For $\mathbf{0} \leq \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} < 1$*, consider*

$$\Delta = (\mathbf{1} - \boldsymbol{abc})^3(\mathbf{1} + \boldsymbol{a}^3)(\mathbf{1} + \boldsymbol{b}^3)(\mathbf{1} + \boldsymbol{c}^3) -$$

$$-(\mathbf{1} + \boldsymbol{abc})^3(\mathbf{1} - \boldsymbol{a}^3)(\mathbf{1} - \boldsymbol{b}^3)(\mathbf{1} - \boldsymbol{c}^3)$$

$$= \begin{vmatrix} (\mathbf{1} - \boldsymbol{abc})^3 & (\mathbf{1} + \boldsymbol{abc})^3 \\ (\mathbf{1} - \boldsymbol{a}^3)(\mathbf{1} - \boldsymbol{b}^3)(\mathbf{1} - \boldsymbol{c}^3) & (\mathbf{1} + \boldsymbol{a}^3)(\mathbf{1} + \boldsymbol{b}^3)(\mathbf{1} + \boldsymbol{c}^3) \end{vmatrix}$$

Use $c_2 \to c_2 - c_1$ *to obtain*

$$\Delta = \begin{vmatrix} (\mathbf{1} - \boldsymbol{abc})^3 & \mathbf{6}\boldsymbol{abc} + \mathbf{2}\boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \\ (\mathbf{1} - \boldsymbol{a}^3)(\mathbf{1} - \boldsymbol{b}^3)(\mathbf{1} - \boldsymbol{c}^3) & \mathbf{2}(\boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3) + \mathbf{2}\boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \end{vmatrix}$$

Use $c_1 \to c_1 + \frac{1}{2}c_2$

$$\Delta = \mathbf{2}\begin{vmatrix} \mathbf{1} + \mathbf{3}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 & \mathbf{3}\boldsymbol{abc} + \boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \\ \mathbf{1} + \boldsymbol{a}^3\boldsymbol{b}^3 + \boldsymbol{b}^3\boldsymbol{c}^3 + \boldsymbol{c}^3\boldsymbol{a}^3 & \boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 + \boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \end{vmatrix}$$

Use $R_2 \to R_1 - R_1$

$$\Delta = \mathbf{2}\begin{vmatrix} \mathbf{1} + \mathbf{3}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 & \mathbf{3}\boldsymbol{abc} + \boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \\ \boldsymbol{a}^3\boldsymbol{b}^3 + \boldsymbol{b}^3\boldsymbol{c}^3 + \boldsymbol{c}^3\boldsymbol{a}^3 - \mathbf{3}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 & \boldsymbol{a}^3 + \boldsymbol{b}^3 + \boldsymbol{c}^3 - \mathbf{3}\boldsymbol{abc} \end{vmatrix}$$

Note that

$$\mathbf{1} + \mathbf{3}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 - (\mathbf{3}\boldsymbol{abc} + \boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3) = (\mathbf{1} - \boldsymbol{abc})^3 > 0$$

$$\Rightarrow \mathbf{1} + \mathbf{3}\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 > 3\boldsymbol{abc} + \boldsymbol{a}^3\boldsymbol{b}^3\boldsymbol{c}^3 \quad \textit{(1)}$$

Also, $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} \geq \boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca}$ *and* $(\boldsymbol{a} - \boldsymbol{b})^2 + (\boldsymbol{b} - \boldsymbol{c})^2 + (\boldsymbol{c} - \boldsymbol{a})^2 \geq$

$$\geq \boldsymbol{c}^2(\boldsymbol{a} - \boldsymbol{b})^2 + \boldsymbol{a}^2(\boldsymbol{b} - \boldsymbol{c})^2 + \boldsymbol{b}^2(\boldsymbol{c} - \boldsymbol{a})^2$$

$$[\because \mathbf{0} \leq \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} < 1]$$

$$\Rightarrow \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})[(\boldsymbol{a} - \boldsymbol{b})^2 + (\boldsymbol{b} - \boldsymbol{c})^2 + (\boldsymbol{c} - \boldsymbol{a})^2]$$

$$\geq \frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{ab} + \boldsymbol{bc} + \boldsymbol{ca})[\boldsymbol{c}^2(\boldsymbol{a} - \boldsymbol{b})^2 + \boldsymbol{a}^2(\boldsymbol{b} - \boldsymbol{c})^2 + \boldsymbol{b}^2(\boldsymbol{c} - \boldsymbol{a})^2]$$

$$\Rightarrow a^3+b^3+c^3-3abc \geq a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2 \quad (2)$$

From (1), (2), we get

$$(1+3a^2b^2c^2)(a^3+b^3+c^3-3abc) \geq 3abc[a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2]$$

$$\Rightarrow \Delta \geq 0 \Rightarrow (1-abc)^3(1+a^3)(1+b^3)(1+c^3) \geq (1+abc)^3(1-a^3)(1-b^3)(1-c^3)$$

Put $a=x, b=y^2, c=z^3$ *to obtain:* $\frac{(1+x^3)(1+y^6)(1+z^3)}{(1-x^3)(1-y^6)(1-z^3)} \geq \frac{(1+xy^2z^3)^3}{(1-xy^2z^3)^3}$

SOLUTION 2.61

Solution by Soumitra Mandal-Chandar Nagore-India

LEMMA: For any concave function $f:[a,b]\to\mathbb{R}$ *and* $x\in(a,b)$ *we have*

$$\frac{f(b)-f(x)}{b-x} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(x)-f(a)}{x-a}$$

Let $f(x)=\tan^{-1}x$ *for any* $x\in[0,d], f'(x)=\frac{1}{1+x^2}, f''(x)=-\frac{2x}{(1+x^2)^2}\leq 0$

so, f *is concave, hence for* $a<b<c$ *we have*

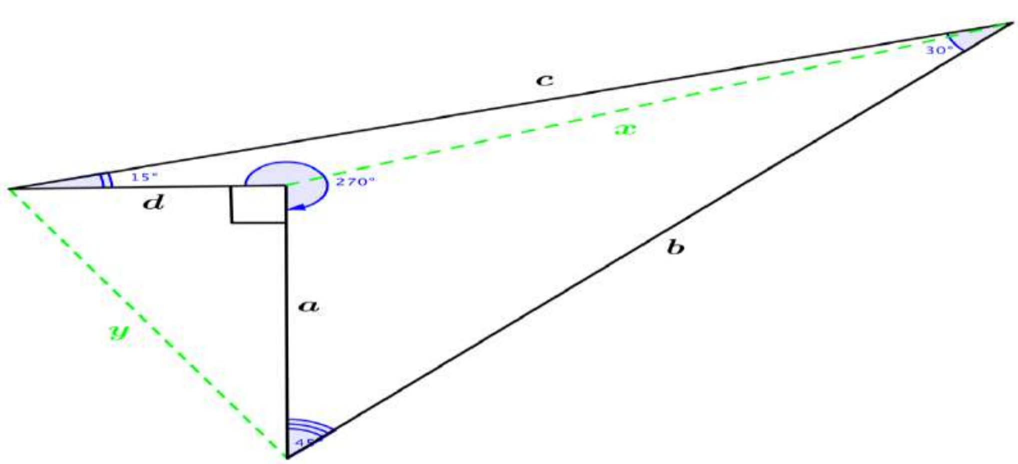
$$\frac{\tan^{-1}c-\tan^{-1}b}{c-b} < \frac{\tan^{-1}c-\tan^{-1}a}{c-a} < \frac{\tan^{-1}b-\tan^{-1}a}{b-a} \quad (1)$$

for $b<c<d, \frac{\tan^{-1}d-\tan^{-1}c}{d-c} < \frac{\tan^{-1}d-\tan^{-1}b}{d-b} < \frac{\tan^{-1}c-\tan^{-1}b}{c-b} \quad (2)$

combining (1) and (2) we have, $\frac{1}{d-b}\tan^{-1}\left(\frac{d-b}{1+bd}\right) < \frac{1}{c-a}\tan^{-1}\left(\frac{c-a}{1+ca}\right)$

SOLUTION 2.62

Solution by Soumava Chakraborty-Kolkata-India



Let us consider a quadrilateral with angle between 'a' and 'b' $= 45°$*, angle between 'b' and 'c'* $= 30°$*, angle between 'c' and 'd'* $= 15°$ *and angle between 'a' and 'd'* $= 270°$

$$\text{Then } x = \sqrt{a^2 + b^2 - 2ab\cos 45°} = \sqrt{c^2 + d^2 - 2cd\cos 15°}$$

$$\left(\cos 15° = \frac{\sqrt{6} + \sqrt{2}}{4}\right)$$

$$y = \sqrt{b^2 + c^2 - 2bc\cos 30°} = \sqrt{a^2 + d^2}$$

$$\therefore x + y + x > y$$

$$\Rightarrow \sqrt{a^2 + b^2 - ab\sqrt{2}} + \sqrt{b^2 + c^2 - bc\sqrt{3}} + \sqrt{c^2 + d^2 - \frac{cd(\sqrt{6} + \sqrt{2})}{4}} > \sqrt{a^2 + d^2}$$

SOLUTION 2.63

Solution by Dat Vo-Quynh Luu – VietNam:

$$25\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3 - 27\left(\frac{1}{c} + \frac{5}{b} - \frac{1}{a}\right)$$

$$= 25\left[\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^3 - \frac{27}{abc}\right] + \frac{27}{abc}[25 - (5ac + ab - bc)]$$

$$= 25\left[3\sum\frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{1}{2}(a + b + c)\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2\right]$$

$$+ \frac{27}{abc}\left[\frac{5}{2}(a^2 + b^2 + c^2) - (5ac + ab - bc)\right]$$

$$= 25\left[3\sum\frac{1}{c}\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{1}{2}(a + b + c)\sum\left(\frac{1}{a} - \frac{1}{b}\right)^2\right]$$

$$+ \frac{27}{2abc}[3(a - c - b)^2 + 2(a - c + b)^2] \geq 0$$

SOLUTION 2.64

Solution by proposer

We have

$$\sum \frac{2a^2+bc}{b+c} = \sum \frac{(a-b)(a-c)+a(a+b+c)}{b+c}$$

$$= \sum \frac{(a-b)(a-c)}{b+c} + \sum \frac{(b-a)(b-c)}{c+a} + (a+b+c)\sum \frac{a}{b+c}$$

$$= \sum \frac{a+b}{(a+c)(b+c)}(a-b)^2 + \frac{3}{2}(a+b+c)$$

$$\frac{a+b}{(a+c)(b+c)}(a-b)^2 = \frac{(a+b)^2}{(a+b)(b+c)(c+a)}(a-b)^2 \geq$$

$$\geq \frac{4ab}{(a+b)(b+c)(c+a)}(a-b)^2 = \frac{4}{(a+b)(b+c)(c+a)} \cdot \frac{(a-b)^2}{\frac{1}{ab}}$$

Similarly $\frac{b+c}{(a+b)(a+c)}(b-c)^2 \geq \frac{4}{(a+b)(b+c)(c+a)} \cdot \frac{(b-c)^2}{\frac{1}{bc}}$,

$$\frac{a+c}{(a+b)(b+c)}(a-c)^2 \geq \frac{4}{(a+b)(b+c)(c+a)} \cdot \frac{(a-c)^2}{\frac{1}{ac}}.$$

$$\sum \frac{a+b}{(a+c)(b+c)}(a-b)^2 \geq \frac{4}{(a+b)(b+c)(c+a)} \cdot \frac{(a-b+b-c+a-c)^2}{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}$$

$$= \frac{16abc(a-c)^2}{(a+b+c)(a+b)(b+c)(c+a)}$$

$$\Rightarrow \frac{2a^2+bc}{b+c} + \frac{2b^2+ca}{c+a} + \frac{2c^2+ab}{a+b} \geq \frac{3}{2}(a+b+c) + \frac{16abc(a-c)^2}{(a+b+c)(a+b)(b+c)(c+a)}$$

SOLUTION 2.65

Solution by Ravi Prakash - New Delhi – India

Let

$$x = b+c+\eta$$

$$y = c+a+\eta$$

$$z = a+b+\eta$$

$$\textit{Now, } a-b=x-y \textit{ and analogous:}$$

$$RHS=\sum|(a-b)(c+b+\eta)(c+a+\eta)|=\sum|(x-y)xy|\ge$$

$$\ge|(x-y)xy+(y-z)yz+(z-x)xz|=\left|x^2(y-z)+y^2(z-x)+z^2(x-y)\right|=$$

$$=\left\|\begin{matrix}1&1&1\\x&y&z\\x^2&y^2&z^2\end{matrix}\right\|=|(x-y)(y-z)(z-x)|=|(a-b)(b-c)(c-a)|$$

SOLUTION 2.66

Solution by Soumava Pal – Kolkata-India

$$\left((|x|^2)^{\frac{1}{n}}+(|y|^2)^{\frac{1}{n}}\right)^n=\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)^n=$$

$$=|x|^2+\binom{n}{1}|x|^{\frac{2(n-1)}{n}}|y|^{\frac{2}{n}}+\cdots+|y|^2>|x|^2+|y|^2=x^2+y^2$$

$\sum_{i=1}^{n-1}\binom{n}{i}|x|^{\frac{2(n-i)}{n}}|y|^{\frac{2i}{n}}>0$ ***- because all terms are positive here***

$$\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)^n>x^2+y^2\Rightarrow(x^2+y^2)^{\frac{1}{n}}<|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}$$

Putting $(x,y,n)=(a,b,6),(b,c,10),(c,a,14)$ ***we get***

$$(a^2+b^2)^{\frac{1}{6}}<|a|^{\frac{1}{3}}+|b|^{\frac{1}{3}},(b^2+c^2)^{\frac{1}{10}}<|b|^{\frac{1}{5}}+|c|^{\frac{1}{5}},(c^2+a^2)^{\frac{1}{14}}<|c|^{\frac{1}{7}}+|a|^{\frac{1}{7}}$$

$$\Rightarrow\frac{\sqrt[6]{a^2+b^2}}{|a|^{\frac{1}{3}}+|b|^{\frac{1}{3}}}<1,\frac{\sqrt[10]{b^2+c^2}}{\sqrt[5]{|b|}+\sqrt[5]{|c|}}<1,\frac{\sqrt[14]{c^2+a^2}}{\sqrt[7]{|c|}+\sqrt[7]{|a|}}<1$$

Adding them we get required inequality. So it is true.

SOLUTION 2.67

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar para todos los números reales x,y,z***:***

$$(x^2+2y^2+2z^2+3xy+5yz+3zx)^2\ge 8(x+y)(y+z)(z+x)(x+y+z)$$

$$x^2+2y^2+2z^2+3xy+5yz+3zx=(x^2+2yx+xz)+(yx+2y^2+yz)+$$

$$+(2zx+4yz+2z^2)$$

$$\Rightarrow x^2+2y^2+2z^2+3xy+5yz+3zx=x(x+2y+z)+y(x+2y+z)+$$

$$+2z(x+2y+z)$$

$$\Rightarrow x^2+2y^2+2z^2+3xy+5yz+3zx=(x+2y+z)(x+y+2z)$$

Realizamos los siguientes cambios de variables:

$$x+y=a, y+z=b, z+x=c,$$

$$2(x+y+z)=a+b+c\Rightarrow \left((a+b)(b+c)\right)^2\geq 4abc(a+b+c)$$

$$\Rightarrow \left(b(b+a)+c(b+a)\right)^2\geq 4a^2bc+4a^2bc+4abc^2$$

$$\Rightarrow b^2(b^2+a^2+2ab)+c^2(b^2+a^2+2ab)+2bc(b^2+a^2+2ab)\geq$$

$$\geq 4a^2bc+4b^2ac+4abc^2$$

$$\Rightarrow a^2(b^2+c^2-2bc)+b^2(b^2+c^2+2bc)+2ab(b^2-c^2)\geq 0$$

$$\Rightarrow \left(a(b-c)+b(b+c)\right)^2\geq 0 \Leftrightarrow \textit{La iguldad se alcanza cuando:}(x,y,z)\rightarrow(0,0,0)$$

SOLUTION 2.68

Solution by Soumava Chakraborty-Kolkata-India

$$a,b,c,d>0\Rightarrow$$

$$\frac{ac+bd+|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}+\frac{(a^2+b^2)(c^2+d^2)}{(ac+bd)|ad-bc|}\underbrace{\geq}_{(1)}2+\sqrt{2}$$

$$\textit{Let } \sqrt{ac+bd}=x, \sqrt{|ad-bc|}=y, \sqrt{(a^2+b^2)(c^2+d^2)}=z$$

$$LHS=\frac{x^2+y^2}{z}+\frac{z^2}{x^2y^2}$$

$$\textit{Now, } x^4+y^4=(ac+bd)^2+(ad-bc)^2=(a^2+b^2)(c^2+d^2)=z^2$$

$$\left.\begin{aligned} x^2 &= z\sin\theta \\ y^2 &= z\cos\theta \end{aligned}\right\} 0 < \theta < \frac{\pi}{2}$$

$$\therefore LHS = \frac{z(\cos\theta + \sin\theta)}{z} + \frac{z^2}{z^2 \sin\theta\cos\theta} = \cos\theta + \sin\theta + \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \cos\theta + \sin\theta + \tan\theta + \cot\theta = f(\theta)$$

$$f'(\theta) = \cos\theta - \sin\theta + \sec^2\theta - csc^2\theta = \cos\theta - \sin\theta + \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta} =$$

$$= \cos\theta - \sin\theta - \frac{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}{\cos^2\theta\sin^2\theta} = (\cos\theta - \sin\theta)\left(1 - \frac{\cos\theta + \sin\theta}{\cos^2\theta\sin^2\theta}\right)$$

$$f'(\theta) \Rightarrow (\cos\theta - \sin\theta)(\cos^2\theta\sin^2\theta - \cos\theta - \sin\theta) = 0$$

If $\cos^2\theta\sin^2\theta = \cos\theta + \sin\theta$***, then***

$$\cos^4\theta\sin^4\theta = 1 + 2\cos\theta\sin\theta$$

$\Rightarrow t^4 - 2t - 1 = 0$ ***(where*** $t = \cos\theta\sin\theta$***)***$\Rightarrow t^4 = 2t + 1$

Now, $\because 0 < \theta < \frac{\pi}{2}, \therefore t > 0 \Rightarrow 2t + 1 > 1$

$\therefore RHS > 1$***. Now,*** $LHS = \left(\frac{1}{2}\sin^2\theta\right)^4 < \frac{1}{16}$

So, $RHS > 1$ ***and*** $LHS < \frac{1}{16} \Rightarrow t^4 - 2t - 1 = 0$ ***has no real root***

$$\Rightarrow \cos^2\theta\sin^2\theta - \cos\theta - \sin\theta \neq 0 \therefore f'(\theta) = 0 \Rightarrow \cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}$$

$$f''(\theta) = -\sin\theta - \cos\theta + 2\sec^2\theta\tan\theta + 2\csc^2\theta\cot\theta$$

At $\theta = \frac{\pi}{4}, f''(\theta) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 4 + 4 = 8 - \sqrt{2}$

$> 0 \Rightarrow$ ***at*** $\theta = \frac{\pi}{4}, f(\theta)$ ***attains a minimal and*** $\because f(\theta)$ ***never attains a maxima,***

$$\therefore f(\theta) \geq f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + 1 = 2 + \sqrt{2}$$

$$\therefore \textit{LHS of (1)} = f(\theta) \geq 2 + \sqrt{2}$$

SOLUTION 2.69

Solution by Ravi Prakash-New Delhi-India

$$\textit{Now, let } \frac{x}{t} = x_1, \frac{y}{t} = y_1, \frac{z}{t} = z_1 \textit{ where } 0 < x_1, y_1, z_1 \leq 1$$

$$\textit{Now, } 4(xyzt)^{\frac{1}{4}} - 3(xyz)^{\frac{1}{3}} = t\left[4(x_1y_1z_1)^{\frac{1}{4}} - 3(x_1y_1z_1)^{\frac{1}{3}}\right] \leq t$$

$$[\textit{as } 0 < x_1y_1z_1 \leq 1]$$

$$\textit{Similarly, } 3(xyz)^{\frac{1}{3}} - 2(xy)^{\frac{1}{2}} \leq z$$

$$\textit{Thus, } \left[4(xyzt)^{\frac{1}{4}} - 3(xyz)^{\frac{1}{3}}\right] \cdot \left[3(xyz)^{\frac{1}{3}} - 2(xy)^{\frac{1}{2}}\right] \leq zt$$

SOLUTION 2.70

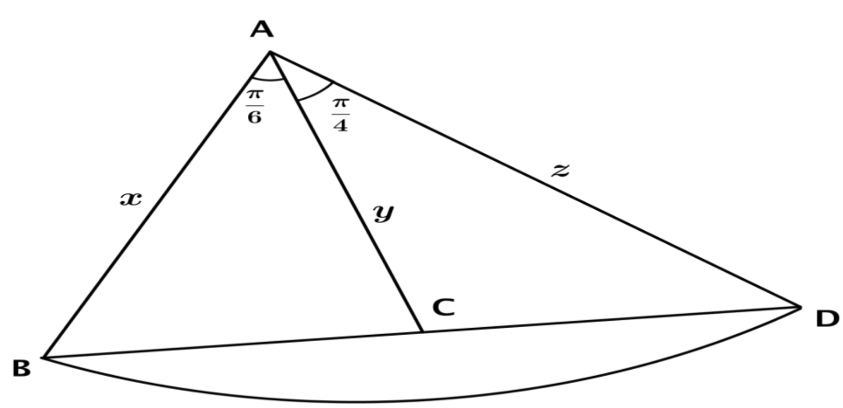
Solution by Ravi Prakash - New Delhi – India

$$\textit{Consider two } \Delta^s \ ABC \textit{ and } ACD$$

such that

$$AB = x, AC = y, AD = z$$

$$\angle BAC = \frac{\pi}{6}, \angle CAD = \frac{\pi}{4}$$



$$\text{Then } BC = \sqrt{x^2 - xy\sqrt{3} + y^2},\ CD = \sqrt{y^2 - \sqrt{2}yz + z^2}$$

$$\text{Also, } LBAD = 75^\circ,\ BD = \sqrt{x^2 - 2\cos 75^\circ\, xz + z^2}$$

$$\text{As } \cos 75^\circ < \cos 60^\circ - 2\cos 75^\circ > -2\cos 60^\circ = -1$$

$$\Rightarrow x^2 - 2\cos 75^\circ\, xz + z^2 > x^2 - xz + z^2$$

$$\text{Now, } BC + CD \geq BD > \sqrt{x^2 - xz + z^2}$$

$$\Rightarrow \sqrt{x^2 - \sqrt{3}xy + y^2} + \sqrt{y^2 - \sqrt{2}yz + z^2} > \sqrt{x^2 - xz + z^2}$$

SOLUTION 2.71

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$x(x^2 - 6x\cdot(y+z) + 9(y+z)^2) +$$

$$+(y+z)\cdot(9x^2 - 6x\cdot(y+z) + (y+z)^2) =$$

$$= x^3 + 3x^2\cdot(y+z) + 3x(y+z)^2 + (y+z)^3 = (x+y+z)^3 \geq \left(3\sqrt[3]{xyz}\right)^3 = 27$$

SOLUTION 2.72

Solution by Abdul Aziz-Semarang-Indonesia

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} + (b-a)(a-c)(b-c) =$$

$$= \frac{a^2b + b^2c + c^2a}{abc} + \frac{abc(b-a)(a-c)(b-c)}{abc}$$

$$\left[\begin{matrix} a+b+c=3 \Rightarrow abc \leq 1 \\ \Downarrow \end{matrix}\right]$$

$$\leq \frac{a^2b + b^2c + c^2a + b^2a - b^2c + bc^2 - a^2b + a^2c - ac^2}{abc}$$

$$= \frac{a^2c + c^2b + b^2a}{abc} = \frac{a}{b} + \frac{c}{a} + \frac{b}{c}$$

SOLUTION 2.73

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$\textit{We have: if } a+b+c=0 \textit{ then } a^3+b^3+c^3=3abc \textit{ so}$$

$$(a+b)+c+d=0 \textit{ then } (a+b)^3+c^3+d^3=3(a+b)cd$$

We have:

$$3(a+b)(ac+ad+bc+bd+4cd)=3(a+b)[(a+b)(c+d)+4cd]$$

$$=3(a+b)[-(a+b)^2+4cd]$$

$$=-3(a+b)^3+4\cdot 3(a+b)cd=-3(a+b)^3+4[(a+b)^3+c^3+d^3]$$

$$=(a+b)^3+4(c^3+d^3)$$

Now, we prove that

$$4(a^3+b^3+c^3)\geq (a+b)^3+4(c^3+d^3)\Rightarrow 4(a^3+b^3)\geq (a+b)^3$$

$$\Leftrightarrow (1^3+1^3)(1^3+1^3)(a^3+b^3)\geq (a+b)^3$$ ***(Right because Hölder's) "=" a=b.***

SOLUTION 2.74

Solution by Lahiru Samarakoon-Sri Lanka

$$LHS-RHS$$

$$4a^2+9b^2+16c^2+25d^2+12ab+16ac+2ad+24bc+30bd+40cd$$

$$-(24ab+40ad+48bc+80cd)$$

$$\Rightarrow 4a^2+9b^2+16c^2+25d^2+12ab+16ac-20ad-24bc+30bd-40cd$$

$$\underbrace{(2a-3b+4c-5d)^2}_{\geq 0}$$

$$LHS-RHS\geq 0$$

SOLUTION 2.75

Solution by Sanong Hauyrerai-Nakon Pathom-Thailand

$$\frac{sin^2x}{a}+\frac{cos^2x}{b}+\frac{sin^2y}{c}+\frac{cos^2y}{d}=\frac{sin^4x}{asin^2x}+\frac{cos^4x}{bcos^2x}+\frac{sin^4y}{csin^2y}+\frac{cos^4y}{dcos^2y}\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(sin^2x+cos^2x)^2}{asin^2x+bcos^2x}+\frac{(sin^2y+cos^2y)^2}{csin^2y+dcos^2y}=$$

$$=\frac{1}{asin^2x+bcos^2x}+\frac{1}{csin^2y+dcos^2y}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{4}{asin^2x+bcos^2x+csin^2y+dcos^2y}>\frac{4}{2(a+b)+2(c+d)}=\frac{2}{a+b+c+d}$$

SOLUTION 2.76

Solution by Sanong Hauyrerai-Nakon Pathom-Thailand

$$x,y>0,n\in\mathbb{N}^*\to\frac{n+1}{n}>1$$

$$(x^n+y^n)^{\frac{n+1}{n}}>(x^n)^{\frac{n+1}{n}}+(y^n)^{\frac{n+1}{n}}\to(x^n+y^n)^{\frac{n+1}{n}}>x^{n+1}+y^{n+1}$$

$$(x^n+y^n)^{n+1}>(x^{n+1}+y^{n+1})^n\to\frac{(x^n+y^n)^{n+1}}{(x^{n+1}+y^{n+1})^n}>1$$

$$\begin{cases}\dfrac{(a^3+b^3)^4}{(a^4+b^4)^3}>1\\ \dfrac{(c^5+d^5)^6}{(c^6+d^6)^5}>1\\ \dfrac{(e^7+f^7)^8}{(e^8+f^8)^7}>1\end{cases}\overset{by\ multipying}{\Rightarrow}\frac{(a^3+b^3)^4}{(c^6+d^6)^5}\cdot\frac{(c^5+d^5)^6}{(e^8+f^8)^7}\cdot\frac{(e^7+f^7)^8}{(a^4+b^4)^3}>1$$

SOLUTION 2.77

Solution by Soumava Chakraborty-Kolkata-India

$$\sum x\geq\sum a\sqrt{\sum a}$$

$$a^3x+b^3y+c^3z=xyz\left(\frac{a^3}{yz}+\frac{b^3}{zx}+\frac{c^3}{xy}\right)$$

$$\overset{Holder}{\geq}xyz\frac{(\sum a)^3}{3\sum xy}\geq xyz\frac{(\sum a)^3}{(\sum x)^2}\ \left(\because 3\sum x\leq\left(\sum x\right)^2\right)$$

$$\Rightarrow xyz \ge xyz\frac{(\sum a)^3}{(\sum x)^2}\ \ (\because xyz = a^3x + b^3y + c^3z)$$

$$\Rightarrow \left(\sum x\right)^2 \ge \left(\sum a\right)^3 \Rightarrow \sum x \ge \left(\sum a\right)\sqrt{\sum a}$$

SOLUTION 2.78

Solution by Soumava Chakraborty-Kolkata-India

$$Let\ e^x = a, e^y = b, e^z = c, 0 \le x \le y \le z \to 1 \le a \le b \le c$$

$$\frac{(2+e^x)^2}{(2+e^y)(2+e^z)} \ge \frac{(1+e^x+e^{2x})^2}{(1+e^y+e^{2y})(1+e^z+e^{2z})} \leftrightarrow$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \ge \frac{(1+a+a^2)^2}{(2+a)^2}, (1)$$

$$1+b+b^2 \ge 1+a+a^2 \leftrightarrow (b-a)(1+b+a) \ge 0, (2)$$

$$b \le c \to 2+b \le 2+c \to \frac{1}{2+b} \ge \frac{1}{2+c} \to \frac{1}{(2+b)(2+c)} \ge \frac{1}{(2+c)^2}, (3)$$

$$\frac{(1+b+b^2)(1+c+c^2)}{(2+b)(2+c)} \overset{(2),(3)}{\ge} \frac{(1+a+a^2)(1+c+c^2)}{(2+c)^2} \ge \frac{(1+a+a^2)^2}{(2+a)^2} \leftrightarrow$$

$$\leftrightarrow \frac{1+c+c^2}{(2+c)^2} \ge \frac{1+a+a^2}{(2+a)^2}, (4)$$

$$f(t) = \frac{1+t+t^2}{(2+t)^2}, \forall t \ge 1, f'(t) = \frac{3t}{(2+t)^2} > 0, \forall t \ge 1$$

$$f - increasing \to f(c) \ge f(a)$$

SOLUTION 2.79

Solution by Le Van-Ho Chi Minh-Vietnam

$$Put\ f(x) = \frac{\ln x}{\ln(x+1)}, x \ge 1$$

$$Then\ f'(x) \cdot [\ln(x+1)]^2 = \frac{\ln(x+1)}{x} - \frac{\ln x}{x+1} = \frac{[(x+1)\ln(x+1) - x\ln(x)]}{x(x+1)} > 0$$

Then $f(x)$ is a positive function, which gives us

$$4f(a) \leq f(a) + f(b) + f(c) + f(d) \leq 4f(d)$$

$\rightarrow$ *Q.E.D. Equality holds when* $a = b = c = d = 1$.

SOLUTION 2.80

Solution by Nguyen Van Nho-Nghe An-Vietnam

Case 1: $y = 0$ *then* $LHS = RHS = x^6 \rightarrow$ *true*

Case 2: $y \neq 0$, *put* $x = ty, t \in \mathbb{R}$

The inequality becomes:

$$\left(y^3(t^3 + 2 - 3t)\right)^2 \leq \left(y^2(t^2 + 2)\right)^3 \leftrightarrow (t^3 + 2 - 3t^2) \leq (t^2 + 2)^3$$

$$\leftrightarrow 12t^4 - 4t^3 + 3t^2 + 12t + 8 \geq 0 \leftrightarrow (2t^2 - t)^2 + 6t^2 + 2t^4 + 2t^2 + 8 + 12t \geq 0 \rightarrow \text{(*)}$$

Using Cauchy's inequality: $2t^4 + 2t^2 + 8 \geq 12\sqrt[12]{t^{12}} = 12|t|$

and $|t| + t \geq 0, \forall t$ *so, (*) is true. Done.*

SOLUTION 2.81

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \leq \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)} \Leftrightarrow \left(\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}}\right)^2 \leq 6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)$$

$$\Leftrightarrow 5\frac{x}{y} + 8\frac{y}{z} + 9\frac{z}{x} \geq 4\sqrt{\frac{x}{2}} + 6\sqrt{\frac{z}{y}} + 12\sqrt{\frac{y}{x}} \quad (1)$$

$$\text{Let } a = \sqrt{\frac{x}{y}}, b = \sqrt{\frac{y}{z}}, c = \sqrt{\frac{z}{x}} \Rightarrow abc = 1$$

$$(1) \Leftrightarrow 5a^2 + 8b^2 + 9c^2 \geq \frac{4}{c} + \frac{6}{b} + \frac{12}{a} \text{ or } 5a^2 + 8b^2 + 9c^2 \geq 4ab + 6ac + 12bc$$

$$\Leftrightarrow 2(a - b)^2 + 3(a - c)^2 + 6(b - c)^2 \geq 0 \text{ “=” } a = b = c = 1 \Leftrightarrow x = y = z.$$

SOLUTION 2.82

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$Let\ x=\sqrt{\frac{a}{b+c}},y=\sqrt{\frac{b}{c+a}},z=\sqrt{\frac{c}{a+b}}$$

Now, we prove that

$$x+2y+4z\le\sqrt{7(x^2+2y^2+4z)^2}$$

$$\Leftrightarrow x^2+4y^2+16z^2+4xy+8xz+16yz\le 7(x^2+2y^2+4z^2)$$

$$\Leftrightarrow 6x^2+10y^2+12z^2\ge 4xy+8xz+16yz\ \Leftrightarrow 2(x-y)^2+4(x-z)^2+8(y-z)^2\ge 0$$

$$\text{“="}\ x=y=z\ or\ a=b=c.$$

SOLUTION 2.83

Solution by Ravi Prakash-New Delhi-India

$$Let\ a=x^2,b=y^2,c=z^2,x,y,z\ge 0$$

$$Also\ 0\le a\le b\le c\Rightarrow 0\le x\le y\le z$$

$$(a-b)c\sqrt{c}+(b-c)a\sqrt{a}+(c-a)b\sqrt{b}=$$

$$=(x^2-y^2)z^3+(y^2-z^2)x^3+(z^2-x^2)y^3$$

$$=\begin{vmatrix}x^3 & y^3 & z^3\\ x^2 & y^2 & z^2\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}x^3-y^3 & y^3-z^3 & z^3\\ x^2-y^2 & y^2-z^2 & z^2\\ 0 & 0 & 1\end{vmatrix}\begin{bmatrix}use\ C_1\to C_1-C_2\\ C_2\to C_2-C_3\end{bmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix}x^2+y^2+xy & y^2+z^2+yz\\ x+y & y+z\end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix}x^2-z^2+(x-z)y & y^2+z^2+yz\\ x-z & y+z\end{vmatrix}$$

$$=(x-y)(y-z)(x-z)\begin{vmatrix}x+y+z & y^2+z^2+yz\\ 1 & y+z\end{vmatrix}$$

$$=(x-y)(y-z)(x-z)\begin{vmatrix}x+y & y^2\\ 1 & y+z\end{vmatrix}=(x-y)(y-z)(x-z)(xy+yz+zx)\le 0$$

$$since\ x \le y \le z$$

SOLUTION 2.84

Solution by Ravi Prakash-New Delhi-India

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$\Leftrightarrow 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge 1+1+1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}$$

$$\Leftrightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \Leftrightarrow \frac{a-c}{b}+\frac{b-a}{c}+\frac{c-b}{a} \ge 0$$

$$\Leftrightarrow ac(a-c)+ab(b-a)+bc(c-b) \ge 0$$

$$\Leftrightarrow \begin{vmatrix} bc & ac & ab \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} \ge 0 \Leftrightarrow (a-b)(b-c)(c-a) \ge 0$$

$$which\ is\ true\ as\ a \le b \le c$$

SOLUTION 2.85

Solution by Ravi Prakash-New Delhi-India

$$2b = a+c, 2c = b+d$$

$$\Rightarrow a, b, c, d\ are\ in\ A.P.\ with\ common\ difference\ \tfrac{1}{3}(d-a)$$

$$\therefore a^2+b^2+c^2+d^2 = a^2+\left\{a+\frac{1}{3}(d-a)\right\}^2+\left\{a+\frac{2}{3}(d-a)\right\}^2+d^2$$

$$= 3a^2+d^2+2(d-a)a+\frac{5}{9}(d-a)^2 = (a+d)^2+\frac{5}{9}(d-a)^2$$

$$= \left\{(a+d)-2e^{\frac{1}{8}}\right\}^2+4e^{\frac{1}{8}}(a+d)-4e^{\frac{1}{4}}+\frac{5}{9}(d-a)^2 \ge 4e^{\frac{1}{8}}\left[a+d-e^{\frac{1}{8}}\right]$$

SOLUTION 2.86

Solution by Do Huu Duc Thinh-Ho Chi Minh-Vietnam

If $x,y,z\in[-5,3]$ *then:* $\sum\sqrt{3x-5y-xy+15}\le 12$

We have: $\sum\sqrt{3x-5y-xy+15}=\sum\sqrt{(3-y)(5+x)}$.

Since $x,y,z\in[-5;3]$ *then* $3-x$;

$3-y; 3-z; 5+x; 5+y; 5+z\ge 0$*, so, by applying Cauchy's inequality:*

$\sum\sqrt{(3-y)(5+x)}\le\sum\left(\frac{3-y+5+x}{2}\right)=\frac{24}{2}=12\Rightarrow$ *Q.E.D. The equality happens iff*

$$\begin{cases}3-y=5+x; 3-z=5+y; 3-x=5+z\\ x,y,z\in[-5;3]\end{cases}\Leftrightarrow x=y=z=-1$$

SOLUTION 2.87

Solution by Le Minh Cuong-Ho Chi Minh-Vietnam

We have $LHS=\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{cd}{c+d}+\frac{da}{d+a}\le\frac{ab}{2\sqrt{ab}}+\frac{bc}{2\sqrt{bc}}+\frac{cd}{2\sqrt{cd}}+\frac{da}{2\sqrt{da}}\le$

$\le\frac{1}{2}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)$. *It need show that:* $\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\le$

$\le ab+bc+cd+da$. *Indeed,* $4(ab+bc+cd+da)\overset{BCS}{\ge}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)^2$

$$\overset{AM-GM}{\ge}4\sqrt[4]{\sqrt{ab}\cdot\sqrt{bc}\cdot\sqrt{dc}\cdot\sqrt{da}}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)\ge$$

$\ge 4\left(\sqrt{ab}+\sqrt{bc}+\sqrt{cd}+\sqrt{da}\right)$. *The equality holds for* $a=b=c=d=1$.

SOLUTION 2.88

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$c+\sqrt{ab}+\sqrt{ab}\ge 3\sqrt[3]{abc};\ c-3\sqrt[3]{abc}\ge -2\sqrt{ab}\Leftrightarrow a+b+c-3\sqrt[3]{abc}\ge$$

$$\ge a+b-2\sqrt{ab}=\left(\sqrt{a}-\sqrt{b}\right)^2\Leftrightarrow\frac{1}{\left(\sqrt{a}-\sqrt{b}\right)^2}+1\ge\frac{1}{a+b+c-3\sqrt[3]{abc}}+1\Leftrightarrow$$

$$\Leftrightarrow \frac{1+\left(\sqrt{a}-\sqrt{b}\right)^2}{\left(\sqrt{a}-\sqrt{b}\right)^2} \geq \frac{1+a+b+c-3\sqrt[3]{abc}}{a+b+c-3\sqrt[3]{abc}} \overset{a<b+c}{>} \frac{1+a+b+c-3\sqrt[3]{abc}}{2b+2c-3\sqrt[3]{abc}} \Leftrightarrow$$

$$\Leftrightarrow \frac{\left(2b+2c-3\sqrt[3]{abc}\right)\left(1+\left(\sqrt{a}-\sqrt{b}\right)^2\right)}{\left(\sqrt{a}-\sqrt{b}\right)^2\left(1+a+b+c-3\sqrt[3]{abc}\right)} > 1$$

SOLUTION 2.89

Solution by proposer

From the hypothesis we have:

$$c\left(\frac{ab}{9}-\frac{2}{3}\right)=\frac{a}{8}+3b-\frac{67}{4a} \Leftrightarrow c=\frac{9(a^2+24ab-134)}{8a(ab-6)}$$

Therefore, we have:

$$P=3a+2b+c=3a+2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}$$

Applying the AM-GM inequality, we have:

$$2b+\frac{9(a^2+24ab-134)}{8a(ab-6)}=2b+\frac{9[a^2+10+24(ab-6)]}{8a(ab-6)}$$

$$=\frac{2(ab-6)}{a}+\frac{9(a^2+10)}{8a(ab-6)}+\frac{39}{a} \geq \frac{2}{a}\cdot\sqrt{2(ab-6)\cdot\frac{9(a^2+10)}{8(ab-6)}}+\frac{36}{a}$$

$$=\frac{3\left(13+\sqrt{a^2+10}\right)}{a} \Rightarrow P \geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)$$

Applying the Cauchy – Schwarz and AM-GM inequality, we have:

$$P \geq 3\left(a+\frac{13+\sqrt{a^2+10}}{a}\right)=3\left(a+\frac{13}{a}+\frac{\sqrt{(15+10)(a^2+10)}}{5a}\right)$$

$$\geq 3\left(a+\frac{13}{a}+\frac{a\sqrt{15}+10}{5a}\right)=3\left(a+\frac{15}{a}+\frac{\sqrt{15}}{5}\right)$$

$$\geq 3\left(2\sqrt{a\cdot\frac{15}{a}}+\frac{\sqrt{15}}{5}\right)=\frac{33\sqrt{15}}{5}\Rightarrow P\geq\frac{33\sqrt{15}}{5}$$

Therefore, $P_{\min} = \frac{33\sqrt{15}}{5}$. *The equality holds for*

$$a=\sqrt{15}, b=\frac{13\sqrt{5}}{20}, c=\frac{23\sqrt{15}}{10}$$

SOLUTION 2.90

Solution by Ravi Prakash-New Delhi-India

$$Let\ f(x)=\frac{1+x+x^2}{1+x^2}, x\geq 1$$

$$f'(x)=\frac{d}{dx}\left[1+\frac{x}{1+x^2}\right]=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}<0, \forall x>1$$

$$\Rightarrow f(x)\ decreases\ on\ [1,\infty)\ \therefore f(x)\leq f(1)\ \forall x\geq 1$$

$$\Rightarrow \frac{1+a+a^2}{1+a^2}\leq\frac{3}{2}\ \forall a\geq 1 \quad (1)$$

$$Let\ g(x)=\frac{1+x+x^2+x^3}{1+x^3}=1+\frac{x+x^2}{1+x^3}$$

$$g'(x)=\frac{(1+x^3)(1+2x)-3x^2(x+x^2)}{(1+x^3)^2}=\frac{1+2x+x^3+2x^4-3x^3-3x^4}{(1+x^3)^2}$$

$$=\frac{1+2x-2x^3-x^4}{(1+x^3)^2}$$

$$g'(x)=\frac{(1-x^4)-2x(1-x^2)}{(1+x^3)^2}=\frac{(1-x^2)(1+x^2-2x)}{(1+x^3)^2}=\frac{(1-x)^3(1+x)}{(1+x^3)^2}<0\ \forall x>1$$

$$\Rightarrow g(x)\ decreases\ on\ [1,\infty)\ \therefore g(x)\leq g(1)\Rightarrow\frac{1+b+b^2+b^3}{1+b^3}\leq\frac{4}{2}=2\ \forall b\geq 1 \quad (2)$$

$$Let\ h(x)=\frac{1+x+x^2+x^3+x^4}{1+x^4}, x\geq 1$$

$$= 1 + \frac{x + x^2 + x^3}{1 + x^4}$$

$$h'(x) = \frac{(1 + 2x + 3x^2)(1 + x^4) - (x + x^2 + x^3)(4x^3)}{(1 + x^4)^2}$$

$$= \frac{1 + 2x + 3x^2 + x^4 + 2x^5 + 3x^6 - 4x^4 - 4x^5 - 4x^6}{(1 + x^4)^2}$$

$$= \frac{1 + 2x + 3x^2 - 3x^4 - 2x^5 - x^6}{(1 + x^4)^2} = \frac{(1 - x^6) + 2x(1 - x^3) + 3x^2(1 - x^2)}{(1 + x^4)^2} < 0\ \forall x \geq 1$$

$$\Rightarrow h(x) \text{ decreases on } [1, \infty) \therefore h(x) \leq h(1)\ \ \forall x \geq 1$$

$$\Rightarrow \frac{1+c+c^2+c^3+c^4}{1+c^4} \leq \frac{5}{2}\ \ \forall c \geq 1\quad (3)$$

Multiplyig (1), (2), (3) we get

$$\frac{(1 + a + a^2)(1 + b + b^2 + b^3)(1 + c + c^2 + c^4)}{(1 + a^2)(1 + b^3)(1 + c^4)} \leq \frac{15}{2}$$

SOLUTION 2.91

Solution by Ravi Prakash-New Delhi-India

For $0 < a \leq b, a(c + 2) \leq a + \sqrt{ab} + b \leq b(c + 2)$

$$\Leftrightarrow c\sqrt{ab} + b - ac - a \geq 0 \text{ and } bc + b - c\sqrt{ab} - a \geq 0$$

$$\Leftrightarrow (c\sqrt{a})(\sqrt{b} - \sqrt{a}) + (b - a) \geq 0 \text{ and } c\sqrt{b}(\sqrt{b} - \sqrt{a}) + (b - a) \geq 0$$

$$\Leftrightarrow (c\sqrt{a} + \sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) \geq 0 \text{ and } (c\sqrt{b} + \sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) \geq 0$$

which is true in view of $b \geq a$.

Thus $a \leq \frac{a+c\sqrt{ab}+b}{c+2} \leq b$. *Similarly for* d *and* e.

Multiplying three inequalities, we get

$$a^3 \le \frac{(a+c\sqrt{ab}+b)(a+d\sqrt{ab}+b)(a+e\sqrt{ab}+b)}{(c+2)(d+2)(e+2)} \le b^3$$

SOLUTION 2.92

Solution by Soumitra Mandal-Chandar Nagore-India

$$\textit{We know for } x, y \ge 0 \textit{ then } x^2+xy+y^2 \ge 3xy \textit{ and } \frac{3}{2}(x^2+y^2) \ge x^2+xy+y^2$$

$$\prod_{cyc} \sqrt[3]{\left(a^3+ab\sqrt{ab}+b^3\right)}$$

$$\Rightarrow \sqrt[3]{\prod_{cyc}\left(3a^{\frac{3}{2}}b^{\frac{3}{2}}\right)} \le \prod_{cyc} \sqrt[3]{a^3+ab\sqrt{ab}+b^3} \le \sqrt[3]{\frac{27}{8}\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3abc \le \prod_{cyc} \sqrt[3]{a^3+ab\sqrt{ab}+b^3} \le \frac{3}{2}\sqrt[3]{\prod_{cyc}(a^3+b^3)}$$

$$\Rightarrow 3ba^2 \le \prod_{cyc} \sqrt[3]{a^3+ab\sqrt{ab}+b^3} \le \frac{3}{2}\sqrt[3]{(2b^3)(2c^2)(2c^3)}[\because a \le b \le c]$$

$$\therefore 3a^2b \le \prod_{cyc} \sqrt[3]{a^3+ab\sqrt{ab}+b^3} \le 3bc^2$$

SOLUTION 2.93

Solution by Marian Ursărescu-Romania

$$\textit{For } a=b=c=0; a \ge 0 \ \textit{(true)}$$

$$\left.\begin{array}{c} a,b,c > 0; 2a^2+6ab+7b^2 \ge 2\sqrt[8]{c}\left(5\sqrt[5]{a^2b^3}-\sqrt[8]{c}\right) \\ \textit{But } 5\sqrt[5]{a^2b^3} \le 2a+3b \end{array}\right\} \Rightarrow$$

$$2\sqrt[8]{c}\left((2a+3b)-\sqrt[8]{c}\right) \le 2a^2+6ab+7b^2 \Leftrightarrow$$

$$-2\sqrt[8]{c^2}+2(2a+3b)\sqrt[8]{c} \le 2a^2+6ab+b^2 \quad \textit{(1)}$$

$$\sqrt[8]{c} = x, x > 0 \Rightarrow -2x^2+2(2a+3b)x = f(x)$$

$$\max f(x)=\frac{-\Delta}{4a}\Leftrightarrow\frac{-4(2a+3b)^2}{-8}=\frac{(2a+3b)^2}{2}\Rightarrow f(x)\le\frac{(2a+3b)^2}{2}\quad (2)$$

From (1)+(2)⇒ we must show: $\frac{(2a+3b)^2}{2}\le 2a^2+6ab+7b^2\Leftrightarrow$

$$4a^2+12ab+9b^2\le 4a^2+12ab+14b^2\Leftrightarrow 9b^2\le 14b^2\Leftrightarrow 5b^2\ge 0 \textit{ true.}$$

SOLUTION 2.94

Solution by Ravi Prakash-New Delhi-India

$$WLOG\ x=\max\{x,z\}$$

$$\sqrt{x^2-xz+z^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}$$
$$=\sqrt{x^2+z(z-x)}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}$$

$$\le\sqrt{x^2}+\sqrt{y^2+z^2}+\sqrt{x^2+xy+y^2}\le\sqrt{a^2}+\sqrt{a^2+a^2}+\sqrt{a^2+a^2+a^2}=a\left(1+\sqrt{2}+\sqrt{3}\right).$$ *Equality holds when* $x=y=z=a$.

SOLUTION 2.95

Solution by Soumava Chakraborty-Kolkata-India

$$\forall x,y,z,t\ge 1, xy+2yz+2zx+ty+tx+9\ge 4x+4y+6z+4t$$

$$\textit{Let } x=a+1, y=b+1, z=c+1, t=d+1\ (a,b,c,d\ge 0)$$

Then, given inequality becomes:

$$(a+1)(b+1)+2(b+1)(c+1)+2(c+1)(d+1)+2(c+1)(a+1)+$$

$$+(d+1)(b+1)+(d+1)(a+1)+9-4(a+1)-4(b+1)-6(c+1)-4(d+1)\ge 0$$

$$\Leftrightarrow ab+2ac+ad+2bc+bd+2cd\ge 0\to \textit{true} \because a,b,c,d\ge 0\ \textit{(proved)}$$

SOLUTION 2.96

Solution by Tran Hong-Vietnam

$$\textit{Let } a=3x, b=2y, c=36z\Rightarrow x=\frac{a}{3}, y=\frac{b}{2}, z=\frac{c}{36}\Rightarrow abc(a+b+c)=36^2$$

$$\textit{Inequality} \Leftrightarrow (a^2b^2+36^2)(b^2c^2+36^2)(c^2a^2+36^2) \ge 64(abc)^4$$

$$\Leftrightarrow (ab+bc+ca+a^2)(ab+bc+ca+b^2)(ab+bc+ca+c^2) \ge 64(abc)^2 \quad (*)$$

(Because: $36^2 = abc(a+b+c)$*)*

$$ab+bc+ca+a^2 \overset{(Cauchy)}{\ge} 4\sqrt[4]{ab\cdot bc\cdot ca\cdot a^2} = 4a\sqrt[4]{(bc)^2} \quad (1)$$

Similarly:

$$ab+bc+ca+b^2 \ge 4b\sqrt[4]{(ac)^2} \quad (2)$$

$$ab+bc+ca+c^2 \ge 4c\sqrt[4]{(ab)^2} \quad (3)$$

$$\overset{(1).(2).(3)}{\Rightarrow} LHS_{(*)} \ge 4^3abc\sqrt[4]{(abc)^4} = 64(abc)^2$$

SOLUTION 2.97

Solution by Chris Kyriazis-Athens-Greece

Let's consider the function $f(x) = \frac{1}{1+e^x}, x>0$*. Easily:* $f'(x) = -\frac{e^x}{(1+e^x)^2} < 0, \forall x>0$ *(*f *strictly decreasing) and* $f''(x) = -e^x\frac{(1-e^x)}{(1+e^x)^3} > 0, \forall x>0$*. So,* f *is convex for every* $x>0$*.*

Working with the fundamental definition of convexity, I have that:

$$\frac{c-b}{c-a}a+\left(1-\frac{c-b}{c-a}\right)\cdot c = \frac{c-b}{c-a}\cdot a+\frac{b-a}{c-a}\cdot c = \frac{ca-ab+bc-ac}{c-a} = b. \text{ And } \frac{c-b}{c-a}+1-\frac{c-b}{c-a} = 1. \text{ So,}$$

$$f(b) = f\left(\frac{c-b}{c-a}\cdot a+\left(1-\frac{c-b}{c-a}\right)c\right) \le \frac{c-b}{c-a}f(a)+\left(1-\frac{c-b}{c-a}\right)f(c) = \frac{c-b}{c-a}f(a)+\frac{b-a}{c-a}f(c) \quad (1)$$

$$\text{Also: } a-b+c = a-\left(\frac{c-b}{c-a}a+\frac{b-a}{c-a}\cdot c\right)+c = \frac{b-a}{c-a}a+\frac{c-b}{c-a}c. \text{ So,}$$

$$f(a-b+c) = f\left(\frac{b-a}{c-a}a+\frac{c-b}{c-a}c\right) \le \frac{b-a}{c-a}f(a)+\frac{c-b}{c-a}f(c) \quad (2)$$

Adding (1) + (2): $f(b)+f(a-b+c) \le f(a)+f(c)$ *as we desire!*

SOLUTION 2.98

Solution by Chris Kyriazis-Athens-Greece

The distance of $M(a,b)$ *from the line:* $3x+4y+2=0$ *is* 1

$$\left(d(M,\varepsilon)=\frac{|3a+4b+2|}{\sqrt{3^2+4^2}}=1\right)$$

I have to prove that: $a^2+b^2+4b+7\geq 4a$*. It suffices to prove that:*

$$(a-2)^2+(b+2)^2\geq 1 \quad \textit{(1)}$$

But its easy to prove that the point $N(2,-2)$ *belong to the straight line*

$$3x+4y+2=0.$$

So, (1) holds becomes: $d(M,\varepsilon)\leq d(M,N)$

SOLUTION 2.99

Solution by Amit Dutta-Jamshedpur-India

Let $P=2\sqrt{ab}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}$*. Now, we have* $\sqrt{ab}\leq\sqrt[3]{abc}$*.*

Because, $(ab)^3\leq(abc)^2\Rightarrow ab\leq c^2$ *(1)*

Now, we have $a\leq c, b\leq c\Rightarrow ab\leq c^2$*. So, (1) is true* $\Rightarrow$ *hence* $\sqrt{ab}\leq\sqrt[3]{abc}$

$$P\leq 2\sqrt[3]{abc}+3\sqrt[3]{abc}+4\sqrt[4]{abcd}$$

$$P\leq 5\sqrt[3]{abc}+4\sqrt[4]{abcd}$$

Also, we have $\sqrt[3]{abc}\leq\sqrt[4]{abcd}$*. Because,* $(abc)^4\leq(abcd)^3\Rightarrow abc\leq d^3$ *(3)*

$\because a\leq d, b\leq d, c\leq d\Rightarrow abc\leq d^3\rightarrow$ *True*

And hence $\sqrt[3]{abc}\leq\sqrt[4]{abcd}$

$$P\leq 5\sqrt[4]{abcd}+4\sqrt[4]{abcd}\leq 9\sqrt[4]{abcd}$$

Also, we have $\sqrt[4]{abcd}=\sqrt[5]{abcde}\Rightarrow(abcd)^5\leq(abcde)^4\Rightarrow abcd\leq e^4$

$\because a\leq e, b\leq e, c\leq e, d\leq e\Rightarrow abcd\leq e^4$ *and hence* $\sqrt[4]{abcd}\leq\sqrt[5]{abcde}\Rightarrow P\leq 9\sqrt[4]{abcde}$

SOLUTION 2.100

Solution by Tran Hong-Vietnam

$$\textit{We have: } 0 \le a^2, b^2, c^2 \le 3 \textit{ then:}$$

$$|a + (a + c)b + c|^2 = |(a + c)(1 + b)|^2$$

$$= (a + c)^2(1 + b)^2 \le [2(a^2 + c^2)][2(1 + b^2)] = 4(3 - b^2)(1 + b^2)$$

$$\overset{(Cauchy)}{\le} (3 - b^2 + 1 + b^2)^2 = 4^2 = 16$$

$$\Rightarrow |a + (a + c) + b| \le 4 \textit{ . Proved.}$$

SOLUTION 2.101

Solution by Amit Dutta-Jamshedpur-India

We use the fundamental inequality

$$\sqrt{x^2 + y^2} + \sqrt{a^2 + b^2} \ge \sqrt{(x + a)^2 + (y + b)^2} \quad (1)$$

Equality holds when $\frac{x}{a} = \frac{y}{b}$

$$2x^2 + (\sqrt{2} - \sqrt{6})x + 2 = 2\left(x^2 + \left(\frac{\sqrt{2} - \sqrt{6}}{2}\right)x + 1\right)$$

$$= 2\left(\left(x + \frac{\sqrt{2} - \sqrt{6}}{4}\right)^2 + 1 - \left(\frac{\sqrt{2} - \sqrt{6}}{4}\right)^2\right)$$

$$= 2\left[\left(x + \left(\frac{\sqrt{2} - \sqrt{6}}{4}\right)\right)^2 + \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)^2\right]$$

$$\Rightarrow \sqrt{2x^2 + (\sqrt{2} - \sqrt{6})x + 2} = \sqrt{2}\sqrt{\left(x + \frac{\sqrt{2}-\sqrt{6}}{4}\right)^2 + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2} \quad (2)$$

Similarly,

$$\sqrt{2x^2 - (\sqrt{2} - \sqrt{6})x + 2} = \sqrt{2}\sqrt{\left(x - \frac{\sqrt{2} + \sqrt{6}}{4}\right)^2 + \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right)^2}$$

$$=\sqrt{2}\sqrt{\left(\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}\quad \textit{(3)}$$

Adding (2) & (3)

$$\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}+\sqrt{6})x+2}=$$

$$=\sqrt{2}\left[\sqrt{\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2}+\sqrt{\left(\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}\right]$$

$$\overset{from\ (i)}{\geq}\sqrt{2}\sqrt{\left(x+\frac{\sqrt{2}-\sqrt{6}}{4}+\frac{\sqrt{2}+\sqrt{6}}{4}-x\right)^2+\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2}$$

$$\geq\sqrt{2}\sqrt{\left(\frac{\sqrt{2}-\sqrt{6}+\sqrt{2}+\sqrt{6}}{4}\right)^2+\left(\frac{\sqrt{3}+1+\sqrt{3}-1}{2\sqrt{2}}\right)^2}$$

$$\geq\sqrt{2}\sqrt{\frac{1}{2}+\frac{3}{2}}\geq\sqrt{2}\times\sqrt{2}\geq 2$$

$$\therefore\sqrt{2x^2+(\sqrt{2}-\sqrt{6})x+2}+\sqrt{2x^2-(\sqrt{2}-\sqrt{6})x+2}\geq 2$$

Equality occurs when

$$\frac{x+\left(\frac{\sqrt{2}-\sqrt{6}}{4}\right)}{\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)-x}=\frac{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}{\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}$$

From (i), equality holds when $\frac{x}{a}=\frac{y}{b}$***. Solving, we get*** $x=\frac{\sqrt{6}}{3}$

$\therefore$ ***Equality holds when*** $x=\frac{\sqrt{6}}{3}$

SOLUTION 2.102

Solution by Omran Kouba-Damascus-Syria

Let $\mathcal{P}_n$ ***be the following property:***

$$\forall(x_1,\dots,x_n)\in[1,+\infty)^n,\sum_{n=1}^{n}x_k\le\prod_{k=1}^{n}x_k+n-1$$

We will prove that $\mathcal{P}_n$ holds true for every positive integer n by induction. Clearly, $\mathcal{P}_1$ is trivially true, and $\mathcal{P}_2$ follows from $(x_1-1)(x_2-1)\ge 0$. Now, suppose we have proved $\mathcal{P}_n$ and consider $(x_1,\dots,x_{n+1})\in[1,+\infty)^{n+1}$.

$$\sum_{k=1}^{n+1}x_k=x_{n+1}+\sum_{k=1}^{n}x_k$$

$$\le x_{n+1}+\prod_{k=1}^{n}x_k+n-1 \quad \textit{using } \mathcal{P}_n$$

$$\le x_{n+1}\cdot\prod_{k=1}^{n}x_k+1+n-1 \quad \textit{using } \mathcal{P}_2$$

$$=\prod_{k=1}^{n+1}x_k+n$$

So, $\mathcal{P}_{n+1}$ is also true, and this completes the proof of $\mathcal{P}_n$ by induction for all $n\ge 1$. Choosing some of the x_k's equal yields the following generalization:

Corollary. Let $x_1,\dots,x_n$ be real numbers greater or equal to 1, and let $m_1,\dots,m_n$ be positive integers, then:

$$\sum_{k=1}^{n}m_k\,x_k\le\prod_{k=1}^{n}x_k^{m_k}+\sum_{k=1}^{n}m_k-1$$

For example, with $(x_1,\dots,x_6)=(a,b,c,d,e,f)$ and $(m_1,\dots,m_6)=(1,1,2,2,1,1)$ we get

$$a+b+2c+2d+e+f\le abc^2d^2ef+7$$

for all $a,b,c,d,e,f\ge 1$

ELEGANT INEQUALITIES AND IDENTITIES

SOLUTIONS

SOLUTION 3.01

Solution by Marian Ursărescu-Romania

Because $a+b+c=3 \Rightarrow \exists m,n,p>0$ *such that:* $a=\frac{3m}{m+n+p}, b=\frac{3n}{m+n+p}, c=\frac{3p}{m+n+p}$.

Inequality becomes:

$$\frac{m}{m+n+p}\cdot\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\cdot\left(\frac{m}{p}\right)^x+\frac{n}{m+n+p}\cdot\left(\frac{m}{n}\right)^x+\frac{p}{m+n+p}\cdot\left(\frac{n}{p}\right)^x+\frac{m}{m+n+p}\cdot\left(\frac{p}{m}\right)^x\le 2 \quad (1)$$

Let $f:(0,+\infty)\to\mathbb{R}, f(\alpha)=\alpha^x; f'(\alpha)=x\alpha^{x-1}, f''(\alpha)=x(x-1)\alpha^{x-2}\Rightarrow f''(x)<0$*, we use Jensen's generalization:* $p_1f(x_1)+p_2f(x_2)+p_3f(x_3)\le f(p_1x_1+p_2x_2+p_3x_3)$ *with* $p_1,p_2,p_3>0\wedge p_1+p_2+p_3=1$. *Let* $p_1=\frac{m}{m+n+p}, p_2=\frac{n}{m+n+p}, p_3=\frac{p}{m+n+p}, x_1=\frac{n}{m}$,

$$x_2=\frac{p}{n}, x_3=\frac{m}{p}\Rightarrow\frac{m}{m+n+p}\left(\frac{n}{m}\right)^x+\frac{n}{m+n+p}\left(\frac{p}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{m}{p}\right)^x\le\left(\frac{n+p+m}{m+n+p}\right)^x=1 \quad (2)$$

Let $p_1=\frac{n}{m+n+p}, p_2=\frac{p}{m+n+p}, p_3=\frac{m}{m+n+p}, x_1=\frac{m}{n}, x_2=\frac{n}{p}, x_3=\frac{p}{m}\Rightarrow$

$$\Rightarrow\frac{n}{m+n+p}\left(\frac{m}{n}\right)^x+\frac{p}{m+n+p}\left(\frac{n}{p}\right)^x+\frac{m}{m+n+p}\cdot\left(\frac{p}{m}\right)^x\le\left(\frac{m+n+p}{m+n+p}\right)^x=1 \quad (3)$$

From (2)+(3) $\Rightarrow$ *(1) its true.*

SOLUTION 3.02

Solution by Soumava Chakraborty-Kolkata-India

Let $\sin^2 x=a, \sin^2 y=b, \cos^2 x=c, \cos^2 y=d$

Then, given inequality $\Leftrightarrow \frac{(a+b)^{a+b}(c+d)^{c+d}}{a^ab^bc^cd^d}\overset{(1)}{\le}4$

Now, $\sqrt[a+b]{a^ab^b}\overset{weighted\ GM-HM}{\ge}\frac{a+b}{\frac{a}{a}+\frac{b}{b}}=\frac{a+b}{2}\Rightarrow a^ab^b\overset{(a)}{\ge}\frac{(a+b)^{a+b}}{2^{a+b}}$. *Similarly,* $c^cd^d\overset{(b)}{\ge}\frac{(c+d)^{c+d}}{2^{c+d}}$

(a).(b) $\Rightarrow a^ab^bc^cd^d\ge\frac{(a+b)^{a+b}\cdot(c+d)^{c+d}}{2^{a+b+c+d}}=\frac{(a+b)^{a+b}(c+d)^{c+d}}{2^{(\sin^2 x+\cos^2 x)+(\sin^2 y+\cos^2 y)}}=\frac{(a+b)^{a+b}(c+d)^{c+d}}{4}\Rightarrow$

$\Rightarrow\frac{(a+b)^{a+b}(c+d)^{c+d}}{a^ab^bc^cd^d}\le 4\Rightarrow$ *(1) is true (Proved)*

SOLUTION 3.03

Solution by Chris Kyriazis-Athens-Greece

I will use that:

1) Function $f(x)=\arcsin\left(\frac{x}{x+1}\right), x>0$ ***is concave (because*** $f''(x)=-\frac{3x+2}{(x+2)^2(2x+1)^{\frac{3}{2}}}<0$

2) Function $\arcsin x$ ***is strictly increasing when*** $0<x<1, \left((\arcsin x)'=\frac{1}{\sqrt{1-x^2}}>0\right)$

3) $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3}\le\frac{1}{2}$***, when*** $a,b,c>0, a+b+c=3$

Proof:

$$a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}\overset{GM-AM}{\le} a\frac{(b+1)^2}{4(b+1)}+b\frac{(c+1)^2}{4(c+1)}+c\frac{(a+1)^2}{4(a+1)}=$$

$$=\frac{ab+a+bc+b+ca+c}{4}\le\frac{\left(\frac{a+b+c}{3}\right)^2+3}{4}=\frac{6}{4}=\frac{3}{2}$$

So, $\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{3}\le\frac{1}{2}$

Now, (using (1)) applying Jensen's inequality with weights a,b,c***, gives then:***

$$LHS\le(a+b+c)\arcsin\left(\frac{a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}}{a+b+c}\right)=$$

$$=3\arcsin\frac{\left(a\frac{b}{b+1}+b\frac{c}{c+1}+c\frac{a}{a+1}\right)}{3}\underset{(2)}{\overset{(3)}{\le}}3\arcsin\left(\frac{1}{2}\right)=3\cdot\frac{\pi}{6}=\frac{\pi}{2}$$

because $\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$

SOLUTION 3.04

Solution by Dimitris Kastriotis-Athens-Greece

$$ex^{\frac{1}{x}}\le e^x, x\in(0,\infty)$$

$$ex^{\frac{1}{x}}\le e^x\Leftrightarrow 1+\frac{1}{x}\log x\le x\Leftrightarrow x+\log x-x^2\le 0, x\in(0,\infty)$$

$$f(x)=x+\log x-x^2, x\in(0,\infty)$$

$$f'(x)=1+\frac{1}{x}-2x, x\in(0,\infty)$$

$$f'(x)=0\Rightarrow 1+\frac{1}{x}-2x=0\Rightarrow x=1$$

$$f''(x)=-\frac{1}{x^2}-2<0,x\in(0,\infty)\Rightarrow \max\{f(x)|0<x<\infty\}=f(1)=0$$

$$\Rightarrow f(x)\le f(1)=0\Rightarrow x\le x+\log x-x^2\le 0,x\in(0,\infty)\Rightarrow ex^{\frac{1}{x}}<e^x,x\in(0,\infty)$$

$$\Rightarrow e^n x_1^{\frac{1}{x_1}}\dots x_n^{\frac{1}{x_n}}\le e^{x_1+\dots+x_n}$$

SOLUTION 3.05

Solution by Amit Dutta-Jamshedpur-India

$$\textit{Let } \sqrt{e^x}=a,\sqrt{e^y}=b\because (a^2+1)\frac{(b-1)^2}{2}+(b^2+1)\frac{(a-1)^2}{2}\ge 0\Rightarrow$$

$$\Rightarrow (a^2+1)\left[\frac{b^2+1}{2}-b\right]+(b^2+1)\left[\frac{a^2+1}{2}-a\right]\ge 0\Rightarrow$$

$$\Rightarrow (a^2+1)\frac{(b^2+1)}{2}-b(a^2+1)+(b^2+1)\frac{(a^2+1)}{2}-a(b^2+1)\ge 0\Rightarrow$$

$$\Rightarrow \frac{(a^2+1)(b^2+1)}{2}+\frac{(b^2+1)(a^2+1)}{2}\ge a(b^2+1)+b(a^2+1)\Rightarrow$$

$$\Rightarrow (a^2+1)(b^2+1)\ge a(b^2+1)+b(a^2+1).\ \textit{Now, put } \sqrt{e^x}=a\Rightarrow e^x=a^2$$

$$\sqrt{e^x}=b\Rightarrow e^y=b^2.\ \textit{So, } (e^x+1)(e^y+1)\ge (e^x+1)\sqrt{e^y}+(e^y+1)\sqrt{e^x}$$

SOLUTION 3.06

Solution by Antonis Anastasiadis-Greece

$$\textit{From well known inequality: } e^x\ge x+1$$

$$\therefore x^{3x^3}=e^{3x^3\ln x}\ge 3x^3\ln x+1\quad (1)$$

$$\textit{It is: } 3\ln x\cdot(x^3-1)\ge 0,\forall x>0$$

$$\textit{So, } 3x^3\ln x\ge 3\ln x,\forall x>0$$

$$(1)\Rightarrow e^{3x^3\ln x}\ge 3\ln x+1=\ln x^3e\Rightarrow x^{3x^3}\ge \ln x^3e\Leftrightarrow e^{x^{3x^3}}\ge x^3e$$

$$\textit{So: } e^{a^{3a^3}}+e^{b^{3b^3}}+e^{c^{3c^3}}\ge a^3e+b^3e+c^3e\Rightarrow LHS\ge e(a^3+b^3+c^3)\overset{AM-GM}{\ge}$$

$$\ge e\cdot 3\sqrt[3]{abc}=3e$$

SOLUTION 3.07

Solution by Ravi Prakash-New Delhi-India

$$A=(a_{ij})_{n\times n},\textit{ where } a_{ij}=10i+j.\ \textit{Let } x=(x_{ij})_{n\times n},\textit{ where } x_{ij}=a_{ij}\textit{ if } i>j$$

$$= \mathbf{0} \text{ if } \boldsymbol{i} < j$$

$$\boldsymbol{x_{11}} = \mathbf{-1}$$

$$\textit{and } \boldsymbol{x_{ii}} = \boldsymbol{a_{ii}} + \mathbf{1}, \forall \boldsymbol{i} \geq \mathbf{2}$$

$$\textit{Let } Y = (y_{ij})_{n\times n}, \textit{ where } y_{ij} = \mathbf{0} \textit{ if } i > j$$

$$= -a_{ij} \textit{ if } i < j$$

$$y_{11} = -12 = -(a_{11} + 1)$$

$$y_{ii} = 1\ \forall i \geq 2$$

$$\textit{Note that } A + Y = X \textit{ and } \det(Y) = -12 < 0 \textit{ and}$$

$$\det(X) = -(23)(34)\dots(10n + n + 1) < 0$$

SOLUTION 3.08

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } k = 2, \ln k - 1 < 0 = \ln 2 - \ln 2 < \frac{1}{2} \quad (1)$$

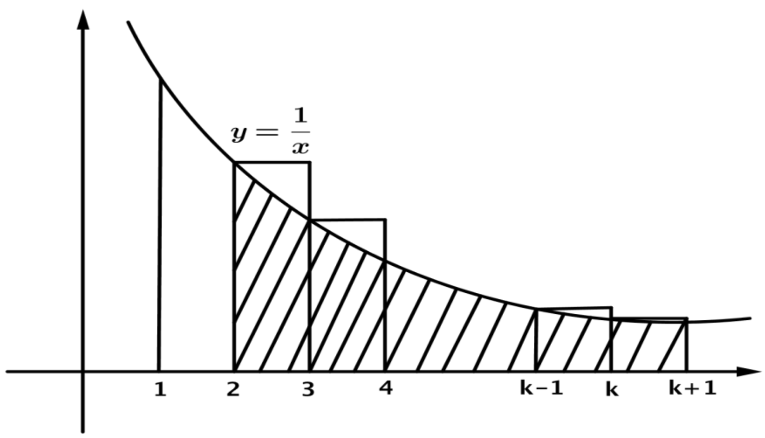


Fig. 1

$$\textit{For } k \geq 3, \ln(k) - 1 < \ln(k) - \ln(2) = \int_2^k \frac{1}{x} dx < \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k-1} \textit{ [see Fig. 1]}$$

$$< \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k-1} + \frac{1}{k} \quad (2)$$

$$\Rightarrow \sum_{k=2}^{n}(\ln k - 1) < \sum_{k=2}^{n}\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right) \quad \textit{[using (1), (2)]}$$

$$\Rightarrow \ln(n!) - (n - 1) < \sum_{k=2}^{n}\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}\right) \quad (3)$$

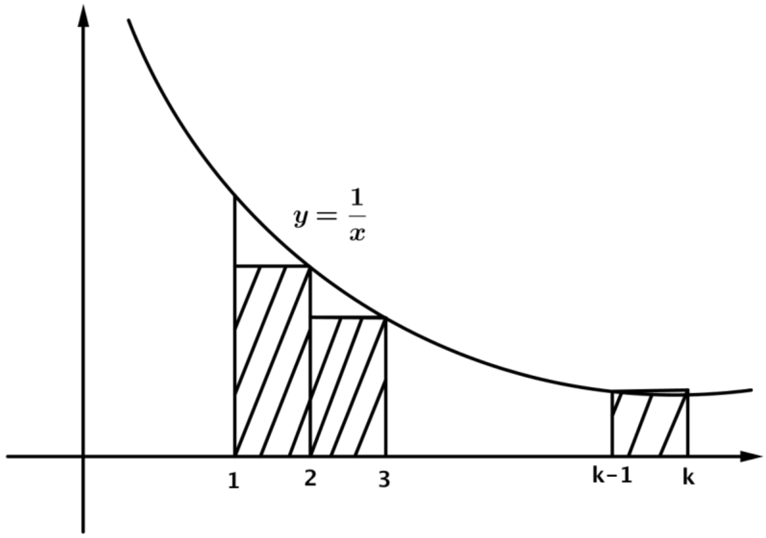


Fig. 2

For $k \geq 2, \ln k = \int_1^k \frac{1}{x} dx > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$ ***[see Fig. 2]***

$$\Rightarrow \ln(n!) = \sum_{k=2}^{n} \ln k > \sum_{k=2}^{n} \left(\frac{1}{2} + \cdots + \frac{1}{k}\right) \quad (4)$$

From (3), (4) the inequality follows.

SOLUTION 3.09

Solution by Soumitra Mandal-Chandar Nagore-India

Definition: A function $f: I \to \mathbb{R}$ ***is said to be a decreasing function on*** I ***if*** $f(y) \geq f(x)$ ***for all*** $x \geq y$ ***where*** $x, y \in I$

Let $f(x) = \frac{\tan^{-1} x}{x}$ ***for all*** $x \in \left[\frac{1}{\sqrt{3}}, 1\right], f'(x) = \frac{1}{x(1+x^2)} - \frac{\tan^{-1} x}{x^2} = \frac{1}{x^2}\left(\frac{x}{1+x^2} - \tan^{-1} x\right)$

Let $\varphi(x) = \frac{x}{1+x^2} - \tan^{-1} x$ ***for all*** $x \in \left[\frac{1}{\sqrt{3}}, 1\right], \varphi'(x) = -\frac{2x^2}{(1+x^2)^2} < 0$

For all $x \in \left[\frac{1}{\sqrt{3}}, 1\right]$***. Hence*** φ ***is decreasing*** $\therefore \varphi(1) \leq \varphi(x) \leq \varphi\left(\frac{1}{\sqrt{3}}\right) \Rightarrow$

$$\Rightarrow \frac{\frac{1}{\sqrt{3}}}{1+\frac{1}{3}} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \geq \varphi(x) \Rightarrow \frac{\sqrt{3}}{4} - \frac{\pi}{6} \geq \varphi(x) \Rightarrow 0 > \frac{\sqrt{3}}{4} - \frac{\pi}{6} \geq \varphi(x)$$

$\therefore \tan^{-1} x > \frac{x}{1+x^2}$ ***hence*** $f'(x) < 0$***. So*** f ***is decreasing on*** $\left[\frac{1}{\sqrt{3}}, 1\right]$

Again, $\sqrt{\frac{ab+bc+ca}{3}} \geq \sqrt[3]{abc}$***, so by definition of decreasing function***

$$\frac{\tan^{-1}\sqrt[3]{abc}}{\sqrt[3]{abc}} \ge \frac{\tan^{-1}\sqrt{\frac{ab+bc+ca}{3}}}{\sqrt{\frac{ab+bc+ca}{3}}}$$

$$\therefore \sqrt{\frac{ab+bc+ca}{3}}\tan^{-1}(\sqrt[3]{abc}) \ge \sqrt[3]{abc}\tan^{-1}\left(\sqrt{\frac{ab+bc+ca}{3}}\right)$$

SOLUTION 3.10

Solution by Soumava Chakraborty-Kolkata-India

$$(a-3)(c-x^2-y^2-z^2) \overset{(1)}{\le} (b-x-y-z)^2$$

$$\textit{(1)} \Leftrightarrow c(a-3)-(a-3)\left(\sum x^2\right)$$

$$\le b^2+\left(\sum x\right)^2-2b\left(\sum x\right)$$

$$\Leftrightarrow (a-3)\left(\sum x^2\right)+\left(\sum x\right)^2-2b\left(\sum x\right)+b^2-c(a-3) \overset{(2)}{\ge} 0$$

$$\because \sum x^2 \ge \frac{(\sum x)^2}{3} \,\&\, a-3 \ge 1 > 0, \therefore \textit{LHS of (2)} \ge \left(\frac{a-3}{3}+1\right)\left(\sum x\right)^2-2b\left(\sum x\right)$$

$$+b^2-c(a-3) = \frac{a}{3}\left(\sum x\right)^2-2b\left(\sum x\right)+b^2-c(a-3)$$

$$\overset{(?)}{\ge} 0 \Leftrightarrow a\left(\sum x\right)^2-6b\left(\sum x\right)+3\{b^2-c(a-3)\} \underset{(3)}{\overset{?}{\ge}} 0$$

$\because a \ge 4 > 0$ ***& LHS of (3) is a quadratic in*** $(\sum x)$ ***&*** $\because \sum x \in \mathbb{R}$ ***(as*** $x, y, z \in \mathbb{R}$***),*** $\therefore$ ***it suffices to prove that the discriminant is*** ≤ 0 ***that is, it suffices to prove:***

$$36b^2-4a\cdot 3\{b^2-c(a-3)\} \le 0 \Leftrightarrow 3b^2-a\{b^2-c(a-3)\} \le 0 \Leftrightarrow$$

$$\Leftrightarrow ac(a-3)-b^2(a-3) \le 0 \Leftrightarrow (a-3)(ac-b^2) \le 0$$

$$\because a-3 \ge 1 > 0, \therefore \textit{it suffices to prove: } ac-b^2 \le 0 \Leftrightarrow 4b^2 \overset{(4)}{\ge} 4ac$$

$$\textit{But LHS of (4)} \ge (a+c)^2 (\because 2b \ge a+c; b \ge 0; a+c \ge 4 > 0)$$

$$\overset{?}{\ge} 4ac \Leftrightarrow (a-c)^2 \ge 0 \to \textit{true} \Rightarrow \textit{(4) is true (proved)}$$

SOLUTION 3.11

Solution by Ravi Prakash-New Delhi-India

$$\textit{Put } x = \cos^2\theta, y = \sin^2\theta, 0 < \theta < \frac{\pi}{2}$$

$$P = (xy)^n + (xy)^{-n} = (\cos\theta\sin\theta)^{2n} + (\cos\theta\sin\theta)^{-2n}$$

$$\frac{dp}{d\theta} = (2n)(\cos\theta\sin\theta)^{2n-1}(\cos 2\theta)$$
$$-2n(\cos\theta\sin\theta)^{-2n-1}(\cos 2\theta)$$
$$= 2n(\cos 2\theta)(\cos\theta\sin\theta)^{-2n-1}[(\cos\theta\sin\theta)^{4n} - 1]$$

As $\cos\theta\sin\theta > 0, 0 < \cos\theta\sin\theta < 1,$

$$\frac{dp}{d\theta} < 0 \text{ if } 0 < \theta < \frac{\pi}{4}$$
$$= 0 \text{ if } \theta = \frac{\pi}{4}$$
$$> 0 \text{ if } \frac{\pi}{4} < \theta < \frac{\pi}{2}$$
$$\Rightarrow P \text{ is least when } \theta = \frac{\pi}{4}$$

Thus, $P \geq P\left(\frac{\pi}{4}\right) = \frac{1}{2^{2n}} + 2^{2n} = \frac{16^n+1}{4^n}$

SOLUTION 3.12

Solution by Amit Dutta-Jamshedpur-India

Let $\tan x = a, \tan y = b, \tan z = c \because x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow a, b, c > 0$

So, to prove $\frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)} \geq 1$ *or* $\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq \left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)$

$$\Rightarrow abc + \frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{1}{abc} \geq abc + a + c + \frac{1}{b} + b + \frac{1}{c} + \frac{1}{a} + \frac{1}{abc} \Rightarrow$$
$$\Rightarrow \frac{ac}{b} + \frac{bc}{a} + \frac{ab}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} \geq (a+b+c) + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \Rightarrow$$
$$\Rightarrow \left(\frac{a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2}{abc}\right) \geq \left(\frac{a^2bc + b^2ac + c^2ab + ab + bc + ac}{abc}\right)$$

or $(a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2) \geq (a^2bc + b^2ac + c^2ab + ab + bc + ac)$ ***(1)***

$\because$ ***we know that***

$$p^2 + q^2 + r^2 \geq pq + qr + pr$$

Taking $p = ab, q = bc, r = ac$***, we get***

$$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + b^2ac + c^2ab \quad (2)$$

Taking $p = a, q = b, r = c$

$$a^2 + b^2 + c^2 \geq ab + bc + ac \quad (3)$$

Adding (2) & (3), we get (1)⇒ (2)+(3)⇒ (1)

$$\textit{So, (1)}\Rightarrow \left(\sum a^2b^2+\sum a^2\right)\geq\left(\sum a^2bc+\sum ab\right)$$

This is true

$$\textit{and hence } \frac{\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)}{\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)}\geq 1 \textit{ or } \frac{(\tan x+\cot x)(\tan y+\cot y)(\tan z+\cot z)}{(\tan x+\cot y)(\tan y+\cot z)(\tan z+\cot x)}\geq 1 \textit{ (proved)}$$

SOLUTION 3.13

Solution by Tran Hong-Vietnam

$$(h_a-h_b+h_c)^{20}\leq h_a^{20}-h_b^{20}+h_c^{20}\quad (*)$$

$$a\leq b\leq c\Rightarrow h_a\geq h_b\geq h_c.\textit{Let } h_a=kh_c; h_b=mh_c(k\geq m\geq 1)$$

$$(*)\Leftrightarrow (k-m+1)^{20}\leq k^{20}-m^{20}+1$$

$$\textit{Let } f(x)=k^{20}-m^{20}+1-(k-m+1)^{20}$$

$$(\textit{with } k\geq m\geq 1)\Rightarrow f'(k)=20k^{19}-20(k-m+1)^{19}$$

$$k^{19}\geq(k-m+1)^{19}\Leftrightarrow k\geq k-m+1\Leftrightarrow m\geq 1\ \textit{(true)}$$

$$\Rightarrow f'(k)\geq 0\Rightarrow f(k)\nearrow[1;+\infty)$$

Then:

$$k\geq m\geq 1\Rightarrow f(k)\geq f(m)=m^{20}-m^{20}+1$$

$$-(m-m+1)^{20}=0\Rightarrow (*)\ \textit{true.}$$

SOLUTION 3.14

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } f(x)=e^{x^2}-e^{(x+1)^2}\ \forall x>0$$

$$f'(x)\overset{(1)}{=}-2\left((x+1)e^{(x+1)^2}-xe^{x^2}\right)$$

$$\textit{Now, } (x+1)^2(\ln e)>x^2(\ln e)(\because 2x+1>0 \textit{ as } x>0)\Rightarrow e^{(x+1)^2}\overset{(i)}{>}e^{x^2}$$

$$\textit{Also, } x+1\overset{(ii)}{>}x\ \&\because x>0\therefore (i).(ii)\Rightarrow (x+1)e^{(x+1)^2}-xe^{x^2}>0\Rightarrow$$

$$\Rightarrow f'(x)<0\ (\textit{by (1)})\therefore f(x)\downarrow\therefore e^{x^2}-e^{(x+1)^2}<e^{(x+2)^2}-e^{(x+3)^2}\Rightarrow$$

$$\Rightarrow e^{x^2}+e^{(x+3)^2}\overset{(a)}{>}e^{(x+1)^2}+e^{(x+2)^2}$$

$$\textit{Now, let } g(x)=\frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}\ \forall x>0$$

$$g'(x)=\frac{e^{x+1}(e^x+1)^2-e^x(e^{x+1}+1)^2}{(e^{x+1}+1)^2(e^x+1)^2}=\frac{et(t+1)^2-t(et+1)^2}{(et+1)^2(t+1)^2}\ (t=e^x)$$

$$=\frac{et(t^2+2t+1)-t(e^2t^2+2et+1)}{(et+1)^2(t+1)^2}=\frac{t(1-e)(et^2-1)}{(et+1)^2(t+1)^2}<0$$

$$\left(\because et^2>1\ as\ t=e^x>1\ (\because x>0)\right)\therefore g(x)\downarrow$$

$$\therefore\frac{1}{1+e^x}-\frac{1}{1+e^{x+1}}>\frac{1}{1+e^{x+2}}-\frac{1}{1+e^{x+3}}\Rightarrow$$

$$\Rightarrow\frac{1}{1+e^x}+\frac{1}{1+e^{x+3}}\overset{(b)}{>}\frac{1}{1+e^{x+1}}+\frac{1}{1+e^{x+2}}$$

(a).(b) ⇒ given inequality is true (proved)

SOLUTION 3.15

Solution by Lahiru Samarakoon-Sri Lanka

$$(a+b+c)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right)\le 3\left(\frac{a}{d}+\frac{b}{e}+\frac{c}{f}\right)$$

We can simplify, $\frac{(b+c)}{d}+\frac{(a+c)}{e}+\frac{(a+b)}{f}\le 2\left(\frac{a}{d}+\frac{b}{e}+\frac{c}{f}\right)$

$$\frac{(5-e+5-f)}{d}+\frac{(5-d+5-f)}{e}+\frac{(5-d+5-e)}{f}\le 2\left(\frac{5}{d}-1+\frac{5}{e}-1+\frac{5}{f}-1\right)$$

$$6\le\left(\frac{e}{d}+\frac{d}{e}\right)+\left(\frac{f}{d}+\frac{d}{f}\right)+\left(\frac{e}{f}+\frac{f}{e}\right)$$

By AM-GM,

$\left(\frac{e}{d}+\frac{d}{e}\right)\ge 2$ ***Similarly,*** $\left(\frac{f}{d}+\frac{d}{f}\right)\ge 2$ ***and*** $\left(\frac{e}{f}+\frac{f}{e}\right)\ge 2$. ***So,*** $\sum\left(\frac{e}{d}+\frac{d}{e}\right)\ge 6$ ***(proved)***

SOLUTION 3.16

Solution by Do Huu Duc Thinh-Vietnam

Let $x,y,z>0$ ***such that*** $x^2+y^2+z^2=3$. ***Find*** $Min: P=\sum\frac{x}{\sqrt{y}+\sqrt{z}}$

By Cauchy-Schwarz we have: $P=\sum\frac{x^2}{x\sqrt{y}+x\sqrt{z}}\ge\frac{(x+y+z)^2}{\sum x\sqrt{y}+\sum y\sqrt{x}}\ge\frac{(x+y+z)^2}{2\sqrt{(x+y+z)(xy+yz+zx)}}$

Let $t=x+y+z$ ***then*** $0<t\le 3$ ***and*** $xy+yz+zx=\frac{t^2-3}{2}$. ***We will prove that:***

$$\frac{t^2}{2\sqrt{t\cdot\frac{t^2-3}{2}}}\ge\frac{3}{2}\Leftrightarrow t^4\ge\frac{9(t^3-3t)}{2}\Leftrightarrow t(2t^3-9t^2+27)\ge 0\Leftrightarrow t(t-3)^2(2t+3)\ge 0\ \textit{(true)}$$

So, $P\ge\frac{3}{2}\Rightarrow P_{Min}=\frac{3}{2}\Leftrightarrow x=y=z=1$.

SOLUTION 3.17

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijian

$$\frac{5\sin^2 x}{1+1-\sin^2 x}=\frac{5\sin^2 x}{2-\sin^2 x};$$

$$\frac{5\cos^2 x\cdot\sin^2 y}{1+\sin^2 x+\cos^2 x\cdot(1-\sin^2 y)}=\frac{5\cos^2 x\cdot\sin^2 y}{2-\cos^2 x\cdot\sin^2 y}$$

$$\frac{5\cos^2 x\cdot\cos^2 y}{1+\sin^2 x+\cos^2 x\cdot(1-\cos^2 y)}=\frac{5\cos^2 x\cdot\cos^2 y}{2-\cos^2 x\cdot\cos^2 y}$$

We take the function $f(x)=\frac{5x}{2-x}$***, this function is convex,*** $f''(x)=\frac{20}{(2-x)^3}>0$

then by Jensen's inequality, we have

$$\frac{f(\sin^2 x)+f(\cos^2 x\sin^2 y)+f(\cos^2 x\cos^2 y)}{3}\geq f\left(\frac{\sin^2 x+\cos^2 x\sin^2 y+\cos^2 x\cos^2 y}{3}\right)$$

or $f(\sin^2 x)+f(\cos^2 x\sin^2 y)+f(\cos^2 x\cos^2 y)\geq 3\cdot f\left(\frac{1}{3}\right)$

(since $\sin^2 x+\cos^2 x\sin^2 y+\cos^2 x\cdot\cos^2 y=1$***)***

$f\left(\frac{1}{3}\right)=\frac{5\cdot\frac{1}{3}}{2-\frac{1}{3}}=1$***, we have:*** $f(\sin^2 x)+f(\cos^2 x\cdot\sin^2 y)+f(\cos^2 x\cdot\cos^2 y)\geq 3$

SOLUTION 3.18

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to\mathbb{R}, f(a)=\frac{9+4a+4a^2}{1+a}, f'(a)=\frac{(2a+5)(2a-1)}{(1+a)^2}$$

$$min(f(a))=f\left(\frac{1}{2}\right)=8\to f(a)\geq 8$$

$$f(a)+f(b)+f(c)\geq 8+8+8=24$$

SOLUTION 3.19

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan

$$bd(2^a-1)(2^c-1)>ac(2^b-1)(2^d-1)\quad \textit{(1)}$$

$$\textit{(1)}\Rightarrow\frac{2^a-1}{a}\cdot\frac{2^c-1}{c}>\frac{2^b-1}{b}\cdot\frac{2^d-1}{d}$$

denote $f(x)=\frac{2^x-1}{x}$

we prove that f increasing function

$$f'(x) = \frac{2^x \cdot \ln 2 \cdot x - 2^x + 1}{x} = \frac{2^x(\ln 2^x - 1) + 1}{x^2} > 0 \Rightarrow f \uparrow$$

$$\textit{then we have } \otimes \begin{cases} \frac{2^a-1}{a} > \frac{2^b-1}{b} & (2) \\ \frac{2^c-1}{c} > \frac{2^d-1}{d} & (3) \end{cases} \Rightarrow f(a) \cdot f(c) > f(b) \cdot f(d)$$

SOLUTION 3.20

Solution by Soumava Chakraborty-Kolkata-India

Let $a = x + 2y$ & $b = 3x + y$. Then, $a \le 5, b \ge 7, ab \ge 20$. We have $b - 2 \ge 5 \ge a \Rightarrow$

$$\Rightarrow b - a \ge 2 \Rightarrow (b-a)^2 \ge 4 \Rightarrow (a+b)^2 - 4ab \ge 4 \Rightarrow (a+b)^2 \ge 4 + 4ab \overset{ab\ge 20}{\ge} 84 \Rightarrow$$

$$\Rightarrow a + b \ge \sqrt{84} > \sqrt{81} = 9 \therefore a + b > 9 \Rightarrow 4x + 3y > 9 \textit{ or, } 4x + 3y \ge 9 \;\; \textit{(proved)}$$

SOLUTION 3.21

Solution by Ravi Prakash-New Delhi-India

For $0 < x < \frac{\pi}{2}; 0 < \cos\left(\frac{x}{2^k}\right) < 1, \forall k \in \mathbb{N}$. Let $a_n = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2^2}\right) \dots \cos\left(\frac{x}{2^n}\right)$

Note $a_{n+1} < a_n \Rightarrow < a_n >$ is a strictly decreasing sequence. Also

$$2^n \sin\left(\frac{x}{2^n}\right) a_n = 2^{n-1}\left[2\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^n}\right)\right]\cos\left(\frac{x}{2^{n-1}}\right) \dots \cos\left(\frac{x}{2}\right) =$$

$$= 2^{n-2}\left[2\sin\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^{n-1}}\right)\right] \dots \cos\left(\frac{x}{2}\right)$$

$$= \dots = \sin x \Rightarrow a_n = \frac{\sin(x)}{2^n \sin\left(\frac{x}{2^n}\right)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin x}{x} \cdot \frac{\frac{n}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}(1) = \frac{\sin x}{x}$$

As $< a_n >$ is strictly increasing and $\lim_{n\to\infty} a_n = \frac{\sin x}{x}$

$$a_n > \frac{\sin x}{x}; \forall n \in \mathbb{N} \quad \textit{(1)}$$

$$\left[\frac{\sin x}{x} = g/b(a_n)\right]$$

Also, for $0 < x < \frac{\pi}{2}$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}(x - \tan x) < 0 \Rightarrow \frac{\sin x}{x} \textit{ is strictly decreasing on } \left(0, \frac{\pi}{2}\right] \Rightarrow$$

$$\Rightarrow \frac{\sin x}{x} > \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \text{ for } 0 < x < \frac{\pi}{2} \text{ (2)}$$

From (1), (2): $a_n > \frac{2}{\pi}, \forall n \in \mathbb{N}$***. Now,***

$$\sum_{k=1}^{n} \log\left(\cos\frac{x}{2^k}\right) = \log a_n > \log\left(\frac{2}{\pi}\right) \Rightarrow \prod e^{\sum_{k=1}^{n} \log\cos\left(\frac{x}{2^k}\right)} > \prod e^{\log\left(\frac{2}{\pi}\right)} = 2$$

SOLUTION 3.22

Solution by Amit Dutta-Jamshedpur-India

Applying Cauchy's Schwarz inequality:

$$\left(\sqrt{2(y^4+z^4)} + 2yz\right)^2 \le (1^2+1^2)(2(y^4+z^4)+4y^2z^2) \le 2\left(2(y^4+z^4+2y^2z^2)\right)$$

$$\le 4(y^2+z^2)^2$$

$$\Rightarrow \sqrt{2(y^4+z^4)} + 2yz \le 2(y^2+z^2)$$

$$\sqrt{2(y^4+z^4)} \le 2(y^2-yz+z^2) \Rightarrow \sqrt{\frac{y^4+z^4}{2}} \le (y^2-yz+z^2) \Rightarrow$$

$$\Rightarrow \sqrt{\frac{y^4+z^4}{2}} + 2yz \le (y^2+yz+z^2) \text{ Similarly, } \sqrt{\frac{x^4+z^4}{2}} + 2xz \le (x^2+xz+z^2)$$

$$\sqrt{\frac{x^4+y^4}{2}} + 2xy \le (x^2+xy+y^2)$$

$$P \ge \frac{x}{y^2+yz+z^2} + \frac{y}{x^2+xz+z^2} + \frac{z}{x^2+xy+y^2} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

$$P \ge \sum \frac{x^2}{xy^2+xyz+xz^2} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

$$P \overset{Bergstrom}{\ge} \frac{(x+y+z)^2}{(xy^2+x^2y+xyz)+(y^2z+z^2y+xyz)+(x^2z+z^2x+xy^2)} +$$

$$+ \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

$$P \ge \frac{(x+y+z)^2}{(x+y+z)(xy+yz+xz)} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

$$P \ge \frac{(x+y+z)}{\sum xy} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18},\ P \ge \frac{3}{\sum xy} + \frac{\sqrt[5]{x}+\sqrt[5]{y}+\sqrt[5]{z}}{18}$$

Using AM-GM

$$\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+\sqrt[5]{x}+x^3+x^3+x^3+1+1\geq 10x \Rightarrow 5(x)^{\frac{1}{5}}+3\cdot x^3+2\geq 10x$$

$$5(y)^{\frac{1}{5}}+3y^3+2\geq 10y,\ \ 5(z)^{\frac{1}{5}}+3z^3+2\geq 10z$$

Adding these: $5\left(x^{\frac{2}{5}}+y^{\frac{2}{5}}+z^{\frac{2}{5}}\right)+3(x^3+y^3+z^3)+6\geq 10(x+y+z)\Rightarrow$

$$\Rightarrow 5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}+z^{\frac{1}{5}}\right)\geq 10\cdot(3)-6-3(x^3+y^3+z^3)\geq 30-6-3(x^3+y^3+z^3)$$

$$5\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}+z^{\frac{1}{5}}\right)\geq 24-3(x^3+y^3+z^3)\quad \textit{(1)}$$

Now, since $x+y+z=3\Rightarrow(x-3)=-y-z\Rightarrow(x-3)<0\because y,z>0$

Similarly, $(y-3)<0,(z-3)<0$

Clearly, $(x-3)(x-1)^2+(y-3)(y-1)^2+(z-3)(z-1)^2\leq 0\Rightarrow$

$$\Rightarrow\sum x^3-5\sum x^2+7\sum x-9\leq 0$$

$$\sum x^3\leq 5\left(\sum x^2\right)+9-7\sum x\leq 5\left[(x+y+z)^2-2\sum xy\right]+9-7\times(3)$$

$$\leq 5\left(3^2-2\sum xy\right)+9-21\leq 45-10\sum xy-12$$

$$\sum x^3\leq 33-10\sum xy$$

$$\because P\geq\frac{3}{\sum xy}+\left\{\frac{(x)^{\frac{1}{5}}+(y)^{\frac{1}{5}}+(z)^{\frac{1}{5}}}{18}\right\},\ P\geq\frac{3}{\sum xy}+\left[\frac{24-3\left(\sum x^3\right)}{90}\right]\ \ \textit{\{From (1)\}}$$

$$P\geq\frac{3}{\sum xy}+\frac{24-3(33-10\sum xy)}{90},\ P\geq\frac{3}{\sum xy}+\left(\frac{30\sum xy-75}{90}\right)$$

$$P\geq\frac{3}{\sum xy}+\frac{\sum xy}{3}-\frac{75}{90},\ P\geq\frac{3}{\sum xy}+\frac{\sum xy}{3}-\frac{5}{6},\ P\overset{AM-GM}{\geq}2-\frac{5}{6}$$

$$P\geq\frac{7}{6}$$

$\therefore$ ***minimum value of P is*** $\left(\frac{7}{6}\right)$***. Equality occurs when*** $(x=y=z=1)$***.***

SOLUTION 3.23

Solution by Soumava Chakraborty-Kolkata-India

Let $\sin x=a,\sin y=b,\sin z=c\because x,y,z\in\left(0,\frac{\pi}{2}\right)\therefore a,b,c\in(0,1)\ \&\sum a=1$***. Now,***

$$\cos^2 x\cos^2 y\cos^2 z=(1-a^2)(1-b^2)(1-c^2)=$$

$$=\{(a+b+c)^2-a^2\}\{(a+b+c)^2-b^2\}\{(a+b+c)^2-c^2\}=$$

$$= (2a+b+c)(2b+c+a)(2c+a+b)(a+b)(b+c)(c+a) \overset{Cesaro}{\geq}$$

$$\geq \{(a+b)+(c+a)\}\{(b+c)+(a+b)\}\{(b+c)+(c+a)\}8abc$$

$$\overset{A-G}{\geq} \left\{2\sqrt{(a+b)(c+a)}\right\}\left\{2\sqrt{(b+c)(a+b)}\right\}\left\{2\sqrt{(b+c)(c+a)}\right\}8abc =$$

$$= 64abc(a+b)(b+c)(c+a) \overset{Cesaro}{\geq} 64abc \cdot 8abc = 512a^2b^2c^2 =$$

$$= 512\sin^2 x \sin^2 y \sin^2 z \quad \textit{(proved)}$$

SOLUTION 3.24

Solution by Soumava Chakraborty-Kolkata-India

$$0 < x < \frac{\pi}{2}, \tan x > x \text{ and } x > \sin x \Rightarrow (\tan x - x)(x - \sin x) > 0$$

$$\Rightarrow x\tan x - x^2 - \sin x \tan x + x\sin x > 0 \Rightarrow x(\tan x + \sin x) > x^2 + \sin x \tan x \Rightarrow$$

$$\Rightarrow \frac{xy(\tan x + \sin x)}{x^2 + \sin x \tan x} \overset{(1)}{>} y$$

$$\textit{Similarly, } \frac{yz(\tan y + \sin y)}{y^2 + \sin y \tan y} \overset{(2)}{>} z \,\&\, \frac{zx(\tan z + \sin z)}{z^2 + \sin z \tan z} \overset{(3)}{>} x$$

$$\textit{(1)+(2)+(3)} \Rightarrow LHS > x + y + z = \pi \quad \textit{(Proved)}$$

SOLUTION 3.25

Solution by Daniel Sitaru-Romania

$$f:(0,\infty) \to \mathbb{R}, f(x) = xe^x, f'(x) = (x+1)e^x > 0, f - increasing,$$

$$f''(x) = (x+2)e^x > 0, f - convexe$$

$$\sum_{i=1}^{n} f(x_i) = \sum x_i e^{x_i} \overset{JENSEN}{\geqq} nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \overset{AM-GM}{\geqq} n \cdot \frac{1}{n}\sum_{i=1}^{n} x_i \cdot e^{\frac{1}{n}\sum_{i=1}^{n} x_i}$$

$$\frac{x_1e^{x_1} + x_2e^{x_2} + \cdots + x_ne^{x_n}}{x_1 + x_2 + \cdots + x_n} \geq e^{\frac{1}{n}\sum_{i=1}^{n} x_i} \overset{AM-GM}{\geq} e^{\sqrt[n]{\prod_1^n x_i}} = e^1 = e$$

SOLUTION 3.26

Solution by Marian Ursărescu – Romania

We must show this:

$$\cos x \cos z \cdot \sin y \sin t\,(\sin x \cos y - \cos x \sin y)(\sin z \cos t - \cos z \sin t) \leq \frac{1}{64} \quad \textit{(1)}$$

We show this: $\cos x\cdot\sin y\,(\sin x\cos y-\cos x\sin y)\le\frac{1}{8}$ *(2)*

$$\left.\begin{array}{r}\cos x=a,\sin y=b\ \ (2)\ \Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)}-ab\right)\le\dfrac{1}{8}\\ But\ \sqrt{(1-a^2)(1-b^2)}\le\dfrac{2-a^2-b^2}{2}\end{array}\right\}\Rightarrow$$

$$\Rightarrow ab\left(\frac{2-a^2-b^2}{2}-ab\right)\le\frac{1}{8}\Leftrightarrow ab(2-a^2-b^2-2ab)\le\frac{1}{4}\Leftrightarrow$$

$$4ab(2-(a+b)^2)\le 1 \quad (3)$$

But $(a+b)^2\ge 4ab\Rightarrow-(a+b)^2\le-4ab$ *(4)*

From (3) + *(4)* $\Rightarrow 4ab(2-4ab)\le 1\Leftrightarrow$

$$8ab-16a^2b^2\le 1\Leftrightarrow 16a^2b^2-8ab+1\ge 0\Leftrightarrow$$

$(4ab-1)^2\ge 0$ *true (equality for* $a=b=\frac{1}{2}$*).*

Similarly: $\cos z\sin t\sin(z-t)\le\frac{1}{8}$ *(5)*

From (2)+*(5)* $\Rightarrow$

$$\cos x\cos z\cdot\sin y\sin t\cdot\sin(x-y)\sin(z-t)\le 1,$$

with equality for $x=z=\frac{\pi}{3}$ *and* $y=t=\frac{\pi}{6}$*.*

SOLUTION 3.27

Solution by Amit Dutta-Jamshedpur-India

$$\because \sin^{-1}x>x\Rightarrow(\sin^{-1}x-x)>0 \quad (1)$$

$$\tan^{-1}x<x\Rightarrow(x-\tan^{-1}x)>0 \quad (2)$$

Multiplying (1) & (2)

$$(\sin^{-1}x-x)(x-\tan^{-1}x)>0\Rightarrow$$

$$\Rightarrow x\sin^{-1}x-\sin^{-1}x\tan^{-1}x-x^2+x\tan^{-1}x>0\Rightarrow$$

$$\Rightarrow x(\sin^{-1}x+\tan^{-1}x)>x^2+\tan^{-1}x\sin^{-1}x$$

$$\Rightarrow\frac{y(\sin^{-1}x+\tan^{-1}x)}{x^2+\tan^{-1}x\sin^{-1}x}>\frac{y}{x} \quad (3)$$

Similarly, $\frac{z(\sin^{-1}y+\tan^{-1}y)}{y^2+\tan^{-1}y\sin^{-1}y}>\frac{z}{y}$ *(4)*

$$\frac{x(\sin^{-1}z+\tan^{-1}z)}{z^2+\tan^{-1}z\sin^{-1}z}>\frac{x}{z} \quad (5)$$

Adding (1), (2), (3):

$$\sum_{cyc(x,y,z)} \frac{y(\sin^{-1}x+\tan^{-1}x)}{x^2+\tan^{-1}x\sin^{-1}x} > \left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right) \overset{AM-GM}{>} 3\sqrt[3]{\frac{xyz}{xyz}} > 3$$

SOLUTION 3.28

Solution by Ravi Prakash-New Delhi-India

For $x=0$, the inequality clearly holds. For $0<x\le 1;\ 0<\sin x<x\le 1$

$$\sin^5 x<\sin^3 x<x^3 \Rightarrow$$

$$\Rightarrow 16\sin^5 x<16x^3<20x^3 \quad (1)$$

Also, for $0<x\le 1, \sin x\le x \Rightarrow$

$$\Rightarrow 5\sin x\le 5x \quad (2)$$

Adding (1) and (2), we get, for $0<x\le 1$

$$16\sin^5 x+5\sin x\le 20x^3+5x \Rightarrow \sin x\,(16\sin^4 x+5)\le 5x(4x^2+1)$$

If $x>1$, then $LHS\le 21$ and $RHS\ge 25$. Thus, for all $x\ge 0$, the inequality holds

SOLUTION 3.29

Solution by Soumava Chakraborty-Kolkata-India

$$\cos 6x=4\cos^3 2x-3\cos 2x \overset{(i)}{=} 4(2\cos^2 x-1)^3-3\cos 2x$$

$$6\cos 4x=6(2\cos^2 2x-1) \overset{(ii)}{=} 12(2\cos^2 x-1)^2-6$$

(i)+(ii)$\Rightarrow 15\cos 2x+6\cos 4x+\cos 6x$

$$=12\cos 2x+4(2\cos^2 x-1)^3+12(2\cos^2 x-1)^2-6$$

$$=24\cos^2 x+4(2\cos^2 x-1)^3+12(2\cos^2 x-1)^2-18 \overset{(a)}{=} 32\cos^2 x-10$$

Similarly, $15\cos 2y+6\cos 4y+\cos 6y \overset{(b)}{=} 32\cos^6 y-10$

& $15\cos 2z+6\cos 4z+\cos 6z \overset{(c)}{=} 32\cos^6 z-10$

(a)+(b)+(c)$\Rightarrow$ *LHS* $=32\sum\cos^6 x-30 \overset{?}{\ge} 18 \Leftrightarrow \sum\cos^6 x \underset{(1)}{\overset{?}{\ge}} \frac{3}{2}$

$$\because x,y,z\in\left(0,\frac{\pi}{2}\right), \therefore \cos^6 x, \cos^6 y, \cos^6 z>0$$

$$\therefore \sum\cos^6 x \overset{A-G}{\ge} 3\sqrt[3]{\cos^6 x\cdot\cos^6 y\cdot\cos^6 z}=3(\cos x\cos y\cos z)^2=3\left(\frac{2}{4}\right)=\frac{3}{2}\Rightarrow$$

$\Rightarrow$ *(1) is true (Proved)*

SOLUTION 3.30

Solution by Tran Hong-Vietnam

$$\frac{\sin^2 x}{1+\sin^2 x}+\frac{\sin^2 y}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{\sin^2 z}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}$$
$$+\frac{1}{8\sin x\sin y\sin z}$$
$$=1-\frac{1}{1+\sin^2 x}+\frac{1}{1+\sin^2 x}-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}+\frac{1}{(1+\sin^2 x)(1+\sin^2 y)}-$$
$$-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}=$$
$$=1-\frac{1}{(1+\sin^2 x)(1+\sin^2 y)(1+\sin^2 z)}+\frac{1}{8\sin x\sin y\sin z}$$
$$\overset{(Cauchy)}{\geq} 1-\frac{1}{2\sin x\cdot 2\sin y\cdot 2\sin z}+\frac{1}{8\sin x\sin y\sin z}=1$$
$$\Rightarrow LHS=RHS\Leftrightarrow \sin x=\sin y=\sin z\Leftrightarrow x=y=z=\frac{\pi}{2}$$

SOLUTION 3.31

Solution by Marian Ursărescu-Romania

We must show: $(1-\sin^2 x)(1-\sin^2 y)(1-\sin^2 z)\geq 512\sin^2 x\cdot\sin^2 y\cdot\sin^2 z$ ***(1)***

Let $\sin x=m,\sin y=n,\sin z=p,m,n,p>0$ ***(2)***

From (1)+(2) we must show:

$(1-m^2)(1-n^2)(1-p^2)\geq 512m^2n^2p^2$***, with*** $m,n,p>0\wedge m+n+p=1$ ***(3)***

Let $m=\frac{a}{a+b+c},n=\frac{b}{a+b+c},n=\frac{c}{a+b+c},a,b,c>0$ ***(4)***

Form (3)+(4) we must show:

$$[(a+b+c)^2-a^2][(a+b+c)^2-b^2][(a+b+c)^2-c^2]\geq 512a^2b^2c^2\Leftrightarrow$$
$$\left[\left(\frac{a+b+c}{a}\right)^2-1\right]\left[\left(\frac{a+b+c}{b}\right)^2-1\right]\left[\left(\frac{a+b+c}{c}\right)^2-1\right]\geq 512\Leftrightarrow$$
$$\left[\left(\frac{b+c}{a}+1\right)^2-1\right]\left[\left(\frac{a+c}{b}+1\right)^2-1\right]\left[\left(\frac{a+b}{c}\right)^2-1\right]\geq 512\Leftrightarrow$$
$$\left[\left(\frac{b+c}{a}\right)^2+2\left(\frac{b+c}{a}\right)\right]\left[\left(\frac{a+c}{b}\right)^2+2\left(\frac{b+c}{a}\right)\right]\left[\left(\frac{a+b}{c}\right)^2+2\left(\frac{a+b}{c}\right)\right]\geq 512\Leftrightarrow$$
$$\left(\frac{b+c}{a}\right)\left(\frac{a+c}{b}\right)\left(\frac{a+b}{c}\right)\left(\frac{b+c+2a}{a}\right)\left(\frac{a+c+2b}{b}\right)\left(\frac{a+b+2c}{c}\right)\geq 512 \quad (5)$$

$$\left.\begin{array}{l} But\ \frac{b+c}{a}\ge 2\sqrt{bc} \\ \frac{a+c}{b}\ge 2\sqrt{ac} \\ \frac{a+b}{c}\ge 2\sqrt{ab}\end{array}\right\}\Rightarrow \frac{b+c}{a}\cdot\frac{a+c}{b}\cdot\frac{a+b}{c}\ge 2^3 \quad (6)$$

$$\frac{b+c+a+a}{a}\ge 4\sqrt[4]{a^2bc}\ \text{ and similarly}\Rightarrow$$

$$\frac{b+c+2a}{a}\cdot\frac{a+c+2b}{b}\cdot\frac{a+b+2c}{c}\ge 2^6 \quad (7)$$

From (6)+(7)⇒ its true.

SOLUTION 3.32

Solution by Ravi Prakash-New Delhi-India

$$\text{For } x, y > 0$$

$$\frac{2x+y}{3}\ge (x^2y)^{\frac{1}{3}}\Rightarrow \frac{2x+y}{3}\cdot\frac{x+2y}{3}\ge [(x^2y)(xy^2)]^{\frac{1}{3}}$$

$$\Rightarrow \frac{2x+y}{3}\cdot\frac{x+2y}{3}\ge xy\Rightarrow \frac{(2x+y)(x+2y)}{9xy}\ge 1\Rightarrow$$

$$\Rightarrow \tan^{-1}\left(\frac{(2x+y)(x+2y)}{9xy}\right)\ge\frac{\pi}{4}$$

Thus,

$$\tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right)+\tan^{-1}\left(\frac{(2b+c)(b+2c)}{9bc}\right)+\tan^{-1}\left(\frac{(2a+c)(a+2c)}{9ac}\right)\ge$$

$$\ge\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{4}=\frac{3}{4}\pi$$

SOLUTION 3.33

Solution by Ravi Prakash-New Delhi-India

$$\text{For } 0<\theta<\frac{\pi}{2}$$

$$f(\theta)=\left(\sin^2\theta+\frac{1}{\sin^2\theta}\right)^3+\left(\cos^2\theta+\frac{1}{\cos^2\theta}\right)^3=$$

$$=\sin^6\theta+\cos^6\theta+3(\sin^2\theta+\cos^2\theta)+3\left(\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}\right)+\left(\frac{1}{\sin^6\theta}+\frac{1}{\cos^6\theta}\right)$$

$$\ge\sin^6\theta+\cos^6\theta+3+\frac{6}{\sin\theta\cos\theta}+\frac{2}{\sin^3\theta\cos^3\theta}=$$

$$=\sin^6\theta+\cos^6\theta+3+\frac{12}{\sin 2\theta}+\frac{16}{(\sin^3 2\theta)}$$

$$\textit{But } \sin^6\theta + \cos^6\theta \geq 2\left(\frac{1}{\sqrt{2}}\right)^6 \textit{ [Using derivatives]}$$

$$\therefore f(\theta) \geq \frac{1}{4} + 3 + 12 + 16 = 31\frac{1}{4} = \frac{125}{4}$$

$$\therefore \textit{For } 0 < x, y, z, t < \frac{\pi}{2}$$

$$\sum\left(\sin x + \frac{1}{\sin x}\right)^3 + \sum\left(\cos x + \frac{1}{\cos x}\right)^3 \geq 4\left(\frac{125}{4}\right) = 125$$

SOLUTION 3.34

Solution by Boris Colakovic-Belgrade-Serbia

$$a = \sin x > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$b = 1 - \sin x > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$$

$$a, b > 0 \textit{ from weighted GM-AM inequality} \Rightarrow$$

$$2a^b \cdot b^a \leq 2\left(\frac{ab+bc}{a+b}\right)^{a+b} = 2\left(\frac{2ab}{a+b}\right)^{a+b} \leq 2\left(\frac{a+b}{2}\right)^{a+b} \textit{ or}$$

$$2(\sin x)^{1-\sin x}(1-\sin x)^{\sin x} \leq 2\left(\frac{\sin x + 1 - \sin x}{2}\right)^{\sin x + 1 - \sin x} = 1$$

SOLUTION 3.35

Solution by Tran Hong-Vietnam

$$\textit{For } x \in \left[0, \frac{\pi}{14}\right) \textit{ we have: } 1 \geq t = \cos x > \cos\frac{\pi}{14} \approx 0,975$$

$$\because \{(t^4-1)^2 + 4(t-1)^2\} \geq 0$$

$$\Leftrightarrow t\{t^8 - 4t^2 + 3\} \geq 0 \Leftrightarrow t^9 \geq 4t^3 - 3t = \cos 3x \quad \textit{(1)}$$

$$t^{25} \geq 16t^5 - 20t^3 + 5t \Leftrightarrow t\{t^{24} - 16t^4 + 20t^2 - 5\} \geq 0$$

$$\Leftrightarrow t(t-1)^2(t+1)^2(t^{20} + 2t^{18} + 3t^{16} + 4t^{14} + 5t^{12} + 6t^{10} + 7t^8 + 8t^6 + 9t^4 + 10t^2 - 5) \geq 0 \textit{ (true)} \Leftrightarrow t^{25} \geq \cos 5x \quad \textit{(2)}$$

$$t^{49} \geq 64t^7 - 112t^5 + 56t^3 - 7t$$

$$\Leftrightarrow t(t-1)^2(t+1)^2(t^{44} + 2t^{42} + 3t^{40} + 4t^{38} + \cdots + 20t^6 + 21t^4 - 42t^2 + 7) \geq 0$$

$$\textit{(true)} \Leftrightarrow t^{49} \geq \cos 7x \quad \textit{(3)}$$

$$\Rightarrow LHS \leq (t^9)^{21} \cdot \left(t^{25}\right)^7 \cdot t^{49} = t^{413} = (\cos x)^{413};$$

$$\textit{Equality} \Leftrightarrow \cos x = 1 \overset{\left[0,\frac{\pi}{14}\right)}{\Leftrightarrow} x = 0.$$

SOLUTION 3.36

Solution by Michael Sterghiou-Greece

$$\pi\left(\frac{\sin x}{x}+\frac{\cos x}{\frac{\pi}{2}-x}\right)>4+(\pi-2)(\sin x+\cos x)\quad (1)$$

Lemma 1. $x\in\left(0,\frac{\pi}{4}\right):\sin x>x-\frac{x^3}{6}$

Lemma 2. $x\in\left(0,\frac{\pi}{4}\right):\cos x>1-\frac{x^2}{2}+\frac{x^4}{48}$

Solution: (1) can be written as: $\overbrace{\left(\frac{\pi}{x}-\pi+2\right)}^{>0}\sin x+\overbrace{\left(\frac{\pi}{\frac{\pi}{2}-x}-\pi+2\right)}^{>0}\cos x>4$ ***(2)***

$f(x)=LHS$ of (2). We observe that $f(x)$ has $x=\frac{\pi}{4}$ as symmetry axis as $f(x)=f\left(\frac{\pi}{2}-x\right)$. We have $\lim_{x\to 0_+}f(x)=\lim_{x\to\frac{\pi}{2}}f(x)=4$, $f\left(\frac{\pi}{4}\right)=\sqrt{2}(6-\pi)>4(\sim 4.042)$. It is easy to show also that $f'\left(\frac{\pi}{4}\right)=0$. We need to prove that $f(x)$ lies on and over the line $y=4$ in the interval $\left(0,\frac{\pi}{4}\right)$ as symmetry will take care of the interval $\left[\frac{\pi}{4},\frac{\pi}{2}\right)$. Consider the function $g(x)$ in $\left(0,\frac{\pi}{4}\right]$: $g(x)=\left(\frac{\pi}{x}-\pi+2\right)\left(x-\frac{x^3}{6}\right)+\left(\frac{\pi}{\frac{\pi}{2}-x}-\pi+2\right)\left(1-\frac{x^2}{2}+\frac{x^4}{48}\right)$. We will show that $g(x)>4$ in $\left(0,\frac{\pi}{4}\right]$. Indeed $g(x)\to 4$ when $x\to 0_+$ and $g\left(\frac{\pi}{4}\right)>4$.

$$g''(x)=\underbrace{-\frac{1}{4}(\pi-2)x^2}_{T_1}+\underbrace{\frac{1}{8}(7\pi-16)x}_{T_2}+\underbrace{\frac{-768\pi+96\pi^3-\pi^5}{48(2x-\pi)^3}}_{T_3}+\underbrace{\frac{1}{48}(-96+32\pi-\pi^2)}_{T_4}$$

$T_1<0, T_3<0$ and $T_4<0$. The max of T_2 is $\frac{1}{8}(7\pi-16)\cdot\frac{\pi}{4}<|T_4|$ therefore $g''(x)<0$ This means $g'(x)\downarrow$ with only one root x_0 in $\left(0,\frac{\pi}{4}\right)$ in which $g''(x_0)<0$ therefore x_0 is a max. Using Lemmas now therefore ≥ 4. The same applies by symmetry in $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$. The proof is complete.

Lemma 2: Consider $h(x)=\cos x-1+\frac{x^2}{2}-\frac{x^4}{48}$ over $\left(0,\frac{\pi}{4}\right]$. $h(0)=0$,

$$h'(x)=-\sin x+x-\frac{4x^3}{48}, h'(0)=0, h''(x)=\cos x+1-\frac{12x^2}{48}>0$$

As for $x\leq\frac{\pi}{4}\cos x>0, 1-\frac{12x^2}{48}>0$. So $h'(x)\uparrow$ and $>h'(0)=0\to h(x)\uparrow$ and

$$h(x)>h(0)=0$$

Lemma 1: Easy in a similar manner.

SOLUTION 3.37

Solution by Khanh Hung Vu-Vietnam

If $2\sin^2 x + 2\sin^2 y = 1, x, y \in \left(0; \frac{\pi}{2}\right)$ then $2\tan x \tan y + 2\tan x + 2\tan y < 3$ *(1)*

We have $2\sin^2 x + 2\sin^2 y = 1 \Rightarrow \sin^2 x + \sin^2 y = \frac{1}{2} \Rightarrow 1 - \cos^2 x + 1 - \cos^2 y = \frac{1}{2}$

$$\Rightarrow \cos^2 x + \cos^2 y = \frac{3}{2} \Rightarrow \frac{1}{1+\tan^2 x} + \frac{1}{1+\tan^2 y} = \frac{3}{2} \quad (2)$$

Put $\tan x = a, \tan y = b \Rightarrow a, b \in (0; +\infty)$

We have the equation (2) equivalent to:

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} = \frac{3}{2} \Rightarrow \frac{a^2+b^2+2}{a^2b^2+a^2+b^2+1} = \frac{3}{2} \Rightarrow$$

$$\Rightarrow 2(a^2+b^2+2) = 3(a^2b^2+a^2+b^2+1) \Rightarrow$$

$$\Rightarrow 3a^2b^2 + a^2 + b^2 = 1 \Rightarrow 3a^2b^2 + (a+b)^2 - 2ab = 1 \quad (3)$$

On the other hand, we have

$$(a+b)^2 \ge 4ab \Rightarrow -3a^2b^2 + 2ab + 1 \ge 4ab \Rightarrow -3a^2b^2 - 2ab + 1 \ge 0 \Rightarrow$$

$\Rightarrow 0 < ab \le \frac{1}{3}$. *That means the equation (3) is equivalent to*

$a + b = \sqrt{-3a^2b^2 + 2ab + 1}$. *We have the inequality (1) equivalent to*

$$2ab + 2a + 2b < 3 \Rightarrow 2ab + 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 \Rightarrow$$

$$\Rightarrow 2\sqrt{-3a^2b^2 + 2ab + 1} < 3 - 2ab \Rightarrow 4(-3a^2b^2 + 2ab + 1) < 4a^2b^2 - 12ab + 9 \Rightarrow$$

$$\Rightarrow 16a^2b^2 - 20ab + 5 > 0 \Rightarrow 16\left(ab - \frac{5+\sqrt{5}}{8}\right)\left(ab - \frac{5-\sqrt{5}}{8}\right) > 0$$

(True since $ab - \frac{5+\sqrt{5}}{8} < 0$ *and* $ab - \frac{5-\sqrt{5}}{8} < 0$ *by* $0 < ab \le \frac{1}{3}$*)*

So, (1) is true $\Rightarrow 2\tan x \tan y + 2\tan x + 2\tan y < 3$

SOLUTION 3.38

Solution by Khanh Hung Vu-Vietnam

By BCS inequality and AM-GM inequality, we have:

$$\frac{x^2}{\sin\frac{3\pi}{11}} + \frac{y^2}{\sin\frac{4\pi}{11}} \ge \frac{(x+y)^2}{\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11}} \ge \frac{4xy}{\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11}} \ge \frac{1}{2\left(\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11}\right)} \quad \text{(Since } xy \ge \tfrac{1}{8}\text{) (1)}$$

We need to prove that $\frac{1}{2\left(\sin\frac{3\pi}{11} + \sin\frac{4\pi}{11}\right)} > \frac{1}{\left(\cos\frac{2\pi}{11} + \cos\frac{5\pi}{11}\right)^2}$ *(2)*

$$\textit{Put } t=\frac{\pi}{11}\Rightarrow 11t=\pi\Rightarrow 4t=\pi-7t\Rightarrow \sin 4t=\sin(\pi-7t)=\sin 7t$$

$$\textit{We have inequality (2) equivalent to } \frac{1}{2(\sin 3t+\sin 4t)}>\frac{1}{(\cos 2t+\sin 5t)^2}\quad \textit{(3)}$$

$$\textit{We have } (\cos 2t+\sin 5t)^2=\left(\sin\left(\frac{\pi}{2}-2t\right)+\sin 5t\right)^2=\left(2\sin\left(\frac{\pi}{4}+\frac{3t}{2}\right)\cos\left(\frac{\pi}{4}-\frac{7t}{2}\right)\right)^2$$

$$\Rightarrow (\cos 2t+\sin 5t)^2=4\sin^2\left(\frac{\pi}{4}+\frac{3t}{2}\right)\cos^2\left(\frac{\pi}{5}-\frac{7t}{2}\right)$$

$$=\left[1-\cos\left(\frac{\pi}{2}+3t\right)\right]\left[1+\cos\left(\frac{\pi}{2}-7t\right)\right]$$

$$\Rightarrow (\cos 2t+\sin 5t)^2=(1+\sin 3t)(1+\sin 7t)\quad \textit{(4)}$$

$$\textit{We have } (\sin\sin 3t-1)(\sin 7t-1)>0\Rightarrow \sin 3t\cdot\sin 7t-\sin 3t-\sin 7t+1>0$$

$$\Rightarrow \sin 3t\cdot\sin 7t+\sin 3t+\sin 7t+1>2(\sin 3t+\sin 7t)$$

$$\Rightarrow (1+\sin 3t)(1+\sin 7t)>2(\sin 3t+\sin 7t)\Rightarrow (1+\sin 3t)(1+\sin 7t)$$
$$>2(\sin 3t+\sin 4t)$$

$$\overset{(4)}{\Rightarrow} (\cos 2t+\sin 5t)^2>2(\sin 3t+\sin 4t)\Rightarrow \textit{(3) true}\Rightarrow \textit{(2) true}$$

$$\textit{From (1) and (2)}\Rightarrow \frac{x^2}{\sin\frac{3\pi}{11}}+\frac{y^2}{\sin\frac{4\pi}{11}}>\frac{1}{\left(\cos\frac{2\pi}{11}+\sin\frac{5\pi}{11}\right)^2}\quad \textit{(Q.E.D.)}$$

SOLUTION 3.39

Solution by Remus Florin Stanca-Romania

$$\sin\alpha-\cos\alpha=\left(\frac{\sqrt{2}}{2}\sin\alpha-\frac{\sqrt{2}}{2}\cos\alpha\right)\cdot\sqrt{2}=\sin\left(\alpha-\frac{\pi}{4}\right)\sqrt{2}\Rightarrow$$

$$\Rightarrow \sin\sqrt{ab}-\cos\sqrt{ab}=\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\sqrt{2}\textit{ and } \sin\left(\frac{a+b}{2}\right)-\cos\left(\frac{a+b}{2}\right)=\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\sqrt{2}$$

$$\textit{The inequality becomes } \sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\sqrt{2}(a+b)\le 2\sqrt{ab}\sqrt{2}\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\Leftrightarrow$$

$$\Leftrightarrow \sin\left(\sqrt{ab}-\frac{\pi}{4}\right)(a+b)\le 2\sqrt{ab}\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\Leftrightarrow$$

$$\Leftrightarrow \sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\cdot\frac{1}{\sqrt{ab}}\le\frac{2}{a+b}\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)$$

$$\textit{Let be the function } f:\left(0;\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\sin\left(x-\frac{\pi}{4}\right)\cdot\frac{1}{x}$$

$$f'(x)=\frac{\cos\left(x-\frac{\pi}{4}\right)x-\sin\left(x-\frac{\pi}{4}\right)}{x^2}=\cos\left(x-\frac{\pi}{4}\right)\cdot\frac{x-\tan\left(x-\frac{\pi}{4}\right)}{x^2}$$

$$x\in\left(0;\frac{\pi}{2}\right)\Rightarrow x-\frac{\pi}{4}\in\left(-\frac{\pi}{4};\frac{\pi}{4}\right)\Rightarrow \cos\left(x-\frac{\pi}{4}\right)\ge 0$$

$$g:\left(0;\frac{\pi}{2}\right)\to\mathbb{R}\ g(x)=x-\tan\left(x-\frac{\pi}{4}\right)\Rightarrow g'(x)=1-\frac{1}{\cos^2\left(x-\frac{\pi}{4}\right)}<0$$

$$\succ g \text{ is a decreasing function and because } g\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-1>0\succ$$

$$\succ g(x)>0\ \forall x\in\left(0;\frac{\pi}{2}\right)\succ f'(x)=\cos\left(x-\frac{\pi}{4}\right)\cdot\frac{x-\tan\left(x-\frac{\pi}{4}\right)}{x^2}>0\succ$$

$$\Rightarrow f \text{ is an increasing function (1)}$$

$$\sqrt{ab} \text{ and } \frac{a+b}{2}\in\left(0;\frac{\pi}{2}\right) \text{ and } \sqrt{ab}\le\frac{a+b}{2}\overset{(1)}{\succ}\sin\left(\sqrt{ab}-\frac{\pi}{4}\right)\frac{1}{\sqrt{ab}}\le\sin\left(\frac{a+b}{2}-\frac{\pi}{4}\right)\frac{2}{a+b}\succ Q.E.D.$$

SOLUTION 3.40

Solution by Soumava Chakraborty-Kolkata-India

$$(1)\Leftrightarrow x^2+y^2+z^2\ge\ln(xyz)+\frac{3}{2}\ln(2e)\Leftrightarrow x^2+y^2+z^2\overset{(2)}{\ge}\ln x+\ln y+\ln z+\frac{3}{2}\ln(2e)$$

$$\text{Let } f(x)=x^2-\ln x-\frac{1}{2}\ln(2e)\ \forall x>0. \text{ Then } f'(x)=2x-\frac{1}{x}=0\Rightarrow x=\frac{1}{\sqrt{2}}$$

$$\text{Also } f''\left(\frac{1}{\sqrt{2}}\right)=\left(\frac{1}{x^2}+2\right)\Big|_{x=\frac{1}{\sqrt{2}}}>0$$

$$\therefore f(x) \text{ attains a minima at } x=\frac{1}{\sqrt{2}}\ \&\because f(x) \text{ never attains a maxima } \forall x>0,$$

$$\therefore f(x)_{\min}=f\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{2}-\ln\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{2}(\ln 2e)=\frac{1}{2}+\frac{1}{2}\ln 2-\frac{1}{2}\ln 2-\frac{1}{2}=0$$

$$\therefore\forall x>0, x^2-\ln x-\frac{1}{2}(\ln 2e)\ge 0\Rightarrow x^2\overset{(i)}{\ge}\ln x+\frac{1}{2}(\ln 2e)$$

$$\text{Similarly, } y^2\overset{(ii)}{\ge}\ln y+\frac{1}{2}(\ln 2e)\ \&\ z^2\overset{(iii)}{\ge}\ln z+\frac{1}{2}(\ln 2e)$$

(i)+(ii)+(iii)⇒ (2) is true (proved)

SOLUTION 3.41

Solution by Michael Sterghiou-Greece

$$3(a+b)+\log(a!\cdot b!)^{10}\ge 6\sqrt{abH_a\cdot H_b}\quad (1)$$

$$H_a=\sum_k^a\frac{1}{k}\le 1+\log a \text{ and } H_b\le 1+\log b$$

$$(1)\to 3(a+b)+10\cdot\log(a!\cdot b!)\ge 6\sqrt{ab}\cdot\sqrt{(1+\log a)(1+\log b)}\quad (2)$$

$$\text{But } 3(a+b)\ge 6\sqrt{ab} \text{ and } \sqrt{(1+\log a)(1+\log b)}\le\frac{1+\log a+1+\log b}{2}=1+\frac{1}{2}(\log a+\log b)$$

From (2) we have stronger inequality

$$10\log(a!\cdot b!)\geq 6\sqrt{ab}\left[\frac{1}{2}(\log a+\log b)+1-1\right]=3\sqrt{ab}(\log a+\log b)$$

and as $\sqrt{ab}\leq\frac{a+b}{2}$ *the even stronger* $20[\log a!+\log b!]\geq 3(a+b)(\log a+\log b)$ ***(3)***

Equality throughout for $a=b=1$***. We observe that if*** $a+b\leq 6$ ***then (3) holds as it can be written as*** $20\left[\sum_{k=1}^{a-1}\log k+\sum_{k=1}^{b-1}\log i\right]+20(\log a+\log b)\geq 18(\log a+\log b)$

So, (3) must be shown for $a+b\geq 7$***. Using the Stirling formula***

$\log a=a\log a-\alpha+\theta\ (\theta>0)$ ***we obtain the stronger inequality***

$$f(a,b)=(17a-3b)\ln a+(17b-3a)\ln b-20(a+b)\quad \textbf{\textit{(4)}}$$

With $a+b\geq 7$***. Assume WLOG*** $a\geq b, a=b+x, x\geq 0$

(4)$\rightarrow f(x,b)=14\log(b+x)-3x\log b+17x\log\,(b+x)-40b+14b\log b-20x$

Assume b fixed and $b,x\in\mathbb{R}^+: f''(x)=\frac{20b+17x}{b+x^2}>0$ ***so***

$$f'(x)\uparrow\rightarrow f'(x)=-\frac{3(x+2b)}{x+b}+17\log(x+b)-3\log b\geq f'(0)=14\log b-6>0$$

for $b\geq 2$***. Thus for*** $b\geq 2$ $\ f'(x)>0\rightarrow f(x)\uparrow\rightarrow f(x)>f(0)\rightarrow$

$\rightarrow f(x)>4b(7\log b-10)>0$ ***for*** $b\geq 5$***. Therefore*** $\forall b\geq 5\ f(x,b)>0$ ***or*** $f(a,b)>0$ ***for*** $a\geq b\geq 5$***. Now we have only the following cases:***

$b=1\rightarrow f(a,b)=(17a-3)\ln a-20(a+1)>0$ ***for*** $a\geq 6$ ***as can easily be shown*** $f'(a)>0$ ***for*** $a\geq 6$ ***and*** $f(\alpha)>f(6)>0$***. In a similar way we meet the cases*** $b=2, a\geq 5$ $b=3, a\geq 4; b=4, a\geq 4$***. All cases have been exhausted and the proof is complete.***

SOLUTION 3.42

Solution by Tran Hong-Vietnam

Inequality $\Leftrightarrow$

$$\frac{4}{\cos x\cos y\cos z\sqrt{\cos(x-y)\cos(y-z)\cos(z-x)}}\geq\sqrt{2}\cdot\frac{(\sin x+\cos x)(\sin y+\cos y)(\sin z+\cos z)}{\cos x\cos y\cos z};$$

$$\Leftrightarrow 8\geq\{\cos(x-y)(\sin x+\cos x)^2\}\times\{\cos(y-z)(\sin y+\cos y)^2\}\times$$
$$\times\{\cos(x-z)(\sin z+\cos z)^2\}\quad \textbf{\textit{(*)}}$$

$$\therefore 0<x,y,z<\frac{\pi}{2}\Rightarrow 0<\cos(x-y),\cos(y-z),\cos(z-x)\leq 1\quad \textbf{\textit{(1)}}$$

$$\therefore(\sin x+\cos x)^2=1+\sin 2x\leq 2\quad \textbf{\textit{(2)}}$$

$$\therefore (\sin y + \cos y)^2 = 1 + \sin 2y \le 2 \quad \textit{(3)}$$

$$\therefore (\sin z + \cos z)^2 = 1 + \sin 2z \le 2 \quad \textit{(4)}$$

From (1), (2), (3), (4) we have

$$RHS_{(*)} \le 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 = 8 \quad \textit{(proved)}$$

SOLUTION 3.43

Solution by Tran Hong-Vietnam

$$LHS \frac{\left(\sqrt{\frac{a}{c}}\right)^2}{\log(eb-\log b)} + \frac{\left(\sqrt{\frac{b}{a}}\right)^2}{\log(ec-\log c)} + \frac{\left(\sqrt{\frac{c}{b}}\right)^2}{\log(ea-\log a)} \overset{(Schwarz)}{\ge}$$

$$\frac{\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}}\right)^2}{\log(eb-\log b) + \log(ec-\log c) + \log(ea-\log a)} \overset{(Cauchy)}{\ge}$$

$$\frac{9}{\log(eb-\log b)+\log(ec-\log c)+\log(ea-\log a)} \quad \textit{(*)}$$

Let $f(x) = x - \log(ex - \log x)$ *with* $x \ge 1$

$$\Rightarrow f'(x) = 1 - \left(\frac{e - \frac{1}{x}}{ex - \log x}\right) = \frac{ex - \log x + \frac{1}{x} - e}{ex - \log x};$$

$$g(x) = ex - \log x + \frac{1}{x} - e \quad (\forall x \ge 1)$$

$$* \; g'(x) = e - \frac{1}{x} - \frac{1}{x^2}; g''(x) = \frac{1}{x^2} + \frac{2}{x^3} > 0$$

$$\Rightarrow g'(x) \nearrow \textit{on } [1; +\infty) \Rightarrow g'(x) \ge g'(1) = e - 2 > 0$$

$$\Rightarrow g(x) \nearrow \textit{on } [1; +\infty) \Rightarrow g(x) \ge g(1) = 0$$

$$\Rightarrow f'(x) \ge 0 \; \forall x \ge 1 \Rightarrow f(x) \nearrow \textit{on } [1; +\infty)$$

$$\Rightarrow f(x) \ge f(1) = 0 \Rightarrow f(x) = x - \log(ex - \log x) \ge 0; \forall x \ge 1 \quad \textit{(**)}$$

Using inequality (**) with $a, b, c \ge 1$ *we have* $f(a) + f(b) + f(c) \ge 0$

$$\Leftrightarrow \sum a \ge \sum \log(ea - \log a) \Rightarrow \textit{(*)} \ge \frac{9}{\sum a} = \frac{9}{a+b+c}$$

Equality $\Leftrightarrow a = b = c = 1.$

SOLUTION 3.44

Solution by Michael Sterghiou-Greece

$$\sum_{cyc} a\left(\frac{b}{a}\right)^x + \sum_{cyc} b\left(\frac{a}{b}\right)^x = \sum_{cyc}\left[a\left(\frac{b}{a}\right)^x + b\left(\frac{a}{b}\right)^x\right] \ge \sum_{cyc} 2\sqrt{ab\left(\frac{b}{a}\cdot\frac{a}{b}\right)^x} = 2\sum_{cyc}\sqrt{ab} = 12$$

Equality for $a = b = c = 2$ ***or*** $x = \frac{1}{2}$

SOLUTION 3.45

Solution by Tran Hong-Vietnam

We have:

Inequality $\Leftrightarrow$

$$\frac{1}{2}\left[a\ln\left(1+\frac{x}{a}\right) + x\ln\left(1+\frac{a}{x}\right) + b\ln\left(1+\frac{y}{b}\right) + y\ln\left(1+\frac{b}{y}\right) + c\ln\left(1+\frac{z}{c}\right) + z\ln\left(1+\frac{c}{z}\right)\right] \le \ln 2 \quad (*)$$

Using Jensen's inequality with $f(u) = \ln u$***:***

$$LHS_{(*)} = \frac{1}{2}af\left(1+\frac{x}{a}\right) + \frac{1}{2}xf\left(1+\frac{a}{x}\right) + \frac{1}{2}bf\left(1+\frac{y}{b}\right) + \frac{1}{2}yf\left(1+\frac{b}{y}\right) +$$

$$+\frac{1}{2}cf\left(1+\frac{z}{c}\right) + \frac{1}{2}zf\left(1+\frac{c}{z}\right) \le$$

$$\le \ln\left\{\frac{1}{2}a\left(1+\frac{x}{a}\right) + \frac{1}{2}x\left(1+\frac{a}{x}\right) + \frac{1}{2}y\left(1+\frac{b}{y}\right) + \frac{1}{2}c\left(1+\frac{z}{c}\right) + \frac{1}{2}z\left(1+\frac{c}{z}\right)\right\}$$

$$= \ln\{(a+x+b+y+c+z)\} = \ln 2$$

Proved. Equality $\Leftrightarrow a = b = c = x = y = z = \frac{1}{3}$.

SOLUTION 3.46

Solution by Ravi Prakash-New Delhi-India

We first show $\tan 10^\circ \tan 50^\circ = \tan 30^\circ \tan 20^\circ \Leftrightarrow \sin 50^\circ \sin 10^\circ \cos 30^\circ \cos 20^\circ$

$$= \sin 30^\circ \cos 10^\circ \sin 20^\circ \cos 50^\circ$$

$$LHS = \frac{\sqrt{3}}{4}[2\sin 50^\circ \cos 20^\circ]\sin 10^\circ = \frac{\sqrt{3}}{4}[\sin 70^\circ + \sin 30^\circ]\sin 10^\circ$$

$$= \frac{\sqrt{3}}{8}[2\sin 70^\circ \sin 10^\circ + \sin 10^\circ] = \frac{\sqrt{3}}{8}[\cos 60^\circ - \cos 80^\circ + \sin 10^\circ] = \frac{\sqrt{3}}{8}\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{16}$$

$$RHS = \frac{1}{4}[2\cos 50^\circ \sin 20^\circ]\cos 10^\circ = \frac{1}{4}[\sin 70^\circ - \sin 30^\circ]\cos 10^\circ$$

$$= \frac{1}{8}[2\sin 70^\circ \cos 10^\circ - \cos 10^\circ] = \frac{1}{8}[\sin 80^\circ + \sin 60^\circ - \cos 10^\circ] = \frac{\sqrt{3}}{16}$$

For $a, b > 0$

$$\frac{4ab}{a\cot 50^\circ + b\cot 10^\circ} \le \frac{2ab}{\sqrt{ab\cot 50^\circ \cot 10^\circ}} = 2\sqrt{ab}\sqrt{\tan 50^\circ \tan 10^\circ}$$

$$= 2\sqrt{ab\tan 30^\circ \tan 20^\circ} \le a\tan 30^\circ + b\tan 20^\circ$$

$$\therefore \sum_{cyc} \frac{4xy}{x\cot 50^\circ + y\cot 10^\circ} \le \sum_{cyc}(x\tan 30^\circ + y\tan 20^\circ)$$

$$= (x+y+z)(\tan 30^\circ + \tan 20^\circ) = (x+y+z)\left(\frac{\sqrt{3}}{3} + \tan 20^\circ\right)$$

SOLUTION 3.47

Solution by Michail Sterghiou-Greece

$$4\left(\frac{3^a}{4^a} - \frac{3^b}{4^b}\right) < 5\left(\frac{4^a}{5^a} - \frac{4^b}{5^b}\right) < 6\left(\frac{5^a}{6^a} - \frac{5^b}{6^b}\right) \quad \textit{(1)}$$

Consider the function $[4,\infty) \to \mathbb{R}: f(x) = \left(\frac{x}{x-1}\right)^{b-a} - \frac{b-a}{a+x-1} - 1 =$

$$= \left(1 + \frac{1}{x-1}\right)^{b-a} - \frac{b-a}{a+x-1} - 1 \overset{Bernoulli}{\ge} 1 + \frac{b-a}{x-1} - \frac{b-a}{a+x-1} - 1 =$$

$$= (b-a)\left(\frac{1}{x-1} - \frac{1}{a+x-1}\right) > 0$$

Consider now the function $[4,\infty) \to \mathbb{R}: g(x) = x\left[\left(\frac{x-1}{x}\right)^a - \left(\frac{x-1}{x}\right)^b\right]$

$$g'(x) = \frac{1}{x-1}\left[\left(\frac{x-1}{x}\right)^a (a+x-1) - \left(\frac{x-1}{x}\right)^b (b+x-1)\right]$$

Assuming $(x-1)g'(x) > 0$

$\left(\frac{x-1}{x}\right)^a (a+x-1) > \left(\frac{x-1}{x}\right)^b (b+x-1) \leftrightarrow \left(\frac{x}{x-1}\right)^{b-a} > \frac{b-a}{a+x-1} + 1 \leftrightarrow f(x) > 0$ ***which is valid. Therefore*** $g'(x) > 0$ ***and*** $g(x) \uparrow$

As $4 < 5 < 6 \to f(4) < f(5) < f(6) \to$ ***(1) is true.***

SOLUTION 3.48

Solution by Lahiru Samarakoon-Sri Lanka

$$\Omega(a,b) = \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots + \frac{b(b-1)\dots 2\cdot 1}{(a+b-1)(a+b-2)\dots a}$$

Then,

$$b\cdot\Omega(a,b) + c\cdot\Omega(b,c) + a\cdot\Omega(c,a) \ge a+b+c$$

By adding last three parts,

$$\Omega(a,b)=\frac{b}{a+b-1}+\cdots+\underbrace{\frac{b(b-1)\dots 2}{(a+b-1)\dots(a+1)}+\frac{b(b-1)\dots 2\cdot 1}{(a+b-1)\dots a}}_{\Downarrow}$$

$$\frac{b}{(a+b-1)}+\cdots+\frac{b(b-1)\dots 2(a+1)}{(a+b-1)(a+b-2)\dots(a+1)a}$$

$$\vdots$$

$$\Omega(a,b)=\frac{b}{(a+b-1)}+\frac{b(b-1)}{(a+b-1)a}=\frac{b(a+b-1)}{(a+b-1)a}=\frac{b}{a}$$

So, similarly,

$$\Omega(b,c)=\frac{c}{b}\textit{ and }\Omega(c,a)=\frac{a}{c}$$

$$\therefore LHS=b\Omega(a,b)+c\Omega(b,c)+a\Omega(c,a)$$

$$=\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}\geq\frac{(b+c+a)^2}{(a+b+c)}=(b+c+a)$$

SOLUTION 3.49

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\csc^4\left(\frac{\pi}{7}\right)}{\sqrt{ab}}+\frac{\csc^4\left(\frac{2\pi}{7}\right)}{\sqrt{bc}}+\frac{\csc^4\left(\frac{3\pi}{7}\right)}{\sqrt{ca}}\overset{(1)}{>}1$$

$$\textit{LHS of (1)}\underset{(a)}{\overset{Bergstrom}{>}}\frac{(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta)^2}{\sum\sqrt{ab}};\left(\theta=\frac{\pi}{7}\right)$$

$$\overset{CBS}{\geq}\frac{(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta)^2}{\sum a}$$

$$=\frac{(\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta)^2}{64}\quad\left(\because\sum a=64\right)$$

$$\textit{Now, }\csc^2\theta+\csc^2 2\theta+\csc^2 3\theta=(\csc\theta+\csc 2\theta+\csc 3\theta)^2-$$

$$-2(\csc\theta\csc 2\theta+\csc 2\theta\csc 3\theta+\csc 3\theta\csc\theta)\overset{(b)}{=}P^2-2Q\textit{ (say)}$$

$$P=\frac{1}{\sin\theta}+\frac{1}{\sin 2\theta}+\frac{1}{\sin 3\theta}$$

$$\overset{(i)}{=}\frac{\sin 2\theta-\sin 3\theta+\sin 3\theta\sin\theta+\sin\theta\sin 2\theta}{\sin\theta\sin 2\theta\sin 3\theta}$$

$$\textit{Numerator of above}=\sin 3\theta(\sin 2\theta+\sin\theta)+\sin\theta\sin 2\theta=$$

$$=\sin 3\theta(\sin 2\theta+\sin 6\theta)+\sin\theta\sin 2\theta\quad(\because\theta=\pi-6\theta)$$

$$=2\sin 3\theta\sin 4\theta\cos 2\theta+\sin\theta\sin 2\theta=(\sin 5\theta+\sin\theta)\sin 4\theta+\sin\theta\sin 2\theta$$

$$= \sin 2\theta \sin 4\theta + \sin\theta \sin 4\theta + \sin\theta \sin 2\theta \quad (\because 5\theta = \pi - 2\theta)$$

$$= \sin 2\theta (\sin 3\theta + \sin\theta) + \sin\theta \sin 4\theta \quad (\because 4\theta = \pi - 3\theta)$$

$$= 2\sin^2 2\theta \cos\theta + 2\sin\theta \sin 2\theta \cos 2\theta = 2\sin 2\theta (\sin 2\theta \cos\theta + \cos 2\theta \sin\theta)$$

$$\overset{(ii)}{=} 2\sin 2\theta \sin 3\theta$$

(i), (ii) $\Rightarrow P^2 \overset{(m)}{=} \frac{4}{\sin^2\theta}$

Now, $Q = \frac{2\cos 2\theta}{\sin^2\theta} \Leftrightarrow \frac{2\cos 2\theta}{\sin^2\theta} = \frac{\sin 3\theta + \sin\theta + \sin 2\theta}{\sin\theta \sin 2\theta \sin 3\theta}$

$$\Leftrightarrow (2^2)\cos 2\theta \sin 2\theta \sin 3\theta = 2\sin 3\theta \sin\theta + 2\sin^2\theta + 2\sin\theta \sin 2\theta$$

$$\Leftrightarrow 2\sin 4\theta \sin 3\theta = \cos 2\theta - \cos 4\theta + 1 - \cos 2\theta + \cos\theta - \cos 3\theta$$

$$\Leftrightarrow \cos\theta - \cos\pi = 1 + \cos\theta - (\cos 3\theta + \cos 4\theta)$$

$$\Leftrightarrow 1 + \cos\theta = 1 + \cos\theta - \{\cos(\pi - 4\theta) + \cos 4\theta\}$$

$\Leftrightarrow 0 = -(-\cos 4\theta + \cos 4\theta) \Leftrightarrow 0 = 0 \rightarrow$ *true*

$$\Rightarrow Q = \frac{2\cos 2\theta}{\sin^2\theta} = \frac{2(1 - 2\sin^2\theta)}{\sin^2\theta} = \frac{2}{\sin^2\theta} - 4 \Rightarrow 2Q\theta \overset{(n)}{=} \frac{4}{\sin^2\theta} - 8$$

(m), (n), (b) $\Rightarrow \csc^2\theta + \csc^2 2\theta + \csc^2 3\theta \overset{(c)}{=} 8$

(a), (c) $\Rightarrow$ *LHS of (1)* $> \frac{8^2}{64} = 1$ *(proved)*

SOLUTION 3.50

Solution by Amit Dutta-Jamshedpur-India

$$x, y, z \in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}, 0 < z < \frac{\pi}{2}$$

$$\Rightarrow x + y + z \in \left(0, \frac{3\pi}{2}\right) \Rightarrow -1 < \sin(x + y + z) < 1 \Rightarrow \sin(x + y + z) > -1$$

$$\Rightarrow \sin x \cos y \cos z + \sin y \cos x \cos z + \sin z \cos y \cos x - \sin x \sin y \sin z > -1$$

Dividing throughout by $\cos x \cos y \cos z$

$$\Rightarrow \tan x + \tan y + \tan z - \tan x \tan y \tan z > -\frac{1}{\cos x \cos y \cos z}$$

$$\Rightarrow \tan x + \tan y + \tan z > \tan x \tan y \tan z - \frac{1}{\cos x \cos y \cos z}$$

SOLUTION 3.51

Solution by Ravi Prakash-New Delhi-India

For $x, y > 0$

$$\frac{2x+y}{3} \geq (x^2y)^{\frac{1}{3}} \Rightarrow \frac{2x+y}{3} \cdot \frac{x+2y}{3} \geq [(x^2y)(xy^2)]^{\frac{1}{3}}$$

$$\Rightarrow \frac{2x+y}{3} \cdot \frac{x+2y}{3} \geq xy \Rightarrow \frac{(2x+y)(x+2y)}{9xy} \geq 1 \Rightarrow \tan^{-1}\left(\frac{(2x+y)(x+2y)}{9xy}\right) \geq \frac{\pi}{4}$$

Thus,

$$\tan^{-1}\left(\frac{(2a+b)(a+2b)}{9ab}\right) + \tan^{-1}\left(\frac{(2b+c)(b+2c)}{9bc}\right) + \tan^{-1}\left(\frac{(2a+c)(a+2c)}{9ac}\right) \geq$$

$$\geq \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} = \frac{3}{4}\pi$$

SOLUTION 3.52

Solution by Lahiru Samarakoon-Sri Lanka

$$(a+b+c)\left(\frac{1}{d}+\frac{1}{e}+\frac{1}{f}\right) \leq 3\left(\frac{a}{d}+\frac{b}{e}+\frac{c}{f}\right)$$

We can simplify, $\frac{(b+c)}{d} + \frac{(a+c)}{e} + \frac{(a+b)}{f} \leq 2\left(\frac{a}{d}+\frac{b}{e}+\frac{c}{f}\right)$

$$\frac{(5-e+5-f)}{d} + \frac{(5-d+5-f)}{e} + \frac{(5-d+5-e)}{f} \leq 2\left(\frac{5}{d}-1+\frac{5}{e}-1+\frac{5}{f}-1\right)$$

$$6 \leq \left(\frac{e}{d}+\frac{d}{e}\right) + \left(\frac{f}{d}+\frac{d}{f}\right) + \left(\frac{e}{f}+\frac{f}{e}\right)$$

By AM-GM,

$$\left(\frac{e}{d}+\frac{d}{e}\right) \geq 2$$

Similarly, $\left(\frac{f}{d}+\frac{d}{f}\right) \geq 2$ ***and*** $\left(\frac{e}{f}+\frac{f}{e}\right) \geq 2$

So, $\sum\left(\frac{e}{d}+\frac{d}{e}\right) \geq 6$ ***(proved)***

SOLUTION 3.53

Solution by Tran Hong-Vietnam

$$\left(\frac{a+b}{2}\right)^2 \sin\frac{2}{a+b} \geq ab\sin\frac{1}{\sqrt{ab}} \ (*); \ (a,b>1)$$

Let $f(t) = t^2\sin\frac{1}{t}\,(t>1)$

$$\Rightarrow f'(t) = 2t\sin\frac{1}{t} - \cos\frac{1}{t} = \cos\frac{1}{t}\left(2t\tan\frac{1}{t}-1\right) > \cos\frac{1}{t} > 0$$

$$\left(\because \tan\frac{1}{t} > \frac{1}{t}; \cos\frac{1}{t} > 0 \ \forall t > 1\right)$$

$$\Rightarrow f(t) \nearrow \text{ on } (1;+\infty)$$

Hence, $\sqrt{ab} \le \frac{a+b}{2} \Rightarrow f(\sqrt{ab}) \le f\left(\frac{a+b}{2}\right) \Rightarrow$ *(*) true.*

$$\Rightarrow \prod \left(\frac{a+b}{2}\right)^2 \sin\frac{2}{a+b} \ge \prod ab \sin\frac{1}{\sqrt{ab}}$$

$$\Leftrightarrow \frac{\prod \sin\frac{2}{a+b}}{\prod \sin\frac{1}{\sqrt{ab}}} \ge 4^3 \frac{a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2}$$

$$\Leftrightarrow \frac{\prod \sin\frac{2}{a+b}}{\prod \sin\frac{1}{\sqrt{ab}}} \ge \frac{(8abc)^2}{[(a+b)(b+c)(c+a)]^2}$$

SOLUTION 3.54

Solution by Daniel Sitaru – Romania

We prove that:

$$\begin{cases} \dfrac{2ab}{a+b} \le \dfrac{a+b}{2} \\ \dfrac{a+b}{2} \le \sqrt{\dfrac{a^2+b^2}{2}} \end{cases} \Leftrightarrow \begin{cases} 4ab \le (a+b)^2 \\ (a+b)^2 \le 2(a^2+b^2) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 0 \le a^2 - 2ab + b^2 \\ a^2 + 2ab + b^2 \le 2a^2 + 2b^2 \end{cases} \Leftrightarrow \begin{cases} 0 \le (a-b)^2 \\ 0 \le (a-b)^2 \end{cases}$$

It follows:

$$0 < a \le \frac{2ab}{a+b} \le \sqrt{\frac{a^2+b^2}{2}} \le b$$

From Schweitzer inequality for $n = 2$***, if*** $x_1, x_2 \in [a,b]$ ***then:***

$$(x_1 + x_2)\left(\frac{1}{x_1} + \frac{1}{x_2}\right) \le \frac{(a+b)^2}{ab}$$

Let be $x_1 = \frac{2ab}{a+b}; x_2 = \sqrt{\frac{a^2+b^2}{2}}$ ***; It follows:***

$$\left(\frac{2ab}{a+b} + \sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{a+b}{2ab} + \sqrt{\frac{2}{a^2+b^2}}\right) \le \frac{(a+b)^2}{ab}$$

(Schweitzer's inequality:

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) \le \frac{(m+M)^2 n^2}{4mM}$$

$$x_1, x_2, \dots, x_n \in [m, M]; 0 < m \le x_k \le M; k \in \overline{1, n}; n \in \mathbb{N}^*)$$

SOLUTION 3.55

Solution by Amit Dutta-Jamshedpur-India

$$x^2 + y^2 = (x+y)^2 - 2xy = (x^2+y^2)^2 - \left(\sqrt{2xy}\right)^2$$

$$x^2 + y^2 = \left(x^2 + y^2 + \sqrt{2xy}\right)\left(x^2 + y^2 - \sqrt{2xy}\right)$$

$$\because GM \ge AM$$

$$\Rightarrow \left[2\left(x^2 + y^2 + \sqrt{2xy}\right)\left(x^2 + y^2 - \sqrt{2xy}\right)\right]^{\frac{1}{2}} \le$$

$$\le \frac{\left(2+\sqrt{2}\right)\left(x + y - \sqrt{2xy}\right) + \left(2 - \sqrt{2}\right)\left(x + y + \sqrt{2xy}\right)}{2}$$

$$\le \frac{4(x+y) - 4\sqrt{xy}}{2} \le 2\left(x + y - \sqrt{xy}\right)$$

$$\Rightarrow 2\left(x^2 + y^2 + \sqrt{2xy}\right)\left(x^2 + y^2 - \sqrt{2xy}\right) \le 4\left(x + y - \sqrt{xy}\right)^2$$

But $x^2 + y^2 = \left(x^2 + y^2 + \sqrt{2xy}\right)\left(x^2 + y^2 - \sqrt{2xy}\right)$

$$\Rightarrow 2(x^2 + y^2) \le 4\left(x + y - \sqrt{xy}\right)^2$$

$$\Rightarrow (x^2 + y^2) \le 2\left(x + y - \sqrt{xy}\right)^2 \quad (1)$$

In this same way, $(z^2 + t^2) \le 2\left(z + t - \sqrt{zt}\right)^2$ ***(2)***

Multiplying (1) & (2):

$$(x^2 + y^2)(z^2 + t^2) \le 4\left[\left(x + y - \sqrt{xy}\right)\left(z + t - \sqrt{zt}\right)\right]^2$$

or $4\left[\left(x - \sqrt{xy} + y\right)\left(z - \sqrt{zt} + t\right)\right]^2 \ge (x^2 + y^2)(z^2 +^2)$

SOLUTION 3.56

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $a, b, c, d > 0$ ***and*** $a + b + c + d = 1$***, we have:***

$$\frac{ab}{1+c+d} + \frac{ac}{1+b+d} + \frac{ad}{1+b+c} + \frac{bc}{1+a+d} + \frac{bd}{1+a+c} + \frac{cd}{1+a+b}$$

$$=\frac{ab}{a+b+c+d+c+d}+\frac{ac}{a+b+c+d+b+d}+\frac{ad}{a+b+c+d+b+c}+$$

$$+\frac{bc}{a+b+c+d+a+d}+\frac{bd}{a+b+c+d+a+c}+\frac{cd}{a+b+c+d+a+b}\le$$

$$\le\frac{1}{4}\left[\begin{array}{l}\frac{ab}{a+c+d}+\frac{ab}{b+c+d}+\frac{ac}{a+b+d}+\frac{ac}{c+b+d}+\frac{ad}{a+b+c}+\frac{ad}{d+b+c}+\\+\frac{bc}{b+a+d}+\frac{bc}{c+a+d}+\frac{bd}{b+a+c}+\frac{bd}{d+a+c}+\frac{cd}{c+a+b}+\frac{cd}{d+a+b}\end{array}\right]$$

$$\frac{1}{4}\left[\frac{cd+ad+bd}{a+b+c}+\frac{ab+bc+bd}{a+c+d}+\frac{ab+ac+ad}{b+c+d}+\frac{bc+cd+ac}{a+b+d}\right]$$

$$=\frac{1}{4}\left[\frac{d(c+a+b)}{(a+b+c)}+\frac{b(a+c+d)}{(a+c+d)}+\frac{a(b+c+d)}{(b+c+d)}+\frac{c(b+d+a)}{(b+d+a)}\right]$$

$$=\frac{1}{4}(a+b+c+d)=\frac{1}{4}$$

SOLUTION 3.57

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For $a,b,c\in\mathbb{N}$ and $a,b,c\ge 4$. We have these facts:

$$\textit{1. } a^{\frac{1}{a+1}}\ge b^{\frac{1}{b+1}}\Leftrightarrow b\ge a\ge 4\because a^{b+1}\ge b^{a+1}, 4\le a\le b$$

$$\textit{2. } a^{\frac{1}{a+1}}+b^{\frac{1}{b+1}}+c^{\frac{1}{c+1}}\ge a^{\frac{1}{b+1}}+b^{\frac{1}{c+1}}+c^{\frac{1}{a+1}}$$

$$\textit{3. } a^{\frac{1}{a+1}}+b^{\frac{1}{b+1}}+c^{\frac{1}{c+1}}\ge a^{\frac{1}{c+1}}+c^{\frac{1}{b+1}}+b^{\frac{1}{a+1}}$$

$$\textit{Consider, } \sqrt[b+1]{a}\sqrt[c+1]{a}\sqrt[a+1]{b}\sqrt[c+1]{b}\sqrt[a+1]{c}\sqrt[b+1]{c}\le 6\sqrt[5]{4}$$

$$\textit{If } a^{\frac{1}{b+1}}a^{\frac{1}{c+1}}b^{\frac{1}{a+1}}b^{\frac{1}{c+1}}c^{\frac{1}{a+1}}c^{\frac{1}{b+1}}\le 6\cdot 4^{\frac{1}{5}}$$

$$\textit{If } \frac{\left(a^{\frac{1}{b+1}}+a^{\frac{1}{c+1}}+b^{\frac{1}{a+1}}+b^{\frac{1}{c+1}}+c^{\frac{1}{a+1}}+c^{\frac{1}{b+1}}\right)^6}{6}\le 6\cdot 4^{\frac{1}{5}}$$

$$\textit{If } \frac{a^{\frac{6}{b+1}}+a^{\frac{6}{c+1}}+b^{\frac{6}{a+1}}+b^{\frac{6}{c+1}}+c^{\frac{6}{a+1}}+c^{\frac{6}{b+1}}}{6}\le 6\cdot 4^{\frac{1}{5}}$$

$$\textit{If } \left(a^{\frac{6}{b+1}}+b^{\frac{6}{c+1}}+c^{\frac{6}{a+1}}\right)+\left(a^{\frac{6}{c+1}}+c^{\frac{b}{b+1}}+b^{\frac{6}{a+1}}\right)\le 36\cdot 4^{\frac{1}{5}}$$

$$\textit{If } \left(a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\right)+\left(a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\right)\le 36\cdot 4^{\frac{1}{5}}$$

$$\textit{If } a^{\frac{6}{a+1}}+b^{\frac{6}{b+1}}+c^{\frac{6}{c+1}}\le 18\cdot 4^{\frac{1}{5}}$$

$$\textit{If } 3a^{\frac{b}{a+1}}\le 18\cdot 4^{\frac{1}{5}}, 4\le a\le b\le c$$

$$\textit{If } a^{\frac{6}{a+1}} \leq 6 \times 4^{\frac{1}{5}}$$

$$\textit{If } a^{30} \leq 6^{5(a+1)} 4^{(a+1)}$$

and it's true because

$$4^{30} \leq b^{25} \cdot 4^{5}$$

$$5^{30} \leq 6^{30} \cdot 4^{6}$$

$$6^{30} \leq 6^{35} \cdot 4^{7}$$

$$\vdots$$

Therefore, it's true.

SOLUTION 3.58

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } f(x) = \sin x\,(\cos x)^{-\frac{1}{2}} - x\ \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$f'(x) = \frac{\sin^2 x}{2\,(\cos x)^{\frac{3}{2}}} + \sqrt{\cos x} - 1\ \&$$

$$f''(x) = \frac{3\sin^3 x + 2\cos^2 x \sin x}{4(\cos x)^{\frac{5}{2}}} \geq 0, \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow f'(x) \geq f'(0) \forall x \in \left[0, \frac{\pi}{2}\right) \Rightarrow f'(x) \geq 0 \forall x \in \left[0, \frac{\pi}{2}\right) \Rightarrow f(x) \geq f(0) \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow f(x) \geq 0\ \ \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\therefore x \in \left[0, \frac{\pi}{4}\right], \sin x\,(\cos x)^{-\frac{1}{2}} - x \geq 0$$

$$\Rightarrow \sin x \geq x\sqrt{\cos x} \Rightarrow \sin^2 x \geq x^2 \cos x \Rightarrow \sin x \tan x \geq x^2 \Rightarrow \sin x \tan x + x^2 \overset{(1)}{\geq} 2x^2$$

$$\textit{Case (1) } x \in \left[\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right] \therefore x \geq \frac{1}{\sqrt{2}} \Rightarrow 2x^2 - 1 \overset{(2)}{\geq} 0$$

$$\textit{(1), (2)} \Rightarrow \sin x \tan x + x^2 \overset{(3)}{\geq} 1 \textit{(3)} \Rightarrow \textit{it suffices to prove:}$$

$$\sin x + \cos x \overset{(4)}{\geq} x \sin x + x \tan x$$

$$\because \frac{1}{\sqrt{2}} \leq x \leq \frac{\pi}{4}, \therefore \cos x \geq \sin x, \therefore \textit{LHS of (4)} \geq 2\sin x \overset{?}{\geq} x \sin x + \frac{x \sin x}{\cos x}$$

$$\Leftrightarrow 2\cos x \overset{?}{\geq} x\cos x + x \Leftrightarrow (2 - x)\cos x \overset{?}{\geq} x$$

$$\Leftrightarrow \cos x \underset{(5)}{\overset{?}{\geq}} \frac{x}{2 - x} \left(\because 2 - x > 0\ as\ \frac{1}{\sqrt{2}} \leq x \leq \frac{\pi}{4}\right)$$

$$\because \frac{1}{\sqrt{2}} \le x \le \frac{\pi}{4}, \therefore \cos x \ge \frac{1}{\sqrt{2}} \overset{?}{\ge} \frac{x}{2-x} \Leftrightarrow \frac{1}{2} \overset{?}{\ge} \frac{x^2}{(2-x)^2} \Leftrightarrow 4 + x^2 - 4x \overset{?}{\ge} 2x^2$$

$$\Leftrightarrow x^2 + 4x - 4 \underset{(6)}{\overset{?}{\le}} 0 \quad \because x \le \frac{\pi}{4}, \therefore \textit{LHS of (6)} \le \frac{\pi^2}{16} + \frac{4\pi}{4} - 4$$

$= \frac{\pi^2+16\pi-64}{16} < 0 \Rightarrow$ ***(6)⇒ (5)⇒ (4) is true⇒ given inequality is true***

Case 2) $x \in \left[0, \frac{1}{\sqrt{2}}\right)$

$$\because x \ge \sin x \therefore x^2 \ge x \sin x \Rightarrow x^2 \cos x \overset{(i)}{\ge} x \sin x \cos x$$

Again, $\cos x > \frac{1}{\sqrt{2}}\left(\because x < \frac{1}{\sqrt{2}} < \frac{\pi}{4}\right) > x \Rightarrow \cos x > 0 \Rightarrow \sin x \cos x \overset{(ii)}{\ge} x \sin x$

Lastly, $1 \overset{(iii)}{\ge} \cos x$

Now, given inequality $\Leftrightarrow$

$$\sin x \cos x + \cos^2 x + \sin^2 x + x^2 \cos x \ge \cos x + x \sin x \cos x + x \sin x$$

$$\Leftrightarrow x^2 \cos x + \sin x \cos x + 1 \overset{(7)}{\ge} x \sin x \cos x + x \sin x + \cos x$$

(i)+(ii)+(iii)⇒ (7) is true ⇒ given inequality is true.

Combining both cases, we conclude that: given inequality is true $\forall x \in \left[0, \frac{\pi}{4}\right]$ ***(proved)***

SOLUTION 3.59

Solution by Soumitra Mandal-Kolkata-India

Let $a + b + c = p, abc + bc + ca = q$ ***and*** $abc = r$

Then $q = 3(1 - x^2)$ ***and*** $0 \le x < 1$

$$\sum_{cyc} a^3 + 3r \ge 6 + 8x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$$

$$\Leftrightarrow p^3 - 3pq + 6r \ge 6 + 8x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$$

$$\Leftrightarrow 27x^2 + 6r \ge 6 + 8x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$$

$$\Leftrightarrow 6r \ge 6 - 19x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$$

again from SCHUR'S INEQUALITY $p^3 + 9r \ge 4pq$

$\therefore 6r \ge 6 - 24x^2$ ***by putting values of p and q***

we need to prove $6 - 24x^2 \ge 6 - 19x^2 - 10x^3 + \log(1 + x^2 - 2x^3)$

$\Leftrightarrow 5(2x^3 - x^2) - \log(1 + x^2 - 2x^3) \ge 0$ ***... (1)***

$$\text{Let } f(x)=5(2x^3-x^2)-\log(1+x^2-2x^3) \text{ for all } 0\le x<1$$

$$\text{Now, } f'(x)=\frac{(6x^2-2x)(6+5x^2-10x^3)}{1+x^2-2x^3}$$

$$\text{Again } f'(x)\le 0 \text{ for all } \frac{1}{2}\ge x\ge 0 \text{ and } f'(x)\ge 0 \text{ for all } 1>x\ge\frac{1}{2}$$

$$\therefore x=\frac{1}{2} \text{ is the global minimum and } f(x)\ge f\left(\frac{1}{2}\right)=0$$

∴ relation (1) is established hence

$$\sum_{cyc} a^3+3abc\ge 6+8x^2-10x^3+\log(1+x^2-2x^3)$$

SOLUTION 3.60

Solution by Rahim Shabazov-Baku-Azerbaidjian

$$\frac{x+y+z}{3}\ge\sqrt[3]{xyz}+k$$

$$k=\frac{3\left|\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{y}-\sqrt[3]{z}\right)\left(\sqrt[3]{z}-\sqrt[3]{x}\right)\right|}{4}$$

$$x=a^3, y=b^3, z=c^3, a,b,c>0$$

$$a^3+b^3+c^3\ge 3abc+\frac{9}{4}|(a-b)(b-c)(c-a)|\Rightarrow$$

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ac)\ge\frac{9}{4}\cdot|(a-b)(b-c)(c-a)|\Rightarrow$$

$$\Rightarrow(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]\ge\frac{9}{2}\cdot|(a-b)(b-c)(c-a)|$$

$$\Rightarrow 2\cdot(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$
$$\ge 9\cdot|(a-b)(b-c)(c-a)|$$

$$2\cdot(a+b+c)\ge 0=a-b+b-c+c-a$$

$$[(a-b)+(b-c)+(c-a)]\cdot[(a-b)^2+(b-c)^2+(c-a)^2]\ge$$
$$\ge 9\cdot|(a-b)(b-c)(c-a)|$$

$$a-b=x, b-c=y, c-a=z$$

$$(x+y+z)(x^2+y^2+z^2)\ge 9\cdot|xyz|$$

SOLUTION 3.61

Solution by Daniel Sitaru – Romania

$$\text{Let be } f:(0,1)\to\mathbb{R}; f(x)=\frac{\ln x}{x+1}$$

$$f'(x)=\frac{(\ln x)'(x+1)-\ln x\cdot(x+1)'}{(x+1)^2}=\frac{x+1-x\ln x}{(x+1)^2}$$

Let be $g:(0,1)\to\mathbb{R};\ g(x)=x+1-x\ln x$

$$g'(x)=1-1-\ln x=-\ln x>0;\ (\forall)\ x\in(0,1)$$

g ***increasing on*** $(0,1)$

$$\inf g(x)=\lim_{\substack{x\to 0\\ x>0}} g(x)=\lim_{\substack{x\to 0\\ x>0}}(x+1-x\ln x)=1-\lim_{\substack{x\to 0\\ x>0}}\frac{\ln x}{\frac{1}{x}}=1+\lim_{\substack{x\to 0\\ x>0}} x=1>0$$

$f'(x)=\frac{g(x)}{(x+1)^2}>0\Rightarrow f$ ***increasing on*** $(0,1)$

$$0<a\le\frac{2}{\frac{1}{a}+\frac{1}{b}}\le\sqrt{ab}\le\frac{a+b}{2}\le b<1$$

from means inequality.

It follows:

$$f\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)\le f\left(\sqrt{ab}\right)\le f\left(\frac{a+b}{2}\right)$$

$$\frac{\ln\frac{2ab}{a+b}}{\frac{2ab}{a+b}+1}\le\frac{\ln\sqrt{ab}}{\sqrt{ab}+1}\le\frac{\ln\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}+1}$$

$$\ln\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{2ab+a+b}}\le\ln\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\le\ln\left(\frac{a+b}{2}\right)^{\frac{1}{\frac{a+b}{2}+1}}$$

$$\left(\frac{2ab}{a+b}\right)^{\frac{a+b}{a+2ab+b}}\le\left(\sqrt{ab}\right)^{\frac{1}{1+\sqrt{ab}}}\le\left(\frac{a+b}{2}\right)^{\frac{2}{a+b+2}}$$

The equality holds if $a=b=c$.

SOLUTION 3.62

Solution by Ngo Minh Ngoc Bao-Ho Chi Minh-Vietnam

We have lemma:

Considering Polynomial

$$f(x,y,z)=\sum x^4+A\sum x^2y^2+Bxyz\sum x+C\sum x^3y+D\sum xy^3$$

(with A,B,C,D ***are the constant).***

$$f(x,y,z)\geq 0\Leftrightarrow\begin{cases}1+A+B+C+D\geq 0\\3(1+A)\geq C^2+CD+D^2\end{cases},(\forall\, x,y,z\geq 0).$$

$$ab+bc+ca\geq 3\sqrt{abc+ab+bc+ca-4}\Leftrightarrow$$

$$\Leftrightarrow(ab+bc+ca)^2\geq 9abc+9(ab+bc+ca)-36$$

$$\Leftrightarrow\left(\sum ab\right)^2-\frac{3}{2}abc\sum a-\frac{1}{4}\left(\sum a\right)^2+\frac{(\sum a)^4}{36}\geq 0$$

$$\Leftrightarrow\frac{(\sum ab)^2}{2}-\frac{3}{2}abc\sum a-\frac{1}{4}\sum a^2\sum ab+\frac{\sum a^4}{36}+\frac{\sum a^3b}{9}+\frac{\sum ab^3}{9}+$$

$$+\frac{abc\sum a}{3}+\frac{\sum a^2b^2}{6}\geq 0$$

$$\Leftrightarrow\frac{\sum a^4}{36}+\frac{\sum a^2b^2}{2}-\frac{abc\sum a}{2}-\frac{\sum a^3b+\sum ab^3+abc\sum a}{4}+\frac{\sum a^3b}{9}+$$

$$+\frac{\sum ab^3}{9}+\frac{abc\sum a}{3}+\frac{\sum a^2b^2}{6}\geq 0$$

$$\Leftrightarrow\frac{\sum a^4}{36}+\frac{2\sum a^2b^2}{3}-\frac{5abc\sum a}{12}-\frac{5\sum a^3b}{36}-\frac{5\sum ab^3}{36}\geq 0\quad (*)$$

Use lemma with $A=\frac{2}{3},B=-\frac{5}{12},C=D=-\frac{5}{36}$,

we have: $\begin{cases}1+A+B+C+D=1+\frac{2}{3}-\frac{5}{12}-\frac{5}{36}-\frac{5}{36}=\frac{35}{36}>0\\3(1+A)=5\geq C^2+CD+D^2=3\cdot\left(\frac{-5}{36}\right)^2\ (true)\end{cases}\Rightarrow$

$$\Rightarrow LHS(*)\geq RHS(*)$$

Equality occur when $a=b=c=2$.

SOLUTION 3.63

Solution by Soumitra Mandal-Chandar Nagore-India

$$(a+1)^a(b+1)^b+2ab\geq 6\Rightarrow(a+1)^a(b+1)^b+2(1+a)(1+b)\geq 12$$

Let $f(x)=\frac{x+2}{3}\ln(1+x)$ *for all* $x\in(0,2)$ *then*

$$f''(x)=\frac{1}{3}\frac{x}{(1+x)^2}\geq 0 \text{ for all } x\in(0,\infty)$$

Applying Jensen's Inequality,

$$\frac{1}{2}\left\{\frac{a+2}{3}\ln(1+a)+\frac{b+2}{3}\ln(1+b)\right\}\geq\frac{\frac{a+b}{2}+2}{3}\ln\left(\frac{a+b}{2}+1\right)$$

$$\therefore (1+a)^{\frac{a+2}{3}}(1+b)^{\frac{b+2}{2}}\geq 4$$

applying A.M $\geq$ *G.M,*

$$(1+a)^a(1+b)^b+2(1+a)(1+b)\geq 3(1+a)^{\frac{a+2}{3}}(1+b)^{\frac{b+2}{3}}\geq 12$$

hence, $(1+a)^a(1+b)^b+2ab\geq 6$ *(proved)*

SOLUTION 3.64

Solution by Soumitra Mandal-Chandar Nagore-India

Let $p=a+b+c, q=ab+bc+ca$ *and* $abc=r$

now, $q+r=4\Rightarrow \frac{p^2}{3}+\frac{p^3}{27}\geq 4\Rightarrow p\geq 3$

$$\therefore a^3+b^3+c^3+abc\geq 4\Leftrightarrow p^3-3pq+4r\geq 4$$

$$\Leftrightarrow p^3-3pq+4(4-q)\geq 4\Leftrightarrow p^3+12\geq q(3p+4)\Leftrightarrow \frac{p^3+12}{3p+4}\geq q$$

again, from Schur's Inequality, $p^3+9r\geq 4pq\Rightarrow p^3+9(4-q)\geq 4pq$

$\Rightarrow \frac{p^3+36}{4p+9}\geq q$. *Hence, we need to show that*

$$\frac{p^3+12}{3p+4}\geq \frac{p^3+36}{4p+9}\Leftrightarrow 4p^4+9p^3+48p+108\geq 3p^4+4p^3+108p+144$$

$$\Leftrightarrow p^4+5p^3-60p-36\geq 0\Leftrightarrow$$

$$\Leftrightarrow p^3(p-3)+8p^2(p-3)+24p(p-3)+12(p-3)\geq 0$$

$\Leftrightarrow (p-3)(p^3+8p^2+24p+12)\geq 0$, *which is true* $\because p\geq 3$

$\therefore a^3+b^3+c^3+abc\geq 4$ *(proved)*

SOLUTION 3.65

Solution by Ravi Prakash - New Delhi – India

For $k\geq 3$,

$$a_k+\frac{k^2-1}{a_k}\geq 2\sqrt{k^2-1}>(2k-1)$$

Also, $a_2+\frac{2^2-1}{a_2}\geq 2\sqrt{3}>1+3$

Now,

$$\sum_{k=1}^{2016}\left(a_k+\frac{k^2}{a_k}\right)=\sum_{k=2}^{2016}\left(a_k+\frac{k^2-1}{a_k}\right)+\sum_{k=1}^{2016}\frac{1}{a_k}$$

$$>\sum_{k=1}^{2016}(2k-1)+2010\left(\frac{1}{a_1a_2\dots a_k}\right)^{\frac{1}{2016}}=2016^2+2016\left(\frac{1}{a_1a_2\dots a_k}\right)^{\frac{1}{2016}}$$

SOLUTION 3.66

Solution by Soumitra Mandal - Chandar Nagore – India

$$\textit{Let } a, b > 0 \textit{ then } b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b$$

$$\textit{Now } b\ln b + a + \frac{1}{a} - \ln 2 - b = b\ln b + \left(a + \frac{1}{a} - 2\right) + 2 - \ln 2 - b$$

$$\geq b\ln b + \left(a + \frac{1}{a} - 2\right) + 2 + 1 - e^{\ln 2} - b \textit{ since, } e^{\ln 2} \geq 1 + \ln 2$$

$$\geq b(\ln b - 1) + \left(a + \frac{1}{a} - 2\right) + 1 \geq b\left(\frac{b-1}{b} - 1\right) + \left(a + \frac{1}{a} - 2\right) + 1$$

$$\because \ln(x+1) \geq \frac{x}{x+1}$$

$$= a + \frac{1}{a} - 2 \geq 0$$

$$\textit{Hence, } b\ln b + a + \frac{1}{a} \geq \ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \geq 2e^b \textit{ (proved)}$$

$$\textit{Let } a > 0, 0 < b \leq 1 \textit{ then } b^b \cdot e^{1+\frac{1}{a}} \geq (2e)^b$$

$$\textit{Now, } b\ln b + a + \frac{1}{a} - b\ln 2 - b$$

$$= b\ln b + 2 + \left(a + \frac{1}{a} - 2\right) - b\ln 2 - b$$

$$\geq b\ln b + 2 + \left(a + \frac{1}{a} - 2\right) + b\left(1 - e^{\ln 2}\right) - b \textit{ since, } e^{\ln 2} \geq 1 + \ln 2$$

$$\geq b\left(\frac{b-1}{b}\right) + 2(1-b) + \left(a + \frac{1}{a} - 2\right) \textit{ since, } \ln(1+x) \geq \frac{x}{x+1} \textit{ for all } x \geq 0$$

$$= 1 - b + \left(a + \frac{1}{a} - 2\right) \geq 0$$

$$\because 0 < b \leq 1 \textit{ and } a + \frac{1}{a} \geq 2$$

$$\textit{Hence, } b\ln b + a + \frac{1}{a} \geq b\ln 2 + b \Rightarrow b^b \cdot e^{a+\frac{1}{a}} \geq (2e)^b \textit{ (proved)}$$

SOLUTION 3.67

Solution by Seyran Ibrahimov-Maasilli-Azerbaidian

Note that $a\sin x + b\cos x \leq \sqrt{a^2+b^2}$ **so that**

$$\frac{a\sin x}{\sqrt{a^2+b^2}} + \frac{b\cos x}{\sqrt{a^2+b^2}} \leq 1,$$

with equality only when $a = \sqrt{a^2+b^2}\sin x$, $b = \sqrt{a^2+b^2}\cos x$, $x \in \left(0, \frac{\pi}{2}\right)$. The given inequality is equivalent to

$$f(x) = \frac{1}{\sqrt{2}}\tan x + \sqrt{2}\cot x + \frac{2}{\cos x} + 2\cos x \geq \frac{9\sqrt{2}}{2}.$$

where $f(x) = 0$

$$f'(x) = \frac{1}{\sqrt{2}\cos^2 x} + \frac{\sqrt{2}}{\sin^2 x} - \frac{2\sin x}{\cos^2 x} - 2\sin x = 0,$$

or, $\left(2\sqrt{2}\sin^3 x - 1\right)(\cos^2 x + 1)$. $\left(2\sqrt{2}\sin^3 x - 1\right) = 0$ implies

$$x = \frac{\pi}{4} \cdot f\left(\frac{\pi}{4}\right) = \frac{9\sqrt{2}}{2}.$$

SOLUTION 3.68

Solution by Daniel Sitaru – Romania

$$a = \sqrt{4x^2+3};\ b = \sqrt{x^2-x+1};\ c = \sqrt{x^2+x+1}$$

$$a + b > c;\ a + c > b;\ b + c > a$$

In $\Delta\ ABC$ with sides a, b, c:

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{-2x^2-1}{2\sqrt{x^4+x^2+1}}$$

$$\sin A = \sqrt{\frac{3}{4(x^4+x^2+1)}}$$

$$S = \frac{1}{2}bc\sin A = \frac{1}{2}\sqrt{x^4+x^2+1} \cdot \frac{\sqrt{3}}{2\sqrt{x^4+x^2+1}} = \frac{\sqrt{3}}{4}$$

By Hadwiger – Finsler's inequality:

$$\sum (a-b)^2 + 4S\sqrt{3} < a^2 + b^2 + c^2$$

By Hadwiger – Finsler's inequality:

$$\sum (a-b)^2 + 4S\sqrt{3} < a^2 + b^2 + c^2$$

$$\sum (a-b)^2 + 4\sqrt{3} \cdot \frac{\sqrt{3}}{4} < x^2 - x + 1 + x^2 + x + 1 + 4x^2 + 3$$

$$\sum (a-b)^2 < 6x^2 + 2$$

$$\left(\sqrt{x^2-x+1}-\sqrt{x^2+x+1}\right)^2+\left(\sqrt{x^2-x+1}-\sqrt{4x^2+3}\right)^2+$$

$$+\left(\sqrt{x^2+x+1}-\sqrt{4x^2+3}\right)^2<6x^2+2$$

SOLUTION 3.69

Solution by Ravi Prakash-New Delhi-India

Suppose $x>y$, then

$$\frac{\ln x-\ln y}{x-y}=\frac{1}{t}<\frac{1}{y}$$

for some t, $y<t<x$

[Lagsange's Mean value Th.]

$$\Rightarrow y\ln\left(\frac{x}{y}\right)<x-y\Rightarrow\left(\frac{x}{y}\right)^y<\exp(x-y)$$

If $y>x$, then

$$\frac{\ln x-\ln y}{x-y}=\frac{1}{t_1}<\frac{1}{x}$$

$$[x<t_1<y]$$

$$\Rightarrow x\ln\left(\frac{x}{y}\right)<x-y\Rightarrow\left(\frac{x}{y}\right)^x<\exp(x-y)$$

As $0<\frac{x}{y}<1$ and $y>x$

$$\left(\frac{x}{y}\right)^y<\left(\frac{x}{y}\right)^x<\exp(x-y)$$

For $x=y$, $\left(\frac{x}{y}\right)^y=\exp(x-y)$

Thus,

$$\exp(x-y)\ge\left(\frac{x}{y}\right)^y\quad\forall x,y>0$$

Take $x=x_i, y=y_i\quad(i=1,2,\dots,n)$

to obtain

$$\exp(x_i-y_i)\ge\left(\frac{x_i}{y_i}\right)^{y_i}\quad i=1,2,\dots,n$$

$$\Rightarrow\prod_{i=1}^{n}\exp(x_i-y_i)\ge\prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{y_i}\Rightarrow\exp\left(\sum_{i=1}^{n}(x_i-y_i)\right)\ge\prod_{i=1}^{n}\left(\frac{x_i}{y_i}\right)^{y_i}$$

SOLUTION 3.70

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

If $a, b, c > 0; m \geq 0$ ***then:***

$$I = \frac{a}{(b+c)^{m+1}} + \frac{b}{(c+a)^{m+1}} + \frac{c}{(a+b)^{m+1}} \geq \frac{3^{m+1}}{2^{m+1} \cdot (a+b+c)^m}$$

$$\left.\begin{matrix} a \geq b \geq c \\ \frac{1}{(b+c)^{m+1}} \geq \frac{1}{(c+a)^{m+1}} \geq \frac{1}{(a+b)^{m+1}} \end{matrix}\right\} \textit{Chebyshev}$$

$$I \geq \frac{1}{3} \cdot (a+b+c) \cdot \left(\sum \frac{1}{(a+b)^{m+1}}\right) \overset{Cauchy}{\geq}$$

$$\geq \frac{1}{3} \cdot (a+b+c) \cdot \frac{3}{\left(\sqrt[3]{(a+b)\cdot(b+c)\cdot(c+a)}\right)^{m+1}} \overset{Cauchy}{\geq}$$

$$\geq (a+b+c) \cdot \frac{1}{\left(\frac{a+b+b+c+c+a}{3}\right)^{m+1}} = \frac{3^{m+1}}{2^{m+1} \cdot (a+b+c)^m}$$

SOLUTION 3.71

Solution by Nirapada Pal-India

We know, $\frac{\sum_{i=1}^n a_i^m}{n} > \left(\frac{\sum_{i=1}^n a_i}{n}\right)^m$ ***for*** $m > 1$

$$\therefore \frac{(\sqrt{a}+\sqrt{b})^2}{4} + \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}{9} + \frac{(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})^2}{16} =$$

$$= \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 + \left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^2 + \left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}}{4}\right)^2 <$$

$$< \frac{a+b}{2} + \frac{a+b+c}{3} + \frac{a+b+c+d}{4}$$

$$= \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)a + \left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)b + \left(\frac{1}{3}+\frac{1}{4}\right)c + \frac{1}{4}d <$$

$$< 4a + 4b + 4c + 4d = 4(a+b+c+d)$$

SOLUTION 3.72

Solution by Marian Dincă – Romania

Apply Popoviciu inequality for $n = 4$ ***and the function convex*** $f(x) = 2^x$***, we obtain:***

$$f(x) + f(y) + f(z) + f(t) + 4(4-2)f\left(\frac{x+y+z+t}{4}\right) \geq \sum_{cycl} 3\, f\left(\frac{x+y+z}{3}\right)$$

SOLUTION 3.73

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{x^3+y^3} \leq \sqrt{x^2+y^2} \Leftrightarrow (x^3+y^3)^2 \leq (x^2+y^2)^3 \quad (\because x, y \geq 0)$$

$$\Leftrightarrow 2x^3y^3 \le 3x^2y^2(x^2+y^2) \Leftrightarrow \{3(x^2+y^2)-2xy\}x^2y^2 \ge 0$$

$$\Leftrightarrow x^2y^2\{2(x-y)^2+(x+y)^2\} \ge 0 \to \textit{true}$$

$$\therefore \sqrt[3]{x^3+y^3} \le \sqrt{x^2+y^2} \quad \textit{(1)}$$

Again, $\sqrt[4]{x^4+y^4} \le \sqrt{x^2+y^2} \Leftrightarrow x^4+y^4 \le (x^2+y^2)^2$

$\Leftrightarrow 2x^2y^2 \ge 0 \to$ *true* $\because \sqrt[4]{x^4+y^4} \le \sqrt{x^2+y^2}$ *(2)*

(1), (2) $\Rightarrow \sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *is true, for* $n=3, n=4$

Let us assume $\sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *holds true*

for $n=k$ *(some integer* ≥ 4*); we shall prove.*

Then show that $\sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *will hold true for* $n=k+1$ *as well*

$$\sqrt[k+1]{x^{k+1}+y^{k+1}} \le \sqrt{x^2+y^2}$$

$$\Leftrightarrow \left(x^{k+1}+y^{k+1}\right)^2 \le (x^2+y^2)^{k+1} \quad \textit{(a)}$$

By our assumption, $\sqrt[k]{x^k+y^k} \le \sqrt{x^2+y^2}$

$$\Rightarrow \left(x^k+y^k\right)^2 \le (x^2+y^2)^k \quad \textit{(3)}$$

Now, $(x^2+y^2)^{k+1} = (x^2+y^2)(x^2+y^2)^k$

$\ge (x^2+y^2)\left(x^k+y^k\right)^2$ *(by our assumption and by using (3))*

$$= (x^2+y^2)\left(x^{2k}+y^{2k}+2x^ky^k\right)$$

$$= x^{2k+2}+y^{2k+2}+x^2y^{2k}+2x^{k+2}y^k+y^2x^{2k}+2x^ky^{k+2} \overset{?}{\ge} \left(x^{k+1}+y^{k+1}\right)^2$$

$$\Leftrightarrow \left(xy^k\right)^2+\left(yx^k\right)^2+2x^ky^k(x^2+y^2-xy) \ge 0$$

$$\Leftrightarrow \left(xy^k\right)^2+\left(yx^k\right)^2+2x^ky^k\left\{\frac{1}{4}(x+y)^2+\frac{3}{4}(x-y)^2\right\} \ge 0 \to \textit{true,}$$

$\because x, y \ge 0 \Rightarrow$ *a is true*

So, whenever $\sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *is true for* $n=k$ $(k \ge 4, k \in \mathbb{N})$*, then,*

$\sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *is true for* $n=k+1$ *as well.*

Hence, by the principle of mathematical induction,

$\sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2}$ *(b)* $(\forall)n \ge 4, n \in \mathbb{N}$

(b), (1) $\Rightarrow \sqrt[n]{x^n+y^n} \le \sqrt{x^2+y^2} \quad \forall\, n \ge 3, n \in \mathbb{N}$

$$\therefore \sum_{i=3}^{n} \sqrt[i]{x^i + y^i} \le (n-2)\sqrt{x^2 + y^2}$$

SOLUTION 3.74

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\left.\begin{matrix} \ln a = x \\ \ln b = y \\ 2016 = k \in \mathbb{N} \end{matrix}\right\} \Rightarrow \left(\frac{1+x}{2}\right)^k + \left(\frac{1+\frac{1}{xy}}{2}\right)^k + \left(\frac{1+y}{2}\right)^k \overset{Cauchy}{\ge}$$

$$\ge \left(\frac{2\sqrt{x}}{2}\right)^k + \left(\frac{2\cdot\frac{1}{\sqrt{xy}}}{2}\right)^k + \left(\frac{2\sqrt{y}}{2}\right)^k = \left(\sqrt{x}\right)^k + \frac{1}{\left(\sqrt{xy}\right)^k} + \left(\sqrt{y}\right)^k \ge$$

$$(AM-GM)y \ge 3\cdot\sqrt[3]{\left(\sqrt{x}\right)^k \cdot \frac{1}{\left(\sqrt{xy}\right)^k}\cdot (y)^k} = 3.$$

SOLUTION 3.75

Solution by Nirapada Pal-India

$$\frac{a+b+c}{3} = \frac{3a+3b+3c}{9} =$$

$$= \frac{a+b+c+2\left(\frac{a+b}{2}\right)+2\left(\frac{b+c}{2}\right)+2\left(\frac{c+a}{2}\right)}{9}$$

$$\overset{AGM}{\ge} \left(abc\left(\frac{a+b}{2}\right)^2\left(\frac{b+c}{2}\right)^2\left(\frac{c+a}{2}\right)^2\right)^{\frac{1}{9}} = \sqrt[9]{\frac{abc(a+b)^2(b+c)^2(c+a)^2}{64}}$$

SOLUTION 3.76

Solution by Nirapada Pal-Jhargram-India

For $x, y, z \in \left(0, \frac{\pi}{2}\right), \cos x < 1, \cos y < 1, \cos z < 1$

So,

$$\frac{\tan x}{\sin y + \sin z} + \frac{\tan y}{\sin z + \sin x} + \frac{\tan z}{\sin x + \sin y}$$

$$= \frac{\sin x}{\sin y + \sin z}\cdot\frac{1}{\cos x} + \frac{\sin y}{\sin z + \sin x}\cdot\frac{1}{\cos y} + \frac{\sin z}{\sin x + \sin y}\cdot\frac{1}{\cos z}$$

$$> \frac{\sin x}{\sin y+\sin z} + \frac{\sin y}{\sin z+\sin x} + \frac{\sin z}{\sin x+\sin y} \quad \left[since\,\frac{1}{\cos x} > 1, etc.\right] \ge \frac{3}{2}$$ ***by Nesbbit.***

$$\therefore \frac{\tan x}{\sin y+\sin z}+\frac{\tan y}{\sin z+\sin x}+\frac{\tan z}{\sin x+\sin y}>\frac{3}{2}$$

SOLUTION 3.77

Solution by Soumitra Mandal-Chandar Nagore-India

We have, $(\tan^{-1} x)'' = -\frac{2x}{(1+x)^2} \le 0$ *for all* $x \ge 0$

So, $\arctan x$ *is concave. Applying Jensen*

$$\frac{1}{3}\sum_{cyc}\tan^{-1}\left(\frac{a+b}{2}\right) \le \tan^{-1}\left(\frac{a+b+c}{3}\right)=\frac{\pi}{6}$$

similarly,

$$\sum_{cyc}\tan^{-1} a \le \frac{\pi}{2}$$

so,

$$\sum_{cyc}\left(2\tan^{-1}\left(\frac{a+b}{2}\right)+\tan^{-1} c\right) \le \frac{3\pi}{2}$$

SOLUTION 3.78

Solution by Nirapada Pal-Jhargram-India

$$\left(\frac{d}{a}\right)^a\left(\frac{e}{b}\right)^b\left(\frac{f}{c}\right)^c$$

$$\overset{Weighted\ GM-AM}{\le}\left[\frac{a\left(\frac{d}{a}\right)+b\left(\frac{e}{b}\right)+c\left(\frac{f}{c}\right)}{a+b+c}\right]^{a+b+c}=\left(\frac{d+e+f}{a+b+c}\right)^{a+b+c}=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

SOLUTION 3.79

Solution by Redwane El Mellas-Casablanca-Morocco

$$\therefore \binom{a+b}{a}=\frac{(a+b)!}{a!\,b!}=\frac{(a+1)\dots(a+b)}{b!}$$

$$=\frac{[(a+1)\dots(a-2+b)]}{[3.4\dots b]}\frac{(a-1+b)(a+b)}{2}$$

Also, $a \ge 2 \Rightarrow (\forall k=3,\dots,b): a-2+k \ge k \Rightarrow \frac{[(a+1)\dots(a-2+b)]}{[3.4\dots b]} \ge 1.$

So $\binom{a+b}{a} \ge \frac{(a-1+b)(a+b)}{2}$ *(*)*

$$\textit{Then } \sum \binom{a+b}{a} \geq \sum \frac{(a-1+b)(a+b)}{2} > \sum \frac{b(2+2)}{2} = 2\sum a = 200$$

A Generalization of Daniel Sitaru's binomial inequality

Let $a_1, \dots, a_n \geq 2 \in \mathbb{N}$ ***such that*** $a_1 + \cdots + a_n = 33n + 1$ ***for*** $n \geq 3$.

$$\textit{So, } \binom{a_1+a_2}{a_1} + \binom{a_2+a_3}{a_2} + \cdots + \binom{a_{n-1}+a_n}{a_{n-1}} > 66n + 2.$$

For a proof, see my proof in the case $n = 3$.

SOLUTION 3.80

Solution by Șerban George Florin – Romania

$$f(x) = a^x + b^x, a^x + b^x \geq a + b \quad (\forall)x \in \mathbb{R}$$

$$\Rightarrow f(x) \geq f(1) \Rightarrow x = 1 \textit{ minimum point.}$$

$$f'(x) = a^x \ln a + b^x \ln b$$

$$\textit{T. Fermat} \Rightarrow f'(1) = 0 = a \ln a + b \ln b$$

$$g(x) = a^x + b^x + c^x, a^x + b^x + c^x \geq a + b + c \quad (\forall)x \in \mathbb{R}$$

$$g(x) \geq g(1) \Rightarrow x = 1 \textit{ minimum point}$$

$$\textit{T. Fermat } g'(1) = 0 \quad g'(1) = a \ln a + b \ln b + c \ln c = 0$$

$$h(x) = a^x + b^x + c^x + d^x, a^x + b^x + c^x + d^x \geq a + b + c + d$$

$$(\forall)x \in \mathbb{R}$$

$$h(x) \geq h(1) \quad | (\forall)x \in \mathbb{R} \Rightarrow x = 1 \textit{ minimum point}$$

$$\textit{T. Fermat} \Rightarrow h'(1) = 0 = a \ln a + b \ln b + c \ln c + d \ln d = 0 \Rightarrow$$

$$d \ln d = 0$$

$$d \neq 0 \Rightarrow \ln d = 0 \Rightarrow d = 1$$

$$\Rightarrow c \ln c = 0, c \neq 0 \Rightarrow \ln c = 0 \Rightarrow c = 1$$

$$a \ln a + b \ln b = 0$$

$$\Rightarrow \ln a^a \cdot b^b = \ln 1 \Rightarrow a^a \cdot b^b = 1$$

$$\Rightarrow a^{3a} \cdot b^{3b} = 1$$

$$\Rightarrow a^{3a} \cdot b^{3b} \cdot c^{2c} \cdot d^d = 1 \cdot 1 \cdot 1 = 1$$

SOLUTION 3.81

Solution by Rozeta Atanasova-Skopje-Macedonia

Let $a = \sinh x$ ***and*** $b = \sinh y$.

$$(\sinh x)' = \cosh x \textit{ and } (\cosh x)' = \sinh x \textit{ and } \cosh x > \sinh x \Rightarrow$$

$$\cosh y-\cosh x<\sinh y-\sinh x \quad \text{... (1)}$$

$$\tanh 0=0 \text{ and } (\tanh x)'=\frac{1}{\cosh^2 x}<1, \forall x\in\mathbb{R}^+ \Rightarrow$$

$$\Rightarrow \tanh\frac{y-x}{2}<\frac{y-x}{2} \text{... (2)}$$

$$LHS=\frac{2\left(\sqrt{b^2+1}-\sqrt{a^2+1}\right)^2}{b^2-a^2}=\frac{2(\cosh y-\cosh x)^2}{\sinh^2 y-\sinh^2 x}\overset{(1)}{<}\frac{2(\cosh y-\cosh x)}{\sinh y+\sinh x}$$

$$=\frac{2\cdot 2\sinh\frac{x+y}{2}\sinh\frac{y-x}{2}}{2\sinh\frac{x+y}{2}\cosh\frac{y-x}{2}}=2\tanh\frac{y-x}{2}\overset{(2)}{<}y-x=\operatorname{arcsinh} a-\operatorname{arcsinh} b$$

$$=\ln\left(b+\sqrt{b^2+1}\right)-\ln\left(a+\sqrt{a^2+1}\right)=\ln\frac{b+\sqrt{b^2+1}}{a+\sqrt{a^2+1}}=RHS$$

SOLUTION 3.82

Solution by Ravi Prakash-New Delhi-India

Put

$$a=x\sin^2 z+y\cos^2 z>0$$

$$b=x\cos^2 z+y\sin^2 z>0\Rightarrow a+b=x+y$$

$$\therefore \frac{(x+y)^2}{(x\sin^2 z+y\cos^2 z)(x\cos^2 z+y\sin^2 z)}+\frac{x}{y}+\frac{y}{x}$$

$$=\frac{(a+b)^2}{ab}+\frac{x}{y}+\frac{y}{x}=\frac{a}{b}+\frac{b}{a}+2+\frac{x}{y}+\frac{y}{x}\geq 6$$

$$\left[\because \frac{a}{b}+\frac{b}{a}\geq 2\ \forall\ a,b>0\right]$$

SOLUTION 3.83

Solution by Soumitra Mandal-Chandar Nagore-India

Applying Weighted AM $\geq$ GM;

$$\frac{c\cdot\left(\frac{a+c}{c}\right)+d\cdot\left(\frac{b+d}{d}\right)}{c+d}\geq\left\{\left(\frac{a+c}{c}\right)^c\cdot\left(\frac{b+d}{d}\right)^2\right\}^{\frac{1}{c+d}}$$

$$\Rightarrow\left(\frac{a+b+c+d}{c+d}\right)^{c+d}\geq\left(\frac{a+c}{c}\right)^c\cdot\left(\frac{b+d}{d}\right)^d$$

$$\Rightarrow (a+b+c+d)^{c+d}\cdot c^c\cdot d^d\geq (c+d)^{c+d}\cdot(c+a)^c\cdot(b+d)^d$$

SOLUTION 3.84

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{2}{y+\sin^2 x}+\frac{2}{y^2+\sin x}\overset{AM\geq GM}{\underbrace{\leq}}\frac{1}{y\sqrt{\sin x}}+\frac{1}{\sqrt{y}\sin x}$$

We need to prove, $\frac{1}{y\sqrt{y}}+\frac{1}{\sin x\sqrt{\sin x}}\geq\frac{1}{y\sqrt{\sin x}}+\frac{1}{\sqrt{y}\sin x}$

$$\Leftrightarrow\left(\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{\sin x}}\right)\left(\frac{1}{\sqrt{y}}-\frac{1}{\sqrt{\sin x}}\right)^2\geq 0\text{, which is true}$$

$$\because y>0,\pi>x>0,\sin x>0$$

SOLUTION 3.85

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } f(x)=6\ln\left(x+\sqrt{x^2+1}\right)-6x+x^3, x\geq 0$$

$$f'(x)=\frac{6}{x+\sqrt{x^2+1}}\left\{1+\frac{x}{\sqrt{x^2+1}}\right\}-6+3x^2, x>0$$

$$=\frac{6}{\sqrt{x^2+1}}+3(x^2+1)-9$$

$$>\frac{9}{(x^2+1)^6}(x^2+1)^3-9=0\quad\forall x>0$$

$$\therefore f(x)\text{ increases on }[0,\infty)$$

$$\Rightarrow f(x)>f(0)\quad\forall x>0\Rightarrow f(x)>0\quad\forall x>0$$

$$\Rightarrow 6\ln\left(x+\sqrt{x^2+1}\right)>6x-x^3\quad\forall x>0$$

$$\Rightarrow\left(x+\sqrt{x^2+1}\right)^6>e^{6x-x^3}\quad\forall x>0\quad (1)$$

Next, let

$$g(x)=6\ln\left(x-\sqrt{x^2+1}\right)+6x+x^3, x\geq 0$$

$$g'(x)=-\frac{6}{\sqrt{x^2+1}}+6+3x^2, x>0$$

$$\Rightarrow g'(x)=6\left(1-\frac{1}{\sqrt{x^2+1}}\right)+3x^2>0\ \forall x>0$$

$$\therefore g(x)\text{ increases on }[0,\infty)\Rightarrow g(x)>g(0)\quad\forall x>0$$

$$\Rightarrow 6\ln\left(x-\sqrt{x^2+1}\right)>-6x-x^3\quad\forall x>0$$

$$\Rightarrow\left(x-\sqrt{x^2+1}\right)^6>e^{-6x-x^3}\quad\forall x>0\quad (2)$$

Putting $x=b$ in (1), $x=a$ in (2),

(with $a, b > 0$) we get

$$\left(a-\sqrt{a^2+1}\right)^6\left(b+\sqrt{b^2+1}\right)^6 > e^{6b-b^3}\cdot e^{-6a-a^3}$$

$$\Rightarrow \frac{\left(b+\sqrt{b^2+1}\right)^6}{\left(a+\sqrt{a^2+1}\right)^6} > e^{(b-a)(6-a^2-b^2-ab)}$$

$$\forall\, a, b > 0$$

SOLUTION 3.86

Solution by Ravi Prakash-New Delhi-India

$$E = (a^2+b^2+c^2)+(ab+bc+ca)(\sin x+\cos x+\sin x\cos x) \ge 0$$

If $ab+bc+ca \ge 0$, then write

$$E = (a^2+b^2+c^2-ab-bc-ca)+(ab+bc+ca)(1+\sin x)(1+\cos x)$$

$$= \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]+(ab+bc+ca)(1+\sin x)(1+\cos x) \ge 0$$

If $ab+bc+ca < 0$ and $\sin x+\cos x+\sin x\cos x < 0$,

still $E \ge 0$.

If $ab+bc+ca < 0$

and $\sin x+\cos x+\sin x\cos x > 0$,

then $\sin x+\cos x+\sin x\cos x \le \sqrt{2}+\frac{1}{2} < 2$

Now, write

$$2E = 2(a^2+b^2+c^2)+(\sin x+\cos x+\sin x\cos x)x$$

$$\{(a+b+c)^2-a^2-b^2-c^2\}$$

$$= (a^2+b^2+c^2)(2-\sin x-\cos x-\sin x\cos x)+$$

$$+(a+b+c)^2(\sin x+\cos x+\sin x\cos x) \ge 0 \Rightarrow E \ge 0 \text{ in this case}$$

Thus, $E \ge 0\ \ \forall\, a, b, c, x \in \mathbb{R}$

SOLUTION 3.87

Solution by Ravi Prakash-New Delhi-India

$$\text{Now, } \Omega_1+2\Omega_2 = x^2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+y^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

$$z^2(a^4+b^4+c^4)$$

$$+2xy\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+2xz(ab+bc+ca)+2yz\left(\frac{b^2}{a}+\frac{c^2}{b}+\frac{a^2}{c}\right)$$

$$=\left(\frac{a^2}{b^2}x^2+\frac{1}{a^2}y^2+b^4z^2+\frac{2}{b}xy+2abxz+\frac{2b^2}{a}yz\right)+$$

$$+\left(\frac{b^2}{c^2}x^2+\frac{1}{b^2}y^2+c^4z^2+\frac{2}{c}xy+2bcxz+\frac{2c^2}{b}yz\right)+$$

$$+\left(\frac{c^2}{a^2}x^2+\frac{1}{c^2}y^2+a^4z^2+\frac{2}{a}xy+2caxz+\frac{2a^2}{c}yz\right)$$

$$=\left(\frac{a}{b}x+\frac{1}{a}y+b^2z\right)^2+\left(\frac{b}{c}x+\frac{1}{b}y+c^2z\right)^2+\left(\frac{c}{a}x+\frac{1}{c}y+a^2z\right)^2\geq 0$$

SOLUTION 3.88

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a,b\in\left(0;\frac{\pi}{2}\right)$ *then* $\frac{\cos a}{1+\cos^4 a}+\frac{\sin a\cos b}{1+\sin^4\cos^4 b}+\frac{\sin a\sin b}{1+\sin^4 a\cdot\sin^4 b}\leq\frac{9\sqrt{3}}{10}$

Put $x=\frac{1}{\cos a},y=\frac{1}{\sin a\cdot\cos b},z=\frac{1}{\sin a\cdot\sin b}$ $(x,y,z>0)$

We have:

$$\cos^2 a+(\sin a\cdot\cos b)^2+(\sin a\cdot\sin b)^2=\cos^2 a+\sin^2 a=1\Rightarrow$$

$$\Rightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1$$

We have $\frac{\cos a}{1+\cos^4 a}+\frac{\sin a\cdot\cos b}{1+\sin^4 a\cos^4 b}+\frac{\sin a\cdot\sin b}{1+\sin^4 a\cdot\sin^4 b}\leq\frac{9\sqrt{3}}{10}$

$$\Rightarrow\frac{\frac{1}{x}}{1+\frac{1}{x^4}}+\frac{\frac{1}{y}}{1+\frac{1}{y^4}}+\frac{\frac{1}{z}}{1+\frac{1}{z^4}}\leq\frac{9\sqrt{3}}{10}\Rightarrow\frac{x^3}{x^4+1}+\frac{y^3}{y^4+1}+\frac{z^3}{z^4+1}\leq\frac{9\sqrt{3}}{10}$$

We have $\frac{x^3}{x^4+1}\leq\frac{27\sqrt{3}}{100}\cdot\frac{1}{x^2}+\frac{21\sqrt{3}}{100}\Rightarrow\frac{x^3}{x^4+1}\leq\frac{27\sqrt{3}+21\sqrt{3}\cdot x^2}{100x^2}\Rightarrow$

$$\Rightarrow 100x^5\leq\left(27\sqrt{3}+21\sqrt{3}x^2\right)(x^4+1)$$

$$\Rightarrow 21\sqrt{3}\cdot x^6-100x^5+27\sqrt{3}\cdot x^4+21\sqrt{3}\cdot x^2+27\sqrt{3}\geq 0\Rightarrow$$

$$\Rightarrow\left(x-\sqrt{3}\right)\left(21\sqrt{3}x^5-37x^4-10\sqrt{3}x^3-30x^2-9\sqrt{3}x-27\right)\geq 0\Rightarrow$$

$$\Rightarrow\left(x-\sqrt{3}\right)^2\left(21\sqrt{3}\cdot x^4+26\cdot x^3+16\sqrt{3}\cdot x^2+18x+9\sqrt{3}\right)\geq 0$$

Similarly, we have $\frac{y^3}{y^4+1}\leq\frac{27\sqrt{3}}{100}\cdot\frac{1}{y^2}+\frac{21\sqrt{3}}{100}$ *and* $\frac{z^3}{z^4+1}\leq\frac{27\sqrt{3}}{100}\cdot\frac{1}{z^2}+\frac{21\sqrt{3}}{100}$

$$\Rightarrow\frac{x^3}{x^4+1}+\frac{y^3}{y^4+1}+\frac{z^3}{z^4+1}\leq\frac{27\sqrt{3}}{100}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{63\sqrt{3}}{100}=\frac{90\sqrt{3}}{100}=\frac{9\sqrt{3}}{10}$$

The equality occurs when

$$x=y=z=\sqrt{3}\Rightarrow \cos a=\sin a\cdot\cos b=\sin a\cdot\sin b=\frac{\sqrt{3}}{3}$$

$$\textit{We have } \cos a=\frac{\sqrt{3}}{3}\Rightarrow \sin a=\sqrt{1-\left(\frac{\sqrt{3}}{3}\right)^2}=\frac{\sqrt{6}}{3}\textit{ and } a=\arccos\left(\frac{\sqrt{3}}{3}\right)$$

$$\textit{We have } \sin a\cdot\cos b=\sin a\cdot\sin b=\frac{\sqrt{3}}{3}\Rightarrow \sin b=\cos b=\frac{\sqrt{2}}{2}\Rightarrow b=\frac{\pi}{4}\Rightarrow QED$$

SOLUTION 3.89

Solution by Boris Colakovic-Belgrade-Serbia

$$be^a+ce^b+ae^c\overset{AM-GM}{\geq}3\sqrt[3]{abce^{a+b+c}}=3\sqrt[3]{e^{a+b+c}}$$

$$\ln\left(be^a+ce^b+ae^c\right)\geq \ln 3+\frac{1}{3}\ln e^{a+b+c}=\ln 3+\frac{a+b+c}{3}\overset{AM-GM}{\geq}\ln 3+1$$

$$be^a+ce^b+ae^c\geq e^{\ln 3+1}=e\cdot e^{\ln 3}=3e>\frac{15}{2}$$

SOLUTION 3.90

Solution by Soumitra Mandal-Chandar Nagore-India

$$(x^2-1)(y^4-1)(z^6-1)+x^2y^4+y^4z^6+z^6x^2-x^2-y^4-z^6=x^2y^4z^6-1$$

$$x^2y^4z^6-1\geq 6yz^2\prod_{cyc}(x^2-1)$$

$$\Leftrightarrow (x^2-1)(y^4-1)(z^6-1)+x^2(y^4-1)+y^4(z^6-1)+z^6(x^2-1)\geq$$
$$6yz^2(x^2-1)(y^2-1)(z^2-1)$$

$$\Leftrightarrow \prod_{cyc}(x^2-1)\{(y^2+1)(z^4+z^2+1)-6yz^2\}+x^2(y^4-1)+$$

$$+y^4(z^6-1)+z^6(x^2-1)\geq 0\textit{, which is true since } x,y,z\geq 1\textit{ and}$$

$$(y^2+1)(z^4+z^2+1)\geq 6yz^2$$

$$\therefore x^2y^4z^6-1\geq 6yz^2\prod_{cyc}(x^2-1)$$

SOLUTION 3.91

Solution by Sanong Hauerai-Nakon Pathom-Thailand

$$arctan\, x^2+arctan\, y^2+arctan\, z^2\geq\frac{(x+y+z)^2}{4}$$

$$x,y,z\geq 0\textit{ and } x^2+y^2+z^2=\frac{\pi}{2}$$

$$\textit{Definition } y=\arctan x\,,x\in\mathbb{R}\textit{ and } -\frac{\pi}{2}<y<\frac{\pi}{2}$$

$$\textit{Iff } x = \tan y, x \in \mathbb{R} \textit{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\textit{Proof give } \arctan x^2 = A \textit{ Iff } \tan A = x^2$$

$$\arctan y^2 = B \textit{ Iff } \tan B = y^2$$

$$\arctan z^2 = C \textit{ Iff } \tan C = y^2$$

$$\textit{consider } \frac{\pi}{2} \le \tan A + \tan B + \tan C = x^2 + y^2 + z^2 \ge \frac{(x+y+z)^2}{3}$$

$$\textit{Iff } \frac{3}{4}(\tan A + \tan B + \tan C) \ge \frac{(x+y+z)^2}{4}$$

$$\textit{and since } x^2 + y^2 + z^2 = \frac{\pi}{2}$$

$$\textit{Hence, } A, B, C \ge 0, \tan A, \tan B, \tan C \ge 0$$

$$\textit{and } A \ge \tan A, B \ge \tan B, C \ge \tan C$$

$$\textit{Hence, } A + B + C \ge \frac{3}{4}(\tan A + \tan B + \tan C) \ge \frac{(x+y+z)^2}{4}$$

$$\textit{Therefore } \arctan x^2 + \arctan y^2 + \arctan z^2 \ge \frac{(x+y+z)^2}{4}$$

SOLUTION 3.92

Solution by Geanina Tudose –Romania

$$\textit{By GM} \le \textit{AM we have}$$

$$\sqrt[n]{\underbrace{1 \cdot 1 \cdot \ldots \cdot 1}_{n-1} \cdot \left(1 + \frac{2\sqrt{ab}}{a+b}\right)} \le \frac{\overbrace{1 + 1 + \ldots + 1}^{n-1} + 1 + \cdots + \frac{2\sqrt{ab}}{a+b}}{n}$$

$$\sqrt[n]{\underbrace{1 \cdot \ldots \cdot 1}_{n-1} \cdot \left(1 - \frac{2\sqrt{ab}}{a+b}\right)} \le \frac{\overbrace{1 + \ldots + 1}^{n-1} + 1 + \frac{2\sqrt{ab}}{a+b}}{n}$$

$$\textit{Summing up, we have } \left(1 + \frac{2\sqrt{ab}}{a+b}\right)^{\frac{1}{n}} + \left(1 - \frac{2\sqrt{ab}}{a+b}\right)^{\frac{1}{4}} \le \frac{n + \frac{2\sqrt{ab}}{a+b} + n - \frac{2\sqrt{ab}}{a+b}}{n} = 2$$

$$\textit{The inequality is strict, since } 1 \pm \frac{2\sqrt{ab}}{a+b} \ne 1, a, b > 0$$

SOLUTION 3.93

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } \theta = \frac{\pi}{7}, 7\theta = \pi \Rightarrow \sin 3\theta = \sin(\pi - 4\theta)$$

$$\Rightarrow 3\sin\theta - 4\sin^3\theta = 4\sin\theta\cos\theta\cos 2\theta$$

$$\sin^2\theta, \sin^2 2\theta, \sin^3 3\theta \textit{ are roots of}$$

$$(3-4t)^2 = 16(1-t)(2t-1)^2 \Rightarrow 64t^3 - 112t^2 + 56t - 7 = 0$$

$$\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 2\theta} + \frac{1}{\sin^2 3\theta} = \frac{\sum \sin^2\theta \sin^2 2\theta}{\sin^2\theta \sin^2 2\theta \sin^2 3\theta} = \frac{\frac{56}{64}}{\frac{7}{64}} = 8$$

Now, by C-S inequality

$$\frac{\sqrt{ab}}{\sin\theta} + \frac{\sqrt{bc}}{\sin 2\theta} + \frac{\sqrt{ca}}{\sin 3\theta} \le\le \sqrt{ab+bc+ca}\sqrt{\frac{1}{\sin^2\theta} + \frac{1}{\sin^2 2\theta} + \frac{1}{\sin^2 3\theta}}$$

$$\le \sqrt{a^2+b^2+c^2}\sqrt{8} = \sqrt{2}\sqrt{8} = 4$$

SOLUTION 3.94

Solution by Chris Kyriazis--Greece

The function $f(x) = 2^x \ln 2 + 4^x \ln 4 + 8^x \ln 8$

$x \in \mathbb{R}$ *is strictly convex and positive for every* $x \in \mathbb{R}$

So, using the Hermite – Hadamard inequality we have: (if $x > y$*)*

$$\frac{\int_y^x f(t)dt}{x-y} > f\left(\frac{x+y}{2}\right) \Leftrightarrow$$

$$\frac{2^x + 4^x + 8^x - (2^y + 4^y + 8^y)}{x-y} > 2^{\frac{x+y}{2}} \ln 2 + 4^{\frac{x+y}{2}} \ln 4 + 8^{\frac{x+y}{2}} \ln 8 =$$

$$= 2^{\frac{x+y}{2}} \ln 2 + \left(2^{\frac{x+y}{2}}\right)^2 \cdot 2\ln 2 + \left(2^{\frac{x+y}{2}}\right)^3 3\ln 3$$

$$\overset{AM-GM}{>} \ln 2 \cdot 6\sqrt[6]{2^{\frac{x+y}{2}}\left(2^{\frac{x+y}{2}}\right)^4 \cdot \left(2^{\frac{x+y}{2}}\right)^9} = \ln 2^6 \cdot \sqrt[6]{\left(2^{\frac{x+y}{2}}\right)^{14}} = \ln 64 \cdot \sqrt[6]{128^{x+y}}$$

SOLUTION 3.95

Solution by Soumava Chakraborty-Kolkata-India

Let's prove that $\forall z \in \left(0, \frac{\pi}{2}\right), \left(\frac{\sin z}{z}\right)^3 > \cos z$

$$(\sin z)^3 > z^3 \cos z \Leftrightarrow \sin^2 z \tan z > z^3$$

Let $f(z) = \sin^2 z \tan z - z^3 \quad f(0) = 0$

$$f'(z) = \sin^2 z \sec^2 z + \tan z\,(2\sin z \cos z) - 3z^2$$

$$= \tan^2 z + 2\sin^2 z - 3z^2 = g(z) \quad g(0) = 0$$

$$g'(z) = 2\tan z \sec^2 z + 4\sin z \cos z - 6z = 2\big(h(z)\big); h(0) = 0$$

$$h'(z)=(\sec^2 z)^2+(\tan z)(2\sec z)(\sec z\tan z)+2(\cos^2 z-\sin^2 z)-3$$
$$=(1+\tan^2 z)^2+2\tan^2 z\,(1+\tan^2 z)+2(2\cos^2 z-1)-3$$
$$=(1+t)^2+2t(1+t)+\frac{4}{1+t}-5=\frac{(1+t)^3+2t(1+t)^2+4-5(1+t)}{1+t}$$
$$=\frac{1+t^3+(3t)+3t^2+(2t)+2t^3+4t^2-(5t)}{1+t}=\frac{3t^3+7t^2}{1+t}>0,$$
$$(t=\tan z>0)$$
$$\therefore h'(z)>0 \text{ and } h(0)=0\Rightarrow h(z)>h(0)=0\Rightarrow g'(z)>0 \text{ and } g(0)=0$$
$$\Rightarrow g(z)>g(0)\Rightarrow f'(z)>0 \text{ and } f(0)=0\Rightarrow f(z)>f(0)=0$$
$$\Rightarrow \sin^2 z\tan z>z^3\Rightarrow\left(\frac{\sin z}{z}\right)^3>\cos z\quad\forall z\in\left(0,\frac{\pi}{2}\right)$$
$$\therefore\left(\frac{\sin y}{y}\right)^3>\cos y \text{ and } \left(\frac{\sin x}{x}\right)^3>\cos x$$
$$\Rightarrow \sin x\left(\frac{\sin y}{y}\right)^3+\sin y\left(\frac{\sin x}{x}\right)^3>\sin x\cos y+\sin y\cos x=\sin(x+y)$$

SOLUTION 3.96

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\textit{If } m\in N, m\geq 2 \textit{ then}$$
$$m+tan^2\left(\frac{\pi}{36}\right)+tan^2\left(\frac{11\pi}{36}\right)+tan^2\left(\frac{13\pi}{36}\right)+tan^2\left(\frac{21\pi}{36}\right)>2+\left(\frac{3}{16}\right)^{\frac{m-2}{m}}$$

We have

$$tan^2\left(\frac{11\pi}{36}\right)>tan^2\left(\frac{\pi}{4}\right)=1\Rightarrow tan^2\left(\frac{\pi}{36}\right)+tan^2\left(\frac{11\pi}{36}\right)+tan^2\left(\frac{13\pi}{36}\right)+tan^2\left(\frac{21\pi}{36}\right)>1$$
$$\Rightarrow m+tan^2\left(\frac{\pi}{36}\right)+tan^2\left(\frac{11\pi}{36}\right)+tan^2\left(\frac{13\pi}{36}\right)+tan^2\left(\frac{21\pi}{36}\right)>m+1\geq 3$$
$$\Rightarrow m+tan^2\left(\frac{\pi}{36}\right)+tan^2\left(\frac{11\pi}{36}\right)+tan^2\left(\frac{13\pi}{36}\right)+tan^2\left(\frac{21\pi}{36}\right)>3\quad (1)$$
$$\textit{On the other hand, we have } \frac{m-2}{m}\geq 0\ \ (\textit{Since } m\geq 2)$$
$$\Rightarrow\left(\frac{3}{16}\right)^{\frac{m-2}{m}}\leq 1\Rightarrow 2+\left(\frac{3}{16}\right)^{\frac{m-2}{2}}\leq 3\quad (2)$$
$$(1)\quad \text{and } (2)\Rightarrow m+tan^2\left(\frac{\pi}{36}\right)+tan^2\left(\frac{11\pi}{36}\right)+tan^2\left(\frac{13\pi}{36}\right)+tan^2\left(\frac{21\pi}{36}\right)>2+\left(\frac{3}{16}\right)^{\frac{m-2}{m}}$$

SOLUTION 3.97

Solution by Ngoc Minh Ngoc Bao-Gia Lang-Vietnam

Use Cauchy – Schwarz inequality:

$$f^2\left(\frac{a_1}{a_2}\right)+f^2\left(\frac{a_2}{a_1}\right)+\cdots+f^2\left(\frac{a_{n-1}}{a_n}\right)+f^2\left(\frac{a_n}{a_1}\right)\ge\frac{1}{n}\left(f\left(\frac{a_1}{a_2}\right)+f\left(\frac{a_2}{a_1}\right)+\cdots+f\left(\frac{a_{n-1}}{a_n}\right)+f\left(\frac{a_n}{a_1}\right)\right)^2$$

(*)

We have:

$$f\left(\frac{a_1}{a_2}\right)+f\left(\frac{a_2}{a_3}\right)+\cdots+f\left(\frac{a_{n-1}}{a_n}\right)+f\left(\frac{a_n}{a_1}\right)$$

$$=\sum_{k=1}^{n}k\left(\frac{a_2}{a_3}\right)^k+\cdots+\sum_{k=1}^{n}k\left(\frac{a_{n-1}}{a_n}\right)^k+\sum_{k=1}^{n}k\left(\frac{a_n}{a_1}\right)^k$$

$$=\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\right)+2\left[\left(\frac{a_1}{a_2}\right)^2+\left(\frac{a_2}{a_3}\right)^2+\cdots+\left(\frac{a_{n-1}}{a_n}\right)^2+\left(\frac{a_n}{a_1}\right)^2\right]+\cdots$$

$$\ldots+n\left[\left(\frac{a_1}{a_2}\right)^n+\left(\frac{a_2}{a_3}\right)^n+\cdots+\left(\frac{a_{n-1}}{a_n}\right)^n+\left(\frac{a_n}{a_1}\right)^n\right]$$

$$\ge n\sqrt[n]{\frac{a_1}{a_2}\cdot\frac{a_2}{a_3}\cdot\ldots\cdot\frac{a_{n-1}}{a_n}\cdot\frac{a_n}{a_1}}+2n\sqrt[n]{\left(\frac{a_1}{a_2}\right)^2\cdot\left(\frac{a_2}{a_3}\right)^2\cdot\ldots\cdot\left(\frac{a_{n-1}}{a_n}\right)^2\cdot\left(\frac{a_n}{a_1}\right)^2}+\cdots$$

$$+n^2\sqrt[n]{\left(\frac{a_1}{a_2}\right)^2\cdot\left(\frac{a_2}{a_3}\right)^n\cdot\ldots\cdot\left(\frac{a_{n-1}}{a_n}\right)^2\cdot\left(\frac{a_n}{a_1}\right)^2}=\frac{n^2(n+1)}{2}$$

$$\Rightarrow LHS\ (*)\ge RHS\ (*)\ge\frac{1}{n}\cdot\left(\frac{n^2(n+1)}{2}\right)^2=\frac{n^3(n+1)^2}{4}$$

Equality when $a_1=a_2=\cdots=a_n$

SOLUTION 3.98

Solution by Abdul Aziz-Semarang-Indonesia

Since $\alpha+\beta+\gamma=\frac{\pi}{2}$ ***then***

$$\tan\alpha\tan\beta+\tan\alpha\tan\gamma+\tan\beta\tan\gamma=1$$

$$\Leftrightarrow A+B+C-15=1\Leftrightarrow A+B+C=16$$

By CS,

$$\sqrt{A}+\sqrt{B}+\sqrt{C}\le\sqrt{(1+1+1)(A+B+C)}$$

$$\Leftrightarrow\sqrt{A}+\sqrt{B}+\sqrt{C}\le\sqrt{3\cdot16}=4\sqrt{3}$$

SOLUTION 3.99

Solution by Rozeta Atanasova-Skopje-Macedonia

$$a\ge b\ge c\Rightarrow m_a\le m_b\le m_c\Rightarrow$$

by Chebyshev's sum inequality

$$LHS \le \frac{1}{3}(m_a + m_b + m_c)\left(cos^2\frac{\pi}{7} + cos^2\frac{2\pi}{7} + cos^2\frac{3\pi}{7}\right) \quad (1)$$

$$\text{But } m_a + m_b + m_c < a + b + c \quad (2)$$

$$\text{because } (2m_c)^2 = a^2 + b^2 - 2ab\,cos(A+B)$$

$$= a^2 + b^2 + 2ab\,cos\,C < (a+b)^2 \Rightarrow$$

$$2m_c < a + b, \text{ and similarly}$$

$$2m_b < a + c$$

$$2m_a < c + b$$

$$-------$$

$$2(m_a + m_b + m_c) < 2(a + b + c)$$

On the other hand

$$cos^2\frac{\pi}{7} + cos^2\frac{2\pi}{7} + cos^2\frac{3\pi}{7} = \frac{3}{2} + \frac{1}{2}\left(cos\frac{2\pi}{7} + cos\frac{4\pi}{7} + cos\frac{6\pi}{7}\right) \quad (3)$$

$$\text{Let's consider the solutions of } z^7 - 1 = 0 \Rightarrow$$

$$\sum_{k=1}^{7} z_k = 0 \Rightarrow \sum_{k=1}^{7} Re(Z_k) = 0 \Rightarrow$$

$$0 = cos\,0 + cos\frac{2\pi}{7} + cos\frac{4\pi}{7} + cos\frac{6\pi}{7} + cos\frac{8\pi}{7} + cos\frac{10\pi}{7} + cos\frac{12\pi}{7}$$

$$= 1 + 2\left(cos\frac{2\pi}{7} + cos\frac{4\pi}{7} + cos\frac{6\pi}{7}\right) \Rightarrow cos\frac{2\pi}{7} + cos\frac{4\pi}{7} + cos\frac{6\pi}{7} = -\frac{1}{2} \quad (4)$$

From (1), (2), (3) and (4) $\Rightarrow$

$$LHS < \frac{1}{3}(a+b+c)\left(\frac{3}{2} - \frac{1}{4}\right) = \frac{2S}{3}\cdot\frac{5}{4} = \frac{5s}{6} = RHS$$

SOLUTION 3.100

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a, b, c \in (4, +\infty)$ *and* $abc = 2^{11}$. *Prove that*

$$\left(a^2\cdot sin\frac{2\pi}{a} + (a+1)^2\cdot sin\frac{2\pi}{a+1}\right)\left(b^2\cdot sin\frac{2\pi}{b} + (b+1)^2\cdot sin\frac{2\pi}{b+1}\right)\left(c^2\cdot sin\frac{2\pi}{c} + (c+1)^2\cdot sin\frac{2\pi}{c+1}\right) > 2^{16}$$

$$\text{Lemma: } sin\,x > \frac{2x}{\pi} \text{ if } x \in \left(0; \frac{\pi}{2}\right)$$

$$\text{Since } \frac{2\pi}{a} \in \left(0, \frac{\pi}{2}\right) \text{ and } \frac{2\pi}{a+1} \in \left(0, \frac{\pi}{2}\right), \text{ applying the lemma, we have:}$$

$$sin\frac{2\pi}{a} > \frac{2}{\pi}\cdot\frac{2\pi}{a} = \frac{4}{a} \text{ and } sin\frac{2\pi}{a+1} > \frac{2}{\pi}\cdot\frac{2\pi}{a+1} = \frac{4}{a+1}$$

$$\textit{We have } a^2\cdot \sin\frac{2\pi}{a}+(a+1)^2\cdot \sin\frac{2\pi}{a+1}>a^2\cdot\frac{4}{a}+(a+1)^2\cdot\frac{4}{a+1}=$$

$$=4a+4(a+1)=8a+4>2\sqrt{8a\cdot 4}$$

$$\textit{Similarly, we have } b^2\cdot \sin\frac{2\pi}{b}+(b+1)^2\cdot \sin\frac{2\pi}{b+1}>2\sqrt{8b\cdot 4} \textit{ and}$$

$$c^2\cdot \sin\frac{2\pi}{c}+(c+1)^2\cdot \sin\frac{2\pi}{c+1}>2\sqrt{8c\cdot 4}$$

So

$$\left(a^2\cdot \sin\frac{2\pi}{a}+(a+1)^2\cdot \sin\frac{2\pi}{a+1}\right)\left(b^2\cdot \sin\frac{2\pi}{b}+(b+1)^2\cdot \sin\frac{2\pi}{b+1}\right)\left(c^2\cdot \sin\frac{2\pi}{c}+(c+1)^2\cdot \sin\frac{2\pi}{c+1}\right)>2\sqrt{8a\cdot 4}\cdot 2\sqrt{8b\cdot 4}\cdot 2\sqrt{8c\cdot 4}=8\sqrt{8^3\cdot abc\cdot 4^3}$$

$$\Rightarrow\left(a^2\cdot \sin\frac{2\pi}{a}+(a+1)^2\cdot \sin\frac{2\pi}{a+1}\right)\left(b^2\cdot \sin\frac{2\pi}{b}+(b+1)^2\cdot \sin\frac{2\pi}{b+1}\right)\left(c^2\cdot \sin\frac{2\pi}{c}+(c+1)^2\cdot \sin\frac{2\pi}{c+1}\right)>8\sqrt{8^3\cdot 2^{11}\cdot 4^3}=2^{16}$$

SOLUTION 3.101

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$\textit{If } n\in\mathbb{N}^*, n\ge 2, a,b,c>1, a+b+c=3^{n+1} \textit{ then}$$

$$\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right)\left(\sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\right)\left(\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\right)<18$$

By AM-GM, we have:

$$\sqrt[n]{a+\sqrt[n]{a}}=\sqrt[n]{\left(\sqrt[n]{a}\right)^{n-1}\left(\sqrt[n]{a}+\frac{1}{\left(\sqrt[n]{a}\right)^{n-2}}\right)}\le\frac{(n-1)\cdot\sqrt[n]{a}+\sqrt[n]{a}+\frac{1}{\left(\sqrt[n]{a}\right)^{n-2}}}{n}=\sqrt[n]{a}+\frac{1}{n\left(\sqrt[n]{a}\right)^{n-2}}$$

$$\textit{Similarly, we have } \sqrt[n]{a-\sqrt[n]{a}}\le\sqrt[n]{a}-\frac{1}{n\left(\sqrt[n]{a}\right)^{n-2}}$$

$$\Rightarrow\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\le 2\cdot\sqrt[n]{a}\quad (1)$$

$$\textit{On the other hand, by AM-GM we have } \sqrt[n]{a\cdot(3^n)^{n-1}}\le\frac{a+(n-1)\cdot 3^n}{n}$$

$$\Rightarrow\sqrt[n]{a}\le\frac{a+(n-1)\cdot 3^n}{n\cdot\sqrt[n]{(3^n)^{n-1}}}=\frac{a+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}\quad (2)$$

$$(1),(2)\Rightarrow\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\le\frac{2a+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}$$

$$\textit{Similarly, we have } \sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\le\frac{2b+(n-1)\cdot 3^n}{n\cdot 3^{n-1}} \textit{ and}$$

$$\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\leq\frac{2c+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}$$

$$\Rightarrow\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right)+\left(\sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\right)+\left(\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\right)\leq$$

$$\leq\frac{2a+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}+\frac{2b+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}+\frac{2c+(n-1)\cdot 3^n}{n\cdot 3^{n-1}}$$

$$\Rightarrow\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right)+\left(\sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\right)+\left(\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\right)\leq$$

$$\leq\frac{2(a+b+c)+3(n-1)\cdot 3^n}{n\cdot 3^{n-1}}=\frac{2\cdot 3^{n+1}+3(n-1)\cdot 3^n}{n\cdot 3^{n-1}}=\frac{(n+1)\cdot 3^{n+1}}{n\cdot 3^{n-1}}=\frac{9(n+1)}{n}$$

Since $\frac{n+1}{n}<2\Rightarrow\frac{9(n+1)}{n}<18$

So $\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right)+\left(\sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\right)+\left(\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\right)\leq 18$

The equality doesn't exist

Therefore,

$$\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right)+\left(\sqrt[n]{b+\sqrt[n]{b}}+\sqrt[n]{b-\sqrt[n]{b}}\right)+\left(\sqrt[n]{c+\sqrt[n]{c}}+\sqrt[n]{c-\sqrt[n]{c}}\right)<18$$

SOLUTION 3.102

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $a,b,c,d>0$***, de tal manera que*** $a+b+c+d=1$***. Probar que***

$$a^3+b^3+c^3+d^3+3(ab+ac+ad+bc+bd+cd)\geq$$

$$\geq 1+6\left(ab\sqrt{cd}+cd\sqrt{ab}\right)$$

Aplicando la siguiente identidad conocida

$(x+y)^3=x^3+y^3+3xy(x+y)$***, donde*** $x=a+b, y=c+d$

$$\Rightarrow(a+b+c+d)^3=(a+b)^3+(c+d)^3+3(a+b+c+d)(a+b)(c+d)$$

$$\Rightarrow 1=a^3+b^3+3ab(a+b)+c^3+d^3+3cd(c+d)+3(a+b)(c+d)$$

$$\Rightarrow 1+3ab(c+d)+3cd(a+b)=a^3+b^3+c^3+d^3+3ab(a+b+c+d)+$$

$$+3cd(c+d+a+b)+3(a+b)(c+d)$$

$$\Rightarrow 1+3ab(c+d)+3cd(a+b)$$

$$=a^3+b^3+c^3+d^3+3ab+3cd+3(ac+ad+bc+bd)$$

$$\Rightarrow 1+3ab(c+d)+3cd(a+b)=a^3+b^3+c^3+d^3+3(ab+ac+ad+bc+bd+cd)$$

Como $a, b, c, d > 0$

Aplicando $MA \geq MG$

$$\Rightarrow a^3 + b^3 + c^3 + d^3 + 3(ab + ac + ad + bc + bd + cd) =$$

$$= 1 + 3ab(c + d) + 3cd(a + b) \geq 1 + 6\left(ab\sqrt{cd} + cd\sqrt{ab}\right)$$

SOLUTION 3.103

Solution by Ravi Prakash-New Delhi-India

Consider the expression

$$E = \prod_{k=1}^{n+1} \frac{1}{a + k}$$

We split this expression into partial fractions.

$$E = \sum_{k=1}^{n+1} \frac{1}{(-k + 1)(k + 2) \dots (-1)(1) \dots (n + 1 - k)} \cdot \frac{1}{a + k}$$

$$E = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{n!} \cdot \frac{n!}{(k - 1)!\,(n + 1 - k)!} \cdot \frac{1}{a + k} = \frac{1}{n!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n}{k - 1} \frac{1}{a + k}$$

Also, $a + n + 1 > 1$

$$\therefore \frac{1}{n!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n}{k - 1} \frac{1}{a + k} = E < \prod_{k=1}^{n} \frac{1}{a + k} \leq \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{a + k}\right)^n$$

$$\Rightarrow \frac{n^n}{n!} \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n}{k - 1} \frac{1}{a + k} < \left(\sum_{k=1}^{n} \frac{1}{a + k}\right)^n$$

SOLUTION 3.104

Solution by Ravi Prakash-New Delhi-India

$$\sum_{k=1}^{n} \binom{n}{k} \cdot x^{2n-2k} \cdot y^{2k} \geq (2^n - 2)x^n y^n \text{ (1)}$$

If $x = 0$ ***or*** $y = 0$***, there is nothing to prove.***

$\therefore$ ***suppose*** $x, y > 0$

Now, (1) can be written as

$$\sum_{k=1}^{n} \binom{n}{k} x^{n-2k} y^{2k-n} \geq 2^n - 2 \Leftrightarrow \sum_{k=1}^{n} \binom{n}{k} \left(\frac{x}{y}\right)^{n-2k} \geq 2^{2n} - 2$$

If n is odd,

$$\sum_{k=1}^{n}\binom{n}{k}\left(\frac{x}{y}\right)^{n-2k} \geq \sum_{1\leq k\leq\left[\frac{n}{2}\right]}\left\{\binom{n}{k}\left(\frac{x}{y}\right)^{n-2k}+\binom{n}{n-k}\left(\frac{x}{y}\right)^{2k-n}\right\}$$

$$=\sum_{1\leq k\leq\left[\frac{n}{2}\right]}\binom{n}{k}\left(\left(\frac{x}{y}\right)^{n-k}+\left(\frac{y}{x}\right)^{n-2k}\right) \geq 2\sum_{1\leq k\leq\left[\frac{n}{2}\right]}\binom{n}{k}=\sum_{k=1}^{n}\binom{n}{k}=2^n-2$$

If n is even,

$$\sum_{k=1}^{n}\binom{n}{k}\left(\frac{x}{y}\right)^{n-2k}$$

$$\geq \sum_{1\leq k\leq\left[\frac{n-1}{2}\right]}\left\{\binom{n}{k}\left(\frac{x}{y}\right)^{n-2k}+\binom{n}{n-k}\left(\frac{y}{x}\right)^{n-2k}\right\}+\binom{n}{\frac{n}{2}}=2\sum_{1\leq k\leq\left[\frac{n}{2}\right]}\binom{n}{k}\left\{\left(\frac{x}{y}\right)^{n-2k}+\left(\frac{y}{x}\right)^{n-2k}\right\}+\binom{n}{\frac{n}{2}}$$

$$\geq 2\sum_{1\leq k\leq\left[\frac{n}{2}\right]}\binom{n}{k}+\binom{n}{\frac{n}{2}}=\sum_{k=1}^{n}\binom{n}{k}=2^n-2$$

SOLUTION 3.105

Solution by Ravi Prakash-New Delhi-India

Let $a,b,c>0, a+b+c+d=0$

$$-\frac{d}{3}=\frac{1}{3}(a+b+c)\geq(abc)^{\frac{1}{3}}\Rightarrow -d^3\geq 27abc$$

$$\Rightarrow -d^3-3abc\geq 24abc\Rightarrow d^3+3abc\leq -24abc<0$$

Also, $bcd+acd+abd+abc$

$$=bc(-a-b-c)+acd+abd+abc=-(b^2c+bc^2)+d(ac+ab)<0$$

∴ the given inequality becomes

$$-3(bcd+acd+abd+abc)\geq -d^3-3abc$$

Now,

$$-3bcd=3bc(-d)\geq b^3+c^3+(-d)^3\Rightarrow -3bcd\geq b^3+c^3-d^3$$

Similarly $-3acd\geq a^3+c^3-d^3$

And $-3abd\geq a^3+b^3-d^3$

Thus,

$$-3bcd-3acd-3abd-3abc\geq 2(a^3+b^3+c^3-d^3)-d^3-3abc \text{ (1)}$$

But

$$\frac{a^3+b^3+c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^3 \Rightarrow a^3+b^3+c^3 \geq -\frac{d^3}{3}$$

$$\Rightarrow 2(a^3+b^3+c^3-d^3) \geq -\frac{2d^3}{3} - 2d^3 > 0 \ \ (2)$$

From (1), (2) we get

$$-3bcd - 3acd - 3abd - 3abc \geq -d^3 - 3abc$$

$$or\ 3|bcd + acd + abd + abc| \geq |d^3 + 3abc|$$

Equality when $a = b = c$

SOLUTION 3.106

Solution by Richdad Phuc-Hanoi-Vietnam

WLOG, assume $a \leq b \leq c$ or $a \geq b \geq c$. We have

$$LHS - RHS = (b-a)\left[(b+1)e^b - (a+1)e^a\right] + (c-a)\left(e^c - e^a - bb^b + ce^c\right)$$

$$Let\ f(x) = (x+1)e^x, x \geq -2$$

$$f'(x) = (x+2)e^x > 0, \forall x > -2$$

f is increasing function on $[-2, +\infty)$

$$\Rightarrow (b-a)\left[(b+1)e^b - (a+1)e^a\right] \geq 0, \forall a, b \geq -2$$

$$Let\ g(x) = xe^x, x \geq -2$$

$$g'(x) = (x+1)e^x > 0, \text{ for all } x \geq -2$$

g is increasing function on $[-2; +\infty)$

case $a \leq b \leq c$

$$\begin{cases} e^c \geq e^a \\ ce^c \geq be^b \end{cases} \Rightarrow (c-a)\left(e^c - e^a + ce^c - be^b\right) \geq 0$$

we get $LHS - RHS \geq 0$

case $a \geq b \geq c$

$$\begin{cases} e^c \leq e^a \\ ce^c \leq be^b \end{cases} LHS - RHS \geq 0 \Rightarrow Q.E.D. \text{ Equality hold if } a = b = c$$

SOLUTION 3.107

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$\forall t > 1: \ln\left(\frac{1}{t}\right) + 1 \leq \frac{1}{t} \Leftrightarrow \frac{1}{t}(t-1) - \ln t \leq 0 \Leftrightarrow \frac{\frac{1}{t}(t-1) - \ln t}{(t-1)^2} \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{d\left(\frac{\ln t}{t-1}\right)}{dt} \le 0$$

$$1 < x < y \Rightarrow \int_1^x \frac{d\left(\frac{\ln t}{t-1}\right)}{dt}dt > \int_1^y \frac{d\left(\frac{\ln t}{t-1}\right)}{dt}dt \Leftrightarrow \frac{\ln x}{x-1} > \frac{\ln y}{y-1} \Leftrightarrow$$

$$\Leftrightarrow x^{\frac{1}{x-1}} > y^{\frac{1}{y-1}}$$

SOLUTION 3.108

Solution by Daniel Sitaru-Romania

$$2^{\sin x} + 2^{\cos x} \overset{AM-GM}{\geq} 2\sqrt{2^{\sin x+\cos x}} = 2\sqrt{2^{\sin x+\tan\frac{\pi}{4}\cos x}} =$$

$$= 2\sqrt{2^{\frac{2}{\sqrt{2}}\sin\left(x+\frac{\pi}{4}\right)}} \ge 2^{1+\frac{1}{2}\cdot\frac{2}{\sqrt{2}}\sin\left(x+\frac{\pi}{4}\right)} \ge 2^{1-\frac{1}{\sqrt{2}}} = 2^{\frac{\sqrt{2}-1}{\sqrt{2}}}$$

SOLUTION 3.109

Solution by Ravi Prakash-New Delhi-India

$$\textit{Note: } \sin(n\theta) \le n\sin\theta \;\; \forall n \in \mathbb{N}, 0 < \theta < \frac{\pi}{2}$$

$$[For\ n = 1, \sin(i\theta) \le i\sin\theta]$$

$$\textit{Assume } \sin(k\theta) \le k\sin\theta \textit{ for some } k \in \mathbb{N}.$$

$$\sin(k+1)\theta = \sin(k\theta+\theta) = \sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta$$

$$\le \sin(k\theta) + \sin\theta \le k\sin\theta + \sin\theta = (k+1)\sin\theta$$

$$\therefore \sin(k^\circ) \le k\sin 1^\circ$$

$$\Rightarrow \sum_{k=1}^{10}\sin(k^\circ) \le \left(\sum_{k=2}^{10} k\right)\sin 1^\circ = 54\sin(1^\circ)$$

SOLUTION 3.110

Solution by Abdelhak Maoukouf-Casablanca-Morocco

$$x \to \frac{\sin x}{x} \textit{ is a descending function on } \left[0; \frac{\pi}{2}\right]$$

So by Chebyshev:

$$\sum_{cyc} x \sum_{cyc} \frac{\sin x}{x} \ge 3\sum_{cyc}\sin x \Leftrightarrow \sum_{cyc}(y+x)\frac{\sin x}{x} \ge 2\sum_{cyc}\sin x$$

$$\Leftrightarrow \sum_{cyc} y\frac{\sin x}{x} + \sum_{cyc} z\frac{\sin x}{x} \ge \sum_{cyc}\sin x + \sum_{cyc}\sin x$$

$$\textit{if } \sum_{cyc} y\frac{\sin x}{x} \le \sum_{cyc} \sin x \textit{ similarly we'll have}$$

$$\sum_{cyc} z\frac{\sin x}{x} \le \sum_{cyc} \sin x$$

$$\Rightarrow \sum_{cyc} y\,\frac{\sin x}{x} + \sum_{cyc} z\frac{\sin x}{x} \le \sum_{cyc} \sin x + \sum_{cyc} \sin x \because \textit{False supposition}$$

$$\textit{So } \sum_{cyc} y\frac{\sin x}{x} \ge \sum_{cyc} \sin x \Leftrightarrow \sum_{cyc} y^2 z \sin x \ge xyz\ \sum_{cyc} \sin x$$

SOLUTION 3.111

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } 0 \le a, b, c < 1\textit{, consider}$$

$$\Delta = (1-abc)^3(1+a^3)(1+b^3)(1+c^3) -$$

$$-(1+abc)^3(1-a^3)(1-b^3)(1-c^3)$$

$$= \begin{vmatrix} (1-abc)^3 & (1+abc)^3 \\ (1-a^3)(1-b^3)(1-c^3) & (1+a^3)(1+b^3)(1+c^3) \end{vmatrix}$$

$$\textit{Use } C_2 \to C_2 - C_1 \textit{ to obtain}$$

$$\Delta = \begin{vmatrix} (1-abc)^3 & 6abc + 2a^3b^3c^3 \\ (1-a^3)(1-b^3)(1-c^3) & 2(a^3+b^3+c^3) + 2a^3b^3c^3 \end{vmatrix}$$

$$\textit{Use } C_1 \to C_1 + \frac{1}{2}C_2$$

$$\Delta = 2\begin{vmatrix} 1 + 3a^2b^2c^2 & 3abc + a^3b^3c^3 \\ 1 + a^3b^3 + b^3c^3 + c^3a^3 & a^3 + b^3 + c^3 + a^3b^3c^3 \end{vmatrix}$$

$$\textit{Use } R_2 \to R_2 - R_1$$

$$\Delta = 2\begin{vmatrix} 1 + 3a^2b^2c^2 & 3abc + a^3b^3c^3 \\ a^3b^3 + b^3c^3 + c^3a^3 - 3a^2b^2c^2 & a^3 + b^3 + c^3 - 3abc \end{vmatrix}$$

$$\textit{Note that}$$

$$1 + 3a^2b^2c^2 - (3abc + a^3b^3c^3) = (1-abc)^3 > 0$$

$$\Rightarrow 1 + 3a^2b^2c^2 > 3abc + a^3b^3c^3 \quad (1)$$

$$\textit{Also, } a + b + c \ge ab + bc + ca$$

$$\textit{and } (a-b)^2 + (b-c)^2 + (c-a)^2 \ge c^2(a-b)^2 + a^2(b-c)^2 + b^2(c-a)^2$$

$$[\because 0 \le a, b, c < 1]$$

$$\Rightarrow \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

$$\ge \frac{1}{2}(ab+bc+ca)[c^2(a-b)^2 + a^2(b-c)^2 + b^2(c-a)^2]$$

$$\Rightarrow a^3+b^3+c^3-3abc \geq a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2 \text{ (2)}$$

From (1), (2) we get

$$(1+3a^2b^2c^2)(a^3+b^3+c^3-3abc) \geq$$

$$\geq 3abc[a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2] \Rightarrow \Delta \geq 0$$

$$\Rightarrow (1-abc)^3(1+a^3)(1+b^3)(1+c^3) \geq (1+abc)^3(1-a^3)(1-b^3)(1-c^3)$$

Put $a=x, b=y^2, c=z^3$ to obtain

$$\frac{(1+x^3)(1+y^6)(1+z^3)}{(1-x^3)(1-y^6)(1-z^3)} \geq \frac{(1+xy^2z^3)^3}{(1-xy^2z^3)^3}$$

SOLUTION 3.112

Solution by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\sum \frac{\sqrt{x}}{3\sqrt{y}+5\sqrt{z}} = \sum \frac{x}{3\sqrt{xy}+5\sqrt{zx}} \geq \frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{8\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)}$$

$$= \frac{x+y+z}{8\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)} + \frac{1}{4}$$

$$\Rightarrow LHS = \frac{x+y+z}{8\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)} + \frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{8(x+y+z)} + \frac{1}{4} \Rightarrow LHS \overset{AM-GM}{\geq} \frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

SOLUTION 3.113

Solution by Ravi Prakash-New Delhi-India

$$\sum \left(a-\frac{b+c+d}{3}\right)\tan^{-1} a = \frac{1}{3}\sum [(a-b)+(a-c)+(a-d)]\tan^{-1} a$$

$$= \frac{1}{3}\sum (a-b)(\tan^{-1} a - \tan^{-1} b)$$

As $\tan^{-1} x$ is increasing on $(0,\infty)$,

$$a \geq b \Rightarrow \tan^{-1} a \geq \tan^{-1} b$$

$$\therefore (a-b)(\tan^{-1} a - \tan^{-1} b) \geq 0, \forall a>0, b>0$$

Thus,

$$\sum a\tan^{-1} a \geq \sum \frac{b+d+c}{3}\tan^{-1} a \geq \sum (bcd)^{\frac{1}{3}}\tan^{-1} a$$

Next,

$$\sum (bcd)^{\frac{1}{3}}\tan^{-1} a \geq 4\left[\prod (bcd)^{\frac{1}{3}}\tan^{-1} a\right]^{\frac{1}{4}} = 4[abcd\tan^{-1} a\tan^{-1} b\tan^{-1} c\tan^{-1} b]^{\frac{1}{4}}$$

SOLUTION 3.114

Solution by Le Minh Cuong-Ho Chi Minh-Vietnam

Apply AM-GM we get:

$$\sqrt{7(a^2-x^2)} = \sqrt{7(a-x)(a+x)} \le \frac{7(a-x)+a+x}{2} = \frac{8a-6x}{2} \quad (1)$$

$$\sqrt{7(a^2-y^2)} \le \frac{8a-6y}{2} \quad (2)$$

$$\sqrt{7(a^2-z^2)} \le \frac{8a-6z}{2} \quad (3)$$

$$\text{and: } 9\sqrt[3]{xyz} \le 3x+3y+3z \quad (4)$$

From (1), (2), (3), (4), we get:

$$\sqrt{7(a^2-x^2)} + \sqrt{7(a^2-y^2)} + \sqrt{7(a^2-z^2)} + 9\sqrt[3]{xyz} \le 12a$$

$$\text{“=”} \Leftrightarrow x = y = z = \frac{3a}{4}$$

SOLUTION 3.115

Solution by Pham Quoc Sang-Ho Chi Minh-Vietnam

$$(ax+by+cz)(ay+bz+cx)(az+bx+cy) \overset{Holder}{\ge}$$

$$\ge \left[\sqrt[3]{axayaz} + \sqrt[3]{bybzbx} + \sqrt[3]{czcxcy}\right]^3 = xyz(a+b+c)^3 \ge 27xyz$$

$$\text{On the other hand, we have } xyz \ge \frac{27}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}$$

$$\text{so } (ax+by+cz)(ay+bz+cx)(az+bx+cy) \ge \frac{729}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}$$

$$\text{“=” } a = b = c = 1 \text{ and } x = y = z$$

SOLUTION 3.116

Solution by Ravi Prakash-New Delhi-India

$$(a+b)(a+c)(a+d) - \left(a + (bcd)^{\frac{1}{3}}\right)^3$$

$$= a^3 + (b+c+d)a^2 + (bc+cd+db)a + bcd - \left[a^3 + 3a^2(bcd)^{\frac{1}{3}} + 3a(bcd)^{\frac{2}{3}} + bcd\right]$$

$$= a^2\left[b+c+d-3(bcd)^{\frac{1}{3}}\right] + a\left[bc+cd+db-3(bcd)^{\frac{2}{3}}\right] \ge 0$$

$$[\because AM \ge GM]$$

$$\Rightarrow (a+b)(a+c)(a+d) \geq \left(a+(bcd)^{\frac{1}{3}}\right)^3 \quad (1)$$

Similarly,

$$(b+a)(b+c)(b+d) \geq \left(b+(acd)^{\frac{1}{3}}\right)^3 \quad (2)$$

$$(c+a)(c+b)(c+d) \geq \left(c(abd)^{\frac{1}{3}}\right)^3 \quad (3)$$

$$\text{and } (d+a)(d+b)(d+c) \geq \left(d+(abc)^{\frac{1}{3}}\right)^3 \quad (4)$$

Multiplying (1), (2), (3), (4) we get the required inequality.

SOLUTION 3.117

Solution by Ravi Prakash-New Delhi-India

$$\text{For } 0 < x < \frac{\pi}{2}; \ln(1+\tan^2 x)\ln(1+\cot^2 x) \leq$$

$$\leq \left\{\frac{\ln(1+\tan^2 x)+\ln(1+\cot^2 x)}{2}\right\}^2 = \left\{\frac{1}{2}\ln(\sec^2 x\csc^2 x)\right\}^2 = \left(\ln\left(\frac{2}{\sin 2x}\right)\right)^2$$

$$\text{Now, } 0 < x,y,z < \frac{\pi}{2}; \prod \ln(1+\tan^2 x)\prod \ln(1+\cot^2 y) =$$

$$= \prod \ln(1+\tan^2 x)(1+\cot^2 x) \leq \prod \left[\ln\left(\frac{2}{\sin 2x}\right)\right]^2$$

SOLUTION 3.118

Solution by Henry Ricardo-Tapan-New York

The power means inequality gives us:

$$\sqrt[n]{\frac{x^n+y^n}{2}} \geq AM \geq GM \leftrightarrow \left(\frac{x^n+y^n}{2}\right)^2 \geq AM^{2n} \geq GM^{2n} \rightarrow$$

$$\rightarrow 2\left(\frac{x^n+y^n}{2}\right)^2 \geq AM^{2n}+GM^{2n} \rightarrow \left(\frac{x^n+y^n}{\sqrt{2}}\right)^2 \geq AM^{2n}+GM^{2n}$$

SOLUTION 3.119

Solution by Ravi Prakash-New Delhi-India

$$\text{Suppose } 0 < a < b\text{, then } a < \sqrt{ab} < \frac{a+b}{2} < b$$

$$\text{Let } f(x) = \ln x, x \in \left[\sqrt{ab}, \frac{a+b}{2}\right]$$

By the first mean value theorem, there exists $c \in \left(\sqrt{ab}, \frac{a+b}{2}\right)$ *such that*

$$\frac{\ln\left(\frac{a+b}{2}\right)-\ln\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}=\frac{1}{c}\Rightarrow\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}\ln\left(\frac{a+b}{2\sqrt{ab}}\right)=\frac{1}{c}$$

$$\Rightarrow\left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}}=e^{\frac{1}{c}}\quad (1)$$

But $a<\sqrt{ab}<c<\frac{a+b}{2}<b\Rightarrow\frac{1}{b}<\frac{1}{c}<\frac{1}{a}$ *(2)*

From (1), (2), we get

$$e^{\frac{1}{b}}<\left(\frac{a+b}{2\sqrt{ab}}\right)^{\frac{2}{\left(\sqrt{b}-\sqrt{a}\right)^2}}<e^{\frac{1}{a}}$$

SOLUTION 3.120

Solution by Ravi Prakash-New Delhi-India

Let $P(x)=A(x-x_1)(x-x_2)\dots(x-x_n)$

$$P'(x)=A(x-x_2)(x-x_3)\dots(x-x_n)+A(x-x_1)(x-x_3)\dots(x-x_n)$$
$$+\dots+A(x-x_1)(x-x_2)\dots(x-x_{n-1})$$

$$P''(x)=\begin{bmatrix}A(x-x_3)(x-x_4)\dots(x-x_n)\\+A(x-x_2)(x-x_4)\dots(x-x_n)\\+\dots+A(x-x_2)\dots(x-x_{n-1})\end{bmatrix}+\begin{bmatrix}A(x-x_3)(x-x_4)\dots(x-x_n)\\+A(x-x_1)(x-x_4)\dots(x-x_n)\\+\dots+A(x-x_1)\dots(x-x_{n-1})\end{bmatrix}$$
$$+\dots+\begin{bmatrix}A(x-x_2)\dots(x-x_{n-1})\\+A(x-x_1)\dots(x-x_{n-1})\\+\dots+A(x-x_1)\dots(x-x_{n-2})\end{bmatrix}$$

$$\frac{P''(x_1)}{P'(x_1)}=\frac{2}{x_1-x_2}+\frac{2}{x_1-x_3}+\dots+\frac{2}{x_1-x_n}$$

Similarly,

$$\frac{P''(x_r)}{P'(x_r)}=2\sum_{\substack{j=1\\j\neq r}}^{n}\frac{1}{x_r-x_j}\Rightarrow\sum_{r=1}^{n}\frac{P''(x_r)}{P'(x_r)}=0$$

Also, $\frac{P''(x)}{P(x)}-\left(\frac{P'(x)}{P(x)}\right)^2=\frac{d}{dx}\left[\frac{P'(x)}{P(x)}\right]=\frac{d}{dx}\left[\frac{d}{dx}\left(\ln\left(P(x)\right)\right)\right]=\frac{d^2}{dx^2}[\ln|A|+\ln|x-x_1|+\dots+$
$\ln|x-x_n|]$

$$=\frac{d}{dx}\left[\frac{1}{x-x_1}+\frac{1}{x-x_2}+\dots+\frac{1}{x-x_n}\right]=-\left[\frac{1}{(x-x_1)^2}+\frac{1}{(x-x_2)^2}+\dots+\frac{1}{(x-x_n)^2}\right]<0$$

Hence, $\frac{P''(x)}{P(x)}<\left(\frac{P'(x)}{P(x)}\right)^2+\sum_{r=1}^{n}\frac{P''(x_r)}{P'(x_r)}$

SOLUTION 3.121

Solution by Serban George Florin-Romania

$$\sqrt[3]{a\cdot\sin^2 x\cdot 1}+\sqrt[3]{b\cdot\cos^2 x\cdot 1}\le\left(\sqrt[3]{a}^3+\sqrt[3]{b}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{\sin^2 x}^3+\sqrt[3]{\cos^2 x}^3\right)^{\frac{1}{3}}\left(\sqrt[3]{1}^3+\sqrt[3]{1}^3\right)^{\frac{1}{3}},$$

(Holder)

$$\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\le\sqrt[3]{2(a+b)},\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le$$

$$\le\sqrt[3]{2(a+b)2(b+c)2(a+c)},$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le$$

$$\le 2\sqrt[3]{(a+b)(b+c)(a+c)}$$

$$2\sqrt[3]{(a+b)(b+c)(a+c)}\le 2\frac{a+b+b+c+c+a}{3}=2\cdot\frac{6}{3}=4,\left(M_a\ge M_g\right)$$

$$\left(\sqrt[3]{a\cdot\sin^2 x}+\sqrt[3]{b\cdot\cos^2 x}\right)\left(\sqrt[3]{b\cdot\sin^2 x}+\sqrt[3]{c\cdot\cos^2 x}\right)\left(\sqrt[3]{c\cdot\sin^2 x}+\sqrt[3]{a\cdot\cos^2 x}\right)\le 4.$$

SOLUTION 3.122

Solution by Ravi Prakash-New Delhi-India

$$(a+b+c+d-1)^2+(a-b)^2+(c-d)^2\ge 0$$

$$\Rightarrow a^2+b^2+c^2+d^2-2(a+b+c+d)+1$$

$$+2(ab+bc+cd+ad+ac+bd)+a^2+b^2-2ab+c^2+d^2-2cd\ge 0$$

$$\Rightarrow 2(a^2+b^2+c^2+d^2)-2(a+b+c+d)+2(a+b)(c+d)+1\ge 0$$

$$\Rightarrow a+b+c+d\le\frac{1}{2}+(a+b)(c+d)+a^2+b^2+c^2+d^2$$

SOLUTION 3.123

Solution by Daniel Sitaru-Romania

$$f:[a,b]\to\mathbb{R},f(x)=ln(tanx)$$

$$f(b)-f(a)\overset{LAGRANGE}{\triangleq}f'(c)(b-a),c\in(a,b)\to ln(tanb)-ln(tana)=\frac{1}{sinccosc}(b-a)$$

$$ln\left(\frac{tana}{tanb}\right)=\frac{2(b-a)}{sin2c}\ge 2(b-a)\to ln\left(\frac{tana}{tanb}\right)\ge lne^{2(b-a)}\to\frac{tana}{tanb}\ge e^{2(b-a)}$$

SOLUTION 3.124

Solution by Daniel Sitaru-Romania

$$\begin{cases} f(x) = 2^x - xln2 \\ g(x) = 3^x - xln3 \\ h(x) = 4^x - xln4 \end{cases} \rightarrow \begin{cases} f'(x) = (2^x - 1)ln2 \\ g'(x) = (3^x - 1)ln3 \\ h'(x) = (4^x - 1)ln4 \end{cases} \rightarrow \begin{cases} f(x) \geq f(0) = 1 \\ g(x) \geq g(0) = 1 \\ h(x) \geq h(0) = 1 \end{cases} \rightarrow$$

$$f(x) + g(x) + h(x) \geq 3 \rightarrow 2^x - xln2 + 3^x - xln3 + 4^x - xln4 \geq 3$$

$$2^x + 3^x + 4^x \geq xln24 + 3, \forall x \in \mathbb{R}$$

SOLUTION 3.125

Solution by Daniel Sitaru-Romania

$$a \leq x_1, x_2, \dots, x_n \leq b \rightarrow a^n \leq \prod_{k=1}^{n} x_k \leq b^n \rightarrow a^\alpha \leq \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \leq b^\alpha \rightarrow$$

$$\left(a^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)\left(b^\alpha - \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right) \leq 0 \rightarrow (ab)^\alpha - (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 \leq 0 \rightarrow$$

$$\left(\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}\right)^2 + (ab)^\alpha \leq (a^\alpha + b^\alpha)\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} \rightarrow \prod_{k=1}^{n} x_k^{\frac{\alpha}{n}} + \frac{(ab)^\alpha}{\prod_{k=1}^{n} x_k^{\frac{\alpha}{n}}} \leq a^\alpha + b^\alpha$$

SOLUTION 3.126

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } k \in \mathbb{N}, n \in \mathbb{N}, \left(1 + \frac{1}{n}\right)^k \geq 1 + \frac{k}{n}$$

$$\therefore P\left(1 + \frac{1}{n}\right) = \sum_{k=0}^{n} a_k \left(1 + \frac{1}{n}\right)^k \geq \sum_{k=0}^{n} a_k \left(1 + \frac{k}{n}\right) \quad [\because a_k > 0]$$

$$= \sum_{k=0}^{n} a_k + \frac{1}{n}\sum_{k=1}^{n} k a_k = P(1) + \frac{1}{n}P'(1)$$

SOLUTION 3.127

Solution by Soumitra Mandal-Chandar Nagore-India

Applying Weighted AM≥GM;

$$\sqrt[a+b]{a^a b^b} \geq \frac{a+b}{2}, \sqrt[b+c]{b^b c^c} \geq \frac{b+c}{2} \textit{ and } \sqrt[c+a]{c^c a^a} \geq \frac{c+a}{2}$$

$$\Rightarrow \prod_{cyc} a^{2a} \geq \prod_{cyc} \left(\frac{a+b}{2}\right)^{a+b} \Rightarrow \prod_{cyc} a^a \geq \prod_{cyc} \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}}$$

Again applying Weighted $AM \geq GM$;

$$\prod_{cyc}\left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \geq \left(\frac{\sum_{cyc}\left(\frac{a+b}{2}\right)}{\frac{(a+b)/_2}{(a+b)/_2}+\frac{(b+c)/_2}{(b+c)/_2}+\frac{(c+a)/_2}{(c+a)/_2}}\right)^{a+b+c} = \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

$$\geq (abc)^{\frac{a+b+c}{3}}$$

SOLUTION 3.128

Solution by Daniel Sitaru-Romania

$$f(x) = tan^{-1}x - \frac{lnx}{2} \to f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = -\frac{(x-1)^2}{2x(1+x^2)} \leq 0 \to f - decreasing$$

$$x_1 \leq x_n \to f(x_1) \geq f(x_n)$$

$$\sum tan^{-1}\frac{d}{1+x_{k-1}x_k} = \sum tan^{-1}\frac{x_k - x_{k-1}}{1+x_{k-1}x_k} = \sum(tan^{-1}x_k - tan^{-1}x_{k-1}) =$$

$$= tan^{-1}x_n - tan^{-1}x_1 \leq ln\sqrt{\frac{x_n}{x_1}} \leftrightarrow tan^{-1}x_1 - \frac{1}{2}lnx_1 \geq tan^{-1}x_n - \frac{1}{2}lnx_n \leftrightarrow$$

$$\leftrightarrow f(x_1) \geq f(x_n)$$

SOLUTION 3.129

Solution by Ravi Prakash-New Delhi-India

Consider

$$(a^3+b^3)(a^6+b^6)(a^8+b^8) - (a+b)(a^5+b^5)(a^{11}+b^{11})$$

$$= (a^3+b^3)(a^{14}+a^8b^6+a^6b^8+b^{14}) - (a+b)(a^{16}+a^5b^{11}+a^{11}b^5+b^{16})$$

$$= a^{17}+a^{11}b^6+a^9b^8+a^3b^{14}+b^{17}+a^6b^{11}+a^8b^9+a^{14}b^3 -$$

$$-[a^{17}+a^6b^{11}+a^{12}b^5+ab^{16}+b^{17}+a^{11}b^6+b^{12}a^5+a^{16}b]$$

$$= a^9b^8+a^8b^9+a^3b^{14}+a^{14}b^3-a^{12}b^5-a^5b^{12}-ab^{16}-a^{16}b$$

$$= a^9b^5(b^3-a^3)+a^5b^9(a^3-b^3)+ab^{14}(a^2-b^2)+a^{14}b(b^2-a^2)$$

$$= a^5b^5(a^3-b^3)(b^3-a^3)+ab(b^{13}-a^{13})(a^2-b^2) \leq 0$$

$$\Rightarrow (a^3+b^3)(a^6+b^6)(a^8+b^8) \leq (a+b)(a^5+b^5)(a^{11}+b^{11})$$

$$\Rightarrow \frac{(a^3+b^3)(a^6+b^6)(a^8+b^8)}{(a+b)(a^5+b^5)(a^{11}+b^{11})} \le 1 \le 1+\sin\theta$$

$$(0 \le \theta \le \pi)$$

SOLUTION 3.130

Solution by Soumava Chakraborty-Kolkata-India

$$RHS = ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)\left(\frac{y+2}{(z-1)^2-2z+5}\right)\left(\frac{x+2}{(y-1)^2-2y+5}\right)+3=$$

$$= ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)\left(\frac{y+2}{(y-1)^2-2y+5}\right)\left(\frac{z+2}{(z-1)^2-2z+5}\right)+3$$

$$= ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)+ln\left(\frac{y+2}{(y-1)^2-2y+5}\right)+ln\left(\frac{z+2}{(z-1)^2-2z+5}\right)+3$$

$$Let\ f(x) = 1-x+ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)\forall x \ge 0$$

$$f(0) = 1-ln\,3 < 0$$

$$f'(x) = \frac{(1-x)(x^2+2)}{(x+2)(x^2-4x+6)} \because x^2-4x+6 = (x-2)^2+2 > 0,$$

$$\therefore f'(x) > 0\ \forall x \in (0,1)$$

$$f'(1) = 0\ and\ f''(1) = \left(\frac{x^4+8x^3-48x^2+32x-20}{(x+2)^2(x-4x+6)^2}\right)|_{x=1} < 0$$

$$\therefore f(x)\ attains\ a\ maxima\ at\ x=1\ and\ f(1)=0\ and\ f'(x)<0\ \forall x \in (1,\infty)$$

$\therefore f(0) < 0$ ***and then*** $f(x)$ ***increases and at*** $x=1$***, it reaches a maxima with*** $f(1)=0$ ***and then*** $f(x)$ ***decreases***

$$\therefore x \in [0,\infty), f(x) \le 0 \Rightarrow \forall x \in (0,\infty), f(x) \le 0\ with\ equality\ at\ x=1$$

$$\Rightarrow \forall x > 0, 1-x+ln\left(\frac{x+2}{(x-1)^2-2x+5}\right) \le 0\ with\ equality\ at\ x=1 \to (1)$$

$$Similarly, \forall y > 0, 1-y+ln\left(\frac{y+2}{(y-1)^2-2y+5}\right) \overset{(2)}{\le} 0\ with\ equality\ at\ y=1$$

$$and, \forall z > 0, 1-z+ln\left(\frac{z+2}{(z-1)^2-2z+5}\right) \overset{(3)}{\le} 0\ with\ equality\ at\ z=1$$

$$(1)+(2)+(3) \Rightarrow 3-\sum x+ln\left(\frac{x+2}{(x-1)^2-2x+5}\right)+ln\left(\frac{y+2}{(y-1)^2-2y+5}\right)+ln\left(\frac{z+2}{(z-1)^2-2z+5}\right) \le 0$$

$$\forall x,y,z > 0$$

$$\Rightarrow \forall x,y,z > 0, x+y+z \ge ln\left(\frac{z+2}{(x-1)^2-2x+5}\right)+ln\left(\frac{y+2}{(z-1)^2-2z+5}\right)+ln\left(\frac{x+2}{(y-1)^2-2y+5}\right)\ (proved)$$

SOLUTION 3.131

Solution by Soumitra Mandal-Chandar Nagore-India

$$Let\ f(x) = 2xe^{x^2}\ for\ all\ x \ge 0$$

$$f'(x) = 2e^{x^2}+4x^2e^{x^2}, f''(x) = 4xe^{x^2}+8xe^{x^2}+8x^3e^{x^2} \ge 0\ for\ all\ x \ge 0$$

Hence f is convex $\therefore$ applying Hermite – Hadamard Inequality.

$$\frac{f(a)+f(b)}{2} \geq \frac{1}{b-a}\int_a^b f(x)\,dx \geq f\left(\frac{a+b}{2}\right) \Rightarrow \frac{1}{b-a}\int_a^b 2xe^{x^2}\,dx \geq 2\left(\frac{a+b}{2}\right)e^{\left(\frac{a+b}{2}\right)^2}$$

$$\Rightarrow \frac{e^{b^2}-e^{a^2}}{b-a} \geq (a+b)\left(1+\left(\frac{a+b}{2}\right)^2\right) \because e^x \geq 1+x$$

$$\therefore \frac{e^{b^2}-e^{a^2}}{b-a} \geq (a+b)(1+ab) \text{ (proved)}$$

SOLUTION 3.132

Solution by Ravi Prakash-New Delhi-India

Let $c > 1$. By the Cauchy's mean value theorem, there exists $\alpha \in (1, c)$ such that

$$\frac{c^9-1}{c^8-1} = \frac{9\alpha^8}{8\alpha^7} = \frac{9}{8}\alpha > \frac{9}{8} \text{ (1)}$$

Case 1 $a = b = 1$, then

$$\frac{\sum_{k=0}^{8} b^{8-k}a^k}{\sum_{k=0}^{7} b^k a^{7-k}} = \frac{9}{8}.$$

Case 2 $a \neq b$. Let $a > b \geq 1$. Put $\frac{a}{b} = c > 1$. Now,

$$\sum_{k=0}^{8} b^{8-k}a^k = \frac{b^8(c^9-1)}{c-1} \text{ and } \sum_{k=0}^{7} a^{7-k}\,b^k = \frac{b^7(c^8-1)}{c-1}$$

Thus,

$$\frac{\sum_{k=0}^{8} b^{8-k}\,a^k}{\sum_{k=0}^{7} b^k a^{7-k}} = \frac{b(c^9-1)}{c^8-1} > \frac{9}{8}b \geq \frac{9}{8}$$

$$[\because b \geq 1]$$

SOLUTION 3.133

Solution by Soumava Chakraborty-Kolkata-India

From the graphs of $y = e^x$ and $y = x + 1$, it is clear that: $\forall x, e^x \geq x + 1 \rightarrow$ (1)

Choosing $x = a^b - 1$ in (1), we get: $e^{a^b} - 1 \geq a^b \Rightarrow \frac{e^{a^b}}{a^b} \overset{(a)}{\geq} e$

Similarly, $\frac{e^{b^c}}{b^c} \overset{(b)}{\geq} e, \frac{e^{c^a}}{c^a} \overset{(c)}{\geq} e, \frac{e^{a^c}}{a^c} \overset{(d)}{\geq} e, \frac{e^{c^b}}{c^b} \overset{(e)}{\geq} e, \frac{e^{b^a}}{b^a} \overset{(f)}{\geq} e$

(a) · (b) · (c) · (d) · (e) · (f) $\Rightarrow \frac{e^{a^b+b^c+c^a+a^c+c^b+b^a}}{a^{b+c}b^{a+c}c^{a+b}} \geq e^6$

SOLUTION 3.134

Solution by Daniel Sitaru-Romania

$$f(x)=e^x-2\sqrt{x}, f'(x)=e^x-\frac{1}{\sqrt{x}}, f''(x)=e^x+\frac{1}{2x\sqrt{x}}>0$$

$$f(a)+f(b)+f(c)\overset{Jensen}{\geq}3f\left(\frac{a+b+c}{3}\right)\leftrightarrow$$

$$\leftrightarrow e^a-2\sqrt{a}+e^b-2\sqrt{b}+e^c-2\sqrt{c}\geq 3e^{\frac{a+b+c}{3}}-6\sqrt{\frac{a+b+c}{3}}>$$

$$>3\left(\frac{a+b+c}{3}+1\right)-6\sqrt{\frac{a+b+c}{3}}=3\left(\sqrt{\frac{a+b+c}{3}}-1\right)^2\geq 0\leftrightarrow$$

$$e^a-2\sqrt{a}+e^b-2\sqrt{b}+e^c-2\sqrt{c}>0\rightarrow\frac{e^a+e^b+e^c}{\sqrt{a}+\sqrt{b}+\sqrt{c}}>2$$

SOLUTION 3.135

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2}{(\sqrt{2})^\theta}\overset{(i)}{\leq}\sin^\theta\alpha+\sin^\theta\beta\overset{(ii)}{\leq}1$$

$$A=\frac{\pi}{2}\Rightarrow B+C=\frac{\pi}{2}\Rightarrow\alpha+\beta=\frac{\pi}{2}\Rightarrow\sin\beta=\cos\alpha\quad(1)$$

$$\because\alpha+\beta=\frac{\pi}{2},\therefore 0<\alpha,\beta<\frac{\pi}{2}\Rightarrow 0<\sin\alpha,\sin\beta<1\because\theta\geq 2$$

$$\therefore\sin^\theta\alpha\overset{(a)}{\leq}\sin^2\alpha\ \ \&\ \ \sin^\theta\beta\overset{(b)}{\leq}\sin^2\beta=\cos^2\alpha$$

$$\text{(a)+(b)}\Rightarrow\sin^\alpha\alpha+\sin^\theta\beta\leq\sin^2\alpha+\cos^2\alpha=1\Rightarrow\text{(ii) is true (*)}$$

$$\text{Let }\alpha=\frac{\pi}{4}+x\ \ \&\ \ \beta=\frac{\pi}{4}-x;\ -\frac{\pi}{4}<x<\frac{\pi}{4}$$

$$\therefore\sin\alpha\overset{(2)}{=}\sin\left(\frac{\pi}{4}+x\right)=\frac{\cos x+\sin x}{\sqrt{2}}\ \ \&\ \sin\beta\overset{(3)}{=}\cos\left(\frac{\pi}{4}+x\right)=\frac{\cos x-\sin x}{\sqrt{2}}$$

$$\text{(2), (3)}\Rightarrow\sin^\theta\alpha+\sin^\theta\beta=\frac{1}{(\sqrt{2})^\theta}\left[\{(\cos x+\sin x)^2\}^{\frac{\theta}{2}}+\{(\cos x-\sin x)^2\}^{\frac{\theta}{2}}\right]$$

$$\overset{(4)}{=}\frac{1}{(\sqrt{2})^\theta}\left[(1+\sin 2x)^{\frac{\theta}{2}}+\left(1+(-\sin 2x)\right)^{\frac{\theta}{2}}\right]$$

From Bernoulli's inequality, we have,

$$\forall r\geq 1\ \&\ \forall t>-1,(1+t)^r\geq 1+rt\quad(5)$$

$$\because-\frac{\pi}{2}<2x<\frac{\pi}{2},\therefore -1<\sin 2x<1$$

$$\textit{So, } \because \sin 2x > -1 \ \& \frac{\theta}{2} \geq 1,$$

$$\therefore (1+\sin 2x)^{\frac{\theta}{2}} \geq 1 + \frac{\theta}{2} \cdot \sin 2x \quad (5)$$

$$\textit{Again, } \because -\sin 2x > -1 \ \& \ \frac{\theta}{2} \geq 1,$$

$$\therefore \left(1+(-\sin 2x)\right)^{\frac{\theta}{2}} \geq 1 + \frac{\theta}{2}(-\sin 2x) \quad (6)$$

$$(5)+(6) \text{ along with } (4) \Rightarrow \sin^\theta \alpha + \sin^\theta \beta \geq \frac{2+\frac{\theta}{2}\sin 2x - \frac{\theta}{2}\sin 2x}{(\sqrt{2})^\theta} = \frac{2}{(\sqrt{2})^\theta} \Rightarrow (i) \text{ is true } (*)$$

SOLUTION 3.136

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } f(x) = x^{\frac{4}{3}};\ g(x) = x^{\frac{3}{2}},\ a \leq x \leq b.$$

By the Cauchy's mean value theorem $\exists c \in (a,b)$*, s.t*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \Rightarrow \frac{b^{\frac{4}{3}} - a^{\frac{4}{3}}}{b^{\frac{3}{2}} - a^{\frac{3}{2}}} = \frac{4}{3} \cdot \frac{2}{3} \cdot \frac{c^{\frac{1}{3}}}{c^{\frac{1}{2}}} = \frac{8}{9}\left(\frac{1}{c^{\frac{1}{6}}}\right) > \frac{8}{9} \quad \left[\because c^{\frac{1}{6}} < b^{\frac{1}{6}} < 1\right]$$

SOLUTION 3.137

Solution by Marian Ursărescu-Romania

$$\textit{Inequality} \Leftrightarrow a\left(\log_{y^b z^c} x + \log_{z^b x^c} y + \log_{x^b y^c} z\right) \geq \frac{3a}{b+c} \Leftrightarrow$$

$$\frac{1}{\log_x y^b z^c} + \frac{1}{\log_y z^b x^c} + \frac{1}{\log_z x^b y^c} \geq \frac{3}{b+c} \Leftrightarrow \frac{1}{b\log_x y + c\log_x z} + \frac{1}{b\log_y z + c\log_y x} +$$

$$+\frac{1}{b\log_z x + c\log_z y} \geq \frac{3}{b+c} \Leftrightarrow$$

$$\frac{\ln x}{b\ln y + c\ln z} + \frac{\ln y}{b\ln z + c\ln x} + \frac{\ln z}{b\ln x + c\ln y} \geq \frac{3}{b+c} \quad (1)$$

$$\textit{Let } \ln x = m, \ln y = n, \ln z = p, m,n,p > 0$$

$$(1) \Leftrightarrow \frac{m}{bn+cp} + \frac{n}{bp+cm} + \frac{p}{bm+cn} \geq \frac{3}{b+c} \quad (2)$$

Inequality (2) is a generalization of Nesbitt inequality (to prove let $bn+cp = x_1$*,*

$bp+cm = x_2$ *and* $bm+cn = x_3$ *and use* $x + \frac{1}{\alpha} \geq 2, \forall \alpha > 0$

SOLUTION 3.138

Solution by Abdallah Almalih-Damascus-Syria

Put $f(x) = (1+\tan^2 x)e^{\tan x + e} + \pi x^{\pi-1}$ *where* $x \in [e,\pi]$*. Clearly, we have* $f(x) > 0$*.*

$$So, \int_e^{\pi} f(x)\,dx > 0. But$$

$$\int_e^{\pi} (1+\tan^2 x)\, e^{\tan x + e} + \pi x^{\pi-1} dx = [e^{\tan x + e} + x^{\pi}]_e^{\pi} = e^{\tan \pi + e} + \pi^{\pi} - (e^{\tan e + e} + e^{\pi})$$

$$= e^e[e^{\tan \pi} - e^{\tan e}] - (e^{\pi} - \pi^{\pi}) = e^e(1 - e^{\tan e}) - (e^{\pi} - \pi^{\pi}) > 0$$

$$\text{Hence } e^e(1 - e^{\tan e}) > e^{\pi} - \pi^{\pi}$$

SOLUTION 3.139

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x) = \cosh x \; \forall x \geq 0$$

$$f'(x) = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}$$

$$= \frac{(e^x+1)(e^x-1)}{2e^x} \geq 0 \; (\because e^x \geq 1 \; as \; x \geq 0)$$

$$\therefore f(x) \text{ is an increasing } f^n\text{, WLOG, we may assume } a \geq b \geq c$$

$$\text{Then, as } \cosh x \text{ is an increasing } f^n, \forall x \geq 0, \therefore \cosh a \geq \cosh b \geq \cosh c$$

$$\Rightarrow \sqrt[3]{\cosh a} \geq \sqrt[3]{\cosh b} \geq \sqrt[3]{\cosh c}$$

$$\therefore \sum a\sqrt[3]{\cosh a} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a\right)\left(\sum \sqrt[3]{\cosh a}\right)$$

$$\Rightarrow \left(\sum a\right)\left(\sum \sqrt[3]{\cosh a}\right) \overset{(1)}{\leq} 3\sum\left(a\sqrt[3]{\cosh a}\right)$$

$$\text{(1)} \Rightarrow \text{it suffices to show: } \sum \sinh a \geq \sum\left(a\sqrt[3]{\cosh a}\right) \quad (i)$$

$$\text{For 2 positive } m \,\&\, n\text{, let}$$

$$A = A(m,n) = \frac{m+n}{2}, G = G(m,n) = \sqrt{mn} \;\; \& \; L = L(m,n) = \frac{m-n}{\ln m - \ln n}$$

$$\text{We have, } \sqrt[3]{G^2 A} \overset{(a)}{<} L \text{ (E.B. Leach \& M.C. Scholander)}$$

$$\text{Now, } A(e^x, e^{-x}) = \cosh x, G(e^x, e^{-x}) = 1, L(e^x, e^{-x}) = \frac{e^x - e^{-x}}{2x} = \frac{\sinh x}{x}$$

$$\therefore \text{applying (a), we get, } \sqrt[3]{\cosh x} < \frac{\sinh x}{x}, \forall x > 0$$

$$\therefore a, b, c > 0, \sinh a > a\sqrt[3]{\cosh a} \text{ etc}$$

$$\Rightarrow \sum \sin h > \sum a\sqrt[3]{\cosh a} \quad (2)$$

$$\text{For } a = 0, \sinh a = 0 \,\&\, a\sqrt[3]{\cosh a} = 0 \Rightarrow \sinh a = a\sqrt[3]{\cosh a}$$

$$\text{Similarly, for } b \,\&\, c = 0, \sinh b = b\sqrt[3]{\cosh b} \;\&\, \sinh c = c\sqrt[3]{\cosh c}$$

$$\therefore \text{ when } a = b = c = 0,$$

$$\sum \sinh a = \sum a\sqrt[3]{\cosh a} \ (= 0) \quad (3)$$

Combining (2) & (3), (i) is true (Proved)

SOLUTION 3.140

Solution by Lahiru Samarakoon-Sri Lanka

$$\Omega(a,b) = \frac{b}{a+b-1} + \frac{b(b-1)}{(a+b-1)(a+b-2)} + \cdots + \frac{b(b-1)\dots 2\cdot 1}{(a+b-1)(a+b-2)\dots a}$$

Then,

$$b\cdot\Omega(a,b) + c\cdot\Omega(b,c) + a\cdot\Omega(c,a) \ge a+b+c$$

By adding last three parts,

$$\Omega(a,b) = \frac{b}{a+b-1} + \cdots + \underbrace{\frac{b(b-1)\dots 2}{(a+b-1)\dots(a+1)} + \frac{b(b-1)\dots 2\cdot 1}{(a+b-1)\dots a}}_{\Downarrow}$$

$$\frac{b}{(a+b-1)} + \cdots + \frac{b(b-1)\dots 2(a+1)}{(a+b-1)(a+b-2)\dots(a+1)a}$$

$$\vdots$$

$$\Omega(a,b) = \frac{b}{(a+b-1)} + \frac{b(b-1)}{(a+b-1)a} = \frac{b(a+b-1)}{(a+b-1)a} = \frac{b}{a}$$

So, similarly,

$$\Omega(b,c) = \frac{c}{b} \text{ and } \Omega(c,a) = \frac{a}{c}$$

$$\therefore LHS = b\Omega(a,b) + c\Omega(b,c) + a\Omega(c,a)$$

$$= \frac{b^2}{a} + \frac{c^2}{b} + \frac{a^2}{c} \ge \frac{(b+c+a)^2}{(a+b+c)} = (b+c+a)$$

SOLUTION 3.141

Solution by Serban George Florin-Romania

$$3 + \frac{1}{(a+b)^4} = 1+1+1+\frac{1}{(a+b)^4} \overset{(Ma\ge Mg)}{\ge} 4\sqrt[4]{1\cdot 1\cdot 1\cdot\frac{1}{(a+b)^4}} = \frac{4}{a+b}$$

$$3 + (\log_a c)^4 = 1+1+1+(\log_a c)^4 \overset{(Ma\ge Mg)}{\ge} 4\sqrt[4]{1\cdot 1\cdot 1(\log_a c)^4} = 4\log_a c$$

$$\sum (3 + (\log_a c)^4)\left(1 + \frac{1}{(a+b)^4}\right) \ge \sum 16\cdot\frac{\log_a c}{a+b} = 16\sum\frac{\log_a c}{a+b}$$

$$16\sum\frac{\log_a c}{a+b}\overset{(Ma\geq Mg)}{\geq}16\cdot 3\sqrt[3]{\frac{\prod\log_a c}{\prod(a+b)}}=$$

$$=\frac{48}{\sqrt[3]{(a+b)(b+c)(a+c)}}\overset{(Ma\geq Mg)}{\geq}\frac{48}{\frac{a+b+b+c+a+c}{3}}=\frac{48}{\frac{2(a+b+c)}{3}}=\frac{48}{\frac{3}{3}}=48$$

$$a,b,c\in(0,1)\Rightarrow\log_a c,\log_b c,\log_a b>0$$

SOLUTION 3.142

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } a=\ln x,b=\ln y,c=\ln z,d=\ln t$$

$$(a,b,c,d\geq 0)$$

Using this substitution, given inequality

$$\Leftrightarrow(a+b)(a^2+b^2-ab-cd)\geq(c+d)(ab+cd-c^2-d^2)$$

$$\Leftrightarrow a^3+b^3+c^3+d^3\overset{(1)}{\geq}abc+bcd+cda+dab$$

$$\textit{Now, } a^3+b^3+c^3=3abc+(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\overset{(2)}{\geq}3abc,$$

$$\forall a,b,c\geq 0$$

$$\textit{Similarly, } b^3+c^3+d^3\overset{(3)}{\geq}3bcd,\forall b,c,d\geq 0, c^3+d^3+a^3\overset{(4)}{\geq}3cda,\forall c,d,a\geq 0\ \&$$

$$d^3+a^3+b^3\overset{(5)}{\geq}3dab,\forall d,a,b\geq 0$$

(2)+(3)+(4)+(5) $\Rightarrow a^3+b^3+c^3+d^3\geq abc+bcd+cda+dab\Rightarrow$ ***(1) is true (Proved)***

SOLUTION 3.143

Solution by Ravi Prakash-New Delhi-India, Generalization by Sagar Kumar-India

$$\textit{Let } 1\leq x<y,n,m\in\mathbb{N},n<m.$$

By the Cauchy's mean value theorem:

$$\frac{y^n-x^n}{y^m-x^m}=\frac{n\alpha^{n-1}}{m\alpha^{m-1}}\ \textit{for some } \alpha\in(x,y)=\frac{n}{m}\alpha^{n-m}=\frac{n}{m}\cdot\frac{1}{\alpha^{m-n}}$$

$$<\frac{n}{m}\ [\because\alpha>x\geq 1\Rightarrow\alpha>1]$$

$$\therefore\frac{y^5-x^5}{y^6-x^6}\cdot\frac{y^7-x^7}{y^8-x^8}\cdot\frac{y^9-x^9}{y^{10}-x^{10}}<\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\left(\frac{9}{10}\right)=\frac{21}{32}$$

Generalization:

$$\Psi = \prod_{r=0}^{n}\left(\frac{y^{2r+1} - x^{2r+1}}{y^{2r+2} - x^{2r+2}}\right) < \frac{1}{4^{n+1}}\binom{n+1}{2n+2}, 1 \le x < y$$

$$\lim_{n\to\infty}(n+1)\Psi \le \frac{1}{\sqrt{\pi}}$$

SOLUTION 3.144

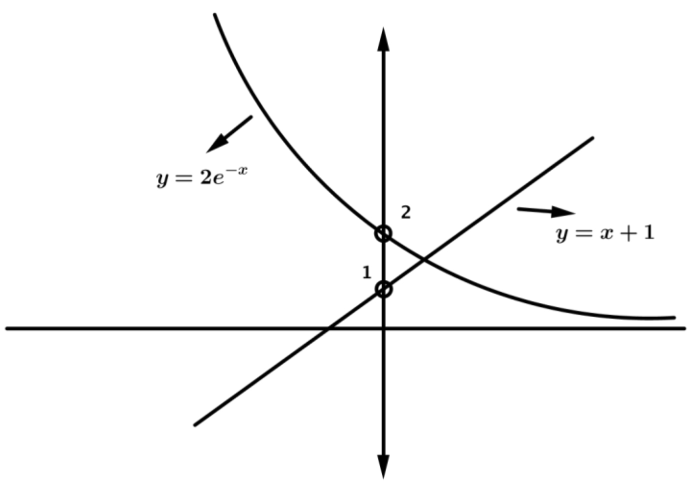
Solution by Soumava Chakraborty-Kolkata-India

We shall show that: $\frac{9a}{1+bcd} + \frac{9}{4}e^{abcd} \overset{(1)}{\le} 2 + \frac{9}{4}e^{16}$

LHS of (1) $\overset{(2)}{\le} \frac{18}{1+bcd} + \frac{9}{4}e^{2bcd}$ $(\because a \le 2 \ \& a \ge 0)$

Let $f(x) = \frac{18}{1+x} + \frac{9}{4}e^{2x}; f'(x) = \frac{9e^{2x}(x+1)^2 - 36}{2(x+1)^2}$ & $f''(x) = 9e^{2x} + \frac{36}{(x+1)^3}$

Now, $f'(x) = 0 \Rightarrow e^{x}(x+1) = 2 \Rightarrow x + 1 = 2e^{-x}$ *(3)*

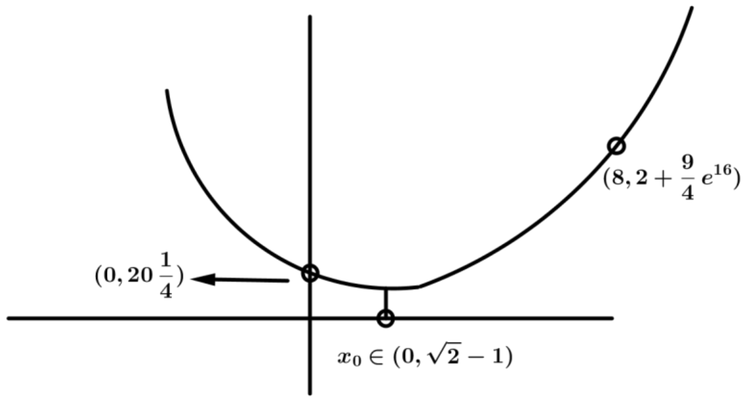


Also, $e^{x} = \frac{2}{x+1} \ge x + 1 \Rightarrow x \le \sqrt{2} - 1 \therefore$ *(3) has only root & it* $\in (0, \sqrt{2} - 1) \Rightarrow$

$\Rightarrow f'(x) = 0$ *at one & only one value* $x_0 \in (0, \sqrt{2} - 1)$

& $\because f''(x) > 0, \forall x \ge 0, \therefore f''(x_0) > 0 \Rightarrow f(x)$ *attains a minima at* $x_0 \in (0, \sqrt{2} - 1)$

Also, $f(0) = 18 + \frac{9}{4} = 20\frac{1}{4}$ & $f(8) = \frac{18}{1+8} + \frac{9}{4}e^{16} = 2 + \frac{9}{4}e^{16} > f(0)$ & $\because f(x)$ *never attains a maxima in* $[0, 8], \therefore$ *the graph of* $f(x)$ *in* $[0, 8]$ *should be like below:*



Hence, it is clear that in $[0,8], f(x)_{\max} = f(8) = 2+\frac{9}{4}e^{16} \Rightarrow \frac{18}{1+x}+\frac{9}{4}e^{2x} \le 2+\frac{9}{4}e^{16}$

$$\Rightarrow \frac{18}{1+bcd}+\frac{9}{4}e^{2bcd} \le 2+\frac{9}{4}e^{16} \text{ (putting } x = bcd \text{ \& } bcd \le 8)$$

$$\left.\begin{aligned} &\Rightarrow \frac{9a}{1+bcd}+\frac{9}{4}e^{abcd} \overset{by\,(2)}{\le} 2+\frac{9}{4}e^{16} \\ &Similarly, \frac{9b}{1+cda}+\frac{9}{4}e^{abcd} \le 2+\frac{9}{4}e^{16} \\ &\frac{9c}{1+dab}+\frac{9}{4}e^{abcd} \le 2+\frac{9}{4}e^{16} \\ &\frac{9d}{1+abc}+\frac{9}{4}e^{abcd} \le 2+\frac{9}{4}e^{16} \end{aligned}\right\}$$

Adding the last 4, we obtain the desired inequality (proved)

SOLUTION 3.145

Solution by Soumitra Mandal-Chandar Nagore-India

We know $x^{2k} \ge x^{2k-1}$ *for all* $x \ge 1$

$$\int_a^b x^{2k} \ge \int_a^b x^{2k-1}\,dx \Rightarrow \frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}} \ge \frac{2k+1}{2k}$$

$$\Rightarrow \prod_{k=1}^{n}\left(\frac{b^{2k+1}-a^{2k+1}}{b^{2k}-a^{2k}}\right) \ge \prod_{k=1}^{n}\left(\frac{2k+1}{2k}\right) = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}{2^n \cdot n!}$$

$$= \frac{1\cdot 2\cdot 3\cdot \ldots \cdot 2n\cdot (2n+1)}{2^n\cdot n!(2\cdot 4\cdot 6\cdot \ldots \cdot 2n)} = \frac{(2n+1)!}{4^n(n!)^2} \text{ (proved)}$$

SOLUTION 3.146

Solution by Daniel Sitaru-Romania

WLOG: $A \le B \le C \to tanA \le tanB \le tanC \to tan^{\alpha}A \le tan^{\alpha}B \le tan^{\alpha}C$

$$\sum Atan^{\alpha}A \overset{CEBYSHEV}{\geq} \frac{1}{3}\sum A\sum tan^{\alpha}A = \frac{\pi}{3}\sum tan^{\alpha}A \leftrightarrow$$

$$\leftrightarrow \frac{1}{\pi}\sum Atan^{\alpha}A \geq \frac{1}{3}\sum tan^{\alpha}A \overset{JENSEN}{\geq} \frac{1}{3}\cdot 3tan^{\alpha}\left(\frac{A+B+C}{3}\right) = tan^{\alpha}\frac{\pi}{3} = 3^{\frac{\alpha}{2}}$$

SOLUTION 3.147

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

If $a, b, c \in (0; 1], x, y > 0$ *then* $\frac{3}{2}\log(x^2 + y^2) > (a + b + c)\log x + (3 - a - b - c)\log y$

(1)

Case 1. $\log\left(\frac{x}{y}\right) > 0$

We have (1) $\Rightarrow (a + b + c - 3)\cdot(\log x - \log y) + 3\log x < \frac{3}{2}\log(x^2 + y^2) \Rightarrow$

$$\Rightarrow (a + b + c - 3)\cdot\log\left(\frac{x}{y}\right) + 3\log x < \frac{3}{2}\log(x^2 + y^2)$$

We have $\log\left(\frac{x}{y}\right) > 0$ *and* $a + b + c - 3 \leq 0$ *so* $(a + b + c - 3)\cdot\log\left(\frac{x}{y}\right) \leq 0$

$$\Rightarrow (a + b + c - 3)\cdot\log\left(\frac{x}{y}\right) + 3\log x \leq 3\log x$$

On the other hand, we have $\frac{3}{2}\log(x^2 + y^2) > \frac{3}{2}\log(x^2) = 3\log x$*. So,*

$(a + b + c - 3)\cdot\log\left(\frac{x}{y}\right) + 3\log x < \frac{3}{2}\log(x^2 + y^2) \Rightarrow$ ***(1) true***

Case 2. $\log\left(\frac{x}{y}\right) < 0$

We have (1) $\Rightarrow (a + b + c)\cdot(\log x - \log y) + 3\log y < \frac{3}{2}\log(x^2 + y^2) \Rightarrow$

$$\Rightarrow (a + b + c)\cdot\log\left(\frac{x}{y}\right) + 3\log y < \frac{3}{2}\log(x^2 + y^2)$$

We have $\log\left(\frac{x}{y}\right) < 0$ *and* $a + b + c > 0$ *so,* $(a + b + c)\cdot\log\left(\frac{x}{y}\right) < 0 \Rightarrow$

$$\Rightarrow (a + b + c)\cdot\log\left(\frac{x}{y}\right) + 3\log y < 3\log y$$

On the other hand, we have $\frac{3}{2}\log(x^2 + y^2) > \frac{3}{2}\log(y^2) = 3\log y$

So $(a + b + c)\cdot\log\left(\frac{x}{y}\right) + 3\log y < \frac{3}{2}\log(x^2 + y^2) \Rightarrow$ ***(1) true. Therefore, we have QED.***

SOLUTION 3.148

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{\sum_{k=0}^{m} a^{m-k}b^{k}}{\sum_{l=0}^{n} a^{n-l}b^{l}} \geq \frac{m+1}{n+1} \Leftrightarrow \frac{\left(\sum_{k=0}^{m} a^{m-k}b^{k}\right)}{(b-a)\left(\sum_{l=0}^{n} a^{n-l}b^{l}\right)} \geq \frac{m+1}{n+1}$$

$$\Leftrightarrow \frac{\frac{b^{m+1}-a^{m+1}}{m+1}}{\frac{b^{n+1}-a^{n+1}}{n+1}} \geq 1 \Leftrightarrow \int_a^b x^m dx \geq \int_a^b x^n\, dx \Leftrightarrow x^{m-n} \geq 1 \Leftrightarrow m \geq n, \text{ which is true}$$

SOLUTION 3.149

Solution by Michael Stergioiu-Greece

For every triad of positive real numbers x, y, z we have:

$x^2y + x^2y + z^2x \geq 3x\sqrt[3]{(xyz)^2}$. ***Cyclic application and addition gives:***

$$x^2y + y^2z + z^2x \geq (x+y+z)\cdot\sqrt[3]{(xyz)^2} \text{ or}$$

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq (x+y+z)\cdot(xyz)^{-\frac{1}{2}} \text{ or } \frac{x+y+z}{(xyz)^{\frac{1}{3}}\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)} \leq 1.$$

The reverse fraction is obviously ≥ 1. For the triads a, b, c and f, d, e we have

$$\frac{a+b+c}{\sqrt[3]{abc}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} \leq 1 \leq \frac{\sqrt[3]{def}\cdot\left(\frac{d}{e}+\frac{e}{f}+\frac{f}{d}\right)}{d+e+f}. \text{ We are done!}$$

SOLUTION 3.150

Solution by Daniel Sitaru-Romania

$$\begin{cases} e^{b+c} > e^a > a+1 \\ e^{c+a} > e^b > b+1 \\ e^{a+b} > e^c > c+1 \end{cases} \to \prod e^{b+c} > \prod(a+1) \to e^{2a+2b+2c} > \prod(a+1) \to$$

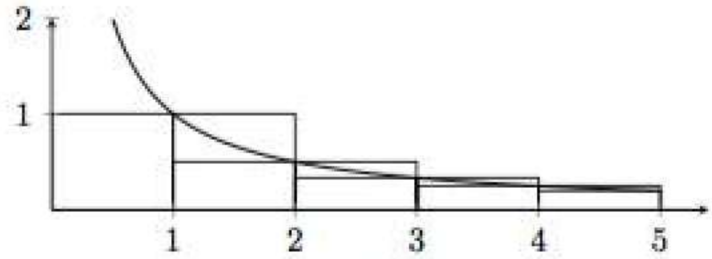
$$\to e^{a+b+c} > \sqrt{(a+1)(b+1)(c+1)} \to \frac{\left((a+1)(b+1)(c+1)\right)^{\frac{1}{2}}}{e^{a+b+c}} < 1$$

SOLUTION 3.151

Solution by Emre Tuvay-Turkey

From Riemann sum of the area of curve $y = \frac{1}{x}$ we have the followings for lower bound.

$$\sum_{k=1}^{n} \frac{1}{k} > \int_1^{n+1} \frac{1}{x} dx > \int_1^{n} \frac{1}{x} dx = \ln n > 0$$



As for upper bound again from Riemann sum keeping $y = \frac{1}{x}$ function's values above the rectangles and adding the area of 1st rectangle we have

$$\mathbf{1} + \int_{1}^{n} \frac{\mathbf{1}}{\boldsymbol{x}} \boldsymbol{dx} > \sum_{k=1}^{n} \frac{\mathbf{1}}{\boldsymbol{k}}$$

hence, $\mathbf{0} < \gamma < 1$.

For convergence, showing the sequence $U_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$ monotonic decreasing should suffice.

$$\boldsymbol{U_{n+1} - U_n} = \frac{\mathbf{1}}{\boldsymbol{n+1}} - \ln(\boldsymbol{n}+\mathbf{1}) + \ln \boldsymbol{n}$$

again, by checking the area under $y = \frac{1}{x}$ curve for $x = n$ and $x = n + 1$ we see that

$$\int_{n}^{n+1} \frac{\mathbf{1}}{\boldsymbol{x}} \boldsymbol{dx} > \frac{\mathbf{1}}{\boldsymbol{n+1}} \Rightarrow \ln(\boldsymbol{n}+\mathbf{1}) - \ln \boldsymbol{n} > \frac{\mathbf{1}}{\boldsymbol{n+1}}$$

hence,

$U_{n+1} - U_n < 0 \Rightarrow U_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$ is monotonic decreasing. Therefore, $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$ converges to γ where $\mathbf{0} < \gamma < 1$. So, $\mathbf{0} < \gamma < 1 < e < \pi$.

Now, for ordering of $\Omega_1 = \gamma^{\sqrt{\pi e}}, \Omega_2 = \pi^{\sqrt{e\gamma}}, \Omega_3 = e^{\sqrt{\gamma\pi}}$

Considering a generic case, $b^{\sqrt{a}}$ and $a^{\sqrt{b}}$ (where $a, b \in \Re_{\geq 0}$ and $b > a$) which can be written as $\left(b^{\frac{1}{\sqrt{b}}}\right)^{\sqrt{a}\sqrt{b}}$ and $\left(a^{\frac{1}{\sqrt{a}}}\right)^{\sqrt{b}\sqrt{a}}$ respectively

. From checking function, $f(x) = \left(x^{\frac{1}{\sqrt{x}}}\right)$,

$f'(x) = x^{\frac{1}{\sqrt{x}}-1}\left(\frac{1}{\sqrt{x}} - \frac{\ln x}{2\sqrt{x}}\right)$. Critical point $f'(x) = \frac{x^{\frac{1}{\sqrt{z}}}}{x^{\frac{3}{2}}}\left(1 - \frac{\ln x}{2}\right) = 0 \Rightarrow x = e^2$.

$$f'(x)=\begin{cases}>0, when\ x<e^2;\\ =0, when\ x=e^2;\\ <0, when\ x>e^2;\end{cases}\ \textit{so, } f(x)=\begin{cases}increasing, when\ x<e^2;\\ maxvalue, when\ x=e^2;\\ decreasing, when\ x>e^2;\end{cases}\ \textit{since,}$$

$$\gamma<e<\pi<e^2\Rightarrow f(\gamma)<f(e)<f(\pi)\ \textit{ hence } \Omega_1<\Omega_3<\Omega_2$$

SOLUTION 3.152

Solution by Le Van-Ho Chi Minh-Vietnam

With $x>1$ and $n\geq 2$, building the function:

$$f(x)=\frac{\sqrt[n]{x+1}}{x+1}-\frac{\sqrt[n]{x}}{x}=(x+1)^{\frac{1}{n}-1}-x^{\frac{1}{n}-1}\Rightarrow$$

$$\Rightarrow f'(x)=\left(\frac{1}{n}-1\right)\left[(x+1)^{\frac{1}{n}-2}-x^{\frac{1}{n}-2}\right]=\left(\frac{1-n}{n}\right)\left[\frac{1}{(x+1)^{\frac{2n-1}{n}}}-\frac{1}{x^{\frac{2n-1}{n}}}\right]>0$$

Accordingly, $f(x)$ is a positive function which gives:

$$f(x)>f(x-1)\Leftrightarrow\frac{\sqrt[n]{x+1}}{x+1}+\frac{\sqrt[n]{x-1}}{x-1}>\frac{2\sqrt[n]{x}}{x}=\frac{2}{\sqrt[n]{x^{n-1}}}$$

Therefore, QED is obtained by AM-GM inequality as:

$$\sum\left(\frac{\sqrt[n]{a^n+1}}{a^n+1}+\frac{\sqrt[n]{a^n-1}}{a^n-1}\right)>\frac{2}{a^{n-1}}+\frac{2}{b^{n-1}}+\frac{2}{c^{n-1}}\geq\frac{6}{\sqrt[3]{a^{n-1}b^{n-1}c^{n-1}}}$$

SOLUTION 3.153

Solution by Marian Ursărescu-Romania

$$\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\geq 2\sqrt{\frac{1}{e^{\sin^2 x+\cos^2 x}}}=\frac{2}{\sqrt{e}}\Rightarrow\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\geq\frac{2}{\sqrt{e}}>\frac{1}{2}+\frac{1}{\sqrt{e}}\ \textit{(because } \frac{1}{\sqrt{e}}>\frac{1}{2}\Leftrightarrow$$

$$\Leftrightarrow 2>\sqrt{e}\Leftrightarrow 4>e\textit{);}\ \left.\begin{array}{l}\frac{1}{e^{\sin^2 y}}+\frac{1}{e^{\cos^2 y}}>\frac{1}{2}+\frac{1}{\sqrt{e}}\\ \frac{1}{e^{\sin^2 z}}+\frac{1}{e^{\cos^2 z}}>\frac{1}{2}+\frac{1}{\sqrt{e}}\end{array}\right\}\Rightarrow\sum\left(\frac{1}{e^{\sin^2 x}}+\frac{1}{e^{\cos^2 x}}\right)>3\left(\frac{1}{2}+\frac{1}{\sqrt{e}}\right)$$

SOLUTION 3.154

Solution by Marian Ursărescu-Romania

We must show this:

$$\cos x\cos z\cdot\sin y\cdot\sin t\,(\sin x\cos y-\cos x\sin y)(\sin z\cot t-\cos z\sin t)\leq\frac{1}{64}\quad\textit{(1)}$$

We show this: $\cos x\sin y\,(\sin x\cos y-\cos x\sin y)\leq\frac{1}{8}$ ***(2)***

$$\left.\begin{array}{c}\cos x=a,\sin y=b\ (2)\Leftrightarrow ab\left(\sqrt{(1-a^2)(1-b^2)}-ab\right)\le\frac{1}{8}\\ But\ \sqrt{(1-a^2)(1-b^2)}\le\frac{2-a^2-b^2}{2}\end{array}\right\}\Rightarrow$$

$$ab\left(\frac{2-a^2-b^2}{2}-ab\right)\le\frac{1}{8}\Leftrightarrow ab(2-a^2-b^2-2ab)\le\frac{1}{4}\Leftrightarrow$$

$$4ab(2-(a+b)^2)\le 1\quad \textit{(3)}$$

$$\textit{But } (a+b)^2\ge 4ab\Rightarrow -(a+b)^2\le -4ab\quad \textit{(4)}$$

$$\textit{From (3)+(4)}\Rightarrow 4ab(2-4ab)\le 1\Leftrightarrow 8ab-16a^2b^2\le 1\Leftrightarrow$$

$$16a^2b^2-8ab+1\ge 0\Leftrightarrow$$

$$\Leftrightarrow (4ab-1)^2\ge 0\ \textit{true (equality for } a=b=\tfrac{1}{2}).$$

$$\textit{Similarly: } \cos z\sin t\sin(z-t)\le\frac{1}{8}\quad \textit{(5)}$$

From (2)+(5)$\Rightarrow \cos x\cos z\cdot\sin y\cdot\sin t\cdot\sin(x-y)\sin(z-t)\le 1$*, with equality for*

$$x=z=\frac{\pi}{3}\ \textit{and}\ y=t=\frac{\pi}{6}.$$

SOLUTION 3.155

Solution by Soumitra Mandal-Chandar Nagore-India

$$\textit{We know, } (1+x)^n=1+C_1^nx+C_2^nx^2+\cdots+x^n\Rightarrow 2^n=\sum_{k=0}^{n}C_k^n$$

$$\frac{1}{n+1}\sum_{k=1}^{n}(C_k^n)^\theta\ge\left(\frac{1}{n+1}\sum_{k=0}^{n}C_k^n\right)^\theta=\left(\frac{2^n}{n+1}\right)^\theta\Bigg|$$

$$\Rightarrow\sum_{k=0}^{n}(C_k^n)^\theta\ge(n+1)\left(\frac{2^n}{n+1}\right)^\theta$$

SOLUTION 3.156

Solution by Serban George Florin – Romania

$$x\cos x+x\cos z+y\cos y+y\cos x+z\cos z+z\cos y\ge x\cos x+x\cos y+y\cos y+$$

$$+y\cos z+z\cos z+z\cos x$$

$$x\cos z+y\cos x+z\cos y\ge x\cos y+y\cos z+z\cos x$$

$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos y)\ge 0$$

$$x(\cos z-\cos y)+y(\cos x-\cos z)-z(\cos x-\cos z+\cos z-\cos y)\ge 0$$

$$x(\cos z-\cos b)+b(\cos x-\cos z)-z(\cos x-\cos z)-z(\cos z-\cos y)\ge 0$$

$$(y-z)(\cos x-\cos z)+(x-z)(\cos z-\cos y)\ge 0$$

$$(x-z)(\cos z-\cos y)\geq(z-y)(\cos x-\cos z)$$

If $x=z\Rightarrow 0\geq 0$ *true. If* $z=y\Rightarrow 0\geq 0$ *true.*

If $x\neq z, z\neq y, x-z>0$ *and* $z-y>0\Rightarrow y<z<x\Rightarrow \cos z<\cos y, \cos x<\cos z$

$$\Rightarrow (x-z)(\cos y-\cos z)\leq(z-y)(\cos z-\cos x)$$

$$\frac{\cos y-\cos z}{z-y}\leq\frac{\cos z-\cos x}{x-z}\Big|\cdot(-1)$$

$$\frac{\cos y-\cos z}{y-z}\geq\frac{\cos z-\cos x}{z-x};\quad f(x)=\cos x$$

T. Lagrange $[x,z],[y,z], f'(x)=-\sin x$

$$-\sin c_1\geq-\sin c_2, \sin c_1\leq\sin c_2$$

$(\exists)c_1\in(y,z),(\exists)c_2\in(z,x), y<z<x\Rightarrow c_1<c_2\Rightarrow\sin c_1<\sin c_2$ *true.*

SOLUTION 3.157

Solution by Soumitra Mandal-Chandar Nagore-India

$$\frac{b\sqrt[m]{b}-a\sqrt[m]{a}}{b\sqrt[n]{b}-a\sqrt[n]{a}}\geq\frac{mn+n}{mn+m}\Leftrightarrow\frac{m}{m+1}\left(b\sqrt[m]{b}-a\sqrt[m]{a}\right)\geq\frac{n}{n+1}\left(b\sqrt[n]{b}-a\sqrt[n]{a}\right)$$

$$\Leftrightarrow\int_a^b\sqrt[m]{x}\,dx\geq\int_a^b\sqrt[n]{x}\,dx\Leftrightarrow x^n\geq x^m\Leftrightarrow\left(\frac{1}{x}\right)^{m-n}\geq 1, \textit{ which is true } \because 1\geq x>0$$

SOLUTION 3.158

Solution by Ravi Prakash-New Delhi-India

$$\textit{For } k\geq 3. \textit{ Let } f_k(x)=\left(\sin^k x+\cos^k x\right)^{\frac{1}{k}}, 0<x<\frac{\pi}{2}$$

$$\ln f_k(x)=\frac{1}{k}\ln\left(\sin^k x+\cos^k x\right)$$

$$\frac{1}{f_k(x)}f'_k(x)=\frac{1}{k}\cdot\frac{k\left[\sin^{k-1}x\cos x-\cos^{k-1}x\sin x\right]}{\sin^k x+\cos^k x}\Rightarrow$$

$$\Rightarrow f'_k(x)=\frac{(\sin x\cos x)\left(\sin^{k-2}x-\cos^{k-2}x\right)}{\sin^k x+\cos^k x}f_k(x)$$

$$f'_k(x)<0 \textit{ for } 0<x<\frac{\pi}{4}$$

$$=0 \textit{ for } x=\frac{\pi}{4},\quad >0 \textit{ for } \frac{\pi}{4}<x<\frac{\pi}{2}$$

$$\therefore f_k(x) \textit{ attains its minimum value at } x=\frac{\pi}{4}\Rightarrow f_k(x)\geq\left(\frac{2}{2^{\frac{k}{2}}}\right)^{\frac{1}{k}}=2^{\frac{1}{k}-\frac{1}{2}}\Rightarrow$$

$$\Rightarrow\prod_{k=3}^n f_k(x)\geq 2^{a_n} \textit{ where } a_n=\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\frac{n-2}{2}=1+\frac{1}{2}+\frac{1}{3}+\cdots-\frac{1}{n}-\frac{n+1}{2}$$

$$\textit{Thus } \prod_{k=3}^{n}\left(\sin^k x+\cos^k x\right)^{\frac{1}{k}} \geq 2^{1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{n+1}{2}}$$

SOLUTION 3.159

Solution by Omran Kouba-Damascus-Syria

Consider $f(x)=\ln(1-x)+x$. *Clearly* $f''(x)=-\frac{1}{(x-1)^2}$ *so* f *is concave.*

Thus the function

$x\to\frac{(f(x)-f(0))}{(x-0)}$ *is decreasing on* $(0,1)$. *Thus, for* $x\in(0,1)$ *and* $n\geq 2$ *we have:* $\frac{f\left(\frac{x}{n}\right)}{\frac{x}{n}}>\frac{f(x)}{x}$.

Consequently $f\left(\frac{x}{n}\right)-\frac{f(x)}{n}>0$. *Applying this to* $x=\frac{1}{k}$ *and adding we get:*

$$0<\sum_{k=2}^{m} f\left(\frac{1}{kn}\right)-\frac{1}{n}\sum_{k=2}^{m} f\left(\frac{1}{k}\right)=\sum_{k=2}^{m}\ln\left(1-\frac{1}{kn}\right)+\frac{1}{n}\sum_{k=2}^{m}\frac{1}{k}-\frac{1}{n}\sum_{k=2}^{m}\ln\frac{k-1}{k}-\frac{1}{n}\sum_{k=2}^{m}\frac{1}{k}=$$

$$=\frac{\ln m}{n}+\ln\frac{\prod_{k=2}^{m}(kn-1)}{n^{m-1}m!}=\ln\left(\frac{m^{\frac{1}{n}}}{n^{m-1}m!}\prod_{k=2}^{m}(kn-1)\right)$$

So, we have proved that for integers $n,m\geq 2$ *the next inequality holds:*

$$\prod_{k=2}^{m}(kn-1)>\frac{n^m m!}{n\cdot m^{\frac{1}{m}}} \quad \textit{(1)}$$

Applying (1) with $n=m=a$ *and* $n=m=b$ *and using the AM-GM inequality we get*

$$\prod_{k=2}^{a}(ka-1)+\prod_{k=2}^{b}(kb-1)\geq 2\sqrt{\prod_{k=2}^{a}(ka-1)\cdot\prod_{k=2}^{b}(kb-1)}>2\sqrt{\frac{a^a a!}{a\cdot a^{\frac{1}{a}}}\cdot\frac{b^b b!}{b\cdot b^{\frac{1}{b}}}}$$

Which is equivalent to the proposed inequality.

SOLUTION 3.160

Solution by Marian Ursărescu-Romania

$$m+a^{m+1}=1+1+\cdots+1+a^{m+1}\geq(m+1)\sqrt[m+1]{1\cdot 1\cdot\ldots\cdot 1\cdot a^{m+1}}\Rightarrow$$

$$m+a^{m+1}\geq(m+1)\,a \quad \textit{(1)}$$

$$n+\frac{1}{(b+c+u)^{n+1}}=1+1+\cdots+1+\frac{1}{(b+c+n)^{n+1}}\geq(n+1)\sqrt[n+1]{\frac{1\cdot 1\cdot\ldots\cdot 1}{(b+c+u)^{n+1}}}$$

$$\Rightarrow n+\frac{1}{(b+c+u)^{n+1}}\geq\frac{n+1}{(b+c+u)} \quad \textit{(2)}$$

From (1) and (2) inequlaity becomes: $\sum(m+a^{m+1})\left(n+\frac{1}{(b+c+u)^{n+1}}\right)\ge$

$\ge(m+1)(n+1)\sum\frac{a}{b+c+u}$*. We must show this:* $\sum\frac{a}{b+c+u}\ge\frac{3(a+b+c)}{2(a+b+c)+3u}$ *(3). From Cauchy's*

inequality $\Rightarrow\sum\frac{a}{b+c+u}=\sum\frac{a^2}{a(b+c+u)}\cdot\sum(ab+ac+au)\ge(a+b+c)^2\Rightarrow$

$$\Rightarrow\sum\frac{a}{b+c+u}\ge\frac{(a+b+c)^2}{2(ab+bc+ac)+(a+b+c)^u}\quad (4)$$

From (3)+(4) we must show: $\frac{(a+b+c)^2}{2(ab+ac+bc)+(a+b+c)u}\ge\frac{3(a+b+c)}{2(a+b+c)+3u}\Leftrightarrow$

$$\Leftrightarrow\frac{(a+b+c)}{2(ab+ac+bc)+(a+b+c)u}\ge\frac{3}{2(a+b+c)+3u}\Leftrightarrow$$

$$\Leftrightarrow 2(a+b+c)^2+3u(a+b+c)\ge 6(ab+ac+bc)+3u(a+b+c)\Leftrightarrow$$

$$\Leftrightarrow(a+b+c)^2\ge 3(ab+ac+bc)\Leftrightarrow a^2+b^2+c^2\ge ab+ac+bc\ \ \textit{(true)}$$

SOLUTION 3.161

Solution by Soumitra Mandal-Chandar Nagore-India

$$\sum_{cyc}\frac{1}{\log_a c+2\log_a b}=\sum_{cyc}\frac{\log a}{\log c+2\log b}=\sum_{cyc}\frac{(\log a)^2}{\log a\log c+2\log a\log b}\ge$$

$$\ge\frac{\left(\sum_{cyc}\log a\right)^2}{3\sum_{cyc}\log a\log b}\ge 1$$

SOLUTION 3.162

Solution by Daniel Sitaru-Romania

$$|2x+3+2y+3+2z+3|+\sum_{cyc(x,y,z)}|2x+3|\overset{HLAWKA}{\ge}\sum_{cyc(x,y,z)}|2x+3+2y+3|$$

$$|2(x+y+z)+9|+\sum_{cyc(x,y,z)}|2x+3|\ge 2\sum_{cyc(x,y,z)}|x+y+3|$$

$$\frac{1}{2}\left(\sum_{cyc(x,y,z)}|2x+3|+9\right)\ge\sum_{cyc(x,y,z)}|-z+3|=\sum_{cyc(x,y,z)}|x-3|$$

SOLUTION 3.163

Solution by Amit Dutta-Jamshedpur-India

$$\textit{Let } P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(1-\tan x)}{1+\tan x}\right|$$

$$P=\left|\frac{2}{\sin x+\cos x}+\frac{(\sin x-\cos x)(\cos x-\sin x)}{\cos x\frac{(\sin x+\cos x)}{\cos x}}\right|$$

$$P=\left|\frac{2-(\cos x-\sin x)^2}{(\sin x+\cos x)}\right|, P=\left|\frac{2-1+\sin 2x}{\sin x+\cos x}\right|, P=\left|\frac{1+\sin 2x}{\sin x+\cos x}\right|$$

$$P=\left|\frac{(\sin x+\cos x)^2}{\sin x+\cos x}\right|, P=|\sin x+\cos x|,\ P=\sqrt{2}\left|\sin\left(x+\frac{\pi}{4}\right)\right|\le\sqrt{2}$$

SOLUTION 3.164

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to(0,\infty), f(a)=a^{-\frac{1}{2}}, f'(a)=-\frac{1}{2}a^{-\frac{3}{2}}, f''(a)=\frac{3}{4}a^{-\frac{5}{2}}>0, f-convexe$$

$$\frac{1}{3}\sum f(a)+f\left(\frac{a+b+c}{3}\right)\overset{POPOVICIU}{\ge}\frac{2}{3}\sum f\left(\frac{a+b}{2}\right)$$

$$a=x+y, b=y+z, c=z+x$$

$$\frac{1}{3}\sum f(x+y)+f\left(\frac{2x+2y+2z}{3}\right)\overset{POPOVICIU}{\ge}\frac{2}{3}\sum f\left(\frac{x+2y+z}{2}\right)$$

$$\frac{1}{3}\sum\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{\frac{2(x+y+z)}{3}}}\ge\frac{2}{3}\sum\frac{1}{\sqrt{\frac{x+2y+z}{2}}}$$

$$\sum\frac{1}{\sqrt{x+y}}+\frac{3\sqrt{3}}{\sqrt{2(x+y+z)}}\ge 2\sqrt{2}\sum\frac{1}{\sqrt{x+2y+z}}$$

SOLUTION 3.165

Solution by Soumava Chakraborty-Kolkata-India

$\because a<b<c<d<e<f<g<h$***, we can consider*** $b=a+x, c=a+x+y,$

$d=a+x+y+z, e=a+x+y+z+u, f=a+x+y+z+u+v,$

$g=a+x+y+z+u+v+w, h=a+x+y+z+u+v+w+t$***, where***

$x,y,z,u,v,w,t>0\ \therefore$ ***by these substitutions, given inequality transforms into:***

$$(8a+7x+6y+5z+4u+3v+2w+t)^2-16a(a+x+y+z+u+v+w+t)-$$
$$-16(a+x)(a+x+y+z+u+v+w)-16(a+x+y)(a+x+y+z+u+v)-$$
$$-16(a+x+y+z)(a+x+y+z+u)\ge 0\Leftrightarrow t^2+8tu+6tv+4tw+14tx+12ty+$$
$$+10tz+16u^2+24uv+16uw+8ux+16uy+24uz+9v^2+12vw+10vx+20vy+$$
$$+30vz+4w^2+12wx+24wy+20wz+x^2+4xy+6xz+4y^2+12yz+9z^2>0\to$$

$\to$ ***true*** $\because x,y,z,u,v,w,t>0$ ***(proved)***

SOLUTION 3.166

Solution by Daniel Sitaru-Romania

$$f:[a,b]\to\mathbb{R}, f(x)=\pi^x, f'(x)=\pi^x\cdot \log\pi,$$

$$g:[a,b]\to\mathbb{R}, g(x)=\log x, g'(x)=\frac{1}{x}$$

$$\frac{\pi^b-\pi^a}{\log\frac{b}{a}}=\frac{f(b)-f(a)}{g(b)-g(a)}\overset{CAUCHY}{\cong}\frac{f'(c)}{g'(c)}=\frac{\pi^c\log\pi}{\frac{1}{c}}>c\cdot\pi^c>e\cdot\pi^e,$$

$$b>c>a\geq e$$

SOLUTION 3.167

Solution by Serban George Florin-Romania

$$\tan x=a, \tan y=b, \tan z=c$$

$$\tan(x+b+z)=\tan\frac{\pi}{4}=1=\frac{\sum\tan x-\prod\tan x}{1-\sum\tan x\tan y}\Rightarrow$$

$$a+b+c-abc=1-ab-bc-ac$$

$$\Rightarrow a+b+c+ba+bc+ac=1+abc$$

$$\sum\tan x(1+\tan y)=\sum a(1+b)=a+b+c+ab+bc+ac=1+abc\overset{(M_a\geq M_g)}{\geq}$$

$$\geq 2\sqrt{1\cdot abc}=2\sqrt{abc}$$

SOLUTION 3.168

Solution by Daniel Sitaru-Romania

$$\sqrt{1+e^x}\overset{QM-AM}{\geq}\frac{1}{\sqrt{2}}\left(1+\sqrt{e^x}\right)\to\prod\sqrt{1+e^x}\geq\frac{1}{2\sqrt{2}}\prod\left(1+\sqrt{e^x}\right)\leftrightarrow$$

$$2\sqrt{2}\prod\sqrt{1+e^x}\geq\frac{1}{\sqrt{e^{x+y+z}}}\cdot\prod\left(1+\sqrt{e^x}\right),(x+y+z=0)\leftrightarrow$$

$$2\sqrt{2(1+e^x)(1+e^y)(1+e^z)}\geq\prod\left(1+\frac{1}{\sqrt{e^x}}\right)$$

SOLUTION 3.169

Solution by Daniel Sitaru-Romania

$$a=y+z, b=z+x, c=x+y, s=x+y+z, S=\sqrt{xyx(x+y+z)}$$

$$s \overset{MITRINOVIC}{\leq} \frac{3\sqrt{3}R}{2} \leftrightarrow \frac{sS}{4RS} \leq \frac{3\sqrt{3}}{8} \leftrightarrow \frac{sS}{abc} \leq \frac{3\sqrt{3}}{8} \leftrightarrow \frac{(x+y+z)\sqrt{xyz(x+y+z)}}{(x+y)(y+z)(z+x)} \leq \frac{3\sqrt{3}}{8}$$

SOLUTION 3.170

Solution by Daniel Sitaru-Romania

$$f: \mathbb{R} \to \mathbb{R}, f(x) = 4^x, f''(x) = 4^x log^2 4 > 0, f - convexe$$

By Popoviciu's inequality:

$$\frac{1}{3}\sum f(x) + f\left(\frac{x+y+z}{3}\right) \geq \frac{2}{3}\sum f\left(\frac{x+y}{2}\right) \leftrightarrow$$

$$\leftrightarrow \frac{1}{3}\sum 4^x + 4^0 \geq \frac{2}{3}\sum 4^{\frac{x+y}{2}} \leftrightarrow \sum 4^x \geq 2\sum 2^{x+y} - 3$$

SOLUTION 3.171

Solution by Tran Hong-Vietnam

We have:Inequality $\Leftrightarrow$

$$\frac{1}{2}\left[a\ln\left(1+\frac{x}{a}\right) + x\ln\left(1+\frac{a}{x}\right) + b\ln\left(1+\frac{y}{b}\right) + y\ln\left(1+\frac{b}{y}\right) + c\ln\left(1+\frac{z}{c}\right) + z\ln\left(1+\frac{c}{z}\right)\right] \leq \ln 2 \quad (*)$$

Using Jensen's inequality with $f(u) = \ln u$:

$$LHS_{(*)} = \frac{1}{2}af\left(1+\frac{x}{a}\right) + \frac{1}{2}xf\left(1+\frac{a}{x}\right) + \frac{1}{2}bf\left(1+\frac{y}{b}\right) + \frac{1}{2}yf\left(1+\frac{b}{y}\right) +$$

$$+\frac{1}{2}cf\left(1+\frac{z}{c}\right) + \frac{1}{2}zf\left(1+\frac{c}{z}\right) \leq$$

$$\leq \ln\left\{\frac{1}{2}a\left(1+\frac{x}{a}\right) + \frac{1}{2}x\left(1+\frac{a}{x}\right) + \frac{1}{2}y\left(1+\frac{b}{y}\right) + \frac{1}{2}c\left(1+\frac{z}{c}\right) + \frac{1}{2}z\left(1+\frac{c}{z}\right)\right\}$$

$$= \ln\{(a+x+b+y+c+z)\} = \ln 2$$

Proved. Equality $\Leftrightarrow a = b = c = x = y = z = \frac{1}{3}$.

GEOMETRICAL INEQUALITIES AND IDENTITIES-SOLUTIONS

SOLUTION 4.01

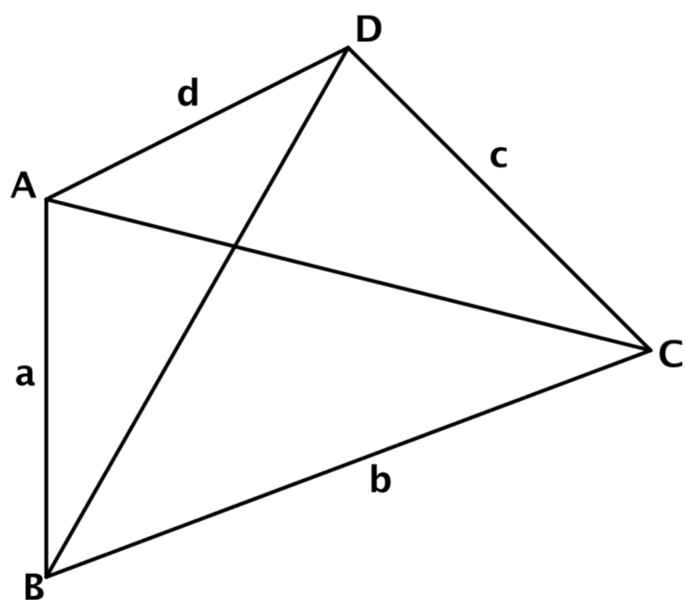
Solution by SK Rejuan-West Bengal-India

$$(a^2+b^2+c^2) > \frac{1}{3}(a+b+c)^2 \quad \textit{[by mth power theorem]}$$

$$\Rightarrow \sqrt{a^2+b^2+c^2} > \frac{1}{\sqrt{3}}(a+b+c)$$

$$\Rightarrow \sum \sqrt{a^2+b^2+c^2} > \frac{1}{\sqrt{3}}\sum(a+b+c) = \frac{3}{\sqrt{3}}(a+b+c+d)$$

$$\Rightarrow \sum \sqrt{a^2+b^2+c^2} > \sqrt{3}(a+b+c) \quad \textit{(1)}$$



$$\textit{For, } \Delta\, ABC, a+b > AC$$

$$\Delta BCD, b+c > BD$$

$$\Delta\, CDA, c+d > AC$$

$$\Delta DAB, d+a > BD$$

Adding the we get, $2(a+b+c+d) > 2(AC+BD)$

$$\Rightarrow \sqrt{3}(a+b+c+d) > \sqrt{3}(AC+BD) \quad (2)$$

Also by $AM > GM$ *we get,* $AC+BD > 2\sqrt{AC\cdot BD} \Rightarrow \sqrt{3}(AC+BD) > 2\sqrt{3\cdot AC\cdot BD} \quad$ *(3)*

From (1), (2) & (3) we get,

$$\sum\sqrt{a^2+b^2+c^2} > \sqrt{3}(a+b+c+d) > \sqrt{3}(AC+BD) > 2\sqrt{3\cdot AC\cdot BD}$$

$$\Rightarrow \sum\sqrt{a^2+b^2+c^2} > 2\sqrt{3\cdot AC\cdot BD}$$

SOLUTION 4.02

Solution by Ravi Prakash - New Delhi – India

$$\sin A = \frac{2S}{ad+bc}$$

$$\sin B = \frac{2S}{ab+cd}$$

Also

$$ad+bc \geq 2\sqrt{adbc}$$

$$\sin A \sin B \leq \frac{4S^2}{4abcd} = \frac{S^2}{abcd} = \frac{(s-a)(s-b)(s-c)(s-d)}{abcd} =$$

$$= \left(1-\frac{s}{a}\right)\left(1-\frac{s}{b}\right)\left(1-\frac{s}{c}\right)\left(1-\frac{s}{d}\right)$$

SOLUTION 4.03

Solution by Adil Abdullayev – Baku – Azerbaidjian

$$\begin{matrix}\sin A = \sin C \\ \sin B = \sin D\end{matrix} \Rightarrow \sin A + \sin B \leq \frac{2S}{\sqrt{abcd}} \ldots (A)$$

$$\begin{matrix}\sin A = \frac{2S}{ad+bc} \\ \sin B = \frac{2S}{ab+cd}\end{matrix} \Rightarrow (A) \Leftrightarrow \frac{1}{ad+bc} + \frac{1}{ab+cd} \leq \frac{1}{\sqrt{abcd}} \ldots (B)$$

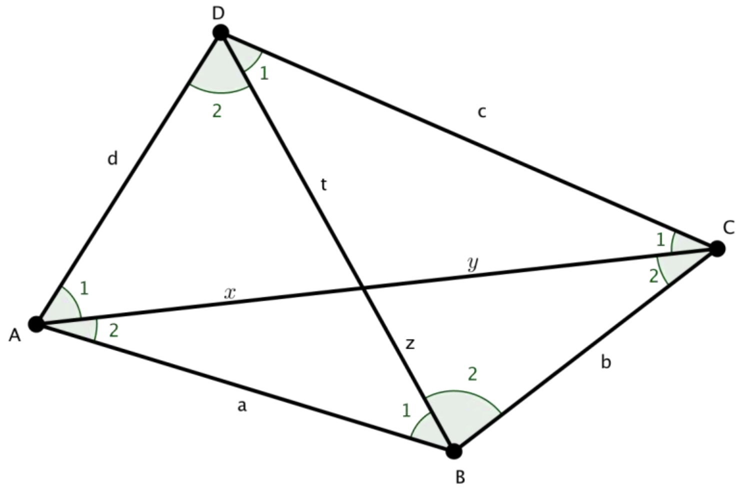
$$ad+bc \overset{AM-GM}{\geq} 2\sqrt{abcd}$$

$$ab+cd \overset{AM-GM}{\geq} 2\sqrt{abcd}$$

$$\frac{1}{ad+bc} + \frac{1}{ab+cd} \leq \frac{1}{2\sqrt{abcd}} + \frac{1}{2\sqrt{abcd}} = \frac{1}{\sqrt{abcd}}.$$

SOLUTION 4.04

Solution by Geanina Tudose – Romania



For convenience denote

$$AB = a; BC = b; CD = c; DA = d, AO = x; CO = y; BO = z; DO = t$$

In Δ ABO by Sine Theorem

$$\frac{\sin B_1}{AO} = \frac{\sin \widehat{O}}{AB} \Rightarrow \sin B_1 = \frac{AO \cdot \sin O}{AB}$$

In ΔBCO, $\sin B_2 = \frac{CO \cdot \sin O}{BC}$

Thus, $\frac{\sin B_1}{\sin B_2} = \frac{AO \cdot BC}{CO \cdot AB} = \frac{x \cdot b}{y \cdot a}$

Similarly, $\frac{\sin C_1}{\sin C_2} = \frac{z \cdot c}{t \cdot b}; \frac{\sin D_1}{\sin D_2} = \frac{y \cdot d}{x \cdot c}; \frac{\sin A_1}{\sin A_2} = \frac{a \cdot t}{dz}$

The inequality becomes

$$\frac{x}{y} \cdot \frac{b}{a} + \frac{z}{t} \cdot \frac{c}{b} + \frac{y}{x} \cdot \frac{d}{c} + \frac{t}{z} \cdot \frac{a}{a} \overset{AM \geq GM}{\geq} 4\sqrt[4]{\frac{x}{y} \cdot \frac{y}{x} \cdot \frac{z}{t} \cdot \frac{t}{z} \cdot \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{d}{c} \cdot \frac{a}{d}} = 4$$

SOLUTION 4.05

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{R}{8\sqrt{2}} \geq \frac{\sqrt[4]{a^3b^3c^3d^3}}{(a+b+c+d)^2} \quad (1)$$

$$\Delta ACB \Rightarrow \frac{a}{2R} = \sin C_1$$

$$\Delta ACD \Rightarrow \frac{d}{2R} = \sin C_2$$

$$\frac{a+d}{2R}=\sin C_1+\sin C_2\le 2\cdot\sin\frac{C_1+C_2}{2}$$

$$\left.\begin{array}{l} a+d\le 4R\cdot\sin\frac{C}{2}\\ b+a\le 4R\cdot\sin\frac{D}{2}\\ c+b\le 4R\cdot\sin\frac{A}{2}\\ d+c\le 4R\cdot\sin\frac{B}{2}\end{array} Similarly\right\} 2(a+b+c+d)\le 4R\left[\left(\sin\frac{A}{2}+\sin\frac{C}{2}\right)+\left(\sin\frac{B}{2}+\sin\frac{D}{2}\right)\right]=$$

$$p\le R\left[\left(\sin\frac{A}{2}+\sin\frac{C}{2}\right)+\left(\sin\frac{B}{2}+\sin\frac{D}{2}\right)\right]\le$$

$$\le 2R\left[\sin\frac{A+C}{4}+\sin\frac{B+D}{4}\right]=2R\cdot 2\cdot\sin\frac{\pi}{4}=2\sqrt{2}R$$

$$p\le 2\sqrt{2}R\Rightarrow\frac{p}{2\sqrt{2}}\le R\quad (*)$$

$$(1)\Rightarrow\frac{R}{2\sqrt{2}}\ge\frac{\left(\sqrt[4]{abcd}\right)^3}{p^2}\Rightarrow\frac{R}{2\sqrt{2}}\overset{(*)}{\ge}\frac{p}{8}\ge\frac{\left(\sqrt[4]{abcd}\right)^3}{p^2}\Rightarrow$$

$$\Rightarrow\left(\frac{p}{2}\right)^3\ge\left(\sqrt[4]{abcd}\right)^3\Leftrightarrow\frac{p}{2}\ge\sqrt[4]{abcd}\Rightarrow\frac{a+b+c+d}{4}\ge\sqrt[4]{abcd}$$

SOLUTION 4.06

Solution by Marian Dincă – Romania

$$Let\ V_p=volume\left[MA_iA_jA_k\right]$$

$$where:\ p\ne i\ne j\ne k\ne p\ and\ (i,j,k,p)=\{1,2,3,4\}$$

result:

$$\sum_{1\le i<j\le 4}S_iS_jd_id_j=\sum_{1\le i<j\le 4}9V_iV_j\le 9\binom{2}{4}\left(\frac{\sum_{k=1}^4 V_k}{4}\right)^2=\frac{27}{8}V^2$$

Mac – Laurin inequality

SOLUTION 4.07

Solution by Kevin Soto Palacios – Huarmey – Peru

A Simple Proof of Euler's Inequality in Space

Zhang Yun – Jinchang City - Gasu Province – China

Let R be the radius of the circumscribed sphere of a tetrahedron and let r be the radius of the inscribed sphere of the tetrahedron. Then Euler's famous inequality in space state that

$$R \geq 3r \qquad (1)$$

We give here a simple proof of this inequality.

Let O be the circumcenter of the tetrahedron $A_1A_2A_3A_4$.

Let s_k $(k = 1, 2, 3, 4)$ denote the area of the face opposite the vertex A_k, let h_k denote the distance from A_k to its opposite face, and let d_k denote the distance from the point O to the face opposite A_k. Then$OA_k + d_k \geq h_k$,and so $R + d_k \geq h_k$.

Thus,$s_kR + s_kd_k \geq s_kh_k$.

Adding the four inequalities, we obtain that

$$R(s_1 + s_2 + s_3 + s_4) + s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4 \geq s_1h_1 + s_2h_2 + s_3h_3 + s_4h_4.$$

Let V denote the volume of the tetrahedron $A_1A_2A_3A_4$.Then

$$V = \frac{1}{3}s_kh_k = \frac{1}{3}(s_1d_1 + s_2d_2 + s_3d_3 + s_4d_4)\text{,so}$$

$$R(s_1 + s_2 + s_3 + s_4) + 3V \geq 4 \times 3V\text{,from which it follows that}$$

$$R(s_1 + s_2 + s_3 + s_4) \geq 9V\text{.Since } V = \frac{r}{3}(s_1 + s_2 + s_3 + s_4)\text{,this gives}$$

$R(s_1 + s_2 + s_3 + s_4) \geq 9 \times \frac{r}{3}(s_1 + s_2 + s_3 + s_4)$.Thus,$R \geq 3r$,so the inequality (1) is proved. In two dimensions rather that three, if R is now the radius of the circumscribed circle of a triangle and r the radius of the inscribed circle, then, by a similar argument, $R \geq 2r$.

Solution by Kevin Soto Palacios – Huarmey – Peru

Ahora bien, por la desigualdad de Cauchy

$$h_A + h_B + h_C + h_D = \frac{1}{\frac{1}{h_A}} + \frac{1}{\frac{1}{h_B}} + \frac{1}{\frac{1}{h_C}} + \frac{1}{\frac{1}{h_D}} \geq \frac{(1+1+1+1)^2}{\frac{1}{h_A} + \frac{1}{h_B} + \frac{1}{h_C} + \frac{1}{h_D}} = 16r$$

Por lo tanto

$$R(h_A + h_B + h_C + h_D) \geq 3r \cdot 16r = 48r^2$$

SOLUTION 4.08

Solution by Marian Ursărescu – Romania

$$\Psi = \begin{vmatrix} \sin A & \sin B & \sin C \\ 2\sin A\cos A & 2\sin B\cos B & 2\sin C\cos C \\ 3\sin A - 4\sin^3 A & 3\sin B - 4\sin^3 B & 3\sin C - 4\sin^3 C \end{vmatrix} =$$

$$= 2\sin A\sin B\sin C\begin{vmatrix} 1 & 1 & 1 \\ \cos A & \cos B & \cos C \\ 3-4\sin^2 A & 3-4\sin^2 B & 3-4\sin^2 C \end{vmatrix} =$$

$$= 2\sin A\sin B\sin C\begin{vmatrix} 1 & 0 & 0 \\ \cos A & \cos B\cos A & \cos C-\cos A \\ 3-4\sin^2 A & 4(\sin^2 A-\sin^2 B) & 4(\sin^2 C-\sin^2 A) \end{vmatrix} =$$

$$= 2\sin A\sin B\sin C\begin{vmatrix} \cos B-\cos A & \cos C-\cos A \\ 4(\cos^2 B-\cos^2 A) & 4(\cos^2 C-\cos^2 A) \end{vmatrix} =$$

$$= 8\sin A\sin B\sin C\,(\cos B-\cos A)(\cos C-\cos A)\begin{vmatrix} 1 & 1 \\ \cos B-\cos A & \cos C-\cos A \end{vmatrix} =$$

$$= 8\sin A\sin B\sin C\,(\cos B-\cos A)(\cos C-\cos A)(\cos C-\cos B) \quad \textit{(1)}$$

From (1) $\Rightarrow$

$$\Rightarrow S[OIH] = \frac{R^6}{abcs}|8\sin A\sin B\sin C\,(\cos B-\cos A)(\cos C-\cos B)(\cos C-\cos A)|$$

$$= \frac{R^6 8\sin A\sin B\sin C}{8R^3\sin A\sin B\sin C\cdot s}|(\cos B+\cos A)(\cos C-\cos B)(\cos C-\cos A)| =$$

$$= \frac{R^3}{s}\left|2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)2\sin\left(\frac{A-C}{2}\right)\sin\left(\frac{A+C}{2}\right)2\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{B+C}{2}\right)\right| =$$

$$= \frac{8R^3}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\right) \quad \textit{(2)}$$

But $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$ ***(3). From (2)+(3)***$\Rightarrow$

$S[OIH] = 2R\left(\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{A-C}{2}\right)\right)$ ***(4). Now, from Sondat theorem*** $\Rightarrow$

$$S[OIH] = \frac{1}{8r}|(a-b)(b-c)(c-a)| =$$

$$= \frac{1}{8r}\cdot 8R^3|(\sin A-\sin B)(\sin B-\sin C)(\sin C-\sin A)| =$$

$$= \frac{R^3}{r}\left|2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)2\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{B+C}{2}\right)2\sin\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C}{2}\right)\right|$$

$$= \frac{8R^3}{r}\sin\frac{A}{2}\cdot\sin\frac{B}{2}\sin\frac{C}{2}\left|\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\right| \quad \textit{(5)}$$

But $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$ ***(6)***

From (5)+(6)$\Rightarrow$ $S[OIH] = 2R^2\left|\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{B-C}{2}\right)\sin\left(\frac{C-A}{2}\right)\right|$ ***(7)***

From (4)+(7) relationship its true.

SOLUTION 4.09

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)} \frac{r+r_a}{h_a-r} = \sum_{cyc(a,b,c)} \frac{\frac{S}{s}+\frac{S}{s-a}}{\frac{2S}{a}-\frac{S}{s}} = \sum_{cyc(a,b,c)} \frac{\frac{1}{s}+\frac{1}{s-a}}{\frac{2}{a}-\frac{1}{s}} = \sum_{cyc(a,b,c)} \frac{2s-a}{s(s-a)}\cdot\frac{sa}{2s-a} =$$

$$= \sum_{cyc(a,b,c)} \frac{a}{s-a} = 2\sum_{cyc(a,b,c)} \frac{S}{s-a}\cdot\frac{a}{2S} = 2\sum_{cyc(a,b,c)} \frac{r_a}{h_a}$$

SOLUTION 4.10

Solution by Daniel Sitaru-Romania

$$\frac{1}{2}\sum\frac{h_a}{AI} = \frac{1}{2}\sum\left(\frac{2S}{a}\cdot\frac{\sin\frac{A}{2}}{r}\right) = \frac{S}{r}\sum\frac{\sin\frac{A}{2}}{a} = \frac{rs}{r}\sum\frac{\sin\frac{A}{2}}{2R\sin\frac{A}{2}\cos\frac{A}{2}} =$$

$$= \frac{s}{2R}\sum\frac{1}{\cos\frac{A}{2}} = \frac{2sS}{4RS}\sum\sqrt{\frac{bc}{s(s-a)}} = \frac{s}{abc}\sum\sqrt{\frac{bcs(s-a)(s-b)(s-c)}{s(s-a)}} =$$

$$= \frac{s}{abc}\sum\sqrt{bc(s-b)(s-c)} = \sum\sqrt{\frac{s^2bc(s-b)(s-c)}{a^2b^2c^2}} =$$

$$= \sum\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}} = \sum\cos\frac{B}{2}\cos\frac{C}{2}$$

SOLUTION 4.11

Solution by Daniel Sitaru-Romania

$$K - Lemoine's\ point \rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \overset{denote}{\equiv} q,$$

$$S = \frac{ax+by+cz}{2} = \frac{(a^2+b^2+c^2)q}{2} \rightarrow q = \frac{2S}{a^2+b^2+c^2}$$

$$\sum_{\substack{cyc(a,b,c)\\cyc(x,y,z)}} \frac{m_a^2}{xh_a} = \sum_{cyc(a,b,c)} \frac{m_a^2}{qa\cdot\frac{2S}{a}} = \frac{1}{2Sq}\cdot\frac{3}{4}(a^2+b^2+c^2) =$$

$$= \frac{3}{8S\cdot\frac{2S}{a^2+b^2+c^2}}\cdot(a^2+b^2+c^2) = \frac{3}{16S^2}\cdot(a^2+b^2+c^2)^2$$

SOLUTION 4.12

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)}\frac{h_a+h_b}{r_a+r_b}=\sum_{cyc(a,b,c)}\frac{\frac{2S}{a}+\frac{2S}{b}}{\frac{S}{s-a}+\frac{S}{s-b}}=2\sum_{cyc(a,b,c)}\frac{a+b}{ab}\cdot\frac{(s-a)(s-b)}{2s-a-b}=$$

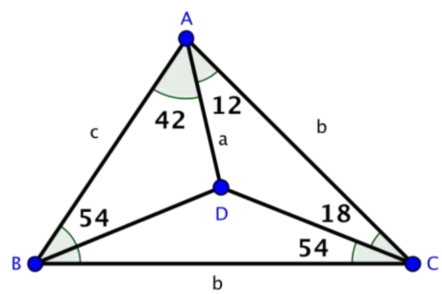
$$=\frac{2}{abc}\sum_{cyc(a,b,c)}\frac{(2s-c)\cdot S^2}{s(s-c)}=\frac{2S^2}{4RSs}\sum_{cyc(a,b,c)}\frac{2s-c}{s-c}=$$

$$=\frac{S}{2Rs}\left(3+s\cdot\sum_{cyc(a,b,c)}\frac{1}{s-c}\right)=\frac{rs}{2Rs}\left(3+s\cdot\frac{4R+r}{rs}\right)=\frac{r}{2R}\cdot\frac{3r+4R+r}{r}$$

$$=\frac{2(R+r)}{R}$$

SOLUTION 4.13

Solution by Seyran Ibrahimov-Maasilli-Azerbaijan



$$BC=AC=b$$

$$\frac{B}{\sin 150}=\frac{a}{\sin 18}\Rightarrow a=2b\cdot\sin 18=b\cdot\frac{\sqrt{5}-1}{2}\quad (1)$$

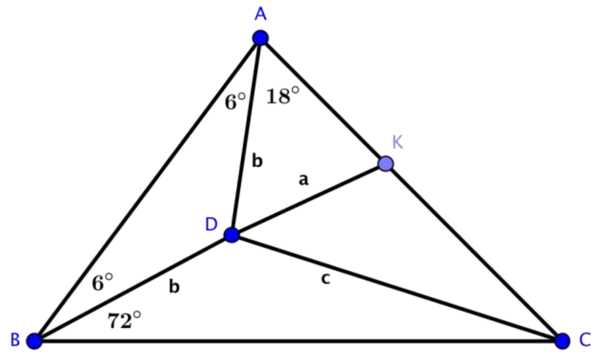
$$\frac{b}{\sin 54}=\frac{c}{\sin 72}\Rightarrow c=\frac{b\sin 72}{\sin 54}=\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{6+2\sqrt{5}}}b\quad (2)$$

$$a^2+b^2\overset{(1)}{=}\frac{6-2\sqrt{5}}{4}b^2+b^2=b^2\cdot\frac{10-2\sqrt{5}}{4}\overset{?}{=}c^2$$

$$c^2\overset{(2)}{=}b^2\cdot\frac{10+2\sqrt{5}}{6+2\sqrt{5}}=b^2\cdot\frac{10-2\sqrt{5}}{4}\quad (=)$$

SOLUTION 4.14

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan



$$\textit{In } \Delta ADB \Rightarrow AB = 2b\cdot\cos 6^\circ$$

$$\textit{In } \Delta BAC \Rightarrow AB = AC = 2b\cdot\cos 6^\circ$$

$$\textit{In } \Delta ADK \Rightarrow \frac{a}{\sin 18^\circ} = \frac{b}{\sin 18^\circ} \Rightarrow b = \frac{a\sin 48^\circ}{\sin 18^\circ}\quad \textit{(1)}$$

By cosine theorem in ΔADC, we have:

$$c^2 = b^2 + 4b^2\cos^2 6^\circ - 2b\cdot 2b\cdot\cos 6^\circ\cdot\cos 18^\circ$$

$$= b^2 + 4b^2\cdot(\cos^2 6^\circ - \cos 6^\circ\cdot\cos 18^\circ)$$

$$\overset{(1)}{=} b^2 + \frac{4a^2\sin^2 48^\circ}{\sin^2 18^\circ}\cdot\left(\frac{1+\cos 12^\circ}{2} - \frac{\cos 12^\circ + \cos 24^\circ}{2}\right) =$$

$$= b^2 + \frac{4a^2\sin^2 18^\circ}{\sin^2 18^\circ}\cdot\frac{1-\cos 24^\circ}{2} = b^2 + \frac{4a^2\sin^2 48^\circ\cdot\sin^2 12^\circ}{\sin^2 18^\circ}, \textit{ we prove that } \frac{4\sin^2 48^\circ\cdot\sin^2 12^\circ}{\sin^2 18^\circ} = 1 \Leftrightarrow$$

$$\Leftrightarrow 4\sin^2 48^\circ\sin^2 12^\circ = \sin^2 18^\circ \Leftrightarrow 4\cdot\frac{1}{4}\cdot(\cos 36^\circ - \cos 60^\circ)^2 = \sin^2 18^\circ \Leftrightarrow$$

$$\Leftrightarrow \cos 36^\circ - \cos 60^\circ = \sin 18^\circ \Leftrightarrow \cos 36^\circ - \sin 18^\circ = \frac{1}{2} \Leftrightarrow \sin 54^\circ - \sin 18^\circ = \frac{1}{2}\quad \textit{(2)}$$

Then (1)(2) $\Rightarrow c^2 = a^2 + b^2$. Q.E.D.

SOLUTION 4.15

Solution by Daniel Sitaru-Romania

$$K - Lemoine's\ point \to \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \overset{denote}{=} q$$

$$LHS = \sum_{cyc(a,b,c)}\frac{xh_a}{r_br_c} = \sum_{cyc(a,b,c)}\frac{qa\cdot\frac{2S}{a}}{\frac{S}{s-b}\cdot\frac{S}{s-c}} = \frac{2q}{S}\sum_{cyc(a,b,c)}(s-b)(s-c) =$$

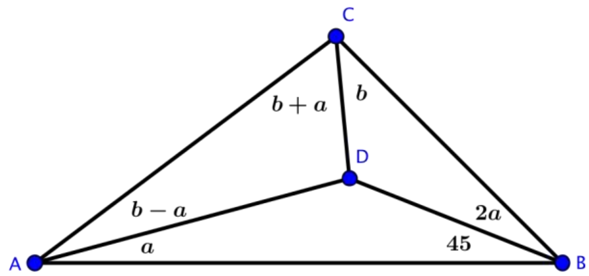
$$= \frac{2q}{S}\left(3s^2 - \sum_{cyc(a,b,c)}s(b+c) + \sum_{cyc(a,b,c)}bc\right) = \frac{2q}{S}(-s^2 + s^2 + r^2 + 4Rr) =$$

$$= \frac{2q}{S}(r^2 + 4Rr) = \frac{q}{S}(2s^2 - 2s^2 + 2r^2 + 8Rr) = \frac{q}{S}\sum_{cyc(a,b,c)}(sa - a^2) =$$

$$= \sum_{cyc(a,b,c)} \frac{qa}{\frac{S}{s-a}} = \sum_{cyc(a,b,c)} \frac{x}{r_a} = RHS$$

SOLUTION 4.16

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan



$$b - a + a + b + a + b + 2a + 45^\circ = 180^\circ \Rightarrow a + b = 45^\circ \quad \textit{(1)}$$

$$1 = \frac{BD}{AD} \cdot \frac{AD}{CD} \cdot \frac{CD}{BD} = \frac{\sin a}{\sin 45^\circ} \cdot \frac{\sin(b+a)}{\sin(b-a)} \cdot \frac{\sin 2a}{\sin b} \quad \textit{(2)}$$

using (1) we have: $1 = \frac{\sin a}{\sin 45^\circ} \cdot \frac{\sin 45^\circ}{\sin(45^\circ - 2a)} \cdot \frac{\sin 2a}{\sin(45^\circ - a)}$

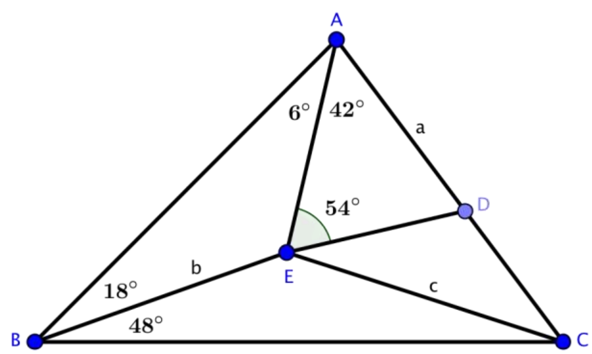
$$\sin(45^\circ - 2a) \cdot \sin(45^\circ - a) = \sin a \cdot \sin 2a$$

$$\cos(45^\circ - 2a - 45^\circ + a) - \cos(45^\circ - 2a + 45^\circ - a) = \cos a - \cos 3a$$

$$\sin 3a - \cos 3a = 0, \sqrt{2}\sin(3a - 45^\circ) = 0, 3a - 45^\circ = 0^\circ \Rightarrow a = 15^\circ$$

SOLUTION 4.17

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan



$$\textit{In } \Delta ABE \Rightarrow \frac{b}{\sin 6^\circ} = \frac{AE}{\sin 18^\circ} \Rightarrow AE = \frac{b\sin 18^\circ}{\sin 6^\circ} \quad \textit{(1)}$$

$$\textit{In } \Delta AED \Rightarrow \frac{a}{\sin 54^\circ} = \frac{AE}{\sin 84^\circ} \Rightarrow AE = \frac{a\sin 84^\circ}{\sin 54^\circ} \quad \textit{(2)}$$

$$(1)=(2)\Rightarrow b\sin 18^\circ\sin 54^\circ=a\cdot\sin 6^\circ\cos 6^\circ\Rightarrow b=2a\sin 12^\circ\quad (3)$$

$$\text{In } \Delta AEB\Rightarrow\frac{AB}{\sin 24^\circ}=\frac{b}{\sin 6^\circ}\Rightarrow AB=\frac{b\sin 24^\circ}{\sin 6^\circ}$$

$$\sphericalangle B=\sphericalangle C\Rightarrow AB=AC\Rightarrow BC=\frac{2b\sin^2 24^\circ}{\sin 6^\circ}\quad (4)$$

By the cosine theorem in $\Delta\, BEC$, we have:

$$c^2=b^2+\frac{4b^2\sin^4 24^\circ}{\sin^2 6^\circ}-2b\cdot\frac{2b\sin^2 24^\circ}{\sin 6^\circ}\cdot\cos 48^\circ=$$

$$=b^2+4b^2\sin^2 24^\circ\left(\frac{\sin^2 24^\circ}{\sin^2 6^\circ}-\frac{\cos 48^\circ}{\sin 6^\circ}\right)\overset{(3)}{=}b^2+a^2\cdot 16\sin^2 24^\circ\cdot\sin^2 12^\circ\left(\frac{\sin^2 24^\circ}{\sin^2 6^\circ}-\frac{\cos 48^\circ}{\sin 6^\circ}\right)$$

(*)

Now we prove that:

$$16\sin^2 24^\circ\cdot\sin^2 12^\circ\left(\frac{\sin^2 24^\circ}{\sin^2 6^\circ}-\frac{\cos 48^\circ}{\sin 6^\circ}\right)=1\quad (5)$$

$$16\sin^2 24^\circ\sin^2 12^\circ\cdot\frac{\sin^2 24^\circ-\sin 6^\circ\cos 48^\circ}{\sin^2 6^\circ}=16\sin^2 24^\circ\sin^2 12^\circ\cdot$$

$$\cdot\frac{\frac{1-\cos 48^\circ}{2}-\frac{\sin 54^\circ-\sin 42^\circ}{2}}{\sin^2 6^\circ}=16\sin^2 24^\circ\sin^2 12^\circ\cdot\frac{\sin^2 18^\circ}{\sin^2 6^\circ}=$$

$$=64\sin^2 24^\circ\cos^2 6^\circ\sin^2 18^\circ=64\left(\frac{\sin 30^\circ+\sin 18^\circ}{2}\right)^2\left(\frac{\sqrt{5}-1}{4}\right)^2=$$

$$=64\cdot\frac{\left(\sqrt{5}+1\right)^2}{64}\cdot\frac{\left(\sqrt{5}-1\right)^2}{16}=1\text{, using (5) in (*)}\Rightarrow \text{Q.E.D.}$$

SOLUTION 4.18

Solution by Daniel Sitaru-Romania

$$K-Lemoine's\ point\rightarrow\frac{x}{a}=\frac{y}{b}=\frac{z}{c}\overset{denote}{=}q$$

$$\frac{xr_a+yr_b+zr_c}{x+y+z}=\frac{qar_a+qbr_b+qcr_c}{qa+qb+qc}=\frac{ar_a+br_b+cr_c}{a+b+c}=\frac{1}{2s}\sum_{cyc(a,b,c)}ar_a=$$

$$=\frac{1}{2s}\sum_{cyc(a,b,c)}\frac{aS}{s-a}=\frac{S}{2s}\sum_{cyc(a.b.c)}\frac{a}{s-a}=\frac{rs}{2s}\cdot\frac{2(2R-r)}{r}=2R-r$$

SOLUTION 4.19

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\sqrt{2s-2\sqrt{a(2s-a)}}$$

$$= \sum \sqrt{\left(\sqrt{2s-a}\right)^2 + \left(\sqrt{a}\right)^2 - 2\sqrt{a(2s-a)}}$$

$$= \sum \sqrt{\left(\sqrt{2s-a} - \sqrt{a}\right)^2} \overset{(1)}{=} \sum\left(\sqrt{2s-a} - \sqrt{a}\right)$$

$$\left(\because \sqrt{2s-a} > \sqrt{a} \; as \; 2s = a+b+c > 2a \because b+c > a\right)$$

(1) ⇒ it suffices to prove:

$$\sum \sqrt{b+c} \geq \sqrt{2}\sum\sqrt{a}$$

$$\Leftrightarrow \sum(b+c) + 2\sum\sqrt{(b+c)(c+a)} \geq 2\sum a + 4\sum\sqrt{ab}$$

$$\Leftrightarrow \sum\sqrt{(b+c)(c+a)} \geq 2\sum\sqrt{ab}$$

$$\Leftrightarrow \sum(b+c)(c+a) + 2\sqrt{(a+b)(b+c)(c+a)}\left(\sum\sqrt{a+b}\right)$$

$$\geq 4\sum ab + 8\sqrt{abc}\left(\sum\sqrt{a}\right)$$

$$\Leftrightarrow \sum a^2 + 3\sum ab + 2\sqrt{(a+b)(b+c)(c+a)}\left(\sum\sqrt{a+b}\right)$$

$$\geq 4\sum ab + 8\sqrt{abc}\left(\sum\sqrt{a}\right)$$

$$\Leftrightarrow \sum a^2 + 2\sqrt{(a+b)(b+c)(c+a)}\left(\sum\sqrt{a+b}\right) \overset{(2)}{\geq} \sum ab + 8\sqrt{abc}\left(\sum\sqrt{a}\right)$$

Let $a+b = x, b+c = y, c+a = z$

Then, $x+y > z, y+z > x, z+x > y \Rightarrow x, y, z \rightarrow$ sides of a Δ

we have $\sqrt{x} + \sqrt{y} + \sqrt{z} \overset{(a)}{\geq} \sqrt{y+z-x} + \sqrt{z+x-y} + \sqrt{x+y-z}$

When, x, y, z are sides of a triangle

Re-substituting the values of x, y, z, (a) ⇒

$$\sum\sqrt{a+b} \geq \sum\sqrt{(b+c)+(c+a)-(a+b)} = \sum\sqrt{2c}$$

$$\Rightarrow \sum\sqrt{a+b} \overset{(i)}{\geq} \sqrt{2}\sum\sqrt{a}$$

Also, $2\sqrt{(a+b)(b+c)(c+a)} \underset{(ii)}{\overset{A-G}{\geq}} 2\sqrt{8abc}$

(i).(ii)⇒ $2\sqrt{(a+b)(b+c)(c+a)}\left(\sum\sqrt{a+b}\right) \overset{(iii)}{\geq} 2\sqrt{8abc}\cdot\sqrt{2}\sum\sqrt{a} = 8\sqrt{abc}\left(\sum\sqrt{a}\right)$

Moreover, $\sum a^2 \overset{(iv)}{\geq} \sum ab$***, (iii)+(iv)⇒ (2) is true (Hence proved)***

SOLUTION 4.20

Solution by Daniel Sitaru-Romania

$$I_aI_b = 4R\cos\frac{C}{2}, I_bI_c = 4R\cos\frac{A}{2}, I_cI_a = 4R\cos\frac{B}{2}$$
$$\sphericalangle(I_bVI_c) = \pi - A, \sphericalangle(I_cVI_a) = \pi - B, \sphericalangle(I_aVI_b) = \pi - C$$
$$R_a = \frac{I_bI_c}{2\sin A} = \frac{4R\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{B}{2}} = \frac{R}{\sin\frac{A}{2}}, R_b = \frac{R}{\sin\frac{B}{2}}, R_c = \frac{R}{\sin\frac{C}{2}}$$
$$\sum_{cyc}\frac{1}{R_a^2} = \frac{1}{R^2}\sum_{cyc}\sin^2\frac{A}{2} = \frac{1}{R^2}\left(1 - \frac{r}{2R}\right) = \frac{2R - r}{2R^3}$$

SOLUTION 4.21

Solution by Daniel Sitaru-Romania

$$\rho(I) = R^2 - OI^2 = 2Rr, \ \rho(I) = AI \cdot IK = BI \cdot IL = CI \cdot IM$$

$$2Rr = \frac{r}{\sin\frac{A}{2}} \cdot IK = \frac{r}{\sin\frac{B}{2}} \cdot IL = \frac{r}{\sin\frac{C}{2}} \cdot IM$$

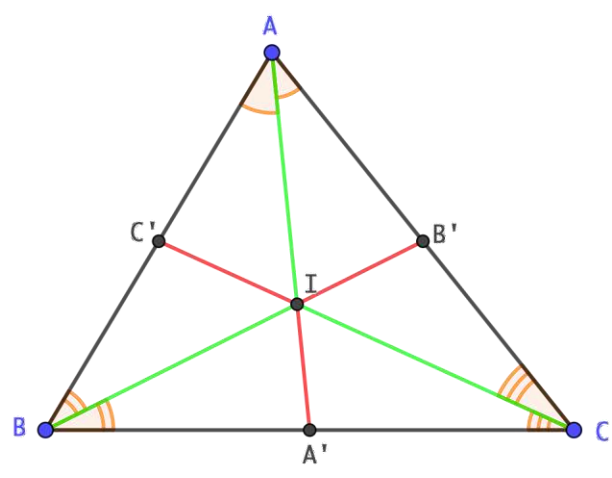
$$IK = 2R\sin\frac{A}{2}, IL = 2R\sin\frac{B}{2}, IM = 2R\sin\frac{C}{2}$$

$$IK \cdot IL \cdot IM = 2R\sin\frac{A}{2} \cdot 2R\sin\frac{B}{2} \cdot 2R\sin\frac{C}{2} =$$

$$= 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 8R^3 \cdot \frac{r}{4R} = 2R^2r = \frac{2R^2s}{s}$$

SOLUTION 4.22

Solution by Soumava Chakraborty-Kolkata-India



Via angle-bisector theorem, on ΔABC,

$$\frac{A'B}{A'C}=\frac{c}{b}\Rightarrow\frac{a}{A'C}=\frac{b+c}{b}\Rightarrow A'C=\frac{ab}{b+c}\Rightarrow A'B=a-\frac{ab}{b+c}\Rightarrow A'B\overset{(1)}{=}\frac{ac}{b+c}$$

Via angle-bisector theorem on $\Delta ABA'$, $\frac{IA}{IA'}\overset{by\,(1)}{=}\frac{c}{\frac{ac}{b+c}}\overset{(a)}{=}\frac{b+c}{a}$

Similarly, $\frac{IB}{IB'}\overset{(b)}{=}\frac{c+a}{b}$ *&* $\frac{IC}{IC'}\overset{(c)}{==}\frac{a+b}{c}$

$\because \frac{IA}{IA'},\frac{IB}{IB'},\frac{IC}{IC'}\in\mathbb{N}^*$*, so, let,* $\frac{b+c}{a}=\sigma_1,\frac{c+a}{b}=\sigma_2$ *&* $\frac{a+b}{c}=\sigma_3$ *(using (a), (b), (c))*

Where $\sigma_1,\sigma_2,\sigma_3\in\mathbb{N}^*-\{1\}$ $\left(\because \frac{b+c}{a},\frac{c+a}{b},\frac{a+b}{c}>1\right)$

$$\therefore b+c\overset{(i)}{=}a\sigma_1,c+a\overset{(ii)}{=}b\sigma_2\ \&\ a+b\overset{(iii)}{=}c\sigma_3$$

(i)+(ii)+(iii)⇒ $2\sum a\overset{(iv)}{=}a\sigma_1+b\sigma_2+c\sigma_3$

If $\sigma_1,\sigma_2,\sigma_3\geq 3$*, then (iv)* $\Rightarrow 2\sum a\geq 3\sum a$*, which is impossible,* $\therefore$ *at least one among* $\sigma_1,\sigma_2,\sigma_3$ *must be* $<3\Rightarrow$ *at least one among* $\sigma_1,\sigma_2,\sigma_3=2$ $(\because \sigma_1,\sigma_2,\sigma_3\in\mathbb{N}-\{1\})$

Case 1) $\sigma_1=2,\sigma_2,\sigma_3\geq 3$ *(i)⇒* $b+c\overset{(2)}{=}2a$

(ii)+(iii) $\overset{by\,(2)}{=}3a=b\sigma_2+c\sigma_3\overset{\sigma_2,\sigma_3\geq 3}{=}3(b+c)\Rightarrow a\geq b+c\rightarrow$ *impossible*

$\Rightarrow$ *at least one of* $\sigma_2,\sigma_3<3\Rightarrow$ *at least one of* $\sigma_2,\sigma_3=2(\because \sigma_2,\sigma_3\geq 2)$

Case 1a) $\sigma_2=2$ *(and of course,* $\sigma_1=2$*)*

(i), (ii)⇒ $b+c=2a\ \&\ c+a=2b\Rightarrow(b+c)-(c+a)=2(a-b)\Rightarrow a=b$

& using $c+a=2b$*, we get* $c=a\therefore a=b=c$

Case 1b) $\sigma_3=2$ *(& of course* $\sigma_1=2$*)*

(i), (iii)⇒ $b+c=2a\ \&\ a+b=2c\Rightarrow(b+c)-(a+b)=2(a-c)\Rightarrow a=c$ *&*

using $a+b=2c$*, we get* $b=a\therefore a=b=c$

Case 2) $\sigma_2=2,\sigma_1,\sigma_3\geq 3$

(ii)⇒ $c+a\overset{(3)}{=}2b\therefore$ *using (3) & (i)+(iii), we get*

$3b=a\sigma_1+c\sigma_3\overset{\sigma_1,\sigma_3\geq 3}{\geq}3(a+c)\Rightarrow b\geq a+c$*, which is impossible ⇒ at least of* $\sigma_1,\sigma_3<3\Rightarrow$ *at least one of* $\sigma_1,\sigma_3=2(\because \sigma_1,\sigma_3\geq 2\ \&\in\mathbb{N}^*)$

Case 2a) $\sigma_1=2$ *(& of course* $\sigma_2=2$*)*

(i), (ii)⇒ $b+c=2a\ \&\ c+a=2b$

$$\Rightarrow (b+c)-(c+a)=2(a-b)\Rightarrow a=b \text{ \& using } c+a=2b\text{, we get,}$$

$$c=a \therefore a=b=c$$

Case 2b) $\sigma_3=2$ *(& of course* $\sigma_2=2$*)*

(ii), (iii) $\Rightarrow c+a=2b \,\&\, a+b=2c$

$$\Rightarrow (c+a)-(a+b)=2(b-c)\Rightarrow b=c \text{ \& using } a+b=2c,$$

we get $a=b \therefore a=b=c$. *Case 3)* $\sigma_3=2,\sigma_1,\sigma_2\geq 3$

(iii)$\Rightarrow a+b\overset{(4)}{=}2c \therefore$ *using (4) & (i)+(ii), we get* $3c=a\sigma_1+b\sigma_2 \overset{\sigma_1,\sigma_3\geq 3}{\geq} 3(a+b)\Rightarrow$

$\Rightarrow c\geq a+b$, *which is impossible* $\Rightarrow$ *at least one of* $\sigma_1,\sigma_2<3\Rightarrow$

at least one of $\sigma_1,\sigma_2=2\ \ (\because \sigma_1,\sigma_2\geq 2 \,\&\in \mathbb{N}^*)$

Case 3a) $\sigma_1=2$ *(& of course* $\sigma_3=2$*). This is same as case 1b)* $\therefore a=b=c.$

Case 3b) $\sigma_2=2$ *(& of course* $\sigma_3=2$*)*

This case is same as case 2b) $\therefore a=b=c$

Combining all cases, we conclude $a=b=c.$

$\therefore h_a=w_a=m_a=h_b=w_b=m_b=h_c=w_c=m_c\ \therefore \Omega=1+1+1=3$ *(answer)*

SOLUTION 4.23

Solution by Fotini Kaldi-Greece

$$\text{Ceva}\Rightarrow \frac{\sin 24}{\sin 42}\cdot\frac{\sin x}{\sin(96-x)}\cdot\frac{\sin 12}{\sin 6}=1\Rightarrow\frac{\sin(84+x)}{\sin x}=\frac{\sin 24}{\sin 42}\cdot\frac{\sin 12}{\sin 6}\Rightarrow$$

$$\frac{\sin(84+x)}{\sin x}=\frac{\sin 54}{\sin 42}\cdot\frac{\sin 24\cdot\sin 12}{\sin 54\cdot\sin 6}\Rightarrow\frac{\sin(84+x)}{\sin x}=\frac{\sin 54}{\sin 42}\cdot\frac{\sin 24\cdot\sin 12}{\cos 36\cdot\sin 6}\Rightarrow$$

$$\frac{\sin 24\cdot\sin 12}{\cos 36\cdot\sin 6}=\frac{2\sin 24\cdot\cos 6}{\cos 36}=\frac{\sin 30+\sin 18}{\cos 36}=1$$

$$\Rightarrow\frac{\sin(84+x)}{\sin x}=\frac{\sin 126}{\sin 42}\Rightarrow x=42, f(x)=\frac{\sin(84+x)}{\sin x}, f'(x)>0$$

SOLUTION 4.24

Solution by Soumava Chakraborty-Kolkata-India

$$\sum a\cos A=\sum(2R\sin A\cos A)=R(\sin 2A+\sin 2B+\sin 2C)$$

$$=R\{2\sin C\cos(A-B)+2\sin C\cos C\}=2R\sin C\{\cos(A-B)-\cos(A+B)\}$$

$$=2R\sin C\cdot 2\sin A\sin B=4R\frac{abc}{8R^3}\overset{(1)}{=}\frac{abc}{2R^2}$$

Now, $b\cos B+c\cos C-a\cos A=R(\sin 2B+\sin 2C-\sin 2A)=$

$$= R\{2\sin A\cos(B-C) + 2\sin A\cos(B+C)\} = 2R\sin A\cdot 2\cos B\cos C$$

$$= 4R\sin A\cos B\cos C$$

$$= 2a\left(\frac{\prod\cos A}{\cos A}\right) \Rightarrow \frac{1}{b\cos B + c\cos C - a\cos A} = \frac{1}{2\cos A\cos B\cos C}\left(\frac{\cos A}{a}\right)$$

$$= \frac{1}{2\cos A\cos B\cos C}\left(\frac{b^2+c^2-a^2}{2abc}\right) \overset{(a)}{=} \frac{b^2+c^2-a^2}{4abcp} \quad \textit{(where } p = \prod\cos A\textit{)}$$

Similarly, $\frac{1}{c\cos C+a\cos A-b\cos B} \overset{(b)}{=} \frac{c^2+a^2-b^2}{4abcp}$ & $\frac{1}{a\cos A+bc\cos B-c\cos C} \overset{(c)}{=} \frac{a^2+b^2-c^2}{4abcp}$

(a)+(b)+(c)$\Rightarrow \sum\frac{1}{bc\cos B+c\cos C-a\cos A} - \frac{1}{\sum a\cos A} \overset{by\,(1)}{=} \frac{\sum a^2}{4abcp} - \frac{2R^2}{abc} = \frac{\sum a^2-8R^2p}{4pabc}$

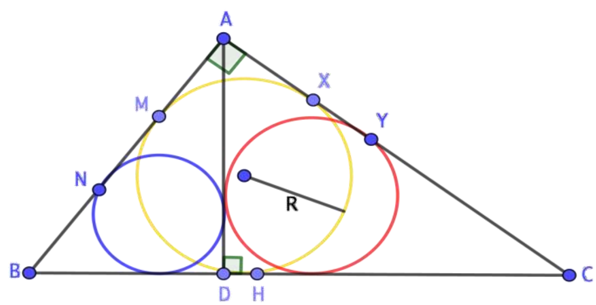
$$= \frac{\sum a^2 - 8R^2\left(\frac{s^2-4R^2-4Rr-r^2}{4R^2}\right)}{4pabc} = \frac{s(s^2-4Rr-r^2)-2(s^2-4R^2-4Rr-r^2)}{4pabc}$$

$$= \frac{8R^2}{4p\cdot 4RS} = \frac{R}{2Sp} = \frac{R}{2S\cos A\cos B\cos C}$$

$$\Rightarrow \sum\frac{1}{b\cos B + c\cos C - a\cos A} = \frac{R}{2S(\cos\prod A)} + \frac{1}{\sum a\cos A}$$

SOLUTION 4.25

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan



Using formula: $x = s - a$*, we have: in* $\Delta ADB \Rightarrow BN = \frac{AB+BD-AD}{2}$,

$$\Delta BAC \Rightarrow BM = \frac{AB+BC-AC}{2}$$

$$MN = BM - BN = \frac{AB+BC-AC-AB-BD+AD}{2} = \frac{\overbrace{BC-BD}^{DC}+AD-AC}{2} = \frac{DC+AD-AC}{2} \quad (1)$$

$$\Delta ADC \Rightarrow CY = \frac{AC+DC-AD}{2},\ \Delta BAC \Rightarrow CX = \frac{AC+BC-AB}{2}$$

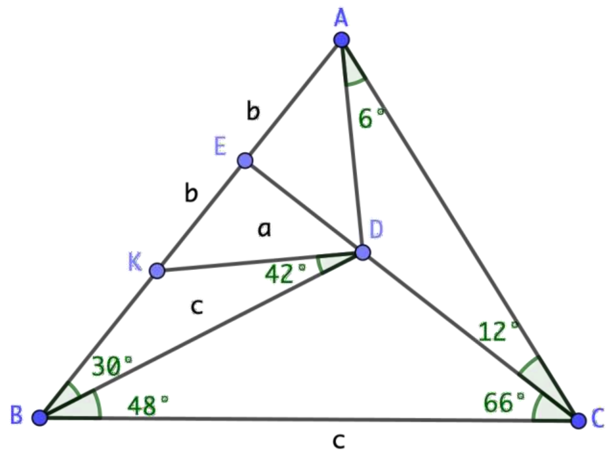
$$XY = \frac{AC+BC-AB-AC-DC+AD}{2} = \frac{\overbrace{BC-DC}^{BD}+AD-AB}{2} = \frac{BD+AD-AB}{2} \quad (2)$$

$$AD-\left(\frac{DC+AD-AC}{2}+\frac{BD+AD-AB}{2}\right)=AD-\frac{BC+2AD-AC-AB}{2}=\frac{AC+AC-BC}{2}=R$$

(using formula $r=\frac{a+b-c}{2}$*) Q.E.D.*

SOLUTION 4.26

Solution by Rovsen Pirguliyev-Sumgait-Azerbaijan



$$\sphericalangle BDC=66^\circ\Rightarrow BD=c$$

$$\sphericalangle EKD=\sphericalangle EDK=72^\circ\Rightarrow EK=ED\ \ (1)$$

$$\sphericalangle EAD=18^\circ,\sphericalangle EDA=18^\circ\Rightarrow AE=ED=b\ \ \ (2)$$

$$\Delta KED\overset{(1)}{\Rightarrow}KE=b$$

$$\sphericalangle KDA=90^\circ,\frac{a}{2b}=\sin 18^\circ\Rightarrow b=\frac{a}{2\sin 18^\circ}\ \ \ (3)$$

$$\text{In }\Delta BKD\Rightarrow\frac{a}{\sin 30^\circ}=\frac{c}{\sin 72^\circ}\Rightarrow c=2a\sin 72^\circ\ \ \ (4)$$

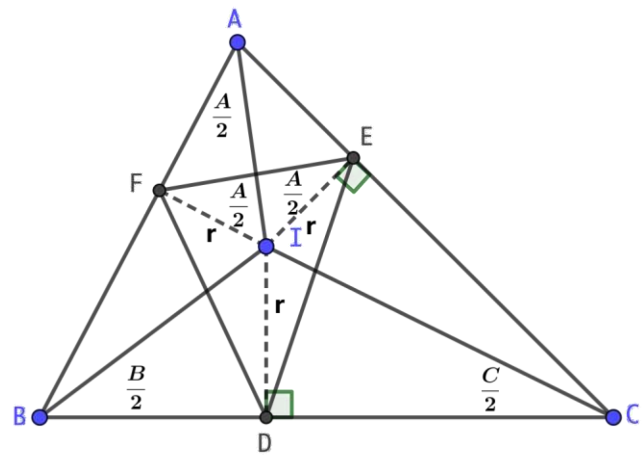
$$a^2+b^2=c^2\Rightarrow a^2+\frac{a^2}{4\sin^2 18^\circ}=4a^2\sin^2 72^\circ\Rightarrow 1+\frac{1}{4\sin^2 18^\circ}=4\cos^2 18^\circ$$

Considering that $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$ *and* $\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}$*, we have:*

$$1+\frac{1}{4\cdot\frac{(\sqrt{5}-1)^2}{16}}=4\cdot\frac{10+2\sqrt{5}}{16},\ 1+\frac{4}{6-2\sqrt{5}}=\frac{10+2\sqrt{5}}{4},\ 1+\frac{6+2\sqrt{5}}{4}=\frac{10+2\sqrt{5}}{4}$$

SOLUTION 4.27

Solution by Lahiru Samarakoon-Sri Lanka



$$\frac{R_a \cdot R_b \cdot R_c}{\varphi_a \cdot \varphi_b \cdot \varphi_c} = \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

Consider, $\Delta AEF,\ 2R_a = \frac{EF}{\sin A} = \frac{2r\cos\frac{A}{2}}{\sin A}$

So, similarly, $R_b = \frac{r\cos\frac{B}{2}}{\sin B}, R_c = \frac{r\cos\frac{C}{2}}{\sin C}$

From, $\Delta BIC,\ 2\varphi_a = \frac{a}{\sin\left(\pi - \frac{B}{2} - \frac{C}{2}\right)} = \frac{a}{\sin\left(\frac{\pi}{2} + \frac{A}{2}\right)} = \frac{a}{\cos\frac{A}{2}}$

Similarly, $\varphi_b = \frac{b}{2\cos\frac{B}{2}}, \varphi_c = \frac{c}{2\cos\frac{C}{2}}$

$$\frac{R_a R_b R_c}{\varphi_a \varphi_b \varphi_c} = \frac{8 \cdot r^3 \prod \cos^2\frac{A}{2}}{abc \prod \sin A} \quad \left(r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= \frac{8 \times (4R)^3 \prod \cos^2\frac{A}{2} \cdot \prod \sin^3\frac{A}{2}}{8R^3 \prod \sin^2 A} = \frac{4^3 \prod \cos^2\frac{A}{2} \cdot \prod \sin^3\frac{A}{2}}{4^3 \cdot \prod \cos^2\frac{A}{2} \cdot \prod \sin^2\frac{A}{2}} = \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}$$

SOLUTION 4.28

Solution by Tran Hong-Vietnam

Let $f(t) = \sqrt[5]{t}\,(t > 0) \Rightarrow f''(t) = -\frac{4}{25}t^{-\frac{9}{5}} < 0\,(t > 0);$

Using Jensen's inequality, we have:

$$LHS \leq 3\sqrt[5]{\frac{2\left(\frac{s-a}{c} + \frac{s-b}{a} + \frac{s-c}{b}\right)}{3}} = \Phi$$

WLOG, suppose: $a \geq b \geq c$. We must show that:

$$\Phi \leq 3 \Leftrightarrow \frac{s-a}{c}+\frac{s-b}{a}+\frac{s-c}{b} \leq \frac{3}{2} \Leftrightarrow \frac{b+c-a}{2c}+\frac{a+c-b}{2a}+\frac{a+b-c}{2b} \leq \frac{3}{2}$$

$$\Leftrightarrow \frac{b-a}{c}+\frac{c-b}{a}+\frac{a-c}{b} \leq 0 \Leftrightarrow \frac{a}{b}-\frac{b}{a}+\frac{b}{c}-\frac{c}{b}+\frac{c}{a}-\frac{a}{c} \leq 0$$

$$\Leftrightarrow \frac{a^2-b^2}{ab}+\frac{b^2-c^2}{cb}+\frac{c^2-a^2}{ac} \leq 0$$

$$\Leftrightarrow c(a^2-b^2)+a(b^2-c^2)+b(c^2-a^2) \leq 0 \Leftrightarrow ca^2-cb^2+ab^2-ac^2+bc^2-ba^2 \leq 0$$

$$\Leftrightarrow (a-c)[b(b-a)-c(b-a)] \leq 0 \Leftrightarrow (a-c)(b-c)(b-a) \leq 0$$

(True: $a-c \geq 0; b-c \geq 0, b-a \leq 0$)

SOLUTION 4.29

Solution by Rozeta Atanasova-Skopje-Macedonia

Equifacial tetrahedrons exist only when the faces are congruent acute triangles, and then

$$R=\sqrt{\frac{a^2+b^2+c^2}{8}} \Rightarrow$$

$$\cos A>0, \cos B>0, \cos C>0 \Rightarrow$$

$$LHS=8(4R^2-a^2)(4R^2-b^2)(4R^2-c^2)$$

$$=8 \cdot \frac{b^2+c^2-a^2}{2} \cdot \frac{a^2+c^2-b^2}{2} \cdot \frac{a^2+b^2-c^2}{2}$$

$$=(b^2+c^2-a^2)(a^2+c^2-b^2)(a^2+b^2-c^2)=$$

$$=8a^2b^2c^2\cos A\cos B\cos C \overset{AM-GM}{\leq} 8a^2b^2c^2\left(\frac{\cos A+\cos B+\cos C}{3}\right)^3$$

$$\overset{Jensen}{\leq} 8a^2b^2c^2\left(\cos\frac{A+B+C}{3}\right)^3=8a^2b^2c^2\left(\frac{1}{2}\right)^3=a^2b^2c^2=RHS$$

SOLUTION 4.30

Solution by Soumava Chakraborty – Kolkata – India

In any scalene acute – angled $\Delta\ ABC$,

$$\sqrt{\sum(\sin A)^{2\cos A}}+\sqrt{\sum(\cos A)^{2\sin A}}>\sqrt{3}$$

$$\{(\sin A)^{\cos A}\}^2 + \{(\sin B)^{\cos B}\}^2 + \{(\sin C)^{\cos C}\}^2 > \frac{1}{3}(\sin A^{\cos A} + \sin B^{\cos B} + \sin C^{\cos C})^2$$

$$\left(\because \sum x^2 > \frac{1}{3}\left(\sum x\right)^2, if\ x \neq y \neq z\right)$$

$$\therefore \sqrt{\sum(\sin A)^{2\cos A}} > \frac{1}{\sqrt{3}}(\sin A^{\cos A} + \sin B^{\cos B} + \sin C^{\cos C}) \quad (1)$$

Again, $\{(\cos A)^{\sin A}\}^2 + \{(\cos B)^{\sin B}\}^2 + \{(\cos C)^{\sin C}\}^2$

$$> \frac{1}{3}(\cos A^{\sin A} + \cos B^{\sin B} + \cos C^{\sin C})^2$$

$$\therefore \sqrt{\sum(\cos A)^{2\sin A}} > \frac{1}{\sqrt{3}}(\cos A^{\sin A} + \cos B^{\sin B} + \cos C^{\sin C}) \quad (2)$$

(∵ Δ ABC is acute – angled, ∴ $\cos A, \cos B, \cos C > 0$*)*

$$(1) + (2) \Rightarrow LHS \overset{(i)}{>}$$

$$> \frac{1}{\sqrt{3}}\{(\cos A^{\sin A} + \sin A^{\cos A}) + (\cos B^{\sin B} + \sin B^{\cos B}) + (\cos C^{\sin C} + \sin C^{\cos C})\}$$

Now, $(\ln \sin A)(\cos A - 2) > 0$

(∵ $\ln \sin A < 0$ *as* $\sin A < 1$ *and* $\cos A - 2 < 0$ *as* $\cos A < 1 < 2$*)*

$$\Rightarrow \cos(\ln \sin A) > 2 \ln \sin A$$

$$\Rightarrow \ln(\sin A^{\cos A}) > \ln(\sin^2 A) \Rightarrow \sin A^{\cos A} > \sin^2 A \quad (3)$$

Also, $(\ln \cos A)(\sin A - 2) > 0$

(∵ $\ln \cos A < 0$ *as* $\cos A < 1$ *and* $\sin A - 2 < 0$ *as* $\sin A < 1 < 2$*)*

$$\Rightarrow \sin A\,(\ln \cos A) > 2 \ln \cos A$$

$$\Rightarrow \ln(\cos A^{\sin A}) > \ln(\cos^2 A) \Rightarrow \cos A^{\sin A} > \cos^2 A \quad (4)$$

$$(3) + (4) \Rightarrow \cos A^{\sin A} + \sin A^{\cos A} \overset{5}{>} \cos^2 A + \sin^2 A = 1$$

Similarly, $\cos B^{\sin B} + \sin B^{\cos B} > 1 \quad (6)$

and,

$$\cos C^{\sin C} + \sin C^{\cos C} > 1 \quad (7)$$

$$(5) + (6) + (7) \Rightarrow LHS > \frac{1}{\sqrt{3}}(1 + 1 + 1) \quad (\text{from } (i)) = \frac{3}{\sqrt{3}} = \sqrt{3}$$

SOLUTION 4.31

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}, \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$$

Por la desigualdad de Cauchy

$$\frac{\sin^2 A}{\sin^{-1}\frac{4}{5}}+\frac{\sin^2 B}{\sin^{-1}\frac{5}{13}}+\frac{\sin^2 C}{\sin^{-1}\frac{16}{65}}\geq\frac{\left(\frac{a+b+c}{2R}\right)^2}{\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}}=\frac{2s^2}{\pi R^2}$$

$$\frac{\sin^2 A}{\tan^{-1}\frac{1}{2}}+\frac{\sin^2 B}{\tan^{-1}\frac{1}{5}}+\frac{\sin^2 C}{\tan^{-1}\frac{1}{8}}\geq\frac{\left(\frac{a+b+c}{2R}\right)^2}{\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}}=\frac{4s^2}{\pi R^2}$$

SOLUTION 4.32

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{h_a}{\sin\frac{A}{2}}=\sum\frac{h_a w_a}{\frac{2bc}{b+c}\cos\frac{A}{2}\sin\frac{A}{2}}=\sum\frac{h_a w_a}{\frac{abc}{2R(b+c)}}=$$

$$=\frac{2R}{4Rrs}\sum h_a w_a(b+c)=\frac{1}{2rs}\sum\frac{2rs}{a}w_a(b+c)\overset{(1)}{=}\sum\left(\frac{b+c}{a}\right)w_a$$

By Bogdan Fustei (see proof below):

$$\sum 3(h_a-2r)w_a\geq r\left(2\sum m_a+\sum w_a\right)$$

$$\sum\left(3\left(\frac{2S}{a}-\frac{2S}{s}\right)w_a\right)\geq r\left(2\sum m_a+\sum w_a\right)$$

$$\sum\left(6rsw_a\left(\frac{1}{a}-\frac{1}{s}\right)\right)\geq 2r\sum m_a+r\sum w_a$$

$$\sum\frac{3r(b+c-a)w_a}{a}\geq 2r\sum m_a+r\sum w_a$$

$$3\sum\frac{(b+c)w_a}{a}\overset{(2)}{\geq}2\sum m_a+4\sum w_a$$

By $(1),(2)$: $\sum \frac{h_a}{\sin\frac{A}{2}} \geq \frac{2}{3}\sum m_a + \frac{4}{3}\sum w_a$

$$\sum 3\,(h_a - 2r)w_a = \sum\left\{3\left(\frac{2\Delta}{a} - \frac{2\Delta}{s}\right)w_a\right\} = \sum\left\{3\cdot 2rs\left(\frac{1}{a} - \frac{1}{s}\right)w_a\right\} =$$

$$= 3r\left\{\sum\frac{2(s-a)}{a}w_a\right\} = 3r\left(\sum\frac{b+c-a}{a}w_a\right) = 3r\sum\frac{b+c}{a}w_a - 3r\sum w_a \geq$$

$$\geq r\left(2\sum m_a + \sum w_a\right) \Leftrightarrow 3\sum\frac{b+c}{a}w_a \overset{(1)}{\geq} 4\sum w_a + 2\sum m_a$$

$$\left.\begin{array}{c} Now, \left(\sum m_a\right)^2 \overset{(a)}{\leq} 4s^2 - 16Rr + 5r^2 (X.G.Chu, X.Z.Yang), \\ \left(\sum w_a\right)^2 \overset{(b)}{\leq} (4R + r)\left(\sum h_a\right)\ (Bogdan\ Fustei), \\ \sum\left(\frac{b+c}{a}\right)w_a \overset{(c)}{\geq} 2s\sqrt{3}\ (Bogdan\ Fustei) \end{array}\right\}$$

Now, $2\sum w_a \leq 2\sum\sqrt{s(s-a)} \overset{C-B-S}{\leq} 2\sqrt{3}\sqrt{S}\sqrt{S} = 2\sqrt{3}S \overset{by\ (c)}{\leq} \sum\left(\frac{b+c}{a}\right)w_a \Rightarrow$

$$\Rightarrow 2\sum w_a \overset{(i)}{\leq} \sum\left(\frac{b+c}{a}\right)w_a$$

(i) $\Rightarrow$ ***in order to prove (1), it suffices to prove:*** $2\sum\frac{b+c}{a}w_a \geq 2\sum w_a + 2\sum m_a \Leftrightarrow$

$\Leftrightarrow \sum\frac{b+c}{a}w_a \overset{(2)}{\geq} \sum w_a + \sum m_a$. ***Now, LHS of (2)*** $\underset{(m)}{\overset{CBS}{\leq}} \sqrt{2}\sqrt{(\sum w_a)^2 + (\sum m_a)^2} \leq$

$$\overset{by\ (a),(b)}{\leq} \sqrt{2}\sqrt{\frac{(4R + r)(s^2 + 4Rr + r^2)}{2R} + 4s^2 - 16Rr + 5r^2} =$$

$$= \sqrt{\frac{(4R + r)(s^2 + 4Rr + r^2) + 2R(4s^2 - 16Rr + 5r^2)}{R}}$$

Again, LHS of (2) $\overset{by\ (c)}{\geq} 2s\sqrt{3}$

(m), (n) $\Rightarrow$ ***in order to prove (2), it suffices to prove:***

$$2s\sqrt{3}\geq\sqrt{\frac{(12R+r)s^2+r(4R+r)^2-2R(16Rr-5r^2)}{R}}\Leftrightarrow 12Rs^2\geq 12rs^2+rs^2+$$

$$+r(4R+r)^2-2R(16Rr-5r^2)\Leftrightarrow r\{2R(16R-5r)-(4R+r)^2\}\geq rs^2\Leftrightarrow$$

$$\Leftrightarrow s^2\overset{(3)}{\leq}16R^2-18Rr-r^2$$

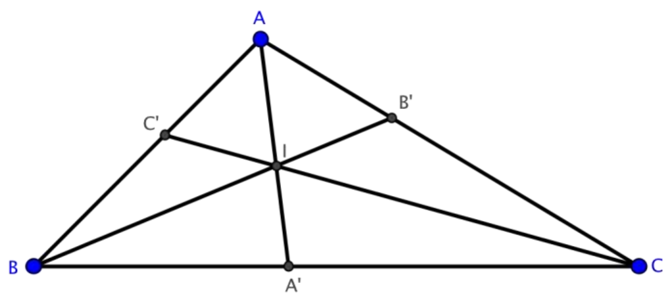
Now, LHS of (3) $\overset{Gerretsen}{\leq} 4R^2+4Rr+3r^2\overset{?}{\leq}16R^2-18Rr-r^2\Leftrightarrow$

$\Leftrightarrow 6R^2-11Rr-2r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(6R+r)\overset{?}{\geq}0\rightarrow$ ***true*** $\because R\overset{Euler}{\geq}2r\Rightarrow$

$\Rightarrow$ ***(3) is true*** $\Rightarrow$ ***(2) is true (proved)***

SOLUTION 4.33

Solution by Marian Ursărescu – Romania



$$\frac{BA'}{A'C}=\frac{c}{b}\Rightarrow\frac{BA'}{BC}=\frac{c}{b+c}\Rightarrow BA'=\frac{ac}{b+c}$$

$\frac{AI}{IA'}=\frac{c}{BA'}=\frac{c}{\frac{ac}{b+c}}\Rightarrow\frac{AI}{IA'}=\frac{b+c}{a}\Rightarrow$ ***we must show:***

$\frac{b+c}{a}\cdot m_a^2+\frac{a+c}{b}m_b^2+\frac{a+b}{c}m_c^2\geq 2(m_am_b+m_bm_c+m_cm_a)$ ***(1)***

Let $a\leq b\leq c\Rightarrow m_a\geq m_b\geq m_c$ ***and*** $\frac{b+c}{a}\geq\frac{a+c}{b}\geq\frac{a+b}{c}\Rightarrow$

From Cebyshev's inequality $\Rightarrow$

$\frac{b+c}{a}m_a^2+\frac{a+c}{b}m_b^2+\frac{a+b}{c}m_c^2\geq\frac{1}{3}\left(\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\right)(m_a^2+m_b^2+m_c^2)$ ***(2)***

From (1)+(2) we must show:

$$\frac{1}{3}\left(\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}\right)\left(m_a^2+m_b^2+m_c^2\right) \geq 2(m_a m_b+m_b m_c+m_c m_a) \quad (3)$$

$$\textit{But } \frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c}=\frac{b}{a}+\frac{a}{b}+\frac{c}{a}+\frac{a}{c}+\frac{b}{c}+\frac{c}{b} \geq 6 \quad (4)$$

From (3)+(4) we must show:

$$2\left(m_a^2+m_b^2+m_c^2\right) \geq 2(m_a m_b+m_b m_c+m_b m_c) \Leftrightarrow$$

$$\Leftrightarrow m_a^2+m_b^2+m_c^2 \geq m_a m_b+m_b m_c+m_b m_c \quad \textit{true.}$$

SOLUTION 4.34

Solution by Ravi Prakash-New Delhi-India

$$3a^2+2b^2-c^2=$$

$$=2a^2+b^2+\left(a^2+b^2-c^2\right)=2a^2+b^2\left(\cos^2 c+\sin^2 c\right)+2ab\cos c=$$

$$=a^2+b^2\cos^2 c+2ab\cos c+a^2+b^2\sin^2 c-2ab\sin c+2ab\sin c=$$

$$=(a+b\cos c)^2+(a-b\sin c)^2+2ab\sin c \geq 2ab\sin c=4S$$

SOLUTION 4.35

Solution by Marian Ursărescu-Romania

$$w_a=\frac{2bc}{b+c}\cdot\sqrt{\frac{s(s-a)}{bc}} \leq \sqrt{bc}\cdot\sqrt{\frac{s(s-a)}{bc}}=\sqrt{s(s-a)} \Rightarrow \textit{we must show:}$$

$$\frac{bc}{a\sqrt{s(s-a)}}+\frac{ac}{b\sqrt{s(s-b)}}+\frac{ab}{c\sqrt{s(s-c)}} \geq \frac{18r}{s} \Leftrightarrow$$

$$\frac{bc}{a\sqrt{s-a}}+\frac{ac}{b\sqrt{s-b}}+\frac{ab}{c\sqrt{s-c}} \geq \frac{18\sqrt{(s-a)(s-b)(s-c)}}{s} \quad (1)$$

$$\textit{Now, let } s-a=x, s-b=y, s-c=z \Rightarrow x+y+z=s \textit{ and}$$

$$a=y+t, b=x+z \textit{ and } c=x+y \ (2)$$

From (1)+(2) we must show: $\sum \frac{(x+z)(x+y)}{(y+z)\sqrt{x}} \geq \frac{18\sqrt{xyz}}{x+y+z}$ ***(3)***

But $x+z \geq 2\sqrt{xz}$ ***and*** $x+y \geq 2\sqrt{xy}$ ***(4)***

From (3)+(4) we must show: $\sum \frac{4\sqrt{xy}\cdot\sqrt{xz}}{(y+z)\sqrt{x}} \geq \frac{18\sqrt{xyz}}{x+y+z} \Leftrightarrow$

$$\sum \frac{1}{y+z} \geq \frac{9}{2(x+y+z)} \quad (5)$$

But $\sum \frac{1}{y+z} \cdot \sum(y+z) \geq 9 \Leftrightarrow \sum \frac{1}{y+z} \geq \frac{9}{2(x+y+z)} \Rightarrow$ ***(5) its true.***

SOLUTION 4.36

Solution by Daniel Sitaru-Romania

$$\sum \frac{a}{bc+r^2} = \sum \frac{a^2}{abc+ar^2} \overset{BERGSTROM}{\geq} \frac{(a+b+c)^2}{3abc+(a+b+c)r^2} = \frac{4s^2}{12Rrs+2sr^2} =$$
$$= \frac{2s}{6Rr+r^2} \overset{MITRINOVIC}{\geq} \frac{6\sqrt{3}r}{r(6R+r)} \overset{EULER}{\geq} \frac{6\sqrt{3}}{6R+\frac{R}{2}} = \frac{12\sqrt{3}}{13R}$$

SOLUTION 4.37

Solution by Soumava Chakraborty-Kolkata-India

By Bogdan Fustei, $\frac{b+c}{2} \overset{(1)}{\geq} \sqrt{2r(r_b+r_c)}$***. Proof of (1):***

$$2r(r_b+r_c) = 2rs\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = 2rs\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = 2rs\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\frac{B}{2}\cos\frac{C}{2}} =$$

$= \frac{2rs\cos^2\frac{A}{2}}{\prod\cos\frac{A}{2}} = \frac{2rs\cos^2\frac{A}{2}}{\frac{s}{4R}} \overset{(a)}{=} 8Rr\cos^2\frac{A}{2}$***. Using (a), (1)*** $\Leftrightarrow \frac{(b+c)^2}{4} \geq 8Rr\cos^2\frac{A}{2} \Leftrightarrow$

$$\Leftrightarrow \frac{(b+c)^2}{4} \geq \frac{8abc}{4\Delta}\cdot\frac{\Delta}{s}\cdot\frac{s(s-a)}{bc} = 2a(s-a) \Leftrightarrow (b+c)^2 \geq 4a(b+c-a) \Leftrightarrow$$

$\Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow$ ***true*** $\Rightarrow$ ***(1) is true. Now,*** $\frac{b+c}{a}w_a \overset{by\ (1)}{\geq} 2\sqrt{2r(r_b+r_c)}\frac{2bc}{a(b+c)}\cos\frac{A}{2}$

$$\stackrel{by\ (a)}{=} 2\sqrt{8Rr\cos^2\frac{A}{2}\cdot\frac{2bc}{a(b+c)}}\cdot\cos\frac{A}{2} = 8\sqrt{2Rr}\cdot\frac{bc\,s(s-a)}{a(b+c)bc} = 4s\sqrt{2Rr}\cdot\frac{b+c-a}{a(b+c)} =$$

$$= 4s\sqrt{2Rr}\left(\frac{1}{a}-\frac{1}{b+c}\right) \Rightarrow \frac{b+c}{a}w_a \stackrel{(i)}{\geq} 4s\sqrt{2Rr}\left(\frac{1}{a}-\frac{1}{b+c}\right).$$ ***Similarly,***

$$\frac{c+a}{b}w_b \stackrel{(ii)}{\geq} 4s\sqrt{2Rr}\left(\frac{1}{b}-\frac{1}{c+a}\right) \;\&\; \frac{a+b}{c}w_c \stackrel{(iii)}{\geq} 4s\sqrt{2Rr}\left(\frac{1}{c}-\frac{1}{a+b}\right)$$

(i)+(ii)+(iii) $\Rightarrow \sum\frac{b+c}{a}w_a \geq 4s\sqrt{2Rr}\left(\sum\frac{1}{a}-\sum\frac{1}{a+b}\right) = 4s\sqrt{2Rr}\left\{\frac{\sum ab}{4Rrs}-\frac{\sum(b+c)(c+a)}{\prod(a+b)}\right\}$

$$= 4s\sqrt{2Rr}\left\{\frac{\sum ab}{4Rrs}-\frac{\sum a^2+3\sum ab}{2abc+\sum ab\,(2s-c)}\right\} =$$

$$= 4s\sqrt{2Rr}\left\{\frac{s^2+4Rr+r^2}{4Rrs}-\frac{4s^2+s^2+4Rr+r^2}{2s(s^2+4Rr+r^2-2Rr)}\right\} =$$

$$= 4s\sqrt{2Rr}\left\{\frac{(s^2+4Rr+r^2)(s^2+2Rr+r^2)-2Rr(5s^2+4Rr+r^2)}{4Rrs(s^2+2Rr+r^2)}\right\}$$

$$= 4s\sqrt{2Rr}\frac{\{s^4-s^2(4Rr-2r^2)+r^2(4R+r)(2R+r)-r^2\cdot 2R(4R+r)\}}{4Rrs(s^2+2Rr+r^2)}$$

$$= \sqrt{2Rr}\left\{\frac{s^4-s^2(4Rr-2r^2)+r^3(4R+r)}{Rr(s^2+2Rr+r^2)}\right\} \stackrel{?}{\geq} 2s\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow \frac{2Rr\{s^4-s^2(4Rr-2r^2)+r^3(4R+r)\}^2}{R^2r^2(s^2+2Rr+r^2)^2} \stackrel{?}{\geq} 4s^2\cdot 3 \Leftrightarrow$$

$$\Leftrightarrow s^8+s^4(4Rr-2r^2)^2+r^6(4R+r)^2-2s^6(4Rr-2r^2)-$$

$$-2s^2r^3(4Rr-2r^2)(4R+r)+2s^4r^3(4R+r) \stackrel{?}{\geq}$$

$$\geq 6s^2Rr\{s^4+r^2(2R+r)^2+2s^2(2Rr+r^2)\} = 6s^6Rr+6s^2Rr^3(2R+r)^2+$$

$$+12s^4Rr(2Rr+r^2) \Leftrightarrow s^8-s^6(14Rr-4r^2)+r^6(4R+r)^2 \underset{(2)}{\stackrel{?}{\geq}}$$

$$\geq s^4r^2(8R^2+20Rr-6r^2)+2s^2r^3[3R(2R+r)^2+(4Rr-2r^2)(4R+r)]$$

$$\because s^8 = s^6 s^2 \overset{Gerretsen}{\geq} s^6(16Rr - 5r^2),$$ $\therefore$ ***LHS of (2)*** $\geq s^6(2Rr - r^2) + r^6(4R + r)^2 \geq$

$$\underset{(3)}{\overset{?}{\geq}} s^4 r^2(8R^2 + 20Rr - 6r^2) + 2s^2 r^3[3R(2R + r)^2 + (4Rr - 2r^2)(4R + r)]$$

Again, LHS of (3) $\overset{Gerretsen}{\geq} s^4(16Rr - 5r^2)(2Rr - r^2) + r^6(4R + r)^2 \overset{?}{\geq}$

$$\geq s^4 r^2(8R^2 + 20Rr - 6r^2) + 2s^2 r^3\,[3R(2R + r)^2 + (4Rr - 2r^2)(4R + r)] \Leftrightarrow$$

$$\Leftrightarrow s^4(24R^2 - 46Rr + 11r^2) + r^4(4R + r)^2 \geq$$

$\underset{(4)}{\overset{?}{\geq}} 2s^2 r[3R(2R + r)^3 + (4Rr - 2r^2)(4R + r)]$. ***Now, LHS of (4)*** $\overset{Gerretsen}{\geq}$

$$\geq s^2(16Rr - 5r^2)(24R^2 - 46Rr + 11r^2) + r^4(4R + r)^2 \overset{?}{\geq}$$

$$\geq s^2 r[6R(2R + r)^3 + (8Rr - 4r^2) \cdot (4R + r)] \Leftrightarrow$$

$$\Leftrightarrow s^2(360R^3 - 912R^2 r + 408Rr^2 - 51r^3) + r^3(4R + r)^2 \overset{?}{\geq} 0 \Leftrightarrow$$

$$\Leftrightarrow s^2(360R^3 - 912R^2 r + 408Rr^2) + r^3(4R + r)^2 \underset{(5)}{\overset{?}{\geq}} 51r^3 s^2$$

$$\because 360R^3 - 912R^2 r + 408Rr^2 = (R - 2r)(360R^2 - 192Rr + 24r^2) + 48r^3 > 0$$

(as $R \geq 2r$***)***

$\therefore$ ***LHS of (5)*** $\overset{Gerretsen}{\geq} (16Rr - 5r^2)(360R^3 - 912R^2 r + 408Rr^2) + r^3(4R + r)^2$ ***&***

RHS of (5) $\overset{Gerretsen}{\leq} 51r^3(4R^2 + 4Rr + 3r^2)$ $\therefore$ ***in order to prove (5), it suffices to prove:***

$$(16R - 5r)(360R^3 - 912R^2 r + 408Rr^2) + r^2(4R + r)^2 \geq 51r^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow$$

$$\Leftrightarrow 1440t^4 - 4098t^3 + 2725t^2 - 559t - 38 \geq 0 \quad \left(t = \frac{R}{r}\right)$$

$\Leftrightarrow (t - 2)\{(t - 2)(1440t^2 + 1662t + 3613) + 7245\} \geq 0 \rightarrow$ ***true***

$$\because t \overset{Euler}{\geq} 2 \Rightarrow \textit{(5) is true (Hence proved)}$$

SOLUTION 4.38

Solution by Soumava Chakraborty-Kolkata-India

$$\sum h_b h_c \cos\frac{A}{2} \leq \frac{\sqrt{3}}{2}\left(\sum h_a^2\right)$$

$h_b h_c \geq h_c h_a \Leftrightarrow h_a \leq h_b \Leftrightarrow a \geq b$. *Similarly,* $h_c h_a \geq h_a h_b \Leftrightarrow h_b \leq h_c \Leftrightarrow b \geq c$. *WLOG, we may assume* $a \geq b \geq c$. *Then,* $h_b h_c \geq h_c h_a \geq h_a h_b$ & $\cos\frac{A}{2} \leq \cos\frac{B}{2} \leq \cos\frac{C}{2}$

$$\therefore \sum h_b h_c \cos\frac{A}{2} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum h_b h_c\right)\left(\sum \cos\frac{A}{2}\right) \leq \left(\frac{1}{3}\sum h_a^2\right)\left(\sum \cos\frac{A}{2}\right) \overset{Jensen}{\leq}$$

$$\leq \frac{\sum h_a^2}{3} \cdot \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}\left(\sum h_a^2\right)}{2} \ \textit{(proved)}$$

SOLUTION 4.39

Solution by Daniel Sitaru-Romania

$$\sum\left(\frac{1}{r_a} + \frac{1}{h_a}\right)m_a^2 = \sum\left(\frac{s-a}{S} + \frac{a}{2S}\right)m_a^2 = \frac{1}{2S}\sum(b+c)m_a^2 =$$

$$= \frac{1}{2S} \cdot \frac{s}{2}\left(5s^2 - 11r^2 - 26Rr\right) \overset{GERRETSEN}{\gtreqless} \frac{s}{4S}\left(80Rr - 25r^2 - 11r^2 - 26Rr\right) =$$

$$= \frac{1}{4r}\left(54Rr - 36r^2\right) = \frac{1}{2}(24R + 3R - 18r) \overset{EULER}{\gtreqless} \frac{1}{2}(24R + 6r - 18r) = 12R - 6r$$

SOLUTION 4.40

Solution by Marian Ursărescu-Romania

In any ΔABC *we have* $(h_a + h_b)(h_b + h_c)(h_c + h_a) = \frac{s^2 r\left(s^2 + r^2 + 2Rr\right)}{R^2}$ *(1)*

(where $s = \frac{a+b+c}{2}$*). From (1) we must show:* $\frac{s^2 r\left(s^2+r^2+2Rr\right)}{27R^2} \geq \frac{16r^4}{R} \Leftrightarrow$

$$s^2\left(s^2 + r^2 + 2Rr\right) \geq 27 \cdot 16Rr^3 \ \textit{(2)}$$

From Mitrinovic inequality $s^2 \geq 27r^2$ *(3)*

From (2) + (3) we must show: $s^2+r^2+2Rr\geq 16Rr$ *(4)*

From Gerretsen inequality we have:

$$s^2\geq 16Rr-5r^2\Rightarrow s^2+r^2+2Rr\geq 18Rr-4r^2 \text{ (5)}$$

From (4) + (5) we must show: $18Rr-4r^2\geq 16Rr\Leftrightarrow 2Rr\geq 4r^2\Leftrightarrow R\geq 2r$, *true because its Euler. From (1) we must show:*

$$\frac{s^2r(s^2+r^2+2Rr)}{27R^2}\leq R^3\Leftrightarrow s^2r(s^2+r^2+2Rr)\leq 27R^5 \text{ (6)}$$

From Gerretsen inequality we have: $s^2\leq 4R^2+4Rr+3r^2\Rightarrow s^2+r^2+2Rr\leq$

$$\leq 4R^2+6Rr+4r^2\leq 4R^2+3R^2+R^2=8R^2 \text{ (7)}$$

Form (6) + (7) we must show: $8R^2s^2r\leq 27R^5\Leftrightarrow 8s^2r\leq\frac{27R^3}{7}$*. But from Mitrinovic inequality:* $s^2\leq\frac{27R^2}{4}$ *and* $r\leq\frac{R}{2}\Rightarrow 8s^2r\leq 8\cdot\frac{27}{4}R^2\cdot\frac{R}{r}=27R^3\Rightarrow (7)$ *its true.*

SOLUTION 4.34

Solution by Soumava Chakraborty-Kolkata-India

$$\sum(a+r+r_a)^2=\sum\left(a^2+r^2+r_a^2+2ar+2rr_a+2ar_a\right)\overset{(1)}{=}2\left(s^2-4Rr-r^2\right)+3r^2+$$

$$+(4R+r)^2-2s^2+2r(2s)+2r(4R+r)+2\sum ar_a. \text{ Now,}$$

$$2\sum ar_a=2\sum 4R\sin\frac{A}{2}\cos\frac{A}{2}s\tan\frac{A}{2}=4RS\sum 2\sin^2\frac{A}{2}=4RS\sum(1-\cos A)=$$

$$=4RS\left(3-1-\frac{r}{R}\right)=4RS\left(\frac{2R-r}{R}\right)\overset{(2)}{=}4S(2R-r)$$

(1),(2)$\Rightarrow\sum(a+r+r_a)^2=-2\left(4Rr+r^2\right)+3r^2+(4R+r)^2+4rs+2\left(4Rr+r^2\right)+$

$$+8Rs-4rs\overset{(3)}{=}3r^2+(4R+r)^2+8RS\overset{Euler}{\geq}3r^2+81r^2+16rs\overset{s\geq 3\sqrt{3}r}{\geq}$$

$$84r^2+48\sqrt{3}r^2=$$

$$= 12r^2(7+4\sqrt{3}) = 12r^2(2+\sqrt{3})^2 = 12r^2\frac{1}{(2-\sqrt{3})^2} = 12r^2\tan^2 75°$$

$$\left(\because \tan 75° = \cot 15° = \frac{1}{2-\sqrt{3}}\right)$$

(3) $\Rightarrow \sum(a+r+r_a)^2 \overset{Euler}{\leq} 3\frac{R^2}{4} + \left(\frac{9R}{2}\right)^2 + 8R\cdot\frac{3\sqrt{3}R}{2} = 21R^2 + 12\sqrt{3}R^2 = 3R^2(7+4\sqrt{3})$

$= 3R^2(2+\sqrt{3})^2 = 3R^2\left(\frac{1}{2-\sqrt{3}}\right)^2 = 3R^2\cot^2 15° = 3R^2\tan^2 75°$ *(Done)*

SOLUTION 4.42

Solution by Soumava Chakraborty-Kolkata-India

$$\sum(s-a)\sin\frac{A}{2} \leq \frac{S\left(\sum r_a^2\right)}{2r_ar_br_c}$$

$$LHS \leq \sum\left(\sqrt{(s-a)(s-b)(s-c)}\sqrt{\frac{s-a}{bc}}\right) = \sum\frac{rs}{\sqrt{S}}\sqrt{\frac{a(s-a)}{4Rs}} = \frac{r}{\sqrt{4Rr}}\sum\sqrt{a(s-a)}$$

$$\underset{(1)}{\overset{C-B-S}{\leq}} \frac{r}{\sqrt{4Rr}}\sqrt{\sum a}\sqrt{\sum(s-a)} = \frac{rs}{\sqrt{2Rr}}$$

$$RHS \overset{(2)}{\geq} \frac{rs\left(\sum r_ar_b\right)}{2rs^2} = \frac{rs^3}{2rs^2} = \frac{S}{2}$$

(1), (2) $\Rightarrow$ *it suffices to prove:* $\frac{1}{2} \geq \frac{r}{\sqrt{2Rr}} \Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r \rightarrow$ *true (Euler)*

SOLUTION 4.43

Solution by Daniel Sitaru-Romania

$$\sqrt{6+\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}} = \sqrt{6+\frac{\frac{2S}{a}}{\frac{S}{s-a}}+\frac{\frac{2S}{b}}{\frac{S}{s-b}}+\frac{\frac{2S}{a}}{\frac{S}{s-a}}} =$$

$$= \sqrt{6+\frac{2(s-a)}{a}+\frac{2(s-b)}{b}+\frac{2(s-c)}{c}} = \sqrt{2s\cdot\sum_{cyc(a,b,c)}\frac{1}{a}} =$$

$$= \sqrt{2} \cdot \sqrt{\sum_{cyc(a,b,c)} (s-a) \cdot \sum_{cyc(a,b,c)} \frac{1}{a}} \overset{CBS}{\geq} \sqrt{2} \cdot \sum_{cyc(a,b,c)} \left(\sqrt{s-a} \cdot \frac{1}{\sqrt{a}}\right) =$$

$$= \sqrt{2}\left(\sqrt{\frac{s-a}{a}} + \sqrt{\frac{s-b}{b}} + \sqrt{\frac{s-c}{c}}\right)$$

SOLUTION 4.44

Solution by Daniel Sitaru-Romania

$$\sum\left(\frac{1}{h_a} + \frac{1}{r_a}\right) bc = \sum\left(\frac{a}{2S} + \frac{s-a}{S}\right) = \frac{1}{2S}\sum(2s-a)bc = \frac{s}{S}\sum bc - \frac{3abc}{2S} =$$

$$= \frac{s}{rs}(s^2 + r^2 + 4Rr) - \frac{12RS}{2S} = \frac{s^2}{r} + r + 4R - 6R \overset{MITRINOVIC}{\geq} \frac{27r^2}{r} + r - 2R = 28r - 2R$$

SOLUTION 4.45

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Firstly, } \sum a\cos A = \sum 2R\sin A\cos A = R(\sin 2A + \sin 2B + \sin 2C) =$$

$$= R\{2\sin C\cos(A-B) + 2\sin C\cos C\} = 2R\sin C\{\cos(A-B) - \cos(A+B)\} =$$

$$= 2R\sin C \cdot 2\sin A\sin B = 4R\frac{abc}{8R^3} \overset{(1)}{=} \frac{abc}{2R^2}. \textit{ Now, } \sum a^2(b\cos B + c\cos C) =$$

$$= \sum a^2\left(\sum a\cos A - a\cos A\right) = \left(\sum a^2\right)\left(\sum a\cos A\right) - \sum a^3\cos A =$$

$$= \frac{abc}{2R^2} \cdot 2(s^2 - 4Rr - r^2) - \sum \frac{a^3(b^2+c^2-a^2)}{2bc} \textit{ (by (1))}$$

$$= \frac{4Rrs(s^2 - 4Rr - r^2)}{R^2} - \sum \frac{a^4(b^2+c^2-a^2)}{2abc} = \frac{4rs(s^2-4Rr-r^2)}{R} -$$

$$- \frac{\sum a^2b^2(\sum a^2 - c^2) - \sum a^6}{8Rrs} \overset{(2)}{=} \frac{32r^2s^2(s^2-4Rr-r^2) - (\sum a^2b^2)(\sum a^2) + 3a^2b^2c^2 + (\sum a^6)}{8Rrs}$$

$$\textit{Numerator of (2)} = 32r^2s^2(s^2-4Rr-r^2) - (\sum a^2b^2)(\sum a^2) + 3a^2b^2c^2 +$$

$$+ \left(3a^2b^2c^2 + \left(\sum a^2\right)\left(\sum a^4 - \sum a^2b^2\right)\right) = 32r^2s^2(s^2-4Rr-r^2) -$$

$$-2\left(\sum a^2b^2\right)\left(\sum a^2\right)+6a^2b^2c^2+\left(\sum a^2\right)\left(\sum a^4\right)=32r^2s^2(s^2-4Rr-r^2)-$$

$$-2\left(\sum a^2b^2\right)\left(\sum a^2\right)+96R^2r^2s^2+\left(\sum a^2\right)\left\{\left(\sum a^2\right)^2-2\sum a^2b^2\right\}=$$

$$=32r^2s^2(s^2-4Rr-r^2)-8\left(\sum a^2b^2\right)(s^2-4Rr-r^2)+8(s^2-4Rr-r^2)^2+96R^2\,r^2s^2=$$

$$=8(s^2-4Rr-r^2)\left\{4r^2s^2+(s^2-4Rr-r^2)^2-(s^2+4Rr+r^2)^2+16Rrs^2\right\}+96R^2r^2s^2=$$

$$=8(s^2-4Rr-r^2)\{16Rrs^2+4r^2s^2+2s^2(-8Rr-2r^2)\}+96R^2r^2s^2\overset{(3)}{=}96R^2r^2s^2$$

(2), (3) $\Rightarrow$ *LHS* $=\frac{96R^2r^2s^2}{8Rrs}=12Rrs\overset{Mitrinovic\,\&\,Euler}{\leq}12R\cdot\frac{R}{2}\cdot\frac{3\sqrt{3}R}{2}=9\sqrt{3}R^3$ ***(Proved)***

SOLUTION 4.46

Solution by Daniel Sitaru-Romania

$$s\geq\frac{9S}{h_a+h_b+h_c}\leftrightarrow s\geq\frac{9rs}{\frac{s^2+r^2+4Rr}{2R}}\leftrightarrow s^2+r^2+4Rr\geq 18Rr\leftrightarrow s^2+r^2\geq 14Rr$$

$$s^2+r^2\overset{GERRETSEN}{\gtreqqless}16Rr-5r^2+r^2\geq 14Rr\leftrightarrow 2Rr\geq 4r^2\leftrightarrow R\geq 2r$$

SOLUTION 4.47

Solution by Soumava Chakraborty-Kolkata-India

Let $x=a\cos A,\ y=b\cos B,\ z=c\cos C;\ x,y,z>0$. ***Then given inequality becomes:***

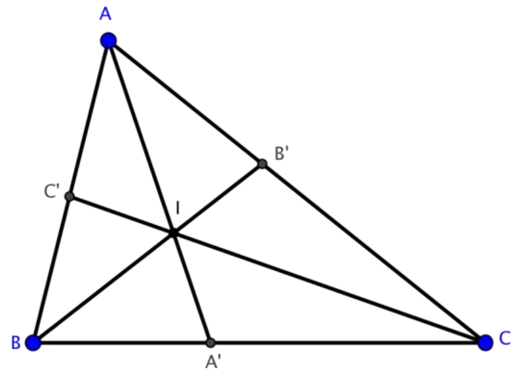
$$2x^2(y+z)^2+2y^2(z+x)^2+2z^2(x+y)^2\leq(x+y+z)(y+z)(z+x)(x+y)\Leftrightarrow$$

$$\Leftrightarrow\left(x^3y+xy^3-2x^2y^2\right)+\left(y^3z+yz^3-2y^2z^2\right)+\left(z^3x+zx^3-2z^2x^2\right)=0\Leftrightarrow$$

$$\Leftrightarrow xy(x-y)^2+yz(y-z)^2+zx(z-x)^2\geq 0\rightarrow$$ ***true (Proved)***

SOLUTION 4.48

Solution by Soumava Chakraborty-Kolkata-India



Angle – bisector theorem $\Rightarrow \frac{BA'}{CA'} = \frac{c}{b} \Rightarrow \frac{CA'+BA'}{BA'} = \frac{b+c}{c} \Rightarrow \frac{a}{BA'} = \frac{b+c}{c} \Rightarrow BA' \overset{(1)}{=} \frac{ac}{b+c}$

Again, angle-bisector theorem $\Rightarrow \frac{AI}{A'I} = \frac{c}{BA'} \Rightarrow \frac{AI}{A'I} = \frac{c(b+c)}{ac} \overset{(a)}{=} \frac{b+c}{a}$*. Similarly,*

$$\frac{BI}{B'I} \overset{(b)}{=} \frac{c+a}{b} \;\&\; \frac{CI}{C'I} \overset{(c)}{=} \frac{a+b}{c}$$

(a), (b), (c) along with $m_a^2 \geq s(s-a)$*, etc* $\Rightarrow$

$$LHS \overset{(i)}{\geq} \sum \frac{s(s-a)bc(b+c)}{abc} = \frac{s}{4Rrs} \sum bc(s-a)(2s-a) =$$

$$= \frac{1}{4Rrs} \sum bc\,(2s^2 - 3sa + a^2) = \frac{2s^2(\sum ab) - 9sabc + abc(2s)}{4Rr} =$$

$$= \frac{2s^2(s^2 + 4Rr + r^2) - 28Rrs^2}{4Rr} = \frac{s^2(s^2 - 10Rr + r^2)}{2Rr}$$

Also, RHS $\overset{22 \leq \frac{2c^2+ab}{4}, etc}{\underset{(ii)}{\leq}} \frac{2}{4}\left(2\sum a^2 + \sum ab\right) = \frac{5s^2 - 12Rr - 3r^2}{2}$

(i),(ii) $\Rightarrow$ *it suffices to prove:*

$$s^4 - s^2(10Rr - r^2) \geq 5Rrs^2 - Rr^2(12R + 3r) \Leftrightarrow$$

$$\Leftrightarrow s^4 - s^2(15Rr - r^2) + 12r^2(12R + 3r) \overset{(2)}{\geq} 0$$

Now, LHS of (2) $\overset{Gerretsen}{\geq} s^2(Rr - 4r^2) + 12R^2(12R + 3r) \overset{?}{\geq} 0 \Leftrightarrow$

$$\Leftrightarrow s^2(R-2r)+Rr(12R+3r)\overset{(3)}{\geq} 2rs^2.$$

Now, LHS of (3) $\geq \{(16R-5r)(R-2r)+R(12R+3r)\}r$ *&*

RHS of (3) $\leq 2r(4R^2+4Rr+3r^2)$

The last two inequalities $\Rightarrow$ *in order to prove (3), it suffices to prove:*

$$(16R-5r)(R-2r)+R(12R+3r)\geq 8R^2+8Rr+6r^2 \Leftrightarrow$$

$$\Leftrightarrow 10R^2-21Rr+2r^2\geq 0 \Leftrightarrow (R-2r)(10R-r)\geq 0 \rightarrow \textit{true (Euler)} \Rightarrow \textit{(3) is true (Done)}$$

SOLUTION 4.49

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)}\frac{\sqrt{b^2+c^2}}{h_a}\overset{QM-AM}{\geq}\sum_{cyc(a,b,c)}\frac{\sqrt{2}\cdot\frac{b+c}{2}}{\frac{2S}{a}}=\frac{\sqrt{2}}{4S}\left(2s\sum_{cyc(a,b,c)}a-\sum_{cyc(a,b,c)}a^2\right)=$$

$$=\frac{\sqrt{2}}{2S}(s^2+r^2+4Rr)\geq\frac{18\sqrt{2}r^2}{S}\ (to\ prove)\leftrightarrow s^2+r^2+4Rr\geq 36r^2$$

$$s^2+r^2+4Rr\overset{GERRETSEN}{\geq}16Rr-5r^2+r^2+4Rr\geq 36r^2\leftrightarrow R\geq 2r$$

SOLUTION 4.50

Solution by Bogdan Fustei-Romania

Knowing the identity: $a\cos A+b\cos B+c\cos C=p$*. The inequality from enuciation becomes:* $\frac{a\cos A}{abc}+\frac{b\cos B}{abc}+\frac{c\cos C}{abc}\geq\frac{1}{2R^2};\frac{1}{abc}\sum a\cos A=\frac{p}{abc}=\frac{p}{4RS}=\frac{1}{4Rr}$*. So,*

$$\frac{1}{4Rr}\geq\frac{1}{2R^2}\Leftrightarrow 4Rr\leq 2R^2\Leftrightarrow 2r\leq R\ \ \textit{(Euler)}$$

SOLUTION 4.51

Solution by Daniel Sitaru-Romania

$$K-Lemoine's\ point\rightarrow\frac{x}{a}=\frac{y}{b}=\frac{z}{c}\overset{denote}{=}q,$$

$$S=\frac{ax+by+cz}{2}=\frac{(a^2+b^2+c^2)q}{2}\to q=\frac{2S}{a^2+b^2+c^2}$$

$$\sum_{\substack{cyc(A,B,C)\\cyc(x,y,z)}}\frac{x}{AH}=\sum_{\substack{cyc(A,B,C)\\cyc(x,y,z)}}\frac{x}{2R\cos A}=\frac{q}{2R}\sum_{cyc(a,b,c)}\frac{a}{\cos A}=$$

$$=\frac{2S}{2R(a^2+b^2+c^2)}\cdot\frac{4SR}{s^2-(2R+r)^2}=\frac{4S^2}{2(s^2-r^2-4Rr)(s^2-(2R+r)^2)}\ge$$

$$\overset{GERRETSEN}{\geq}\frac{4S^2}{2(4R^2+3r^2+4Rr-r^2-4Rr)(4R^2+3r^2+4Rr-(2R+r)^2)}=$$

$$=\frac{4S^2}{2(4R^2+2r^2)\cdot 2r^2}\overset{EULER}{\geq}\frac{4S^2}{8\cdot\left(2R^2+\frac{R^2}{4}\right)\cdot\frac{R^2}{4}}=\frac{8S^2}{9R^4}$$

SOLUTION 4.52

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)}\frac{(R-r_a)^2}{h_a}\overset{BERGSTROM}{\geq}\frac{(3R-r_a-r_b-r_c)^2}{h_a+h_b+h_c}=\frac{(R+r)^2}{\frac{s^2+r^2+4Rr}{2R}}\ge$$

$$\overset{GERRETSEN}{\geq}\frac{2R(R+r)^2}{4R^2+4Rr+3r^2+r^2+4Rr}=\frac{R}{2}\ge\frac{13r^2-3R^2}{r}\leftrightarrow(R-2r)(16R+13r)\ge 0$$

SOLUTION 4.53

Solution by Soumava Chakraborty-Kolkata-India

$$3\sum\frac{h_a}{\sin\frac{B}{2}\sin\frac{C}{2}}=3\sum\frac{\sin\frac{A}{2}}{\left(\frac{r}{4R}\right)}\left(\frac{2rs}{4R\sin\frac{A}{2}\cos\frac{A}{2}}\right)=6s\sum\frac{1}{\cos\frac{A}{2}}$$

$$\overset{Berstrom}{\geq}\frac{54s}{\sum\cos\frac{A}{2}}\overset{Jensen}{\geq}\frac{54s}{3\frac{\sqrt{3}}{2}}\ \left(\because f(x)=\cos\frac{x}{2}\,\forall x\in(0,\pi)\text{ is concave as }f''(x)<0\right)$$

$$=12\sqrt{3}s\ \therefore LHS\overset{(1)}{\geq}12\sqrt{3}s$$

Now, $4\sum m_a+8\sum w_a=4\{(w_b+w_c+m_a)+(w_c+w_a+m_b)+(w_a+w_b+m_c)\}$

$$\overset{Lessel-Pelling}{\leq} 4(3\sqrt{3}s) = 12\sqrt{3}s \overset{by\ (1)}{\leq} LHS \text{ (Proved)}$$

SOLUTION 4.54

Solution by Marian Ursărescu-Romania

First, we show: $\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\geq\frac{24r^2}{R}$

From Bergström's inequality, we have:

$$\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\geq\frac{(a+b+c)^2}{m_a+m_b+m_c}=\frac{4s^2}{m_a+m_b+m_c} \quad (1)$$

But in any ΔABC *we have:* $m_a+m_b+m_c\leq\frac{9R}{2}$ *(2)*

From (1)+(2) $\Rightarrow \frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\geq\frac{8s^2}{9R}$ *(3)*

From (3) we must show: $\frac{8s^2}{9R}\geq\frac{24r^2}{R}\Leftrightarrow s^2\geq 27r^2$*, which its true, because its Mitrinovic's inequality. Second, we show:* $\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\leq\frac{4R^2-2Rr}{r}$*. We know:* $m_a\geq\frac{b^2+c^2}{4R}\geq\frac{bc}{2R}\Rightarrow$

$$\Rightarrow\frac{1}{m_a}\leq\frac{2R}{bc}\Rightarrow\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c}\leq 2R\left(\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\right) \quad (4)$$

From (4) we must show:

$$2R\left(\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\right)\leq\frac{4R^2-2Rr}{r}\Leftrightarrow\frac{a^3+b^3+c^3}{abc}\leq\frac{2R-r}{r} \quad (5)$$

But: $a^3+b^3+c^3=2s(s^2-3r^2-6Rr)$ *and* $abc=4sRr$ *(6)*

From (5)+(6) we must show: $\frac{2s(s^2-3r^2-6Rr)}{4sRr}\leq\frac{2R-r}{r}\Leftrightarrow s^2-3r^2-6Rr\leq 4R^2-2Rr\Leftrightarrow$

$\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2$*, which its true, because its Gerretsen's inequality.*

SOLUTION 4.55

Solution by Daniel Sitaru-Romania

$$f(x)=\frac{1-sinx}{1+sinx}, f'(x)=\frac{-2cosx}{(1+sinx)^2}, f''(x)=\frac{2sinx(1+sinx)+4cos^2x}{(1+sinx)^3}$$
$$>0, f-convexe$$

$$\sum\frac{2R-a}{2R+a}=\sum\frac{1-sinA}{1+sinA}\overset{JENSEN}{\geq}3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)=3\cdot\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}=$$

$$=3\left(7-4\sqrt{3}\right)=3\left(2-\sqrt{3}\right)^2=3tan^2 15^\circ$$

SOLUTION 4.56

Solution by Soumava Chakraborty-Kolkata-India

$$\sum r_a^2\tan A\geq\sqrt{3}s^2$$

Let $f(x)=\tan x; \forall x\in\left(0,\frac{\pi}{2}\right); f''(x)=2\sec^2 x\tan x>0 \therefore f(x)=\tan x$ ***is convex on*** $\left(0,\frac{\pi}{2}\right)$***. WLOG, we may assume*** $a\geq b\geq c$***. Then,*** $r_a^2\geq r_b^2\geq r_c^2$ ***&*** $\tan A\geq\tan B\geq\tan C$

$$\left(\because f(x)=\tan x \text{ is increasing on } \left(0,\frac{\pi}{2}\right)\right)$$

$$\therefore\sum r_a^2\tan A\geq\frac{1}{3}\left(\sum r_a^2\right)\left(\sum\tan A\right)\overset{Jensen}{\geq}\frac{1}{3}\left(\sum r_a^2\right)3\sqrt{3}$$

$$\left(\because f(x)=\tan x \text{ is convex on } \left(0,\frac{\pi}{2}\right)\right)$$

$$=\sqrt{3}\left(\sum r_a^2\right)\geq\sqrt{3}\left(\sum r_a r_b\right)=\sqrt{3}s^2 \textit{ (proved)}$$

SOLUTION 4.57

Solution by Daniel Sitaru-Romania

$$\sum\frac{1}{[A\Omega B]}=\sum\frac{1}{\frac{1}{2}A\Omega\cdot B\Omega\cdot sinB}=\sum\frac{2}{\frac{b}{a}\cdot 2Rsin\omega\cdot\frac{c}{b}\cdot 2Rsin\omega\cdot sinB}=$$

$$=\frac{1}{2R^2sin^2\omega}\cdot\sum\frac{a\cdot 2R}{bc}=\frac{1}{Rsin^2\omega}\cdot\frac{1}{abc}\cdot\sum a^2=\frac{1}{4R^2rs\cdot sin^2\omega}\cdot\sum a^2=$$

$$=\frac{1}{2[I_aI_bI_c]\cdot sin^2\omega\cdot Rr}\sum a^2\geq\frac{9}{[I_aI_bI_c]\cdot sin^2\omega}\leftrightarrow\sum a^2\geq 18Rr$$

$$\sum a^2=2s^2-2r^2-8Rr\geq 18Rr\leftrightarrow s^2\geq 13Rr+r^2$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 13Rr+r^2\leftrightarrow 3Rr\geq 6r^2\leftrightarrow R\geq 2r$$

SOLUTION 4.58

Solution by Daniel Sitaru-Romania

$$\sum\frac{h_a}{r_b+r_c}=\sum\frac{\frac{2S}{a}}{\frac{S}{s-b}+\frac{S}{s-c}}=2\sum\frac{(s-b)(s-c)}{a^2}=2\sum\frac{s^2-(b+c)s+bc}{a^2}=$$

$$=2s^2\sum\frac{1}{a^2}-2s\sum\frac{2s-a}{a^2}+2\sum\frac{bc}{a^2}\overset{AM-GM}{\geq}-2s^2\sum\frac{1}{a^2}+2s\sum\frac{1}{a}+6\sqrt[3]{\prod\frac{bc}{a^2}}\geq$$

$$\geq-2s^2\frac{1}{4r^2}+2s\cdot\frac{9r}{2S}+6=15-\frac{s^2}{2r^2}$$

SOLUTION 4.59

Solution by Daniel Sitaru-Romania

$$\sum\frac{a^3}{h_b+h_c}=\sum\frac{a^3}{\frac{2S}{b}+\frac{2S}{c}}=\frac{1}{2S}\sum\frac{a^3bc}{b+c}=\frac{abc}{2S}\sum\frac{a^2}{b+c}\overset{BERGSTROM}{\geq}$$

$$\geq\frac{4RS}{2S}\cdot\frac{(a+b+c)^2}{2(a+b+c)}=R(a+b+c)=2sR$$

SOLUTION 4.60

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)}\frac{1}{a^3}=\sum_{cyc(a,b,c)}\frac{1^4}{a^3}\overset{RADON}{\geq}\frac{(1+1+1)^4}{(a+b+c)^3}=\frac{81}{8s^3}\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{81}{8\cdot\left(3\sqrt{3}\cdot\frac{R}{2}\right)^3}=\frac{81}{8\cdot27\cdot3\sqrt{3}\cdot\frac{R^3}{8}}=\frac{1}{\sqrt{3}R^3}=\frac{\sqrt{3}}{3R^3}$$

SOLUTION 4.61

Solution by Daniel Sitaru-Romania

$$\sum\frac{a^2}{h_b+h_c}\overset{BERGSTROM}{\geq}\frac{(a+b+c)^2}{2(h_a+h_b+h_c)}=\frac{4s^2}{2\cdot\frac{s^2+r^2+4Rr}{2R}}=\frac{4s^2R}{s^2+r^2+4Rr}\geq$$

$$\overset{GERRETSEN}{\geq}\frac{4s^2R}{4R^2+4Rr+3r^2+r^2+4Rr}=\frac{s^2R}{(R+r)^2}\overset{EULER}{\geq}\frac{s^2R}{\left(R+\frac{R}{2}\right)^2}=\frac{4s^2}{9R}$$

SOLUTION 4.62

Solution by Daniel Sitaru-Romania

$$K - Lemoine's\ point \rightarrow \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \overset{denote}{\cong} q$$

$$S = \frac{ax + by + cz}{2} = \frac{q}{2} \sum_{cyc(a,b,c)} a^2 \rightarrow \sum_{cyc(a,b,c)} a^2 = \frac{2S}{q}$$

$$\sum_{\substack{cyc(a,b,c)\\cyc(x,y,z)}} \frac{x}{a} = \sum_{cyc(a,b,c)} \frac{aq}{a} = \frac{3}{q} = \frac{3(a^2 + b^2 + c^2)}{2S} \geq$$

SOLUTION 4.63

Solution by Daniel Sitaru-Romania

$$\sum \frac{a^2}{m_a^2} \overset{BERGSTROM}{\geq} \frac{(a + b + c)^2}{m_a^2 + m_b^2 + m_c^2} = \frac{4s^2}{\frac{3}{4}(a^2 + b^2 + c^2)} =$$

$$= \frac{16s^2}{3 \cdot 4R^2(sin^2A + sin^2B + sin^2C)} \geq \frac{16s^2}{12R^2 \cdot \frac{9}{4}} = \frac{16s^2}{27R^2}$$

SOLUTION 4.64

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum a^2 - 8R^2}{8R^2} = \frac{s^2 - 4Rr - r^2 - 4R^2}{4R^2} = \cos A \cos B \cos C \Rightarrow 3abc\left(\frac{\sum a^2 - 8R^2}{8R^2}\right) \overset{(1)}{=}$$

$$= 3(a\cos A)(b\cos B)(c\cos C)$$

(1) ⇒ given inequality $\Leftrightarrow \sum a^3\cos^3 A + 3\prod(a\cos A) \overset{(2)}{\geq} 2\sum(a\cos A)^2\,(b\cos B)$

Now, $b\cos B + c\cos C - a\cos A = R(\sin 2B + \sin 2C - \sin 2A) =$

$$= R\{2\sin A\cos(B - C) + 2\sin A\cos(B + C)\}$$

$$(\because \sin(B + C) = \sin A\,;\cos A = -\cos(B + C))$$

$= 4R\sin A\cos B\cos C > 0$ ***(∵*** ΔABC ***is acute-angled). Similarly,*** $c\cos C + a\cos A -$ $-b\cos B > 0$ ***&*** $a\cos A + b\cos B - c\cos C > 0$***. Let*** $b\cos B + c\cos C - a\cos A = x,$ $c\cos C + a\cos A - b\cos B = y$ ***&*** $a\cos A + b\cos B - c\cos C$ ***(of course*** $x, y, z > 0$***)*** ***Then,*** $a\cos A = \frac{y+z}{2}, b\cos B = \frac{z+x}{2}$ ***&*** $c\cos C = \frac{x+y}{2}$***. Via above substitution & (2), given inequality*** $\Leftrightarrow \sum\frac{(y+z)^3}{8} + \frac{3}{8}\prod(x + y) \geq \frac{2}{8}\sum(y + z)^2\,(z + x) \Leftrightarrow 2\sum x^3 + 3\sum x^2y +$

$$+3\sum xy^2 + 6xyz + 3\sum x^2y + 3\sum xy^2 \geq$$

$$\geq 2\sum(y^2z+xy^2+z^3+z^2x+2yz^2+2xyz)\Leftrightarrow 2\sum x^3+6\sum x^2y+6\sum xy^2+$$

$$+6xyz\geq 4\sum x^2y+6\sum xy^2+2\sum x^3+12xyz\Leftrightarrow 2\sum x^2y\geq 6xyz\Leftrightarrow$$

$$\Leftrightarrow \sum x^2y\geq 3xyz\rightarrow \textit{true by A-G (proved)}$$

SOLUTION 4.65

Solution by Daniel Sitaru-Romania

$$f:(0,1)\rightarrow\mathbb{R}, f(x)=sinx, f''(x)=-sinx, f-concave\rightarrow$$

$$\frac{1}{3}\sum_{cyc(A,B,C)} sinA+sin\left(\frac{A+B+C}{3}\right)\leq\frac{2}{3}\sum_{cyc(A,B,C)} sin\left(\frac{B+C}{2}\right)$$

$$\leftrightarrow \sum_{cyc(A,B,C)} sinA+3sin\frac{\pi}{3}\leq 2\sum_{cyc(A,B,C)} sin\left(\frac{\pi-A}{2}\right)$$

$$\leftrightarrow \sum_{cyc(A,B,C)} sinA+\frac{3\sqrt{3}}{2}\leq \sum_{cyc(A,B,C)} cos\frac{A}{2}$$

SOLUTION 4.66

Solution by Daniel Sitaru-Romania

$$\sum\frac{h_a}{bx+cy}=2S\sum\frac{1}{a(bx+cy)}\overset{BERGSTROM}{\geq} 2S\cdot\frac{(1+1+1)^2}{a(bx+cy)+b(cx+ay)+c(ax+by)}=$$

$$=\frac{18S}{(x+y)(ab+bc+ca)}\overset{IPOTHESSIS}{\geq}\frac{18S}{ab+bc+ca}\overset{GORDON}{\geq}\frac{18S}{4\sqrt{3}S}=\frac{3\sqrt{3}}{2}=\frac{1}{R}\cdot\frac{3\sqrt{3}R}{2}\geq$$

$$\overset{MITRINOVIC}{\geq}\frac{s}{R}=\frac{S}{Rr}\overset{EULER}{\geq}\frac{S}{R}\cdot\frac{2}{R}=\frac{2S}{R^2}$$

SOLUTION 4.67

Solution by Daniel Sitaru-Romania

$$\sum m_a\overset{TERESHIN}{\geq}\frac{1}{4R}\sum(b^2+c^2)=\frac{2}{\frac{abc}{S}}\sum a^2=2S\sum\frac{a}{bc}=\sum\frac{\frac{2S}{b}\cdot\frac{2S}{c}}{\frac{2S}{a}}=\sum\frac{h_ch_b}{h_a}$$

SOLUTION 4.68

Solution by Daniel Sitaru-Romania

$$\sum\frac{h_a^2}{r_br_c}=4\sum\frac{(s-b)(s-c)}{a^2}\overset{AM-GM}{\geq}12\sqrt[3]{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}=12\sqrt[3]{\frac{S^4}{s^2\cdot 16R^2S^2}}=$$

$$= 12\sqrt[3]{\frac{r^2}{16R^2}} \geq 12\frac{r^2}{R^2} \leftrightarrow \frac{r^2}{16R^2} \geq \frac{r^6}{R^6} \leftrightarrow R^4 \geq 16r^4 \leftrightarrow R \geq 2r$$

$$\sum \frac{h_a^2}{r_b r_c} = 4\sum \frac{(s-b)(s-c)}{a^2} \overset{AM-GM}{\leq} 4\sum \frac{\left(\frac{s-b+s-c}{2}\right)^2}{a^2} = 4 \cdot \frac{1}{4} \cdot 3 = 3$$

SOLUTION 4.69

Solution by Bogdan Fustei-Romania

We know: $\frac{R}{2r} \geq \frac{m_a}{h_a}$ *and the analogs* $\Rightarrow \frac{3R}{2r} \geq \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c}$ *(1)*

$l_a \leq h_a$ *(and the analogs)* $\Rightarrow \frac{1}{l_a} \leq \frac{1}{h_a}$ *(and the analogs)*

$\Rightarrow \sum \frac{m_a}{l_a} \leq \sum \frac{m_a}{h_a}$ *(2). From (1) and (2) we have inequality from enunciation:*

Namely: $\frac{m_a}{l_a} + \frac{m_b}{l_b} + \frac{m_c}{l_c} \leq \frac{3R}{2r}$ *Q.E.D.*

SOLUTION 4.70

Solution by Soumava Chakraborty-Kolkata-India

$$\sum s_a^2 \cot\frac{A}{2} \geq \sqrt{3}\left(\sum s_a s_b\right)$$

WLOG, we may assume $a \geq b \geq c$ *we shall prove* $s_a^2 \leq s_b^2 \leq s_c^2$

$$s_a^2 \leq s_b^2 \Leftrightarrow \frac{4b^2c^2}{(b^2+c^2)^2} \cdot \frac{2b^2+2c^2-a^2}{4} \leq \frac{4c^2a^2}{(c^2+a^2)^2} \cdot \frac{2c^2+2a^2-b^2}{4} \Leftrightarrow$$

$$\Leftrightarrow \frac{b^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2} \leq \frac{a^2(2c^2+2a^2-b^2)}{(c^2+a^2)^2} \Leftrightarrow \frac{2b^2}{b^2+c^2} - \frac{a^2b^2}{(b^2+c^2)^2} \leq$$

$$\leq \frac{2a^2}{c^2+a^2} - \frac{a^2b^2}{(c^2+a^2)^2} \Leftrightarrow \frac{2b^2}{b^2+c^2} + \frac{a^2b^2}{(c^2+a^2)^2} \overset{(1)}{\leq} \frac{2a^2}{c^2+a^2} + \frac{a^2b^2}{(b^2+c^2)^2}$$

Now, $(b^2+c^2)^2 \leq (c^2+a^2)^2 \quad (\because a \geq b) \Rightarrow \frac{a^2b^2}{(b^2+c^2)^2} \geq \frac{a^2b^2}{(c^2+a^2)^2} \Rightarrow$

$\Rightarrow \frac{a^2b^2}{(c^2+a^2)^2} \overset{(a)}{\leq} \frac{a^2b^2}{(b^2+c^2)^2}$*. Also,* $2b^2(c^2+a^2) \leq 2a^2(b^2+c^2) \ (\because a \geq b) \Rightarrow$

$$\Rightarrow \frac{2b^2}{b^2+c^2} \overset{(b)}{\leq} \frac{2a^2}{c^2+a^2}$$

(a)+(b)⇒(1) is true $\Rightarrow s_a^2 \leq s_b^2$*. Similarly,* $s_b^2 \leq s_c^2 \therefore s_a^2 \leq s_b^2 \leq s_c^2$*. Also,* $a \geq b \geq c \Rightarrow$

$$\Rightarrow \cot\frac{A}{2} \leq \cot\frac{B}{2} \leq \cot\frac{C}{2} \therefore \sum s_a^2 \cot\frac{A}{2} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum s_a^2\right)\left(\sum \cot\frac{A}{2}\right) \geq$$

$$\overset{Jensen}{\geq} \frac{\sum s_a^2}{3}\left(3\cot\frac{\pi}{6}\right)\ \left(\because f(x)=\cot\frac{x}{2}\forall x\in(0,\pi)\ is\ convex\right)$$

$$\geq\left(\frac{\sum s_a s_b}{3}\right)3\sqrt{3}\ \ \left(\because \sum x^2\geq\sum xy\right)=\sqrt{3}\left(\sum s_a s_b\right)$$

SOLUTION 4.71

Solution by Daniel Sitaru-Romania

$$\sum\frac{a}{h_b+h_c}=\sum\frac{a}{\frac{2S}{b}+\frac{2S}{c}}=\frac{1}{2S}\sum\frac{abc}{b+c}=\frac{abc}{2S}\sum\frac{1}{b+c}=$$

$$=\frac{4RS}{2S}\sum\frac{1}{2R(sinB+sinC)}=\sum\frac{1}{sinB+sinC}\overset{BERGSTROM}{\geq}\frac{(1+1+1)^2}{2(sinA+sinB+sinC)}\geq$$

$$\overset{JENSEN}{\geq}\frac{9}{2\cdot\frac{3\sqrt{3}}{2}}=\sqrt{3}$$

SOLUTION 4.72

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality}\Leftrightarrow\frac{c\sin B\sin C+a\sin C\sin A+b\sin A\sin B}{\sin A\sin B\sin C}\leq\frac{a^2b+b^2c+c^2a}{2S}\Leftrightarrow$$

$$\Leftrightarrow\frac{ab^2+bc^2+ca^2}{4R^2\sin A\sin B\sin C}\leq\frac{a^2b+b^2c+c^2a}{4R^2\sin A\sin B\sin C}\Leftrightarrow$$

$$\Leftrightarrow ab(a-b)+bc(b-c)+ca(c-a)\overset{?}{\geq}0\because A\geq B\geq C\therefore a\geq b\geq c$$

Let $b=m+c$ ***&*** $a=m+n+c\quad(m,n\geq 0)$***. Using the above substitution (1)*** $\Leftrightarrow$

$$\Leftrightarrow(m+n+c)(m+c)n+c(m+c)m+c(m+n+c)(-m-n)\geq 0\Leftrightarrow$$

$$\Leftrightarrow n(m+n+c)(m+c)-nc(m+n+c)+mc(m+c)-mc(m+n+c)\geq 0\Leftrightarrow$$

$$\Leftrightarrow n(m+n+c)m-mnc\geq 0\Leftrightarrow mn(m+n)\geq 0\to \textit{true}\because m,n\geq 0\ \ \textit{(Proved)}$$

SOLUTION 4.73

Solution by Tran Hong-Vietnam

Using Schwarz's inequality we have:

$$LHS\geq\frac{16\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)^2}{4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)}=4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)$$

$$4\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=4\left(\frac{a+b+c}{abc}\right)=\frac{2s}{RS}=\frac{2}{Rr}\overset{(Euler)}{\geq}\frac{4}{R^2}$$

$$Equality \Leftrightarrow a = b = c.$$

SOLUTION 4.74

Solution by Soumava Chakraborty-Kolkata-India

$$Mitrinovic \Rightarrow \frac{1}{s} \ge \frac{2\sqrt{3}}{9R}, \text{ which} \Rightarrow \text{it suffices to prove: } \frac{s\prod(a+b)+8Rr(\sum ab)}{\{\prod(a+b)\}\sum ab} \overset{(a)}{\ge} \frac{1}{s}$$

$$Now, \prod(a+b) = 2abc + \sum ab\,(2s-c) = 2s(s^2+4Rr+r^2) - 4Rrs \overset{(1)}{=} 2s(s^2+2Rr+r^2)$$

$$(1) \Rightarrow \text{LHS of (a)} = \frac{2s^2(s^2+2Rr+r^2)+8Rr(s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)(s^2+4Rr+r^2)} \overset{?}{\ge} \frac{1}{s}$$

$$\Leftrightarrow s^4 + s^2(6Rr+r^2) + 4R(4R+r)r^2 \overset{?}{\ge} s^4 + s^2(6Rr+2r^2) + r^2(2R+r)(4R+r)$$

$$\Leftrightarrow r^2(4R+r)(2R-r) \overset{?}{\ge} s^2r^2 \Leftrightarrow s^2 \underset{(b)}{\overset{?}{\le}} 8R^2 - 2Rr - r^2$$

$$Now, \text{ LHS of (b)} \overset{Gerretsen}{\le} 4R^2 + 4Rr + 3r^2 \overset{?}{\le} 8R^2 - 2Rr - r^2$$

$$\Leftrightarrow (R-2r)(2R+r) \overset{?}{\ge} 0 \rightarrow true \because R \overset{Euler}{\ge} 2r \;\; (Proved)$$

SOLUTION 4.75

Solution by Marian Ursărescu-Romania

$$\frac{2(m_a+m_b+m_c)}{\sqrt{3(a^2+b^2+c^2)}} + \frac{\sqrt{3(a^2+b^2+c^2)^3}}{8m_am_bm_c} + \frac{\sqrt{3(a^2+b^2+c^2)^3}}{8m_am_bm_c} + \frac{\sqrt{3(a^2+b^2+c^2)^3}}{8m_am_bm_c} \ge$$

$$\ge 4\sqrt[4]{\frac{2(m_a+m_b+m_c)\sqrt{2+(a^2+b^2+c^2)^9}}{\sqrt{3(a^2+b^2+c^2)}\cdot 8^3m_a^3m_b^3m_c^3}} \Rightarrow$$

We must show:

$$4\sqrt[4]{\frac{2(m_a+m_b+m_c)\cdot 3(a^2+b^2+c^2)\sqrt[4]{3(a^2+b^2+c^2)}}{\sqrt{3(a^2+b^2+c^2)}\cdot 2^9m_a^3m_b^3m_c^3}} \ge 4\sqrt{3} \Leftrightarrow$$

$$\Leftrightarrow \frac{(m_a+m_b+m_c)3(a^2+b^2+c^2)^4}{2^8m_a^3m_b^3m_c^3} \ge 9 \Rightarrow$$

$$(m_a+m_b+m_c)(a^2+b^2+c^2)^4 \ge 3\cdot 2^8m_a^3m_b^3m_c^3 \quad (1)$$

$$But\; m_a^2+m_b^2+m_c^2 = \frac{3(a^2+b^2+c^2)}{9} \Rightarrow a^2+b^2+c^2 = \frac{4}{3}(m_a^2+m_b^2+m_c^2) \;\; (2)$$

From (1)+(2) we must show:

$$(m_a+m_b+m_c)\cdot\frac{2^8}{3^4}\left(m_a^2+m_b^2+m_c^2\right)^4\geq 3\cdot 2^8\cdot m_a^3\cdot m_b^3\cdot m_c^3\Leftrightarrow$$

$$(m_a+m_b+m_c)\left(m_a^2+m_b^2+m_c^2\right)^4\geq 3^5m_a^3m_b^3m_c^3\quad (3)$$

From Cauchy inequality we have:

$$3\left(m_a^2+m_b^2+m_c^2\right)\geq(m_a+m_b+m_c)^2\Rightarrow$$

$$\left(m_a^2+m_b^2+m_c^2\right)^4\geq\frac{1}{3^4}(m_a+m_b+m_c)^8\quad (4)$$

From (3)+(4) we must show:

$$(m_a+m_b+m_c)\frac{1}{3^4}(m_a+m_b+m_c)^8\geq 3^5m_a^3m_b^3m_c^3\Leftrightarrow$$

$\Leftrightarrow(m_a+m_b+m_c)^9\geq 3^9m_a^3m_b^3m_c^3\Leftrightarrow m_a+m_b+m_c\geq 3\sqrt[3]{m_am_bm_c}$***, which its true.***

SOLUTION 4.76

Solution by Daniel Sitaru-Romania

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\leq\frac{1}{4r^2}-Cucurezeanu's\ inequality\ (1989)$$

$$\sum_{cyc}\frac{h_a}{h_bh_c}=\sum_{cyc}\frac{\frac{2S}{a}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{2S}\sum_{cyc}\frac{bc}{a}=\frac{1}{2S}\sum_{cyc}\frac{abc}{a^2}=$$

$$=\frac{abc}{2S}\sum_{cyc}\frac{1}{a^2}\overset{CUCUREZEANU}{\overbrace{\leq}}\frac{abc}{2S}\cdot\frac{1}{4r^2}=\frac{4RS}{2S}\cdot\frac{1}{4r^2}=\frac{R}{2r^2}$$

SOLUTION 4.77

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{h_b+h_c}{h_a}=\sum_{cyc}\frac{\frac{2S}{b}+\frac{2S}{c}}{\frac{2S}{a}}=\sum_{cyc}\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{a}}=\sum_{cyc}\frac{a(b+c)}{bc}=\frac{1}{abc}\sum_{cyc}a^2(b+c)=$$

$$=\frac{1}{abc}\left(\sum_{cyc}a^2(2s-a)\right)=\frac{1}{abc}\left(2s\sum_{cyc}a^2-\sum_{cyc}a^3\right)=$$

$$=\frac{1}{abc}(2s\cdot 2(s^2-r^2-4Rr)-2s(s^2-3r^2-6Rr))=\frac{2s}{abc}(s^2+r^2-2Rr)\geq$$

$$\overset{GERRETSEN}{\overbrace{\geq}}\frac{2s}{4Rrs}(16Rr-5r^2+r^2-2Rr)=\frac{1}{R}(7R-2r)\leq\frac{3R}{r}\leftrightarrow$$

$$\leftrightarrow 7Rr-2r^2\leq 3R^2\leftrightarrow(R-2r)(3R-r)\geq 0$$

SOLUTION 4.78

Solution by Soumava Chakraborty-Kolkata-India

Let $x=\frac{1}{ab\sin\frac{A}{2}\sin\frac{B}{2}}, y=\frac{1}{bc\sin\frac{B}{2}\sin\frac{C}{2}}, z=\frac{1}{ca\sin\frac{C}{2}\sin\frac{A}{2}}$*. Using the substitution, given inequality becomes:* $\sum\frac{x^7}{y^6+z^6}\geq\frac{1}{2r^2}$*. WLOG, we may assume* $x\geq y\geq z$*.*

$$\frac{x^6}{y^6+z^6}\geq\frac{y^6}{z^6+x^6}\Leftrightarrow x^{12}+z^6x^6\geq y^{12}+y^6z^6\Leftrightarrow(x^{12}-y^{12})+z^6(x^6-y^6)\geq 0\to \textit{true}$$

$$\because x\geq y$$

$$\therefore\frac{x^6}{y^6+z^6}\geq\frac{y^6}{z^6+x^6}\text{. Similarly, }\frac{y^6}{z^6+x^6}\geq\frac{z^6}{x^6+y^6}\Rightarrow\frac{x^6}{y^6+z^6}\geq\frac{y^6}{z^6+x^6}\geq\frac{z^6}{x^6+y^6}$$

$$\therefore\sum\frac{x^7}{y^6+z^6}=\sum x\left(\frac{x^6}{y^6+z^6}\right)\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum x\right)\left(\sum\frac{x^6}{y^6+z^6}\right)$$

$$\overset{Nesbitt}{\geq}\frac{1}{3}\left(\sum x\right)\left(\frac{3}{2}\right)=\frac{\sum x}{2}=\frac{1}{2}\sum\frac{1}{ab\sin\frac{A}{2}\sin\frac{B}{2}}$$

$$=\frac{1}{2}\cdot\frac{\sum c\sin\frac{C}{2}}{abc\left(\prod\sin\frac{A}{2}\right)}=\frac{1}{2}\cdot\frac{\sum a\sin\frac{A}{2}}{4Rrs\left(\frac{r}{4R}\right)}=\frac{\sum a\sin\frac{A}{2}}{2sr^2}$$

$$\overset{?}{\geq}\frac{1}{2r^2}\Leftrightarrow\sum a\sin\frac{A}{2}\underset{(a)}{\overset{?}{\geq}}s$$

$$\textit{Now, }\sum a\sin\frac{A}{2}=\sum 4R\cos\frac{A}{2}\left(\sin^2\frac{A}{2}\right)=2R\sum\left[\frac{(s-b)(s-c)}{bc}\left(\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right)\right]$$

$$\geq 2R\sum\frac{a(s-b)(s-c)}{4Rrs}(\sin B+\sin C)$$

$$\left(\because 0<\cos\frac{B-C}{2}\leq 1\ as\ -\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}\right)$$

$$=2R\sum\left[\left(\frac{b+c}{2R}\right)\left(\frac{a(s-b)(s-c)}{4Rrs}\right)\right]$$

$$=\frac{\sum a(b+c)(s-b)(s-c)}{4Rrs}\therefore\sum a\sin\frac{A}{2}\overset{(1)}{\geq}\frac{\sum a(b+c)(s-b)(s-c)}{4Rrs}$$

$$\textit{Now, }\sum a(b+c)(s-b)(s-c)=\sum\left(a(b+c)(s^2-s(b+c)+bc)\right)$$

$$=s^2\sum a(b+c)-s\sum a(b+c)^2+abc\sum(b+c)$$

$$=2s^2(s^2+4Rr+r^2)+16Rrs^2-s\sum a(2s-a)^2$$

$$= 2s^2(s^2+12Rr+r^2) - s\sum a(4s^2-4sa+a^2)$$

$$= 2s^2(s^2+12Rr+r^2) - 4s^3(2s) + 4s^2\sum a^2 - s\sum a^3$$

$$= 2s^2(s^2+12Rr+r^2) - 8s^4 + 8s^2(s^2-4Rr-r^2) - 2s^2(s^2-6Rr-3r^2)$$

$$= s^2(4Rr) = 4Rrs^2 \Rightarrow \sum a(b+c)(s-b)(s-c) \overset{(2)}{=} 4Rrs^2$$

(1),(2)$\Rightarrow \sum a\sin\frac{A}{2} \geq \frac{4Rrs^2}{4Rrs} = s \Rightarrow$ ***(a) is true***

$\Rightarrow \sum\frac{x^7}{y^6+z^6} \geq \frac{1}{2r^2}$ ***(Hence proved)***

Now, $\sum a\sin\frac{A}{2} = \sum\frac{a\cdot 2\sin\frac{A}{2}\cos\frac{B-C}{2}}{2\cos\frac{B-C}{2}} \geq \sum\frac{a\,2\sin\frac{A}{2}\cos\frac{B-C}{2}}{2}$ $\left(\because 0 < \cos\frac{B-C}{2} \leq 1\ as -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2}\right)$

$$= \frac{1}{2}\sum a\left(2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\right) = \frac{1}{2}\sum a(\cos B + \cos C)$$

$$= \frac{1}{2}\sum\left(a\left(\sum\cos A - \cos A\right)\right) = \frac{1}{2}\left(\sum\cos A\right)\left(\sum a\right) - \frac{1}{2}\sum a\cos A$$

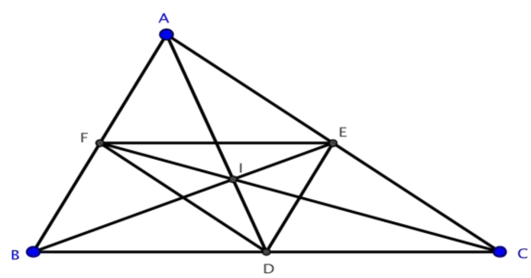
$$= \frac{1}{2}\left(\frac{R+r}{R}\right)(2s) - \frac{R}{2}\sum\sin 2A$$

$$= \left(\frac{R+r}{R}\right)s - \frac{R}{2}\cdot 4\left(\frac{abc}{8R^3}\right) = \left(\frac{R+r}{R}\right)s - \frac{R\cdot 4Rrs}{4R^3}$$

$= \left(\frac{R+r}{R}\right)s - \frac{rs}{R} = s \Rightarrow$ ***(a) is true (Proved)***

SOLUTION 4.79

Solution by Soumava Chakraborty-Kolkata-India



Angle bisector theorem $\Rightarrow \frac{BD}{CD} = \frac{c}{b} \Rightarrow \frac{BD+CD}{CD} = \frac{b+c}{b} \Rightarrow \frac{a}{CD} = \frac{b+c}{b} \Rightarrow CD = \frac{ab}{b+c}$

Similarly, $BD = \frac{ac}{b+c}, BF = \frac{ca}{a+b}, AF = \frac{bc}{a+b}, AE = \frac{bc}{c+a}, CE = \frac{ab}{c+a}$

$$S[FBD] = \frac{1}{2}BF\cdot BD\sin B = \frac{1}{2}\cdot\frac{ca}{a+b}\cdot\frac{ac}{b+c}\cdot\frac{b}{2R} \overset{(1)}{=} \frac{abc}{4R}\cdot\frac{ac}{(a+b)(b+c)}$$

$$\textit{Similarly, } S[FAE] \overset{(2)}{=} \frac{abc}{4R} \cdot \frac{bc}{(a+b)(c+a)} \;\&\; S[ECD] \overset{(3)}{=} \frac{abc}{4R} \cdot \frac{ab}{(c+a)(b+c)}$$

$$\text{(1)+(2)+(3)} \Rightarrow S - S[DEF] = \frac{abc}{4R} \cdot \frac{\sum ab(a+b)}{\prod(a+b)} = S\frac{\sum ab(2s-c)}{2abc+\sum ab(2s-c)} = \frac{2s(s^2+4Rr+r^2)-12Rrs}{2s(s^2+4Rr+r^2)-4Rrs} \cdot S$$

$$= S\left(\frac{s^2-2Rr+r^2}{s^2+2Rr+r^2}\right) \Rightarrow S[DEF] = S\left(1 - \frac{s^2-2Rr+r^2}{s^2+2Rr+r^2}\right) = S\left(\frac{4Rr}{s^2+2Rr+r^2}\right) \Rightarrow$$

$$\Rightarrow \frac{S[ABC]}{S[DEF]} = \frac{s^2+2Rr+r^2}{4Rr} \le \frac{R^2+r^2}{Rr} + \frac{3}{2} \Leftrightarrow \frac{s^2+2Rr+r^2}{4Rr} - \frac{3}{2} \le \frac{R^2+r^2}{Rr} \Leftrightarrow$$

$$\Leftrightarrow s^2 - 4Rr + r^2 \le 4R^2 + 4r^2 \Leftrightarrow s^2 \le 4R^2 + 4Rr + 3r^2 \rightarrow \textit{true (Gerretsen) (Proved)}$$

SOLUTION 4.80

Solution by Daniel Sitaru-Romania

Known:

$$m_a m_b m_c \ge sS \;\; (1)$$

$$w_a w_b w_c \ge 27r^3 \;\; (2)$$

$$\sum_{cyc} \frac{m_a w_b}{h_c} = \sum_{cyc} \frac{m_a w_b}{\frac{2S}{c}} = \frac{1}{2S}\sum_{cyc} c m_a w_b \overset{AM-GM}{\ge}$$

$$\ge \frac{3}{2S}\sqrt[3]{abc \cdot m_a m_b m_c \cdot w_a w_b w_c} \overset{(1),(2)}{\ge} \frac{3}{2S}\sqrt[3]{4RS \cdot sS \cdot 27r^3} = \frac{9r}{2S}\sqrt[3]{4RS^2 s} \ge$$

$$\overset{EULER}{\ge} \frac{9r}{2S}\sqrt[3]{8r \cdot r^2 s^2 \cdot s} = \frac{9r}{2rs}\sqrt[3]{(2rs)^3} = 9r \ge \frac{2\sqrt{3}S}{R} \leftrightarrow$$

$$\leftrightarrow 9r \ge \frac{2\sqrt{3}rs}{R} \leftrightarrow R \ge \frac{2\sqrt{3}s}{9} \leftrightarrow s \le \frac{9R}{2\sqrt{3}} \leftrightarrow s \le \frac{3\sqrt{3}R}{2} \;\; (MITRINOVIC)$$

SOLUTION 4.81

Solution by Marian Ursărescu – Romania

$$a^2\cos^2 A + b^2\cos^2 B + c^2\cos^2 C \ge 3\sqrt[3]{a^2b^2c^2\cos^2 A\cos^2 B\cos^2 C} \Rightarrow$$

We must show:

$$27a^2b^2c^2\cos^2 A\cos^2 B\cos^2 C \ge 8^3 3\sqrt{3}S^3\cos^3 A\cos^3 B\cos^3 C \Leftrightarrow$$

$$\Leftrightarrow 9a^2b^2c^2 \ge 8^3\sqrt{3}S^3\cos A\cos B\cos C \;\; (1)$$

But $abc = 4sRr$ *(2),* $S = sr$ *(3) and* $\cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$ *(4).*

From (1)+(2)+(3)+(4) we must show: $9 \cdot 16s^2R^2r^2 \ge 8^3\sqrt{3}s^3r^3\frac{(s^2-(2R+r)^2)}{4R^2} \Leftrightarrow$

$$\Leftrightarrow 9R^4 \geq 8\sqrt{3}sr(s^2-(2R+r)^2)$$ *(5). But* $s \leq \frac{3\sqrt{3}}{2}R$ *(6). From (5)+(6) we must show:*

$$R^3 \geq 4r(s^2-(2R+r)^2)$$ *(7). From Gerretsen's inequality:* $s^2 \leq 4R^2+4Rr+3r^2 \Rightarrow$

$$\Rightarrow s^2-(2R+r)^2 \leq 2r^2$$ *(8). From (7)+(8) we must show:*

$$R^3 \geq 8r^2 \Leftrightarrow R \geq 2r$$ *true (Euler)*

SOLUTION 4.82

Solution by Marian Ursărescu-Romania

We must show $(AI+BI+CI)^2 \leq 6R(h_a+h_b+h_c-6r)$ *(1)*

But form Cauchy's inequality $(AI+BI+CI)^2 \leq 3(AI^2+BI^2+CI^2)$ *(2)*

From (1)+(2) we must show: $AI^2+BI^2+CI^2 \leq 2R(h_a+h_b+h_c-6r)$ *(3)*

But $AI^2 = 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2}$ *(4). From (3)+(4) we must show:*

$$16R^2\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2} \leq 2R(h_a+h_b+h_c-6r) \Leftrightarrow$$

$$\Leftrightarrow 8R\cdot\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2} \leq (h_a+h_b+h_c-6r) \quad (5)$$

But $\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{s^2+r^2-8Rr}{16R^2} \Rightarrow 8R\sum\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{s^2+r^2-8Rr}{2R}$ *(6)*

Now, $h_a+h_b+h_c-6r = \frac{s^2+r^2+4Rr}{2R}-6r = \frac{s^2+r^2+4Rr-12Rr}{2R} = \frac{s^2+r^2-8Rr}{2R}$ *(7)*

From (6)+(7)⇒ (5) its true.

SOLUTION 4.83

Solution by Soumava Chakraborty-Kolkata-India

Let $a^2AN^2 = x, b^2BN^2 = y$ *&* $c^2CN^2 = z$*. Then, given inequality becomes:*

$$\frac{x}{5y+5z-x}+\frac{y}{5z+5x-y}+\frac{z}{5x+5y-z} \geq \frac{1}{3} \Leftrightarrow 3x(5z+5x-y)(5x+5y-z)+$$

$$+3y(5x+5y-z)(5y+5z-x)+3z(5y+5z-x)(5z+5x-y) \geq$$

$$\geq (5y+5z-x)(5z+5x-y)(5x+5y-z) \Leftrightarrow 5\sum x^3+3xyz \overset{(1)}{\geq} 3\sum x^2y+\sum xy^2$$

Now, $5\sum x^3+15xyz \overset{Schur}{\geq} 5\sum x^2y+\sum xy^2$ *&* $2(\sum x^2y+\sum xy^2) \overset{A-G}{\geq} 12xyz$

Adding the last two inequalities, (1) is true (proved)

SOLUTION 4.84

Solution by Daniel Sitaru-Romania

$$\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right)\overset{CESARO}{\geq}\frac{8}{r_ar_br_c}=\frac{8}{rs^2}\overset{MITRINOVIC}{\geq}\frac{8}{r\cdot\left(\frac{3\sqrt{3}R}{2}\right)^2}=$$

$$=\frac{8}{r\cdot\frac{27R^2}{4}}=\frac{32}{27R^2r}$$

$$\left(\frac{1}{r_a}+\frac{1}{r_b}\right)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\left(\frac{1}{r_c}+\frac{1}{r_a}\right)\overset{AM-GM}{\leq}\left(\frac{\left(\frac{1}{r_a}+\frac{1}{r_b}\right)+\left(\frac{1}{r_b}+\frac{1}{r_c}\right)+\left(\frac{1}{r_c}+\frac{1}{r_a}\right)}{3}\right)^3$$

$$=\frac{8}{27}\cdot\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)^3=\frac{8}{27r^3}=\frac{8r}{27r^4}\overset{EULER}{\leq}\frac{4R}{27r^4}$$

SOLUTION 4.85

Solution by Soumava Chakraborty-Kolkata-India

$$\because a^2x+b^2y+c^2z\geq 4S\sqrt{\sum xy},\forall x,y,z\geq 0$$

$$\therefore a^2\cos 7^\circ+b^2\cos 65^\circ+c^2\cos 79^\circ\geq 4S\sqrt{\cos 7^\circ\cos 65^\circ\cos 79^\circ+\cos 79^\circ\cos 7^\circ}$$

$$>4rs\sqrt{\cos 10^\circ\cos 70^\circ+\cos 70^\circ\cos 80^\circ+\cos 80^\circ\cos 10^\circ}$$

$$=4rs\sqrt{\frac{1}{2}(\cos 80^\circ+\cos 60^\circ+\cos 150^\circ+\cos 10^\circ+\cos 90^\circ+\cos 70^\circ)}$$

$$=4rs\sqrt{\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}+\cos 80^\circ+\cos 10^\circ+\cos 70^\circ\right)}$$

$$=4rs\sqrt{\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}+2\cos 45^\circ\cos 35^\circ+2\cos^2 35^\circ-1\right)}$$

$$\overset{(1)}{>}4rs\sqrt{\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}+\frac{2}{\sqrt{2}}\cos 36^\circ+2\cos^2 36^\circ-1\right)}$$

$$=4rs\sqrt{\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}+\sqrt{2}\left(\frac{\sqrt{5}+1}{4}\right)+2\left(\frac{6+2\sqrt{5}}{16}\right)-1\right)}$$

$$=4rs\sqrt{\frac{1}{2}\left(\frac{2-2\sqrt{3}+\sqrt{10}+\sqrt{2}+3+\sqrt{5}-4}{4}\right)}=4rs\sqrt{\frac{1+\sqrt{2}+\sqrt{5}+\sqrt{10}-2\sqrt{3}}{8}}$$

$$Also, \cos 29^\circ < \cos 22\frac{1}{2}^\circ = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} \Rightarrow \frac{1}{\cos 29^\circ} \overset{(2)}{>} \sqrt{\frac{2\sqrt{2}}{\sqrt{2}+1}}$$

$$\& \because \cos 35^\circ, \cos 43^\circ < \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\therefore \frac{5}{\cos 35^\circ} + \frac{1}{\cos 43^\circ} \overset{(3)}{>} 6 \cdot \frac{2}{\sqrt{3}} = \frac{12}{\sqrt{3}}$$

(1), (2), (3) ⇒ LHS

$$> 4rs\sqrt{\frac{1+\sqrt{2}+\sqrt{5}+\sqrt{10}-2\sqrt{3}}{8}}\left(\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}+\frac{12}{\sqrt{3}}\right)$$

$$\overset{s\ge 3\sqrt{3}r}{>} 12\sqrt{3}\sqrt{\frac{1+\sqrt{2}+\sqrt{5}+\sqrt{10}-2\sqrt{3}}{8}}\left(\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}+\frac{12}{\sqrt{3}}\right)r^2$$

$$> 120r^2 > 108r^2 \quad \textit{(Proved)}$$

SOLUTION 4.86

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)} \frac{\sqrt{(r_b-r)(r_c-r)}}{a} = \sum_{cyc(a,b,c)} \frac{1}{a}\sqrt{\left(\frac{S}{s-b}-\frac{S}{s}\right)\left(\frac{S}{s-c}-\frac{S}{s}\right)} =$$

$$= \sum_{cyc(a,b,c)} \frac{S}{as}\sqrt{\frac{bc}{(s-b)(s-c)}} = r\sum_{cyc(a,b,c)} \frac{1}{\sin\frac{A}{2}} \cdot \frac{1}{a} \overset{AM-GM}{\ge}$$

$$\ge 3r\sqrt[3]{\frac{1}{abc \cdot \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}} = 3\sqrt[3]{\frac{r^3}{abc \cdot \frac{r}{4R}}} = 3\sqrt[3]{\frac{r^2}{4Rrs \cdot \frac{1}{4R}}} =$$

$$= 3\sqrt[3]{\frac{r}{s}} \overset{MITRINOVIC}{\ge} 3\sqrt[3]{\frac{r}{3\sqrt{3}r}} = 3\sqrt[3]{\frac{1}{\left(\sqrt{3}\right)^3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

SOLUTION 4.87

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2 \overset{(1)}{\ge} \left(\frac{b^2+c^2}{4R}\right)^2 + \frac{(b-c)^2(a^2-b^2-c^2)^2}{16b^2c^2}$$

$$\textit{(1)} \Leftrightarrow \frac{2b^2+2c^2-a^2}{4} \ge \frac{\Delta^2}{a^2b^2c^2}(b^2+c^2)^2 + \frac{(b-c)^2\left(c^2+b^2-a^2\right)^2}{16b^2c^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{(b+c)^2}{4}+\frac{(b-c)^2}{4}-\frac{a^2}{4} \geq \frac{1}{4}\cdot\frac{b^2c^2\sin^2 A}{a^2b^2c^2}(b^2+c^2)^2+\frac{(b-c)^2\cdot 4b^2c^2\cos^2 A}{16b^2c^2}$$

$$\Leftrightarrow \frac{(b+c)^2-a^2}{4}+\frac{(b-c)^2}{4} \geq \frac{(b^2+c^2)\sin^2 A}{4a^2}+\frac{(b-c)^2}{4}(1-\sin^2 A)$$

$$\Leftrightarrow 4s(s-a) \geq \sin^2 A\left[\frac{(b^2+c^2)^2}{a^2}-(b-c)^2\right] \Leftrightarrow$$

$$\Leftrightarrow 4bc\cos^2\frac{A}{2} \geq 4\sin^2\frac{A}{2}\cos^2\frac{A}{2}\left[\frac{(b^2+c^2)^2}{a^2}-(b^2+c^2)+2bc\right] \Leftrightarrow$$

$$\Leftrightarrow \cos^2\frac{A}{2}\left[bc-\sin^2\frac{A}{2}\left\{(b^2+c^2)\left(\frac{b^2+c^2-a^2}{a^2}\right)+2bc\right\}\right] \geq 0$$

$$\Leftrightarrow \cos^2\frac{A}{2}\left[bc-\sin^2\frac{A}{2}\left\{(b^2+c^2)\frac{2bc\cos A}{a^2}+2bc\right\}\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow bc\cos^2\frac{A}{2}\left[1-2\sin^2\frac{A}{2}\left\{1+\frac{b^2+c^2}{a^2}\cos A\right\}\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow bc\cos^2\frac{A}{2}\left[1-(1-\cos A)\left(1+\frac{b^2+c^2}{a^2}\cos A\right)\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow bc\cos^2\frac{A}{2}\left[1-1-\frac{b^2+c^2}{a^2}\cos A+\cos A+\cos^2 A\left(\frac{b^2+c^2}{a^2}\right)\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow bc\cos^2\frac{A}{2}\left[\cos^2 A\left(\frac{b^2+c^2}{a^2}\right)-\cos A\left(\frac{b^2+c^2}{a^2}-1\right)\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow bc\cos^2\frac{A}{2}\left[\cos^2 A\frac{b^2+c^2}{a^2}-\frac{2bc\cos^2 A}{a^2}\right] \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{bc\cos^2\frac{A}{2}\cos^2 A}{a^2}(b-c)^2 \geq 0 \rightarrow \textit{true} \Rightarrow \textit{(1) is true (Proved)}$$

SOLUTION 4.88

Solution by Soumava Chakraborty-Kolkata-India

$$RHS=\sqrt{6+\sum\frac{2\Delta}{a}\cdot\frac{s-a}{\Delta}}=\sqrt{6+2\sum\frac{s-a}{a}}=\sqrt{6+2s\sum\frac{1}{a}-6}\overset{(1)}{=}\sqrt{2s\sum\frac{1}{a}}$$

$$LHS=\sum\sqrt{\frac{2\Delta}{a}\cdot\frac{s}{\Delta}-2}=\sum\sqrt{2\left(\frac{s}{a}-1\right)}=\sum\sqrt{2\left(\frac{s-a}{a}\right)}\overset{CBS}{\leq}\sqrt{\sum\{2(s-a)\}}\sqrt{\sum\frac{1}{a}}$$

$$=\sqrt{2s\sum\frac{1}{a}}=RHS\ \ \textit{(by (1)) (Proved)}$$

SOLUTION 4.89

Solution by Daniel Sitaru-Romania

Known:

$$m_a + m_b + m_c \ge 9r \quad (1)$$

$$\sum_{cyc} \frac{m_a^2}{a} \overset{BERGSTROM}{\ge} \frac{(m_a + m_b + m_c)^2}{a + b + c} \overset{(1)}{\ge}$$

$$\ge \frac{81r^2}{2s} = \frac{81r^2 s}{2s^2} \overset{MITRINOVIC}{\ge} \frac{81r^2 s}{2 \cdot \frac{27R^2}{4}} = 6s\left(\frac{r}{R}\right)^2$$

SOLUTION 4.90

Solution by Marian Ursărescu – Romania

$$\sum a^4 \cdot 2bc\cos A \ge 32 \cdot \frac{abc}{4S} \cdot S^2\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C} \Leftrightarrow$$

$$\Leftrightarrow 2abc\sum a^3\cos A \ge 8abcS\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C} \Leftrightarrow$$

$$\Leftrightarrow \sum a^3\cos A \ge 4S\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C} \quad \textit{(1)}$$

$$\sum a^3\cos A \ge 3abc\sqrt[3]{\cos A\cos B\cos C} \quad \textit{(2)}$$

From (1)+(2) we must show:

$$3abc\sqrt[3]{\cos A\cos B\cos C} \ge 4 \cdot \frac{abc}{4R}\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C}$$

$$\Leftrightarrow 3\sqrt[3]{\cos A\cos B\cos C} \ge \frac{1}{R}\sqrt{2(a^2+b^2+c^2)\cos A\cos B\cos C} \Leftrightarrow$$

$$\Leftrightarrow 3^6R^6\cos^2 A\cos^2 B\cos^2 C \ge 8(a^2+b^2+c^2)^3\cos^3 A\cos^3 B\cos^2 C \Leftrightarrow$$

$$3^6R^6 \ge 8(a^2+b^2+c^2)^3\cos A\cos B\cos C \quad \textit{(3)}$$

But $\cos A\cos B\cos C \le \frac{1}{8}$ ***(4) From (3)+(4) we must show:***

$3^6R^6 \ge (a^2+b^2+c^2)^3 \Leftrightarrow 9R^2 \ge a^2+b^2+c^2$***, which its true.***

SOLUTION 4.91

Solution by Daniel Sitaru – Romania

Known: $4m_bm_c \le 2a^2 + bc$ ***(1)***

$$4\sum_{cyc} m_bm_c - 4R\sum_{cyc}\frac{h_bh_c}{h_a} \overset{(1)}{=} \sum_{cyc}(2a^2+bc) - 4R\sum_{cyc}\frac{\frac{2S}{b}\cdot\frac{2S}{c}}{\frac{2S}{a}} =$$

$$= 2\sum_{cyc} a^2 + \sum_{cyc} bc - 8RS\sum_{cyc}\frac{a}{bc} = 2\sum_{cyc} a^2 + \sum_{cyc} bc - \frac{8RS}{abc}\sum_{cyc} a^2 =$$

$$= 2\sum_{cyc} a^2 + \sum_{cyc} bc - 2\sum_{cyc} a^2 = \sum_{cyc} bc = s^2 + 4Rr + r^2$$

SOLUTION 4.92

Solution by Mohamed Alhafi-Aleppo-Syria

$$\textit{Let } \sqrt[3]{\frac{\sin A}{\sin B}} = x, \sqrt[3]{\frac{\sin A}{\sin C}} = y \textit{ then our inequality is:}$$

$$x + \frac{y}{x} + \frac{1}{y} - y - \frac{1}{x} - \frac{x}{y} < 1 \Leftrightarrow \frac{(xy + 1 - x - y)(x - y)}{xy} < 1 \Leftrightarrow$$

$$\Leftrightarrow (x-1)(y-1)(x-y) < xy$$

$$\textit{Note that: } x = \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}, y = \sqrt[3]{\frac{a}{c}} = \frac{\sqrt[3]{a}}{\sqrt[3]{c}}$$

But since a, b, c are lenghts of sides of a triangle then $\alpha = \sqrt[3]{a}, \beta = \sqrt[3]{b}, \gamma = \sqrt[3]{c}$ are lenghts of sides of a triangle too

$$\textit{Note that } (x-1)(y-1) = \left(\frac{\alpha}{\beta} - 1\right)\left(\frac{\alpha}{\beta} - 1\right) = \left(\frac{\alpha-\beta}{\beta}\right)\left(\frac{\alpha-\gamma}{\gamma}\right) < \frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma} = 1$$

$$x - y = \frac{\alpha}{\beta} - \frac{\alpha}{\gamma} = \frac{\alpha(\gamma - \beta)}{\beta\gamma} < \frac{\alpha^2}{\beta\gamma} = xy$$

$$\textit{So: } (x-1)(y-1)(x-y) < xy$$

SOLUTION 4.93

Solution by Bogdan Fustei-Romania

$$\left.\begin{array}{l} R_a = 2R\sin\frac{A}{2} \ (and\ analogous) \\ \sin\frac{A}{2} = \sqrt{\frac{r_a - r}{4R}} \ (and\ analogous) \end{array}\right\rangle R_a = \sqrt{R(r_a - r)} \textit{ (and analogous)}$$

$$R_a^4 = R^2(r_a - r)^2 \textit{ (and analogous)} \Rightarrow R_a^4 + R_b^4 + R_c^4 = R^2 \cdot \sum (r_a - r)^2$$

$$R_a^4 + R_b^4 + R_c^4 = R^2\left[\sum r_a^2 + 3r^2 - 2r(r_a + r_b + r_c)\right]$$

$$r_a r_b + r_b r_c + r_a r_c = s^2 \Rightarrow \sum r_a^2 = (r_a + r_b + r_c)^2 - 2\sum r_a r_b$$

$$\sum r_a^2 = (r_a + r_b + r_c)^2 - 2s^2$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(r_a + r_b + r_c)^2 - 2s^2 - 2r(r_a + r_b + r_c) + 3r^2]$$

$$R_a^4+R_b^4+R_c^4=R^2[(R_a+R_b+R_c)(R_a+R_b+R_c-2r)-2s^2+3r^2]$$

$$R_a^4+R_b^4+R_c^4=R^2[(4R+r)(4R-r)-s^2+3r^2]$$

$$R_a^4+R_b^4+R_c^4=R^2(16R^2-r^2-2s^2+3r^2)=2R^2(8R^2-s^2+r^2)$$

$\frac{R_a^4+R_b^4+R_c^4}{4R^2}=\frac{2R^2(8R^2-s^2+r^2)}{4R^2}=\frac{8R^2-s^2+r^2}{2}$. ***The inequality from enunciation becomes:***

$$2R^2-2Rr-r^2\le\frac{8R^2-s^2+r^2}{2}\le 4R^2-8Rr+3r^2$$

$$4R^2-4Rr-2r^2\le 8R^2-s^2+r^2\Rightarrow s^2\le 8R^2+r^2-4R^2+4Rr+2r^2=$$

$$=4R^2+4Rr+3r^2 \quad \textit{(Gerretsen's inequality)}$$

$$8r^2-s^2+r^2\le 8R^2-16Rr+6r^2\Rightarrow 16Rr-5r^2\le s^2 \quad \textit{(Gerretsen's inequality)}$$

From the above the inequality from enunciation is proved.

SOLUTION 4.94

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sum(r_a-r_b)^2}{3s^2}\le\frac{R-2r}{r}$$

$$\textit{Given inequality}\Leftrightarrow r\sum\left(r_a^2+r_b^2-2r_ar_b\right)\le 3s^2(R-2r)\Leftrightarrow$$

$$\Leftrightarrow 2r\sum r_a^2-2r\sum r_ar_b\le 3s^2(R-2r)\Leftrightarrow 2r(4R+r)^2-4rs^2-2rs^2\le$$

$$\le 3s^2(R-2r)\Leftrightarrow 3Rs^2\overset{(1)}{\ge}2r(4R+r)^2$$

$$\textit{Now, LHS of (1)}\overset{Gerretsen}{\ge}3R(16Rr-5r^2)\overset{(1)}{\ge}2r(4R+r)^2\Leftrightarrow$$

$$\Leftrightarrow 16R^2-31Rr-2r^2\overset{?}{\ge}0\Leftrightarrow$$

$$\Leftrightarrow(R-2r)(16R+r)\overset{?}{\ge}0\to \textit{true (Euler)}\Rightarrow \textit{(1) is true (Proved)}$$

SOLUTION 4.95

Solution by Daniel Sitaru-Romania

Panaitopol's inequality (1980): $m_aw_a\ge s(s-a)$

$$2R\sum_{cyc}m_aw_ah_a=2R\sum_{cyc}m_aw_a\cdot\frac{2S}{a}=4RS\sum_{cyc}\frac{m_aw_a}{a}\ge$$

$$\overset{PANAITOPOL}{\ge}4RS\sum_{cyc}\frac{1}{a}\cdot s(s-a)=4R\cdot rs\cdot s\sum_{cyc}\frac{s-a}{a}=$$

$$= 4Rrs^2 \cdot \frac{s^2 + r^2 - 8Rr}{4Rr} \geq s^2(s^2 + r^2 - 8Rr) \overset{MITRINOVIC}{\overbrace{\geq}}$$

$$\geq 27r^2(s^2 + r^2 - 8Rr) \geq 9r^2(s^2 + r^2 + 4Rr) \leftrightarrow$$

$$3s^2 + 3r^2 - 24Rr \geq s^2 + r^2 + 4Rr \leftrightarrow 2s^2 \geq 28Rr - 2r^2$$

$$\leftrightarrow s^2 \geq 14Rr - r^2 (to\ prove)$$

$$s^2 \overset{GERRETSEN}{\overbrace{\geq}} 16Rr - 5r^2 \geq 14Rr - r^2 \leftrightarrow 2Rr \geq 4r^2 \leftrightarrow R \geq 2r$$

SOLUTION 4.96

Solution by Soumava Chakraborty-Kolkata-India

$$\sum ab \overset{Gordon}{\geq} 4\sqrt{3}S$$

Applying the above inequality on a triangle with sides $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$ whose area of course will be $\frac{S}{3}$, we get,

$$\frac{4}{9}\sum m_a m_b \geq 4\sqrt{3}\frac{S}{3} \Rightarrow \sum m_a m_b \overset{(1)}{\geq} 3\sqrt{3}S$$

Now, $\left(\sum m_a\right)^2 \geq 3\sum m_a m_b \overset{by\ (1)}{\geq} 9\sqrt{3}S$ ***(Proved)***

SOLUTION 4.97

Solution by Marian Ursărescu-Romania

$$\sqrt{\frac{a}{r_a}} + \sqrt{\frac{b}{r_b}} + \sqrt{\frac{c}{r_c}} \geq 3\sqrt[3]{\frac{abc}{r_a r_b r_c}} \quad (1)$$

But $abc = 4pRr$ ***and*** $r_a r_b r_c = p^2 r$ ***(2)***

From (1)+(2) $\Rightarrow \sqrt{\frac{a}{r_a}} + \sqrt{\frac{b}{r_b}} + \sqrt{\frac{c}{r_c}} \geq 3\sqrt[6]{\frac{4R}{p}}$ ***(3)***

But $p \leq \frac{3\sqrt{3}}{2}R \Rightarrow \frac{R}{p} \geq \frac{2}{3\sqrt{3}}$ ***(4)***

From (3)+(4)$\Rightarrow \sqrt{\frac{a}{r_a}} + \sqrt{\frac{b}{r_b}} + \sqrt{\frac{c}{r_c}} \geq 3\sqrt[6]{\frac{8}{3\sqrt{3}}} = 3\sqrt[6]{\frac{2^3}{\sqrt{3^3}}}$

$$= 3\sqrt{\frac{2}{\sqrt{3}}} = \sqrt{\frac{18}{\sqrt{3}}} = \sqrt[4]{\frac{18^2}{3}} = \sqrt[4]{108}$$

Now from Cauchy's inequality $\Rightarrow$

$$\sqrt{\frac{a}{r_a}}+\sqrt{\frac{b}{r_b}}+\sqrt{\frac{c}{r_c}}\le\sqrt{3\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)}\ \ (5)$$

From (5) we must show: $\sqrt{3\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)}\le\sqrt{3\sqrt{3}\frac{R}{r}}\Rightarrow$

$$\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\le\sqrt{3}\frac{R}{r}\ \ (6)$$

But $\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=\frac{2(4R+r)}{p}$ *(7)*

From (6)+(7)$\Rightarrow\sqrt{3}\frac{R}{r}\ge\frac{2(4R+r)}{p}\Leftrightarrow\sqrt{3}pR\ge 2(4R+r)r$ *(8)*

But $p\ge 3\sqrt{3}r\Rightarrow 9Rr\ge 2r(4R+r)\Leftrightarrow 9R\ge 8R+2r\Leftrightarrow R\ge 2r$ *(true)*

SOLUTION 4.98

Solution by Daniel Sitaru-Romania

$$I_aI_b=4Rcos\frac{C}{2},I_bI_c=4Rcos\frac{A}{2},I_cI_a=4Rcos\frac{B}{2}$$

$$\sphericalangle(I_bVI_c)=\pi-A,\sphericalangle(I_cVI_a)=\pi-B,\sphericalangle(I_aVI_b)=\pi-C$$

$$R_a=\frac{I_bI_c}{2sinA}=\frac{4rcos\frac{A}{2}}{4sin\frac{A}{2}cos\frac{A}{2}}=\frac{R}{sin\frac{A}{2}},R_b=\frac{R}{sin\frac{B}{2}},R_c=\frac{R}{sin\frac{C}{2}}$$

$$\sum_{cyc}\frac{w_a}{R_a}=\frac{1}{R}\sum_{cyc}\frac{2}{b+c}\sqrt{bcs(s-a)}\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}=$$

$$=\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{R}\sum_{cyc}\frac{1}{b+c}\overset{BERGSTROM}{\geq}\frac{2S}{R}\cdot\frac{(1+1+1)^2}{b+c+c+a+a+b}=$$

$$=\frac{2rs}{R}\cdot\frac{9}{4s}=\frac{9r}{2R}$$

SOLUTION 4.99

Solution by Soumava Chakraborty-Kolkata-India

In any acute-angled $\Delta ABC,\sum a^3\cos^3 A+\frac{3abc(\sum a^2-8R^2)}{8R^2}\ge 2\sum b\cos B\,a^2\cos^2 A$

$$\frac{\sum a^2-8R^2}{8R^2}=\frac{s^2-4Rr-r^2-4R^2}{4R^2}=\cos A\cos B\cos C$$

$$\Rightarrow 3abc\left(\frac{\sum a^2-8R^2}{8R^2}\right)\overset{(1)}{=}3(a\cos A)(b\cos B)(c\cos C)$$

(1) $\Rightarrow$ *given inequality* $\Leftrightarrow \sum a^3\cos^3 A + 3\prod(a\cos A) \overset{(2)}{\geq} 2\sum(a\cos A)^2(b\cos B)$

Now, $b\cos B + c\cos C - a\cos A = R(\sin 2B + \sin 2C - \sin 2A) =$

$= R\{2\sin A\cos(B-C) + 2\sin A\cos(B+C)\}$

$(\because \sin(B+C) = \sin A \ \& \cos A = -\cos(B+C))$

$= 4R\sin A\cos B\cos C > 0$ *($\because$ ΔABC is acute-angled)*

Similarly, $c\cos C + a\cos A - b\cos B > 0$ *&* $a\cos A + b\cos B - \cos C > 0$

Let $b\cos B + c\cos C - a\cos A = x, c\cos C + a\cos A - b\cos B = y$

& $a\cos A + b\cos B - c\cos C$ *(of course* $x, y, z > 0$*)*

Then, $a\cos A = \frac{y+z}{2}, b\cos B = \frac{z+x}{2}$ *&* $c\cos C = \frac{x+y}{2}$

Via above substitution & (2) , given inequality $\Leftrightarrow \sum\frac{(y+z)^3}{8} + \frac{3}{8}\prod(x+y) \geq$

$$\geq \frac{2}{8}\sum(y+z)^2(z+x) \Leftrightarrow 2\sum x^3 + 3\sum x^2y + 3\sum xy^2 + 6xyz + 3\sum x^2y +$$

$$+3\sum xy^2 \geq 2\sum(y^2z + xy^2 + z^3 + z^2x + 2yz^2 + 2xyz) \Leftrightarrow$$

$$\Leftrightarrow 2\sum x^3 + 6\sum x^2y + 6\sum xy^2 + 6xyz \geq 4\sum x^2y + 6\sum xy^2 + 2\sum x^3 + 12xyz$$

$\Leftrightarrow 2\sum x^3y \geq 6xyz \Leftrightarrow \sum x^2y \geq 3xyz \rightarrow$ *true by A-G (proved)*

SOLUTION 4.100

Solution by Marian Ursărescu – Romania

$\frac{a}{\sin A} = 2R \Rightarrow \frac{a}{\frac{\sqrt{3}}{2}} = 2R \Rightarrow R = \frac{9}{\sqrt{3}} \Rightarrow \sqrt{3}R = a \Rightarrow 3a + a \geq \frac{4bc}{a} \Leftrightarrow 4a^2 \geq 4bc \Leftrightarrow a^2 \geq bc$ *(1)*

But $\sqrt{bc} \leq \frac{b+c}{2}$ *(2). From (1)+(2) we must show* $\frac{b+c}{2} \leq a \Leftrightarrow b + c \leq 2a \Leftrightarrow$

$$\Leftrightarrow \sin B + \sin C \leq 2\sin A \Leftrightarrow 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \leq 2\sin A \Leftrightarrow$$

$$\Leftrightarrow \sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right) \leq \sin A \Leftrightarrow \cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right) \leq 2\sin\frac{A}{2}\cos\frac{A}{2} \Leftrightarrow$$

$\Leftrightarrow \cos\left(\frac{B-C}{2}\right) \leq 2\sin\frac{A}{2} \Leftrightarrow \cos\left(\frac{B-C}{2}\right) \leq 1$*, true with equality for* $B = C$*. i.e equilateral* Δ

SOLUTION 4.101

Solution by Daniel Sitaru-Romania

$$I_aI_b = 4R\cos\frac{C}{2}, I_bI_c = 4R\cos\frac{A}{2}, I_cI_a = 4R\cos\frac{B}{2}$$

$$\sphericalangle(I_bVI_c)=\pi-A, \sphericalangle(I_cVI_a)=\pi-B, \sphericalangle(I_aVI_b)=\pi-C$$

$$R_a=\frac{I_bI_c}{2\sin A}=\frac{4R\cos\frac{A}{2}}{4\sin\frac{A}{2}\cos\frac{B}{2}}=\frac{R}{\sin\frac{A}{2}}, R_b=\frac{R}{\sin\frac{B}{2}}, R_c=\frac{R}{\sin\frac{C}{2}}$$

$$\sum_{cyc}\frac{h_a}{R_a^2}=\sum_{cyc}\frac{\frac{2S}{a}}{\frac{R^2}{\sin^2\frac{A}{2}}}=\frac{2S}{R^2}\sum_{cyc}\frac{\sin^2\frac{A}{2}}{a}=\frac{2S}{R^2}\sum_{cyc}\frac{(s-a)(s-b)}{abc}=$$

$$=\frac{2S}{abcR^2}\sum_{cyc}(s-a)(s-b)=\frac{2S}{4RS\cdot R^2}\sum_{cyc}(s-a)(s-b)=$$

$$=\frac{1}{2R^3}\cdot r(4R+r)=\frac{r}{2R^3}(r_a+r_b+r_c)$$

SOLUTION 4.102

Solution by Tran Hong-Vietnam

We have:

$$\frac{r}{R}=\frac{(a+b-c)(b+c-a)(c+a-b)}{2abc};$$

Let $f(c)=\frac{(a+b-c)(b+c-a)(c+a-b)}{2abc}$,

$$\Rightarrow f'(c)=\frac{(a+b)c^2-2c^3+(a+b)(a-b)^2}{2abc^2}\geq\frac{(a+b-2c)c^2}{2abc^2}=\frac{a+b-2c}{2ab}\geq 0;$$

$$\Rightarrow f(c)\leq f\left(\frac{a+b}{3}\right)=\frac{4(2b-a)(2a-b)}{9ab}=\frac{4}{9}-\frac{2}{9}\cdot\frac{(a-b)^2}{ab}\leq\frac{4}{9};$$

$\Leftrightarrow \frac{r}{R}\leq\frac{4}{9}\Leftrightarrow 9r\leq 4R.$ ***(Proved)***

SOLUTION 4.103

Solution by Marian Ursărescu-Romania

We have: $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}\Leftrightarrow m_a\geq\frac{b+c}{2}\sqrt{\frac{p(p-a)}{bc}}\Rightarrow$

$m_a\geq\sqrt{bc}\sqrt{\frac{p(p-a)}{bc}}\Rightarrow m_a\geq\sqrt{p(p-a)}\Rightarrow$ ***we must show:***

$$\frac{1}{p}\left(\frac{a^2}{p-a}+\frac{b^2}{p-b}+\frac{c^2}{p-c}\right)\leq 4\left(\frac{R}{r}-1\right)\quad (1)$$

But in ΔABC ***we have:*** $\frac{a^2}{p-a}+\frac{b^2}{p-b}+\frac{c^2}{p-c}=\frac{4p(R-r)}{r}$ ***(2)***

From (1)+(2)⇒ $\frac{1}{p}\left(\frac{a^2}{p-a}+\frac{b^2}{p-b}+\frac{c^2}{p-c}\right)=\frac{4p(R-r)}{pr}=4\left(\frac{R}{r}-1\right)$ ⇒ *relationship it's true.*

SOLUTION 4.104

Solution by Marian Ursărescu-Romania

$$\frac{1}{a\cos B\cos C}+\frac{1}{b\cos C\cos A}+\frac{1}{c\cos A\cos B}=\frac{bc\cos A+ac\cos B+ab\cos C}{abc\cos A\cos B\cos C}\quad (1)$$

$$\text{But } a^2=b^2+c^2-2bc\cos A\Rightarrow bc\cos A=\frac{b^2+c^2-a^2}{2}\Rightarrow$$

$$bc\cos A+ac\cos B+ab\cos C=\frac{1}{2}(a^2+b^2+c^2)\quad (2)$$

$$\text{From (1)+(2) we must show this: }\frac{a^2+b^2+c^2}{2abc\cos A\cos B\cos C}\ge\frac{18}{S}\quad (3)$$

$$\text{But }\cos A\cos B\cos C\le\frac{1}{8}\Rightarrow\frac{1}{\cos A\cos B\cos C}\ge 8\quad (4)$$

$$\text{From (3)+(4) we must show: }\frac{4(a^2+b^2+c^2)}{abc}\ge\frac{18}{S}\quad (5)$$

$$\text{But } a^2+b^2+c^2=2(s^2-r^2-4Rr)\quad (6)$$

$$\text{and } abc=4SRr\quad (7)$$

$$\text{From (5)+(6)+(7) we must show this: }\frac{4\cdot 2(S^2-r^2-4Rr)}{4SRr}\ge\frac{18}{S}\Leftrightarrow$$

$$\frac{s^2-r^2-4Rr}{Rr}\ge 9\Leftrightarrow S^2-r^2-4Rr\ge 9Rr\Leftrightarrow S^2\ge 13Rr+r^2\quad (8)$$

But from Gerretsen's inequality: $S^2\ge 16Rr-5r^2$ *and from (8) we must show this:*

$$16Rr-5r^2\ge 13Rr+r^2\Leftrightarrow 3Rr\ge 6r^2\Leftrightarrow R\ge 2r\ \ \text{(true)}$$

SOLUTION 4.105

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{r_a}{h_a}=\sum_{cyc}\frac{\frac{S}{s-a}}{\frac{2S}{a}}=\frac{1}{2}\sum_{cyc}\frac{a}{s-a}=\frac{1}{2}\cdot\frac{2(2R-r)}{r}=\frac{2R-r}{r}$$

$$\frac{2R-r}{r}\ge 3\leftrightarrow 2R-r\ge 3r\leftrightarrow R\ge 2r$$

$$\frac{2R-r}{r}\le\left(\frac{R}{r}\right)^2-\frac{R}{2r}\leftrightarrow 2r(2R-r)\le 2R^2-rR\leftrightarrow 2R^2-5Rr+2r^2\ge 0\leftrightarrow$$

$$\leftrightarrow 2R^2-4Rr-Rr+2r^2\ge 0\leftrightarrow 2R(R-2r)-r(R-2r)\ge 0\leftrightarrow$$

$$\leftrightarrow (R-2r)(2R-r)\ge 0$$

SOLUTION 4.106

Solution by Tran Hong-Vietnam

$$64rr'\sqrt{ss'}+4\left(\sqrt{Rrs}-\sqrt{R'r's'}\right)^2$$

$$=64\sqrt{rr'\cdot\frac{abc}{4R}\cdot\frac{a'b'c'}{4R'}}+4\left(\sqrt{R\cdot\frac{abc}{4R}}-\sqrt{R'\cdot\frac{a'b'c'}{4R'}}\right)^2$$

$$=16\sqrt{\frac{r}{R}\cdot\frac{r'}{R'}}\sqrt{abc\cdot a'b'c'}+\left(\sqrt{abc}-\sqrt{a'b'c'}\right)^2$$

$$\overset{Euler}{\leq}8\sqrt{abc\cdot a'b'c'}+abc-2\sqrt{abc\cdot a'b'c'}+a'b'c'$$

$$=abc+a'b'c'+6\sqrt{abc\cdot a'b'c'}\quad (1)$$

$$(a+a')(b+b')(c+c')=(abc+a'b'c'+a'bc+ab'c+abc'+a'b'c+a'bc'+ab'c')$$

$$\overset{Cauchy}{\geq}\left\{abc+a'b'c'+6\sqrt[6]{(abc)^4\cdot(a'b'c')^3}\right\}$$

$$=abc+a'b'c'+6\sqrt{abc\cdot a'b'c'}\quad (2)$$

From (1) and (2) ⇒ Proved. Equality ⇔ $a=a',b=b',c=c'$.

SOLUTION 4.107

Solution by Lahiru Samarakoon-Sri Lanka

$$2R\cos A\cos A+2R\cos B\cos C+2R\cos C\cos C\geq 12\sqrt{3}R\cos A\cos B\cos C$$

$$\underbrace{R(\sin 2A+\sin 2B+\sin 2C)}_{4R\sin A\sin B\sin C}\geq\underset{\geq 12\sqrt{3}\cos A\cos B\cos C\times R}{12\sqrt{3}\cos A\cos B\cos C\times\ R}$$

We have to prove, $\tan A\tan B\tan C\geq 3\sqrt{3}$

$$A=\frac{\sum\tan A}{3}\geq\tan\left(\frac{A+B+C}{3}\right)=\sqrt{3}$$

So, it's true.

$$\left(\because\sum\tan A=\tan A\tan B\tan C\right)$$

SOLUTION 4.108

Solution by Lahiru Samarakoon-Sri Lanka

AM-GM

$$(m_a+m_b+m_c)\geq 3\sqrt[3]{m_am_bm_c}$$

So, $m_am_bm_c(m_a+m_b+m_c)\geq 3\left[m_a^4m_b^4m_c^4\right]^{\frac{1}{3}}$ *but,* $m_a\geq\sqrt{p(p-a)}$*. So,*

$$\geq 3[p^6(p-a)^2(p-b)^2(p-c)^2]^{\frac{1}{3}}=3[S^4p^4]^{\frac{1}{3}}$$

But, $p^2\geq 3\sqrt{3}S$*. So,* $\geq 3[S^4+27S^2]^{\frac{1}{3}}=9S^2$

SOLUTION 4.109

Solution by Soumava Chakraborty-Kolkata-India

In any $\left(\sum\sqrt{\sin A}\right)\left(\sum\frac{1}{\sqrt{\sin A}}\right)\leq\frac{9m_am_bm_c}{h_ah_bh_c}$

$$LHS=3+\sum\left(\sqrt{\frac{\sin A}{\sin B}}\right)+\sum\left(\sqrt{\frac{\sin B}{\sin A}}\right)=3+\sum\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)$$

$$\overset{CBS}{\leq}3+\sqrt{6}\sqrt{\sum\left(\frac{a}{b}+\frac{b}{a}\right)}=3+\sqrt{6}\sqrt{\frac{\sum a^2b+\sum ab^2}{abc}}=3+\sqrt{\frac{6}{4Rrs}}\sqrt{\sum ab\,(2s-c)}$$

$$=3+\sqrt{\frac{3}{2Rrs}}\sqrt{2s(s^2+4Rr+r^2)-12Rrs}=3+\sqrt{\frac{3}{Rr}}\sqrt{s^2-2Rr+r^2}$$

$$\therefore LHS\overset{(1)}{\leq}3+\sqrt{\frac{3}{Rr}}\sqrt{s^2-2Rr+r^2}$$

Now, RHS $=\frac{9m_am_bm_c}{h_ah_bh_c}\overset{m_a\geq\sqrt{s(s-a)}}{\geq}\frac{9s\cdot rs}{\frac{a^2b^2c^2}{8R^3}}=\frac{72R^3rs^2}{16R^2r^2s^2}=\frac{9R}{2r}\therefore RHS\overset{(2)}{\geq}\frac{9R}{2r}$

(1), (2) $\Rightarrow$ *it suffices to prove:* $\frac{9R-6r}{2r}\geq\sqrt{\frac{3}{Rr}}\sqrt{s^2-2Rr+r^2}$

$$\Leftrightarrow\frac{9(3R-2r)^2}{4r^2}\geq\frac{3(s^2-2Rr+r^2)}{Rr}\Leftrightarrow 3R(3R-2r)^2\overset{(3)}{\geq}4r(s^2-2Rr+r^2)$$

$\because R\overset{Euler}{\geq}2r\therefore$ *(3)* $\Rightarrow$ *it suffices to prove:* $2(s^2-2Rr+r^2)\overset{(4)}{\leq}3(3R-2r)^2$

Now, LHS of (4) $\overset{Gerretsen}{\leq}8R^2+4Rr+8r^2\overset{?}{\leq}27R^2-36Rr+12r^2$

$\Leftrightarrow 19R^2-40Rr+4r^2\overset{?}{\geq}0\Leftrightarrow(R-2r)(19R-2r)\overset{?}{\geq}0\rightarrow$ *true (Euler) (Proved)*

SOLUTION 4.110

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{w_a}{h_a} + 2\sum \frac{m_a}{w_a} \overset{(1)}{\leq} \frac{\sum w_a}{r}$$

$$\text{(1)} \Leftrightarrow \sum w_a\left(\frac{1}{r} - \frac{1}{h_a}\right) \overset{(2)}{\geq} 2\sum \frac{m_a}{w_a}$$

$$\text{LHS of (2)} = \sum w_a\left(\frac{1}{r} - \frac{a}{2rs}\right) = \sum w_a\left(\frac{2s-a}{2rs}\right) = \sum \frac{2bc}{b+c}\cos\frac{A}{2}\left(\frac{b+c}{2rs}\right) = \sum \frac{abc}{rsa}\cos\frac{A}{2}$$

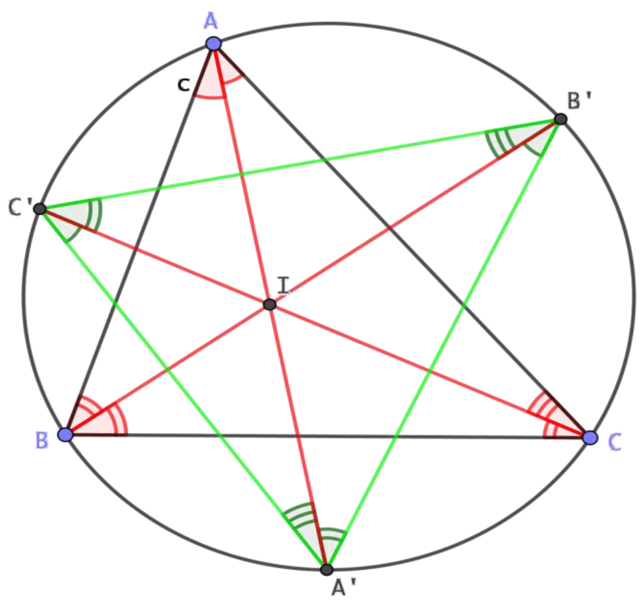
$$= \sum \frac{4Rrs\cos\frac{A}{2}}{4Rrs\sin\frac{A}{2}\cos\frac{A}{2}} \overset{(a)}{=} \sum \frac{1}{\sin\frac{A}{2}}$$

$$\text{RHS of (2)} \overset{Tsintsifas}{\leq} \sum \frac{b^2+c^2}{bc} = \sum\left(\frac{b}{c} + \frac{c}{b}\right) = \sum\left(\frac{c}{a} + \frac{b}{a}\right) = \sum \frac{b+c}{a} = \sum \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{2R\sin\frac{A}{2}\cos\frac{A}{2}} \leq \sum \frac{1}{\sin\frac{A}{2}}$$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1\ etc\ as\ -\frac{\pi}{2} < \frac{B-C}{2} < \frac{\pi}{2}\ etc\right) \overset{by\,(a)}{=} \text{LHS of (2)} \Rightarrow \text{(2) is true (Proved)}$$

SOLUTION 4.111

Solution by Soumava Chakraborty-Kolkata-India



$$\because I \text{ is the orthocenter of } \Delta A'B'C' \ \&\ LA' = \frac{B+C}{2}, LB' = \frac{C+A}{2}\ \&\ LC' = \frac{A+B}{2}$$

$$\therefore IA' = 2R\cos\frac{B+C}{2} \overset{(1)}{=} 2R\sin\frac{A}{2}, IB' \overset{(2)}{=} 2R\sin\frac{B}{2}\ \&\ IC' \overset{(3)}{=} 2R\sin\frac{C}{2}$$

$$(\because \text{circumradius of } \Delta A'B'C' = R)$$

$$\text{Also, } RHS \underset{(4)}{\overset{A-G}{\leq}} \frac{48\sqrt{3}r^3}{4}\sum\frac{1}{ab} = 12\sqrt{3}r^3\left(\frac{2S}{4Rrs}\right) = 6\sqrt{3}\left(\frac{r^2}{R}\right)$$

(1), (2), (3), (4)⇒ it suffices to prove:

$$2R^2\sum\sin\frac{A}{2}\overset{(5)}{\geq}6\sqrt{3}r^2$$

$$\textit{Now, LHS of (5)} = 2R^2\sum\frac{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}{2\cos\frac{B-C}{2}}\geq R^2\sum(\cos B+\cos C)$$

$$\left(\because 0<\cos\frac{B-C}{2}\leq 1\ as -\frac{\pi}{2}<\frac{B-C}{2}<\frac{\pi}{2}\right)$$

$$=2R^2\left(1+\frac{r}{R}\right)=2R(R+r)\overset{Euler}{\geq}4r\cdot 3r=12r^2\overset{?}{\geq}6\sqrt{3}r^2\Leftrightarrow 2\overset{?}{\geq}\sqrt{3}\to \textit{true (proved)}$$

SOLUTION 4.112

Solution by Urfan Aliyev-Baku-Azerbaijan

$$2\sqrt[3]{abc}\leq\sqrt{3}(3R-2r)$$

$$2\sqrt[3]{(2R\sin A)(2R\sin B)(2r\sin C)}\leq\sqrt{3}(3R-2r)$$

$$4R\sqrt[3]{\sin A\sin B\sin C}\leq 3\sqrt{3}R-2\sqrt{3}r$$

$$\sqrt[3]{\sin A\sin B\sin C}\leq\frac{3\sqrt{3}}{4}-\frac{\sqrt{3}r}{2R}\ \left(\frac{r}{R}\leq\frac{1}{2}\right)$$

$$\sqrt[3]{\sin A\sin B\sin C}\leq\frac{3\sqrt{3}}{4}-\frac{\sqrt{3}r}{2R}\leq\frac{3\sqrt{3}}{4}-\frac{\sqrt{3}}{4}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$$

$$\sqrt[3]{\sin A\sin B\sin C}\leq\frac{\sqrt{3}}{2}$$

$$\sin A\sin B\sin C\leq\frac{\sqrt{27}}{8}=\frac{3\sqrt{3}}{8}\ \ \textit{(True)}$$

SOLUTION 4.113

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{2a(s-a)}{h_a}=\sum\frac{2a(s-a)a}{2rs}=\frac{1}{rs}\sum a^2(s-a)=$$

$$\frac{s\cdot 2(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{rs}=\frac{2s(2Rr+2r^2)}{rs}=4(R+r)$$

$$\therefore\frac{1}{6}\sum\frac{2a(s-a)}{h_a}=\frac{2}{3}(R+r)$$

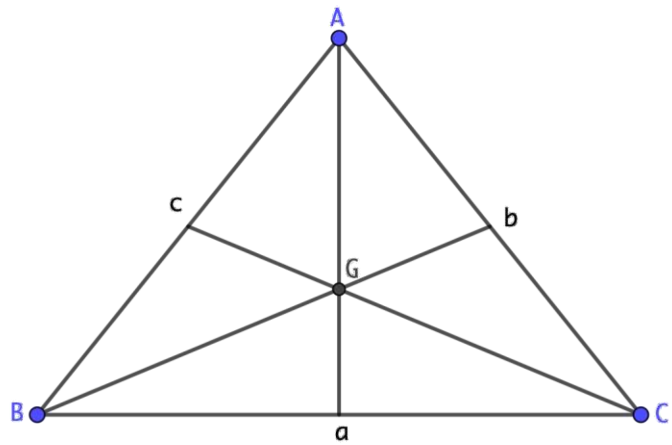
$$\therefore R\geq\frac{1}{6}\sum\frac{2a(s-a)}{h_a}\Leftrightarrow R\geq\frac{2}{3}(R+r)\Leftrightarrow R\geq 2r\to \textit{true}$$

$$\&\ \frac{1}{6}\sum\frac{2a(s-a)}{h_a}\geq 2r\Leftrightarrow\frac{R+r}{3}\geq r\Leftrightarrow R\geq 2r\to \textit{true}$$

SOLUTION 4.114

Solution by Lahiru Samarakoon-Sri Lanka

$$\sum\left(\frac{2m_a+2m_b}{m_c}\right)^7>\sum\left(\frac{3a}{m_a}\right)^7$$



$$\Delta ACG, AG=\frac{2}{3}m_a \text{ and } CG=\frac{2}{3}m_a$$

$$\text{So, to have:} AG+GC>AC,\ \text{So, } \frac{2m_a}{3}+\frac{2}{3}m_c>b$$

$$(2m_a+2m_c)>3b$$

$$\left(\frac{2m_a+2m_c}{m_b}\right)^7>\left(\frac{3b}{m_b}\right)^7\ (\because m_b>0)$$

So, similarly, from ΔAGB and ΔBGC, and by summation:

$$\sum\left(\frac{2m_a+2m_c}{m_b}\right)^7>\sum\left(\frac{3b}{m_b}\right)^7$$

SOLUTION 4.115

Solution by Soumava Chakraborty-Kolkata-India

(1) $$\Leftrightarrow 3\left(\frac{\sum a^2}{4S^2}\right)\le\frac{m_a^2m_b^2m_c^2}{S^4}\Leftrightarrow 4m_a^2m_b^2m_c^2\overset{(2)}{\ge}6r^2s^2(s^2-4Rr-r^2)$$

$$\text{Now, } m_a^2m_b^2m_c^2=\frac{(2b^2+2c^2-a^2)(2c^2+2a^2-b^2)(2a^2+2b^2-c^2)}{64}=$$

$$\overset{(a)}{=}\frac{-4\sum a^6+6(\sum a^4b^2+\sum a^2b^4)+3a^2b^2c^2}{64}$$

$$\text{Now, } \sum a^6=(\sum a^2)^3-3(a^2+b^2)(b^2+c^2)(c^2+a^2)=$$

$$=\left(\sum a^2\right)^3-3\left(\sum a^2-c^2\right)\left(\sum a^2-a^2\right)\left(\sum a^2-b^2\right)$$

$$= \left(\sum a^2\right)^3 - 3\left\{\left(\sum a^2\right)^3 - \left(\sum a^2\right)^3 + \left(\sum a^2\right)\left(\sum a^2b^2\right) - a^2b^2c^2\right\} =$$

$$\overset{(b)}{=} \left(\sum a^2\right)^3 - 3\left(\sum a^2\right)\left(\sum a^2b^2\right) + 3a^2b^2c^2$$

Also, $\sum a^4b^2 + \sum a^2b^4 = \sum a^2b^2\left(\sum a^2 - c^2\right) \overset{(c)}{=} \left(\sum a^2\right)\left(\sum a^2b^2\right) - 3a^2b^2c^2$

(a), (b), (c) $\Rightarrow m_a^2m_b^2m_c^2 = \frac{1}{64}\{-4(\sum a^2)^3 + 18(\sum a^2)(\sum a^2b^2) - 27a^2b^2c^2\}$

$$= \frac{1}{64}\left[\begin{matrix}-32(s^2-4Rr-r^2)^3 + 36(s^2-4Rr-r^2)\{(s^2+4Rr+r^2)^2 - 2abc(2s)\} \\ -432R^2r^2s^2\end{matrix}\right]$$

$$\overset{(d)}{=} \frac{1}{16}\left\{\begin{matrix}s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - 64R^3r^3 - 48R^2r^4 \\ -12Rr^5 - r^6\end{matrix}\right\}$$

(d) $\Rightarrow$ *(2)* $\Leftrightarrow s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - 64R^3r^3 -$

$$-48R^2r^4 - 12Rr^5 - r^6 \ge 24r^2s^2(s^2 - 4Rr - r^2)$$

$$\Leftrightarrow s^6 - s^4\left(12Rr - 9r^2\right) - s^2\left(60R^2r^2 + 24Rr^3 + 9r^4\right) - 64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \overset{(3)}{\ge} 0$$

Now, LHS of (3) $\overset{Gerretsen}{\ge} s^4(4Rr + 4r^2) - s^2(60R^2r^2 + 24Rr^3 + 9r^4) - 64R^3r^3 -$

$$-48R^2r^4 - 12Rr^5 - r^6 \underset{(4)}{\overset{Gerretsen}{\ge}} 0$$

Now, LHS of (4) $\overset{Gerretsen}{\ge} s^2\{(16Rr - 5r^2)(4Rr + 4r^2) - (60R^2r^2 + 24Rr^3 + 9r^4)\} -$

$$-64R^3r^3 - 48R^2r^4 - 12Rr^5 - r^6 \overset{?}{\ge} 0 \Leftrightarrow$$

$$\Leftrightarrow s^2(4R^2 + 20Rr - 29r^2) - 64R^3r - 48R^2r^2 - 12Rr^3 - r^4 \underset{(5)}{\overset{?}{\ge}} 0$$

Now, $4R^2 + 20Rr - 29r^2 \overset{Euler}{\ge} 4R^2 + 40r^2 - 29r^2 > 0 \therefore$ *LHS of (5)*

$$\overset{Gerretsen}{\ge} (16Rr - 5r^2)(4R^2 + 20Rr - 29r^2) - 64R^3r - 48R^2r^2 - 12Rr^3 - r^4 \overset{?}{\ge} 0$$

$\Leftrightarrow 7R^2 - 16Rr + 4r^2 \overset{?}{\ge} 0 \Leftrightarrow (R - 2r)(7R - 2r) \overset{?}{\ge} 0 \to$ *true (Euler) (Proved)*

SOLUTION 4.116

Solution by Daniel Sitaru-Romania

$$f: (0,\pi) \to \mathbb{R}, f(x) = (\sin x)^{\frac{1}{2}}, f''(x) = -\frac{1}{2}\sin x\,(\sin x)^{-\frac{1}{2}} - \frac{1}{4}\cos^2 x\,(\sin x)^{-\frac{3}{2}} < 0,$$

$$f - \textit{concave}$$

$$\frac{1}{3}\sum_{cyc(A,B,C)} f(A)+f\left(\frac{A+B+C}{3}\right)\le\frac{2}{3}\sum_{cyc(A,B,C)} f\left(\frac{B+C}{2}\right)$$

$$\frac{1}{3}\sum_{cyc(A,B,C)}\sqrt{\sin A}+\sin\left(\frac{\pi}{3}\right)\le\frac{2}{3}\sum_{cyc(A,B,C)}\sqrt{\sin\left(\frac{B+C}{2}\right)}$$

$$\sum_{cyc(A,B,C)}\sqrt{\sin A}+3\sqrt{\frac{\sqrt{3}}{2}}\le 2\sum_{cyc(A,B,C)}\sqrt{\sin\left(\frac{\pi-A}{2}\right)}$$

$$2\sum_{cyc(A,B,C)}\sqrt{\cos\frac{A}{2}-\sum_{cyc(A,B,C)}\sqrt{\sin A}}\ge\frac{3^{\frac{5}{4}}}{2^{\frac{1}{2}}}$$

SOLUTION 4.117

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{m_a}{AI^2}\le\frac{4R+r}{4r^2}$$

$$\because m_a\le R(1+\cos A),\ \textit{etc},\ \therefore\sum\frac{m_a}{AI^2}\le\sum\frac{R\cdot 2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}{r^2}=\sum\frac{R\sin^2 A}{2r^2}=\sum\frac{R\cdot a^2}{2r^2\cdot 4R^2}=$$

$$=\frac{1}{8Rr^2}\sum a^2\overset{?}{\le}\frac{4R+r}{4r^2}\Leftrightarrow\sum a^2\overset{?}{\le}8R^2+2Rr\Leftrightarrow$$

$$\Leftrightarrow s^2-4Rr-r^2\overset{?}{\le}4R^2+Rr\Leftrightarrow s^2\underset{(1)}{\overset{?}{\le}}4R^2+5Rr+r^2$$

$$\textit{Now, LHS of (1)}\overset{Gerretsen}{\le}4R^2+4Rr+3r^2\overset{?}{\le}4R^2+5Rr+r^2\Leftrightarrow Rr\overset{?}{\ge}2r^2\Leftrightarrow$$

$$\Leftrightarrow R\overset{?}{\ge}2r\to\textit{true (Euler)}$$

SOLUTION 4.118

Solution by Marian Ursărescu-Romania

$$\textit{From Cauchy's inequality}\Rightarrow\left(\sum\sqrt{r_a(r_b+r_c)}\right)^2\le 3\sum r_a(r_b+r_c)$$

$$\Rightarrow\sum\sqrt{r_a(r_b+r_c)}\le\sqrt{6\sum r_ar_b}\quad (1)$$

$$\textit{But}\sum r_ar_b=s^2\quad (2)$$

$$\textit{From (1)+(2)}\Rightarrow\sum\sqrt{r_a(r_b+r_c)}\le\sqrt{6}s\ (3)$$

$$\left.\begin{array}{c} m_a+m_b+m_c\ge 3\sqrt[3]{m_am_bm_c}\\ m_a\ge\sqrt{s(s-a)}\end{array}\right\}\Rightarrow m_a+m_b+m_c\ge 3\sqrt[3]{sS}\Rightarrow$$

$$m_a+m_b+m_c\ge 3\sqrt[3]{s^2r}\quad (4)$$

From (3)+(4) we must show:

$$3\sqrt[3]{s^2r}\cdot\sqrt{\frac{R}{r}}\ge\sqrt{6}s\Leftrightarrow 3^6s^4r^2\cdot\frac{R^3}{r^3}\ge 6^3s^6\Leftrightarrow 3^6\frac{R^3}{r}\ge 3^3\cdot 2^3\cdot s^2\Leftrightarrow 27R^3\ge 8s^2r\quad (5)$$

From Mitrinovic's inequality: $27R^2\ge 4s^2\Rightarrow 27R^3\ge 4Rs^2$ *(6)*

From (5)+(6) we must show: $4Rs^2\ge 8s^2r\Leftrightarrow R\ge 2r$*, true (Euler)*

SOLUTION 4.119

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $\sum_{cyc} r_a=4R+r$ *and* $\sum_{cyc} r_ar_b=s^2$

$$\sum_{cyc}\frac{x}{y+z}r_a^2=(x+y+z)\sum_{cyc}\frac{r_a^2}{y+z}-\sum_{cyc}r_a^2$$

$$\overset{\substack{BERGSTROM'S\\ INEQUALITY}}{\ge}\frac{(r_a+r_b+r_c)^2}{2}-\left(\sum_{cyc}r_a\right)^2+2\sum_{cyc}r_ar_b=2s^2-\frac{(4R+r)^2}{2}$$

We need to prove, $2s^2-\frac{(4R+r)^2}{2}\ge\frac{91r^2-16R^2}{2}\Leftrightarrow s^2\ge 23r^2+2Rr$

We know, $s^2\ge 16Rr-5r^2$ *we need to prove,* $16Rr-5r^2\ge 23r^2+2Rr$

$\Leftrightarrow 14R(R-2r)\ge 0$*, which is true*

SOLUTION 4.120

Solution by Marian Ursărescu-Romania

We must show:

$$\frac{2r^3}{27}(AI+BI+CI)^3+r^2(AI^4+BI^4+CI^4)\ge(AI\cdot BI\cdot CI)^2\quad (1)$$

But $AI=\frac{r}{\sin\frac{A}{2}}$ *and* $AI\cdot BI\cdot CI=4Rr^2$ *(2)*

From (1)+(2) we must show:

$$\frac{2r^3}{27}(AI+BI+CI)^3+r^2(AI^4+BI^4+CI^4)\ge 16R^2r^4\Leftrightarrow$$

$$\frac{2r}{27}(AI+BI+CI)^3+(AI^4+BI^4+CI^4)\ge 16R^2r^2\ (3)$$

$$AI+BI+CI\ge 3\sqrt[3]{AI\cdot BI\cdot CI}\quad (4)$$

From (3)+ (4) we must show:

$$2r\cdot AI\cdot BI\cdot CI+(AI^4+BI^4+CI^4)\ge 16R^2r^2\overset{(2)}{\Leftrightarrow}$$

$$AI^4 + BI^4 + CI^4 \geq 8Rr^2(2R-r) \text{ (5)}$$

From Cauchy's inequality: $AI^4 + BI^4 + CI^4 \geq \frac{(AI^2+BI^2+CI^2)^2}{3}$ *and*

$$AI^2 + BI^2 + CI^2 = s^2 + r^2 - 8Rr \Rightarrow$$

$$AI^4 + BI^4 + CI^4 \geq \frac{(s^2+r^2-8Rr)^2}{3} \quad \text{(6)}$$

From (5)+(6) we must show:

$$(s^2 + r^2 - 8Rr)^2 \geq 24Rr^2(2R-r) \quad \text{(7)}$$

From Gerretsen's inequality we have: $s^2 \geq 16Rr - 5r^2$ *(8)*

From (7)+(8): $(8Rr - 4r^2)^2 \geq 24Rr^2(2R-r) \Leftrightarrow R \geq 2r$ *true.*

SOLUTION 4.121

Solution by Bogdan Fustei-Romania

In ΔABC *the following relationship:* $\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} \leq 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2$

(I – incenter in ΔABC*)*

R_a, R_b, R_c *– circumradii* $\Delta BIC, \Delta CIA, \Delta AIB$)

Using two additional inequalities:

1) $\frac{R}{r} \geq \frac{abc+a^2+b^3+c^3}{2abc}$

2) $x, y, z > 0: \frac{x^3+y^3+z^3}{4xyz} + \frac{1}{4} \geq \left(\frac{x^2+y^2+z^2}{xy+yz+zx}\right)^2$

From the two inequalities from above we can write the following:

$\frac{R}{2r} \overset{(1)}{\geq} \frac{a^3+b^3+c^3}{4abc} + \frac{1}{4} \overset{(2)}{\geq} \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2$*. So, finally:*

$$\frac{R}{2r} \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2$$

$R_a = 2R\sin\frac{A}{2}$ *(and the analogs)*

$\sin\frac{A}{2} = \sqrt{\frac{r_a-r}{4R}}$ *(and the analogs)*

$a^2 = (r_b + r_c)(r_a - r)$ *(and the analogs)*

$\Rightarrow R_a = 2R \cdot \sqrt{\frac{r_a-r}{R}} = \sqrt{4R^2\frac{(r_a-r)}{4R}} = \sqrt{R(r_a-r)}$ *(and the analogs)*

$$R_a^2 = R(r_a - r) \text{ (and the analogs)} \Rightarrow \frac{a^2}{R_a^2} = \frac{(r_b+r_c)(r_a-r)}{R(r_a-r)} = \frac{r_b+r_c}{R}$$

$$\text{So, } \frac{a^2}{R_a^2} = \frac{r_b+r_c}{R} \text{ (and the analogs)}$$

$$\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{r_b + r_c}{R} + \frac{r_a + r_c}{R} + \frac{r_a + r_b}{R} = \frac{2(r_a + r_b + r_c)}{R} = \frac{2(4R + r)}{R}$$

$$(r_a + r_b + r_c = 4R + r) \Rightarrow \frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{8R + 2r}{R} = 8 + \frac{2r}{R}$$

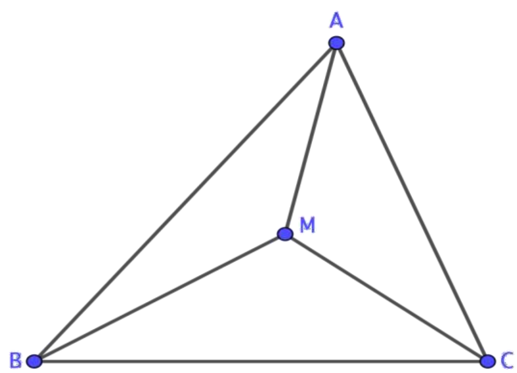
The inequality from enunciation becomes: $8 + \frac{2r}{R} \leq 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2 \Rightarrow$

$$\Rightarrow \frac{R}{2r} \geq \left(\frac{a^2 + b^2 + c^2}{ab + bc + ac}\right)$$

From the above, the inequality from enunciation is proved.

SOLUTION 4.122

Solution by Mehmet Sahin-Ankara-Turkey



Let (x, y, z) *be the barycentric coordinates of* M.

$$x + y + z = 1 \text{ and}$$

$$[MBC] = x \cdot [ABC]$$

$$[MCA] = y \cdot [ABC]$$

$$[MAB] = z \cdot [ABC]$$

$$[MAB] \cdot [MBC] \cdot [MCA] = xyz[ABC]^3 \quad (1)$$

Using Arithmetic and Geometric Mean inequality:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \Rightarrow \sqrt[3]{xyz} \leq \frac{1}{3} \Rightarrow xyz \leq \frac{1}{27} \quad (2)$$

From (1) and (2): $27[MAB] \cdot [MBC] \cdot [MCA] \leq [ABC]^3$

SOLUTION 4.123

Solution by Marian Ursărescu-Romania

In any ΔABC *we have:* $\sum \frac{r_a}{\sin^2\frac{A}{2}} = \frac{s^2+r^2+4Rr}{r} \Rightarrow$ *we must show:*

$$4(m_a + m_b + m_c) \le \frac{s^2+r^2+4Rr}{r} \text{ (1)}$$

But in any ΔABC *we have:* $m_a + m_b + m_c \le 4R + r$ *(2)*

From (1)+(2) we must show:

$$16R + 4r \le \frac{s^2+r^2+4Rr}{r} \Leftrightarrow 16Rr + 4r^2 \le s^2 + r^2 + 4Rr \Leftrightarrow s^2 \ge 12Rr + 3r^2 \text{ (3)}$$

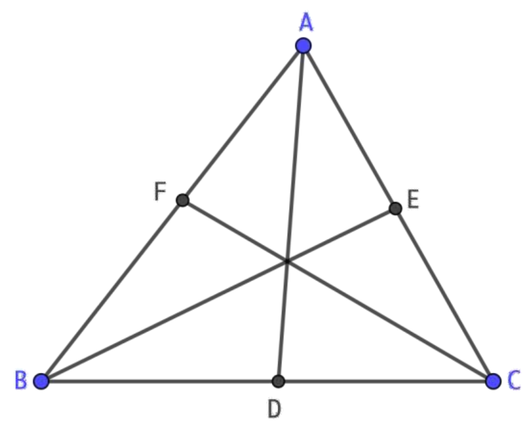
Form Gerretsen's inequality: $s^2 \ge 16Rr - 5r^2$ *(4)*

From (3)+(4) we must show: $16Rr - 5r^2 \ge 12Rr + 3r^2 \Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow R \ge 2r$*, true*

(Euler)

SOLUTION 4.124

Solution by Lahiru Samarakooon-Sri Lanka



Because CF, AD *bisectors:*

$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b} \Rightarrow BD = \frac{ac}{b+c}$$

So, $BD \cdot AC = \frac{ac}{b+c} b = \frac{abc}{b+c}$

$\therefore$ *similarly, for* AF, BC *and* CE, AB *set summating*

$$LHS = \sum BDAC = abc \sum \left[\frac{1}{b+c}\right]$$

$$= abc\left[\frac{12}{b+c} + \frac{12}{a+c} + \frac{2}{b+c}\right] \ge abc \times \frac{(1+1+1)^2}{2(a+b+c)}$$

$$= 4RSr \times \frac{9}{4S} \ (\because \sum a = 2s) = 9Rr, \textit{ but } R \geq 2r$$

$$\textit{So, } \geq 18r^2 \quad \textit{(proved)}$$

$$\sum BD \cdot AC \geq 18r^2$$

SOLUTION 4.125

Solution by Marian Ursărescu-Romania

$$\textit{From AM-GM} \Rightarrow \frac{1}{a\cos A} + \frac{1}{b\cos B} + \frac{1}{c\cos C} \geq 3\sqrt[3]{\frac{1}{abc\cos A\cos B\cos C}} \Rightarrow$$

$$\textit{We must show this: } \frac{3}{\sqrt[3]{abc\cos A\cos B\cos C}} \geq \frac{2\sqrt{3}}{R} \Leftrightarrow$$

$$\Leftrightarrow \frac{ab\cos A\cos B\cos C}{27} \leq \frac{R^3}{8\cdot 3\sqrt{3}} \Leftrightarrow ab\cos A\cos B\cos C \leq \frac{3\sqrt{3}}{7}R^3 \quad (1)$$

$$\left.\begin{matrix} But\ abc \leq 3\sqrt{3}R^3 \\ and \cos A\cos B\cos C \leq \frac{1}{8} \end{matrix}\right\} \Rightarrow \textit{(1) it's true.}$$

Let $a \leq b \leq c \Rightarrow \cos A \geq \cos B \geq \cos C$***. From Chebyshev's inequality*** $\Rightarrow$

$$\frac{1}{a\cos A} + \frac{1}{b\cos B} + \frac{1}{c\cos C} \leq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) \Rightarrow$$

$$\textit{We must show this: } \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) \leq \frac{\sqrt{3}}{4R\cos A\cos B\cos C}$$

$$\Leftrightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)(\cos A\cos B + \cos A\cos C + \cos C\cos A) \leq \frac{3\sqrt{3}}{4R} \quad (2)$$

$$\textit{But } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r} \ \textit{(3). From (2)+(3) we must show:}$$

$$\sum \cos A\cos B \leq \frac{3r}{2R} \quad (4)$$

$$\textit{But } \sum \cos A\cos B = \frac{s^2 + r^2 - 4R^2}{4R^2} \quad (5)$$

From (4)+(5) we must show:

$$\frac{s^2 + r^2 - 4R^2}{4R^2} \leq \frac{3r}{2R} \Leftrightarrow s^2 + r^2 - 4R^2 \leq 6Rr \ (6)$$

From Gerretsen's inequality:

$$s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow s^2 + r^2 - 4R^2 \leq 4Rr + 4r^2 \quad (7)$$

Form (6)+(7) we must show: $4Rr + 4r^2 \leq 6Rr \Leftrightarrow 4r^2 \leq 2Rr \Leftrightarrow 2r \leq R$ ***(true Euler)***

SOLUTION 4.126

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$A; B; C \in \left(0; \frac{\pi}{2}\right)$$

$$f(x) = \cos x \cdot \sin(\sin x)$$

$$f'(x) = -\sin x \cdot \sin(\sin x) + \cos^2 x \cdot \cos(\sin x)$$

$$f''(x) = -\cos x \cdot \sin(\sin x) - \sin^2 x \cdot \cos(\sin x) - 2 \cdot \cos x$$

$$\cdot \sin x \cdot \cos(\sin x) - \cos^3 x \cdot \sin(\sin x) =$$

$$= -\left((\cos x + \cos^3 x) \cdot \sin(\sin x) + (\sin^2 x + 2\cos x \cdot \sin x) \cdot \cos(\sin x)\right) < 0$$

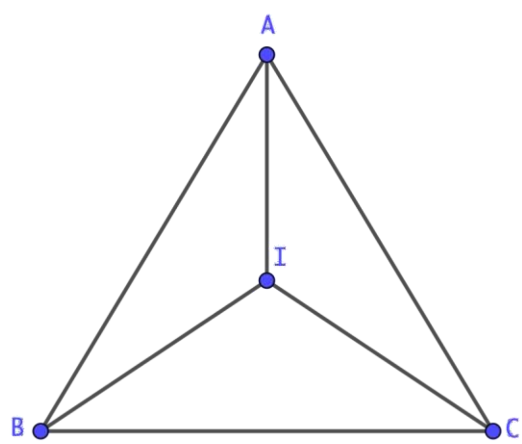
$$f''(x) < 0$$

$$\sum \cos A \cdot \sin(\sin A) \leq 3 \cdot \cos \frac{A+B+C}{3} \cdot \sin\left(\sin \frac{A+B+C}{3}\right) =$$

$$= \frac{3}{2} \cdot \sin\left(\sin \frac{\pi}{3}\right) = \frac{3}{2} \sin\left(\frac{\sqrt{3}}{2}\right) \overset{\substack{Acute \\ Euler}}{\leq} \frac{3}{2} \cdot \sin\left(\frac{\sqrt{3}R}{4r}\right)$$

SOLUTION 4.127

Solution by Lahiru Samarakoon-Sri Lanka



For ABI triangle, $AI + BI > AB$, $\left(\frac{AI+BI}{CI}\right) > \left(\frac{AB}{CI}\right)$ $(\because CI > 0)$. ***So,*** $\left(\frac{AI+BI}{CI}\right)^5 > \left(\frac{AB}{CI}\right)^5$

$\therefore$ ***similarly, from*** ΔBIC ***and*** ΔAIC***, and get summation,***

$$\sum \left(\frac{AI+BI}{CI}\right)^5 > \sum \left(\frac{BC}{AI}\right)^5$$

SOLUTION 4.128

Solution by Soumava Chakraborty-Kolkata-India

$$\because m_a \leq \frac{R}{2r} h_a \text{ etc.}, \therefore \sqrt{\frac{r_a}{m_a}} \geq \sqrt{\frac{2r}{R} \cdot \frac{r_a}{h_a}} \text{ etc.}$$

$$\Rightarrow \sum \sqrt{\frac{r_a}{m_a}} \overset{(1)}{\geq} \sum \sqrt{\frac{2r}{R} \cdot \frac{\Delta}{s-a} \cdot \frac{a}{2\Delta}} = \sum \sqrt{\frac{r}{R}} \sqrt{\frac{abcs}{s(s-a)bc}} = \sum \sqrt{\frac{r}{R}} \sqrt{\frac{a^2 s}{4Rrs}} \cdot \frac{1}{\cos\frac{A}{2}}$$

$$= \sum \sqrt{\frac{r}{R}} \sqrt{\frac{1}{4Rr}} \cdot \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}} = 2\sum \sin\frac{A}{2}$$

Now, $\frac{h_b+h_c}{w_a} \geq 4\sin\frac{A}{2}$

$$\Leftrightarrow \frac{ca+ab}{2R} \cdot \frac{(b+c)}{2bc\cos\frac{A}{2}} \geq 4\sin\frac{A}{2} \Leftrightarrow a(b+c)^2 \geq \left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)(4bc)$$

$$\Leftrightarrow a(b+c)^2 \geq 4abc \Leftrightarrow (b+c)^2 \geq 4bc \rightarrow \textit{true} \Rightarrow \frac{h_b+h_c}{w_a} \overset{(a)}{\geq} 4\sin\frac{A}{2}$$

Similarly, $\frac{h_c+h_a}{w_b} \overset{(b)}{\geq} 4\sin\frac{B}{2}$ *&* $\frac{h_a+h_b}{h_c} \overset{(c)}{\geq} 4\sin\frac{C}{2}$

(a)+(b)+(c)$\Rightarrow \sum\frac{h_b+h_c}{w_a} \overset{(2)}{\geq} 4\sum\sin\frac{A}{2}$

(1)+(2)$\Rightarrow \sum\sqrt{\frac{r_a}{m_a}} + \sum\frac{h_b+h_c}{w_a} \geq 6\sum\sin\frac{A}{2}$

SOLUTION 4.129

Solution by Marian Ursărescu-Romania

$$\frac{a}{b+c-a} = \frac{2R\sin A}{2R(\sin B+\sin C-\sin A)} = \frac{\sin A}{\sin B+\sin C-\sin A} \quad \textit{(1)}$$

But if $A+B+C=\pi$ *then:* $\sin B+\sin C-\sin A = 4\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}$ *(2)*

From (1)+(2)$\Rightarrow \frac{a}{b+c-a} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{4\sin\frac{B}{2}\sin\frac{B}{2}\cos\frac{A}{2}} = \frac{\sin\frac{A}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}$ *(3)*

From (3)$\Rightarrow \sum\frac{a}{b+c-a} = \sum\frac{\sin\frac{A}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{1}{2}\sum\frac{\sum\sin^2\frac{A}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$ *(4)*

But $\sum\sin^2\frac{A}{2} = \frac{2R-r}{2R}$ *and* $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R}$ *(5)*

From (4)+(5)$\Rightarrow \sum\frac{a}{b+c-a} = \frac{2R-r}{r} = 2\frac{R}{r} - 1$ *(6)*

From (6) inequality becomes: $2\frac{R}{r} - 1 - 2\sum\left(\frac{a-b}{a+b}\right)^2 \geq 3 \Leftrightarrow$

$\frac{R}{r} - 2 - \sum\left(\frac{a-b}{a+b}\right)^2 \geq 0$ *(7)*

But $(a+b)^2 \geq 4ab \Rightarrow$ *(7) becomes:*

$$\frac{R}{r}-2-\sum\frac{(a-b)^2}{4ab}\geq 0\Leftrightarrow\frac{R}{r}-2-\sum\frac{a^2-2ab+b^2}{ab}\geq 0\Leftrightarrow$$

$$\Leftrightarrow\frac{R}{r}-2-\frac{1}{4}\sum\frac{a^2+b^2}{ab}+\frac{3}{2}\geq 0\Leftrightarrow\frac{R}{r}-\frac{1}{2}-\frac{1}{4}\sum\frac{a^2+b^2}{ab}\geq 0\Leftrightarrow$$

$$\Leftrightarrow\frac{4R}{r}-2-\sum\frac{a^2+b^2}{ab}\geq 0\quad\textit{(8). But }\sum\frac{a^2+b^2}{ab}=\frac{s^2+r^2-2Rr}{2Rr}\quad\textit{(9)}$$

From (8)+(9) we must show this:

$$4\frac{R}{r}-2-\frac{s^2+r^2-2Rr}{2Rr}\geq 0\quad\textit{(10)}$$

But from Gerretsen $s^2\leq 4R^2+4Rr+3r^2$ ***(11)***

From (10)+(11) we must show: $4\frac{R}{r}-2-\frac{4R^2+2Rr+4r^2}{2Rr}\geq 0$. ***Let*** $\frac{R}{r}=x, x\geq 2$

$$\Rightarrow 4x-2-\frac{4x^2+2x+4}{2x}\geq 0\Leftrightarrow 2x^2-3x-2\geq 0\Leftrightarrow(2x+1)(x-2)\geq 0\textit{ true.}$$

SOLUTION 4.130

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\frac{r_a}{r_b+r_c}+\frac{r_b}{r_c+r_a}+\frac{r_c}{r_a+r_b}+\frac{3}{2}\leq\frac{12}{6-\frac{R}{r}}$$

$$\sum\left(\frac{r_a}{r_b+r_c}+1\right)+\frac{3}{2}-3\leq\frac{12r}{6r-R}$$

$$\sum r_a\cdot\sum\frac{1}{r_a+r_b}\leq\frac{12r}{6r-R}+\frac{3}{2}=\frac{42r-3R}{2(6r-R)}$$

$$\sum r_a\cdot\frac{\sum(r_a+r_b)(r_b+r_c)}{\prod(r_a+r_b)}\leq\frac{42r-3R}{2(6r-R)}$$

$$\sum r_a\cdot\frac{(\sum r_a)^2+\sum r_a r_b}{\sum r_a\cdot\sum r_a\cdot r_b-r_ar_br_c}\leq\frac{42r-3R}{2(6r-R)}$$

a) $\sum r_a=4R+r$

b) $\sum r_ar_b=s^2$

c) $r_ar_br_c=r\cdot s^2$

$$(4R+r)\left[\frac{(4R+r)^2+s^2}{(4R+r)s^2-rs^2}\right]=(4R+r)\left[\frac{(4R+r)^2+s^2}{4Rs^2}\right]\leq\frac{42r-3R}{2(6r-R)}$$

$$(6r-R)(4R+r)^3+(4R+r)(6r-R)\cdot s^2\leq 2R(42r-3r)s$$

$$(6r-R)(4R+r)^3\leq(61Rr-2R^2-6r^2)s^2$$

$$6(Rr-2r^2-6r^2)>0$$

$$(6r-R)(4R+r)^3 \le (61Rr-2R^2-6r^2)(16Rr-5r^2)$$

$$\frac{R}{2}=t$$

$$(6-t)(4t+1)^3 \le (61t-2t^2-6)(16t-5)$$

$$-64t^4+336t^3+276t^2+71t+6 \le -32t^3+986t^2-410t+30$$

$$32t^4+184t^3+355t^2-236t+12 \ge 0$$

$$\underbrace{(t-2)^2}_{\ge 0}\underbrace{(32t^2-56+2)^2}_{\ge 0} \ge 0$$

SOLUTION 4.131

Solution by Tran Hong-Vietnam

$$\sin\omega=\frac{2S}{\sqrt{\sum a^2b^2}}$$

$$\textit{Inequality} \Leftrightarrow \frac{S}{\sqrt{\sum a^2b^2}} \le \frac{\sum a^2+\sum ab}{8\sum a^2}$$

$$\Leftrightarrow \frac{2\sum a^2b^2-\sum a^4}{\sum a^2b^2} \le \left(\frac{\sum a^2+\sum ab}{2\sum a^2}\right)^2 \quad \textit{(1)}$$

$$\textit{Let } p=\sum a,\ q=\sum ab,\ r=abc, \textit{ suppose } c \le b \le a$$

$$\textit{(1)}\Leftrightarrow \{8(q^2-2pr)-4(p^4-4p^2q+2q^2+4pr)\}(p^2-2q)^2 \le (q^2-2pr)(p^2-q)^2;$$

$$\Leftrightarrow \{-2p(p^2-q)^2+32p(p^2-2q)^2\}r+g(p,q) \ge 0$$

$$\Leftrightarrow 2p\{16(p^2-2q)^2-(p^2-q)^2\}r+g(p,q) \ge 0$$

$$\Leftrightarrow 2p\{15p^4-62p^2q+63q^2\}r+g(p,q) \ge 0$$

$$\textit{Let } f(r)=2p\{15p^4-62p^2q+63q^2\}r+g(p,q)$$

$$15p^4-62p^2q+63q^2=(3p^2-7q)(5p^2-9q)>0 \ \textit{(because } p^2 \ge 3q\textit{)}$$

$\Rightarrow$ ***The function f increasing of $r=abc$, by ABC Theorem we just check:***

$$\because c=0, 0<a\le b:$$

$$\textit{(1)}\Leftrightarrow \frac{2a^2b^2-(a^4+b^4)}{2} \le \left(\frac{a^2+b^2+ab}{2a^2+2b^2}\right)^2$$

$$\Leftrightarrow 4(a^4-b^4)^2+a^2b^2(a^2+ab+b^2)^2 \ge 0 \ \textit{(true)}$$

$$\because a=b, c\le a:$$

$$\textit{(1)}\Leftrightarrow \frac{4a^2c^2-c^4}{a^4+2a^2c^2} \le \left(\frac{3a^2+c^2+2ac}{4a^2+2c^2}\right)^2$$

$$\Leftrightarrow (a-c)^2\left(9a^6+30a^5c+15a^4c^2+28a^3c^3+14a^2c^4+8ac^5+4c^6\right) \ge 0$$

It is true. Proved. Equality $\Leftrightarrow a = b = c$.

SOLUTION 4.132

Solution by Soumava Chakraborty-Kolkata-India

$$AM \geq GM \Rightarrow \sqrt{\sum ab\sin^2 A}\sqrt{\sum ab\cos^2 A} \leq \frac{\sum ab\sin^2 A + \sum ab\cos^2 A}{2}$$

$$= \frac{ab(\sin^2 A + \cos^2 A) + bc(\sin^2 B + \cos^2 B) + ca(\sin^2 C + \cos^2 C)}{2}$$

$$= \frac{\sum ab}{2} (\because \sin^2 A + \cos^2 A = 1, etc.)$$

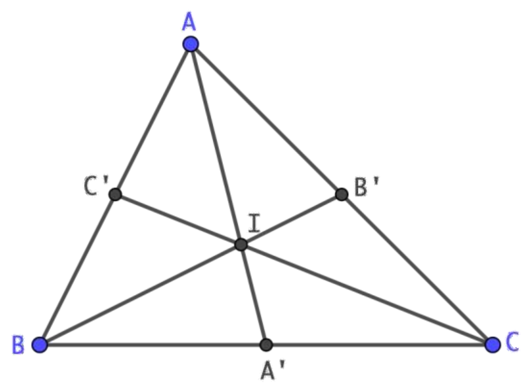
$$\therefore \left(\sum ab\sin^2 A\right)\left(\sum ab\cos^2 A\right) \leq \frac{(\sum ab)^2}{4}$$

$$\Rightarrow 16\left(\sum ab\sin^2 A\right)\left(\sum ab\cos^2 A\right) \leq 4\left(\sum ab\right)^2 \overset{?}{\leq} 324R^4$$

$$\Leftrightarrow \sum ab \leq 9R^2 \rightarrow true \because \sum ab \leq \sum a^2 \overset{Leibnitz}{\leq} 9R^2$$

SOLUTION 4.133

Solution by Soumava Chakraborty-Kolkata-India



$$Angle - bisector\ theorem \Rightarrow \frac{A'C}{A'B} = \frac{b}{c} \Rightarrow \frac{a}{A'B} = \frac{b+c}{c} \Rightarrow A'B \overset{(1)}{=} \frac{ac}{b+c}$$

$$Angle - bisector\ on\ \Delta ABA' \Rightarrow \frac{IA'}{IA} \overset{by\ (1)}{=} \frac{\frac{ac}{b+c}}{c} = \frac{a}{b+c} \Rightarrow IA' = \frac{a}{b+c}IA \Rightarrow IA' \cdot IA = \frac{a}{b+c}IA^2$$

$$= \frac{a}{b+c} \cdot \frac{r^2}{\sin^2\frac{A}{2}} \Rightarrow \frac{IA \cdot IA'}{w_a} = \frac{ar^2bc(b+c)}{(b+c)(s-b)(s-c)2bc\cos\frac{A}{2}}$$

$$= \frac{4R\sin\frac{A}{2}\cos\frac{A}{2}r^2}{2(s-b)(s-c)\cos\frac{A}{2}} = \frac{2Rr^2}{(s-b)(s-c)}\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$=\frac{2Rr^2}{\sqrt{bc(s-b)(s-c)}}=\frac{2Rr^2\sqrt{a(s-a)}}{\sqrt{4Rrs\cdot r^2S}}=\sqrt{\frac{4R^2r^4}{4Rr^3s^2}}\sqrt{a(s-a)}\overset{(a)}{=}\sqrt{\frac{Rr}{s^2}}\sqrt{a(s-a)}$$

Similarly, $\frac{IB\cdot IB'}{w_b}\overset{(b)}{=}\sqrt{\frac{Rr}{s^2}}\sqrt{b(s-b)}$ & $\frac{IC\cdot IC'}{w_c}\overset{(c)}{=}\sqrt{\frac{Rr}{s^2}}\sqrt{c(s-c)}$

(a)+(b)+(c)⇒ $LHS=\sqrt{\frac{Rr}{s^2}}\sum\sqrt{a(s-a)}$

$$\overset{CBS}{\le}\frac{\sqrt{Rr}}{s}\sqrt{3}\sqrt{\sum a(s-a)}=\frac{\sqrt{3Rr}}{s}\sqrt{s(2s)-2(s^2-4Rr-r^2)}$$

$$\overset{(i)}{=}\frac{\sqrt{3Rr}}{s}\sqrt{2(4Rr+r^2)}=\frac{r}{s}\sqrt{6R(4R+r)}$$

Now, $RHS=\frac{3\sqrt{3}}{4rs}\cdot\frac{r^3}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{3\sqrt{3}r^2}{s\left(\frac{r}{R}\right)}\overset{(ii)}{=}\frac{3\sqrt{3}Rr}{s}$

(i), (ii) ⇒ it suffices to prove: $6R(4R+r)\le 27R^2\Leftrightarrow 3R^2\ge 6Rr\Leftrightarrow R\ge 2r\rightarrow$ *true (Euler)*

SOLUTION 4.134

Solution by Sagar Kumar-Patna Bihar-India

$$P=e^{(\sin A+2\sin B)(\sin B+2\sin C)(\sin C+2\sin A)}\Rightarrow \cos 0<A,B,C<\pi\Rightarrow$$

$$\Rightarrow \sin A,\sin B,\sin C>0\Rightarrow(\sin A+2\sin B)(\sin B+2\sin C)(\sin C+2\sin A)$$

$$\le\left(\frac{3(\sin(A)+\sin B+\sin C)}{3}\right)^3$$

AM ≥ GM

$$\Rightarrow LHS\le(\sin(A)+\sin B+\sin C)^3$$

and we know that in a ΔABC*:* $\sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}$

$$\Rightarrow LHS\le\left(\frac{3\sqrt{3}}{2}\right)^3=\frac{81\sqrt{3}}{8}$$

Hence $P_{\max}\le e^{\left(\frac{81\sqrt{3}}{8}\right)}$

Equality holds when $A=B=C=\frac{\pi}{3}$

SOLUTION 4.135

Solution by Lahiru Samarakoon-Sri Lanka

$$\sum 2R\sin A\sin B\le\frac{3\sqrt{3}}{2}R$$

$$R\sum\sin 2A \le \frac{3\sqrt{3}}{2}R \Rightarrow 4R\sin A\cos B\cos C \le \frac{3\sqrt{3}}{2}R$$

We have to prove, $\sin A\cos B\cos C \le \frac{3\sqrt{3}}{8}$

But, $\frac{\sum\sin A}{3} \le \cos\left(\frac{A+B+C}{3}\right) = \frac{\sqrt{3}}{2}$

$GM \le AM$

$\frac{\sum\cos A}{3} \ge \sqrt[3]{\sin A\sin B\cos C}$. *So,* $\sin A\sin B\cos C \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$

SOLUTION 4.136

Solution by Soumava Chakraborty-Kolkata-India

In any $\Delta ABC, \sum\frac{m_a}{h_a} \ge \frac{1}{2}\sum\left(\frac{h_b+h_c}{h_a}\right)$

$$RHS = \frac{1}{2}\sum\left(\frac{\frac{ca+ab}{2R}}{\frac{bc}{2R}}\right) = \frac{1}{2}\sum\left(\frac{ca+ab}{bc}\right) \overset{(1)}{=} \frac{\sum a^2b + \sum ab^2}{2ab}$$

$$LHS \overset{Tereshin}{\ge} \sum\left(\frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}}\right) = \frac{1}{2}\sum\left(\frac{b^2+c^2}{bc}\right) = \frac{\sum a^2b + \sum ab^2}{2ab} \overset{by\ (1)}{=} RHS$$

SOLUTION 4.137

Solution by Bogdan Fustei-Romania

$$\left.\begin{array}{l} R_a = 2R\sin\frac{A}{2}\ (and\ analogous) \\ \sin\frac{A}{2} = \sqrt{\frac{r_a-r}{4R}}\ (and\ analogous)\end{array}\right\rangle R_a = \sqrt{R(r_a-r)} \text{ (and analogous)}$$

$$R_a^4 = R^2(r_a-r)^2 \text{ (and analogous)} \Rightarrow R_a^4 + R_b^4 + R_c^4 = R^2\cdot\sum(r_a-r)^2$$

$$R_a^4 + R_b^4 + R_c^4 = R^2\left[\sum r_a^2 + 3r^2 - 2r(r_a+r_b+r_c)\right]$$

$$r_ar_b + r_br_c + r_ar_c = s^2 \Rightarrow \sum r_a^2 = (r_a+r_b+r_c)^2 - 2\sum r_ar_b$$

$$\sum r_a^2 = (r_a+r_b+r_c)^2 - 2s^2$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(r_a+r_b+r_c)^2 - 2s^2 - 2r(r_a+r_b+r_c) + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(R_a+R_b+R_c)(R_a+R_b+R_c-2r) - 2s^2 + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2[(4R+r)(4R-r) - s^2 + 3r^2]$$

$$R_a^4 + R_b^4 + R_c^4 = R^2(16R^2 - r^2 - 2s^2 + 3r^2) = 2R^2(8R^2 - s^2 + r^2)$$

$\frac{R_a^4+R_b^4+R_c^4}{4R^2}=\frac{2R^2(8R^2-s^2+r^2)}{4R^2}=\frac{8R^2-s^2+r^2}{2}$*. The inequality from enunciation becomes:*

$$2R^2-2Rr-r^2\le\frac{8R^2-s^2+r^2}{2}\le 4R^2-8Rr+3r^2$$

$$4R^2-4Rr-2r^2\le 8R^2-s^2+r^2\Rightarrow s^2\le 8R^2+r^2-4R^2+4Rr+2r^2=$$

$$=4R^2+4Rr+3r^2 \quad \textit{(Gerretsen's inequality)}$$

$$8r^2-s^2+r^2\le 8R^2-16Rr+6r^2\Rightarrow 16Rr-5r^2\le s^2 \quad \textit{(Gerretsen's inequality)}$$

From the above the inequality from enunciation is proved.

SOLUTION 4.138

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{abc}{2R^2}\sum a^2-\sum a^3\cos A=\frac{4Rrs}{R^2}(s^2-4Rr-r^2)-\sum a^3\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{4rs}{R}(s^2-4Rr-r^2)-\sum\frac{a^4(b^2+c^2-a^2)}{2abc}$$

$$=\frac{4rs(s^2-4Rr-r^2)}{R}-\frac{\sum a^2b^2\left(\sum a^2-c^2\right)-\sum a^6}{8Rrs}$$

$$\overset{(1)}{=}\frac{32r^2s^2(s^2-4Rr-r^2)-(\sum a^2b^2)(\sum a^2)+3a^2b^2c^2+\sum a^6}{8Rrs}$$

Numerator $=32r^2s^2(s^2-4Rr-r^2)-(\sum a^2b^2)(\sum a^2)+3a^2b^2c^2+3a^2b^2c^2+$

$$+\sum a^2\left(\sum a^4-\sum a^2b^2\right)=$$

$$=32r^2s^2(s^2-4Rr-r^2)-2\left(\sum a^2b^2\right)\left(\sum a^2\right)+96R^2r^2s^2+$$

$$+\left(\sum a^2\right)\left\{\left(\sum a^2\right)^2-2\sum a^2b^2\right\}$$

$$=32r^2s^2(s^2-4Rr-r^2)-8\left(\sum a^2b^2\right)(s^2-4Rr-r^2)+$$

$$+96R^2r^2S^2+8(s^2-4Rr-r^2)^2$$

$$=8(s^2-4Rr-r^2)\left\{(s^2-4Rr-r^2)^2-\left(\sum ab\right)^2+16Rrs^2+4r^2s^2\right\}+$$

$$+96R^2r^2S^2$$

$$=8(s^2-4Rr-r^2)\{(2s^2)(-8Rr-2r^2)+16Rrs^2+4r^2s^2+96R^2r^2S^2\}$$

$$\overset{(2)}{=}96R^2r^2S^2$$

$$\textit{(1), (2)} \Rightarrow \frac{abc}{2R^2}\sum a^2 - \sum a^3 \cos A = \frac{96R^2r^2S^2}{8Rrs} \overset{(3)}{=} 12Rrs$$

$$\textit{Now, } \sum a^3 \cos B \cos C = \frac{1}{2}\sum a^3 \, (2\cos B \cos C) =$$

$$= \frac{1}{2}\sum a^3 \{\cos(B+C) + \cos(B-C)\} =$$

$$= -\frac{1}{2}\sum a^3 \cos A + \frac{1}{2}\sum a^2 \cdot 2R\sin(B+C)\cos(B-C)$$

$$= -\frac{1}{2}\sum a^3 \cos A + \frac{R}{2}\sum a^2 \,(\sin 2B + 2\sin 2C)$$

$$= -\frac{1}{2}\sum a^3 \cos A + \frac{R}{2}\sum a^2 \left(\sum \sin 2A - \sin 2A\right)$$

$$= -\frac{1}{2}\sum a^3 \cos A + \frac{R}{2}\left(\sum a^2\right)\left(4\frac{abc}{8R^3}\right) - \frac{R}{2}\sum a^2 \cdot 2\sin A \cos A$$

$$= -\frac{1}{2}\sum a^3 \cos A + \frac{R}{2}\left(\sum a^2\right)\left(\frac{abc}{2R^3}\right) - \frac{1}{2}\sum a^2 \cdot a\cos A = -\sum a^3 \cos A + \frac{abc}{4R^2}\left(\sum a^2\right)$$

$$= \left(\frac{abc}{2R^2}\left(\sum a^2\right) - \sum a^3 \cos A\right) - \frac{abc}{4R^2}\left(\sum a^2\right) \overset{by\,(3)}{=} (12Rrs) - \frac{4Rrs}{4R^2}\cdot 2(s^2 - 4Rr - r^2)$$

$$= 12Rrs - \frac{2rs(s^2 - 4Rr - r^2)}{R}$$

$$= \frac{12R^2rs - 2rs(s^2 - 4Rr - r^2)}{R} \overset{(4)}{=} \frac{2rs(6R^2 - s^2 + 4Rr + r^2)}{R}$$

$$\textit{Now, } 6R^2 - s^2 + 4Rr + r^2 > 0 \Leftrightarrow s^2 < 6R^2 + 4Rr + r^2$$

$$\textit{But, } s^2 \overset{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \overset{?}{<} 6R^2 + 4Rr + r^2$$

$$\Leftrightarrow R > r \rightarrow \textit{true} \therefore 6R^2 - s^2 + 4Rr + r^2 > 0$$

$$\textit{(4)} \Rightarrow \textit{given inequality} \Leftrightarrow \frac{2rs(6R^2-s^2+4Rr+r^2)}{R}\left(\sum \frac{1}{\cos A}\right)^2 \overset{(5)}{\geq} 27abc$$

$$\because \left(\sum \frac{1}{\cos A}\right)^2 \overset{(6)}{\geq} 3\sum \frac{1}{\cos A \cos B} = \frac{3\sum \cos A}{\prod \cos A}$$

$$= \frac{3\left(\frac{R+r}{R}\right)}{\frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}} = \frac{12R(R+r)}{s^2 - 4R^2 - 4Rr - r^2},$$

$$\therefore \textit{(6)} \Rightarrow \textit{(5)} \Leftrightarrow \frac{12(R+r)\cdot 2rs(6R^2-s^2+4Rr+r^2)}{R(s^2-4R^2-4Rr-r^2)} \geq 108Rrs$$

$$\Leftrightarrow 2(R+r)(6R^2 - s^2 + 4Rr + r^2) \geq 9R(s^2 - 4R^2 - 4Rr - r^2)$$

$$\Leftrightarrow 2(R+r)6R^2 - 2(R+r)s^2 + 2(R+r)(4Rr + r^2) \geq$$

$$\geq 9Rs^2 - 36R^3 - 9R(4Rr + r^2)$$

$$\Leftrightarrow 48R^3 + 12R^2r + (11R + 2r)(4Rr + r^2) \overset{(7)}{\geq} (11R + 2r)s^2$$

$$\textit{Now, RHS of (7)} \overset{Gerretsen}{\leq} (11R + 2r)(4R^2 + 4Rr + 3r^2)$$

$$\overset{?}{\leq} 48R^3 + 12R^2r + (4Rr + r^2)(11R + 2r) \Leftrightarrow 2t^3 + 2t^2 - 11t - 2 \overset{?}{\geq} 0 \ \left(\textit{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)(2t^2 + 6t + 1) \overset{?}{\geq} 0 \to \textit{true because } t \overset{Euler}{\geq} 2 \Rightarrow \textit{(7) is true} \Rightarrow \textit{(5) is true}$$

SOLUTION 4.139

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\left(\frac{h_b h_c}{h_a}\right)^2 = \sum_{cyc}\left(\frac{\frac{2S}{b}\cdot\frac{2S}{c}}{\frac{2S}{a}}\right)^2 = 4S^2\sum_{cyc}\frac{a^2}{b^2c^2} =$$

$$= 4S^2 \cdot \frac{1}{a^2b^2c^2}\sum_{cyc} a^4 \overset{GOLDNER(1949)}{\geq} \frac{4S^2}{a^2b^2c^2}\cdot 16S^2 = \frac{4S^2}{16R^2S^2}\cdot 16S^2 = \left(\frac{2S}{R}\right)^2$$

SOLUTION 4.140

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{1}{\cos A} > \sum A^2 + \sum\cos A \Leftrightarrow \sum\left(\frac{1}{\cos A} - \cos A\right) > \sum A^2 \Leftrightarrow \sum\frac{\sin^2 A}{\cos A} \overset{(1)}{>} \sum A^2$$

$$\textit{Let } f(x) = \sin^2 x - x^2\cos x, \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$f'(x) = 2\sin x\cos x + x^2\sin x - 2x\cos x \overset{(2)}{\geq} 2\sin x\cos x + x^2\sin x - 2\sin x$$

$$\left(\because x\cos x \leq \sin x \ \textit{as } x \leq \tan x; \forall x \in \left[0, \frac{\pi}{2}\right)\right) = \sin x\,(2\cos x + x^2 - 2)$$

$$\textit{Let } g(x) = 2\cos x + x^2 - 2 \ \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$g'(x) = -2\sin x + 2x \geq 0 \ \textit{as } \forall x \in \left[0, \frac{\pi}{2}\right), x \geq \sin x$$

$$\therefore g(x) \overset{(3)}{>} g(0) = 0$$

$$\textit{(2), (3)} \Rightarrow f'(x) \geq 0 \therefore f(x) \geq f(0) = 0$$

$$\Rightarrow \forall x \in \left[0, \frac{\pi}{2}\right), \sin^2 x \geq x^2\cos x, \textit{ with equality at } x = 0$$

$$\therefore \forall x \in \left(0, \frac{\pi}{2}\right), \sin^2 x > x^2\cos x \Rightarrow \frac{\sin^2 x}{\cos x} \overset{(a)}{>} x^2$$

$$\because A, B, C \in \left(0, \frac{\pi}{2}\right) \therefore \textit{(a)} \Rightarrow \frac{\sin^2 A}{\cos A} > A^2 \text{ etc}$$

$$\Rightarrow \sum \frac{\sin^2 A}{\cos A} > \sum A^2 \Rightarrow \textit{(1) is true (Proved)}$$

SOLUTION 4.141

Solution by Marian Ursărescu-Romania

$$b^2 + c^2 \geq 2bc \text{ and } h_a = \frac{2S}{a} \Rightarrow \frac{b^2+c^2}{h_a} \geq \frac{abc}{S} \Rightarrow \sum \frac{b^2+c^2}{h_a} \geq \frac{3abc}{S} \quad (1)$$

$$\text{But } abc = 4sRr \text{ and } S = sr \text{ (2). From (1)+(2)} \Rightarrow \sum \frac{b^2+c^2}{h_a} \geq \frac{12sRr}{sr} = 12R$$

$$\text{Now: } \sum \frac{b^2+c^2}{h_a} \geq \frac{9\sqrt{3}R^3}{S} \Leftrightarrow \sum \frac{b^2+c^2}{\frac{2S}{a}} \geq \frac{9\sqrt{3}R^2}{S} \Leftrightarrow \sum a(b^2 + c^2) \geq 18\sqrt{3}R^3 \Leftrightarrow$$

$$\Leftrightarrow \sum a^2 (b + c) \geq 18\sqrt{3}R^3 \quad (3)$$

$$\text{But } \sum a^2 (b + c) = 2s(s^2 + r^2 - 2Rr) \quad (4)$$

SOLUTION 4.142

Solution by Marian Ursărescu-Romania

$$\frac{am_a}{\sin \frac{A}{2}} = \frac{2R \sin A \, m_a}{\sin \frac{A}{2}} = \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} m_a}{\sin \frac{A}{2}} = 4R \cos \frac{A}{2} m_a \Rightarrow$$

We must show this:

$$m_a \cos \frac{A}{2} + m_b \cos \frac{B}{2} + m_c \cos \frac{C}{2} \geq \frac{3}{2} s \quad (1)$$

$$\text{But } m_a \geq \frac{b+c}{2} \cos \frac{A}{2} \quad (2).$$

From (1)+(2) we must show:

$$\sum (b + c) \cos^2 \frac{A}{2} \geq 3s \quad (3)$$

$$\text{But } \cos^2 \frac{A}{2} = \frac{s(s-a)}{bc} \quad (4)$$

From (3)+(4) we must show:

$$\sum \frac{(b+c)(s-a)}{bc} \geq 3 \Leftrightarrow \sum \frac{(b+c)(b+c-a)}{bc} \geq 6 \quad (5)$$

$$\text{But } \sum \frac{(b+c)(b+c-a)}{bc} = \sum \frac{a(b+c)(b+c-a)}{abc} =$$

$$= \frac{\sum a(b + c)^2 - \sum a^2(b + c)}{abc} = \frac{\sum(ab^2 + ac^2 + 2abc) - \sum a^2 b - \sum a^2 c}{abc}$$

$$= \frac{\sum ab^2 + \sum ac^2 + 6abc - \sum a^2 b - \sum a^2 c}{abc} = \frac{6abc}{abc} = 6 \quad (6)$$

From (6) $\Rightarrow$ it's true.

SOLUTION 4.143

Solution by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a \geq b \geq c$

Then, $\sqrt{b+c} \leq \sqrt{c+a} \leq \sqrt{a+b}$ *&* $\frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c}$

$$\therefore LHS \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum \sqrt{b+c}\right)\left(\sum \frac{1}{r_a}\right)$$

$$\overset{CBS}{\leq} \frac{\sqrt{3}}{3}\sqrt{4s}\left(\frac{1}{r}\right) = \frac{1}{r}\sqrt{\frac{4s}{3}} \overset{?}{\leq} \frac{4R-2r}{r\sqrt[4]{27r^2}}$$

$$\Leftrightarrow \frac{4s}{3} \overset{?}{\leq} \frac{4(2R-r)^2}{3\sqrt{3}r} \Leftrightarrow sr\sqrt{3} \underset{(1)}{\overset{?}{\leq}} (2R-r)^2$$

$$\text{Now, LHS of (1)} \overset{Mitrinovic}{\leq} \frac{3\sqrt{3}R}{2} \cdot r\sqrt{3} = \frac{9Rr}{2}$$

$$\overset{?}{\leq} (2R-r)^2 \Leftrightarrow 8R^2 - 17Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (8R-r)(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{Euler}{\geq} 2$$

SOLUTION 4.144

Solution by Soumava Chakraborty-Kolkata-India

We shall first prove: $\left(\sum a\right)\left(\sum \frac{1}{a}\right) + \frac{16S^2}{abc(\sum a)} \geq 10$

$$\Leftrightarrow \left(\frac{2s}{4Rrs}\right)(s^2+4Rr+r^2) + \frac{16r^2s^2}{8Rrs^2} \geq 10 \Leftrightarrow \frac{s^2+4Rr+5r^2}{2Rr} \geq 10$$

$$\Leftrightarrow s^2 \geq 16Rr - 5r^2 \rightarrow \text{true (Gerretsen)} \therefore \left(\sum a\right)\left(\sum \frac{1}{a}\right) + \frac{16S^2}{abc(\sum a)} \overset{(1)}{\geq} 10$$

Applying (1) on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ *and whose area*

of course, will be $\frac{S}{3}$*, we get:* $\left(\frac{2}{3}\sum m_a\right)\left(\frac{3}{2}\sum \frac{1}{m_a}\right) + \frac{16\left(\frac{S^2}{9}\right)}{\left(\frac{8}{27}\prod m_a\right)\left(\frac{2}{3}\sum m_a\right)} \geq 10$

$$\Leftrightarrow \left(\sum m_a\right)\left(\sum \frac{1}{m_a}\right) + \frac{9S^2}{(\prod m_a)(\sum m_a)} \geq 10$$

SOLUTION 4.145

Solution by Soumava Chakraborty-Kolkata-India

$$a\cos A, b\cos B, c\cos C > 0$$

$$a\cos A + b\cos B - c\cos C$$