \frac{3}{4}(a^2+b^2)+\frac{1}{2}ab\right) \Rightarrow$$

$$\prod(a^2+ab+b^2) \geq \prod \frac{3}{4}(a+b)^2 \Rightarrow$$

$$\Rightarrow 8\sqrt{\prod(a^2+ab+b^2)} \geq 8\sqrt{\prod \frac{3}{4}(a+b)^2} = 8\left(\frac{\sqrt{3}}{2}\right)^3 \prod(a+b) =$$

$$= 3\sqrt{3}(a+b)(b+c)(c+a)$$

SOLUTION 1.12

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4}{\sum a}\left(\sum a^2\right)\left(\sum a^4\right) \overset{(1)}{\leq} 3\sum a^5$$

(1) $\Leftrightarrow 3\left(\sum a^5\right)\left(\sum a\right)^3+\left(\sum a^2\right)^4 \geq 4\left(\sum a^2\right)\left(\sum a^4\right)\left(\sum a\right)^2$

$$\Leftrightarrow \sum a^7b+\sum ab^7+5\sum a^6b^2+5\sum a^2b^6+10abc\left(\sum a^5\right)+$$

$$+abc\left(\sum a^4b+\sum ab^4\right)+4a^2b^2c^2\left(\sum a^2\right)\overset{(2)}{\geq}5\sum a^5b^3+5\sum a^3b^5+2\sum a^4b^4+$$

$$+8abc\left(\sum a^3b^2+\sum a^2b^3\right)$$

Now, $\sum a^7b+\sum ab^7=\sum(a^7+ab^7)\underset{(a)}{\overset{A-G}{\geq}}2\sum a^4b^4$

Also, $5\sum a^6b^2+5\sum a^2b^6=5\sum a^2b^2\,(a^4+b^4)\overset{Chebyshev}{\geq}5\sum\frac{1}{2}a^2b^2(a^2+b^2)(a^2+b^2)$

$$\underset{(b)}{\overset{A-G}{\geq}}5\sum a^3b^3\,(a^2+b^2)=5\sum a^5b^3+5\sum a^3b^5$$

Schur $\Rightarrow a^3(a-b)(a-c)+b^3(b-c)(b-a)+c^3(c-a)(c-b)\geq 0$

$$\Rightarrow\sum a^5+abc\left(\sum a^2\right)\geq\sum a^4b+\sum ab^4$$

$$\Rightarrow 4abc\left(\sum a^5\right)+4a^2b^2c^2\left(\sum a^2\right)\overset{(c)}{\geq}4abc\left(\sum a^4b+\sum ab^4\right)$$

(a)+(b)+(c)$\Rightarrow LHS\geq 2\sum a^4b^4+5\left(\sum a^5b^3+\sum a^3b^5\right)+6abc\left(\sum a^5\right)+$

$$+5abc\left(\sum a^4b+\sum ab^4\right)\overset{?}{\geq}5\left(\sum a^5b^3+\sum a^3b^5\right)+2\sum a^4b^4+$$

$$+8abc\left(\sum a^3b^2+\sum a^2b^3\right)\Leftrightarrow$$

$$\Leftrightarrow 6abc\left(\sum a^5\right)+5abc\left(\sum a^4b+\sum ab^4\right)\overset{?}{\geq}8abc\left(\sum a^3b^2+\sum a^2b^3\right)\quad (3)$$

Now, $a^5+b^5\overset{Chebyshev}{\geq}\frac{1}{2}(a^2+b^2)(a^3+b^3)\overset{A-G}{\geq}ab(a^3+b^3)\geq a^2b^2(a+b)=$

$$=a^3b^2+a^2b^3$$

$$\Rightarrow\sum\left(a^5+b^5\right)\geq\sum a^3b^2+\sum a^2b^3\Rightarrow 2\sum a^5\geq\sum a^3b^2+\sum a^2b^3$$

$$\Rightarrow 6abc\left(\sum a^5\right)\geq 3abc\left(\sum a^3b^2+\sum a^2b^3\right)\quad (d)$$

Again, $\sum a^4b+\sum ab^4=\sum ab\,(a^3+b^3)\geq\sum a^2b^2\,(a+b)=\sum a^3b^2+\sum a^2b^3$

$$\Rightarrow 5abc\left(\sum a^4b+\sum ab^4\right)\geq 5abc\left(\sum a^3b^2+\sum a^2b^3\right)\quad (e)$$

(d)+(e)⇒(3) is true (proved)

SOLUTION 1.13

Solution by Boris Colakovic-Belgrade-Serbia

$$\frac{1}{\sqrt{x+y^2+z^2}}\leq\frac{1}{\sqrt{x+\frac{(3-x)^2}{2}}}=\frac{\sqrt{2}}{\sqrt{x^2-4x+9}}\leq\frac{x+5}{6\sqrt{3}}\Leftrightarrow(x-1)^2(x^2+8x+9)\geq 0\ \textit{true}$$

$$\textit{Similarly } \frac{1}{\sqrt{x^2+y+z^2}} \le \frac{y+5}{6\sqrt{3}}; \frac{1}{\sqrt{x^2+y^2+z}} \le \frac{z+5}{6\sqrt{3}}$$

$$\sum \frac{1}{\sqrt{x+y^2+z^2}} \le \frac{1}{6\sqrt{3}} \sum (x+5) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

SOLUTION 1.14

Solution by Do Quoc Chinh-Vietnam

By Cauchy-Schwarz's inequality, we have:

$$\left(\sum \frac{a^3(a+b)}{a^2+ab+b^2}\right)\left(\sum \frac{a(a^2+ab+b^2)}{a+b}\right) \ge (a^2+b^2+c^2)^2$$

We have:

$$\sum \frac{a(a^2+ab+b^2)}{a+b} = \sum \frac{a[(a+b)^2-ab]}{a+b} = \sum a(a+b) - \sum \frac{a^2b}{a+b}$$

By Cauchy-Schwarz's inequality, we have:

$$\sum \frac{a^2b}{a+b} = \sum \frac{a^2b^2}{ab+b^2} \ge \frac{(\sum ab)^2}{\sum a^2 + \sum ab}$$

$$\Rightarrow \sum \frac{a(a^2+ab+b^2)}{a+b} \le \sum a(a+b) - \frac{(\sum ab)^2}{\sum a^2 + \sum ab}$$

$$= \sum a^2 + \sum ab - \frac{(\sum ab)^2}{\sum a^2 + \sum ab} =$$

$$= \frac{(\sum a^2 + \sum ab)^2 - (\sum ab)^2}{\sum a^2 + \sum ab} = \frac{(\sum a^2)^2 + 2(\sum a^2)(\sum ab)}{\sum a^2 + \sum ab} = \frac{(\sum a^2)(\sum a)^2}{\sum a^2 + \sum ab}$$

$$\Rightarrow LHS \ge \frac{(a^2+b^2+c^2)^2}{\sum \frac{a(a^2+ab+b^2)}{a+b}} \ge \frac{(\sum a^2 + \sum ab)(\sum a^2)}{(\sum a)^2}$$

$$\ge \frac{(\sum a^2 + \sum ab)(\sum a)^2}{3(\sum a)^2} = \frac{\sum (a+b)^2}{6} \ge \frac{4(\sum a)^2}{18} = \frac{2(\sum a)^2}{9}$$

The equality holds for $a = b = c$.

SOLUTION 1.15

Solution by Marian Ursărescu-Romania

$$x^{12} + 2y^4 + 1 = x^{12} + y^4 + y^4 + 1 \ge 4\sqrt[4]{x^{12} \cdot y^8 \cdot 1} \Rightarrow$$

$$x^{12} + 2y^4 + 1 \ge 4x^3y^2 \Rightarrow \frac{1}{x^{12}+2y^4+1} \le \frac{1}{4x^3y^2}$$

Inequality becomes:

$$\sum \frac{x}{x^{12}+2y^4+1} \le \frac{1}{4}\sum \frac{x}{x^3y^2} = \frac{1}{4}\sum \frac{1}{x^2y^2} = \frac{1}{4}\sum \frac{x^2y^2z^2}{x^2y^2} = \frac{1}{4}\sum z^2$$

We must show this: $x^2+y^2+z^2 \le x^8+y^8+z^8$ *(1)*

Now, we use: $a^2+b^2+c^2 \ge ab+bc+ac \Rightarrow$

$$x^8+y^8+z^8 \ge x^4y^4+y^4z^4+z^4x^4 \ge x^2y^2z^4+x^2y^4z^2+x^4y^2z^2$$

$= x^2y^2z^2(x^2+y^2+z^2) = x^2+y^2+z^2$*. So, (1) is true*

SOLUTION 1.16

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

$$x,y,z \ge 0 \Rightarrow \frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy} \ge \frac{x}{1}+\frac{y}{1}+\frac{z}{1} = 1 \quad (1)$$

We prove: $\frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy} \le \frac{9}{8}$

$$\sum \frac{x}{1-yz} \le \sum \frac{x}{1-\frac{(y+z)^2}{4}} = \sum \frac{x}{1-\frac{(1-x)^2}{4}} = 4\sum \frac{x}{3+2x-x^2}$$

$\frac{x}{3+2x-x^2} \le \frac{63x+3}{256} \Leftrightarrow (7x-9)(3x-1)^2 \le 0$ *(true,* $x \le 1$*)*

$$\Rightarrow \sum \frac{x}{1-yz} \le 4\sum \frac{63x+3}{156} = \frac{63\sum x+9}{64} = \frac{63\cdot 1+9}{64} = \frac{9}{8}$$

$$\Rightarrow \sum \frac{x}{1-yz} \le \frac{9}{8} \Leftrightarrow x=y=z=\frac{1}{3} \quad (2)$$

(1), (2) $\Rightarrow 1 \le \frac{x}{1-yz}+\frac{y}{1-xz}+\frac{z}{1-xy} \le \frac{9}{8}$

SOLUTION 1.17

Solution by Soumava Chakraborty-Kolkata-India

Firstly $\left(\sqrt{2}+1\right)^3 = 2\sqrt{2}+1+3\left(\sqrt{2}\right)^2\cdot 1+3\sqrt{2} = 7+5\sqrt{2} \Rightarrow \sqrt[3]{7+5\sqrt{2}} = \sqrt{2}+1$

$\therefore xyz \ge 7+5\sqrt{2} \Rightarrow \sqrt[3]{xyz} \ge \sqrt{2}+1$*. Now,* $t = \sum x \overset{A-G}{\ge} 3\sqrt[3]{xyz} = 3\left(\sqrt{2}+1\right)$

Now, $t^2-6t-9 \ge 0 \Leftrightarrow t \le \frac{6-\sqrt{72}}{2} = 3-3\sqrt{2}$ *or* $t \ge \frac{6+\sqrt{72}}{2} = 3+3\sqrt{2}$

$\because t > 0, \therefore t^2-6t-9 \ge 0 \Leftrightarrow t \ge 3\left(\sqrt{2}+1\right)$*, that is,*

$$t \ge 3\left(\sqrt{2}+1\right) \Rightarrow t^2-6t-9 \ge 0 \Rightarrow\Rightarrow 6t+9 \le t^2 \Rightarrow$$

$$\Rightarrow 6\sum x+9 \le \left(\sum x\right)^2 \quad (1)$$

But $\left(\sum x\right)^2 \le 3\sum x^2$ *(2)*

(1), (2) $\Rightarrow 3\sum x^2 \ge 6\sum x+9 \Rightarrow \sum x^2-2\sum x \ge 3$

Thus, it is established that $\forall x,y,z>0$ *such that* $xyz\geq 7+5\sqrt{2}$*, we have*

$$x^2+y^2+z^2-2(x+y+z)\geq 3$$

SOLUTION 1.18

Solution by Soumava Chakraborty-Kolkata-India

Numerator of LHS $=a^2b(b^2-c^2)(c+a)+b^2c(c^2-a^2)(a+b)+c^2a(a^2-b^2)(b+c)$

$$=\sum a^3b^3-abc\left(\sum a^2b\right)\to \textbf{(a)}$$

Let us consider $x,y,z>0$

$$x^3+y^3+y^3\overset{A-G}{\geq}3xy^2\to \textbf{(1)}$$

$$y^3+z^3+z^3\overset{A-G}{\geq}3yz^2\to \textbf{(2)}$$

$$z^3+x^3+x^3\overset{A-G}{\geq}3zx^2\to \textbf{(3)}$$

$$\textbf{(1)+(2)+(3)}\Rightarrow\sum x^3\geq\sum xy^2\to \textbf{(i)}$$

Putting $x=ab,y=bc,z=ca$ *and applying (i), we get*

$$\sum a^3b^3\geq abc\left(\sum a^2b\right)\Rightarrow\sum a^3b^3-abc\left(\sum a^2b\right)\geq 0$$

$\Rightarrow$ *numerator of LHS* ≥ 0 *(by(a)).Also, denominator of LHS* $=abc\prod(a+b)>0$

$$\therefore LHS\geq 0\ \ \textit{(Proved)}$$

SOLUTION.1.19

Solution by Christos Eythimiou-Greece

$$x,y,z>0\Rightarrow\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)}\geq\frac{8}{9}$$

$$x,y,z>0\Rightarrow\frac{(x+y)(y+z)(z+x)}{(x+y+z)(xy+yz+zx)}=$$

$$=\frac{(x+y+z)(xy+yz+zx)-\sqrt[3]{xyz}\sqrt[3]{xyyzzx}}{(x+y+z)(xy+yz+zx)}\geq$$

$$\geq\frac{(x+y+z)(xy+yz+zx)-\frac{x+y+z}{3}\cdot\frac{xy+yz+zx}{3}}{(x+y+z)(xy+yz+zx)}=\frac{8}{9}$$

SOLUTION 1.20

Solution by Abdul Aziz-Semarang-Indonesia

$$\sum\frac{1}{a^a+a}=\sum\frac{1}{(1+a-1)^a+a}\overset{Bernoulli}{\geq}\sum\frac{1}{1+a(a-1)+a}=\sum\frac{1}{a^2+1}$$

$$\overset{CS}{\geq}\frac{(1+1+1)^2}{a^1+1+b^2+1+c^2+1}=\frac{9}{a^2+b^2+c^2+3}$$

Equality holds when $a=b=c=1$.

SOLUTION 1.21

Solution by Marian Ursărescu-Romania

$$ab+bc+ca=6abc\Leftrightarrow\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=6$$

Let $x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\Rightarrow x+y+z=6,x,y,z>0$

$$\frac{1}{\sqrt{ab(a+b)}}=\frac{1}{\sqrt{\frac{1}{xy}\left(\frac{1}{x}+\frac{1}{y}\right)}}=\frac{xy}{\sqrt{x+y}}$$

We must show this:

$$\sum\frac{xy}{\sqrt{x+y}}\leq 3+\frac{1}{4}(xy+xz+yz)\text{ with }x+y+z=6\text{ (1)}$$

We show this: $\frac{xy}{\sqrt{x+y}}\leq\frac{x+y+xy}{4}\Leftrightarrow$ ***(1')***

$$(x+y+z)\sqrt{x+y}\geq 4xy\quad(2)$$

Let $x+y=S,xy=p.\ S=x+y\geq 2\sqrt{xy}\Rightarrow S\geq 2\sqrt{p}$

$$(2)\Leftrightarrow(S+p)\sqrt{S}\geq 4p\quad(3)$$

But $(S+p)\sqrt{S}\geq 2S\sqrt{p}\geq 4\sqrt{p}\cdot\sqrt{p}=4p\Rightarrow$ ***then (3) its true.***

From (1') $\Rightarrow\sum\frac{xy}{\sqrt{x+y}}\leq\frac{2(x+y+z)}{4}+\frac{1}{4}(xy+xz+yz)=3+\frac{1}{4}(xy+xz+yz).$

SOLUTION 1.22

Solution by Ravi Prakash-New Delhi-India

$$\left(2+x+y+\frac{1}{x}+\frac{1}{y}\right)^2-4\left(x+\frac{x+1}{y}\right)\left(y+\frac{y+1}{x}\right)=$$

$$=4+(x+y)^2+\left(\frac{x+y}{xy}\right)^2+4(x+y)+4\left(\frac{x+y}{xy}\right)+\frac{2(x+y)^2}{xy}-$$

$$-4\left[xy+x+1+y+1+\frac{(x+1)(y+1)}{xy}\right]=$$

$$=(x+y)^2-4xy+\left(\frac{x+y}{xy}\right)^2-4\left(\frac{1}{x}+\frac{1}{y}\right)+4\left(\frac{1}{x}+\frac{1}{y}\right)-\frac{4}{xy}+4(x+y)-4(x+y)+$$

$$+4-12+\frac{2(x+y)^2}{xy}=(x-y)^2+\frac{(x-y)^2}{x^2y^2}+\frac{2(x-y)^2}{xy}\ge 0$$

SOLUTION 1.23

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{c^c(ab)+a^a(bc)+b^b(ca)}{\sum ab}\overset{weighted\ A-G}{\ge}\sqrt[(\sum ab)]{(c^c)^{ab}(a^a)^{bc}(b^b)^{ca}}$$

$$=\sqrt[\sum ab]{(abc)^{abc}}=\sqrt[abc]{(abc)^{abc}}\quad\left(\because\sum ab=abc\right)=(abc)$$

$$\Rightarrow\sum c^c\cdot ab\ge\left(\sum ab\right)(abc)=a^2b^2c^2\quad\left(\because\sum ab=abc\right)$$

SOLUTION 1.24

Solution by Marian Ursărescu-Romania

From Hölder inequality $\Rightarrow$

$$\Rightarrow\left(a+\frac{b}{c}\right)^4+\left(a+\frac{b}{d}\right)^4+\left(a+\frac{b}{e}\right)^4\ge\frac{\left(a+\frac{b}{c}+a+\frac{b}{d}+a+\frac{b}{c}\right)^4}{27}\quad (1)$$

From (1) we must show this:

$$\frac{\left(3a+\frac{b}{c}+\frac{b}{d}+\frac{b}{e}\right)^4}{27}\ge 3(a+3b)^4\Leftrightarrow\left[3a+b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\right]^4\ge 81(a+3b)^4\Leftrightarrow$$

$$\Leftrightarrow 3a+b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\ge 3(a+3b)\Leftrightarrow b\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\ge 9b\Leftrightarrow\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\ge 9\ \textit{true}$$

$$\textit{because }(c+d+e)\left(\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)\ge 9\textit{, but }c+d+e=1\Rightarrow\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\ge 9.$$

SOLUTION 1.25

Solution by Marian Ursărescu-Romania

$$\textit{Inequality}\Leftrightarrow\sum\left(\frac{\sqrt[3]{yzt}}{\sqrt[3]{ztx}+\sqrt[3]{txy}+\sqrt[3]{xyz}}\right)^3\ge\frac{4}{27}\quad (1)$$

$$\textit{Let }\sqrt[3]{yzt}=x_1,\ \sqrt[3]{ztx}=x_2,\ \sqrt[3]{txy}=x_3,\ \sqrt[3]{xyz}=x_4$$

$$(1)\qquad\textit{becomes}\sum\left(\frac{x_1}{x_2+x_3+x_4}\right)^3\ge\frac{4}{27}\ (2)$$

$$\textit{From Holder we have:}\sum\left(\frac{x_1}{x_2+x_3+x_4}\right)^3\ge\frac{\left(\sum\frac{x_1}{x_2+x_3+x_4}\right)^3}{16}\quad (3)$$

$$\textit{From (2)+(3) we must show}\left(\sum\frac{x_1}{x_2+x_3+x_4}\right)^3\ge\frac{64}{27}\Leftrightarrow$$

$$\sum\frac{x_1}{x_2+x_3+x_4}\geq\frac{4}{3}\quad (4)$$

$$Let\ \left.\begin{matrix}x_2+x_3+x_4=y_1\\x_1+x_3+x_4=y_2\\x_1+x_2+x_4=y_3\\x_1+x_2+x_3=y_4\end{matrix}\right\}\Rightarrow\begin{matrix}x_1=\frac{-2y_1+y_2+y_3+y_4}{3}\\x_2=\frac{y_1-2y_2+y_3+y_4}{3}\\x_3=\frac{y_1+y_2-2y_3+y_4}{3}\\x_4=\frac{y_1+y_2+y_3-2y_4}{3}\end{matrix}$$

Inequality (4) becomes: $\frac{1}{3}\sum(-2y_1+y_2+y_3+y_4)\geq\frac{4}{3}\Leftrightarrow$

$$\sum(-2y_1+y_2+y_3+y_4)\geq 4\quad (5)$$

But $\sum(-2y_1+y_2+y_3+y_4)=-8+\sum\left(\frac{y_1}{y_2}+\frac{y_2}{y_1}\right)\geq -8+12=4\Rightarrow$*(5) is true.*

SOLUTION 1.26

Solution by Rovsen Pirguliyev-Sumgait-Azerbaidian

It is known that if $x>3$ *then* $\sqrt{x^2+9}\sin\frac{\pi}{x}>3, x\to 3a\Rightarrow\sin\frac{\pi}{3a}>\frac{1}{\sqrt{a^2+1}}$

$$LHS>\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{(a^2+1)}}\cdot\frac{1}{\sqrt{(b^2+1)}}\cdot\frac{1}{\sqrt{(c^2+1)}}$$

Now, we prove that

$$\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{(a^2+1)(b^2+1)(c^2+1)}}>\frac{1}{\sqrt{(a^2+b^2+2)(b^2+c^2+1)(c^2+a^2+2)}}$$

$$(a^2+b^2+2)(b^2+c^2+2)(c^2+a^2+2)>8(a^2+1)(b^2+1)(c^2+1)$$

$$a^2+b^2+2=(a^2+1)+(b^2+1)>2\sqrt{(a^2+1)(b^2+1)}$$

$$b^2+c^2+2=(b^2+1)+(c^2+1)>2\sqrt{(b^2+1)(c^2+1)}$$

$$c^2+a^2+2=(c^2+1)+(a^2+1)>2\sqrt{(c^2+1)(a^2+1)}$$

SOLUTION 1.27

Solution by Soumitra Mandal-Chandar Nagore-India

Let $a\geq b\geq c\Rightarrow\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$

$$\sum_{cyc}\frac{a^x}{(b+c)^y}\geq\frac{1}{3}\left(\sum_{cyc}a^x\right)\left(\sum_{cyc}\frac{1}{(a+b)^y}\right)\geq\left(\frac{a+b+c}{3}\right)^x\cdot 3\left(\frac{3}{2(a+b+c)}\right)^y$$

$$[\because x,y\in[1,+\infty)]=\frac{(a+b+c)^{x-y}}{2^y3^{x-y-1}}$$

SOLUTION 1.28

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c \in (0,1) \mid \sum a^2 = 3, (1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}} < \frac{1}{e^3}$$

$$\because a,b,c \in (0,1), 0 < (1-a^2),(1-b^2)(1-c^2) < 1$$

$$\textit{Now, } (1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}} < \frac{1}{e^3}$$

$$\Leftrightarrow \ln\left((1-a^2)^{\frac{1}{a}}(1-b^2)^{\frac{1}{b}}(1-c^2)^{\frac{1}{c}}\right) < \ln\left(\frac{1}{e^3}\right)$$

$$\Leftrightarrow \left(\frac{1}{a}\right)\ln(1-a^2) + \left(\frac{1}{b}\right)\ln(1-b^2) + \left(\frac{1}{c}\right)\ln(1-c^2) < -3 = -\sqrt{3}\left(\sum a^2\right)$$

$$\Leftrightarrow \sum\left[\left(\frac{1}{a}\right)\ln(1-a^2) + \sqrt{3}a^2\right] \overset{(1)}{<} 0$$

$$\textit{Let } f(x) = \ln(1-x^2) + \sqrt{3}x^3 \ \forall x \in [0,1)$$

$$\textit{We have } f'(x) = x\left(3\sqrt{3}x - \frac{2}{1-x^2}\right) = \left(\frac{x}{1-x^2}\right)\left(3\sqrt{3}x(1-x^2) - 2\right) \overset{?}{\le} 0$$

$$\Leftrightarrow 3\sqrt{3}x - 3\sqrt{3}x^3 - 2 \overset{?}{\le} 0 \Leftrightarrow 9x^3 - 9x + 2\sqrt{3} \overset{?}{\ge} 0$$

$$\Leftrightarrow \left(3x + 2\sqrt{3}\right)\left(\sqrt{3}x - 1\right)^2 \overset{?}{\ge} 0 \to \textit{true} \Rightarrow f'(x) \le 0 \ \forall x \in [0,1)$$

$$\Rightarrow f(x) \le f(0) = 0, \forall x \in [0,1) \Rightarrow \forall x \in [0,1), f(x) \le 0, \textit{ equality at } x = 0$$

$$\therefore x \in (0,1), f(x) < 0 \Rightarrow \ln(1-x^2) + \sqrt{3}x^3 < 0$$

$$\Rightarrow \left(\frac{1}{x}\right)\ln(1-x^2) + \sqrt{3}x^2 \overset{(a)}{<} 0 \ \forall x \in (0,1)$$

$$\therefore \left(\frac{1}{a}\right)\ln(1-a^2) + \sqrt{3}a^2 \underset{(i)}{\overset{by\ (a)}{<}} 0, \left(\frac{1}{b}\right)\ln(1-b^2) + \sqrt{3}b^2 \underset{(ii)}{\overset{by\ (a)}{<}} 0 \ \&$$

$$\left(\frac{1}{c}\right)\ln(1-c^2) + \sqrt{3}c^2 \underset{(iii)}{\overset{by\ (a)}{<}} 0 \quad \textit{(i)+(ii)+(iii)} \Rightarrow \textit{(1) is true (Proved)}$$

SOLUTION 1.29

Solution by Omran Kouba-Damascus-Syria

Let a, b, c be positive numbers and suppose that $1 \le p < q$,

by Hölder's inequality we have

$$a^p + b^p + c^p \le \left((a^p)^{\frac{q}{p}} + (b^p)^{\frac{q}{p}} + (c^p)^{\frac{q}{p}}\right)^{\frac{p}{q}}(1+1+1)^{1-\frac{p}{q}} \le 3(a^q + b^q + c^q)^{\frac{p}{q}}3^{-\frac{p}{q}}$$

Equivalently

$$3^p(a^p+b^p+c^p)^q \le 3^q(a^q+b^q+c^q)^p$$

And the desired inequality follows by taking $(p,q)=(e,\pi)$*, since* $e<\pi$.

SOLUTION 1.30

Solution by Soumava Chakraborty-Kolkata-India

Upon squaring, given inequality becomes

$$\sum a^4b^2+\sum a^2b^4+2a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)}+2b^2c^2\sqrt{(c^2+a^2)(a^2+b^2)}+$$

$$+2c^2a^2\sqrt{(a^2+b^2)(b^2+c^2)} \overset{(1)}{\ge} 2a^2b^2c^2+\sum a^4b^2+\sum a^2b^4$$

Now, $b^2+c^2\ge\frac{1}{2}(b+c)^2$ & $c^2+a^2\ge\frac{1}{2}(b+c)(c+a)$

$$\Rightarrow \sqrt{(b^2+c^2)(c^2+a^2)}\ge\frac{1}{2}(b+c)(c+a)$$

$$\therefore 2a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)} \overset{(a)}{\ge} a^2b^2(b+c)(c+a)=a^2b^2\left(\sum ab\right)+a^2b^2c^2$$

$(\because a,b,c\ge 0)$*. Similarly,* $2b^2c^2\sqrt{(c^2+a^2)(a^2+b^2)} \overset{(b)}{\ge} b^2c^2(\sum ab)+a^2b^2c^2$ &

$$2c^a a^2\sqrt{(a^2+b^2)(b^2+c^2)} \overset{(c)}{\ge} c^2a^2\left(\sum ab\right)+a^2b^2c^2$$

(a)+(b)+(c)$\Rightarrow 2\sum a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)}\ge(\sum ab)(\sum a^2b^2)+3a^2b^2c^2\ge$

$$\ge 2a^2b^2c^2(\because a,b,c\ge 0)$$

$$\Leftrightarrow \sum a^4b^2+\sum a^2b^4+2\sum a^2b^2\sqrt{(b^2+c^2)(c^2+a^2)}\ge$$

$\ge\sum a^4b^2+\sum a^2b^4+2a^2b^2c^2\Rightarrow$ *(1) is true (Done)*

SOLUTION 1.31

Solution by Soumava Chakraborty-Kolkata-India

$$6+\sum\frac{x^3+z^3}{x^2+z^2} \underset{(1)}{\overset{Chebyshev}{\ge}} 6+\sum\frac{(x+z)(x^2+z^2)}{2(x^2+z^2)}=6+\frac{\sum(x+z)}{2}$$

$$=6+\sum x=6+3=9$$

$$\sum\sqrt{(x+y+1)(y+z+1)} \overset{C-B-S}{\le} \sqrt{\sum(x+y)+3}\sqrt{\sum(x+y)+3}=$$

$$=2\sum x+3=2\cdot 3+3=9 \overset{by\ (1)}{\le} 6+\sum\frac{x^3+z^3}{x^2+z^2}$$

SOLUTION 1.32

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x^4+y^4=a, y^4+z^4=b, z^4+x^4=c.$$

$$\text{Then } a+b>c, b+c>a, c+a>b \Rightarrow a,b,c$$

are three sides of a triangle with circumradius, inradius & semi-perimeter $= R, r, s$ *(say).*

Now, $2\sum x^4=\sum a=2s \Rightarrow \sum x^4=s \therefore z^4=s-a, x^4=s-b, y^4=s-c$. *Using the above transformation, given inequality becomes* $\frac{\sum a^2}{\sqrt{s}}\geq 4\sqrt{3}\sqrt{(s-a)(s-b)(s-c)}\Leftrightarrow$

$$\Leftrightarrow \sum a^2\geq 4\sqrt{3}S \rightarrow \text{true (Ionescu – Weitzenbock) (proved)}$$

SOLUTION 1.33

Solution by Chris Kyriazis-Athens-Greece

The range of the function $e^{\frac{t}{4}}, t\geq 0$ *is* $[1,+\infty)$. *So, we can find* t_1,t_2,t_3,t_4 *such that* $e^{\frac{t_1}{4}}=a, e^{\frac{t_2}{4}}=b, e^{t_3}=c, e^{t_4}=d$. *Considering the function* $g(t)=\frac{1}{1+e^{\frac{t}{4}}}$ *when* $t\geq 0$*, it's easy to check that is convex in* $[0,+\infty)$. *(*$g'(1)=\frac{e^{\frac{t}{2}}-e^{\frac{4}{4}}}{16\left(e^{\frac{t}{4}}+1\right)^3}>0$ *, when* $t>0$*)*

Then, Jensen inequality gives us that

$$g(t_1)+g(t_2)+g(t_3)+g(t_4)\geq 4g\left(\frac{t_1+t_2+t_3+t_4}{4}\right)\Rightarrow$$

$$\Rightarrow \frac{1}{1+e^{\frac{t_1}{4}}}+\frac{1}{1+e^{\frac{t_2}{4}}}+\frac{1}{1+e^{\frac{t_3}{4}}}+\frac{1}{1+e^{\frac{t_4}{4}}}\geq \frac{4}{1+e^{\frac{t_1+t_2+t_3+t_4}{4}}}\Rightarrow$$

$$\Rightarrow \frac{4}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\geq \frac{4}{1+abcd}$$

SOLUTION 1.34

Solution by Marian Ursărescu-Romania

$$\text{If } x,y,z,t\geq 1 \text{ then:}$$

$$x^x\cdot y^y\cdot z^z\cdot t^t\geq x^{\sqrt[3]{yzt}}\cdot y^{\sqrt[3]{ztx}}\cdot z^{\sqrt[3]{txy}}\cdot t^{\sqrt[3]{xyt}}$$

$$\sqrt[3]{yzt}\leq \frac{y+z+t}{3}\Rightarrow x^{\sqrt[3]{yzt}}\leq x^{\frac{y+z+t}{3}}\quad (1)$$

From (1) and similarly we must show: $x^x\cdot y^y\cdot z^z\cdot t^t\geq x^{\frac{y+z+t}{3}}\cdot y^{\frac{z+t+x}{3}}\cdot z^{\frac{t+x+y}{3}}\cdot t^{\frac{x+y+t}{3}}$

$$\Leftrightarrow x^{3x}\cdot y^{3y}\cdot z^{3z}\cdot t^{3t}\geq x^{x+z+t}\cdot y^{z+t+x}\cdot z^{t+x+y}\cdot t^{x+y+z}\Leftrightarrow$$

$$\Leftrightarrow x^{4x}\cdot y^{4y}\cdot z^{4z}\cdot t^{4t}\geq(xyzt)^{x+y+z+t}\Leftrightarrow x^x y^y z^z t^t\geq\sqrt[4]{xyzt}^{x+y+z+t}\quad (2)$$

$$\text{But } \sqrt[4]{xyzt}\leq\frac{x+y+z+t}{4}\quad (3)$$

From (2)+(3) we must show: $x^x y^y z^z t^t\geq\left(\frac{x+y+z+t}{4}\right)^{x+y+z+t}\Leftrightarrow$

$$\Leftrightarrow \ln(x^x y^y z^z t^t)\geq\ln\left(\frac{x+y+z+t}{4}\right)^{x+y+z+t}\Leftrightarrow$$

$$\Leftrightarrow x\ln x+y\ln y+z\ln z+\ln t\geq(x+y+z+t)\ln\left(\frac{x+y+z+t}{4}\right)\quad (4)$$

Let $f(\alpha)=\alpha\ln\alpha$

$f'(\alpha)=\ln\alpha+1, f''(\alpha)=\frac{1}{\alpha}>0,\forall\alpha>1\Rightarrow f$ *convex from Jensen's inequality* $\Rightarrow$

$$\Rightarrow f\left(\frac{x+y+z+t}{4}\right)\leq\frac{f(x)+f(y)+f(z)+f(t)}{4}\Leftrightarrow$$

$$\Leftrightarrow\frac{x\ln x+y\ln y+z\ln z+t\ln t}{4}\geq\frac{x+y+z+t}{4}\ln\left(\frac{x+y+z+t}{4}\right)\Leftrightarrow 4 \text{ its true.}$$

SOLUTION 1.35

Solution by Le Van-Ho Chi Minh-Vietnam

By geometrizing, we may transform $(a;b;c)=(x^2+y^2;y^2+z^2;z^2+x^2)$ *of which* a,b *and* c *are three sides of triangle, namely* ΔABC.

Hence, the to-prove problem becomes:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{2\sqrt{3s(s-a)(s-b)(s-c)}}{abc}=\frac{2\sqrt{3}s}{abc}=\frac{\sqrt{3}}{2R}$$

By $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$, *it is enough to prove that:*

$$\frac{1}{\sin A+\sin B}+\frac{1}{\sin B+\sin C}+\frac{1}{\sin C+\sin A}\geq\sqrt{3}$$

Indeed, applying Schwarz's inequality:

$$\sum\frac{1}{\sin A+\sin B}\geq\frac{9}{2(\sin A+\sin B+\sin C)}\geq\sqrt{3}$$

Note that in any given triangle $ABC, \sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$.

Q.E.D. Equality holds when triangle ABC *is equilateral, in other words* $x=y=z$.

SOLUTION 1.36

Solution by Marian Ursărescu-Romania

$$x+y+z\geq 3\sqrt[3]{xyz}\Rightarrow(x+y+z)^9\geq 3^9x^3y^3z^3\quad (1)$$

From (1) inequality becomes:

$$8\sum\left(\frac{yz}{xy+xz}\right)^3 \geq 3 \Leftrightarrow \sum\left(\frac{yz}{xy+xz}\right)^3 \geq \frac{3}{8} \text{ (2)}$$

From Hölder's inequality we have: $\sum\left(\frac{yz}{xy+xz}\right)^3 \geq \frac{1}{9}\left(\sum\frac{yz}{xy+xz}\right)^3$ *(3)*

From (2) + (3) we must show: $\left(\sum\frac{yz}{xy+xz}\right)^3 \geq \frac{27}{8} \Leftrightarrow \sum\frac{yz}{xy+xz} \geq \frac{3}{2}$ *(4)*

Let $yz = a, xy = b, xz = c, a, b, c > 0$.

(4) $\Leftrightarrow \sum\frac{a}{b+c} \geq \frac{3}{2}$ *(true because is Nesbitt's inequality)*

SOLUTION 1.37

Solution by Marian Ursărescu – Romania

First: $3 + \sqrt{y(2-x)} + \sqrt{z(2-y)} + \sqrt{x(2-z)} \leq 2\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)$ *(1)*

But $\left.\begin{array}{l}\sqrt{y(2-x)} \leq \frac{y+2-x}{2}\\ \sqrt{z(2-y)} \leq \frac{z+2-y}{2}\\ \sqrt{x(2-z)} \leq \frac{x+2-t}{2}\end{array}\right\} \Rightarrow \sum\sqrt{y(2-x)} \leq 3$ *(2)*

From (1) + (2) we must show: $6 \leq 2\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right) \Leftrightarrow$

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq 3 \quad (3)$$

From $\sqrt{xy} + \sqrt{yz} + \sqrt{xz} = 3 \Rightarrow \exists a, b, c > 0$ *such that*

$$x = \frac{3bc}{a(a+b+c)}, y = \frac{3ac}{b(a+b+c)}, z = \frac{3ab}{c(a+b+c)} \quad (4)$$

From (3) + (4) we must show:

$$\sqrt{\frac{3bc}{a(a+b+c)}} + \sqrt{\frac{3ac}{b(a+b+c)}} + \sqrt{\frac{3ab}{c(a+b+c)}} \geq 3 \Leftrightarrow$$

$\Leftrightarrow ab + bc + ac \geq \sqrt{3abc(a+b+c)} \Leftrightarrow (ab + ac + bc)^2 \geq 3abc(a+b+c)$ *which its true because* $(m+n+p)^2 \geq 3(ma + mp + np)$

Second: $\sqrt{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right) < 3 + \sqrt{y(2-x)} + \sqrt{z(2-y)} + \sqrt{x(2-z)}$

$$\Leftrightarrow \sqrt{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right) < \sqrt{xy} + \sqrt{yz} + \sqrt{zx} + \sqrt{y(2-x)} + \sqrt{z(2-y)} + \sqrt{x(2-z)} \Leftrightarrow$$

$$\Leftrightarrow \sqrt{y}\left(\sqrt{x} + \sqrt{2-x}\right) + \sqrt{z}\left(\sqrt{y} + \sqrt{2-y}\right) + \sqrt{x}\left(\sqrt{z} + \sqrt{2-z}\right) > \sqrt{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right) \text{ (5)}$$

Let $f: [0, 2] \to \mathbb{R}, f(x) = \sqrt{x} + \sqrt{2-x}; f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{2-x}} = \frac{\sqrt{2-x}-\sqrt{x}}{2\sqrt{x(2-x)}}$

$$f'(x) = 0 \Rightarrow x = 1$$

x	0	1	2
$f'(x)$	$1 + + + + + + + +$	$0 - - - - - - - -$	-1
$f(x)$	$\sqrt{2}$ $\sqrt{2}$ m ↗	2 M ↘	m

$$\Rightarrow \sqrt{x} + \sqrt{2-x} \ge \sqrt{2} \quad \textit{(6)}$$

From (6) $\Rightarrow \sqrt{y}\left(\sqrt{x} + \sqrt{2-x}\right) \ge \sqrt{2}y$ *and similarly (7)*

From (7) $\Rightarrow \sum \sqrt{y}\left(\sqrt{x} + \sqrt{2-x}\right) > \sqrt{2}\left(\sqrt{x} + \sqrt{y} + \sqrt{z}\right)$ *(strictly)*

SOLUTION 1.38

Solution by Chris Kyriazis-Athens-Greece

The range of the function $g(t) = e^{-\frac{t}{2}}$ ***is*** $(0, 1]$ ***when*** $t \ge 0$***. So, there are unique*** t_1, t_2, t_3 ***such that*** $e^{-\frac{t_1}{2}} = a^m, e^{-\frac{t_2}{2}} = b^m, e^{-\frac{t_3}{2}} = c^m$ ***(since*** $0 < a^m, b^m, c^m \le 1$***). Considering the function***

$$f(x) = \frac{1}{\sqrt{1+e^{-\frac{x}{2}}}}, x \ge 0 \textit{ we have that } f''(x) < 0 \textit{ for every}$$

$x > 0$ ***because*** $2e^x - 1 > 0$ ***when*** $x > 0$***. So t is concave in*** $[0, +\infty)$***. By Jensen's inequality, we have that*** $\frac{f(t_1)+f(t_2)+f(t_3)}{3} \le f\left(\frac{t_1+t_2+t_3}{3}\right) \Rightarrow \frac{1}{\sqrt{1+e^{-\frac{t_1}{2}}}} + \frac{1}{\sqrt{1+e^{-\frac{t_2}{2}}}} + \frac{1}{\sqrt{1+e^{-\frac{t_3}{2}}}} \le \frac{3}{\sqrt{1+e^{-\frac{t_1+t_2+t_3}{6}}}}$ ***or***

$$\frac{1}{\sqrt{1+a^m}} + \frac{1}{\sqrt{1+b^m}} + \frac{1}{\sqrt{1+c^m}} \le \frac{3}{\sqrt{1+\sqrt[3]{a^m b^m c^m}}}. \textit{ It suffices to prove that:}$$

$$\frac{3}{\sqrt{1+\sqrt[3]{a^m b^m c^m}}} \le \frac{3\sqrt{2}}{1+\sqrt[6]{a^m b^m c^m}} \textit{ or } 1 + \sqrt[6]{a^m b^m c^m} \le \sqrt{2\left(1 + \sqrt[3]{a^m b^m c^m}\right)} \textit{ or}$$

$$1 + \sqrt[3]{a^m b^m c^m} + 2\sqrt[6]{a^m b^m c^m} \le 2 + 2\sqrt[3]{a^m b^m c^m} \textit{ or } \left(\sqrt[3]{a^m b^m c^m}\right)^2 \ge 0 \textit{ which holds!!!}$$

SOLUTION 1.39

Solution by Marian Ursărescu-Romania

Because $ab + bc + ac = 3 \Rightarrow \exists x, y, z > 0$ *such that:*

$$a = \frac{\sqrt{3}x}{\sqrt{xy+xz+yt}}, b = \frac{\sqrt{3}y}{\sqrt{xy+xt+yt}}, c = \frac{\sqrt{3}z}{\sqrt{xy+xz+yt}}. \textit{ Inequality becomes:}$$

$$\sum \frac{\sqrt[6]{xy+xz+yt}}{\sqrt[6]{3}\sqrt[6]{x^2+xy+xz+yt}} \le \frac{\sqrt[3]{36}}{2} \cdot \frac{\sqrt[6]{xy+xz+yz}}{3}\left(\sqrt[3]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \Leftrightarrow$$

$$\Leftrightarrow \sum \frac{1}{\sqrt[6]{(x+y)(x+z)}} \le \frac{\sqrt[3]{36}}{2}\left(\sqrt[6]{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) \quad \textit{(1)}$$

$$\textit{(1)} \Leftrightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le \frac{9}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \quad \textit{(2)}$$

From Hölder's inequality $\Rightarrow \left(\sum \frac{1}{\sqrt[6]{(x+y)(x+z)}}\right)^3 \le 9\sum \frac{1}{\sqrt{(x+y)(x+z)}}$ *(3)*

From (2)+(3) we must show: $\sum \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ *(4)*

But $\begin{matrix} x+y \ge 2\sqrt{xy} \\ x+z \ge 2\sqrt{xz} \end{matrix} \Rightarrow (x+y)(x+z) \ge 4x\sqrt{yz} \Rightarrow \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2\sqrt{x\sqrt{yz}}} \Rightarrow$

$$\Rightarrow \sum \frac{1}{\sqrt{(x+y)(x+z)}} \le \frac{1}{2}\sum \frac{1}{\sqrt{x\sqrt{yz}}} \quad \textit{(5)}$$

From (4)+(5) we must show: $\sum \frac{1}{\sqrt{x\sqrt{yz}}} \le \sum \frac{1}{x}$ *(6)*

Now use $\alpha^2+\beta^2+\gamma^2 \ge \alpha\beta+\alpha\gamma+\beta\gamma \Rightarrow \sum \frac{1}{x} = \sum \frac{1}{(\sqrt{x})^2} \ge \sum \frac{1}{\sqrt{xy}} = \sum \frac{1}{\left(\sqrt{\sqrt{xy}}\right)^2} \ge$

$\ge \sum \frac{1}{\sqrt{x\sqrt{yz}}} \Rightarrow$ *(6) its true.*

SOLUTION 1.40

Solution by Soumava Chakraborty-Kolkata-India

$\sum a^3 - 3abc = (\sum a)(\sum a^2 - \sum ab)$ ∴ *given inequality becomes:*

$$\left(\sum a\right)^2\left(\sum a^2 - \sum ab\right)^2 \le \left(\sum a^2\right)^3 \Leftrightarrow \left(\sum a^2 + 2\sum ab\right)\left(\sum a^2 - \sum ab\right)^2 \le$$

$\le (\sum a^2)^3 \Leftrightarrow (x+2y)(x^2+y^2-2xy) \le x^3$ *(where* $\sum a^2 = x, \sum ab = y$*)* $\Leftrightarrow$

$$\Leftrightarrow y^2(2y-3x) \le 0 \Leftrightarrow 3x \ge 2y \Leftrightarrow x + 2(x-y) \ge 0 \Leftrightarrow$$

$\Leftrightarrow \sum a^2 + 2(\sum a^2 - \sum ab) \ge 0 \Leftrightarrow \sum a^2 + \sum (a-b)^2 \ge 0 \rightarrow$ *true (Hence proved)*

SOLUTION 1.41

Solution by Daniel Sitaru-Romania

$$f(a,b,c) = a^4 + b^4 + c^4 + \lambda(a+b+c-3)$$

$$\begin{cases} f'_a = 4a^3 + \lambda = 0 \\ f'_b = 4b^3 + \lambda = 0 \\ f'_c = 4c^3 + \lambda = 0 \\ f'_\lambda = a + b + c - 3 = 0 \end{cases} \rightarrow a = b = c = -\sqrt[3]{\frac{\lambda}{4}} \rightarrow -3\sqrt[3]{\frac{\lambda}{4}} - 3 = 0 \rightarrow \lambda = -4 \rightarrow \begin{cases} a = 1 \\ b = 1 \\ c = 1 \end{cases}$$

$$H_f(1,1,1,-4) = \begin{pmatrix} 12 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \Delta_1 = 12 > 0, \Delta_2 = 144 > 0, \Delta_3 = 12^3 > 0$$

$$\min f(a,b,c,\lambda) = f(1,1,1-4) = 3 \rightarrow a^4 + b^4 + c^4 \geq 3$$

SOLUTION 1.42

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

Because $abc = 1, a, b, c > 0$ ***we have***

$ab + bc + ca \geq 3 \Rightarrow (ab)^2 + (bc)^2 + (ca)^2 \geq 3 \Rightarrow 2((ab)^2 + (bc)^2 + (ca)^2) \geq 6$ ***and***

$a^4 + b^4 + c^4 \geq \frac{(a^3+b^3+c^3)(a+b+c)}{3} \geq a^3 + b^3 + c^3$ ***and since for*** $x, y, z > 0$***, we get*** $\frac{1}{x^4+y^4+z} \leq \frac{1+1+z^3}{(x^2+y^2+z^2)^2}$***. Hence*** $\frac{1}{a^4+b^4+c} + \frac{1}{b^4+c^4+a} + \frac{1}{c^4+a^4+b} \leq \frac{1+1+1+1+1+1+a^3+b^3+c^3}{(a^2+b^2+c^2)^2} =$

$= \frac{a^3+b^3+c^3+6}{a^4+b^4+c^4+2((ab)^2+(bc)^2+(ca)^2)} \leq \frac{a^3+b^3+c^3+6}{a^3+b^3+c^3+6} = 1$***. Therefore it is to be true.***

Because $abc = 1, a, b, c > 0$***, we have:*** $\frac{1}{a^4+b^4+c} + \frac{1}{b^4+c^4+a} + \frac{1}{c^4+a^4+b} \leq$

$$\leq \frac{1}{a^3b + b^2a + abc^2} + \frac{1}{b^3c + c^3b + a^2bc} + \frac{1}{c^3a + a^3c + ab^2c} =$$

$$= \frac{1}{ab(a^2 + b^2 + c^2)} + \frac{1}{bc(a^2 + b^2 + c^2)} + \frac{1}{ca(a^2 + b^2 + c^2)} =$$

$$= \frac{c}{a^2 + b^2 + c^2} + \frac{a}{a^2 + b^2 + c^2} + \frac{b}{a^2 + b^2 + c^2} = \frac{a + b + c}{a^2 + b^2 + c^2} \leq$$

$\leq \frac{a+b+c}{\frac{(a+b+c)^2}{3}} = \frac{3}{a+b+c} \leq 1$***. Therefore it is to be true.***

SOLUTION 1.43

Solution by Amit Dutta-Jamshedpur-India

Using AM of m^th power ≥ m^th power of AM, i.e. $\frac{a_1^n+a_2^n}{2} \geq \left(\frac{a_1+a_2}{2}\right)^m, \forall m \in R - (0,1)$

Put $a_1 = a^2, a_2 = b^2, m = 8 \Rightarrow \frac{(a^2)^8+(b^2)^8}{2} \geq \left(\frac{a^2+b^2}{2}\right)^8 \Rightarrow \frac{a^{16}+b^{16}}{2} \geq \frac{(a^2+b^2)^8}{2^8} \Rightarrow$

$\Rightarrow \frac{a^{16}+b^{16}}{a^2+b^2} \geq \left(\frac{a^2+b^2}{2}\right)^2$ ***(1)***

Also, putting $a_1 = a^4, a_2 = b^4, m = 8 \Rightarrow \frac{(a^4)^8+(b^4)^8}{2} \ge \left(\frac{a^4+b^4}{2}\right)^8 \Rightarrow \frac{a^{32}+b^{32}}{a^4+b^4} \ge \left(\frac{a^4+b^4}{2}\right)^7$ *(2)*

From (1) & (2):

$$LHS = \sum \frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^2+b^2)(a^4+b^4)} \ge \sum \left[\frac{(a^2+b^2)(a^4+b^4)}{4}\right]^7$$

$$LHS \ge \sum \left[\frac{(a^2+b^2)(b^4+b^4)}{4}\right]^7 \overset{AM-GM}{\ge} \sum \left[\frac{2ab\cdot 2a^2b^2}{4}\right]^7 \ge \sum \left(\frac{4a^3b^3}{4}\right)^7 \ge \sum a^{21}b^{21} \ge$$

$$\ge a^{21}b^{21} + b^{21}c^{21} + c^{21}a^{21} \ge a^{21}b^{21}c^{21}\left(\frac{1}{a^{21}} + \frac{1}{b^{21}} + \frac{1}{c^{21}}\right)$$

$$LHS = \sum \frac{(a^{16}+b^{16})(a^{32}+b^{32})}{(a^4+b^4)(a^2+b^2)} \ge \left(\frac{1}{a^{21}} + \frac{1}{b^{21}} + \frac{1}{c^{21}}\right) \{\because abc = 1\}$$

SOLUTION 1.44

Solution by Daniel Sitaru-Romania

$$a(a-1)^2 \ge 0 \to a(a^2-2a+1) \ge 0 \to 0 \ge 2a^2 - a(a^2+1) \to$$

$$\to 2 \ge 2(a^2+1) - a(a^2+1) \to \frac{2}{a^2+1} \ge 2 - a \to \sum \frac{2}{a^2+1} \ge 6 - \sum a \to$$

$$\to \sum \frac{2}{a^2+1} \ge 6 - 3 \to \sum \frac{1}{a^2+1} \ge \frac{3}{2}$$

SOLUTION 1.45

Solution by Le Van-Ho Chi Minh-Vietnam

Applying Schwarz's inequality: $LHS = \sum \frac{a^8}{a^4+abcd} \ge \frac{(a^4+b^4+c^4+d^4)^2}{a^4+b^4+c^4+d^4+4abcd}$*. Hence, it is enough to show that:* $(a^4+b^4+c^4+d^4)^2 \ge 2abcd(a^4+b^4+c^4+d^4) + 8(abcd)^2$

Indeed, by AM-GM inequality: $\begin{cases} 2abcd \le \frac{a^4+b^4+c^4+d^4}{2} \\ 8(abcd)^2 \le 8\left(\frac{a^4+b^4+c^4+d^4}{4}\right)^2 = \frac{(a^4+b^4+c^4+d^4)^2}{2} \end{cases}$

Q.E.D. Equality holds when $a = b = c$*.*

SOLUTION 1.46

Solution by Amit Dutta-Jamshedpur-India

$$LHS = \frac{1}{\sqrt[3]{xyzt}}\left(\frac{x^2}{x+1} + \frac{y^2}{y+1} + \frac{z^2}{z+1} + \frac{t^2}{t+1}\right)$$

$$LHS = \left(\frac{\sqrt[3]{x^2}}{x+1}\right)\left(\frac{x}{\sqrt[3]{yzt}}\right) + \left(\frac{\sqrt[3]{y^2}}{y+1}\right)\left(\frac{y}{\sqrt[3]{xzt}}\right) + \left(\frac{\sqrt[3]{z^2}}{z+1}\right)\left(\frac{z}{\sqrt[3]{xyt}}\right) + \left(\frac{\sqrt[3]{t^2}}{t+1}\right)\left(\frac{t}{\sqrt[3]{xyz}}\right)$$

Now, using Chebyshev's inequality {assume $x \geq y \geq z \geq t > 0$}

$$\left(\frac{LHS}{4}\right) \geq \sum_{cyclic} \frac{\left(\frac{\sqrt[3]{x^2}}{x+1}\right)}{4} \sum_{cyclic} \frac{\left(\frac{x}{\sqrt[3]{yzt}}\right)}{4}$$

AM – GM

$$\frac{x}{\sqrt[3]{yzt}} + \frac{y}{\sqrt[3]{xzt}} + \frac{z}{\sqrt[3]{xyt}} + \frac{t}{\sqrt[3]{xyz}} \geq 4\sqrt[4]{\frac{xyz}{xyz}} \geq 4 \Rightarrow \left(\frac{LHS}{4}\right) \geq \sum_{cyclic} \frac{\left(\frac{\sqrt[3]{x^2}}{x+1}\right)}{4} \times \left(\frac{4}{4}\right) \Rightarrow$$

$$\Rightarrow LHS \geq \sum_{cyclic}\left(\frac{\sqrt[3]{x^2}}{x+1}\right) \text{ (proved)}$$

SOLUTION 1.47

Solution by Soumava Chakraborty-Kolkata-India

$$1 \overset{(A)}{\leq} \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + 2^{abc} \overset{(B)}{\leq} 8^9 + \frac{9}{10}$$

In order to prove (B), we shall first prove: $\frac{a}{1+bc} + \frac{2^{abc}}{3} \overset{(i)}{\leq} \frac{8^9}{3} + \frac{3}{10} \because a \leq 3 \ \& \ bc \geq 0$

$$\therefore (a-3)bc \leq 0 \Rightarrow (abc - 3bc)\ln 2 \leq 0 \Rightarrow abc \ln 2 \leq 3bc \ln 2 \Rightarrow \ln 2^{abc} \leq \ln 2^{3bc} \Rightarrow$$

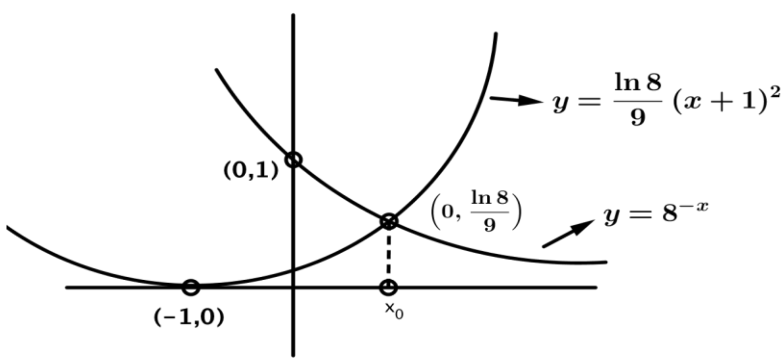
$\Rightarrow 2^{abc} \leq 2^{3bc}$. *Also,* $a \leq 3 \Rightarrow \frac{a}{1+bc} \leq \frac{3}{1+bc}$ $(\because 1 + bc > 0)$

$\therefore \frac{a}{1+bc} + \frac{2^{abc}}{3} \overset{(a)}{\leq} \frac{3}{1+bc} + \frac{2^{3bc}}{3} = \frac{3}{1+bc} + \frac{8^{bc}}{3}$. *Let* $f(x) = \frac{3}{1+x} + \frac{8^x}{3} \ \forall x \in [0,9]$

$$f'(x) = \frac{(\ln 8)8^x}{3} - \frac{3}{(x+1)^2} \ \& \ f''(x) = \frac{(\ln^2 8)8^x}{3} + \frac{6}{(x+1)^3} > 0$$

$\therefore f'(x)$ *is an increasing* f^n *in* $[0,9]$. *Now,* $f'(0) = \frac{\ln 8}{3} - 3 < 0 \ \&\& \ f'(x) = 0 \Leftrightarrow$

$$\frac{\ln 8}{9}(x+1)^2 = 8^{-x}$$



We see that $\frac{\ln 8}{9}(x+1)^2$ & 8^{-x} *intersect at only one point* $x_0 > 0$ & $\forall x \in [0, x_0)$,

$8^{-x} > \frac{\ln 8}{9}(x+1)^2$ & $\forall x \in (x_0, 9], \frac{\ln 8}{9}(x+1)^2 > 8^{-x}$. *So,* $\because \frac{\ln 8}{9}(x+1)^2\Big|_{x=1} > 8^{-x}|_{x=1}$,

$\therefore 1 > x_0 \Rightarrow x_0 \in (0,1)$ *(1)*

$\because f'(x) = 0$ *for some* $x_0 \in (0,1) \because f'(x)$ *is an increasing* f' *in* $[0,9] \therefore \forall x \geq x_0$,

$f'(x) \geq f'(x_0) = 0 \therefore \forall x \in [0, x_0), f'(x) < 0$ & $\forall x \in [x_0, 9], f'(x) \geq 0 \Rightarrow$

$\Rightarrow \forall x \in [0, x_0], f(x)$ *is a decreasing* f^n & $\forall x \in [x_0, 9], f(x)$ *is an increasing* f^n *(2)*

$\because f(0) = \frac{10}{3}$ & $f(9) = \frac{3}{10} + \frac{8^9}{3} \therefore f(9) > f(0) \rightarrow$ *(3)*

Combining (1), (2), (3) $\forall x \in [0,9], f_{\max} = f(9) = \frac{8^9}{3} + \frac{3}{10} \therefore x \in [0,9]$,

$\frac{3}{1+x} + \frac{8^x}{3} \leq \frac{8^9}{3} + \frac{3}{10}$. *Putting* $x = bc$ (& $bc \leq 9$), $\frac{3}{1+bc} + \frac{8^{bc}}{3} \overset{(b)}{\leq} \frac{8^9}{3} + \frac{3}{10}$

(a), (b) $\Rightarrow \frac{a}{1+bc} + \frac{2^{abc}}{3} \leq \frac{8^9}{3} + \frac{3}{10} \Rightarrow$ *(i) is true. Similarly,* $\frac{b}{1+ca} + \frac{2^{abc}}{3} \overset{(ii)}{\leq} \frac{8^9}{3} + \frac{3}{10}$ &

$$\frac{c}{1+ab} + \frac{2^{abc}}{3} \overset{(iii)}{\leq} \frac{8^9}{3} + \frac{3}{10}$$

(i) + (iii) + (iii) $\Rightarrow$ *(B) is true. Also,* $\because a, b, c \geq 0, \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + 2^{abc} \geq$

$\geq 0 + 0 + 0 + 2^0 = 1 \Rightarrow$ *(A) is true (proved)*

SOLUTION 1.48

Solution by Ravi Prakash-New Delhi-India

For $a, b > 0$. *Let* $z_1 = a + i, z_2 = b + i$. *Now,* $(a^2+1)(b^2+1) = |a+1|^2|b+i|^2 =$

$= |(a+i)(b+i)|^2 = |(ab-1) + i(a+b)|^2 = (ab-1)^2 + (a+b)^2 \geq$

$\geq (ab-1)^2 + \left(2\sqrt{ab}\right)^2 = (ab-1)^2 + 4ab = (ab+1) \Rightarrow$

$$(a^2+1)(b^2+1) \geq (ab+1)^2$$

$\therefore$ *for* $x, y, z > 0$; $(x^2+1)(y^2+1) \geq (xy+1)^2$; $(y^2+1)(z^2+1) \geq (yz+1)^2$

$(z^2+1)(x^2+1) \geq (zx+1)^2$. *Multiplying above inequalities, we get*

$$[(x^2+1)(y^2+1)(z^2+1)]^2 \geq [(xy+1)(yz+1)(zx+1)]^2 \Rightarrow$$

$$\Rightarrow (x^2+1)(y^2+1)(z^2+1) \geq (xy+1)(yz+1)(zx+1) \Rightarrow$$

$$\Rightarrow (x^4+1)(y^4+1)(z^4+1) \geq (x^2y^2+1)(y^2z^2+1)(z^2x^2+1)$$

Multiplying above two inequalities, we get

$$(x^2+1)(y^2+1)(z^2+1)(x^4+1)(y^4+1)(z^4+1) \geq$$

$$\geq (xy+1)(yz+1)(zx+1)(x^2y^2+1)(y^2z^2+1)(z^2x^2+1) \Rightarrow$$

$$\Rightarrow \frac{(x^2+1)(y^2+1)(z^2+1)}{(x^2y^2+1)(y^2z^2+1)(z^2x^2+1)} \geq \frac{(xy+1)(yz+1)(zx+1)}{(x^4+1)(y^4+1)(z^4+1)}$$

SOLUTION 1.49

Solution by Soumava Chakraborty-Kolkata-India

Given inequality $\Leftrightarrow$

$$3\sum a^6 + 2\sum a^4b^2 + 2\sum a^2b^4 + 2abc\left(\sum a^3\right) + 2\sum a^3b^3 + 6a^2b^2c^2 \overset{(1)}{\geq}$$

$$\geq 2abc(\sum a^2b + \sum ab^2). \textit{ Now, } \sum a^6 + 2\sum a^3b^3 = (\sum a^3)^2 \overset{(a)}{\geq} 0. \textit{ Also,}$$

$$(a^4b^2 + a^4c^2 + 2a^4bc) + (b^4c^2 + b^4c^2 + 2b^4ac) + (c^4a^2 + c^4b^2 + 2c^4ab) =$$

$$= (a^2b + a^2c)^2 + (b^2c + b^2a)^2 + (c^2a + c^2b)^2 \geq 0 \Rightarrow \sum a^4b^2 + \sum a^2b^4 +$$

$$+2abc(\sum a^3) \overset{(b)}{\geq} 0. \textit{ Again, } \because a^2, b^2, c^2 \geq 0, \therefore \textit{ applying Schur, } \sum (a^2)^3 + 3a^2b^2c^2 \geq$$

$$\geq \sum a^4b^2 + \sum a^2b^4 = (a^4b^2 + b^4c^2 + c^4a^2) + (a^2b^4 + b^2c^4 + c^2a^4) \geq$$

$$\geq (a^2bb^2c + b^2c \cdot c^2a + c^2aa^2b) + (ab^2bc^2 + bc^2ca^2 + ca^2ab^2)$$

$$\left(\because \forall x, y, z \in \mathbb{R}, \sum x^2 \geq \sum xy \; as \; \sum x^2 - \sum xy = \frac{1}{2}\sum (x-y)^2 \geq 0\right)$$

$$= abc(\sum a^2b + \sum ab^2) \Rightarrow 2\sum a^6 + 6a^2b^2c^2 \overset{(c)}{\geq} 2abc(\sum a^2b + \sum ab^2). \textit{ Moreover,}$$

$$\sum a^4b^2 + \sum a^2b^2 \overset{(d)}{\geq} 0$$

(a)+(b) +(c) +(d) $\Rightarrow$ ***(1) is true (proved)***

SOLUTION 1.50

Solution by Marian Ursărescu – Romania

Inequality becomes: $\frac{x}{x+y+z}\left(\frac{8}{3y+5z}\right)^7 + \frac{y}{x+y+z}\left(\frac{8}{3z+5x}\right)^7 + \frac{z}{x+y+z}\left(\frac{8}{3x+5y}\right) \geq \frac{3^7}{(x+y+z)^7}$ ***(1)***

But $f: (0, +\infty) \to \mathbb{R}; f(x) = x^7$ ***is a convex function. From Jensen's inequality (general form)***

$$\Rightarrow p_1f(x_1) + p_2f(x_2) + p_3f(x_3) \geq f(p_1x_1 + p_2x_2 + p_3x_3), p_1 + p_2 + p_3 = 1 \Rightarrow$$

$$\Rightarrow \frac{x}{x+y+z}\left(\frac{8}{3y+5z}\right)^7 + \frac{y}{x+y+z}\left(\frac{8}{3z+5x}\right)^7 + \frac{z}{x+y+z}\left(\frac{8}{3x+5y}\right) \geq$$

$$\geq \left(\frac{x}{x+y+z} \cdot \frac{8}{3y+5z} + \frac{y}{x+y+z} \cdot \frac{8}{3z+5x} + \frac{z}{x+y+z} \cdot \frac{8}{3x+5y}\right)^7 \quad \textit{(2)}$$

$$\text{From (1) + (2) we must show: } \left(\frac{8x}{3y+5z}+\frac{8y}{3z+5x}+\frac{8z}{3x+5y}\right)^7 \ge 3^7 \Leftrightarrow$$

$$\Leftrightarrow \frac{x}{3y+5z}+\frac{y}{3z+5x}+\frac{z}{3x+5y} \ge \frac{3}{8} \quad (3)$$

$$\text{But from Cauchy's inequality we have: } \frac{x}{3y+5z}+\frac{y}{3z+5x}+\frac{z}{3x+5y} = \frac{x^2}{3xy+5xz}+\frac{y^2}{3yz+5xy}+$$

$$+\frac{z^2}{3xz+5yz} \ge \frac{(x+y+z)^2}{8(xy+xz+yz)} \quad (4)$$

$$\text{From (3)+(4) we must show: } \frac{(x+y+z)^2}{8(xy+xz+yz)} \ge \frac{3}{8} \Leftrightarrow (x+y+z)^2 \ge 3(xy+xz+yz) \Leftrightarrow$$

$$\Leftrightarrow x^2+y^2+z^2 \ge xy+xz+yz \text{ which its true.}$$

SOLUTION 1.51

Solution by Amit Dutta-Jamshedpur-India

Using Power mean AM of m^th^ power ≥ m^th^ power of AM

$$\Rightarrow \frac{a_1^m+a_2^m+\cdots+a_n^m}{n} \ge \left(\frac{a_1+a_2+\cdots+a_n}{n}\right)^m; \forall m \in \mathbb{R} \setminus (0,1)$$

$$\Rightarrow \frac{(a+b-c)^3+(b+c-a)^3+(c+a-b)^3}{3} \ge \left(\frac{a+b+c}{3}\right)^3 \Rightarrow$$

$$\Rightarrow \sum(a+b-c)^3 \ge 3\left(\frac{a+b+c}{3}\right)^3 \overset{AM-GM}{\ge} 3abc \quad (1)$$

$$\text{Again using power mean, } \frac{(a+b-c)^5+(b+c-a)^5+(c+a-b)^5}{3} \ge \left(\frac{a+b+c}{3}\right)^5 \Rightarrow$$

$$\Rightarrow \sum(a+b-c)^5 \ge 3\left(\frac{a+b+c}{3}\right)^5$$

$$\sum(a+b-c)^5 \ge 3 \quad (2) \{\because a+b+c=3\}$$

$$\text{Multiplying (1) \& (2): } \sum(a+b-c)^3 \cdot \sum(a+b-c)^5 \ge 9abc$$

SOLUTION 1.52

Solution by Soumava Chakraborty-Kolkata-India

$$RHS \ge \frac{1}{\sqrt{2}}(abc+xyz) \overset{?}{\ge} \frac{\sqrt{2}(a+x)(b+y)(c+z)}{(a+1)(b+1)(c+1)} \Leftrightarrow (abc+xyz)(a+1)(b+1)(c+1) \ge$$

$$\underset{(1)}{\overset{?}{\ge}} 2(a+x)(b+y)(c+z) \because a,b,c \ge 1, \text{ we can let } a=1+m, b=1+n, c=1+p$$

$$(m,n,p \ge 0) \ \& \because x \ge a, y \ge b, z \ge c, \text{ hence, we can let } x=a+u, y=b+v, z=c+w$$

$$(u,v,w \ge 0) \therefore x=1+m+u, y=1+n+v, z=1+p+w$$

$$\therefore (1) \Leftrightarrow \{(1+m)(1+n)(1+p)+(1+m+u)(1+n+v)(1+p+w)\}$$

$$(2+m)(2+n)(2+p)\geq 2(2+2m+u)(2+2n+v)(2+2p+w)\Leftrightarrow$$

$$\Leftrightarrow 2m^2n^2p^2+m^2n^2pw+m^2np^2v+m^2npvw+mn^2p^2u+mn^2puw+mnp^2uv+$$

$$+mnpuvw+6m^2n^2p+2m^2n^2w+6m^2np^2+3m^2npv+3m^2npw+2m^2nvw+$$

$$+2n^2p^2v+2m^2pvw+6mn^2p^2+3mn^2pu+3mn^2pw+2mn^2uw+$$

$$+3mnp^2u+3mnp^2v+3mnpuw+3mnpvw+$$

$$+2mnuvw+2mp^2uv+2npuvw+2n^2p^2u+2n^2puw+2np^2uv+2npuvw+$$

$$+4m^2n^2+18m^2np+2m^2nv+6m^2nw+4m^2p^2+6m^2pv+2m^2pw+4m^2vw+$$

$$+18mn^2p+2mn^2u+6mn^2w+18mnp^2+9mnpu+9mnpv+9mnpw+2mnuv+$$

$$+6mnuw+4muvw+4n^2p^2+6n^2pu+2n^2pw+4n^2uw+6np^2u+2np^2v$$

$$+6npuv6npuw+2npvw+4nuvw+4p^2uv+4puvw+$$

$$12m^2n+12m^2p+4m^2v+4m^2w12mn^2+38mnp+6mnu+6mnv+10mnw$$

$$+12mp^2+6mpu+10mpv+6mpw+$$

$$+4muv+8mvw+12n^2p+4n^2u+4n^2w+12np^2+10npu+6npv+6npw+$$

$$+4nuv+8nuw+4nvw+4p^2u+4p^2v+8puv+4puw+4pvw+6uvw+8m^2+$$

$$+20mn+20mp+4mu+4mv+4mw+8n^2+20np+4nu+4nv+4nw+8p^2+$$

$$+4pu+4pv+4pw+4uv+4uw+4vw+8m+8n+8p\geq 0\rightarrow \textit{true}$$

$$\because m,n,p,u,v,w\geq 0 \quad \textit{(proved)}$$

SOLUTION 1.53

Solution by Marian Ursărescu-Romania

Let $a=\frac{x}{x+y+z+t}, b=\frac{y}{x+y+z+t}, c=\frac{z}{x+y+z+t}, d=\frac{t}{x+y+z+t}$*. We must show:*

$$2^{16}xyzt(y+z+t)(x+z+t)(x+y+t)(x+y+z)\leq 81(x+y+z+t)^8 \quad \textit{(1)}$$

But $\sqrt[4]{xyzt}\leq\frac{x+y+z+t}{4}\Leftrightarrow 2^8xyzt\leq(x+y+z+t)^4$ *(2). From (1)+(2) we must show:*

$$2^8(y+z+t)(x+z+t)(x+y+t)(x+y+z)\leq 81(x+y+z+t)^4\Leftrightarrow$$

$$\sqrt[4]{(y+z+t)(x+z+t)(x+y+t)(x+y+z)}\leq\frac{3}{4}(x+y+z+t) \quad \textit{(3)}$$

But $\sqrt[4]{(y+z+t)(x+z+t)(x+y+z)(x+y+t)}\leq\frac{3x+3y+3t+4z}{4}\Rightarrow$ *(3) its true.*

SOLUTION 1.54

Solution by Marian Ursărescu-Romania

Because $2\sqrt{x}\leq x+1\Rightarrow$ *we must show:*

$$\frac{1}{(x+y+1)^\theta}+\frac{1}{(y+yz+1)^\theta}+\frac{1}{(z+zx+1)^\theta}\ge\frac{1}{3^{\theta-1}} \quad (1)$$

Because $xyz=1\Rightarrow \exists a,b,c>0$ *such that* $x=\frac{a}{b}, y=\frac{b}{c}, z=\frac{c}{a}$ *(2)*

From (1)+(2) we must show:

$$\frac{1}{\left(\frac{a}{b}+\frac{a}{b}\cdot\frac{b}{c}+1\right)^\theta}+\frac{1}{\left(\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+1\right)^\theta}+\frac{1}{\left(\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}+1\right)^\theta}\ge\frac{1}{3^{\theta-1}}\Leftrightarrow$$

$$\frac{(bc)^\theta}{(ac+ab+bc)^\theta}+\frac{(ac)^\theta}{(ab+bc+ac)^\theta}+\frac{(ab)^\theta}{(bc+ac+ab)^\theta}\ge\frac{1}{3^{\theta-1}}\Leftrightarrow$$

$$\Leftrightarrow (bc)^\theta+(ac)^\theta+(ab)^\theta\ge\frac{(ab+bc+ac)^\theta}{3^{\theta-1}} \quad (3)$$

Let $ab=m, bc=n, ac=p, m,n,p>0$ *(4)*

From (3)+(4) we must show: $m^\theta+n^\theta+p^\theta\ge\frac{(m+n+p)^\theta}{3^{\theta-1}}$*, which its true, because its Hölder's inequality (generalization).*

SOLUTION 1.55

Solution by Soumava Chakraborty-Kolkata-India

$$\because a+b+c=8 \therefore (a+1)+(b+1)+(c+1)=11$$

$$\therefore \sqrt[3]{(a+1)(b+1)(c+1)}\overset{G-A}{\le}\frac{(a+1)+(b+1)+(c+1)}{3}=\frac{11}{3}\Rightarrow$$

$$\Rightarrow (a+1)(b+1)(c+1)\le\frac{11^3}{27}\Rightarrow\frac{81}{(a+1)(b+1)(c+1)}\ge\frac{81\cdot 27}{11^3}$$

$$\approx 1.643 \text{ and } \because \frac{1}{\sqrt[4]{27}}\approx 0.439 \therefore \frac{81}{(a+1)(b+1)(c+1)}>\frac{1}{\sqrt[4]{27}} \quad \text{(Done)}$$

SOLUTION 1.56

Solution by Boris Colakovic-Belgrade-Serbie

$$2(a^4+b^4+c^4)+12\ge 3(a^3+b^3+c^3+a+b+c)\Leftrightarrow$$

$$\Leftrightarrow 2(a^4+b^4+c^4)+4(a^2+b^2+c^2)\ge 3(a^3+b^3+c^3+a+b+c)\Leftrightarrow$$

$$\Leftrightarrow (2a^4-3a^3+4a^2-3a)+(2b^4-3b^3+4b^2-3b)+(2c^4-3c^3+4c^2-3c)\ge 0 \text{ or}$$

$$2\sum a^4-3\sum a^3+4\sum a^2-3\sum a\ge 0 \quad (1)$$

How is $2a^4-3a^3+4a^2-3a\ge 2a^2-2\Leftrightarrow 2a^4-3a^3+2a^2-3a+2\ge 0\Leftrightarrow$

$$\Leftrightarrow (a-1)^2(2a^2+a+2)\ge 0 \text{ true } \forall a\in\mathbb{R}$$

$$2a^4-2a^3+4a^2-3a\ge 2a^2-2 \quad (2)$$

$$2b^4-3b^3+4b^2-3b\geq 2b^2-2 \quad \textit{(3)}$$

$$2c^4-3c^3+4c^2-3c\geq 2c^2-2 \quad \textit{(4)}$$

(2)+(3)+(4)$\Rightarrow 2\sum a^4-3\sum a^3+4\sum a^2-3\sum a\geq 2\sum a^2-6=2\cdot 3-6=0\Rightarrow$ *(1) true*

SOLUTION 1.57

Solution by Serban George Florin-Romania

$$\ln a=x,\ln b=y,\ln c=z\Rightarrow x,y,z\geq 1\Rightarrow$$

$$\Rightarrow(\ln a+\ln e)(\ln b+\ln e)(\ln c+\ln e)+4\geq 4(\ln a+\ln b+\ln c)$$

$$(x+1)(y+1)(z+1)+4\geq 4x+4y+4z$$

$$x+1=\alpha,y+1=\beta,z+1=\gamma\Rightarrow\alpha,\beta,\gamma\geq 2$$

$$\alpha\beta\gamma+4\geq 4(\alpha-1)+4(\beta-1)+4(\gamma-1)$$

$$\alpha\beta\gamma+4-4\alpha-4\beta-4\gamma+12\geq 0$$

$$\alpha\beta\gamma-4(\alpha+\beta+\gamma)+16\geq 0$$

$$(\alpha-2)(\beta-2)(\gamma-2)=(\alpha-2)(\beta\gamma-2\beta-2\gamma+4)=\alpha\beta\gamma-2\sum\alpha\beta+4\sum\alpha-8\Rightarrow$$

$$\Rightarrow\alpha\beta\gamma-4(\alpha+\beta+\gamma)+16=\prod_{\alpha,\beta,\gamma}(\alpha-2)+2\sum\alpha\beta-8\sum\alpha+24=\prod_{\alpha,\beta,\gamma}(\alpha-2)+$$

$$+2\sum\alpha\beta-4\sum\alpha-4\sum\alpha+8+8+8=\prod_{\alpha,\beta,\gamma}(\alpha-2)+2\alpha(\beta-2)+2\beta(\gamma-2)+$$

$$+2\gamma(\alpha-2)-4(\alpha-2)-4(\beta-2)-4(\gamma-2)=\prod_{\alpha,\beta,\gamma}(\alpha-2)+(\alpha-2)(2\gamma-4)+$$

$$+(\beta-2)(2\alpha-4)+(\gamma-2)(2\beta-4)=(\alpha-2)(\beta-2)(\gamma-2)+2(\alpha-2)(\gamma-2)+$$

$$+2(\beta-2)(\alpha-2)+2(\gamma-2)(\beta-2)\geq 0 \quad \textit{true}$$

$$\alpha-2\geq 0,\beta-2\geq 0,\gamma-2\geq 0$$

SOLUTION 1.58

Solution by Ravi Prakash-New Delhi-India

Consider $2(a^6+b^6)-(a^3+b^3)^2=a^6+b^6-2a^3b^3=(a^3-b^3)^2\geq 0\Rightarrow$

$$\Rightarrow\sqrt{2}\sqrt{a^6+b^6}\geq a^3+b^3\Rightarrow\sqrt{2}\sum_{cyc}\sqrt{a^6+b^6}\geq 2(a^3+b^3+c^3) \quad \textit{(1)}$$

Also, $4(a^6+b^6)-(a^2+b^2)^3=3(a^6+b^6-a^4b^2-a^2b^4)=$

$$=3(a^4)(a^2-b^2)+3b^4(b^2-a^2)=3(a^2-b^2)^2(a^2+b^2)\geq 0\Rightarrow$$

$$\Rightarrow 4^{\frac{1}{3}}(a^6+b^6)^{\frac{1}{3}}\geq a^2+b^2\Rightarrow 4^{\frac{1}{3}}\sum_{cycl}(a^6+b^6)^{\frac{1}{3}}\geq 2(a^2+b^2+c^2) \quad \textit{(2)}$$

Adding (1), (2), we get: $2(a^2+b^2+c^2+a^3+b^3+c^3)\le$

$$\le\sqrt{2}\sum_{cycl}\sqrt{a^6+b^6}+4^{\frac{1}{3}}\sum_{cycl}(a^6+b^6)^{\frac{1}{3}}$$

SOLUTION 1.59

Solution by Marian Ursărescu-Romania

From Cauchy inequality we have:

$$\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{a^2}}+e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}\right)=\left(e^{a^2}+e^{b^2}+e^{c^2}\right)\left(e^{\frac{1}{b^2}}+e^{\frac{1}{c^2}}+e^{\frac{1}{a^2}}\right)\ge$$

$$\ge\left(e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}+e^{\frac{1}{2}\left(b^2+\frac{1}{c^2}\right)}+e^{\frac{1}{2}\left(c^2+\frac{1}{a^2}\right)}\right)^2\Rightarrow \textit{we must show:}$$

$$e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}+e^{\frac{1}{2}\left(b^2+\frac{1}{c^2}\right)}+e^{\frac{1}{2}\left(c^2+\frac{1}{a^2}\right)}\ge e^{\frac{a}{b}}+e^{\frac{b}{c}}+e^{\frac{c}{a}}\quad (1)$$

But $e^{\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)}\ge e^{\frac{a}{b}}\Leftrightarrow\frac{1}{2}\left(a^2+\frac{1}{b^2}\right)\ge\frac{a}{b}\Leftrightarrow a^2b^2+1\ge 2ab\Leftrightarrow(ab-1)^2\ge 0$

and similarly $(bc-1)^2\ge 0$,, $(ac-1)^2\ge 0\Rightarrow$ *(1) its true*

SOLUTION 1.60

Solution by Soumitra Mandal-Chandar Nagore-India

$abc=1$ *and* $x\in(0,1)$

$$\sum_{cyc}\frac{1}{(a^2+2ab+3)^x}\overset{AM\ge GM}{\le}\sum_{cyc}\frac{1}{(2a+2ab+2)^x}=\frac{1}{2^x}\sum_{cyc}\frac{1}{(a+ab+1)^x}\le$$

$$\le\frac{3}{2^x}\left(\frac{1}{3}\sum_{cyc}\frac{1}{a+ab+1}\right)^x\ \textit{[since } x\in(0,1)\textit{]}=\frac{3}{6^x}\left(\sum_{cyc}\frac{1}{a+ab+1}\right)^x$$

$$=\frac{3}{6^x}\left(\frac{1}{a+ab+abc}+\frac{1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x$$

$$=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+1}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{bc+1}{b+bc+abc}+\frac{1}{c+ca+1}\right)^x=$$

$$=\frac{3}{6^x}\left(\frac{ac(bc+1)}{1+c+ac}+\frac{1}{c+ca+1}\right)^x=\frac{3}{6^x}\left(\frac{c+ac+1}{c+ca+1}\right)^x=\frac{3}{6^x}\quad \textit{(proved)}$$

SOLUTION 1.61

Solution by Marian Ursărescu-Romania

We must show:

$$z^2\left(\frac{x^2+y^2}{x^4+y^4}\right)^2+x^2\left(\frac{y^2+z^2}{y^4+z^4}\right)^2+y^2\left(\frac{z^2+x^2}{z^4+x^4}\right)^2\le 1\quad (1)$$

$$\textit{But } x^4+y^4 \ge xy(x^2+y^2) \quad \textit{(2) because} \Leftrightarrow x^4-x^3y+y^4-xy^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)^2(x^2+xy+y^2) \ge 0 \textit{ true (2)} \Rightarrow \frac{x^2+y^2}{x^4+y^2} \le \frac{1}{xy} \quad (2)$$

$$\textit{From (1)+(2) we must show: } \frac{z^2}{x^2y^2}+\frac{x^2}{y^2z^2}+\frac{y^2}{x^2z^2} \le 1 \quad (3)$$

$$\textit{But } x^4+y^4+z^4 = x^2y^2z^2 \Leftrightarrow \frac{x^2}{y^2z^2}+\frac{y^2}{x^2z^2}+\frac{z^2}{x^2y^2} = 1 \Rightarrow \textit{(3) its true.}$$

SOLUTION 1.62

Solution by Boris Colakovic-Belgrade-Serbie

$$\textit{From Bernoulli's inequality } x^y > 1+y(x-1) \overset{AM-GM}{\ge} 2\sqrt{y(x-1)} > 2\sqrt{y}$$

$$\textit{Similarly } y^z > 2\sqrt{z};\ z^x > 2\sqrt{x};\ y^x > 2\sqrt{x};\ z^y > 2\sqrt{y};\ x^z > 2\sqrt{z}$$

$$\textit{Therefore } LHS > 4\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right) \overset{AM-GM}{\ge} 12(xyz)^{\frac{1}{6}} = 12\sqrt[4]{2} > 9$$

SOLUTION 1.63

Solution by Marian Ursărescu-Romania

$$\sum\left(\frac{y^3+z^3}{x^3}+\frac{3}{x^3}\right) \ge 3xyz \Leftrightarrow \sum\frac{y^3+z^3}{x^3}+3 \ge 3xyz \quad (1)$$

$$\textit{Let } x=\sqrt[3]{\frac{a+b+c}{a}},\ y=\sqrt[3]{\frac{a+b+c}{b}},\ z=\sqrt[3]{\frac{a+b+c}{c}} \quad (2)$$

$$\textit{From (1)+(2) we must show: } \sum\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}}+3 \ge \frac{3(a+b+c)}{\sqrt[3]{abc}} \Leftrightarrow \sum\frac{a(b+c)}{bc}+3 \ge \frac{3(a+b+c)}{\sqrt[3]{abc}} \quad (3)$$

$$\textit{But } \sqrt[3]{abc} \ge \frac{3abc}{ab+ac+bc} \Leftrightarrow \frac{1}{\sqrt[3]{abc}} \le \frac{ab+ac+bc}{3abc} \quad (4)$$

$$\textit{From (3)+(4) we must show: } \sum\frac{a(b+c)}{bc}+3 \ge \frac{(ab+ac+bc)(a+b+c)}{abc} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sum a^2(b+c)}{abc}+3 \ge \frac{(ab+bc+ac)(a+b+c)}{abc} \Leftrightarrow$$

$$\sum a^2(b+c)+3abc \ge (ab+bc+ac)(a+b+c) \quad (5)$$

$$\textit{But (5)} \Leftrightarrow a^2(b+c)+b^2(a+c)+c^2(a+b)+3abc \ge a^2b+ab^2+abc+abc+$$

$$+b^2c+bc^2+a^2c+abc+ac^2 \Leftrightarrow 0 \ge 0 \Leftrightarrow \textit{(5) its true.}$$

SOLUTION 1.64

Solution by Daniel Sitaru-Romania

$$a,b,c \in [0,3],$$

$$[0,3]X[0,3]X[0,3] \textit{ – convexe domain with vertex: } (0,0,0),(3,0,0),$$

$$(0,3,0),(0,0,3),(3,3,0),(3,0,3),(0,3,3),(3,3,3)$$

$$f:[0,3]X[0,3]X[0,3]\to\mathbb{R}, f(a,b,c)=a^4+b^4+c^4-abc$$

$$\begin{cases} f'_a=4a^3-bc, f''_{aa}=12a^2\ge 0 \\ f'_b=4b^3-ac, f''_{bb}=12b^2\ge 0 \\ f'_a=4c^3-ba, f''_{cc}=12c^2\ge 0 \end{cases}$$

***f* – convexe in each variable – max is attained in a vertex**

$$\max\{f(a,b,c)|a+b+c=3\}=\max\{f(3,0,0),f(0,3,0),f(0,0,3)\}=81$$

$$f(a,b,c)\le 81, a^4+b^4+c^4-abc\le 81$$

SOLUTION 1.65

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum\frac{bc^2(ab+1)}{a(b^2c^2+1)}=2\sum\left(\frac{b^2c^2}{b^2c^2+1}\right)\left(\frac{ab+1}{ab}\right)=2\sum\left\{\frac{(b^2c^2+1)-1}{b^2c^2+1}\right\}\left(1+\frac{1}{ab}\right)$$

$$=2\sum\left(1-\frac{1}{b^2c^2+1}\right)\left(1+\frac{1}{ab}\right)=2\sum\left\{1+\frac{1}{ab}-\frac{1}{b^2c^2+1}-\frac{1}{ab(b^2c^2+1)}\right\}$$

$$\underset{(1)}{\overset{A-G}{\ge}} 2\sum\left(1+\frac{1}{ab}-\frac{1}{2bc}-\frac{1}{2ab^2c}\right)=6+2\sum\frac{1}{ab}-\sum\frac{1}{ab}-\frac{1}{abc}\sum\frac{1}{a}$$

$$\textit{(1)}\Rightarrow LHS\ge\sum\frac{1}{a^2b^2}-\sum\frac{1}{ab}+6+\sum\frac{1}{ab}-\frac{1}{abc}\sum\frac{1}{a}\overset{?}{\ge}6$$

$$\Leftrightarrow\sum\frac{1}{a^2b^2}\overset{?}{\ge}\frac{1}{abc}\sum\frac{1}{a}$$

$$\Leftrightarrow\sum x^2y^2\overset{?}{\ge}xyz\left(\sum x\right)\left(x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)$$

$$\Leftrightarrow\sum u^2\overset{?}{\ge}\sum uv\quad(xy=u,yz=v,zx=w)\to\textit{true (Proved)}$$

SOLUTION 1.66

Solution by Tran Hong-Vietnam

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=3\sqrt{xyz}\Leftrightarrow\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=3$$

$$\textit{Let } u=\frac{1}{\sqrt{x}}; v=\frac{1}{\sqrt{y}}; w=\frac{1}{\sqrt{z}}\Rightarrow u+v+w=3$$

$$a(1+a^2)^2\le(1+a^3)^2\quad(*)\Leftrightarrow a(1+a^2)^2-(1+a^3)^2\le 0$$

$$\Leftrightarrow a^5+a-a^6-1\le 0\Leftrightarrow(a-1)^2(a^4+a^3+a^2+1)\ge 0\ \textit{(true for all } a>0\textit{)}$$

Using (*) we have:LHS $\le uv+vw+wu$

We need to prove: $uv + vw + wu \le 3 = \frac{(u+v+w)^2}{3}$

$$\Leftrightarrow (u-v)^2 + (v-w)^2 + (w-u)^2 \ge 0. \text{ True.}$$

Equality $\Leftrightarrow u = v = w = 1 \Leftrightarrow x = y = z = 1.$

SOLUTION 1.67

Solution by Tran Hong-Vietnam

$$\left(\sqrt{x^2+y^2} + \sqrt{y^2+z^2} + \sqrt{z^2+x^2}\right)^2 \ge 3\sum\left(\sqrt{x^2+y^2}\right)\left(\sqrt{y^2+z^2}\right) \ge$$

$$\ge \frac{3}{2}\sum(x+y)(y+z) = \frac{3}{2}\left[\sum x^2 + 3\sum xy\right] \ge \frac{3}{2}\left[\sum xy + 3\sum xy\right] = 6\sum xy$$

We need to prove: $6\sum xy \ge 2\sqrt{3\sum x^2y^2} \Leftrightarrow 3(\sum xy)^2 \ge \sum x^2y^2 \Leftrightarrow$

$$\Leftrightarrow 3(xy+yz+zx)^2 \ge x^2y^2 + y^2z^2 + z^2x^2 \Leftrightarrow 2\sum x^2y^2 + 6xyz(x+y+z) \ge 0$$

True because $x, y, z \ge 0$. *Equality* $\Leftrightarrow x = y = z = 0.$

SOLUTION 1.68

Solution by Tran Hong-Vietnam

Let $x = a^3, y = b^3, c = z^3 \Rightarrow \begin{cases} a = \sqrt[3]{x}, b = \sqrt[3]{y}, c = \sqrt[3]{z} \\ x + y + z = 3 \end{cases} (x, y, z > 0)$

LHS$= \sum\left(\frac{x^{\frac{2}{3}}+1}{\frac{1}{x^{\frac{1}{3}}+1}}\right)^3$. *Let* $f(t) = \left(\frac{t^{\frac{2}{3}}+1}{\frac{1}{t^{\frac{1}{3}}+1}}\right)^3, 0 < t < 3$

$$\Rightarrow f''(t) = \frac{2\left(\sqrt[3]{t}-1\right)^2\left(t^{\frac{2}{3}}+1\right)\left(t^{\frac{2}{3}}+4\sqrt[3]{t}+1\right)}{3\left(\sqrt[3]{t}+1\right)^5 \cdot t^{\frac{5}{3}}} \ge 0, \forall t \in (0,3)$$

Using Jensen's inequality:

LHS$= f(x) + f(y) + f(z) \ge 3f\left(\frac{x+y+z}{3}\right) = 3f(1) = 3.$

$\Rightarrow$ *Proved. Equality* $\Leftrightarrow x = y = z = 1 \Leftrightarrow a = b = c = 1.$

SOLUTION 1.69

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$ *and* $x + y + z \le 1$, *we have* $xyz \le \frac{1}{27}$ *and* $xyz \le 1 \Rightarrow \left(xyz - \frac{1}{27}\right) \le 0$ *and*

$$(xyz-1) \le 0 \Rightarrow \left(xyz - \frac{1}{27}\right)(xyz-1) \ge 0 \Rightarrow (27xyz-1)(xyz-1) \ge 0 \Rightarrow$$

$$\Rightarrow 27xyz^2 - 28xyz + 1 \geq 0 \Rightarrow 27xyz + \frac{1}{xyz} \geq 28 \Rightarrow (x+y+z)^3 + \frac{1}{xyz} \geq 28$$

Therefore, it is to be true.

SOLUTION 1.70

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } a \geq 4, \left(1+\frac{1}{a}\right)^a < 3 \leq a-1$$

$$\Rightarrow \left(1+\frac{1}{a}\right)^a \left(1+\frac{1}{a}\right) < (a-1)\left(1+\frac{1}{a}\right) \Rightarrow \left(1+\frac{1}{a}\right)^{a+1} < a - \frac{1}{a} < a$$

$$\Rightarrow (1+a)^{a+1} < a^{a+2} \Rightarrow a^{\frac{1}{(a+1)}} > (1+a)^{\frac{1}{(a+2)}}$$

Similarly, for b and c. Multiplying the inequalities, we get:

$$a^{\frac{1}{(a+1)}} b^{\frac{1}{(b+1)}} c^{\frac{1}{(c+1)}} > (1+a)^{\frac{1}{(a+2)}} (1_b)^{\frac{1}{(b+2)}} \times (1+c)^{\frac{1}{(c+2)}}$$

SOLUTION 1.71

Solution by Amit Dutta-Jamshedpur-India

Using Cauchy's Schwarz inequality:

$$(1^2+1^2+1^2+1^2)(25-x^2+25-y^2+25-z^2+25-t^2) \geq$$

$$\geq \left(\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right)^2$$

$$4\left(100-(x^2+y^2+z^2+t^2)\right) \geq \left[\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right]^2 \quad \textit{(1)}$$

Using Cauchy's Schwarz inequality:

$$(1^2+1^2+1^2+1^2)(x^2+y^2+z^2+t^2) \geq (x+y+z+t)^2$$

$$\because (x+y+z+t) = 0$$

$$\Rightarrow 4\left(\sum x^2\right) \geq 0 \Rightarrow \sum x^2 \geq 0 \Rightarrow x^2+y^2+z^2+t^2 \geq 0 \Rightarrow -(x^2+y^2+z^2+t^2) \leq 0$$

From (i):

$$\left(\sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2}\right) \leq \{4(100+0)\}^{\frac{1}{2}} \leq (400)^{\frac{1}{2}} \leq 20$$

$$\therefore \sqrt{25-x^2}+\sqrt{25-y^2}+\sqrt{25-z^2}+\sqrt{25-t^2} \leq 20$$

SOLUTION 1.72

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\left(\frac{1}{a+1}-\frac{1}{a+4}\right)-3\sum\left(\frac{1}{a+2}-\frac{1}{a+3}\right)=$$

$$=3\sum\frac{1}{(a+1)(a+4)}-3\sum\frac{1}{(a+2)(a+3)}=$$

$$=3\sum\left(\frac{1}{(a+1)(a+4)}-\frac{1}{(a+2)(a+3)}\right)=\sum\frac{6}{(a+1)(a+2)(a+3)(a+4)}$$

$$\sum\frac{6}{(a+1)(a+2)(a+3)(a+4)}<\frac{\sqrt{6}}{8}\sum\frac{\sqrt{a}}{a^2}$$

$\frac{6}{(a+1)(a+2)(a+3)(a+4)}<\frac{\sqrt{6}}{8}\cdot\frac{\sqrt{a}}{a^2}$ ***(ASSURE)***

$a>0\Rightarrow a+4>1$ ***(TRUE)***

$$4\sqrt{6}a(a+4)>4\sqrt{6}a$$

$$4\sqrt{6}a<4\sqrt{6}a(a+4)=2\sqrt{2a}\cdot 2\sqrt{3a}(a+4)\overset{M_g\le M_a}{\le}(a+2)(a+3)(a+4)$$

$$4\sqrt{6}a<(a+2)(a+3)(a+4)|\cdot 2\sqrt{a}$$

$$8\sqrt{6}a\sqrt{a}<2\sqrt{a}(a+2)(a+3)(a+4)\overset{M_g\le M_a}{\le}(a+1)(a+2)(a+3)(a+4)$$

$$8\sqrt{6}a\sqrt{a}<(a+1)(a+2)(a+3)(a+4)|\cdot\sqrt{6a}$$

$$8\cdot 6a^2<\sqrt{6}\cdot\sqrt{a}(a+1)(a+2)(a+3)(a+4)$$

$$\frac{6}{(a+2)(a+3)(a+4)}<\frac{\sqrt{6}}{8}\cdot\frac{\sqrt{a}}{a^2}$$

SOLUTION 1.73

Solution by Nguyen Van Nho-Nghe An-Vietnam

Hope: $\left(\frac{2}{1-x^2}\right)^6\ge 3^9x^6\leftrightarrow(1-x^2)^6(2x^2)^3\le\left(\frac{2}{3}\right)^9\to$ ***(*)***

Use AM-GM: LHS (*) $=(1-x^2)^6(2x^2)^3\le\left(\frac{6(1-x^2)+3(2x^2)}{9}\right)^9=\left(\frac{2}{3}\right)^9\to$ ***(*) is true***

Similarly: $\left(\frac{2}{1-y^2}\right)^6\ge 3^9y^6$ ***and*** $\left(\frac{2}{1-z^2}\right)^6\ge 3^9z^6$

So: LHS$\ge 3^9\sum x^6=3^9\frac{1}{9}=3^7$ ***(done)***

SOLUTION 1.74

Solution by Marian Ursărescu-Romania

We use this inequality: $\frac{a^4+1}{a^6+1}\le\frac{1}{a}\overset{\forall a>0}{\Leftrightarrow}a^5+a\le a^6+1\Leftrightarrow a^6-a^5-a+1\ge 0\Leftrightarrow$

$$a^5(a-1)-(a-1)\geq 0 \Leftrightarrow (a-1)\left(a^5-1\right)\geq 0 \Leftrightarrow$$

$$(a-1)^2(a^4+a^3+a^2+a+1)\geq 0, \forall a\geq 0 \text{ true}$$

(with equality for $a=1$) $\Rightarrow \frac{(x+y)^4+1}{(x+y)^6+1}\leq \frac{1}{x+y}\Rightarrow$

$$\sum \frac{(x+y)^4+1}{(x+y)^6+1}\leq \sum \frac{1}{x+y} \quad (1)$$

But $\frac{1}{x+y}\leq \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right), \forall x,y>0$ *(2) because*

$\Leftrightarrow \frac{1}{x+y}\leq \frac{x+y}{4xy}\Leftrightarrow (x+y)^2\geq 4xy \Leftrightarrow (x-y)^2\geq 0$ *with equality* $x=y$.

From (1)+(2)$\Rightarrow \sum \frac{(x+y)^4+1}{(x+y)^6+1}\leq \frac{1}{2}\sum \frac{1}{x}$, *with equality for* $x=y=z=\frac{1}{2}$.

SOLUTION 1.75

Solution by Nguyen Tan Path-Vietnam

Using Cauchy-Schwarz's Inequality we have:

$$a^6+b^6+c^6\geq \frac{(a^3+b^3+c^3)^2}{3}$$

$$(3-a)^6+(3-b)^2+(3-c)^6\geq \frac{[(3-a)^3+(3-b)^3+(3-c)^3]^2}{3}$$

Using Holder's Inequality, we have:

$$a^3+b^3+c^3\geq \frac{(a+b+c)^3}{9}=3$$

$$(3-a)^3+(3-b)^3+(3-c)^3\geq \frac{(3-a-b-c)^3}{9}=24$$

So, $a^6+b^6+c^6\geq \frac{3^2}{3}=3$

$$(3-a)^6+(3-b)^6+(3-c)^6\geq \frac{24^2}{3}=192$$

$$\Rightarrow a^6+b^6+c^6+\frac{1}{32}((3-a)^6+(3-b)^6+(3-c)^6)\geq 3+\frac{192}{39}=9$$

SOLUTION 1.76

Solution by Marian Ursărescu-Romania

Inequality $\Leftrightarrow \left(\frac{2a}{a+b}\right)^2+ab+\left(\frac{a+b}{2}\right)^2-\frac{2\cdot 2ab}{a+b}\sqrt{ab}-2\sqrt{ab}\frac{a+b}{2}$

$$+2\cdot\frac{2ab}{a+b}\cdot\frac{a+b}{2}+ab\leq \left(\frac{2ab}{a+b}\right)^2+\left(\frac{a+b}{2}\right)^2 \Leftrightarrow$$

$$2ab+2ab\le\frac{4ab\sqrt{ab}}{a+b}+\sqrt{ab}(a+b)\Leftrightarrow 4ab\le\frac{4ab\sqrt{ab}}{a+b}+\sqrt{ab}(a+b)\Leftrightarrow$$

$$4\sqrt{ab}\le\frac{4ab}{a+b}+a+b\quad (1)$$

$$But\ \frac{4ab}{a+b}+a+b\ge 2\sqrt{\frac{4ab}{a+b}(a+b)}\Rightarrow\frac{4ab}{a+b}+a+b\ge 4\sqrt{ab}\Rightarrow (1)\ \text{it's true.}$$

SOLUTION 1.77

Solution by Tran Hong-Vietnam

Inequality $\Leftrightarrow$

$$\{(a+b)(b+c)(c+a)\}^4\ge 2\cdot 4^4(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$\Leftrightarrow (a+b)^4(b+c)^4(c+a)^2\ge 512(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a^2)\quad (*)$$

$$(a+b)^4\ge 8ab(a^2+b^2)\ \ (1)\Leftrightarrow (a-b)^4\ge 0\ \ (\text{true})$$

$$\text{Same:}(b+c)^4\ge 8bc(b^2+c^2)\quad (2);\ (c+a)^4\ge 8ca\ge (c^2+b^2)\ \ (3);$$

From (1), (2), (3) we have:

$$LHS_{(*)}\ge 8^3(abc)^2(a^2+b^2)(b^2+c^2)(c^2+a)^2=RHS_{(*)}$$

SOLUTION 1.78

Solution by Soumava Chakraborty-Kolkata-India

$$8\left(\sum x\right)\sqrt{\sum x}\le 3\sqrt{3}\prod(x+y)\Leftrightarrow 27\left(\prod(x+y)\right)^2\overset{(1)}{\ge} 64\left(\sum x\right)^3$$

$$\because\prod(x+y)\ge\frac{8}{9}\left(\sum x\right)\left(\sum xy\right)\therefore 27\left(\prod(x+y)\right)^2\ge\frac{27\cdot 64}{81}\left(\sum x\right)^2\left(\sum xy\right)^2$$

$$\ge\frac{64}{3}(\textstyle\sum x)^2\left(3xyz(\sum x)\right)=64(\sum x)^3(\because\prod x=1)\Rightarrow (1)\ \text{is true (Proved)}$$

SOLUTION 1.79

Solution by Soumava Chakraborty-Kolkata-India

$$8\left(\sum x\right)\sqrt{\sum x}\le 3\sqrt{3}\prod(x+y)\Leftrightarrow 27\left(\prod(x+y)\right)^2\overset{(1)}{\ge} 64\left(\sum x\right)^3$$

$$\because\prod(x+y)\ge\frac{8}{9}\left(\sum x\right)\left(\sum xy\right)\therefore 27\left(\prod(x+y)\right)^2\ge\frac{27\cdot 64}{81}\left(\sum x\right)^2\left(\sum xy\right)^2$$

$$\ge\frac{64}{3}(\textstyle\sum x)^2\left(3xyz(\sum x)\right)=64(\sum x)^3(\because\prod x=1)\Rightarrow (1)\ \text{is true (Proved)}$$

SOLUTION 1.80

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2=\sum\left(\frac{x^{16}}{y^{16}}+\frac{y^{16}}{x^{16}}+2\right)\overset{(a)}{=}\sum\frac{x^{16}}{y^{16}}+\sum\frac{y^{16}}{x^{16}}+6$$

Now, $\sum\frac{x^{16}}{y^{16}}=\frac{x^{16}}{y^{16}}+\frac{y^{16}}{z^{16}}+\frac{z^{16}}{x^{16}}\geq\frac{x^8}{z^8}+\frac{y^8}{x^8}+\frac{z^8}{y^8}(\because a^2+b^2+c^2\geq ab+bc+ca)$

$$\geq\frac{y^4}{z^4}+\frac{z^4}{x^4}+\frac{x^4}{y^4}\left(\because\sum a^2\geq\sum ab\right)\overset{(1)}{\geq}\frac{y^2}{x^2}+\frac{z^2}{y^2}+\frac{x^2}{z^2}\left(\because\sum a^2\geq\sum ab\right)=\sum\frac{y^2}{x^2}$$

Similarly, $\sum\frac{y^{16}}{x^{16}}\overset{(2)}{\geq}\sum\frac{x^2}{y^2}$ $\quad\therefore$ *(1)+(2)*$\Rightarrow\sum\frac{x^{16}}{y^{16}}+\sum\frac{y^{16}}{x^{16}}+6$

$$\overset{(b)}{\geq}\sum\frac{x^2}{y^2}+\sum\frac{y^2}{x^2}+6=\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+2\right)=\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

(a), (b)$\Rightarrow\sum\left(\frac{x^8}{y^8}+\frac{y^8}{x^8}\right)^2\overset{(i)}{\geq}\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$

Again, $\sum\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}\right)^2=\sum\frac{x^8}{y^8}+\sum\frac{y^8}{x^8}+6\overset{(ii)}{\geq}\sum\frac{x^2}{y^2}+\sum\frac{y^2}{x^2}+6$ *(proceeding in previous fashion)*$=\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$

Also, $\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2=\sum\frac{x^4}{y^4}+\sum\frac{y^4}{x^4}+6\geq\sum\frac{y^2}{x^2}+\sum\frac{x^2}{y^2}+6$

$$\Rightarrow\sum\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)^2\overset{(iii)}{\geq}\sum\left(\frac{x}{y}+\frac{y}{x}\right)^2$$

(i).(ii).(iii)$\Rightarrow$ given inequality is true (proved)

SOLUTION 1.81

Solution by Marian Ursărescu-Romania

From Cauchy's inequality we have: $2(x+y)\geq\left(\sqrt{x}+\sqrt{y}\right)^2\Rightarrow$

$\frac{x+y}{\left(\sqrt{x}+\sqrt{y}\right)^2}\geq\frac{1}{2}$ *(1). From (1) inequality becomes:*

$$\frac{\sqrt{z}(x+y)}{2}+\frac{\sqrt{x}(y+z)}{2}+\frac{\sqrt{y}(z+x)}{2}\geq 9\quad (2)$$

But $x+y\geq 2\sqrt{xy}$ *(3). Form (2)+(3) we must show:*

$$\left.\begin{matrix}\sqrt{xyz}+\sqrt{xyz}+\sqrt{xyz}\geq 9\\ \text{But } xyz=9\end{matrix}\right\}\Rightarrow \text{it's true.}$$

SOLUTION 1.82

Solution by Marian Ursărescu – Romania

$$\textit{We must show: } (a+1)(b+1)(c+1)(d+1) \geq 16abcd \quad (1)$$

$$\textit{Let } a = \frac{4x}{x+y+z+t}, b = \frac{4y}{x+y+z+t}, c = \frac{4z}{x+y+z+t}, d = \frac{4t}{x+y+z+t}$$

$$(1) \qquad \Leftrightarrow \prod\left(\frac{4x}{x+y+z+t}+1\right) \geq 16 \cdot 4^4 \frac{xyzt}{(x+y+z+t)^4} \Leftrightarrow$$

$$\prod(5x+y+z+t) \geq 4^6 xyzt \quad (2)$$

$$\text{But } \left.\begin{array}{l} 5x+y+z+t \geq 8\sqrt[8]{x^5yzt} \\ x+5y+z+t \geq 8\sqrt[8]{xy^5zt} \\ x+y+5z+t \geq 8\sqrt[8]{xyz^5t} \\ x+y+z+5t \geq 8\sqrt[8]{xyzt^5} \end{array}\right\} \Rightarrow$$

$$\prod(5x+y+z+t) \geq 8^4 xyzt \Rightarrow \textit{(2) it's true.}$$

SOLUTION 1.83

Solution by Hoang Le Nhat Tung-Hanoi-Vietnam

By Minkowski inequality:

$$\sqrt{a^8+\frac{1}{a^2}+\frac{1}{a}}+\sqrt{b^8+\frac{1}{b^2}+\frac{1}{b}}+\sqrt{c^8+\frac{1}{c^2}+\frac{1}{c}} \geq$$

$$\geq \sqrt{(a^4+b^4+c^4)^2+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2+\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^2} \quad (1)$$

$$a^4+b^4+c^4 \geq \frac{(a+b+c)^4}{27} = \frac{3^4}{27} = 3$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq \frac{81}{(a+b+c)^2} = 9$$

$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}} \geq \frac{9}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \geq \frac{9}{\sqrt{3(a+b+c)}} = \frac{9}{\sqrt{3\cdot 3}} = 3$$

$$\Rightarrow LHS \geq \sqrt{3^2+9+3^2} = 3\sqrt{3} \leftrightarrow a=b=c=1.$$

SOLUTION 1.84

Solution by Amit Dutta-Jamshedpur-India

$$\textit{Let } F(x) = e^{-\frac{2x}{\sqrt{3}}}(x^2+x+1)$$

$$F'(x) = e^{-\frac{2x}{\sqrt{3}}}(2x+1)+(x^2+x+1)e^{-\frac{2x}{\sqrt{3}}}\left(-\frac{2}{\sqrt{3}}\right) = e^{-\frac{2x}{\sqrt{3}}}\{(2x+1)-2\sqrt{3}\,(x^2+x+1)\}$$

$$= -e^{-\frac{2x}{\sqrt{3}}}\{2x^2+(2-2\sqrt{3})x+(2-\sqrt{3})\} = -\frac{2}{\sqrt{3}}e^{-\frac{2x}{\sqrt{3}}}\left\{x^2+(1-\sqrt{3})x+\left(\frac{2-\sqrt{3}}{2}\right)\right\}$$

$$=-\frac{2}{\sqrt{3}}e^{-\frac{2x}{\sqrt{3}}}\left\{x-\left(\frac{\sqrt{3}-1}{2}\right)\right\}^2\le 0\Rightarrow F'(x)\le 0$$

$$\Rightarrow F(x)\ \textit{is a decreasing function}\ \ x\ge 0\Rightarrow F(x)\le F(0)$$

$$\Rightarrow e^{-\frac{2x}{\sqrt{3}}}(x^2+x+1)\le 1\Rightarrow (x^2+x+1)\le e^{\frac{2x}{\sqrt{3}}}$$

Putting $x=a,b,c$ ***and multiplying, then, we get:***

$$(a^2+a+1)(b^2+b+1)(c^2+c+1)\le e^{\frac{2}{\sqrt{3}}(a+b+c)}$$

$$or\ \{(a^2+a+1)(b^2+b+1)(c^2+c+1)\}^3\le e^{2\sqrt{3}(a+b+c)}$$

SOLUTION 1.85

Solution by Soumitra Mandal - Chandar Nagore – India

Let $a+b+c=1, ab+bc+ca=\frac{1-q^2}{3}$ ***and*** $abc=r$

$$\sum_{cyc}ab(a^2+b^2)=\frac{(1+2q^2)(1-q^2)}{9}-r,\sum_{cyc}a^4=\frac{-1+8q^2+2q^4}{9}+4r$$

and

$$\sum_{cyc}a^2b^2=\frac{(1-q^2)^2}{9}-2r$$

VQBC inequality relation, $r\le\frac{(1+2q)(1-q)^2}{27}$

$$\therefore 9\left(\sum_{cyc}a^2\right)^2\ge 8\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^3\right)\Rightarrow\sum_{cyc}a^4+18\left(\sum_{cyc}a^2b^2\right)\ge 8\sum_{cyc}ab(a^2+b^2)$$

$$\Leftrightarrow\frac{-1+8q^2+2q^4}{9}+4r+18\left(\frac{1-2q^2+q^4}{9}-2r\right)\ge 8\left(\frac{1+q^2-2q^4}{9}-r\right)$$

$$\Leftrightarrow\frac{-1+8q^2+2q^4+18-36q^2+18q^4-8-8q^2+16q^4}{9}\ge 24r$$

$$\Leftrightarrow\frac{36q^4-36q^2+9}{9}\ge 24\Leftrightarrow 4q^4-4q^2+1\ge\frac{8}{9}(1-3q^2+2q^3)$$

$$\Leftrightarrow 1-12q^2+36q^4-16q^3\ge 0.\ \textit{Let}\ f(q)=36q^4-16q^3-12q^2+1$$

for all $1>q\ge 0$

$$f'(q)=144q^3-48q^2-24q\Rightarrow f''(q)=24(18q^2-4q-1)<0$$

for all $1>q\ge 0$

$$\therefore f \text{ is concave. Hence, } f(q) \geq f(1) = 9 > 0. \therefore 1 - 12q^2 + 36q^4 - 16q^3 > 0$$

$$\therefore 9\left(\sum_{cyc} a^2\right)^2 \geq 8\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^3\right)$$

SOLUTION 1.86

Solution by Le Khansy Sy-Long An-Vietnam

1) Using the Cauchy – Schwarz inequality, we have

$$\sum_{cyc} \frac{ab(1+k)^2}{b+2kc+k^2a} \leq \sum_{cyc}\left(\frac{ab}{b+kc} + \frac{abk}{c+ka}\right) =$$

$$= \sum_{cyc}\left(\frac{ab}{b+kc} + \frac{ack}{b+kc}\right) = \sum_{cyc}\left[\frac{a(b+kc)}{b+kc}\right] = a+b+c$$

Or

$$\frac{ab}{b+2kc+k^2a} + \frac{bc}{c+2ka+k^2b} + \frac{ca}{a+2kb+k^2c} \leq \frac{a+b+c}{(1+k)^2}.$$

The equality holds for $a = b = c$, and for $a = 0$ and $c = kb$ (or any cyc permuation)

2) Case 1 $4ab + c^2$ We have a previous case.

Case 2 $4ab > c^2$

Using the identity

$$\frac{xy}{ax+by+cz} = \frac{4bxy}{(4ab-c^2)x + c(cx+2bz) + 2b(cz+2by)}$$

Using the Cauchy – Schwarz inequality gives

$$\frac{xy}{ax+by+cz} \leq \frac{4bxy}{(4ab-c^2+c^2+2bc+2bc+4b^2)^2}\left[\frac{(4ab-c^2)^2}{(4ab-c^2)x} + \frac{(c^2+2bc)^2}{c(cx+2bz)} + \frac{(2bc+4b^2)^2}{2b(cz+2by)}\right]$$

$$= \frac{1}{4b(a+b+c)^2}\left[y(4ab-c^2) + (c+2b)^2\left(\frac{cxy}{cx+2bz} + \frac{2bxy}{cz+2by}\right)\right],$$

hence

$$\sum_{cyc} \frac{xy}{ax+by+cz} \leq \sum_{cyc}\left\{\frac{1}{4b(a+b+c)^2}\left[y(4ab-c^2) + (c+2b)^2\left(\frac{cxy}{cx+2bz} + \frac{2bxy}{cz+2by}\right)\right]\right\}$$

$$= \sum_{cyc} \left\{ \frac{1}{4b(a+b+c)^2} \left[y(4ab - c^2) + (c + 2b^2) \left(\frac{cxy}{cx + 2bz} + \frac{2byz}{cx + 2bz} \right) \right] \right\}$$

$$= \sum_{cyc} \frac{y}{a+b+c} = \frac{x+y+z}{a+b+c}$$

SOLUTION 1.87

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{c+a}{c+b} = \frac{c \cdot \left(1 + \frac{a}{c}\right)}{c \cdot \left(1 + \frac{b}{c}\right)} = \frac{1 + \frac{a}{c}}{1 + \frac{b}{c}}$$

$$\frac{a}{b} = x; \frac{b}{c} = y; \frac{c}{a} = z \Rightarrow \begin{matrix} xyz = 1 \\ x + y + z \geq 3 \end{matrix}$$

$$\frac{b}{a} = \frac{1}{x}; \frac{c}{b} = \frac{1}{y}; \frac{a}{c} = \frac{1}{z}$$

$$\frac{b}{a} = \frac{1}{x} = \frac{xyz}{x} = yz; \frac{c}{b} = zx; \frac{a}{c} = xy$$

$$x + y + z \geq \frac{1 + xy}{1 + y} + \frac{1 + yz}{1 + z} + \frac{1 + zx}{1 + x}$$

$$\underbrace{(x+y+z)(1+y)(1+z)(1+x)}_{LHS} \geq$$

$$\geq \underbrace{(1+xy)(1+z)(1+x) + (1+yz)(1+x)(1+y) + (1+zx)(1+y)(1+z)}_{RHS}$$

1) $LHS = 2 \cdot (x+y+z) + (x+y+z)^2 + (x+y+z)(xy+yz+zx)$

2) $RHS \Rightarrow \left. \begin{matrix} (1+xy)(1+z)(1+z) = 2 + z + 2x + xz + yx + x^2y \\ (1+yz)(1+x)(1+y) = 2 + x + 2y + xy + yz + y^2z \\ (1+zx)(1+y)(1+z) = 2 + y + 2z + zx + zy + z^2x \end{matrix} \right\}$

$$xyz = 1$$

$$RHS = 6 + 3(x+y+z) + 2(xy+yz+zx) + x^2y + y^2z + z^2x$$

$$LHS = 2(x+y+z) + x^2 + y^2 + z^2 + 2(xy+yz+zx) + 3xyz +$$

$$+(x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2) \geq$$

$$\geq 6 + 3(x+y+z) + 2(xy+yz+zx) + (x^2y + y^2z + z^2x)$$

$$x^2 + y^2 + z^2 + (xy^2 + yz^2 + zx^2) \geq 6 - 3xyz + (x+y+z)$$

$$x^2 + y^2 + z^2 + (xy^2 + yz^2 + zx^2) \geq 3 + (x+y+z)$$

a) $xy^2 + yz^2 + zx^2 \underset{Cauchy}{\geq} 3 \cdot \sqrt[3]{(xyz)^3} = 3xyz = 3$

$$b)\ x^2+y^2+z^2 \underset{Chebyshev}{\geq} \frac{1}{3}\cdot(x+y+z)\cdot(x+y+z) \overset{Cauchy}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\sqrt[3]{xyz}\cdot(x+y+z)=x+y+z$$

$$x^2+y^2+z^2+(xy^2+yz^2+zx^2)\geq x+y+z+3$$

SOLUTION 1.88

Solution by Soumitra Mandal-Chandar Nagore-India

Let $a,b,c\geq 0$ ***then***

$$27\prod_{cyc}(a^2+ab+b^2)\geq\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6$$

We know, $a^2+ab+b^2=\frac{3}{4}(a+b)^2+\frac{(a-b)^2}{4}\geq\frac{3}{4}(a+b)^2$

Similarly, $b^2+bc+c^2\geq\frac{3}{4}(b+c)^2$ ***and*** $c^2+ca+a^2\geq\frac{3}{4}(c+a)^2$

$$\therefore 27\prod_{cyc}(a^2+ab+b^2)\geq 27\cdot\left(\frac{3}{4}\right)^3\prod_{cyc}(a+b)^2$$

$$\geq 27\cdot\left(\frac{3}{4}\right)^3\cdot\frac{64}{81}(a+b+c)^2(ab+bc+ca)^2\left[\therefore 9\prod_{cyc}(a+b)\geq 8\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\right]$$

$$\geq 27(ab+bc+ca)^3\ \textit{[since, } (a+b+c)^2\geq 3(ab+bc+ca)\textit{]}$$

$$\geq\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^6\left[\because\frac{ab+bc+ca}{3}\geq\left(\frac{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}{3}\right)^2\right]$$

SOLUTION 1.89

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{a^2+b^2+c^2}{2}\quad (1)$$

**** Since inequality Buniakovski we have:***

$$\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}=\frac{\left(\frac{a^2}{b^2}\right)^2}{\sqrt{2c(a^3+1)}}+\frac{\left(\frac{b^2}{c^2}\right)^2}{\sqrt{2a(b^3+1)}}+\frac{\left(\frac{c^2}{a^2}\right)^2}{\sqrt{2b(c^3+1)}}$$

$$\geq\frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2}{\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}}\quad (2)$$

- Other, since AM-GM for 3 positive real numbers:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}=\frac{\frac{a^2}{b^2}+\frac{a^2}{b^2}+\frac{b^2}{c^2}}{3}+\frac{\frac{b^2}{c^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}{3}+\frac{\frac{c^2}{a^2}+\frac{c^2}{a^2}+\frac{a^2}{b^2}}{3}\geq$$

$$\geq\frac{3\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}}}{3}+\frac{3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}}}{3}+\frac{3\sqrt[3]{\frac{c^2}{a^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}}{3}=\sqrt[3]{\frac{a^4}{b^2c^2}}+\sqrt[3]{\frac{b^4}{c^2a^2}}+\sqrt[3]{\frac{c^4}{a^2b^2}}$$

$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}$. ***Because:*** $3=a+b+c\geq 3\cdot\sqrt[3]{abc}\Rightarrow$

$$\sqrt[3]{abc}\leq 1\Leftrightarrow\sqrt[3]{a^2b^2c^2}\leq 1$$

$$\Rightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq\frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}\geq\frac{a^2+b^2+c^2}{1}=a^2+b^2+c^2\Leftrightarrow\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geq a^2+b^2+c^2\quad\textit{(3)}$$

+ Since (2), (3):

$$\Rightarrow\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{(a^2+b^2+c^2)^2}{\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}}$$

- Since AM-GM for 2 positive real numbers

$$\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}$$

$$=\sqrt{(ca+c)(2a^2-2a+2)}+\sqrt{(ab+a)(2b^2-2b+2)}+\sqrt{(bc+b)(2c^2-2c+2)}\leq$$

$$\leq\frac{(ca+c)+(2a^2-2a+2)}{2}+\frac{(ab+a)+(2b^2-2b+2)}{2}+\frac{(bc+b)+(2c^2-2c+2)}{2}$$

$$\Rightarrow\sqrt{2c(a^3+1)}+\sqrt{2a(b^3+1)}+\sqrt{2b(c^3+1)}\leq a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3$$

(5)

- Since (4), (5):

$$\Rightarrow\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3}\quad\textit{(6)}$$

We will prove that: $\frac{(a^2+b^2+c^2)^2}{a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3}\geq\frac{a^2+b^2+c^2}{2}$ ***(7)***

$$\Leftrightarrow 2(a^2+b^2+c^2)\geq a^2+b^2+c^2+\frac{ab+bc+ca}{2}-\frac{a+b+c}{2}+3$$

$\Leftrightarrow a^2+b^2+c^2+\frac{a+b+c}{2}\geq\frac{ab+bc+ca}{2}+3\Leftrightarrow a^2+b^2+c^2+\frac{3}{2}\geq\frac{ab+bc+ca}{2}+3$ ***(Do***

$a+b+c=3$***)***

$$\Leftrightarrow a^2+b^2+c^2\geq\frac{ab+bc+ca}{2}+\frac{3}{2}\Leftrightarrow 2(a^2+b^2+c^2)\geq ab+bc+ca+3\quad\textit{(8)}$$

- Other, such that: $a+b+c=3$*. We have:*

$$2(a^2+b^2+c^2)+3=\frac{a^2+b^2}{2}+\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}+(a^2+1)+(b^2+1)+(c^2+1)\geq$$

$$\geq\frac{2ab}{2}+\frac{2bc}{2}+\frac{2ca}{2}+2\sqrt{a^2}+2\sqrt{b^2}+2\sqrt{c^2}=ab+bc+ca+2(a+b+c)$$

$$=ab+bc+ca+6$$

$\Rightarrow 2(a^2+b^2+c^2)\geq ab+bc+ca+3\Rightarrow$ *Inequality (8) True* $\Rightarrow$ *(7) True.*

- Since (6), (7): $\Rightarrow\frac{a^4}{b^4\sqrt{2c(a^3+1)}}+\frac{b^4}{c^4\sqrt{2a(b^3+1)}}+\frac{c^4}{a^4\sqrt{2b(c^3+1)}}\geq\frac{a^2+b^2+c^2}{2}$

$\Rightarrow$ *Inequality (1) true and we get the result:*

+ The occurs if: $\begin{cases} a,b,c>0; a+b+c=3 \\ \frac{\frac{a^2}{b^2}}{\sqrt{2c(a^3+1)}}=\frac{\frac{b^2}{c^2}}{\sqrt{2a(b^3+1)}}=\frac{\frac{c^2}{a^2}}{\sqrt{2b(c^3+1)}} \\ \frac{a^2}{b^2}=\frac{b^2}{c^2}=\frac{c^2}{a^2}; a=b=c=1 \\ ca+c=2a^2-2a+2 \\ ab+a=2b^2-2b+2 \\ bc+b=2c^2-2c+2 \end{cases}\Leftrightarrow a=b=c=1.$

SOLUTION 1.90

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum\frac{a^7+b^7}{ab(a+b)}\geq 3\cdot\sum a^2b^2-2$$

$$3\sum a^2b^2-2(a^4+b^4+c^4)\leq 3\cdot\sum a^2b^2-2\cdot\sum a^2b^2=\sum a^2b^2$$

$\sum\left(\frac{a^7+b^7}{ab(a+b)}-a^2b^2\right)\geq 0$ *(ASSURE)*

$$\sum\left(\frac{(a+b)\cdot(a^6-a^3b+\cdots+b^6)}{(a+b)\cdot ab}-a^2b^2\right)=$$

$$=\sum\left(\frac{a^5(a-b)+a^3b^2(a-b)-a^2b^3(a-b)-b^5(a-b)}{ab}\right)=$$

$$=\sum\frac{(a-b)\cdot(a^3+a^3b^2-a^2b^3-b^3)}{ab}=\sum\frac{(a-b)^2\cdot(a^4+a^3b+2a^2b^2+ab^3+b^4)}{ab}\geq 0$$

SOLUTION 1.91

Solution by Soumava Chakraborty-Kolkata-India

$$a^2-ab+b^2=\frac{3}{4}(a-b)^2+\frac{1}{4}(a+b)^2\overset{(1)}{\geq}\frac{(a+b)^2}{4}$$

Similarly, $b^2-bc+c^2\overset{(2)}{\geq}\frac{(b+c)^2}{4}$*, and* $c^2-ca+a^2\overset{(3)}{\geq}\frac{(c+a)^2}{4}$

(1) $\times$ *(2)* $\times$ *(3)* $\Rightarrow\prod(a^2-ab+b^2)\geq\frac{(a+b)^2(b+c)^2(c+a)^2}{64}$

$$=\frac{\big((a+b)(b+c)(c+a)\big)\{(a+b)(b+c)(c+a)\}}{64}$$

$$\overset{AM-GM}{\geq}\frac{(8abc)}{64}(a+b)(b+c)(c+a)=\frac{(a+b)(b+c)(c+a)abc}{8}$$

∴ it suffices to prove:

$$\frac{\{\prod(a+b)\}abc}{8}\geq\frac{abc}{7\sqrt{7}}\prod\sqrt{a^2+5ab+b^2}\Leftrightarrow\prod\left\{\frac{\sqrt{7}(a+b)}{2}\right\}\geq\prod\sqrt{a^2+5ab+b^2}\quad (a)$$

Now, $\frac{\sqrt{7}(a+b)}{2}\geq\sqrt{a^2+5ab+b^2}\Leftrightarrow 7(a^2+b^2+2ab)\geq 4(a^2+5ab+b^2)$

$\Leftrightarrow 3(a-b)^2\geq 0\rightarrow$ *true* $\Rightarrow\frac{\sqrt{7}(a+b)}{2}\overset{(4)}{\geq}\sqrt{a^2+5ab+b^2}$

Similarly, $\frac{\sqrt{7}(b+c)}{2}\overset{(5)}{\geq}\sqrt{b^2+5bc+c^2}$ *and* $\frac{\sqrt{7}(c+a)}{2}\overset{(6)}{\geq}\sqrt{c^2+5ca+a^2}$

(4) $\times$ *(5)* $\times$ *(6)* $\Rightarrow\prod\left\{\frac{\sqrt{7}(a+b)}{2}\right\}\geq\prod(a^2+5ab+b^2)\Rightarrow$ *(a) is true*

SOLUTION 1.92

Solution by Soumitra Mandal-Chandar Nagore-India

$$2\left(\sum_{cyc}\frac{a+b}{\left(a^3\sqrt{b}+b^3\sqrt{a}\right)^2}\right)\overset{Cauchy-Schwarz}{\geq}2\left(\sum_{cyc}\frac{a+b}{(a+b)(a^6+b^6)}\right)$$

$$=2\sum_{cyc}\frac{1}{a^6+b^6}\geq 2\frac{9}{\sum_{cyc}(a^6+b^6)}\left[\because\frac{1}{3}\left(\sum_{cyc}\frac{1}{x}\right)\geq\frac{3}{x+y+z}\right]=1\text{ equality at } a=b=c=\sqrt[6]{3}$$

SOLUTION 1.93

Solution by Anas Adlany-El Zemmara-Morocco

We have by AM-GM inequality

$$\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\leq\sum\frac{(a+4b)+(2a+3b)+(3a+2b)+(4a+b)}{4}=5,$$

Also, by AM-GM inequality we have

$$\sum\sqrt[4]{(a+4b)(2a+3b)(3a+2b)(4a+b)}\geq 5\sum\sqrt[4]{\sqrt[5]{ab^4a^2b^2a^4b}}=5\sum\sqrt{ab}$$

SOLUTION 1.94

Solution by Le Khanh Sy-Long An-Vietnam

The inequality becomes as follows.

$$\Leftrightarrow (a+b+c)^2\sum_{cyc}\frac{a}{b}+(k-1)^2\sum_{cyc}ab \ge 2(k+3)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left[\frac{a^3}{b}+\frac{a^2b}{c}+ac+2a^2+\frac{2ab^2}{c}+2cb\right]+(k-1)^2\sum_{cyc}ab \ge 2(k+3)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left[\frac{a^3}{b}+\frac{a^2b}{c}+\frac{2ab^2}{c}\right]+(k^2-2k+4)\sum_{cyc}ab \ge 2(k+2)\sum_{cyc}a^2$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^3}{b}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)+\sum_{cyc}\left[\frac{4a^2c}{b}-(4k-k^2)ab\right] \ge 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{b^3}{c}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)+\sum_{cyc}\left(\frac{4a^2c}{b}-4kac+k^2cb\right) \ge 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

Using the AM-GM inequality, we have.

$$\sum_{cyc}\left[\frac{b(a-b)^2}{c}+\frac{c(2a-kb)^2}{b}\right] \ge 2\sum_{cyc}[(a-b)(2a-kb)] = 2(k+2)\left[\sum_{cyc}a^2-\sum_{cyc}ab\right]$$

SOLUTION 1.95

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{a^5}{(2a+3b)^3}+\sum_{cyc}\frac{a^5}{(2a+3c)^3} \ge \frac{2}{125}(a^2+b^2+c^2)$$

$$\sum_{cyc}\frac{a^5}{(2a+3b)^3}+\sum_{cyc}\frac{a^5}{(2a+3c)^3} = \sum_{cyc}\frac{a^8}{(2a^2+3ab)^3}+\sum_{cyc}\frac{a^8}{(2a^2+3ac)^3}$$

$$\overset{RADON'S\ INEQUALITY}{\ge} \frac{(a^2+b^2+c^2)^4}{(2a^2+2b^2+2c^2+3ab+3bc+3ca)^3}$$

$$+\frac{(a^2+b^2+c^2)^4}{(2a^2+2b^2+2c^2+3ab+3bc+3ca)^3} \ge \frac{2(a^2+b^2+c^2)^4}{125(a^2+b^2+c^2)^3} = \frac{2}{125}(a^2+b^2+c^2)$$

SOLUTION 1.96

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{16}{a^2+b^2+c^2} > \frac{9}{a^2+b^2+c^2} \text{ (TRUE)}$$

$$\frac{4\cdot 9}{a^2+b^2+c^2} > \frac{81}{4\cdot(a^2+b^2+c^2)}$$

$$\frac{4\cdot 9}{a^2+b^2+c^2} = 4\cdot\frac{27}{3(a^2+b^2+c^2)} =$$

$$= 4\cdot\frac{27}{3\cdot(a^2+b^2+c^2)} \le 4\cdot\frac{27}{2\cdot(a^2+b^2+c^2)+ab+bc+ca} \le$$

$$\le 4\cdot\left(\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\right) =$$

$$= \frac{4}{a^2+ab+b^2}+\frac{4}{b^2+bc+c^2}+\frac{4}{c^2+ca+a^2} <$$

$$< \left(\frac{4}{a^2-ab+b^2}\right)+\left(\frac{4}{b^2-bc+c^2}\right)+\left(\frac{4}{c^2-ca+a^2}\right) \le$$

$$\le \left(\frac{1}{(a-b)^2}+\frac{1^2}{ab}\right)+\left(\frac{1^2}{(b-c)^2}+\frac{1^2}{bc}\right)+\left(\frac{1^2}{(c-a)^2}+\frac{1^2}{ca}\right) =$$

$$= \frac{1}{(a-b)^2}+\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

SOLUTION 1.97

Solution by Ravi Prakash-New Delhi-India

Thus, for $x < 0 \le y$

$$|x+y| \le \frac{x|x|-y|y|}{-x+y} \le |x|+|y|$$

For $x, y \ge 0, x \ne y$

$$\frac{x|x|-y|y|}{x-y} = \frac{x^2-y^2}{x-y} = x+y \Rightarrow |x+y| = \frac{x|x|-y|y|}{x-y} = |x|+|y|$$

For $x, y < 0, x \ne y$

$$\frac{x|x|-y|y|}{x-y} = \frac{-x^2+y^2}{x-y} = -x-y$$

$$\therefore |x+y| = \frac{x|x|-y|y|}{x-y} = |x|+|y|$$

Thus, for $x, y \in R\ x \ne y$

$$|x+y| \le \frac{x|y|-y|y|}{x-y} \le |x|+|y|$$

$$\textit{For } x < 0 \le y$$

$$\frac{x|x| - y|y|}{x - y} = \frac{-x^2 - y^2}{x - y} = \frac{y^2 + x^2}{y - x}$$

$$\textit{As } xy < 0, (y - x)^2 = y^2 + x^2 - 2xy > x^2 + y^2$$

$$\Rightarrow \frac{x^2 + y^2}{-x + y} < y - x = |y| + |x| \Rightarrow \frac{x^2 + y^2}{y - x} < |x| + |y|$$

$$\textit{If } x + y \ge 0,$$

$$(y - x)(y + x) = y^2 - x^2 < y^2 + x^2 \Rightarrow |y + x| = y + x \le \frac{y^2 + x^2}{y - x}$$

$$\textit{If } x + y < 0, \textit{ then } -(x + y)(y - x) = x^2 - y^2 < x^2 + y^2$$

$$\Rightarrow -(x + y) < \frac{x^2 + y^2}{y - x} \Rightarrow |x + y| < \frac{x^2 + y^2}{y - x}$$

$$\textit{Now, for } a, b, c \in R, a, b, c \textit{ distinct}$$

$$3\omega < |a + b| + |b + c| + |c + a| \le$$

$$\le \frac{a|a| - b|b|}{a - b} + \frac{b|b| - c|c|}{b - c} + \frac{c|c| - a|a|}{c - a} \le$$

$$\le (|a| + |b|) + (|b| + |c|) + (|c| + |a|) < 6\Omega$$

SOLUTION 1.98

Solution by Soumitra Mandal-Chandar Nagore-India

$$\textit{Applying Weighted A.M} \ge \textit{G.M;}$$

$$\frac{\sum_{cyc} a(a^2 + 2bc)}{\sum_{cyc}(a^2 + 2bc)} \ge \left(\prod_{cyc} a^{a^2+2bc}\right)^{\frac{1}{\sum_{cyc}(a^2+2bc)}} \Rightarrow \frac{a^3 + b^3 + c^3 + 6abc}{(a + b + c)^2} \ge \left(\prod_{cyc} a^{a^2+2bc}\right)^{\frac{1}{(a+b+c)^2}}$$

$$\therefore \sum_{cyc} a^3 + 6abc \ge \prod_{cyc} a^{a^2+2bc}. \textit{(proved) equality at } a = b = c = \frac{1}{3}$$

SOLUTION 1.99

Solution by Ravi Prakash-New Delhi-India

$$(a^2 + b^2 + c^2)^2 + \sum (a^2 + b^2 - c^2)^2 =$$

$$= (a^2 + b^2 + c^2)^2 + (a^2 + b^2 - c^2)^2 + (a^2 - b^2 + c^2)^2 + (-a^2 + b^2 + c^2)^2 =$$

$$= 2(a^2 + b^2)^2 + 2c^4 + 2(a^2 - b^2)^2 + 2c^4 = 4(a^4 + b^4 + c^4)$$

$$\therefore \sum (a^2 + b^2 - c^2)^2 + 8 \sum a^2 b^2$$

$$= 4(a^4+b^4+c^4)+8\sum a^2b^2-(a^2+b^2+c^2)^2$$

$$= 4(a^2+b^2+c^2)^2-(a^2+b^2+c^2)^2 = 3(a^2+b^2+c^2)^2 \geq 3\left[3|abc|^{\frac{2}{3}}\right]^2 = 27|abc|^{\frac{4}{3}}$$

$$\Rightarrow \sum(a^2+b^2-c^2)^2+8\sum a^2b^2 \geq 27(abc)(abc)^{\frac{1}{3}}$$

SOLUTION 1.100

Solution by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\textit{Chebyshev: } x^3+y^3 \geq \frac{1}{2}(x+y)(x^2+y^2)$$

$$x^2-xy+y^2 \geq xy \textit{ (AM-GM)}$$

$$(x^3+y^3)^3(x^2-\overset{\geq xy}{xy}+y^2) \geq \frac{xy}{8}(x+y)^3(x^2+y^2)^3 \overset{AM-GM}{\geq} RHS$$

$$(x+y)^3 \geq \left(2\sqrt{xy}\right)^3 \geq 8xy\sqrt{xy}$$

SOLUTION 1.101

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$a^4 = x,\ b^4 = y,\ c^4 = z$$

$$x^2y^2+y^2z^2+z^2x^2 \geq xyz\cdot\sqrt[4]{27\,xyz\cdot(x+y+z)}$$

$$(x^2y^2+y^2z^2+z^2x^2)^4 \geq (xyz)^4\cdot 27\cdot xyz\cdot(x+y+z)$$

$$(x^2y^2+y^2z^2+z^2x^2)\cdot(x^2y^2+y^2z^2+z^2x^2)^3 \geq (xyz)^4\cdot 27xyz\cdot(x+y+z)\ \textit{(*)}$$

(ASSURE)

$$\textit{a) } x^2y^2+y^2z^2+z^2x^2 \geq (xy)(yz)+(yz)(zx)+(zx)(xy) =$$

$$= xyz(x+y+z)$$

$$\textit{b) } (x^2y^2+y^2z^2+z^2x^2)^3 \overset{AM-GM}{\geq} \left(3\cdot\sqrt[3]{(xyz)^4}\right)^3 = 27(xyz)^4$$

$$\textit{a); b)} \Rightarrow (x^2y^2+y^2z^2+z^2x^2)(x^2y^2+y^2z^2+z^2x^2)^3 \overset{a);b)}{\geq}$$

$$\geq (xyz)^4\cdot 27xyz\cdot(x+y+z)\ \textit{(*)}$$

SOLUTION 1.102

Solution by Marjan Milanovic-Nis-Serbia

By Jensen, since $x^{\left(-\frac{1}{2}\right)}$ ***is convex,***

$$\sum (a+b^2)^{\left(-\frac{1}{2}\right)} \geq 3\left(\frac{a+b+c+a^2+b^2+c^2}{3}\right)^{\left(-\frac{1}{2}\right)} =$$

$$= 3\left(\frac{27(a+b+c)}{3}\right)^{\left(-\frac{1}{2}\right)} = (a+b+c)^{\left(-\frac{1}{2}\right)}$$

SOLUTION 1.103

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } x, y > 0$$

$$\textit{Put } x = r\cos\theta,\ y = r\sin\theta,\ 0 < \theta < \frac{\pi}{2}$$

Now, consider

$$a(x^6+y^6) - 2\left(x^2y + xy\sqrt{xy} + xy^2\right)^2 = r^6 E \textit{ where}$$

$$E = a(\cos^6\theta + \sin^6\theta) - 2\sin^2\theta\cos^2\theta\left(\cos\theta + \sqrt{\cos\theta\sin\theta} + \sin\theta\right)^2$$

$$= a[(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)] -$$

$$-2\sin^2\theta\cos^2\theta\left\{\cos^2\theta + \sin^2\theta + 3\cos\theta\sin\theta + 2(\cos\theta + \sin\theta)\sqrt{\cos\theta\sin\theta}\right\}$$

$$= 9[1 - 3\cos^2\theta\sin^2\theta]$$

$$-2\sin^2\theta\cos^2\theta\left[1 + 3\cos\theta\sin\theta + 2(\cos\theta\sin\theta)\sqrt{\cos\theta\sin\theta}\right]$$

$$= 9 - \frac{29}{4}\sin^2 2\theta - \frac{6}{8}\sin^3 2\theta - \frac{4}{4\sqrt{2}}(\sin 2\theta)^{\frac{5}{2}}\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$$

$$= \frac{29}{4}(1 - \sin^2 2\theta) + \frac{3}{4}(1 - \sin^3 2\theta) + \left(1 - (\sin 2\theta)^{\frac{5}{2}}\sin\left(\theta + \frac{\pi}{4}\right)\right) \geq 0$$

$$\Rightarrow 3\sqrt{x^6+y^6} \geq \sqrt{2}\left(x^2y + xy\sqrt{xy} + xy^2\right)\quad \forall x, y > 0.$$

Equality when $x = y$. Put $x = a, y = b + 0$ get

$$a^2b + ab\sqrt{ab} + ab^2 \leq \frac{3}{\sqrt{2}}\sqrt{a^6+b^6}\ \textit{. Similarly, for other expressions.}$$

$$\Rightarrow \sum\left(a^2b + ab\sqrt{ab} + ab^2\right) \leq \frac{3}{2}\sqrt{2}\sum\sqrt{a^6+b^6}$$

SOLUTION 1.104

Solution by Daniel Sitaru-Romania

$$abcd = e^4 \rightarrow \sum \ln a = 4\ \ (1)$$

$$\sum \frac{lnd}{log_d(ab^2c^3)} = \sum \frac{lnd}{\frac{lna + 2lnb + 3lnc}{lnd}} = \sum \frac{ln^2 d}{lna + 2lnb + 3lnc} \geq$$

$$\overset{BERGSTROM}{\geq} \frac{(\sum lna)^2}{6\sum lna} = \frac{\sum lna}{6} \overset{(1)}{=} \frac{4}{6} = \frac{2}{3}$$

SOLUTION 1.105

Solution by Daniel Sitaru-Romania

$$a = y + z, b = z + x, c = x + y, s = x + y + z, S = \sqrt{xyz(x + y + z)}$$

$$\sum a^2 \overset{IONESCU-WEITZENBOCK}{\geq} 4\sqrt{3}S \leftrightarrow s^2 - r^2 - 4Rr \geq 2\sqrt{3}S \leftrightarrow$$

$$-2s^2 + 4s^2 - \sum bc \geq 2\sqrt{3}S \leftrightarrow s^2 - 3s^2 + \sum s(b + c) - \sum bc \geq 2\sqrt{3}S \leftrightarrow$$

$$s^2 - \sum (s - b)(s - c) \geq 2\sqrt{3}S \leftrightarrow \left(\sum x\right)^2 - \sum xy \geq 2\sqrt{3xyz(x + y + z)} \leftrightarrow$$

$$x^2 + y^2 + z^2 + xy + yz + zx \geq 2\sqrt{3xyz(x + y + z)}$$

SOLUTION 1.106

Solution by Daniel Sitaru-Romania

$$f(x) = x^{\frac{1}{2}}, f''(x) = \frac{1}{4}x^{-\frac{3}{2}} > 0, f: (0, \infty) \to \mathbb{R}, f - convexe$$

$$\frac{1}{3}\sum f(a) + f\left(\frac{a + b + c}{3}\right) \geq \frac{2}{3}\sum f\left(\frac{a + b}{2}\right) \to \frac{1}{3}\sum \sqrt{a} + f\left(\frac{3}{3}\right) \geq \frac{2}{3}\sum \sqrt{\frac{a + b}{2}} \to$$

$$\sum \sqrt{a} + 3f(1) \geq \frac{2}{\sqrt{2}}\sum \sqrt{a + b} \to \frac{\sqrt{2}}{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c} + 3\right) \geq \sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a}$$

SOLUTION 1.107

Solution by Daniel Sitaru-Romania

$$f: (0, \infty) \to (0, \infty), f(x) = x^{-3}, f'(x) = -3x^{-4}, f''(x) = 12x^{-5} > 0, f - convexe$$

By Popoviciu's inequality:

$$\frac{1}{3}(f(a) + f(b) + f(c)) + f\left(\frac{a + b + c}{3}\right) \geq \frac{2}{3}\left(f\left(\frac{a + b}{2}\right) + f\left(\frac{b + c}{2}\right) + f\left(\frac{c + a}{2}\right)\right)$$

For $a = x + y, b = y + z. c = z + x$:

$$\frac{1}{3}\sum f(x + y) + f\left(\frac{2(x + y + z)}{3}\right) \geq \frac{2}{3}\sum f\left(\frac{x + y + y + z}{2}\right)$$

$$\frac{1}{3}\sum\frac{1}{(x+y)^3}+\frac{1}{2^3}\geq\frac{2}{3}\sum\frac{1}{\left(\frac{x+2y+z}{2}\right)^3}$$

$$\frac{1}{(x+y)^3}+\frac{1}{(y+z)^3}+\frac{1}{(z+x)^3}+\frac{3}{8}\geq 16\left(\frac{1}{(2x+y+z)^2}+\frac{1}{(2y+z+x)^2}+\frac{1}{(2z+x+y)^2}\right)$$

SOLUTION 1.108

Proposed by Marian Ursărescu – Romania

$$\textit{We must show: } (a+1)(b+1)(c+1)(d+1)\geq 16abcd \quad \textit{(1)}$$

$$\textit{Let } a=\frac{4x}{x+y+z+t}, b=\frac{4y}{x+y+z+t}, c=\frac{4z}{x+y+z+t}, d=\frac{4t}{x+y+z+t}$$

$$\textit{(1)} \qquad \Leftrightarrow \prod\left(\frac{4x}{x+y+z+t}+1\right)\geq 16\cdot 4^4\frac{xyzt}{(x+y+z+t)^4}\Leftrightarrow$$

$$\prod(5x+y+z+t)\geq 4^6xyzt \quad \textit{(2)}$$

$$\textit{But } \left.\begin{array}{r} 5x+y+z+\geq 8\sqrt[8]{x^5yzt} \\ x+5y+z+t\geq 8\sqrt[8]{xy^5zt} \\ x+y+5z+t\geq 8\sqrt[8]{xyz^5t} \\ x+y+z+5t\geq 8\sqrt[8]{xyzt^5} \end{array}\right\}\Rightarrow \prod(5x+y+z+t)\geq 8^4xyzt\Rightarrow \textit{(2) it's true.}$$

SOLUTION 1.109

Solution by Remus Florin Stanca – Romania

$$\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)(a+b+c)\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2$$

$$\Rightarrow\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\geq\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}\Rightarrow\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c\geq$$

$$\geq\frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}+1-a-b-c \quad \textit{(1)}$$

$$\textit{We note } x=\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \textit{ and } y=a+b+c$$

$$y\leq 1\Rightarrow 1-y\geq 0\Rightarrow(x-y)^2(1-y)\geq 0\Rightarrow(x^2+y^2-2xy)(1-y)\geq 0\Rightarrow$$

$$\Rightarrow x^2(1-y)+y^2(1-y)+2xy(y-1)\geq 0\Rightarrow$$

$$\Rightarrow x^2-x^2y+y^2-y^3+2xy^2-2xy\geq 0\succ x^2\geq x^2y+2xy+y^3-2xy^2-y^2\succ$$

$$x^2+y-y^2\geq x^2y+y+y^3+2xy-2xy^2-2y^2$$

$$\Rightarrow\frac{x^2}{y}+1-y\geq x^2+1+y^2+2x-2xy-2y=(x+1-y)^2\Rightarrow$$

$$\Rightarrow \frac{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2}{a+b+c}+1-a-b-c \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2 \quad (2)$$

$$\overset{(1)(2)}{\Rightarrow} \frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}+1-a-b-c \geq \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1-a-b-c\right)^2$$

SOLUTION 1.110

Solution by Michael Sterghiou-Greece

$$\left|\left(\sqrt[3]{a^2b}-\sqrt[3]{ab^2}\right)\left(\sqrt[5]{a^4b}-\sqrt[5]{ab^4}\right)\right| \leq (a-b)^2 \quad (1)$$

LHS of (1) is always ≥ 0 so, we can get rid of the absolute value.

(1) is homogeneous so, WLOG, assume $ab=1$. Then (1) becomes

$$\left(a^{\frac{1}{3}}-\frac{1}{a^{\frac{1}{3}}}\right)\left(a^{\frac{1}{5}}-\frac{1}{a^{\frac{1}{5}}}\right)-\left(a-\frac{1}{a}\right)^2 \leq 0 \quad (2)$$

Let $a^{\frac{1}{15}}=x\geq 1$ (2)→

$$\left(x^5-\frac{1}{x^5}\right)\left(x^3-\frac{1}{x^3}\right)-\left(a^{15}-\frac{1}{a^{15}}\right)^2 \leq 0 \to x^{30}+\frac{1}{x^{30}}+x^2+\frac{1}{x^2}-x^8-\frac{1}{x^8}+2 \geq 0$$

But $x^{30}\geq x^8, x^2 \geq \frac{1}{x^8}$ so, we are done!

SOLUTION 1.111

Solution by Marian Ursărescu-Romania

We must show:

$$(a+b+c)(a^{11}c^{10}+b^{11}a^{10}+c^{11}b^{10}) \geq \left(a^6c^5+b^6a^5+c^6b^5\right)^2 \quad (1)$$

From Cauchy's inequality we have:

$$\left(\left(a^{\frac{1}{2}}\right)^2+\left(b^{\frac{1}{2}}\right)^2+\left(c^{\frac{1}{2}}\right)^2\right)\left(\left(a^{\frac{11}{2}}\right)^2(c^5)^2+\left(b^{\frac{11}{2}}\right)^2(a^5)^2+\left(c^{\frac{11}{2}}\right)^2(b^5)^2\right) \geq$$

$$\geq \left(a^6b^5+b^6a^5+c^6c^5\right)^2 \Rightarrow \textit{(1) it's true.}$$

SOLUTION 1.112

Solution by Tran Hong-Vietnam

$$\textit{Let } f(x)=\frac{\sin x}{x}\left(0<x<\frac{\pi}{2}\right)$$

$$\Rightarrow f'(x)=\frac{x\cos x-\sin x}{x^2}<0 \ \left(0<x<\frac{\pi}{2}\right) \Rightarrow f(x)\searrow\left(0;\frac{\pi}{2}\right)$$

(Because: $g(x) = x\cos x - \sin x\left(0 < x < \frac{\pi}{2}\right) \Rightarrow g'(x) = -x\sin x < 0 \Rightarrow g(x) \searrow \left(0, \frac{\pi}{2}\right)$

$\Rightarrow g(x) < g(0) = 0$*)*

$$\Rightarrow f''(x) = -\frac{(x^2-2)\sin x + 2x\cos x}{x^3} < 0 \ \left(0 < x < \frac{\pi}{2}\right)$$

(Because: $h(x) = -[(x^2-2)\sin x + 2x\cos x] \Rightarrow h'(x) = -x^2\cos x < 0\left(0 < x < \frac{\pi}{2}\right)$

$\Rightarrow h(x) \searrow \left(0; \frac{\pi}{2}\right) \Rightarrow h(x) < h(0) = 0$*)*

Now, inequality $\Leftrightarrow (a+b+c)\log u \geq a\log v + b\log w + c\log t$

$$\left[u = \frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\left(\frac{ab+bc+ca}{a+b+c}\right)}, v = \frac{\sin b}{b}, w = \frac{\sin c}{c}, t = \frac{\sin a}{a}\right]$$

Using Jensen's inequality with $\varphi(x) = \log x \ (x > 0)$

$$a\varphi(v) + b\varphi(w) + c\varphi(t) \leq (a+b+c)\cdot\varphi\left(\frac{av+bw+ct}{a+b+c}\right)$$
$$= (a+b+c)\cdot\log\frac{av+bw+ct}{a+b+c}$$

We must show that

$$u \geq \frac{av+bw+ct}{a+b+c} \Leftrightarrow (a+b+c)u \geq av+bw+ct \Rightarrow av+bw+ct$$
$$= a\cdot\frac{\sin b}{b} + b\cdot\frac{\sin c}{c} + c\cdot\frac{\sin a}{a} \overset{(Jensen)}{\leq} (a+b+c)\cdot\frac{\sin\left(\frac{ab+bc+ca}{a+b+c}\right)}{\left(\frac{ab+bc+ca}{a+b+c}\right)}$$

SOLUTION 1.113

Solution by Tran Hong-Vietnam

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3} \Rightarrow \sqrt{a^2+b^2+c^2} \geq \frac{a+b+c}{\sqrt{3}}$$

$$a^3+b^3+c^3 \geq \frac{(a+b+c)^3}{3^2} \Rightarrow \sqrt[3]{a^3+b^3+c^3} \geq \frac{a+b+c}{\sqrt[3]{3^2}}$$

$$a^5+b^5+c^5 \geq \frac{(a+b+c)^5}{3^4} \Rightarrow \sqrt[5]{a^5+b^5+c^5} \geq \frac{a+b+c}{\sqrt[5]{3^4}}$$

$$\Rightarrow LHS \geq \frac{(a+b+c)^3}{\sqrt{3}\cdot\sqrt[3]{3^2}\cdot\sqrt[5]{3^4}} = \frac{(a+b+c)^3}{3^{\frac{1}{2}+\frac{2}{3}+\frac{4}{5}}} = \frac{(a+b+c)^3}{3^{\frac{59}{30}}}$$

We must show that:

$$\frac{(a+b+c)^3}{3^{\frac{59}{30}}} \geq 3abc$$

$$3abc \leq 3\cdot\frac{(a+b+c)^3}{27} = \frac{(a+b+c)^3}{3^2} \Rightarrow \frac{(a+b+c)^3}{3^{\frac{59}{30}}} \geq \frac{(a+b+c)^3}{3^2}$$

$$\Leftrightarrow (a+b+c)^3\left[3^2 - 3^{\frac{59}{30}}\right] \geq 0$$

(true: $a,b,c \geq 0, 3^2 - 3^{\frac{59}{30}} > 0 \left(\because 2 > \frac{59}{30}\right)$).Proved. Equality $\Leftrightarrow a=b=c=0$.

SOLUTION 1.114

Solution by Tran Hong-Vietnam

$$\therefore \sum_{cyc}\frac{3x^2+xy+2y^2}{2x^2+y^2} \leq 6$$

$$\Leftrightarrow \sum_{cyc}\frac{3x^2+xy+2y^2}{2x^2+y^2} \leq 3\cdot\sum_{cyc}\frac{x+y}{2x+y} \quad (1)$$

$\because$ ***Must show that:*** $\frac{3x^2+xy+2y^2}{2x^2+y^2} \leq 3\cdot\frac{x+y}{2x+y}$ ***(2)***

$$\Leftrightarrow (3x^2+xy+2y^2)(2x+y) \leq 3(x+y)(2x^2+y^2)$$

$$\Leftrightarrow 6x^3+5yx^2+5xy^2+2y^3 \leq 3(2x^3+xy^2+2yx^2+y^3)$$

$$\Leftrightarrow 2xy^2 \leq yx^2+y^3 \Leftrightarrow y(y^2-2xy+x^2) \geq 0 \Leftrightarrow y(x-y)^2 \geq 0 \text{ (true because } y>0\text{).}$$

Similarly:

$$\frac{3y^2+yz+2z^2}{2y^2+z^2} \geq 3\cdot\frac{y+z}{2y+z} \quad (3)$$

$$\frac{3z^2+xz+2x^2}{2z^2+x^2} \geq 3\cdot\frac{z+x}{2z+x} \quad (4)$$

From (2)+(3)+(4)⇒ (1) true.

SOLUTION 1.115

Solution by Tran Hong-Vietnam

$$\textit{Inequality} \Leftrightarrow a^a\cdot b^b\cdot c^c\cdot(a+b+c)^{a+b+c}\geq\left(\frac{3}{4}\right)^{a+b+c}(a+b)^{a+b}(b+c)^{b+c}(c+a)^{c+a}\quad (*)$$

$$\textit{Let } f(x)=x\log x\ (x>0)\Rightarrow f''(x)=\frac{1}{x}>0\ (\forall x>0)$$

Using Popoviciu's inequality, with $f(x)=x\log x\ (x>0)$ we have: Δ

$$\Leftrightarrow\sum a\log a+3\cdot\frac{a+b+c}{3}\log\left(\frac{a+b+c}{3}\right)\geq 2\sum\left(\frac{a+b}{2}\cdot\log\frac{a+b}{2}\right)$$

$$\Leftrightarrow\sum a\log a+\log\left(\frac{a+b+c}{3}\right)^3\geq\sum\log\left(\frac{a+b}{2}\right)^{a+b}$$

$$\Leftrightarrow\log\left[a^a\cdot b^b\cdot c^c\cdot\left(\frac{a+b+c}{3}\right)^{a+b+c}\right]\geq\log\left[\left(\frac{a+b}{2}\right)^{a+b}\left(\frac{b+c}{2}\right)^{b+c}\left(\frac{c+a}{2}\right)^{c+a}\right]$$

$$\Leftrightarrow a^a\cdot b^b\cdot c^c\cdot(a+b+c)^{a+b+c}\cdot\frac{1}{3^{a+b+c}}\geq(a+b)^{a+b}(b+c)^{b+c}(c+a)^{c+a}\cdot\frac{1}{4^{a+b+c}}$$

$$\Leftrightarrow a^a\cdot b^b\cdot c^c\cdot(a+b+c)^{a+b+c}\geq\left(\frac{3}{4}\right)^{a+b+c}\cdot(a+b)^{a+b}\cdot(b+c)^{b+c}\cdot(c+a)^{c+a}$$

⇒ (*) true. Proved. Equality $\Leftrightarrow a=b=c$.

SOLUTION 1.116

Solution by Michael Sterghiou-Greece

$$\frac{2}{5}\leq\sum_{cyc}\frac{x}{1+x^2}\leq\frac{18}{13}\quad (1)$$

$$\sum_{cyc}\frac{x}{1+x^2}=\sum_{cyc}\frac{x^2}{x+x^3}\overset{BCS}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\left(\sum_{cyc}x\right)+\left(\sum_{cyc}x^3\right)}=\frac{4}{2+\sum_{cyc}x^3}\overset{?}{\geq}\frac{2}{5}\to\sum_{cyc}x^3\leq 8$$

This is true because $x\leq 2\to x^2\leq 4\to x^2-4\leq 0\to x(x^2-4)\leq 0\to$

$$\to x^3-4x\leq 0\to\sum_{cyc}x^3\leq 4\sum_{cyc}x=8$$

Consider the function $f(t)=\frac{t}{1+t^2}$ on $[0,2]$ $f'(t)=\frac{1-t^2}{(1+t^2)^2}$ with

root $t=1$ in $[0,2]$ $f''(t)=\frac{2t(t^3-3)}{(t^2+1)^3}$ with root $\sqrt{3}$. $t=1$ is a max for $f(t)$ and also $f(t)$ is

concave in $[0,\sqrt{3}]$ as $f''(t)\leq 0$ in this interval. Assume $\max\{x,y,z\}=x\leq\sqrt{3}$.

Then by Jensen we have:

$\sum_{cyc}\frac{x}{1+x^2}\leq 3\cdot\frac{\frac{1}{3}\sum_{cyc}x}{1+\left(\frac{5x}{3}\right)^2}=\frac{2}{1+\frac{4}{9}}=\frac{18}{13}$ and we are done. Assume $x>\sqrt{3}$ then

$y+z \le 2-\sqrt{3} < \sqrt{3}$ ***and by Jensen*** $\frac{y}{1+y^2}+\frac{z}{1+z^2} \le 2\cdot\frac{\frac{y+z}{2}}{1+\left(\frac{y+z}{2}\right)^2} \le 2\cdot\frac{\frac{2-\sqrt{3}}{2}}{1+\left(\frac{2-\sqrt{3}}{2}\right)^2} < 0{,}3(2)$

because $f(t)$ ***is*** $\uparrow$ ***in*** $[0,1]$ ***and*** $2-\sqrt{3}<1$***. Also,*** $f(t)$ ***is*** $\downarrow$ ***in*** $[1,2]$ $(f'(t)\le 0)$ ***so***

$f(x) < f(1)$ ***as*** $x>\sqrt{3}>1$ ***or*** $\frac{x}{1+x^2}\le\frac{1}{2}$***. Combining this with (2) we have***

$\sum_{cyc}\frac{x}{1+x^2} < 0{,}3+0{,}5 = 0{,}8 < \frac{18}{13}$***. We are done.***

SOLUTION 1.117

Solution by Tran Hong-Vietnam

$$\because a^5+b^5 \ge ab(a^3+b^3) = ab(a+b)[(a+b)^2-3ab];$$

We must show that:

$$(ab)^3\left[(ab)^6+\left(\frac{a+b}{2}\right)^{12}\right]\left[ab+\left(\frac{a+b}{2}\right)^2\right] \le (a+b)^2[(a+b)^2-3ab]^2 \times$$

$$\times\left[(ab)^3\sqrt{ab}+\left(\frac{a+b}{2}\right)^7\right]^2 \quad (*)$$

Let $u=\sqrt{ab};\ v=\frac{a+b}{2}\ (v\ge u>0)$

(*) $\Leftrightarrow u^6[u^{12}+v^{12}][u^2+v^2] \le v^2(4v^2-3u^2)^2(u^7+v^7)^2$

(Let $u=tv, 0<u\le v \Rightarrow 0<t\le 1$***)***

$$\Leftrightarrow t^6(1+t^{12})(1+t^2) \le (4-3t^2)^2(1+t^7)^2$$

$$\Leftrightarrow [t^3(1+t^{12})][t^3(1+t^2)] \le [(4-3t^2)(1+t^7)][(4-3t^2)(1+t^7)] \quad (**)$$

$$t^3(1+t^{12}) \le (4-3t^2)(1+t^7) \quad (1)$$

(1) ***true because:*** $\begin{cases} 0<t\le 1 \Rightarrow t^3 \le 1 \le 4-3t^2 \\ 0\le t\le 1 \Rightarrow t^5\le 1 \Rightarrow 1+t^{12}\le 1+t^7 \end{cases}$

$$t^3(1+t^2) \le (4-3t^2)(1+t^7) \quad (2)$$

$\because t\le 1 \Rightarrow 4-3t^2 \ge 1$***. We must show that:***

$$t^3(1+t^2) \le 1+t^7$$

$$\Leftrightarrow t^3+t^5 \le 1+t^7 \Leftrightarrow (t-1)\left(t^6-t^5-t^2-t-1\right) \ge 0$$

It is true because:

$$\because t\le 1 \Rightarrow t-1\le 0, t^6\le t^2, t^5\le t \Rightarrow t^5 < t+1$$

From (1) and (2) $\Rightarrow$ ***(*) true.***

Proved.

ACYCLIC,ASYMMETRICAL INEQUALITIES SOLUTIONS

SOLUTION 2.01

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$\textit{We have: if } a+b+c=0 \textit{ then } a^3+b^3+c^3=3abc \textit{ so}$$

$$(a+b)+c+d=0 \textit{ then } (a+b)^3+c^3+d^3=3(a+b)cd$$

We have:

$$3(a+b)(ac+ad+bc+bd+4cd)=3(a+b)[(a+b)(c+d)+4cd]$$

$$=3(a+b)[-(a+b)^2+4cd]$$

$$=-3(a+b)^3+4\cdot 3(a+b)cd=-3(a+b)^3+4[(a+b)^3+c^3+d^3]$$

$$=(a+b)^3+4(c^3+d^3)$$

Now, we prove that

$$4(a^3+b^3+c^3)\geq(a+b)^3+4(c^3+d^3)\Rightarrow 4(a^3+b^3)\geq(a+b)^3$$

$$\Leftrightarrow(1^3+1^3)(1^3+1^3)(a^3+b^3)\geq(a+b)^3$$ ***(Right because Hölder's). "=" a=b.***

SOLUTION 2.02

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$xa^xb^yc^zd^t+yb^xc^yd^za^t+zc^xd^ya^zb^t+td^xa^yb^zc^y\geq$$

$$\overset{AM-GM}{\gtrapprox}(x+y+z+t)(abcd)^{\frac{x+y+z+t}{x+y+z+t}}=abcd(x+y+z+t)$$

$$2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\geq$$

$$\geq\left(2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}\right)abcd=\frac{73}{12}abcd\,,(1)$$

$$\left(\sum\frac{1}{a}\right)^4\overset{AM-GM}{\gtrapprox}\frac{256}{abcd},(2)$$

$$\textit{By multiplying } (1),(2):$$

$$\left(2a^2\sqrt{b^3}\sqrt[3]{c^4}\sqrt[4]{d^5}+\frac{3}{2}b^2\sqrt{c^3}\sqrt[3]{d^4}\sqrt[4]{a^5}+\frac{4}{3}c^2\sqrt{d^3}\sqrt[3]{a^4}\sqrt[4]{b^5}+\frac{5}{4}d^2\sqrt{a^3}\sqrt[3]{b^4}\sqrt[4]{c^5}\right)\left(\sum\frac{1}{a}\right)^4\geq$$

$$\geq\frac{73}{12}abcd\cdot\frac{256}{abcd}=\frac{4672}{3}$$

SOLUTION 2.03

Solution by Sanong Hauyrerai-Nakon Pathom-Thailand

$$\frac{\sin^2 x}{a}+\frac{\cos^2 x}{b}+\frac{\sin^2 y}{c}+\frac{\cos^2 y}{d}=\frac{\sin^4 x}{a\sin^2 x}+\frac{\cos^4 x}{b\cos^2 x}+\frac{\sin^4 y}{c\sin^2 y}+\frac{\cos^4 y}{d\cos^2 y}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(\sin^2 x+\cos^2 x)^2}{a\sin^2 x+b\cos^2 x}+\frac{(\sin^2 y+\cos^2 y)^2}{c\sin^2 y+d\cos^2 y}=$$

$$=\frac{1}{a\sin^2 x+b\cos^2 x}+\frac{1}{c\sin^2 y+d\cos^2 y}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{4}{a\sin^2 x+b\cos^2 x+c\sin^2 y+d\cos^2 y}>\frac{4}{2(a+b)+2(c+d)}=\frac{2}{a+b+c+d}$$

SOLUTION 2.04

Solution by Sanong Hauyrerai-Nakon Pathom-Thailand

$$x,y>0,n\in\mathbb{N}^*\to\frac{n+1}{n}>1$$

$$(x^n+y^n)^{\frac{n+1}{n}}>(x^n)^{\frac{n+1}{n}}+(y^n)^{\frac{n+1}{n}}\to(x^n+y^n)^{\frac{n+1}{n}}>x^{n+1}+y^{n+1}$$

$$(x^n+y^n)^{n+1}>(x^{n+1}+y^{n+1})^n\to\frac{(x^n+y^n)^{n+1}}{(x^{n+1}+y^{n+1})^n}>1$$

$$\begin{cases}\dfrac{(a^3+b^3)^4}{(a^4+b^4)^3}>1\\ \dfrac{(c^5+d^5)^6}{(c^6+d^6)^5}>1\\ \dfrac{(e^7+f^7)^8}{(e^8+f^8)^7}>1\end{cases}\overset{by\ multipying}{\Rightarrow}\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{(c^5+d^5)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

SOLUTION 2.05

Solution by Ravi Prakash-New Delhi-India

$$f(x)=(1+x)^{\frac{1}{x}},x>0,\ln f(x)=\frac{\ln(1+x)}{x}$$

$$\frac{f'(x)}{f(x)}=\frac{x-(x+1)\ln(x+1)}{x^2(x+1)},g(x)=x-(x+1)\ln(x+1)$$

$$g'(x) = -ln(x+1) < 0, x > 0 \to g(x) < g(0) = 0, \forall x > 0$$

$$f'(x) < 0, \forall x > 0, f - strictly\ decreasing$$

$$0 < a \le b \to \frac{a+3b}{4} \ge \frac{3a+b}{4} \to f\left(\frac{a+3b}{4}\right) \le f\left(\frac{3a+b}{4}\right)$$

$$\left(1+\frac{a+3b}{4}\right)^{\frac{4}{a+3b}} \le \left(1+\frac{4a+3}{4}\right)^{\frac{4}{4a+b}} \to \left(1+\frac{a+3b}{4}\right)^{3a+b} \le \left(1+\frac{3a+b}{4}\right)^{a+3b}$$

SOLUTION 2.06

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } e^x = a, e^y = b, e^z = c, 0 \le x \le y \le z \to 1 \le a \le b \le c$$

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)} \ge \frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})} \leftrightarrow$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \ge \frac{(1+a+a^2)^2}{(2+a)^2}, (1)$$

$$1+b+b^2 \ge 1+a+a^2 \leftrightarrow (b-a)(1+b+a) \ge 0, (2)$$

$$b \le c \to 2+b \le 2+c \to \frac{1}{2+b} \ge \frac{1}{2+c} \to \frac{1}{(2+b)(2+c)} \ge \frac{1}{(2+c)^2}, (3)$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \overset{(2),(3)}{\ge} \frac{(1+a+a^2)(1+c+c^2)}{(2+c)^2} \ge \frac{(1+a+a^2)^2}{(2+a)^2} \leftrightarrow$$

$$\leftrightarrow \frac{1+c+c^2}{(2+c)^2} \ge \frac{1+a+a^2}{(2+a)^2}, (4)$$

$$f(t) = \frac{1+t+t^2}{(2+t)^2}, \forall t \ge 1, f'(t) = \frac{3t}{(2+t)^2} > 0, \forall t \ge 1$$

$$f - increasing \to f(c) \ge f(a)$$

SOLUTION 2.07

Solution by Le Van-Ho Chi Minh-Vietnam

$$\textit{Put } f(x) = \frac{\ln x}{\ln(x+1)}, x \ge 1$$

$$\textit{Then } f'(x) \cdot [\ln(x+1)]^2 = \frac{\ln(x+1)}{x} - \frac{\ln x}{x+1} = \frac{[(x+1)\ln(x+1) - x\ln(x)]}{x(x+1)} > 0$$

Then $f(x)$ is a positive function, which gives us:

$$4f(a) \leq f(a) + f(b) + f(c) + f(d) \leq 4f(d)$$

→ *Q.E.D. Equality holds when $a = b = c = d = 1$.*

SOLUTION 2.08

Solution by Lazaros Zachariadis-Thessaloniki-Greece

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \leq \frac{a+b}{2} \Rightarrow \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{2^2} \leq \frac{a+b}{2} \Rightarrow \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{a+b} \leq 2 \quad (1)$$

$$\left(\frac{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}{3}\right)^3 \leq \frac{a+b+c}{3} \Rightarrow \frac{\left(\sum_{cyc}\sqrt[3]{a}\right)^3}{3^3} \leq \frac{a+b+c}{3} \Rightarrow \frac{\left(\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\right)^3}{a+b+c} \leq 9 \quad (2)$$

$$\left(\frac{\sum_{cyc}\sqrt[4]{a}}{4}\right)^4 \leq \frac{a+b+c+d}{4} \Rightarrow \frac{\left(\sum_{cyc}\sqrt[4]{a}\right)^4}{\sum_{cyc} a} \leq 4^3 = 64 \quad (3)$$

$$(1)+(2)+(3) \Rightarrow \frac{\left(\sum_{cyc}\sqrt{a}\right)^2}{b+a+c} + \frac{\left(\sum_{cyc}\sqrt[3]{a}\right)^3}{a+b+c} + \frac{\left(\sum_{cyc}\sqrt[4]{a}\right)^4}{a+b+c+d} \leq 2 + 9 + 64 = 75$$

SOLUTION 2.09

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}$$

$$\Leftrightarrow \sqrt[3]{\frac{1}{a} \cdot \frac{2}{a+b} \cdot \frac{3}{a+b+c}}\left(a + \sqrt{ab} + \sqrt[3]{abc}\right) \leq 3$$

$$\Leftrightarrow \sqrt[3]{\frac{2a}{a+b} \cdot \frac{3a}{a+b+c} \cdot 1} + \sqrt[3]{\frac{2\sqrt{ab}}{a+b} \cdot \frac{3b}{a+b+c} \cdot 1} + \sqrt[3]{\frac{2b}{a+b} \cdot \frac{3c}{a+b+c} \cdot 1} \leq 3$$

Now, $\sqrt[3]{\frac{2a}{a+b} \cdot \frac{3a}{a+b+c} \cdot 1} \overset{AM \geq GM}{\leq} \frac{\frac{2a}{a+b} + \frac{3a}{a+b+C} + 1}{3}$

$\sqrt[3]{\frac{2\sqrt{ab}}{a+b} \cdot \frac{3b}{a+b+c} \cdot 1} \overset{AM \geq GM}{\leq} \frac{\frac{2\sqrt{ab}}{a+b} + \frac{3b}{a+b+c} + 1}{3} \leq \frac{2 + \frac{3b}{a+b+c}}{3}$ *and*

$$\sqrt[3]{\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot 1}\overset{AM\geq GM}{\leq}\frac{\frac{2b}{a+b}+\frac{3c}{a+b+c}+1}{3}$$

$$\therefore\sqrt[3]{\frac{1}{a}\cdot\frac{2}{a+b}\cdot\frac{3}{a+b+c}}\left(a+\sqrt{ab}+\sqrt[3]{abc}\right)$$

$$\leq\frac{4+2+3}{3}=3\Rightarrow\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3}\leq\sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{3}\right)}$$

SOLUTION 2.10

Solution by Le Minh Cuong-Ho Chi Minh-Vietnam

$$\textit{Apply Schwarz we get: } (LHS)^2=\left(\sqrt{\frac{x}{y}}+\sqrt{2}\sqrt{\frac{2y}{z}}+\sqrt{3}\sqrt{\frac{3z}{x}}\right)^2$$

$$\leq\left(1^2+\left(\sqrt{2}\right)^2+\left(\sqrt{3}\right)^2\right)\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)\leq 6\left(\frac{x}{y}+\frac{2y}{z}+\frac{3z}{x}\right)\leq(RSH)^2$$

SOLUTION 2.11

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$\textit{Let } x=\sqrt{\frac{a}{b+c}},y=\sqrt{\frac{b}{c+a}},z=\sqrt{\frac{c}{a+b}}$$

$$\textit{Now, we prove that: } x+2y+4z\leq\sqrt{7(x^2+2y^2+4z)^2}$$

$$\Leftrightarrow x^2+4y^2+16z^2+4xy+8xz+16yz\leq 7(x^2+2y^2+4z^2)$$

$$\Leftrightarrow 6x^2+10y^2+12z^2\geq 4xy+8xz+16yz\Leftrightarrow 2(x-y)^2+4(x-z)^2+8(y-z)^2\geq 0$$

$$\text{“=”}\ x=y=z \textit{ or } a=b=c.$$

SOLUTION 2.12

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } a=x^2,b=y^2,c=z^2,x,y,z\geq 0$$

$$\textit{Also } 0 \leq a \leq b \leq c \Rightarrow 0 \leq x \leq y \leq z$$

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}=$$

$$=(x^2-y^2)z^3+(y^2-z^2)x^3+(z^2-x^2)y^3$$

$$=\begin{vmatrix} x^3 & y^3 & z^3 \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} x^3-y^3 & y^3-z^3 & z^3 \\ x^2-y^2 & y^2-z^2 & z^2 \\ 0 & 0 & 1 \end{vmatrix}\begin{bmatrix} use\ C_1 \to C_1-C_2 \\ C_2 \to C_2-C_3 \end{bmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x^2+y^2+xy & y^2+z^2+yz \\ x+y & y+z \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x^2-z^2+(x-z)y & y^2+z^2+yz \\ x-z & y+z \end{vmatrix}$$

$$=(x-y)(y-z)(x-z)\begin{vmatrix} x+y+z & y^2+z^2+yz \\ 1 & y+z \end{vmatrix}$$

$$=(x-y)(y-z)(x-z)\begin{vmatrix} x+y & y^2 \\ 1 & y+z \end{vmatrix}=(x-y)(y-z)(x-z)(xy+yz+zx)\leq 0$$

$$\textit{since } x \leq y \leq z$$

SOLUTION 2.13

Solution by Ravi Prakash-New Delhi-India

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\Leftrightarrow 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \geq 1+1+1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$$

$$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \frac{a-c}{b}+\frac{b-a}{c}+\frac{c-b}{a} \geq 0$$

$$\Leftrightarrow ac(a-c)+ab(b-a)+bc(c-b) \geq 0$$

$$\Leftrightarrow \begin{vmatrix} bc & ac & ab \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \geq 0 \Leftrightarrow (a-b)(b-c)(c-a) \geq 0$$

$$\textit{which is true as } a \le b \le c$$

SOLUTION 2.14

Solution by Ravi Prakash-New Delhi-India

$$2b = a + c, 2c = b + d$$

$$\Rightarrow a, b, c, d \textit{ are in A.P. with common difference } \frac{1}{3}(d-a)$$

$$\therefore a^2 + b^2 + c^2 + d^2 = a^2 + \left\{a + \frac{1}{3}(d-a)\right\}^2 + \left\{a + \frac{2}{3}(d-a)\right\}^2 + d^2$$

$$= 3a^2 + d^2 + 2(d-a)a + \frac{5}{9}(d-a)^2 = (a+d)^2 + \frac{5}{9}(d-a)^2$$

$$= \left\{(a+d) - 2e^{\frac{1}{8}}\right\}^2 + 4e^{\frac{1}{8}}(a+d) - 4e^{\frac{1}{4}} + \frac{5}{9}(d-a)^2 \ge 4e^{\frac{1}{8}}\left[a + d - e^{\frac{1}{8}}\right]$$

SOLUTION 2.15

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

$$\textit{If } x, y, z \in [-5, 3] \textit{ then: } \sum \sqrt{3x - 5y - xy + 15} \le 12$$

$$\textit{We have: } \sum \sqrt{3x - 5y - xy + 15} = \sum \sqrt{(3-y)(5+x)}.$$

$$\textit{Since } x, y, z \in [-5; 3] \textit{ then } 3 - x;$$

$$3 - y; 3 - z; 5 + x; 5 + y; 5 + z \ge 0, \textit{ so, by applying Cauchy's inequality:}$$

$$\sum \sqrt{(3-y)(5+x)} \le \sum \left(\frac{3-y+5+x}{2}\right) = \frac{24}{2} = 12 \Rightarrow \textit{Q.E.D. The equality happens iff}$$

$$\begin{cases} 3 - y = 5 + x; 3 - z = 5 + y; 3 - x = 5 + z \\ x, y, z \in [-5; 3] \end{cases} \Leftrightarrow x = y = z = -1$$

SOLUTION 2.16

Solution by Le Minh Cuong-Ho Chi Minh-Vietnam

$$\textit{We have } LHS = \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{cd}{c+d} + \frac{da}{d+a} \le \frac{ab}{2\sqrt{ab}} + \frac{bc}{2\sqrt{bc}} + \frac{cd}{2\sqrt{cd}} + \frac{da}{2\sqrt{da}} \le$$

$$\leq \frac{1}{2}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)\text{. It need show that: } \sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da} \leq$$

$$\leq ab+bc+cd+da\text{. Indeed, } 4(ab+bc+cd+da) \overset{BCS}{\geq} \left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)^2$$

$$\overset{AM-GM}{\geq} 4\sqrt[4]{\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{dc}\cdot\sqrt{da}}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right) \geq$$

$$\geq 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)\text{. The equality holds for } a=b=c=d=1.$$

SOLUTION 2.17

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$c+\sqrt{ab}+\sqrt{ab} \geq 3\sqrt[3]{abc};\ c-3\sqrt[3]{abc} \geq -2\sqrt{ab} \Leftrightarrow a+b+c-3\sqrt[3]{abc} \geq$$

$$\geq a+b-2\sqrt{ab} = \left(\sqrt{a}-\sqrt{b}\right)^2 \Leftrightarrow \frac{1}{\left(\sqrt{a}-\sqrt{b}\right)^2}+1 \geq \frac{1}{a+b+c-3\sqrt[3]{abc}}+1 \Leftrightarrow$$

$$\Leftrightarrow \frac{1+\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} \geq \frac{1+a+b+c-3\sqrt[3]{abc}}{a+b+c-3\sqrt[3]{abc}} \overset{a<b+c}{>} \frac{1+a+b+c-3\sqrt[3]{abc}}{2b+2c-3\sqrt[3]{abc}} \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)} > 1$$

SOLUTION 2.18

Solution by proposer

From the hypothesis we have:

$$c\left(\frac{ab}{9}-\frac{2}{3}\right) = \frac{a}{8}+3b-\frac{67}{4a} \Leftrightarrow c = \frac{9(a^2+24ab-134)}{8a(ab-6)}$$

Therefore, we have:

$$P = 3a+2b+c = 3a+2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}$$

Applying the AM-GM inequality, we have:

$$2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}=2b+\frac{9[a^2+10+24(ab-6)]}{8a(ab-6)}$$

$$=\frac{2(ab-6)}{a}+\frac{9(a^2+10)}{8a(ab-6)}+\frac{39}{a}\geq\frac{2}{a}\cdot\sqrt{2(ab-6)\cdot\frac{9(a^2+10)}{8(ab-6)}}+\frac{36}{a}$$

$$=\frac{3\left(13+\sqrt{a^2+10}\right)}{a}\Rightarrow P\geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)$$

Applying the Cauchy – Schwarz and AM-GM inequality, we have:

$$P\geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)=3\left(a+\frac{13}{a}+\frac{\sqrt{(15+10)(a^2+10)}}{5a}\right)$$

$$\geq 3\left(a+\frac{13}{a}+\frac{a\sqrt{15}+10}{5a}\right)=3\left(a+\frac{15}{a}+\frac{\sqrt{15}}{5}\right)$$

$$\geq 3\left(2\sqrt{a\cdot\frac{15}{a}}+\frac{\sqrt{15}}{5}\right)=\frac{33\sqrt{15}}{5}\Rightarrow P\geq\frac{33\sqrt{15}}{5}$$

Therefore, $P_{\min}=\frac{33\sqrt{15}}{5}$***. The equally holds for*** $a=\sqrt{15}, b=\frac{13\sqrt{5}}{20}, c=\frac{23\sqrt{15}}{10}$

SOLUTION 2.19

Solution by Ravi Prakash-New Delhi-India

$$Let\ f(x)=\frac{1+x+x^2}{1+x^2}, x\geq 1$$

$$f'(x)=\frac{d}{dx}\left[1+\frac{x}{1+x^2}\right]=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}<0,\forall x>1$$

$$\Rightarrow f(x)\ decreases\ on\ [1,\infty)\therefore f(x)\leq f(1)\ \forall x\geq 1\Rightarrow\frac{1+a+a^2}{1+a^2}\leq\frac{3}{2}\ \forall a\geq 1\quad (1)$$

$$Let\ g(x)=\frac{1+x+x^2+x^3}{1+x^3}=1+\frac{x+x^2}{1+x^3}$$

$$g'(x) = \frac{(1+x^3)(1+2x) - 3x^2(x+x^2)}{(1+x^3)^2} = \frac{1+2x+x^3+2x^4-3x^3-3x^4}{(1+x^3)^2}$$

$$= \frac{1+2x-2x^3-x^4}{(1+x^3)^2}$$

$$g'(x) = \frac{(1-x^4)-2x(1-x^2)}{(1+x^3)^2} = \frac{(1-x^2)(1+x^2-2x)}{(1+x^3)^2} = \frac{(1-x)^3(1+x)}{(1+x^3)^2} < 0\ \forall x > 1$$

$$\Rightarrow g(x) \text{ decreases on } [1,\infty) \therefore g(x) \le g(1)$$

$$\Rightarrow \frac{1+b+b^2+b^3}{1+b^3} \le \frac{4}{2} = 2\ \forall b \ge 1 \quad \textit{(2). Let } h(x) = \frac{1+x+x^2+x^3+x^4}{1+x^4}, x \ge 1$$

$$= 1 + \frac{x+x^2+x^3}{1+x^4}$$

$$h'(x) = \frac{(1+2x+3x^2)(1+x^4)-(x+x^2+x^3)(4x^3)}{(1+x^4)^2}$$

$$= \frac{1+2x+3x^2+x^4+2x^5+3x^6-4x^4-4x^5-4x^6}{(1+x^4)^2}$$

$$= \frac{1+2x+3x^2-3x^4-2x^5-x^6}{(1+x^4)^2} = \frac{(1-x^6)+2x(1-x^3)+3x^2(1-x^2)}{(1+x^4)^2} < 0\ \forall x \ge 1$$

$$\Rightarrow h(x) \text{ decreases on } [1,\infty) \therefore h(x) \le h(1)\ \ \forall x \ge 1 \Rightarrow \frac{1+c+c^2+c^3+c^4}{1+c^4} \le \frac{5}{2}\ \ \forall c \ge 1 \quad \textit{(3)}$$

Multiplying (1), (2), (3) we get

$$\frac{(1+a+a^2)(1+b+b^2+b^3)(1+c+c^2+c^4)}{(1+a^2)(1+b^3)(1+c^4)} \le \frac{15}{2}$$

SOLUTION 2.20

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } 0 < a \le b, a(c+2) \le a + \sqrt{ab} + b \le b(c+2)$$

$$\Leftrightarrow c\sqrt{ab} + b - ac - a \ge 0 \text{ and } bc + b - c\sqrt{ab} - a \ge 0$$

$$\Leftrightarrow (c\sqrt{a})(\sqrt{b}-\sqrt{a})+(b-a)\ge 0 \text{ and } c\sqrt{b}(\sqrt{b}-\sqrt{a})+(b-a)\ge 0$$

$$\Leftrightarrow (c\sqrt{a}+\sqrt{b}+\sqrt{a})(\sqrt{b}-\sqrt{a})\ge 0 \text{ and } (c\sqrt{b}+\sqrt{b}+\sqrt{a})(\sqrt{b}-\sqrt{a})\ge 0$$

which is true in view of $b\ge a$.

Thus $a\le \frac{a+c\sqrt{ab}+b}{c+2}\le b$. Similarly for d and e.

Multiplying three inequalities, we get

$$a^3\le \frac{(a+c\sqrt{ab}+b)(a+d\sqrt{ab}+b)(a+e\sqrt{ab}+b)}{(c+2)(d+2)(e+2)}\le b^3$$

SOLUTION 2.21

Solution by Soumitra Mandal-Chandar Nagore-India

We know for $x,y\ge 0$ then $x^2+xy+y^2\ge 3xy$ and $\frac{3}{2}(x^2+y^2)\ge x^2+xy+y^2$

$$\prod_{cyc}\sqrt[3]{(a^3+ab\sqrt{ab}+b^3)}$$

$$\Rightarrow \sqrt[3]{\prod_{cyc}\left(3a^{\frac{3}{2}}b^{\frac{3}{2}}\right)}\le \prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le \sqrt[3]{\frac{27}{8}\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3abc\le \prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le \frac{3}{2}\sqrt[3]{\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3ba^2\le \prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le \frac{3}{2}\sqrt[3]{(2b^3)(2c^2)(2c^3)}[\because a\le b\le c]$$

$$\therefore 3a^2b\le \prod_{cyc}\sqrt[3]{a^3+ab\sqrt{ab}+b^3}\le 3bc^2$$

SOLUTION 2.22

Solution by Marian Ursărescu-Romania

For $a=b=c=0; a\ge 0$ (true)

$$\left.\begin{array}{c} a,b,c>0;\, 2a^2+6ab+7b^2\ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right) \\ But\ 5\sqrt[5]{a^2b^3}\le 2a+3b \end{array}\right\}\Rightarrow$$

$$2\sqrt[8]{c}\left((2a+3b)-\sqrt[8]{c}\right)\le 2a^2+6ab+7b^2\Leftrightarrow$$

$$-2\sqrt[8]{c^2}+2(2a+3b)\sqrt[8]{c}\le 2a^2+6ab+b^2 \quad (1)$$

$$\sqrt[8]{c}=x, x>0\Rightarrow -2x^2+2(2a+3b)x=f(x)$$

$$\max f(x)=\frac{-\Delta}{4a}\Leftrightarrow \frac{-4(2a+3b)^2}{-8}=\frac{(2a+3b)^2}{2}\Rightarrow f(x)\le\frac{(2a+3b)^2}{2} \quad (2)$$

From (1)+(2)⇒ we must show: $\frac{(2a+3b)^2}{2}\le 2a^2+6ab+7b^2\Leftrightarrow$

$$4a^2+12ab+9b^2\le 4a^2+12ab+14b^2\Leftrightarrow 9b^2\le 14b^2\Leftrightarrow 5b^2\ge 0 \textit{ true.}$$

SOLUTION 2.23

Solution by Ravi Prakash-New Delhi-India

$$\textit{WLOG } x=\max\{x,z\}$$

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}$$
$$=\sqrt{x^2+z(z-x)}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}$$

$$\le\sqrt{x^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}\le\sqrt{a^2}+\sqrt{a^2+a^2}+\sqrt{a^2+a^2+a^2}=$$

$$=a\left(1+\sqrt{2}+\sqrt{3}\right). \textit{ Equality holds when } x=y=z=a.$$

SOLUTION 2.24

Solution by Soumava Chakraborty-Kolkata-India

$$\forall x,y,z,t\ge 1, xy+2yz+2zx+ty+tx+9\ge 4x+4y+6z+4t$$

$$\textit{Let } x=a+1, y=b+1, z=c+1, t=d+1\ (a,b,c,d\ge 0)$$

Then, given inequality becomes:

$$(a+1)(b+1)+2(b+1)(c+1)+2(c+1)(d+1)+2(c+1)(a+1)+$$

$$+(d+1)(b+1)+(d+1)(a+1)+9-4(a+1)-4(b+1)-6(c+1)-4(d+1)\geq 0$$

$$\Leftrightarrow ab+2ac+ad+2bc+bd+2cd\geq 0 \to \textit{true} \therefore a,b,c,d\geq 0 \ \textit{(proved)}$$

SOLUTION 2.25

Solution by Marian Ursărescu-Romania

From Hölder's inequality, we have: $a_1^p+a_2^p+\cdots+a_n^p\geq \frac{(a_1+a_2+\cdots+a_n)^p}{n^{p-1}}, p\in\mathbb{N}^*$

$$x^{12}+y^{12}=(x^6)^2+(y^6)^2\geq \frac{(x^6+y^6)^2}{2}\Rightarrow \frac{(x^6+y^6)^2}{x^{12}+y^{12}}\leq 2 \quad (1)$$

$$x^{12}+y^{12}+z^{12}=(x^4)^3+(y^4)^3+(z^4)^3\geq \frac{(x^4+y^4+z^4)^3}{9}\Rightarrow$$

$$\frac{x^4+y^4+z^4}{x^{12}+y^{12}+z^{12}}\leq 9 \quad (2)$$

$$x^{12}+y^{12}+z^{12}+t^{12}=(x^3)^4+(y^3)^4+(z^3)^4+(t^3)^4\geq \frac{(x^3+y^3+z^3+t^3)^4}{64}$$

$$\Rightarrow \frac{(x^3+y^3+z^3+t^3)^4}{x^{12}+y^{12}+z^{12}+t^{12}}\leq 64 \quad (3)$$

$$\text{From (1)+(2)+(3)}\Rightarrow \frac{(x^6+y^6)^2(x^4+y^4+z^4)^3(x^3+y^3+z^3+t^3)^4}{(x^{12}+y^{12})(x^{12}+y^{12}+z^{12})(x^{12}+y^{12}+z^{12}+t^{12})}\leq 1152$$

SOLUTION 2.26

Solution by Chris Kyriazis-Athens-Greece

Let's consider the function $f(x)=\frac{1}{1+e^x}, x>0$***. Easily:*** $f'(x)=-\frac{e^x}{(1+e^x)^2}<0,\forall x>0$ ***(f strictly decreasing) and*** $f''(x)=-e^x\frac{(1-e^x)}{(1+e^x)^3}>0,\forall x>0$***. So, f is convex for every*** $x>0$***.***

Working with the fundamental definition of convexity, I have that:

$\frac{c-b}{c-a}a+\left(1-\frac{c-b}{c-a}\right)\cdot c=\frac{c-b}{c-a}\cdot a+\frac{b-a}{c-a}\cdot c=\frac{ca-ab+bc-ac}{c-a}=b$***. And*** $\frac{c-b}{c-a}+1-\frac{c-b}{c-a}=1$***. So,***

$$f(b)=f\left(\frac{c-b}{c-a}\cdot a+\left(1-\frac{c-b}{c-a}\right)c\right)\le\frac{c-b}{c-a}f(a)+\left(1-\frac{c-b}{c-a}\right)f(c)=\frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c)\quad (1)$$

Also: $a-b+c=a-\left(\frac{c-b}{c-a}a+\frac{b-a}{c-a}\cdot c\right)+c=\frac{b-a}{c-a}a+\frac{c-b}{c-a}c$. *So,*

$$f(a-b+c)=f\left(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\right)\le\frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c)\quad (2)$$

Adding (1) + (2): $f(b)+f(a-b+c)\le f(a)+f(c)$ *as we desire!*

SOLUTION 2.27

Solution by Chris Kyriazis-Athens-Greece

The distance of $M(a,b)$ *from the line:* $3x+4y+2=0$ *is* 1

$$\left(d(M,\varepsilon)=\frac{|3a+4b+2|}{\sqrt{3^2+4^2}}=1\right)$$

I have to prove that: $a^2+b^2+4b+7\ge 4a$. *It suffices to prove that:*

$$(a-2)^2+(b+2)^2\ge 1\quad (1)$$

But its easy to prove that the point $N(2,-2)$ *belong to the straight line*

$$3x+4y+2=0.$$

So, (1) holds becomes: $d(M,\varepsilon)\le d(M,N)$

SOLUTION 2.28

Solution by Mohamed Alhafi-Aleppo-Syria

Since $a^2+b^2=1, c^2+b^2=1$ *we must have:* $a=c$. *So, our inequality is:*

$2a+2b+\frac{2}{ab}\ge 4+2\sqrt{2}\Leftrightarrow a+b+\frac{1}{ab}\ge 2+\sqrt{2}$. *Let* $s=a+b, p=ab$ *then:*

$s^2=1+2p\Rightarrow\frac{1}{p}=\frac{2}{s^2-1}$ *so, we need to show:* $s+\frac{2}{s^2-1}\ge 2+\sqrt{2}$ *or*

$s^3-(2+\sqrt{2})s^2-s+4+\sqrt{2}\ge 0$. *Let* $f(x)=x^3-(2+\sqrt{2})x^2-x+4+\sqrt{2}$

$$f'(x) = 3x^2 - (4 + 2\sqrt{2})x - 1 = x(3x - 4 - 2\sqrt{2}) - 1$$

Clearly $f'(x) < 0$ for $0 < x \le \sqrt{2}$ so, f is decreasing on the interval $]0, \sqrt{2}]$. Now, by Titu's inequality we have: $1 = a^2 + b^2 \ge \frac{(a+b)^2}{2} \Rightarrow \sqrt{2} \ge s$. So, $f(s) \ge f(\sqrt{2}) = 0$ and we are done.

SOLUTION 2.29

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\frac{a^4}{4} + \frac{b^8}{8} + \frac{5\sqrt[5]{c^8}}{8}\right)\left(\frac{5\sqrt[5]{a^8}}{8} + \frac{b^8}{8} + \frac{c^4}{4}\right) \overset{(1)}{\ge} \frac{27(abc)^4}{(ab + bc + ca)^3}$$

$$\frac{a^4}{4} + \frac{b^8}{8} + \frac{5c^{\frac{8}{5}}}{8} = \frac{a^4}{8} + \frac{a^4}{8} + \frac{b^8}{8} + \frac{c^{\frac{8}{5}}}{8} + \frac{c^{\frac{8}{5}}}{8} + \frac{c^{\frac{8}{5}}}{8} + \frac{c^{\frac{8}{5}}}{8} + \frac{c^{\frac{8}{5}}}{8} \underset{(i)}{\overset{A-G}{\ge}} 8\sqrt[8]{\frac{a^4 \cdot a^4 \cdot b^8 \left(c^{\frac{8}{5}}\right)^5}{8^8}} = abc$$

$$\textit{Also, } \frac{5a^{\frac{8}{5}}}{8} + \frac{b^8}{8} + \frac{c^4}{4} = \frac{a^{\frac{8}{5}}}{8} + \frac{a^{\frac{8}{5}}}{8} + \frac{a^{\frac{8}{5}}}{8} + \frac{a^{\frac{8}{5}}}{8} + \frac{a^{\frac{8}{5}}}{8} + \frac{b^8}{8} + \frac{c^4}{8} + \frac{c^4}{8} \underset{(ii)}{\overset{A-G}{\ge}} 8\sqrt[8]{\frac{\left(a^{\frac{8}{5}}\right)^5 b^8 c^4 c^4}{8^8}} = abc$$

(i), (ii) ⇒ LHS of (1)

$$\ge a^2b^2c^2 \overset{?}{\ge} \frac{27(abc)^4}{(\sum ab)^3} \Leftrightarrow (\sum ab)^3 \overset{?}{\ge} 27a^2b^2c^2 \Leftrightarrow \sum ab \overset{?}{\ge} 3\sqrt[3]{ab \cdot bc \cdot ca}$$

→ true by A-G (proved)

SOLUTION 2.30

Solution by Sarah El-Kenitra-Morocco

$$\left(\sqrt{a^2 - b^2} + \sqrt{2}b\right)^2 = a^2 + b^2 + 2b\sqrt{2(a^2 - b^2)} \ge a^2 + b^2 \textit{ hence}$$

$$\sqrt{a^2 - b^2} + \sqrt{2b} \ge \sqrt{a^2 + b^2}$$

Using the same method, we get $\sqrt{b^2 - c^2} + \sqrt{2}c \ge \sqrt{b^2 + c^2}$ and

$$\sqrt{a^2 - c^2} + \sqrt{2}c \ge \sqrt{a^2 + c^2}$$

After the sum we get

$$\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(b+2c)\ge\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

But we have $a\ge c$ ***therefore*** $\sqrt{a^2-b^2}+\sqrt{b^2-c^2}+\sqrt{a^2-c^2}+\sqrt{2}(a+b+c)\ge$

$$\ge\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{a^2+c^2}$$

SOLUTION 2.31

Solution by Amit Dutta-Jamshedpur-India

Let $P=2\sqrt{ab}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}$. ***Now, we have*** $\sqrt{ab}\le\sqrt[3]{abc}$.

Because, $(ab)^3\le(abc)^2\Rightarrow ab\le c^2$ ***(1)***

Now, we have $a\le c, b\le c\Rightarrow ab\le c^2$. ***So, (1) is true*** $\Rightarrow$ ***hence*** $\sqrt{ab}\le\sqrt[3]{abc}$

$$P\le 2\sqrt[3]{abc}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}$$

$$P\le 5\sqrt[3]{abc}+4\sqrt[4]{abcd}$$

Also, we have $\sqrt[3]{abc}\le\sqrt[4]{abcd}$. ***Because,*** $(abc)^4\le(abcd)^3\Rightarrow abc\le d^3$ ***(3)***

$\because a\le d, b\le d, c\le d\Rightarrow abc\le d^3\rightarrow$ ***True***

And hence $\sqrt[3]{abc}\le\sqrt[4]{abcd}$

$$P\le 5\sqrt[4]{abcd}+4\sqrt[4]{abcd}\le 9\sqrt[4]{abcd}$$

Also, we have $\sqrt[4]{abcd}=\sqrt[5]{abcde}\Rightarrow(abcd)^5\le(abcde)^4\Rightarrow abcd\le e^4$

$\because a\le e, b\le e, c\le e, d\le e\Rightarrow abcd\le e^4$ ***and hence*** $\sqrt[4]{abcd}\le\sqrt[5]{abcde}\Rightarrow P\le 9\sqrt[4]{abcde}$

SOLUTION 2.32

Solution by Tran Hong-Vietnam

$$a,b,c,d\ge 1\Rightarrow ab\le abc\le abcd$$

$a+b-2\sqrt{ab}\ge 0\Rightarrow z\ge 1$, ***similarly:*** $x,y\ge 1$. ***We have:*** $z\le y\le x$

$$\textit{In fact: } \frac{(a+b)^2}{4ab} \le \frac{(a+b+c)^3}{27abc} \Leftrightarrow 27c(a+b)^2 \le 4(a+b+c)^3 \quad (1)$$

$$\because 2c(a+b)(a+b) \overset{AM-GM}{\le} \frac{(2c+2a+2b)^3}{27} \Leftrightarrow 2 \cdot 27c(a+b)^2 \le 8(a+b+c)^3 \Leftrightarrow \textit{(1) true.}$$

$$\frac{(a+b+c)^3}{27abc} \le \frac{(a+b+c+d)^4}{256abcd} \quad (2) \Leftrightarrow 256d(a+b+c)^3 \le 27(a+b+c+d)^4$$

$$\therefore 3d(a+b+c)(a+b+c)(a+b+c) \overset{AM-GM}{\le} \frac{(3d+3a+3b+3c)^4}{256}$$

$$\Leftrightarrow 3 \cdot 256d(a+b+c)^3 \le 3^4(a+b+c+d)^4 \Leftrightarrow \textit{(2) true.}$$

Now, using Chebyshev's inequality:

$$(abcdx + abcy + abz) \ge \frac{1}{3}(abcd + abc + ab)(x+y+z)$$

$$\Leftrightarrow 3(abcdx + abcy + abz) \ge ab(1+c+cd)(x+y+z)$$

SOLUTION 2.33

Solution by Tran Hong-Vietnam

We prove that: $4x \ge 3y \ge 2z \ge 0$

$$\because 4x \ge 3y \Leftrightarrow a+b+c+d - \sqrt[4]{abcd} \ge a+b+c - \sqrt[3]{abc} \Leftrightarrow d + 3\sqrt[3]{abc} \ge 4\sqrt[4]{abcd}$$

It is true because:

$$d + 3\sqrt[3]{abc} = d + \sqrt[3]{abc} + \sqrt[3]{abc} + \sqrt[3]{abc} \overset{AM-GM}{\ge} 4\sqrt[4]{d\sqrt[3]{[abc]^3}} = 4\sqrt[4]{abcd}$$

$$\because 3y \ge 2z \Leftrightarrow a+b+c - 3\sqrt[3]{abc} \ge a+b - 2\sqrt{ab} \Leftrightarrow c + 2\sqrt{ab} \ge 3\sqrt[3]{abc}$$

It is true because $c + 2\sqrt{ab} = c + \sqrt{ab} + \sqrt{ab} \ge 3\sqrt[3]{c\sqrt{(ab)^2}} = 3\sqrt[3]{abc}$

$\because 2z \ge 0 \Leftrightarrow z \ge 0 \Leftrightarrow a + b \ge 2\sqrt{ab}$ *(true). Similarly:* $3y, 4x \ge 0$

Hence: $4x \ge 3y \ge 2z \ge 0$

More, $p \ge q \ge r \ge 0$ *then using Chebyshev's inequality:*

$$4xp + 3yq + 2zr \geq \frac{1}{3}(4x + 3y + 2z)(p + q + r)$$

$$\Leftrightarrow 3(4px + 3qy + 2rz) \geq (4x + 3y + 2z)(p + q + r)$$

SOLUTION 2.34

Solution by Tran Hong-Vietnam

$$\frac{\sin x}{\sin y} + \frac{\sin x + \sin y}{\sin z} = \frac{\sin x}{\sin y} + \frac{\sin x}{\sin z} + \frac{\sin y}{\sin z} \overset{(AM-GM)}{\geq} 3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2}$$

We must show that:

$$3\sqrt[3]{\left(\frac{\sin x}{\sin z}\right)^2} > \frac{6}{\pi}\sqrt[3]{\left(\frac{x}{z}\right)^2} \Leftrightarrow \left(\frac{\sin x}{\sin z}\right)^2 > \frac{8}{3} \cdot \left(\frac{x}{z}\right)^2 \Leftrightarrow \frac{\sin x}{\sin z} > \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \cdot \frac{x}{z}$$

$$\Leftrightarrow \pi\sqrt{\pi} \cdot \frac{\sin x}{x} > 2\sqrt{2} \cdot \frac{\sin z}{z} \quad (*)$$

Because: $\sin z < z$ ***and*** $\sin x > \frac{2x}{\pi}$ ***for*** $x, z \in \left(0, \frac{\pi}{2}\right)$

$$\pi\sqrt{\pi} \cdot \frac{\sin x}{x} > \pi\sqrt{\pi} \cdot \frac{2}{\pi} = 2\sqrt{\pi},\ \ 2\sqrt{2} \cdot \frac{\sin z}{z} < 2\sqrt{2} \cdot 1 = 2\sqrt{2}$$

We have: $2\sqrt{\pi} > 2\sqrt{2} \Rightarrow$ ***(*) true.***

SOLUTION 2.35

Solution by Anas Adlany-El Zemamra-Morocco

First, note that:

$$(a + b + c)^5 = a^4 + b^4 + c^4 + 5(a + b)(b + c)(c + a)(a^2 + b^2 + c^2 + ab + bc + ca)$$

Then when $a + b + c = 0$***, we will have the following***

$$a^5 + b^5 + c^5 = 5abc(a^2 + b^2 + c^2 + ab + bc + ca);$$

Now, let's go back to the main problem and check out what we are really dealing with the problem asks us to show that:

$$6\left(a^5+b^5+c^5\right) \geq 5(2ab+c^2)\left(2ab\sqrt{ab}+c^2\right) \textit{ whenever } a,b>0$$

We have: $6\left(a^5+b^5+c^5\right) \geq 5(2ab+c^2)\left(2ab\sqrt{ab}+c^3\right)$

$$\Leftrightarrow 6abc(a^3+b^3+c^3+ab+bc+ca) \geq (2ab+c^2)\left(2ab\sqrt{ab}+c^3\right)$$

$$\Leftrightarrow 2(a^3+b^3+c^3)(a^2+b^2+c^2+ab+bc+ca) \geq (2ab+c^2)\left(2ab\sqrt{ab}+c^2\right)$$

and the last step can be explained as follows:

$$a+b+c=0 \Rightarrow (a+b)^3+c^3=0 \Rightarrow a^3+b^3+c^3+3ab(a+b)=0$$

$$\Rightarrow a^3+b^3+c^3=3abc \qquad (a+b=-c)$$

Now, since $a,b>0$***; by the AM-GM inequality we get:***

$$a^3+b^3+c^3 \geq 2\sqrt{a^3b^3}+c^3-2ab\sqrt{ab}+c^3 \quad \textit{(1)}$$

Also,

$$2\sum(ab+c^2)-2ab-c^2=2ca+2bc+c^2+2a^2+2b^2$$

$$=2(a^2+b^2)+c^3+2c(a+b) \qquad (a+b=-c)$$

$$=2((a+b)^2-2ab)-c^2=2c^4-4ab-c^2=c^2-4ab=(a-b)^2 \geq 0$$

Which prove that : $2\sum(ab+c^2) \geq 2ab+c^2$ ***(2)***

Finaly, from results (1) & (2) the proof is completed.

SOLUTION 2.36

Solution by Marian Dincă – Romania

$$xyz=x+27y+125z \Leftrightarrow \frac{1}{yz}+\frac{27}{xz}+\frac{125}{xy}=1$$

$$1=\left(\frac{1}{yz}+\frac{27}{xz}+\frac{125}{xy}\right) \geq \frac{1^3}{\left(\frac{y+z}{2}\right)^2}+\frac{3^3}{\left(\frac{x+z}{2}\right)^2}+\frac{5^3}{\left(\frac{x+y}{2}\right)^2} \geq \frac{(1+3+5)^3}{\left(\frac{y+z}{2}+\frac{x+z}{2}+\frac{x+y}{2}\right)^2}$$

$AM - GM$ **and Radon inequality result:**

$$1 \geq \frac{(1+3+5)^3}{\left(\frac{y+z}{2}+\frac{x+z}{2}+\frac{x+y}{2}\right)^2} = \frac{9^3}{(x+y+z)^2}$$

$$(x+y+z)^2 \geq 9^3 \Leftrightarrow x+y+z \geq 27$$

SOLUTION 2.37

Solution by Soumitra Moukherjee - Chandar Nagore – India

$$e^a - e^c + e^b - e^d \geq 2\left(\sqrt{e^{a+b}} - \sqrt{e^{c+d}}\right) \Leftrightarrow e^a + e^b - 2\sqrt{e^{a+b}} \geq e^c + e^d - 2\sqrt{e^{c+d}}$$

$$\Leftrightarrow \left(\sqrt{e^b} - \sqrt{e^a}\right)^2 \geq \left(\sqrt{e^c} - \sqrt{e^d}\right)^2 \Leftrightarrow \sqrt{e^b} - \sqrt{e^a} \geq \sqrt{e^c} - \sqrt{e^d}$$

$$\Leftrightarrow \sqrt{e^d} - \sqrt{e^a} \geq \sqrt{e^c} - \sqrt{e^b} \qquad (1)$$

Let $f: \mathbb{R} \to \mathbb{R}$ ***be defined as*** $f(x) = \sqrt{e^x} \ \ \forall x \in \mathbb{R}$

$$f'(x) = \frac{\sqrt{e^x}}{2} \Rightarrow f''(x) = \frac{\sqrt{e^x}}{4} > 9 \ \ \forall x \in \mathbb{R}$$

$f(x)$ ***is a convex function:*** $\sqrt{e^d} - \sqrt{e^a} \geq \sqrt{e^c} - \sqrt{e^b}$

Hence statement (1) is true.

$$e^a - e^c + e^b - e^d \geq 2\left(\sqrt{e^{a+b}} - \sqrt{e^{c+d}}\right)$$

SOLUTION 2.38

Solution by Abdallah El Farissi – Bechar – Algerie

Let $f(x) = sh(x) + sh(a+b-c), x \in [a,b]$ ***we have***

$f''(x) = sh\,(x) + sh\,(a+b-x) = sh\,\left(\frac{a+b}{2}\right) ch\left(x - \frac{a+b}{2}\right)$ ***then f is a convex function, for all c*** $\in [a,b]$ ***there is*** $\lambda \in [0,1]$ ***such that*** $c = \lambda a + (1-\lambda)b$***, now***

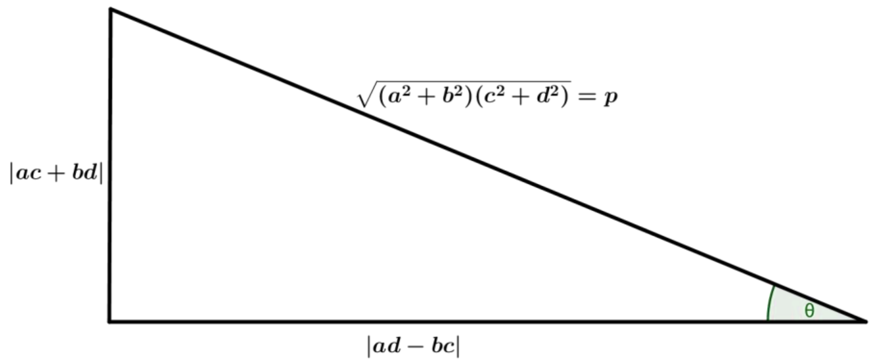
$$2sh\left(\frac{a+b}{2}\right) = sh\,\left(\frac{a+b-c+c}{2}\right)$$

$$\leq sh(c) + sh(a+b-c) = f(c) = f(\lambda + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) = sh(a) + sh(b)$$

SOLUTION 2.39

Solution by Soumava Chakraborty-Kolkata-India

$$(ac+bd)^2+(ad-bc)^2=(a^2+b^2)(c^2+d^2)$$



Given inequality $\Leftrightarrow 2p^4\sin^4\theta+2p^4\cos^4\theta\geq p^4\Leftrightarrow 2(\sin^4\theta+\cos^4\theta)\geq 1$

But, $2(\sin^4\theta+\cos^4\theta)\geq\frac{2}{2}(\sin^2\theta+\cos^2\theta)^2=1\ \left(\because x^2+y^2\geq\frac{1}{2}(x+y)^2\right)$

SOLUTION 2.40

Solution by Nirapada Pal-Jhargram-India

$$A=\frac{\sum a}{3}\overset{AM-HM}{\geq}\frac{3}{\sum\frac{1}{a}}=\frac{3abc}{\sum ab}=C$$

$$A-B=\frac{\sum a}{3}-\frac{\sum ab}{\sum a}=\frac{(\sum a)^2-3\sum ab}{3\sum a}=\frac{\sum a^2-\sum ab}{2\sum a}\geq 0$$

$$B-C=\frac{\sum ab}{\sum a}-\frac{3abc}{\sum ab}=\frac{(\sum ab)^2-3abc\sum a}{\sum a\sum ab}=\frac{\sum(ab)^2-\sum(ab)(bc)}{\sum a\sum ab}\geq 0$$

since $P^2+Q^2+R^2\geq PQ+QR+RS$, *So* $A\geq B\geq C$

And $0<x\leq y\Rightarrow x\leq\sqrt{xy}\leq y$

So, $\frac{3(Ax+B\sqrt{xy}+Cy)}{x+y+\sqrt{xy}}\overset{CHEBISHEV}{\leq}\frac{(A+B+C)(x+y+\sqrt{xy})}{x+y+\sqrt{xy}}=A+B+C$

SOLUTION 2.41

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\textit{Condición } c \geq b \geq a \Leftrightarrow 5c - 4a - b = 4(c - a) + (c - b) \geq 0$$

Desarrollan do la desigualdad

$$9ax^2 + 9ay^2 + 12bx^2 + 12by^2 + 18cx^2 + 18cy^2 + 18axy + 12bxy \geq$$

$$\geq 13ax^2 + 13bx^2 + 13cx^2 + 10xya + 10xyb + 10xyc +$$

$$+13ay^2 + 13by^2 + 13cy^2$$

$$\Leftrightarrow (-4ax^2 + 8axy - 4ay^2) + (-bx^2 + 2bxy - by^2) + (5cx^2 - 10xy + 5cy^2) \geq 0$$

$$\Leftrightarrow -4a(x-y)^2 - b(x-y)^2 + 5c(x-y)^2 = (x-y)^2(5c - 4a - b) \geq 0 \textit{ (LQQD)}$$

SOLUTION 2.42

Solution by Chris Kyriazis-Greece

$$\textit{The function } f(a,b) = \frac{a+1}{2^b} + \frac{b+3}{3^a} + (1-a)e^{1-b} - e - 4$$

$$0 \leq a \leq 1, 0 \leq b \leq 1, \textit{ is convex due to } a \textit{ or } b$$

(it's easy to check it with positive derivatives)

So the function achieves its maximum to one of the vertices of the square

$$[0,1] \times [0,1]$$

$$f(0,0) = 1 + 3 + e - e - 4 = 0$$

$$f(1,1) = \frac{2}{2} + \frac{4}{3} + 0 - e - 4 = \frac{4}{3} - e - 3 < 0$$

$$f(1,0) = 2 + 1 + 0 - e - 4 = -e - 1 < 0$$

$$f(0,1) = \frac{1}{2} + 4 + 1 - e - 4 = \frac{3}{2} - e < 0$$

So the maximum is zero when $a = 0$ and $b = 0$.

SOLUTION 2.43

Solution by proposer

$$\big((a+b)+(b+c)\big)\big((a+b)^2-(a+b)(b+c)+(b+c)^2\big)+4\geq$$

$$\geq 3b+3a+3b+3c$$

$$a+b=x$$

$$b+c=y$$

$$(x+y)(x^2-xy+y^2)+4\geq 3x+3y$$

$$x^3+y^3+4\geq 3x+3y$$

$$\left.\begin{array}{l} x^3+1+1\geq 3\sqrt[3]{x^3}=3x \\ y^3+1+1\geq 3\sqrt[3]{y^3}=3y \end{array}\right\}+$$

$$x^3+y^3+4\geq 3x+3y$$

SOLUTION 2.44

Solution by Ravi Prakash-New Delhi-India

Let $f(x)=x^2+\frac{1}{x^3}, x\geq 2$, $f'(x)=2x-\frac{3}{x^4}=\frac{2}{x^4}\left(x^5-\frac{3}{2}\right)>0, \forall x>2$

Thus, f increases on $[2,\infty)$.

For $2<a<b<c$, ***then*** $f(2)<f(a)<f(b)<f(c)$

We have

$$(\sqrt{b}+\sqrt{c})\left(a^2+\frac{1}{a^3}\right)+(\sqrt{a}-\sqrt{c})\left(b+\frac{1}{b^3}\right)=(\sqrt{b}+\sqrt{c})f(a)+(\sqrt{a}-\sqrt{c})f(b)$$

$$<(\sqrt{b}+\sqrt{c})f(b)+(\sqrt{a}-\sqrt{c})f(b)$$

$$=(\sqrt{b}+\sqrt{a})f(b)<(\sqrt{a}+\sqrt{b})f(c)=(\sqrt{a}+\sqrt{b})\left(c^2+\frac{1}{c^3}\right)$$

SOLUTION 2.45

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$LHS = 3\sqrt{a} + 3\sqrt[4]{b} + 2\sqrt[6]{c} + \sqrt[8]{d} > \sqrt[6]{ab} + \sqrt[12]{abc} + \sqrt[20]{abcd}$$

$$\Delta: \left.\begin{matrix} 3\sqrt{a} > \frac{3\sqrt{a}}{5} \\ 3\sqrt[4]{b} > \frac{6\sqrt[4]{b}}{5} \\ \sqrt[6]{c} > \frac{4}{5}\sqrt[6]{c} \\ \sqrt[8]{d} > \frac{2}{5}\sqrt[8]{d} \end{matrix}\right\} \text{(*)}, LHS > \frac{3\sqrt{a}}{5} + \frac{6\sqrt[4]{b}}{5} + \frac{4\sqrt[6]{c}}{5} + \frac{2\sqrt[8]{d}}{5} \text{ (**)}, \left.\begin{matrix} \frac{3\sqrt{a}}{5} = \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{10}\right)\sqrt{a} \\ \frac{6\sqrt[4]{b}}{5} = \left(\frac{2}{3} + \frac{2}{6} + \frac{2}{10}\right) \cdot \sqrt[4]{b} \\ \frac{4\sqrt[6]{c}}{5} = \left(\frac{3}{6} + \frac{3}{10}\right) \cdot \sqrt[6]{c} \\ \frac{2\sqrt[8]{d}}{5} = \frac{4}{10}\sqrt[8]{d} \end{matrix}\right\} \text{(***)}$$

$$\text{(**), (***)} \Rightarrow LHS > \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{10}\right) \cdot \sqrt{a} + \left(\frac{2}{3} + \frac{2}{6} + \frac{2}{10}\right) \cdot \sqrt[4]{b} +$$

$$+\left(\frac{3}{6} + \frac{3}{10}\right) \cdot \sqrt[6]{c} + \frac{4}{10} \cdot \sqrt[8]{d} = \left(\frac{\sqrt{a}}{3} + \frac{\sqrt[4]{b}}{3} + \frac{\sqrt[4]{b}}{3}\right) +$$

$$+\left(\frac{\sqrt{a}}{6} + \frac{2}{6} \cdot \sqrt[4]{b} + \frac{3}{6}\sqrt[6]{c}\right) + \left(\frac{1}{10}\sqrt{a} + \frac{2}{10}\sqrt[4]{b} + \frac{3}{10}\sqrt[6]{c} + \frac{4}{10}\sqrt[8]{d}\right)$$

$$\overset{AM \geq GM}{\geq} 3 \cdot \sqrt[3]{\frac{\sqrt{a} \cdot \sqrt[4]{b} \cdot \sqrt[4]{b}}{3^3}} + 6 \cdot \sqrt[6]{\frac{\sqrt{a}}{6} \cdot \frac{\sqrt[4]{b}}{6} \cdot \frac{\sqrt[4]{b}}{6} \cdot \frac{\sqrt[6]{c}}{6} \cdot \frac{\sqrt[6]{c}}{6} \cdot \frac{\sqrt[6]{c}}{6}} +$$

$$+10\sqrt[10]{\frac{\sqrt{a}}{10} \cdot \frac{\sqrt[4]{b}}{10} \cdot \frac{\sqrt[4]{b}}{10} \cdot \frac{\sqrt[6]{c}}{10} \cdot \frac{\sqrt[6]{c}}{10} \cdot \frac{\sqrt[6]{c}}{10} \cdot \frac{\sqrt[8]{d}}{10} \cdot \frac{\sqrt[8]{d}}{10} \cdot \frac{\sqrt[8]{d}}{10} \cdot \frac{\sqrt[8]{d}}{10}} = \sqrt{ab} + \sqrt[12]{abc} + \sqrt[20]{a \cdot b \cdot c \cdot d}$$

SOLUTION 2.46

Solution by Dat Vo-Quynh Luu-Vietnam

$$\frac{6}{a} - \frac{1}{b} + \frac{1}{c} = \frac{6bc + ab - ac}{abc}$$

$$\rightarrow \textit{inequality} \leftrightarrow 6bc + ab - ac < 3(a^2 + b^2 + c^2)$$

$$\leftrightarrow 3(b-c)^2 - a(b-c) + 3a^2 > 0 \leftrightarrow \frac{5}{4}(b-c+a)^2 + \frac{7}{4}(b-c-a)^2 > 0$$

SOLUTION 2.47

Solution by Nirapada Pal-Jhargram-India

$$\sqrt[3]{a+1}+\sqrt[5]{b+1}+\sqrt[7]{c+1}<6$$

$$\overset{GM-AM}{\leq} \frac{a+1+1+1}{3}+\frac{b+1+1+1+1+1}{5}+\frac{c+1+1+1+1+1+1+1}{7}=\frac{a+3}{3}+\frac{b+5}{5}+\frac{c+7}{7}$$

$$<\frac{3+3}{3}+\frac{5+5}{5}+\frac{7+7}{7}. \text{ As } 0\leq a<3, 0\leq b<5, 0\leq c<7$$

$$=2+2+2=6$$

SOLUTION 2.48

Solution by Ravi Prakash-New Delhi-India

For $0<a\leq b, k\in\mathbb{N}$*,*

$$a(a^k+b^k)\leq a^{k+1}+ab^k\leq a^{k+1}+b^{k+1}\leq a^kb+b^{k+1}=b(a^k+b^k)$$

$$\Rightarrow a\leq\frac{a^{k+1}+b^{k+1}}{a^k+b^k}\leq b$$

Taking $k=2,4,6$ ***and adding we get the desired inequality.***

SOLUTION 2.49

Solution by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a=x^4,\qquad b=y^4,\qquad c=z^4$$

$$x^4+y^4+z^4\geq(x+y)xyz$$

$$x^4+y^4+z^4\overset{Chebyshev}{\geq}\frac{1}{3}(x+y+z)(x^3+y^3+z^3)\overset{AM-GM}{\geq}xyz(x+y+z)$$

$$xyz(x+y+z)\geq(x+y)xyz\Rightarrow z\geq 0$$

SOLUTION 2.50

Solution by Ravi Prakash-New Delhi-India

Let $m\in\mathbb{N}$ ***and*** $\alpha>1$***.Consider*** $f(t)=t^{m+1}-1, t\in[1,\alpha]$

$$g(t)=t^m-1, t\in[1,\alpha]$$

By the Cauchy's mean value theorem $\exists c \in (1,\alpha)$ *such that*

$$\frac{f(\alpha)-f(1)}{g(\alpha)-g(1)}=\frac{f'(c)}{g'(c)}\Rightarrow\frac{\alpha^{m+1}-1}{\alpha^m-1}=\frac{(m+1)c^m}{mc^{m-1}}$$

$$=\frac{m+1}{m}c>\frac{m+1}{m}[\because c>1]\Rightarrow\frac{\alpha^{m+1}-1}{m+1}>\frac{\alpha^m-1}{m}$$

If $b>a\geq 1$***, then***

$$\frac{\left(\frac{b}{a}\right)^{m+1}-1}{m+1}>\frac{\left(\frac{b}{a}\right)^{m}-1}{m}\Rightarrow\frac{b^{m+1}-a^{m+1}}{m+1}>\frac{b^m-a^m}{m}a\geq\frac{b^m-a^m}{m}$$

Thus,

$$\frac{b^{m+1}-a^{m+1}}{m+1}>\frac{b^m-a^m}{m},\ \forall b>a\geq 1, m\in N$$

$\therefore$ ***if*** $1\leq a<b<c$***, then***

$$\frac{b^3-a^3}{3}+\frac{c^5-a^5}{5}+\frac{c^7-a^7}{7}<\frac{b^4-a^4}{4}+\frac{c^6-b^6}{6}+\frac{c^8-a^8}{8}$$

SOLUTION 2.51

Solution by Geanina Tudose-Romania

$$9(a+b)\sqrt{ab}+6(a+b+c)\sqrt[3]{abc}+18c^2\geq$$

$$\geq(5a+5b+8c)\left(c+\sqrt{ab}+\sqrt[3]{ab}\right)$$

$$\Leftrightarrow 9(a+b)\sqrt{ab}+6(a+b+c)\sqrt[3]{abc}+18c^2\geq 5c(a+b+c)+5(a+b)\sqrt{ab}$$

$$+5c\sqrt{ab}+5(a+b+c)\sqrt[3]{abc}+3c^2+3c\sqrt{ab}+3c\sqrt[3]{abc}$$

$$\Leftrightarrow 4(a+b)\sqrt{ab}+(a+b+c)\sqrt[3]{abc}+15c^2\geq 5c(a+b+c)+8c\sqrt{ab}+3c\sqrt[3]{abc}$$

$$\Leftrightarrow 4(a+b-2c)\sqrt{ab}+(a+b-2c)\sqrt[3]{abc}+5c(2c-a-b)\geq 0$$

$$\Leftrightarrow (2c-a-b)\left(5c-\sqrt[3]{abc}-4\sqrt{ab}\right)\geq 0\quad \textit{(1)}$$

AM ≥ GM we have

$$4\sqrt{ab} \le 4 \cdot \frac{a+b}{2} = 2(a+b) \le 4c,\ \sqrt[3]{abc} \le \frac{a+b+c}{3} \le \frac{3c}{3} = c$$

Hence $5c - \sqrt[3]{abc} - 4\sqrt[3]{ab} \ge 0$ *and* $2c - a - b \ge 0$

Therefore (1) is true

SOLUTION 2.52

Solution by Abdallah El Farissi-Bechar-Algerie

If $abc = 1$*, then* $\frac{a}{1+a} + \frac{b}{(1+a)(1+b)} + \frac{c}{(1+a)(1+b)(1+c)} \ge \frac{7}{8}$

We have $(1+a)(1+b)(1+c) \ge 8\sqrt{abc} = 8$ *then* $1 - \frac{1}{(1+a)(1+b)(1+c)} \ge \frac{7}{8}$

and $\frac{a}{1+a} + \frac{b}{(1+a)(1+b)} + \frac{c}{(1+a)(1+b)(1+c)} = 1 - \frac{1}{(1+a)(1+b)(1+c)}$

generalization

if $x_i \ge 0\ (i = 1, 2, \dots, n) - \prod_{i=1}^{n} x_i = 1$*, then*

$$\sum_{i=1}^{n} \frac{x_i}{\prod_{k=1}^{i}(x_k + 1)} \ge \frac{2^n - 1}{2^n}$$

SOLUTION 2.53

Solution by Geanina Tudose-Romania

Rewritting the inequality

$$(1+a)^3 \cdot (1+b)^3 \cdot (1+c)^2 \cdot (1+d) \le 64(1+ab)(1+abc)(1+abcd)$$

We show: $(1+a)(1+b) \le 2(1+ab)$ *(1)*

$(1+a)(1+b)(1+c) \le 4(1+abc)$ *(2)*

and $(1+a)(1+b)(1+c)(1+d) \le 8(1+abcd)$ *(3)*

In fact, we have more generally

$$(1+a_1)(1+a_2)\dots(1+a_n)\le 2^{n-1}(1+a_1\dots a_n)$$

for $(\forall)n\ge 2, a_i\ge 1, t>i$***, we show it by induction:***

$$P(2):(1+a_1)(1+a_2)\le 2(1+a_1a_2)$$

$$\Leftrightarrow 1+a_1a_2+a_1a_2\le 2+2a_1a_2 \Leftrightarrow (a-1)(a_2-1)\ge 0 \text{ true } P(n)\to P(n+1)$$

$$(1+a_1)\dots(1+a_n)(1+a_{n+1}) \overset{P(n)}{\lesssim} 2^{n-1}(1+a_1\dots a_n)(1+a_{n+1}) \overset{P(2)}{\lesssim}$$

$$\le 2^{n-1}\cdot 2\cdot(1+a_1\dots a_n)$$

Thus multiplying (1), (2), (3) we have the desired inequality

SOLUTION 2.54

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$0<x\le y\le z<1$$

$$A=(y-x)arc\tan x+(z-x)arc\tan y+(z-y)arc\tan z$$

$$=(y-x)arc\tan x+(z-y+y-x)arc\tan y+(z-y)arc\tan z$$

$$=(y-x)(arc\tan x+arc\tan y)+(z-y)(arc\tan z+arc\tan y)$$

$$=(y-x)arc\tan\left(\frac{x+y}{1-xy}\right)+(z-y)arc\tan\left(\frac{z+y}{1-zy}\right)$$

$$\text{“ } xy\le zy<1\text{”}$$

$$\text{“ } \frac{x+y}{1-xy}<1;\frac{z+y}{1-zy}<1\text{”}$$

$$<(y-x)arc\tan 1+(z-y)arc\tan 1=(z-x)\frac{\pi}{4}<\frac{\pi}{4}<\frac{\pi}{2}-\log 2$$

SOLUTION 2.55

Solution by Kevin Soto Palacios – Huarmey – Peru

Es suficiente probar

$$\frac{1}{1+a}+\frac{1}{1+b}\geq\frac{2}{1+\sqrt{ab}}\ \ \textit{(A)}$$

$$\Leftrightarrow\frac{1}{1+a}-1+\frac{1}{1+b}\geq\frac{2}{1+\sqrt{ab}}-1\Leftrightarrow\frac{1}{1+b}-\frac{a}{1+b}\geq\frac{1-\sqrt{ab}}{1+\sqrt{ab}}\Leftrightarrow$$

$$\Leftrightarrow\frac{1-ab}{(1+a)(1+b)}\geq\frac{1-\sqrt{ab}}{1+\sqrt{ab}}$$

$$\Leftrightarrow\frac{(1+\sqrt{ab})(1-\sqrt{ab})}{(1+a)(1+b)}\geq\frac{1-\sqrt{ab}}{1+\sqrt{ab}}\Leftrightarrow\left(1-\sqrt{ab}\right)\left[\frac{(1+\sqrt{ab}}{(1+a)(1+b)}-\frac{1}{1+\sqrt{ab}}\right]=$$

$$=\left(1-\sqrt{ab}\right)\left[\frac{\left(1+\sqrt{ab}\right)^2-(1+a)(1+b)}{(1+a)(1+b)\left(1+\sqrt{ab}\right)}\right]=$$

$$=\frac{(\sqrt{ab}-1)(\sqrt{a}-\sqrt{b})^2}{(1+a)(1+b)(1+\sqrt{ab})}\geq 0\text{, lo cual es cierto ya que}\rightarrow ab\geq 1$$

Analogamente para los siguientes términos

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{\sqrt[3]{abc}}\geq\frac{2}{1+\sqrt{ab}}+\frac{2}{1+\sqrt{c\cdot\sqrt[3]{abc}}}\geq\frac{4}{1+\sqrt[3]{abc}}\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\geq\frac{3}{1+\sqrt[3]{abc}}\ \ \textit{(B)}$$

$$\Rightarrow\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\geq\frac{2}{1+\sqrt{ab}}+\frac{2}{1+\sqrt{ca}}\geq\frac{4}{1+\sqrt[4]{abcd}}\ \ \textit{(C)}$$

Sumando (A)+(B)+(C)

$$\Rightarrow\frac{3}{1+a}+\frac{3}{1+b}+\frac{2}{1+c}+\frac{1}{1+d}\geq\frac{2}{1+\sqrt{ab}}+\frac{3}{1+\sqrt[3]{abc}}+\frac{4}{1+\sqrt[4]{abcd}}$$

Por lo tanto

$$3\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}\right)=\left(\frac{3}{1+a}+\frac{3}{1+b}+\frac{2}{1+c}+\frac{1}{1+d}\right)+\left(\frac{1}{1+c}+\frac{2}{1+d}\right)\geq$$

$$\geq \frac{2}{1+\sqrt{ab}}+\frac{3}{1+\sqrt[3]{abc}}+\frac{4}{1+\sqrt[4]{abcd}}+\left(\frac{1}{1+c}+\frac{2}{1+d}\right)>$$

$$>\frac{2}{1+\sqrt{ab}}+\frac{3}{1+\sqrt[3]{abc}}+\frac{4}{1+\sqrt[4]{abcd}}$$

SOLUTION 2.56

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\forall\ 2<x\leq y \quad f'_y(x)=\ln(x-1)\ln y-\ln(y-1)\ln x$$

$$\Rightarrow f'_y(x)=\frac{\ln y}{x-1}-\frac{\ln(y-1)}{x}=\frac{x\ln y-(x-1)\ln(y-1)}{x(x-1)}>0$$

$$x\leq y\Leftrightarrow f_y(x)\leq f_y(y)\Leftrightarrow f_y(x)\leq 0$$

$$2<a\leq b\leq c\Rightarrow f_b(a)\leq 0\ \&\ f_c(b)\leq 0\ \&\ f_c(a)\leq 0$$

$$\Rightarrow f_b(a)+f_c(b)+f_c(a)\leq 0$$

$$\Leftrightarrow [\ln(a-1)\ln b-\ln(b-1)\ln a]+[\ln(b-1)\ln c-\ln(c-1)\ln b]+$$

$$+[\ln(a-1)\ln c-\ln(c-1)\ln a]\leq 0$$

$$\Leftrightarrow \ln(a-1)\ln(bc)+\ln(b-1)\ln c\leq \ln(c-1)\ln(ab)+\ln(b-1)\ln a$$

SOLUTION 2.57

Solution by Anas Adlany-El Zemamra-Morocco

First, note that:

$$(a+b+c)^5=a^4+b^4+c^4+5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+bc+ca)$$

Then when $a+b+c=0$, we will have the following

$$a^5+b^5+c^5=5abc(a^2+b^2+c^2+ab+bc+ca);$$

Now, let's go back to the main problem and check out what we are really dealing with the problem asks us to show that:

$$6\left(a^5+b^5+c^5\right)\geq 5(2ab+c^2)\left(2ab\sqrt{ab}+c^2\right)$$

whenever $a, b > 0$

We have

$$6\left(a^5+b^5+c^5\right)\geq 5(2ab+c^2)\left(2ab\sqrt{ab}+c^3\right)$$

$$\Leftrightarrow 6abc(a^3+b^3+c^3+ab+bc+ca)\geq (2ab+c^2)\left(2ab\sqrt{ab}+c^3\right)$$

$$\Leftrightarrow 2(a^3+b^3+c^3)(a^2+b^2+c^2+ab+bc+ca)\geq (2ab+c^2)\left(2ab\sqrt{ab}+c^2\right)$$

and the last step can be explained as follows:

$$a+b+c=0\Rightarrow (a+b)^3+c^3=0\Rightarrow a^3+b^3+c^3+3ab(a+b)=0$$

$$\Rightarrow a^3+b^3+c^3=3abc \qquad (a+b=-c)$$

Now, since $a, b > 0$***; by the AM-GM inequality we get:***

$$a^3+b^3+c^3\geq 2\sqrt{a^3b^3}+c^3-2ab\sqrt{ab}+c^3 \quad \textit{(1). Also,}$$

$$2\sum(ab+c^2)-2ab-c^2=2ca+2bc+c^2+2a^2+2b^2$$

$$=2(a^2+b^2)+c^3+2c(a+b) \qquad (a+b=-c)$$

$$=2((a+b)^2-2ab)-c^2=2c^4-4ab-c^2=c^2-4ab=(a-b)^2\geq 0$$

Which prove that: $2\sum(ab+c^2)\geq 2ab+c^2$ ***(2)***

Finaly, from results (1) & (2) the proof is completed.

SOLUTION 2.58

Solution by Chris Kyriazis-Greece

If $a=b$ ***or*** $b=c$ ***or*** $a=c$***, the inequality is obvious***

Let's suppose that $a<b<c$

The function $f(x) = e^x$ is strictly convex, so using the Hermite-Hadamard inequality we have that:

$$\frac{\int_a^b f(x)dx}{b-a} \geq f\frac{(a+b)}{2} \Leftrightarrow \frac{e^b - e^a}{b-a} \geq e^{\frac{a+b}{2}}$$

$$\overset{\frac{a+b}{2}\geq\sqrt{ab}}{\Rightarrow} \quad e^b - e^a \geq (b-a)e^{\sqrt{ab}}$$ ***. Doing exactly the same:***

$$e^c - a^a \geq (c-a)e^{\sqrt{ac}}$$

$$e^a - e^b \geq (c-b)e^{\sqrt{cb}}$$

Multiplying those three inequalities (everything is positive) we have that

$$\left(e^b - e^a\right)(e^c - e^a)\left(e^c - e^b\right) \geq (b-a)(c-a)(c-b)e^{\sqrt{ab}+\sqrt{bc}+\sqrt{ca}}$$

SOLUTION 2.59

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\therefore \textit{let} \quad f(x;z) = \frac{\ln x - \ln z}{x-z};\ (x;z) \in (\mathbb{R}_+^*)^2$$

$$\Rightarrow f'_x(x,z) = \frac{\frac{1}{x}(x-z) - (\ln x - \ln z)}{(x-z)^2} = \frac{1 - \left(\frac{z}{x}\right) + \ln\left(\frac{z}{x}\right)}{(x-z)^2} \leq 0$$

$$\because \ln X + 1 \leq X: \forall X > 0$$

$$* f(x;z) = f(z;x) \Rightarrow f'_z(x;z) \leq 0$$

$$\because \begin{cases} 0 < a < b \\ 0 < c < d \end{cases} \Rightarrow f(a;c) > f(b;d) \Leftrightarrow \frac{\ln a - \ln c}{a-c} > \frac{\ln b - \ln d}{b-d}$$

$$\Leftrightarrow \ln\left(\left(\frac{a}{c}\right)^{\frac{1}{a-c}}\right) > \ln\left(\left(\frac{b}{d}\right)^{\frac{1}{b-d}}\right) \Leftrightarrow \left(\frac{a}{c}\right)^{\frac{1}{a-c}} > \left(\frac{b}{d}\right)^{\frac{1}{b-d}}$$

SOLUTION 2.60

Solution by Ravi Prakash-New Delhi-India