$$= R(\sin 2A + \sin 2B) - 2R\sin C\cos C = R\cdot 2\sin C\cos(A-B) - 2R\sin C\cos C$$

$$= 2R\sin C\{\cos(A-B)+\cos(A+B)\} = 2R\sin C\cdot 2\cos A\cos B$$

$$= 4R\sin C\cos A\cos B > 0 \ \ (\because \cos A, \cos B > 0)$$

Similarly, $b\cos B + c\cos C - a\cos A > 0$ *&* $c\cos C + a\cos A - b\cos B > 0$

$\therefore$ $a\cos A, b\cos B, c\cos C$ *are sides of a triangle.*

Let $a\cos A = x, b\cos B = y, c\cos C = z.$

Then, given inequality becomes:

$xyz \ge (x+y-z)(y+z-x)(z+x-y)$*, which, of course holds true when* x, y, z *are 3 sides of a triangle (proved).*

SOLUTION 4.146

Solution by Serban George Florin-Romania

$$\left(\frac{bm_c}{cm_b}-\frac{cm_b}{bm_c}\right)+\left(\frac{am_b}{bm_a}-\frac{bm_a}{am_b}\right)+\left(\frac{cm_a}{am_c}-\frac{am_c}{cm_a}\right)\ge 0$$

$$\frac{b^2m_c^2-c^2m_b^2}{bcm_bm_c}+\frac{a^2m_b^2-b^2m_a^2}{abm_am_b}+\frac{c^2m_a^2-a^2m_a^2}{acm_am_c}\ge 0$$

$$\Rightarrow \frac{\left(\frac{m_c}{c}\right)^2-\left(\frac{m_b}{b}\right)^2}{\frac{m_b}{b}\cdot\frac{m_c}{c}}+\frac{\left(\frac{m_b}{b}\right)^2-\left(\frac{m_a}{a}\right)^2}{\frac{m_a}{a}\cdot\frac{m_b}{b}}+\frac{\left(\frac{m_a}{a}\right)^2-\left(\frac{m_c}{c}\right)^2}{\frac{m_a}{a}\cdot\frac{m_c}{c}}\ge 0$$

If $a \le b$ *then* $\frac{m_a}{a}\ge\frac{m_b}{b} \Leftrightarrow \frac{m_a^2}{a^2}\ge\frac{m_b^2}{b^2}$

$$\frac{b^2(2b^2+2c^2-a^2)}{4}\ge\frac{a^2(2a^2+2c^2-b^2)}{4}, 2b^4+2b^2c^2-a^2b^2\ge 2a^4$$

$$+2a^2c^2-a^2b^2, (b^4-a^4)+c^2(b^2-a^2)\ge 0, (b^2-a^2)(b^2+a^2)+c^2(b^2-a^2)\ge 0$$

$(b^2-a^2)(b^2+a^2+c^2)\ge 0$ *(true)* $b^2\ge a^2, b^2-a^2\ge 0$

Note $\frac{m_a}{a}=x, \frac{m_b}{b}=y, \frac{m_c}{c}=z, a\le b\le c \Rightarrow x\ge y\ge z$

$$\Rightarrow \frac{z^2-y^2}{yz}+\frac{y^2-x^2}{xy}+\frac{x^2-z^2}{xz}\ge 0$$

$$\Rightarrow \frac{x^2-z^2}{xz}\ge\frac{y^2-z^2}{yz}+\frac{x^2-y^2}{xy}, \frac{(x^2-y^2)+(y^2-z^2)}{xz}\ge\frac{y^2-z^2}{yz}+\frac{x^2-b^2}{xy}$$

$$\frac{x^2-y^2}{xz}+\frac{y^2-z^2}{xz}\ge\frac{y^2-z^2}{yz}+\frac{x^2-y^2}{xy}$$

$$(x^2-y^2)\left(\frac{1}{xz}-\frac{1}{xy}\right)\geq (y^2-z^2)\left(\frac{1}{yz}-\frac{1}{xz}\right)$$

$$\frac{(x-y)(x+y)(y-z)}{xyz}\geq\frac{(y-z)(y+z)(x-y)}{xyz}$$

$$\Rightarrow (x-y)(x+y)(y-z)\geq (y-z)(y+z)(x-y)$$

$$\Rightarrow (x-y)(x+y)(y-z)-(y-z)(y+z)(x-y)\geq 0$$

$$\Rightarrow (x-y)(y-z)(x+y-y-z)\geq 0, (x-y)(y-z)(x-z)\geq 0$$

True

$$x\geq y\Rightarrow x-y\geq 0$$

$$y\geq z\Rightarrow y-z\geq 0$$

$$x\geq z\Rightarrow x-z\geq 0$$

SOLUTION 4.147

Solution by Kevin Soto Palacios – Huarmey-Peru

$$(\tan x+2\,sen\,x-3x)+(\tan y+2\,sen\,y-3y)+(\tan z+2\,sen\,z-3z)>0$$

Consideremos: $f(x)=\tan x+2\,sen\,x-3x$***. Realizamos la primera derivada:***

$f'(x)=sec^2x+2\cos x-3$***. Realizamos la segunda derivada.***

$$f''(x)=-2\,sen\,x+\frac{2\,sen\,x}{cos^3x}=2\,sen\,x\frac{(1-cos^3x)}{cos^3x}>0\ \forall x\in<0,\frac{\pi}{2}>$$

Desde que: $f(0)=f'(0)=0$ ***y*** $f''(x)>0$***, se concluye que:*** $f(x)>0$

$$\tan x+2\,sen\,x-3x>0\quad \textit{(A)}$$

$$\tan y+2\,sen\,y-3y>0\quad \textit{(B)}$$

$$\tan z+2\,sen\,z-3z>0\quad \textit{(C)}$$

$$(\tan x+2\,sen\,x-3x)+(\tan y+2\,sen\,y-3y)+(\tan z+2\,sen\,z-3z)>0$$

SOLUTION 4.148

Solution by Lahiru Samarakoon-Sri Lanka

Lets consider, $\cos^{-1}\left(\frac{5}{2\sqrt{13}}\right)=\theta\Leftrightarrow\cos\theta=\frac{5}{2\sqrt{13}}$

$$\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}=\frac{\sqrt{13}}{6}\cos\left(\frac{\theta}{3}\right)+\frac{1}{12}$$

$$\frac{\left(12\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}-1\right)}{2\sqrt{13}}=\cos\frac{\theta}{3}$$

$$\textit{But, } \cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$$

$\therefore$ ***we have to prove,***

$$4\left(\frac{12\cos\frac{3\pi}{13}\cos\frac{3\pi}{13} - 1}{2\sqrt{13}}\right)^3 - 3\left(\frac{12\cos\frac{2\pi}{13}\cos\frac{3\pi}{3} - 1}{2\sqrt{13}}\right) = \frac{5}{2\sqrt{13}}$$

$$\textit{So, } \left(12\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} - 1\right)^3 - 39\left(12\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} - 1\right) = 65$$

$$12^3\cos^3\frac{2\pi}{13}\cos^3\frac{3\pi}{13} - 12^2 \times 3\cos^2\frac{2\pi}{13}\cdot\cos^2\frac{3\pi}{13} + 12 \times 3\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}$$

$$-1 - 39 \times 12\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} + 39 = 65$$

$$12^3\cos^3\frac{2\pi}{13}\cos^3\frac{3\pi}{13} - 12^2 \times 3\cos^2\frac{2\pi}{13}\cos^2\frac{3\pi}{13} = 12 \times 36\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} = 27$$

Therefore, now we have to prove,

$$64\cos^3\frac{2\pi}{13}\cos^3\frac{3\pi}{13} - 16\cos^2\frac{2\pi^2}{13}\cos^2\frac{3\pi}{13} - 16\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} = 1$$

Consider L.H.S,

$$\textit{L.H.S} = 64\cos^3\frac{2\pi}{13}\cos^3\frac{3\pi}{13} - 16\cos^2\frac{2\pi}{13}\cos^2\frac{3\pi}{13} - 16\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}$$

$$= 8\underbrace{\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}}_{A}\left(8\cos^2\frac{2\pi}{13}\cos^2\frac{3\pi}{13} - 2\cos\frac{2\pi}{13}\cos\frac{2\pi}{13} - 2\right)$$

$$= 8A\left[\frac{8\left(1+\cos\frac{4\pi}{13}\right)}{2}\cdot\frac{\left(1+\cos\frac{6\pi}{13}\right)}{2} - 2\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} - 2\right]$$

$$\textit{L.H.S} = 8A\left(2 + 2\cos\frac{4\pi}{13} + 2\cos\frac{6\pi}{13} + 2\cos\frac{4\pi}{13}\cos\frac{6\pi}{13} - 2\cos\frac{2\pi}{13}\cos\frac{3\pi}{13} - 2\right)$$

$$= 8A\left(2\cos\frac{4\pi}{13} + 2\cos\frac{6\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{2\pi}{13} - \cos\frac{2\pi}{13} - \cos\frac{\pi}{13}\right)$$

$$[\because \cos(n+y) + \cos(n-y) = 2\cos n\cos y]$$

$$= 8A\left[\left(\cos\frac{4\pi}{13} + \cos\frac{6\pi}{13}\right) + \left(\cos\frac{4\pi}{13} + \cos\frac{2\pi}{13}\right) - \left(\cos\frac{\pi}{13} + \cos\frac{3\pi}{13}\right)\right.$$

$$\left. - \left(\cos\frac{5\pi}{13} + \cos\frac{7\pi}{13}\right)\right]$$

$$\left(\because \cos\frac{10\pi}{3} = -\cos\frac{3\pi}{13} \textit{ and } \cos\frac{6\pi}{13} = -\cos\frac{7\pi}{13}\right)$$

$$= 8A\left[2\cos\frac{\pi}{13}\cos\frac{5\pi}{13} + 2\cos\frac{\pi}{13}\cos\frac{3\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{2\pi}{13} - 2\cos\frac{\pi}{13}\cos\frac{6\pi}{13}\right]$$

$$= 16\underbrace{\cos\frac{\pi}{13}\cdot\cos\frac{2\pi}{13}\cdot\cos\frac{3\pi}{13}}_{B}\left[\left(\cos\frac{5\pi}{13} + \cos\frac{3\pi}{13}\right) - \left(\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13}\right)\right]$$

$$= 16B\left[2\cos\frac{4\pi}{13}\cdot\cos\frac{\pi}{13} - 2\cos\frac{4\pi}{13}\cos\frac{2\pi}{13}\right]$$

$$= 32\cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}\left(\cos\frac{\pi}{13} + \cos\frac{11\pi}{13}\right)$$

$$\left(\because \cos\frac{2\pi}{13} = -\cos\frac{11\pi}{13}\right)$$

So,

$$L.H.S = 64\cos\frac{\pi}{13}\cos\frac{2\pi}{13}\cos\frac{3\pi}{13}\cos\frac{4\pi}{13}\cos\frac{5\pi}{13}\cos\frac{6\pi}{13} =$$

$$= \frac{32}{\sin\frac{\pi}{13}}\underbrace{\left(2\sin\frac{\pi}{13}\cos\frac{\pi}{13}\right)}_{\sin\frac{2\pi}{13}}\cdot\cos\frac{2\pi}{13}\cos\frac{4\pi}{13}\cos\frac{3\pi}{13}\cos\frac{5\pi}{13}\cos\frac{6\pi}{13}$$

$$= \frac{16}{\sin\frac{\pi}{3}}\left(2\sin\frac{2\pi}{13}\cos\frac{2\pi}{13}\right)\cdot\cos\frac{3\pi}{13}\ldots.\cos\frac{6\pi}{13} \Leftrightarrow \textit{using similar way,}$$

$$L.H.S = \frac{\sin\frac{12\pi}{3}}{\sin\frac{\pi}{13}} = \frac{\sin\left(\pi - \frac{\pi}{13}\right)}{\sin\frac{\pi}{13}} = 1$$

$$L.H.S. = R.H.S.$$

Therefore, it's true

$$\therefore \cos\frac{2\pi}{13}\cos\frac{3\pi}{13} = \frac{\sqrt{13}}{6}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{5}{2\sqrt{13}}\right)\right) + \frac{1}{12}$$

SOLUTION 4.149

Solution by Kevin Soto Palacios –Huarmey-Peru:

$$\frac{tan(A+B) + tan(C+D)}{1 - tan(A+B)\,tan(C+D)} = 1$$

$$\Rightarrow tan(A+B) + tan(C+D) = 1 - tan(A+B)\,tan(C+D)$$

$$\frac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B} + \frac{tan\,C + tan\,D}{1 - tan\,C\,tan\,D} = 1 - \left(\frac{tan\,A + tan\,B}{1 - tan\,A\,tan\,B}\right)\left(\frac{tan\,C + tan\,D}{1 - tan\,C\,tan\,D}\right)$$

Multiplicamos: $(1 - tan\,A\,tan\,B)(1 - tan\,C\,tan\,D) \neq 0$

$$(tan\,A + tan\,B)(1 - tan\,C\,tan\,D) + (tan\,C + tan\,D)(1 - tan\,A\,tan\,B) =$$

$$= (1 - tan\,A\,tan\,B)(1 - tan\,C\,tan\,D) - (tan\,A + tan\,B)(tan\,C + tan\,D)$$

$$\rightarrow A_1 = \sum \tan A + \sum \tan A \tan B - \sum \tan A \tan B \tan C$$
$$= \tan A \tan B \tan C \tan D + 1$$
$$\Rightarrow 16(A_1 - 1) \leq A_2 \rightarrow 16 \tan A \tan B \tan C \tan D \leq (\tan A + \tan B)^2 (\tan C + \tan D)^2$$
$$\rightarrow (\text{Válido por}: MA \geq MG)$$

SOLUTION 4.150

Solution 1 by Tran Hong-Vietnam

$$\left[\sum m_a(h_b - h_c)\right]^2 \leq \left[\sum m_a|h_b - h_c|\right]^2 \overset{BCS}{\leq} \sum m_a^2 \cdot \sum (h_b - h_c)^2$$

$$= \frac{9}{4}\left(\sum a^2\right)\sum (h_b - h_c)^2 = \frac{3}{4}\left(\sum a^2\right)\left\{2\left(\sum h_a^2 - \sum h_a h_b\right)\right\} \ (*)$$

We must show that

$$2\left(\sum h_a^2 - \sum h_a h_b\right) < 3\sum h_a^2 \Leftrightarrow -2\sum h_a h_b < \sum h_a^2$$

(It is true because: $h_a, h_b, h_c > 0$***)***$\Rightarrow$ ***(*)*** $< \frac{9}{4}\left(\sum a^2\right)\sum h_a^2$

$$\Rightarrow 4\left[\sum m_a(h_b - h_c)\right]^2 < 9\left(\sum a^2\right)\left(\sum h_a^2\right)$$

SOLUTION 4.151

Solution by Soumava Chakraborty-Kolkata-India

Let $f(x) = \operatorname{cosec} x + 2\sqrt{2}\operatorname{cosec} x - 3\sqrt{3}$

$$f'(x) = -\operatorname{cosec} x \cot x + 2\sqrt{2}\sec x \tan x$$

$$f''(x) = \operatorname{cosec}^3 x + \cot^2 x \operatorname{cosec} x + 2\sqrt{2}(\sec^3 x + \sec x \tan^2 x) > 0, \forall\, x \in \left(0, \frac{\pi}{2}\right)$$

when $f'(x) = 0$, $f(x)$ **attains a minima and** $f(x)$ **never attains a maxima in** $\left(0, \frac{\pi}{2}\right)$**,**

point at which $f(x)$ **attains a minima is the point at which** $f(x)$ **attains its minimum value**

$$f'(x) = 0 \Rightarrow 2\sqrt{2} \cdot \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$\Rightarrow 2\sqrt{2}\sin^3 x = \cos^3 x \Rightarrow \sqrt{2}\sin x = \cos x \Rightarrow \tan x = \frac{1}{\sqrt{2}} \Rightarrow \cos^2 x = \frac{2}{3}$ **and** $\sin^2 x = \frac{1}{3}$

$$f_{\min} = \frac{1}{1\sqrt{3}} + \frac{2\sqrt{2}}{\frac{\sqrt{2}}{\sqrt{3}}} = 3\sqrt{3} \quad \text{at } x = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \frac{1}{\sin x} + \frac{2\sqrt{2}}{\cos x} \geq 3\sqrt{3},$$

$$\text{equality at } x = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

SOLUTION 4.152

Solution by Kunihiko Chikaya-Tokyo-Japan

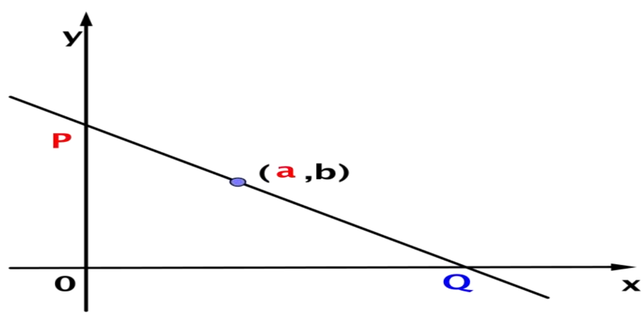
$$\frac{p^3}{cos\theta} + \frac{q^3}{sin\theta} \geq p^3\left(\frac{2\sqrt{p^2+q^2}}{p} - \frac{p^2+q^2}{p^2}\cos\theta\right)$$

$$\text{Equality } \binom{p}{q} = \binom{\cos\theta}{\sin\theta} + q^3\left(\frac{2\sqrt{p^2+q^2}}{q} - \frac{p^2+q^2}{q^2}\sin\theta\right)$$

$$\Leftrightarrow \tan\theta = \frac{q}{p} = \sqrt[3]{\frac{b}{a}} = 2(p^2+q^2)\sqrt{p^2+q^2} - (p^2+q^2)(p\cos\theta + q\sin\theta)$$

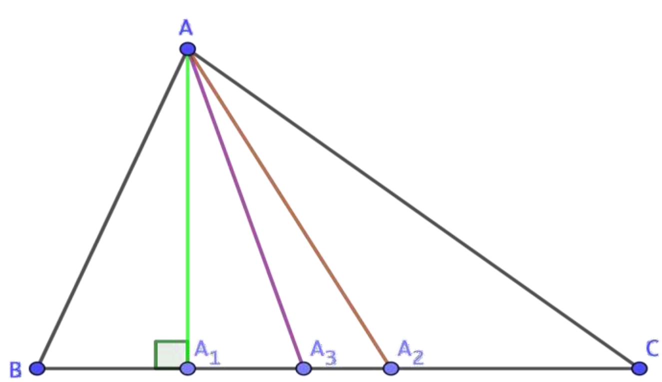
$$\geq (p^2+q^2)^{\frac{3}{2}}$$

$$p = a^{\frac{1}{3}}, q = b^{\frac{1}{3}} \ (a, b > 0) \Rightarrow \frac{a}{\cos\theta} + \frac{b}{\sin\theta} \geq \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}} = \left|\overrightarrow{PQ}\right|_{min}$$



SOLUTION 4.153

Solution by Soumava Chakraborty-Kolkata-India



$$\text{Let } BA_3 = m \ \&\ CA_3 = n. \text{ Then, } \frac{m}{n} = \frac{c^2}{b^2} \ (\&\ m + n = a)$$

$$\therefore \frac{m+n}{n} = \frac{c^2+b^2}{b^2}$$

$$\Rightarrow \frac{a}{n} = \frac{c^2+b^2}{b^2} \Rightarrow n = \frac{ab^2}{c^2+b^2} \Rightarrow m = \frac{c^2}{b^2} n = \frac{c^2}{b^2} \cdot \frac{ab^2}{b^2+c^2} = \frac{ac^2}{b^2+c^2}$$

$$\Rightarrow BA_3 \overset{(i)}{=} \frac{ai^2}{b^2+c^2} \therefore A_2A_3 = BA_1 - BA_3$$

$$\overset{by\,(i)}{=} \frac{a}{2} - \frac{ai^2}{b^2+c^2} = \frac{a(b^2+c^2) - 2ai^2}{2(b^2+c^2)} \overset{(1)}{=} \frac{a(b^2-c^2)}{2(b^2+c^2)}$$

$$\text{From } \Delta ABA, \frac{BA_1}{C} = \cos B \Rightarrow BA_1 = c\cos B = \frac{c(c^2+a^2-b^2)}{2ca} \overset{(ii)}{=} \frac{c^2+a^2-b^2}{2a}$$

$$\therefore A_2A_1 = BA_2 - BA_1 \overset{by\,(ii)}{=} \frac{a}{2} - \frac{c^2+a^2-b^2}{2a}$$

$$= \frac{a^2-(c^2+a^2-b^2)}{2a} \overset{(2)}{=} \frac{b^2-c^2}{2a}$$

$$\text{(1), (2)} \Rightarrow \frac{A_2A_3}{A_2A_1} \overset{(a)}{=} \frac{a^2}{b^2+c^2}$$

$$\text{Similarly, } \frac{B_2B_3}{B_2B_1} \overset{(b)}{=} \frac{b^2}{c^2+a^2} \ \&\ \frac{c_2c_3}{c_2c_1} \overset{(c)}{=} \frac{c^2}{a^2+b^2}$$

$$\text{(a)+(b)+(c)} \Rightarrow LHS = \sum \frac{a^2}{b^2+c^2} \overset{Nesbitt}{>} 3 \overset{?}{>} \frac{108r^2}{\sum a^2} \Leftrightarrow \sum a^2 \underset{(3)}{\overset{?}{>}} 36r^2$$

$$\text{But } \sum a^2 \overset{\substack{Ionescu-\\Weitzenbock}}{>} 4\sqrt{3}rs \overset{Mitrinovic}{>} 4\sqrt{3}r\left(3\sqrt{3}r\right) = 36r^2 \Rightarrow \text{(3) is true (Proved)}$$

SOLUTION 4.154

Solution by Soumava Chakraborty-Kolkata-India

$$\because m_a \geq \sqrt{s(s-a)}, \text{ etc } \therefore LHS \geq \frac{2\sqrt{s(s-a)s(s-b)s(s-c)}}{\frac{16R^2r^2s^2}{8R^3}}$$

$$= \frac{16R^3rs^2}{16R^2r^2s^2} = \frac{R}{r} \therefore \text{it suffices to prove: } \frac{R}{r} \geq 1 + \frac{\sum r_a^2}{\sum r_ar_b}$$

$$\Leftrightarrow \frac{R-r}{r} \geq \frac{(4R+r)^2 - 2s^2}{s^2} \Leftrightarrow (R-r)s^2 + 2rs^2 \geq r(4R+r)^2$$

$$\Leftrightarrow (R+r)s^2 \overset{(1)}{\geq} r(4R+r)^2$$

$$\text{Now, LHS of (1)} \overset{Gerretsen}{\geq} (R+r)(16Rr - 5r^2) \overset{?}{\geq} r(4R+r)^2$$

$$\Leftrightarrow 16R^2+11Rr-5r^2 \overset{?}{\geq} 16R^2+8Rr+r^2 \Leftrightarrow 3Rr \overset{?}{\geq} 6r^2 \to \textit{true (Euler) (Done)}$$

SOLUTION 4.155

Solution by Tran Hong-Vietnam

$$\because f(x)=x^5 (x>0) \Rightarrow f''(x)=20x^3>0\ (x>0)$$

Using Jensen's inequality:

$$\sum am_a^5 = 2s\sum \frac{a}{2s}m_a^5 \geq 2s\sum\left(\frac{a}{2s}\cdot m_a\right)^5 = \frac{1}{(2s)^4}\sum (am_a)^5 \Leftrightarrow \frac{\sum am_a^5}{\sum (am_a)^5} \geq \frac{1}{16s^4}$$

Must show that: $\frac{1}{16s^4} \geq \frac{1}{729R^4} \Leftrightarrow 729R^4 \geq 16s^4$

It is true because $\because s \leq \frac{3\sqrt{3}}{2}R \Rightarrow s^4 \leq \frac{729}{16}R^4 \Leftrightarrow 729R^4 \geq 16s^4$

SOLUTION 4.156

Solution by Soumava Chakraborty-Kolkata-India

$$\sum \frac{\sin\frac{A}{2}}{m_a} = \frac{\sum m_a^2}{2m_am_bm_c} \Leftrightarrow \sum m_bm_c \sin\frac{A}{2} \overset{(1)}{\leq} \frac{\sum m_a^2}{2} = \frac{3\sum a^2}{8}$$

$$\sum m_bm_c\sin\frac{A}{2} \overset{CBS}{\leq} \sqrt{\sum m_b^2m_c^2}\sqrt{\sum \sin^2\frac{A}{2}} = \sqrt{\frac{9\sum a^2b^2}{16}}\sqrt{\frac{\sum(1-\cos A)}{2}}$$

$$= \sqrt{\frac{9\sum a^2b^2}{16}}\sqrt{\frac{2R-r}{2R}} \overset{?}{\leq} \frac{3\sum a^2}{8} \Leftrightarrow \frac{9\sum a^2b^2}{16}\left(\frac{2R-r}{2R}\right) \overset{?}{\leq} \frac{9}{64}\left(\sum a^2\right)^2$$

$$\Leftrightarrow 2(2R-r)\left(\left(\sum ab\right)^2 - 2abc(2s)\right) \overset{?}{\leq} 4R(s^2-4Rr-r^2)^2$$

$$\Leftrightarrow 2(2R-r)(s^2+4Rr+r^2)^2 - 4R(s^2-4Rr-r^2)^2 \overset{?}{\leq} 32(2R-r)Rrs^2$$

$$\Leftrightarrow 2R((s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2) \overset{?}{\leq} 16(2R-r)Rrs^2 + r(s^2+4Rr+r^2)^2$$

$$\Leftrightarrow 2R(2s^2)(8Rr+2r^2) \overset{?}{\leq} 16(2R-r)Rrs^2 + r(s^2+4Rr+r^2)^2$$

$$\Leftrightarrow s^4 + r^2(4R+r)^2 + 2s^2(4Rr+r^2) \underset{(2)}{\overset{?}{\geq}} 24Rrs^2$$

Now, LHS of (2) $\overset{Gerretsen}{\leq} s^2(16Rr-5r^2) + r^2(4R+r)^2 + 2s^2(4Rr+r^2) \overset{?}{\geq} 24Rrs^2$

$\Leftrightarrow r^2(4R+r)^2 \overset{?}{\geq} 3r^2s^2 \Leftrightarrow 4R+r \overset{?}{\geq} \sqrt{3}s \to$ ***true (Trucht)⇒ (1) is true (Done).***

SOLUTION 4.157

Solution by Soumava Chakraborty-Kolkata-India

$$ax=u, by=v, cz=w$$

The inequality to prove can be written:

$$\sum \frac{u}{u+v+98w} \ge \frac{3}{100} \leftrightarrow 100\sum u(v+w+98u)(w+u+98v) \ge 3\prod(u+v+98w)$$

$$9506\sum u^3 + 931491\sum u^2v \ge 2794764uvw + 9409\sum uv^2 \ (a)$$

$$u^3+v^3+v^3 \overset{AM-GM}{\ge} 3uv^2$$

$$v^3+w^3+w^3 \overset{AM-GM}{\ge} 3vw^2$$

$$w^3+u^3+u^3 \overset{AM-GM}{\ge} 3wu^2$$

$$\sum u^3 \ge \sum uv^2 \to 9506\sum u^3 \ge 9506\sum uv^2 \ (1)$$

$$97\sum uv^2 \overset{AM-GM}{\ge} 291uvw \ (2)$$

$$931491\sum u^2v \overset{AM-GM}{\ge} 2794473uvw \ (3)$$

By adding $(1),(2),(3) \to (a)$

SOLUTION 4.158

Solution by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{m_a}{h_a} \ge \sum \frac{\frac{b^2+c^2}{4R}}{\frac{bc}{2R}} = \frac{1}{2}\sum \frac{b^2+c^2}{bc} = \frac{1}{2}\sum \frac{ab^2+ac^2}{abc} =$$

$$= \frac{1}{2}\sum \frac{bc^2+c^2a}{abc} = \frac{1}{2}\sum \frac{bc+ca}{ab} = \frac{1}{2}\sum \frac{\frac{bc}{2R}+\frac{ca}{2R}}{\frac{ab}{2R}} = \frac{1}{2}\sum \frac{h_a+h_b}{h_c}$$

SOLUTION 4.159

Solution by Bogdan Fustei-Romania

In ΔABC the following relationship: $\frac{a^2}{R_a^2}+\frac{b^2}{R_b^2}+\frac{c^2}{R_c^2} \le 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2$

(I – incenter in ΔABC); R_a, R_b, R_c – circumradii $\Delta BIC, \Delta CIA, \Delta AIB$)

Using two additional inequalities:

1) $\frac{R}{r} \ge \frac{abc+a^2+b^3+c^3}{2abc}$

2) $x,y,z>0: \frac{x^3+y^3+z^3}{4xyz}+\frac{1}{4} \ge \left(\frac{x^2+y^2+z^2}{xy+yz+zx}\right)^2$

From the two inequalities from above we can write the following:

$$\frac{R}{2r} \overset{(1)}{\geq} \frac{a^3+b^3+c^3}{4abc} + \frac{1}{4} \overset{(2)}{\geq} \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2. \text{ So, finally: } \frac{R}{2r} \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)^2$$

$$R_a = 2R\sin\frac{A}{2} \quad \text{(and the analogs)}$$

$$\sin\frac{A}{2} = \sqrt{\frac{r_a-r}{4R}} \quad \text{(and the analogs)}$$

$$a^2 = (r_b+r_c)(r_a-r) \quad \text{(and the analogs)}$$

$$\Rightarrow R_a = 2R \cdot \sqrt{\frac{r_a-r}{R}} = \sqrt{4R^2\frac{(r_a-r)}{4R}} = \sqrt{R(r_a-r)} \text{ (and the analogs)}$$

$$R_a^2 = R(r_a-r) \text{ (and the analogs)} \Rightarrow \frac{a^2}{R_a^2} = \frac{(r_b+r_c)(r_a-r)}{R(r_a-r)} = \frac{r_b+r_c}{R}$$

$$\text{So, } \frac{a^2}{R_a^2} = \frac{r_b+r_c}{R} \quad \text{(and the analogs)}$$

$$\frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{r_b+r_c}{R} + \frac{r_a+r_c}{R} + \frac{r_a+r_b}{R} = \frac{2(r_a+r_b+r_c)}{R} = \frac{2(4R+r)}{R}$$

$$(r_a+r_b+r_c = 4R+r) \Rightarrow \frac{a^2}{R_a^2} + \frac{b^2}{R_b^2} + \frac{c^2}{R_c^2} = \frac{8R+2r}{R} = 8 + \frac{2r}{R}$$

$$\text{The inequality from enunciation becomes: } 8 + \frac{2r}{R} \leq 8 + \left(\frac{ab+bc+ac}{a^2+b^2+c^2}\right)^2 \Rightarrow$$

$$\Rightarrow \frac{R}{2r} \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ac}\right)$$

From the above, the inequality from enunciation is proved.

SOLUTION 4.160

Solution by Mehmet Sahin-Ankara-Turkey

$$(am_a+bm_b+cm_c)^2 \leq (a^2+b^2+c^2)(m_a^2+m_b^2+m_c^2)$$

$$am_a+bm_b+cm_c \leq \sqrt{9R^2 \cdot \frac{3}{4} \cdot 9R^2} = \frac{9\sqrt{3}R^3}{2} \quad (1)$$

$$s_a \leq m_a, s_b \leq m_b, s_c \leq m_c$$

$$s_am_a+s_bm_b+s_cm_c \leq m_a^2+m_b^2+m_c^2 = \frac{3}{4}\cdot 9R^2 \quad (2)$$

From (1) and (2):

$$(am_a+bm_b+cm_c)(s_am_b+s_bm_b+s_cm_c) \leq \frac{9\sqrt{3}}{2}\cdot\frac{27}{4}R^4 \leq \frac{243\sqrt{3}}{8}R^4$$

SOLUTION 4.161

Solution by Marian Ursărescu-Romania

$$\textit{Inequality} \Leftrightarrow \left(\sqrt{h_a+h_b}+\sqrt{h_b+h_c}+\sqrt{h_c+h_a}\right)^2 \le \frac{4s^2}{R} \quad (1)$$

$$\textit{From Cauchy's Inequality} \Rightarrow \left(\sqrt{h_a+h_b}+\sqrt{h_b+h_c}+\sqrt{h_c+h_a}\right)^2 \le 6(h_a+h_b+h_c) \quad (2)$$

From (1)+(2) we must show:

$$3(h_a+h_b+h_c) \le \frac{2s^2}{R} \quad (3)$$

But $h_a+h_b+h_c = \frac{s^2+r^2+4Rr}{2R}$ *(4) From (3)+(4) we must show:*

$$\frac{3(s^2+r^2+4Rr)}{2R} \le \frac{2s^2}{R} \Leftrightarrow 3(s^2+r^2+4Rr) \le 4s^2 \Leftrightarrow$$

$$s^2 \ge 3r^2+12Rr \quad (5)$$

From Gerretsen's inequality we have: $s^2 \ge 16Rr-5r^2$ *(6)*

From (5)+(6) we must show:

$$16Rr-5r^2 \ge 3r^2+12Rr \Leftrightarrow 4Rr \ge 8r^2 \Leftrightarrow$$

$R \ge 2r$*, true because it's Euler's inequality.*

SOLUTION 4.162

Solution by Marian Ursărescu-Romania

We must show:

$$a(b+c)\cos\frac{A}{2}+b(a+c)\cos\frac{B}{2}+c(a+b)\cos\frac{C}{2} \ge 36\sqrt{3}r^2 \quad (1)$$

But

$$a(b+c)\cos\frac{A}{2}+b(a+c)\cos\frac{B}{2}+c(a+b)\cos\frac{C}{2} \ge$$

$$3\sqrt[3]{abc(a+b)(b+c)(a+c)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \quad (2)$$

From (1)+(2) we must show:

$$\sqrt[3]{abc(a+b)(b+c)(a+c)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \ge 12\sqrt{3}r^2 \quad (3)$$

But $abc = 4sRr$ *(4),* $(a+b)(b+c)(a+c) = 2s(s^2+r^2+2Rr)$ *(5)*

and $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$ *(6). From (4)+(5)+(6) we must show:*

$$\sqrt[3]{4sRr\cdot 2s(s^2+r^2+2Rr)\cdot\frac{s}{4R}} \ge 12\sqrt{3}r^2 \Leftrightarrow$$

$$\left.\begin{array}{c} s\sqrt[3]{2r(s^2+r^2+2Rr)} \ge 12\sqrt{3}r^2 \ (7) \\ \textit{From Mitrinovic } s \ge 3\sqrt{3}r \ (8) \end{array}\right\} \textit{we must show}$$

$$\sqrt[3]{2r(s^2+r^2+2Rr)} \ge 4r \Leftrightarrow 2r(s^2+r^2+2Rr) \ge 64r^3 \Leftrightarrow$$

$$s^2+r^2+2Rr \ge 32r^2 \quad (9)$$

From Gerretsen we have $s^2 \ge 16Rr-5r^2$ *(10)*

From (9)+(10) we must show: $18Rr-4r^2 \ge 32r^2 \Leftrightarrow$

$\Leftrightarrow 18Rr \ge 36r^2 \Leftrightarrow R \ge 2r$ *true (Euler)*

SOLUTION 4.163

Solution by Soumitra Mandal-Chandar Nagore-India

We know, $\frac{3\sqrt{3}}{2} \ge \sum_{cyc}\sin A$ *and* $\frac{3\sqrt{3}}{8} \ge \prod_{cyc}\sin A$

$$\prod_{cyc}\left(1+\frac{1}{\sin A}+\frac{1}{\sin B+\sin C}\right) \overset{\substack{HOLDER'S\\INEQUALITY}}{\ge} \left(1+\frac{1}{\sqrt[3]{\sin A\sin B\sin C}}+\frac{1}{\sqrt[3]{\prod_{cyc}(\sin A+\sin B)}}\right)^3$$

$$\overset{\substack{REVERSE\\AM\ge GM}}{\ge} \left(1+\frac{2}{\sqrt{3}}+\frac{3}{2\sum_{cyc}\sin A}\right)^3 \ge \left(1+\frac{2}{\sqrt{3}}+\frac{3}{3\sqrt{3}}\right)^3 = \left(1+\sqrt{3}\right)^3$$

SOLUTION 4.164

Solution by Soumava Chakraborty-Kolkata-India

$$ab = 12R^2\sin^2\frac{C}{2}$$

$$\Rightarrow ab = 12\left(\frac{abc}{4\Delta}\right)^2\frac{(s-a)(s-b)}{ab} \Rightarrow a^2b^2 = \frac{3}{4}\cdot\frac{a^2b^2c^2(s-a)(s-b)}{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow 4s(s-c) = 3c^2 \Rightarrow (a+b+c)(a+b-c) = 3c^2$$

$$\Rightarrow (a+b)^2-c^2 = 3c^2 \Rightarrow a+b = 2c \Rightarrow a+b+c = 3c$$

$$\Rightarrow s = \frac{3c}{2} \overset{s\ge 3\sqrt{3}r}{\ge} 3\sqrt{3}r \Rightarrow c \ge 2\sqrt{3}r \Rightarrow \frac{c\sqrt{3}}{6} \ge r \Rightarrow r \le \frac{c\sqrt{3}}{6}$$

SOLUTION 4.165

Solution by Marian Ursărescu-Romania

We must show: $3\sqrt[3]{a^2b^2c^2} \le (8R-10r)^2$ *(1)*

But $\sqrt[3]{a^2b^2c^2} \le \frac{a^2+b^2+c^2}{3}$ *(2)*

Form (1)+(2) we must show: $a^2+b^2+c^2 \le (8R-10r)^2$ *(3)*

But $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$ *(4)*

From (3)+(4) we must show:

$$s^2 - r^2 - 4Rr \le 2(4R - 5r)^2 \quad (5)$$

From Gerretsen's inequality: $s^2 \le 4R^2 + 4Rr + 3r^2$ *(6)*

From (5)+(6) we must show:

$$4R^2 + 2r^2 \le 2(4R - 5r)^2 \Leftrightarrow 2R^2 + r^2 \le 16R^2 - 40Rr + 25r^2 \Leftrightarrow$$

$$\Leftrightarrow 14R^2 - 40Rr + 24r^2 \ge 0 \Leftrightarrow 7R^2 - 20Rr + 12r^2 \ge 0$$

Which is true because $R \ge 2r \Rightarrow 7R^2 - 20Rr + 12r^2 \ge 28r^2 - 40r^2 + 12r^2 = 0$

SOLUTION 4.166

Solution by Marian Ursărescu-Romania

We must show:

$$\frac{1}{2S}\left(a^2w_a^2 + b^2w_b^2 + c^2w_c^2\right) \ge 18r^2\sqrt{\frac{6r}{R}} \quad (1)$$

But $r \le \frac{R}{2} \Rightarrow 6r \le 3R \Rightarrow \frac{6r}{R} \le 3$ *(2)*

From (1)+(2):

We must show: $a^2w_a^2 + b^2w_b^2 + c^2w_c^2 \ge 36Sr^2\sqrt{3}$ *(3)*

$$\left.\begin{array}{c} a^2w_a^2 + b^2w_b^2 + c^2w_c^2 \ge 3\sqrt[3]{(abc)^2(w_aw_bw_c)^2} \\ \text{But } \sqrt[3]{w_aw_bw_c} \ge 3r \end{array}\right\} \Rightarrow$$

$$a^2w_a^2 + b^2w_b^2 + c^2w_c^2 \ge 27r^2\sqrt[3]{(abc)^2} \quad (4)$$

From (3)+(4) we must show:

$$2 + r^2\sqrt[3]{(abc)^2} \ge 36Sr^2\sqrt{3} \Leftrightarrow 3\sqrt[3]{(abc)^2} \ge 4S\sqrt{3} \Leftrightarrow$$

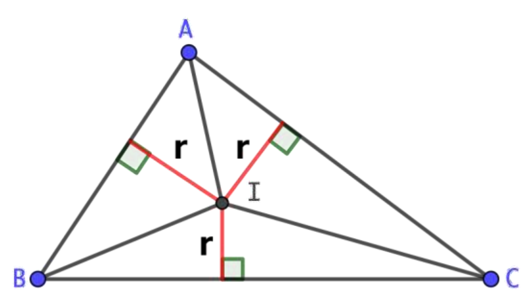
$$3\sqrt[3]{(4RS)^2} \ge 4S\sqrt{3} \Leftrightarrow 27 \cdot 16R^2S^2 \ge 64S^3 3\sqrt{3} \Leftrightarrow$$

$$3\sqrt{3}R^2 \ge 4S \Leftrightarrow 3\sqrt{3}R^2 \ge 4sr \quad (5)$$

$$\left.\begin{array}{r} \text{But } R \ge 2r \\ r \ge \frac{2s}{3\sqrt{3}} \end{array}\right\} \Rightarrow R^2 \ge \frac{4sr}{3\sqrt{3}} \Rightarrow 3\sqrt{3}R^2 \ge 4sr \Rightarrow \text{(5) it's true.}$$

SOLUTION 4.167

Solution by Soumava Chakraborty-Kolkata-India



$$R_a=\frac{BI\cdot CI\cdot BC}{4\cdot\frac{1}{2}BC\cdot r}=\frac{\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}a}{2ar}=\frac{r\sin\frac{A}{2}}{2\left(\pi\sin\frac{A}{2}\right)}=\frac{r\sin\frac{A}{2}}{2\left(\frac{r}{4R}\right)}\overset{(1)}{=}2R\sin\frac{A}{2}$$

$$\therefore\sqrt{\frac{R_a}{h_a}}\overset{by\,(1)}{=}\sqrt{2R\sin\frac{A}{2}\cdot\frac{2R_a}{abc}}=\sqrt{\frac{4R^2}{4Rrs}a\sin\frac{A}{2}}\overset{(a)}{=}\sqrt{\frac{R}{rs}}\sqrt{a\sin\frac{A}{2}}$$

Similarly, $\sqrt{\frac{R_b}{h_b}}\overset{(b)}{=}\sqrt{\frac{R}{rs}}\sqrt{a\sin\frac{B}{2}}$ & $\sqrt{\frac{R_c}{h_c}}\overset{(c)}{=}\sqrt{\frac{R}{rs}}\sqrt{c\sin\frac{C}{2}}$

(a)+(b)+(c) $\Rightarrow\sum\sqrt{\frac{R_a}{h_a}}\overset{(2)}{=}\sqrt{\frac{R}{rs}}\sum\sqrt{a\sin\frac{A}{2}}\overset{A-G}{\geq}3\sqrt{\frac{R}{rs}}\sqrt[6]{4Rrs\left(\frac{r}{4R}\right)}\overset{?}{\geq}\sqrt{6}$

$$\Leftrightarrow 27R^3\overset{?}{\geq}8rs^2\to\ (i)$$

Now, $R^2\overset{Mitrinovic}{\geq}\frac{4S^2}{27}$ & $R\overset{Euler}{\geq}2r$

$\therefore 27R^3\geq 8rs^2$ *(multiplying the above two)⇒ (i) is true* $\therefore\sum\sqrt{\frac{R_a}{h_a}}\geq\sqrt{6}$

Also, using (2), $\sum\sqrt{\frac{R_a}{h_a}}\overset{CBS}{\leq}\sqrt{\frac{R}{rs}}\sqrt{2s}\sqrt{\sum\sin\frac{A}{2}}$

$\overset{Jensen}{\leq}\sqrt{\frac{R}{rs}}\sqrt{2S}\sqrt{3\sin\left(\frac{\pi}{6}\right)}$ *(∵* $f(x)=\sin\frac{x}{2}\ \forall x\in(0,\pi)$ *is concave)*

$$=\sqrt{\frac{3R}{r}}\ \therefore\sum\sqrt{\frac{R_a}{h_a}}\overset{(ii)}{\leq}\sqrt{\frac{3R}{r}}$$

Now, $\sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}}\overset{m_a\geq\sqrt{s(s-a)},etc}{\geq}\sqrt{\frac{6Srs}{\frac{16R^2r^2S^2}{8R^3}}}=\sqrt{\frac{3R}{r}}\overset{by\,(ii)}{\geq}\sum\sqrt{\frac{R_a}{h_a}}\Rightarrow\sum\sqrt{\frac{R_a}{h_a}}\leq\sqrt{\frac{6m_am_bm_c}{h_ah_bh_c}}$

SOLUTION 4.168

Solution by Serban George Florin-Romania

$$\Omega=\sum\left(\frac{b\cos B}{c\cos C}+\frac{c\cos C}{b\cos B}\right)\cdot\cos 2A=\sum\left(\frac{2R\sin B\cos B}{2R\sin C\cos C}+\frac{2R\sin C\cos C}{2R\sin B\cos B}\right)\cdot\cos 2A=$$

$$=\sum\left(\frac{\sin 2B}{\sin 2C}+\frac{\sin 2C}{\cos 2C}\right)\cdot\cos 2A$$

$$\Omega=\frac{\sin 2B\cos 2A}{\sin 2C}+\frac{\sin 2C\cos 2A}{\cos 2C}+\frac{\sin 2A\cdot\cos 2B}{\sin 2C}+\frac{\sin 2C\cos 2B}{\sin 2A}+$$

$$+\frac{\sin 2A\cos 2C}{\sin 2B}+\frac{\sin 2B\cos 2C}{\sin 2A}=\sum\left(\frac{\sin 2A\cos 2B}{\sin 2C}+\frac{\sin 2B\cos 2A}{\sin 2C}\right)=$$

$$= \sum \frac{\sin(2A+2B)}{\sin 2C} = \sum \frac{\sin(2A-2C)}{\sin 2C}$$

$$\Omega = \sum -\frac{\sin 2C}{\sin 2C} = \sum(-1) = -3$$

SOLUTION 4.169

Solution by Daniel Sitaru-Romania

$$\sum_{cyc(a,b,c)} \frac{\sqrt{b^2+c^2}}{h_a} \overset{CBS}{\leq} \sqrt{\sum_{cyc(a,b,c)} (b^2+c^2) \cdot \sum_{cyc(a,b,c)} \frac{1}{h_a^2}} = \sqrt{2 \sum_{cyc(a,b,c)} a^2 \cdot \sum_{cyc(a,b,c)} \frac{a^2}{4S^2}} =$$

$$= \frac{1}{\sqrt{2} \cdot S} \cdot \sum_{cyc(a,b,c)} a^2 \overset{LEIBNIZ}{\leq} \frac{9R^2}{\sqrt{2} \cdot S}$$

SOLUTION 4.170

Solution by Marian Ursărescu-Romania

$$\frac{am_a}{h_a} + \frac{bm_b}{h_b} + \frac{cm_c}{h_c} \geq 3\sqrt[3]{\frac{abc\, m_a m_b m_c}{h_a h_b h_c}} \quad (1)$$

But $m_a \geq \frac{b+c}{2}\cos\frac{A}{2} \geq \sqrt{bc}\cos\frac{A}{2} \Rightarrow m_a m_b m_c \geq abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ ***(2)***

From (1)+(2)$\Rightarrow \frac{am_a}{h_a} + \frac{bm_b}{h_b} + \frac{cm_c}{h_c} \geq 3\sqrt[3]{\frac{a^2b^2c^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{h_a h_b h_c}}$ ***(3)***

$abc = 4sRr, \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{s}{4R}$ ***and*** $h_a h_b h_c = \frac{2s^2r^2}{R}$ ***(4)***

$\frac{am_a}{h_a} + \frac{bm_b}{h_b} + \frac{cm_c}{h_c} \geq 3\sqrt[3]{\frac{16s^2R^2r^2 \cdot s \cdot R}{4R \cdot 2s^2r^2}} \Rightarrow$ ***we must show:***

$$3\sqrt[3]{2R^2s} \geq 2\sqrt{3\sqrt{3}S} \Leftrightarrow 3^6 2^2 R^4 s^2 \geq 2^6 3^3 3\sqrt{3} s^3 r^3 \Leftrightarrow 9R^4 \geq 16\sqrt{3} s r^3 \quad (5)$$

$\left.\begin{matrix} R^3 \geq 8r^3 \\ R \geq \frac{2}{3\sqrt{3}} s \end{matrix}\right\} \Rightarrow R^4 \geq \frac{16}{3\sqrt{3}} s r^3 \Leftrightarrow 9R^4 \geq 16\sqrt{3} s r^3 \Rightarrow$ ***(5) it's true.***

SOLUTION 4.171

Solution by Soumitra Mandal-Chandar Nagore-India

$$\Delta = \frac{a \cdot h_a}{2} = \frac{b \cdot h_b}{2} = \frac{c \cdot h_c}{2}$$

$$\sum_{cyc} \sqrt{h_a - 2r} = \sum_{cyc} \sqrt{\frac{2\Delta}{a} - 2r} = \sum_{cyc} \sqrt{\frac{2r}{a}(s-a)}$$

$$\overset{\substack{Cauchy\\Schwarz}}{\leq}\sqrt{\left(\sum_{cyc}\frac{2r}{a}\right)\left(\sum_{cyc}(s-a)\right)}=\sqrt{\sum_{cyc}\frac{2r}{a}}=\sqrt{\sum_{cyc}h_a}$$

SOLUTION 4.172

Solution by Lahiru Samarakoon-Sri Lanka

$$(a+a')(b+b')(c+c')\geq 24\sqrt{RR'SS'}+4RS+4R'S'$$

$$\textit{but, } R=\frac{abc}{4S}$$

$$(a+a')(b+c')(c+c')\geq 6\sqrt{aa'bb'cc'}+abc+a'b'c'\Rightarrow$$

$$\Rightarrow(abc+a'bc+b'ac+c'ab+a'b'c+b'c'a+a'c'b+a'b'c')\geq 6\sqrt{aa'bb'cc'}$$

So, we have to prove,

$$ab'c'+bc'a'+ca'b'+abc'+bca'+acb'\geq 6\sqrt{aa'bb'cc'}$$

$$\textit{Then, } AM\geq GM$$

$$\frac{a'b'c'+b'c'a'+ca'b'+abc'+bca'+acb'}{6}\geq 6\sqrt{a^3a'^3b^3b'^3c^3c'^3}=\sqrt{aa'bb'cc'}$$

So, it's true.

SOLUTION 4.173

Solution by Soumava Chakraqborty-Kolkata-India

$$LHS=\sum\sqrt{\frac{r_ar_br_c}{as\tan\frac{A}{2}}}=\sum\sqrt{\frac{rs^2}{4Rs\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2}}}=\sum\sqrt{\frac{rs^2}{4Rs}}\csc\frac{A}{2}$$

$$=\sum\sqrt{\frac{rs^2}{4Rs}}\sqrt{\frac{bc(s-a)}{(s-b)(s-c)(s-a)}}$$

$$=\sum\sqrt{\frac{rs^2}{4RS\cdot r^2s}}\sqrt{bc(s-a)}\overset{CBS}{\leq}\sqrt{\frac{1}{4Rr}}\sqrt{\sum ab}\sqrt{\sum(s-a)}$$

$$=\sqrt{\frac{2R}{4Rr}\cdot\frac{\sum ab}{2R}\cdot s}=\sqrt{\frac{s}{2r}\left(\sum h_a\right)}\ \textit{(Proved)}$$

SOLUTION 4.174

Solution by Boris Colakovic-Belgrade-Serbie

$$s-a=\frac{a+b+c}{2}-a=\frac{b+c-a}{2};\frac{a(s-a)}{b+c}=\frac{1}{2}\frac{a(b+c-a)}{b+c}=\frac{1}{2}\left(a-\frac{a^2}{b+c}\right)$$

$$s-b=\frac{a+b+c}{2}-b=\frac{a+c-b}{2};\frac{b(s-b)}{c+a}=\frac{1}{2}\frac{b(a+c-b)}{c+a}=\frac{1}{2}\left(b-\frac{b^2}{c+a}\right)$$

$$s-c=\frac{a+b+c}{2}-c=\frac{a+b-c}{2};\frac{c(s-c)}{a+b}=\frac{1}{2}\frac{c(a+b-c)}{a+b}=\frac{1}{2}\left(c-\frac{c^2}{a+b}\right)$$

LHS$=\frac{1}{2}(a+b+c)-\frac{1}{2}\left(\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\right)\le\frac{1}{2}(a+b+c)-\frac{1}{2}\cdot\frac{(a+b+c)^2}{2(a+b+c)}=\frac{1}{2}\cdot 2s-\frac{1}{4}\cdot\frac{4s^2}{2s}=$

$$=s-\frac{s}{2}=\frac{s}{2}\le\frac{1}{2}\cdot\frac{3\sqrt{3}}{2}R=\frac{3\sqrt{3}}{4}R$$

SOLUTION 4.175

Solution by Marian Ursărescu-Romania

$$\left(\frac{h_a}{aw_a^2}\right)^2+\left(\frac{h_b}{bw_b^2}\right)^2+\left(\frac{h_c}{cw_c}\right)^2\ge 3\sqrt[3]{\frac{(h_ah_bh_c)^2}{a^2b^2c^2(w_aw_bw_c)^4}}\ \textit{(1)}$$

But $w_a\le\sqrt{s(s-a)}\Rightarrow w_a^4\le s^2(s-a)^2\Rightarrow\frac{1}{w_a^4}\ge\frac{1}{s^2(s-a)^2}$ *(2)*

From (1)+(2)$\Rightarrow\sum\left(\frac{h_a}{aw_a^2}\right)^2\ge 3\sqrt[3]{\frac{(h_ah_bh_c)^2}{a^2b^2c^2s^6(s-a)^2(s-b)^2(s-c)^2}}$ *(3)*

$$(h_ah_bh_c)^2=\frac{4s^4r^4}{R^2}\ \textit{(4)}$$

$(abc)^2=16s^2R^2r^2$ *(5) and* $\left((s-a)(s-b)(s-c)\right)^2=s^2r^4$ *(6)*

From (3)+(4)+(5)+(6)$\Rightarrow\sum\left(\frac{h_a}{aw_a^2}\right)^2\ge\frac{3}{\sqrt[3]{4R^4r^2s^6}}$ *(7)*

From (7) we must show this:

$$\frac{3}{\sqrt[3]{4R^4s^2s^6}}\ge\frac{1}{R^2(2R^2+r^2)}\Leftrightarrow\frac{27}{4R^4r^2s^6}\ge\frac{1}{R^6(2R^2+r^2)^3}\Leftrightarrow$$

$27R^2(2R^2+r^2)^3\ge 4r^2s^6$ *(8) But* $R\ge 2r\Rightarrow R^2\ge 4r^2$ *(9)*

Form (8)+(9) we must show this:

$27(2R^2+r^2)^3\ge s^6\Leftrightarrow 3(2R^2+r^2)\ge s^2$ *(10)*

But from Gerretsen we have: $s^2\le 4R^2+4Rr+3r^2\Rightarrow$

$s^2\le 4R^2+4Rr+3r^2\le 6R^2+3r^2\Leftrightarrow 4Rr\le 2R^2\Leftrightarrow 2r\le R$ *true.*

SOLUTION 4.176

Solution by Soumava Chakraborty-Kolkata-India

$$4\sqrt{3}\overset{(a)}{\leq}\frac{b^2+c^2}{ar_a}+\frac{c^2+a^2}{br_b}+\frac{a^2+b^2}{cr_c}\overset{(b)}{\leq}\frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3-8\sqrt{3}$$

$$\frac{b^2+c^2}{ar_a}+\frac{c^2+a^2}{br_b}+\frac{a^2+b^2}{cr_c}$$

$$=\left(\sum a^2\right)\left(\sum\frac{1}{ar_a}\right)-\sum\frac{a}{r_a}=\left(\sum a^2\right)\left(\sum\frac{s-a}{a\Delta}\right)-\sum\frac{a(s-a)}{\Delta}$$

$$=\frac{\sum a^2}{\Delta}\left(s\sum\frac{1}{a}-3\right)-\frac{s(2s)-2(s^2-4Rr-r^2)}{\Delta}$$

$$=\frac{\sum a^2}{\Delta}\left\{\frac{S(S^2+4Rr+r^2)}{4Rrs}-3\right\}-\frac{2(4Rr+r^2)}{\Delta}$$

$$=\frac{(s^2-4Rr-r^2)(s^2-8Rr+r^2)}{2Rr\Delta}-\frac{2(4Rr+r^2)}{\Delta}$$

$$=\frac{s^4-12Rrs^2+r^2(4R+r)(8R-r)-4R(4R+r)r^2}{2Rr\Delta}$$

$$\overset{(c)}{=}\frac{s^4-12Rrs^2+r^2(4R+r)(4R-r)}{2sRr^2}$$

$$\overset{Mitrinovic}{\leq}\frac{S^4-12Rrs^2+r^2(16R^2-r^2)}{2sr^2\frac{2S}{3\sqrt{3}}}\overset{?}{\leq}\frac{3\sqrt{3}}{2}\left(\frac{R}{r}\right)^3-8\sqrt{3}$$

$$\Leftrightarrow\frac{3\{S^4-2Rrs^2+r^2(16R^2-r^2)\}}{4S^2r^2}\overset{?}{\leq}\frac{3R^3}{2r^3}-8=\frac{3R^3-16r^3}{2r^3}$$

$$\Leftrightarrow 3r\{S^4-12Rrs^2+r^2(16R^2-r^2)\}\underset{(1)}{\overset{?}{\leq}}2S^2(3R^3-16r^3)$$

Now, LHS of (1) $\overset{Gerretsen}{\leq}3r\{S^2(4R^2-8Rr+3r^2)+r^2(16R^2-r^2)\}$

$$\overset{?}{\leq}2S^2(3R^3-16r^3)$$

$$\Leftrightarrow S^2\left(6R^3-12R^2r+24Rr^2-\frac{4}{r^3}\right)\underset{(2)}{\overset{?}{\geq}}3r^3(16R^2-r^2)$$

$$\because 6R^3-12R^2r+24Rr^2-\frac{4}{r^3}$$

$$=(R-2r)(6R^2+24r^2)+7r^3>0\ \left(\because R\overset{Euler}{\geq}2r\right)$$

$\therefore$ *LHS of (2)* $\overset{Gerretsen}{\geq}(16Rr-5r^2)\left(6R^3-12R^2r+24Rr^2-\frac{4}{r^3}\right)\overset{?}{\geq}3r^3(16R^2-r^2)$

$$\Leftrightarrow 48t^3-111t^3+198t^2-388t+104\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\{(t-2)(48t^2+81t+330)+608\} \overset{?}{\geq} 0$$

$$\rightarrow \textit{true} \because t \overset{Euler}{\geq} 2 \Rightarrow \textit{(b) is true}$$

Also, using (c) & $2s \overset{Mitrinovic}{\leq} 3\sqrt{3}R$

$$\sum \frac{b^2+c^2}{ar_a} \geq \frac{S^4-12Rrs^2+r^2(16R^2-r^2)}{3\sqrt{3}R^2r^2} \overset{?}{\geq} 4\sqrt{3}$$

$$\Leftrightarrow S^4-12Rrs^2+r^2(16R^2-r^2) \overset{?}{\geq} 36R^2r^2 \Leftrightarrow S^4-12Rrs^2 \underset{(3)}{\overset{?}{\geq}} r^2(20R^2+r^2)$$

Now, LHS of (3) $\overset{Gerretsen}{\geq} S^2(4Rr-5r^2) \overset{Gerretsen}{\geq} r^2(16R-5r)(4R-5r) \overset{?}{\geq} r^2(20R^2+r^2)$

$$\Leftrightarrow 11R^2-25Rr+6r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(11R-2r) \overset{?}{\geq} 0 \rightarrow \textit{true}$$

$\Rightarrow$ *(a) is true (Done).*

SOLUTION 4.177

Solution by Marian Ursărescu-Romania

$r_a = \frac{S}{s-a}, h_a = \frac{2S}{a} \Rightarrow$ *inequality becomes:*

$$2S^2\sum\frac{1}{a^2(s-a)} \leq \frac{3\cdot 2s}{4} \Leftrightarrow s^2r^2\sum\frac{1}{a^2(s-a)} \leq \frac{3s}{4} \quad (1)$$

But $\sum\frac{1}{a^2(s-a)} = \frac{s^4-2s^2(2Rr-r^2)+(4R+r)^3}{16R^2r^2s^3} \quad (2)$

From (1)+(2) we must show:

$$s^2r^2\frac{s^4-2s^2(2Rr-r^2)+r(4R+r)^3}{16R^2r^2s^3} \leq \frac{3s}{4} \Leftrightarrow$$

$$s^4-2s^2(2Rr-r^2)+r(4R+r)^3 \leq 12s^2R^2 \Leftrightarrow$$

$$s^2(12R^2-s^2+4Rr-2r^2) \geq r(4R+r)^3 \quad (3)$$

Now, from Doucet's inequality, we have: $s^2 \geq 3r(4R+r)$ *(4)*

From (3)+(4) we must show this:

$$3r(4R+r)(12R^2-s^2+4Rr-2r^2) \geq r(4R+r)^3 \Leftrightarrow$$

$$3(12R^2-s^2+4Rr-2r^2) \geq (4R+r)^2 \Leftrightarrow 36R^2-3s^2+12Rr-6r^2 \geq 16R^2+8Rr+ r^2 \Leftrightarrow 20R^2+4Rr \geq 3s^2+7r^2 \quad (5)$$

Now, form Doucet's inequality we have:

$$3s^2 \leq (4R+r)^2 \quad (6) \Leftrightarrow 3s^2 \leq 16R^2+8Rr+r^2 \Rightarrow$$

$$3s^2+7r^2 \leq 16R^2+8Rr+8r^2 \quad (7)$$

From (5)+(6) +(7) we must show this:

$$20R^2 + 4Rr \geq 16R^2 + 8Rr + 8r^2 \Leftrightarrow 4R^2 \geq 4Rr + 8r^2 \Leftrightarrow R^2 \geq r(R + 2r) \text{ (8)}$$

But from Euler's inequality we have $R \geq 2r \Rightarrow$

$$R^2 \geq 2Rr \text{ (9)}$$

From (8)+(9) we must show:

$$2R \geq r(R + 2r) \Leftrightarrow 2R \geq R + 2r \Leftrightarrow R \geq 2r \text{ (true)}$$

Observation: Relationship (2) it's from Viète and Newton relations from the equation with the roots a, b, c.

ANALYTICAL INEQUALITIES AND IDENTITIES-SOLUTIONS

SOLUTION 5.01

Solution by Dimitris Kastriotis-Athens-Greece

$$\Omega(x) = -1 + 4 \cdot \sum_{n=0}^{\infty} \frac{n+x}{(n+1)(n+2)(n+3)}, x \in \mathbb{R}$$

$$\Rightarrow \left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < \Omega(a)\Omega(b) + 1, 0 < a < 1, b > 1$$

$$\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{n+2} \cdot \frac{1}{(n+1)(n+3)} = \frac{1}{n+2}\left(\frac{1}{2(n+1)} - \frac{1}{2(n+3)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$$

$$S_1 = \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{2}\lim_{N\to\infty} \sum_{n=0}^{N}\left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+2)(n+3)}\right)$$

$$= \frac{1}{2}\lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{(N+2)(N+3)}\right) = \frac{1}{4}$$

$$\frac{n}{(n+1)(n+2)(n+3)} = \frac{1}{(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$S_2 = \sum_{n=0}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)} - S_1$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - S_1 = \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{N+3}\right) - \frac{1}{4} = \frac{1}{4}$$

$$\Omega(x) = -1 + 4 \cdot \sum_{n=0}^{\infty} \frac{n+x}{(n+1)(n+2)(n+3)} = -1 + 4(S_2 \cdot x + S_1) =$$

$$= -1 + 4\left(\frac{1}{4} + \frac{1}{4}x\right) = x$$

$$\left(\Omega(a)\right)^{\Omega(b)} + \left(\Omega(b)\right)^{\Omega(a)} < \Omega(a)\Omega(b) + 1$$

$$\Leftrightarrow a^b + b^a - ab - 1 < 0, 0 < a < 1, b > 1$$

$$\text{Let } f(b) = a^b + b^a - ab - 1, 0 < a < 1, b > 1$$

$$f'(b) = a^b\log(a) + ab^{a-1} - a = a^b\log(a) - a(1 - b^{a-1}) < 0\ \forall b > 1 \Rightarrow f \searrow (1, \infty)$$

$$\text{For } b > 1 \Leftrightarrow f(b) < f(1) = 0 \Leftrightarrow a^b + b^a < 1 + ab, 0 < a < 1, b > 1$$

SOLUTION 5.02

Solution by Ravi Prakash-New Delhi-India

$$\text{Let } a_n = \frac{n^{10}}{(10^n)(n!)}, \text{ then } \frac{a_n}{a_{n+1}} = \frac{n^{10}}{(10^n)(n!)} \cdot \frac{10^{n+1}(n+1)!}{(n+1)^{10}} = \left(1 - \frac{1}{n+1}\right)^{10}(10)(n+1)$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty \therefore \sum a_n \text{ converges. Let } S = \sum_{n=1}^{\infty} a_n > 0$$

Now,

$$\lim_{k\to\infty}\left(1 + \frac{1}{k^2}\left(\sum_{n=1}^{\infty} a_n\right)\right)^{k^4} = \lim_{k\to\infty}\left(1 + \frac{S}{k^2}\right)^{k^4} =$$

$$\lim_{k\to\infty}\left[\left(1 + \frac{S}{k^2}\right)^{k^2}\right]^{k^2} = (e^S)^{\infty} = \infty$$

SOLUTION 5.03

Solution by Igor Soposki-Skopje-Macedonia

$$I = \int \frac{x^2+1}{x^4+x^2+1}dx = \int \frac{\frac{x^2+1}{x^2}}{\frac{x^4+x^2+1}{x^2}}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + \left(\sqrt{3}\right)^2}dx =$$

$$= \left\{\begin{matrix} x - \frac{1}{x} = t \\ \left(1 + \frac{1}{x^2}\right)dx = dt \end{matrix}\right\} = \int \frac{dt}{t^2 + \left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} = \frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}$$

$$I = \int_0^{\frac{1}{n^5}} \frac{x^2+1}{x^4+x^2+1}dx = \frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{\sqrt{3}x}\Bigg|_0^{\frac{1}{n^5}} = \frac{1}{\sqrt{3}}\arctan\frac{n^5\sqrt{3}}{n^{10}-1}$$

$$L = \lim_{n\to\infty} n^8 \cdot \frac{1}{\sqrt{3}} \cdot \arctan\frac{n^5\sqrt{3}}{n^{10}-1} = \frac{1}{\sqrt{3}}\lim_{n\to\infty}\frac{\arctan\frac{n^5\sqrt{3}}{n^{10}-1}}{\frac{1}{(n)^8}} = \frac{0}{0}$$

$$\lim_{n\to\infty}\frac{\frac{1}{1+\left(\frac{n^5\sqrt{3}}{n^{10}-1}\right)^2}\cdot\frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^5n^9}{(n^{10}-1)^2}}{-\frac{8n^7}{n^{16}}}=$$

$$=\lim_{n\to\infty}\frac{\frac{5\sqrt{3}n^4(n^{10}-1)-10\sqrt{3}n^{14}}{(n^{10}-1)^2+3n^{10}}}{-\frac{8}{n^9}}=-\lim_{n\to\infty}\frac{n^9\left[5\sqrt{3}n^{14}-5\sqrt{3}n^4-10\sqrt{3}n^{14}\right]}{8[n^{20}-2n^{10}+1+3n^{10}]}=$$

$$=\lim_{n\to\infty}\frac{n^9 5\sqrt{3}[n^{14}+n^4]}{8[n^{20}+n^{10}+1]}=\frac{5\sqrt{3}}{8}\lim_{n\to\infty}\frac{n^9n^{14}\left[1+\frac{1}{n^{10}}\right]}{n^{20}\left[1+\frac{1}{n^{10}}+\frac{1}{n^{20}}\right]}=\frac{5\sqrt{3}}{8}\lim_{n\to\infty}n^3=+\infty$$

$$L=\frac{1}{\sqrt{3}}\cdot(+\infty)=+\infty$$

SOLUTION 5.04

Solution by Srinivasa Raghava-AIRMC-India

$$\Omega=\sum_{m=1}^{\infty}\frac{(2m-1)!!}{(2m+2)!!}$$

We know that

$$(2m-1)!!=\frac{\Gamma\left(m+\frac{1}{2}\right)2^n}{\sqrt{\pi}}=\frac{2}{2m+1}\left(\frac{2m+1}{2}\right)!$$

$$(2m+2)!!=2^m(2m+2)m!$$

then the above sum becomes $\omega=\sum_{m=1}^{\infty}\frac{1}{2^{m+1}}=\frac{1}{2}$

$$\lim_{n\to\infty}(\pi+n\omega)^{1+\frac{1}{n\omega}}-(n\omega)^{1+\frac{1}{\pi+n\omega}}=\lim_{n\to\infty}\left(\pi+\frac{n}{2}\right)^{1+\frac{2}{n}}-\left(\frac{n}{2}\right)^{1+\frac{1}{\pi+\frac{n}{2}}}$$

(substituting $\omega=\frac{1}{2}$***)***

$$=\lim_{n\to\infty}\pi+\frac{2\pi-4\pi\ln(2)-4\pi\ln\left(\frac{1}{2}\right)}{n}+O\left(\frac{1}{n^2}\right)$$ ***(series expansion around*** $n=\infty$***)***

hence the answer is π***.***

SOLUTION 5.05

Solution by Feti Sinani-Kosovo

When $x\to 0^+$ ***we have***

$$x^x = e^{x\ln x} = e^{\sqrt{x}+o(\sqrt{x})} = 1+\sqrt{x}+o(\sqrt{x})$$

$$x^{x^x} = x^{1+\sqrt{x}+o(\sqrt{x})} = xe^{\left(\sqrt{x}+o(\sqrt{x})\right)\ln(x)} = x\left(1+\sqrt[3]{x}+o(\sqrt[3]{x})\right) = x+o(x)$$

$$x^{x^{x^x}} = x^{x+o(x)} = e^{(x+o(x))\ln x} = e^{\sqrt{x}+o(\sqrt{x})} = 1+\sqrt{x}+o(\sqrt{x})$$

$$\left.x^{x^{x^{\dots^{x^x}}}}\right\} n \ \ times = \begin{cases} 1+\sqrt{x}+o(\sqrt{x}) & n-even \\ x+o(x) & n-odd \end{cases}$$

$$(\sin x)^{(\sin x)^{\dots(\sin x)^{(\sin x)}}} = \left(x+o(x)\right)^{(x+o(x))^{\dots(x+o(x))^{(x+o(x))}}} \qquad x\to 0^+$$

$$\left(x+o(x)\right)^{(x+o(x))} = e^{(x+o(x))\ln(x+o(x))} = e^{(x+o(x))\ln(x)+o(x)} = 1+\sqrt{x}+o(\sqrt{x})$$

$$\left(x+o(x)\right)^{(x+o(x))^{(x+o(x))}} = \left(x+o(x)\right)^{1+\sqrt{x}+o(\sqrt{x})} = x^{1+\sqrt{x}+o(\sqrt{x})}e^{\left(1+\sqrt{x}+o(\sqrt{x})\right)\ln\left(1+o(1)\right)}$$

$$= \left(x+o(x)\right)\left(1+o(1)\right) = x+o(x)$$

$$\left.(\sin x)^{(\sin x)^{\dots(\sin x)^{(\sin x)}}}\right\} n \ \ times = \begin{cases} 1+\sqrt{x}+o(\sqrt{x}) & n-even \\ x+o(x) & n-odd \end{cases}$$

$$\lim_{n\to\infty}\left(\lim_{x\to 0^+}\frac{(\sin x)^{(\sin x)^{\dots(\sin x)^{(\sin x)}}}}{x^{x^{x\dots x^x}}}\right) = \lim_{n\to\infty}\left(\lim_{x\to 0^+}\left(1+o(1)\right)\right) = 1$$

SOLUTION 5.06

Solution by Remus Florin Stanca-Romania

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^4} = \frac{\pi^4}{90} \Rightarrow \Omega_1 \overset{1^\infty}{=} \lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}}\cdot\left(\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)n} =$$

$$= e^{\lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k^4}-\frac{\pi^4}{90}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^4}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty} -\frac{1}{(n+1)^4}\cdot(n^2+n)} = e^0 = 1$$

$$\succ \Omega_1 = 1$$

$$\Omega_2 = \lim_{n\to\infty}\left(4-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^n$$

$$\sum_{k=1}^{n}\frac{1}{(k^2+k)^2} = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2 = \sum_{k=1}^{n}\frac{1}{k^2}+\sum_{k=1}^{n}\frac{1}{(k+1)^2}-2\sum_{k=1}^{n}\frac{1}{k(k+1)} =$$

$$\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{(k^2+k)^2} = \frac{\pi^2}{6}+\frac{\pi^2}{6}-1-2 = \frac{\pi^2}{3}-3 \Rightarrow$$

$$\succ \Omega_2 \overset{1^\infty}{=} \lim_{n\to\infty}\left(1+3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)^{\frac{1}{\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)}\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)n} =$$

$$= e^{\lim_{n\to\infty} n\left(3-\frac{\pi^2}{3}+\sum_{k=1}^{n}\frac{1}{(k^2+k)^2}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n^2+3n+2)^2}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim_{n\to\infty} -\frac{1}{(n^2+3n+2)^2}\cdot(n^2+n)} = e^0 = 1$$

$$\Rightarrow \Omega_2 = 1$$

$$\Omega_3 = \lim_{n\to\infty}\left(5-4\ln 2-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)^n$$

$$\frac{1}{k^2(2k+1)} = \frac{Ak+B}{k^2}+\frac{C}{2k+1} = \frac{2Ak^2+Ak+2Bk+B+Ck^2}{k^2(2k+1)}\ \forall k$$

$$\Leftrightarrow 2A=-C, A=-2B, B=1 \Rightarrow A=-2, C=4$$

$$\Leftrightarrow \frac{1}{k^2(2k+1)} = \frac{-2k+1}{k^2}+\frac{4}{2k+1} = -\frac{2}{k}+\frac{1}{k^2}+\frac{4}{2k+1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k^2(2k+1)} = \frac{\pi^2}{6}-4\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2k(2k+1)} = l$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k}-\frac{1}{2k+1}\right) = -\lim_{n\to\infty}\left(-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}\right) =$$

$$= -\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}-2\left(\frac{1}{2}+\cdots+\frac{1}{2n}\right)\right) =$$

$$= -\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n+1}-1-\cdots-\frac{1}{n}\right) = -\lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2n+1}-1\right) =$$

$$= -\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{2n+1}-\ln(2n+1)+\ln(2n+1)-\left(1+\cdots+\frac{1}{n}-\ln(n)+\ln(n)\right)-1\right) =$$

$$= -4\ln(2)+4 \Rightarrow l = \frac{\pi^2}{6}+4\ln(2)-4$$

$$\Rightarrow \Omega_3 \overset{1^\infty}{=} \lim_{n\to\infty} e^{n\left(4-4\ln(2)-\frac{\pi^2}{6}+\sum_{k=1}^{n}\frac{1}{k^2(2k+1)}\right)} \overset{Stolz\ Cesaro}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2(2n+3)}}{-\frac{1}{n(n+1)}}} = e^0 = 1$$

$$\Rightarrow \Omega_3 = 1$$

$$\Omega_4 = \lim_{n\to\infty}\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{1}{(2n-1)^2}\right)=\lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{(2n)^2}-\frac{1}{2^2}-\cdots-\frac{1}{4n^2}\right)=$$

$$=\frac{\pi^2}{6}-\lim_{n\to\infty}\frac{1}{4}\left(\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)=\frac{\pi^2}{8}$$

$$\Rightarrow \Omega_4 \stackrel{1^\infty}{=} \lim_{n\to\infty} e^{n\left(1-\frac{\pi^2}{8}+\sum_{k=1}^{n}\frac{1}{(2k-1)^2}\right)} \stackrel{Stolz\ Cesaro}{=} e^{\lim\limits_{n\to\infty}\frac{\frac{1}{(2n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}} = e^{\lim\limits_{n\to\infty}-\frac{n^2+n}{4n^2+4n+1}} = e^{\frac{-1}{4}}$$

$$\succ \Omega_4=\frac{1}{\sqrt[4]{e}}$$

SOLUTION 5.07

Solution by Remus Florin Stanca-Romania

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n]{1+\sin x}+\sqrt[n]{1-\sin x}-2\right)\cdot\lim_{n\to\infty}\frac{1}{n\left(\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2\right)}$$

$$\lim_{n\to\infty} n\left(\sqrt[n+1]{1+\sin x}+\sqrt[n+1]{1-\sin x}-2\right)=$$

$$=\lim_{n\to\infty} n\left(\sqrt[n+1]{1+\sin x}-1\right)+\lim_{n\to\infty} n\left(\sqrt[n+1]{1-\sin x}-1\right)=$$

$$=\lim_{n\to\infty} n\frac{e^{\frac{\ln(1+\sin x)}{n+1}}-1}{\frac{\ln(1+\sin x)}{n+1}}\cdot\frac{\ln(1+\sin x)}{n+1}+\lim_{n\to\infty} n\frac{e^{\frac{\ln(1-\sin x)}{n+1}}}{\frac{\ln(1-\sin x)}{n+1}}\cdot\frac{\ln(1-\sin x)}{n+1}=$$

$$=\ln(1+\sin x)+\ln(1-\sin x)=\ln\left(\cos^2 x\right)=2\ln|\cos x|\succ\Omega=0\cdot\frac{1}{2\ln|\cos x|}=0\succ\Omega=0$$

SOLUTION 5.08

Solution by Remus Florin Stanca-Romania

$$Let\ \left[\sqrt{n}\right]=k\Rightarrow k\le\sqrt{n}<k+1\Rightarrow k^2\le n<(k+1)^2\Rightarrow k^2+1\le n+1<(k+1)^2+1$$

$$\succ k^2+2\le n+2<(k+1)^2+2$$

$$k^2\le n<(k+1)^2$$

$$k^2+1\le n+1<(k+1)^2+1$$

$$k^2+2\le n+2<(k+1)^2+2$$

$$-----------------\text{“}\cdot\text{”}$$

$$\succ k^2(k^2+1)(k^2+2)\le n(n+1)(n+2)<(k+1)^2((k+1)^2+1)((k+1)^2+2)$$

$$\Rightarrow\sqrt[6]{k^2(k^2+1)(k^2+2)}\le\sqrt[6]{n(n+1)(n+2)}<\sqrt[6]{(k+1)^2((k+1)^2+1)((k+1)^2+2)}$$

$$\sqrt[6]{k^2(k^2+1)(k^2+2)} = \sqrt[6]{k^6\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} = k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)}$$

We prove that $k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} < k+1\ \forall k \in \mathbb{N}$

$$\Leftrightarrow \left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right) < \left(\frac{k+1}{k}\right)^6$$

$$1+\frac{1}{k^2} < \left(\frac{k+1}{k}\right)^3 (?) \Leftrightarrow \frac{k^2+1}{k^2} < \frac{(k+1)^3}{k^3} \Leftrightarrow k^3+k < k^3+3k^2+3k+1 \text{ (true)} \succ$$

$$\Rightarrow \left(1+\frac{1}{k^2}\right) < \left(\frac{k+1}{k}\right)^3 \text{ (proved)}$$

$$1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3 (?) \Leftrightarrow \frac{k^2+2}{k^2} < \frac{(k+1)^3}{k^3} \Leftrightarrow k^3+2k < k^3+3k^2+3k+1 \text{ (true)} \Rightarrow$$

$$\Rightarrow 1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3 \text{ (proved), so:}$$

$$1+\frac{1}{k^2} < \left(\frac{k+1}{k}\right)^3$$

$$1+\frac{2}{k^2} < \left(\frac{k+1}{k}\right)^3$$

$$- - - - - - - - - - - \text{ "."}$$

$$\Rightarrow \left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right) < \left(\frac{k+1}{k}\right)^6 \Leftrightarrow k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} < k+1 \text{ (proved)}$$

$$k \le k\sqrt[6]{\left(1+\frac{1}{k^2}\right)\left(1+\frac{2}{k^2}\right)} < k+1 \Rightarrow \left[\sqrt[6]{k^2(k^2+1)(k^2+2)}\right] = k \forall k \in \mathbb{N}$$

$$\Rightarrow \left[\sqrt[6]{(k+1)^2((k+1)^2+1)((k+1)^2+2)}\right] = k+1 \Rightarrow$$

$$\left[\sqrt[6]{n(n+1)(n+2)}\right] \in \{k; k+1\}$$

$$\textit{If } \left[\sqrt[6]{n(n+1)(n+2)}\right] = k \Rightarrow \left[\sqrt[6]{n(n+1)(n+2)}\right] = [\sqrt{n}]$$

$$\Rightarrow \Omega = \lim_{n\to\infty} \frac{[\sqrt{n}]}{n[\sqrt{n}]} = \frac{1}{\infty} = 0 \quad (1)$$

$$\textit{if } \left[\sqrt[6]{n(n+1)(n+2)}\right] = [\sqrt{n}]+1 \Rightarrow \Omega = \lim_{n\to\infty} \frac{[\sqrt{n}]+1}{n[\sqrt{n}]} = \frac{1}{\infty}+\frac{1}{\infty} = 0+0 = 0 \quad (2)$$

from (1) and (2) we obtain that $\Omega = 0$.

SOLUTION 5.09

Solution by Avishek Mitra-India

$$f(x)=\frac{1}{(x^2+2x+3)}$$

$$\Rightarrow [f(x)\cdot f(x^2+2x+3)]_{n+1}=0$$

$$\Rightarrow f^{(n+1)}(x)\cdot(x^2+2x+3)+(n+1)\cdot f^n(x)(2x+2)+{}^{n+1}C_2\cdot f^{(n-1)}(x)\cdot 2=0$$

$$\Rightarrow 3f^{(n+1)}(0)+2(n+1)\cdot f^n(0)+2\cdot {}^{n+1}C_2 f^{(n-1)}(0)=0$$

$$\Rightarrow 2(n+1)f^n(0)=-2\cdot {}^{n+1}C_2 f^{(n-1)}(0)-3f^{(n+1)}(0)$$

$$\Rightarrow \lim_{n\to\infty}\frac{f^n(0)}{n!}=-\lim_{n\to\infty}\frac{{}^{n+1}C_2\cdot f^{(n-1)}(0)}{n!}-\frac{3}{2}\cdot\lim_{n\to\infty}\frac{f^{(n+1)}(0)}{n!}$$

$$=-\lim_{n\to\infty}\frac{(n+1)!\,f^{(n-1)}(0)}{2!\,(n-1)!\,(n)!}-0=-\lim_{n\to\infty}\frac{(n+1)f^{(n-1)}(0)}{2(n-1)(n-2)!}-0$$

$$=-\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)f^{(n-1)}(0)}{2\left(1-\frac{1}{n}\right)(n-2)!}-0=0$$

SOLUTION 5.10

Solution by Remus Florin Stanca-Romania

$$x_{n+1}=a^{-(x_1+\cdots x_{n-1})}\cdot a^{-x_n}\succ x_{n+1}=x_n\cdot a^{-x_n}$$

$$y_{n+1}=a^{\frac{1}{y_1}+\cdots+\frac{1}{y_{n-1}}}\cdot a^{\frac{1}{y_n}}\succ y_{n+1}=y_n\cdot a^{\frac{1}{y_n}}$$

we prove by using the Mathematical induction that $x_n>0$***:***

1) we prove that $P(1)$: ***"***$x_1>0$***" is true (true).***

2) we suppose that $P(n)$: ***"***$x_n>0$***" is true.***

3) we prove that $P(n+1)$: ***"***$x_{n+1}>0$***" is true by using*** $P(n)$***:***

$x_{n+1}=x_n\cdot a^{-x_n}, a>1, x_n>0\succ x_{n+1}>0\succ x_n>0\forall n\in\mathbb{N}$ ***(proved)***

$x_{n+1}=x_n\cdot a^{-x_n}\Rightarrow\frac{x_{n+1}}{x_n}=a^{-x_n}$ ***(1). Also, we know that*** $a>1$ ***and*** $x_n>0$***.***

$\overset{(1)}{\Rightarrow}\frac{x_{n+1}}{x_n}<1\Rightarrow x_{n+1}<x_n\Rightarrow(x_n)_{n\in\mathbb{N}}$ ***is a decreasing sequence.***

$x_n>0$ ***and*** $(x_n)_{n\in\mathbb{N}}$ ***is decreasing sequence*** $\succ|l\in\mathbb{R}$ ***such that***

$$\lim_{n\to\infty}(x_n)=l$$

$$x_{n+1}=x_n\cdot a^{-x_n}\succ l=l\cdot a^{-l}\succ l\left(1-a^{-l}\right)=0\succ l=0$$

$$\succ\lim_{n\to\infty}(x_n)=0$$

we prove by using the Mathematical induction that $y_n>0$***:***

1) we prove that $P(1)$: "$y_1 > 0$" is true (true)

2) we suppose that $P(n)$: "$x_n > 0$" is true

3) we prove that $P(n+1)$: "$x_{n+1} > 0$" is true by using $P(n)$:

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}}, y_n > 0 \succ y_{n+1} > 0 \succ y_n > 0 \forall n \in \mathbb{N} \text{ (proved)}$$

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}} \succ \frac{y_{n+1}}{y_n} = a^{\frac{1}{y_n}}, \text{ we know also that } a > 1 \text{ and } y_n > 0 \succ \frac{y_{n+1}}{y_n} > 1$$

$$\Rightarrow y_{n+1} > y_n \Rightarrow (y_n)_{n\in\mathbb{N}} \text{ is an increasing sequence (2)}$$

$$\text{we suppose that } y_n \text{ is verged} \overset{(2)}{\Rightarrow} \exists l \in \mathbb{R} \text{ such that } \lim_{n\to\infty}(y_n) = l$$

$$y_{n+1} = y_n \cdot a^{\frac{1}{y_n}} \Rightarrow l = l \cdot a^{\frac{1}{l}} \Rightarrow l\left(1 - a^{\frac{1}{l}}\right) = 0 \Rightarrow l = 0, \text{ but } y_n > 0 \text{ and}$$

$$\text{increasing} \Rightarrow \text{contradiction} \Rightarrow \lim_{n\to\infty} y_n = \infty$$

$$\Omega = \lim_{n\to\infty}(nx_n) \cdot \lim_{n\to\infty}\left(\frac{y_n}{n}\right) (a)$$

$$\lim_{n\to\infty}\frac{y_n}{n} \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty}(y_{n+1} - y_n) = \lim_{n\to\infty}\left(y_n\left(a^{\frac{1}{y_n}} - 1\right)\right) =$$

$$= \lim_{n\to\infty}\frac{a^{\frac{1}{y_n}} - 1}{\frac{1}{y_n}} = \ln(a)(3)$$

$$\lim_{n\to\infty}(nx_n) = \lim_{n\to\infty}\frac{n}{\frac{1}{x_n}} \overset{\text{Stolz Cesaro}}{=} \lim_{n\to\infty}\frac{1}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty}\frac{1}{\frac{a^{x_n}}{x_n} - \frac{1}{x_n}}$$

$$= \lim_{n\to\infty}\frac{1}{\frac{a^{x_n}-1}{x_n}} = \frac{1}{\ln(a)} \text{ (4)}$$

$$\overset{(a);(3);(4)}{\Rightarrow} \lim_{n\to\infty}(x_n y_n) = \ln(a) \cdot \frac{1}{\ln(a)} = 1 \Rightarrow \Omega = 1.$$

SOLUTION 5.11

Solution by Pierre Mounir-Cairo-Egypt

$$\Omega = \lim_{n\to\infty}\left\{\left[\left(\frac{1+H_n}{H_n}\right)^{H_n} - \ln\left(\frac{1+H_n}{H_n}\right)^{eH_n}\right]\right\}$$

$$\because (H_n)_{n\geq 1} \text{ is a divergent sequence (partial sums).}$$

$$\therefore x = \frac{1}{H_n} \to 0^+ \text{ as } n \to \infty \Rightarrow$$

$$\Omega = \lim_{x\to 0^+} \frac{(1+x)^{\frac{1}{x}} - e\ln(1+x)^{\frac{1}{x}}}{x^2} = e \lim_{x\to 0^+} \frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1 - \ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]}{x^2}$$

$$\textit{Let } y = \frac{(1+x)^{\frac{1}{x}}}{e} - 1 \to 0^+ \textit{ as } x \to 0^+ \Rightarrow$$

$$\Omega = e \lim_{y\to 0^+} \frac{y - \ln(y+1)}{y^2} \times \lim_{x\to 0^+} \frac{\left[\frac{(1+x)^{\frac{1}{x}}}{e} - 1\right]^2}{x^2}$$

$$= e \lim_{y\to 0^+} \frac{y - \ln(y+1)}{y^2} \times \left[\lim_{x\to 0^+} \frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{x}\right]^2$$

$$\textit{Let } M = \lim_{x\to 0^+} \frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{x}$$

$$= \lim_{x\to 0^+} \frac{\frac{(1+x)^{\frac{1}{x}}}{e} - 1}{\ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]} \times \frac{\ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]}{x}$$

$$\textit{Let } z = \frac{(1+x)^{\frac{1}{x}}}{e} \to 1^- \textit{ as } x \to 0^+ \Rightarrow M = \lim_{z\to 1^-} \frac{z-1}{\ln z} \times \lim_{x\to 0^+} \frac{\frac{1}{x}\ln(1+x) - 1}{x}$$

$$M = 1 \times \lim_{x\to 0^+} \frac{\ln(1+x) - x}{x^2} = -L$$

$$\therefore \Omega = e \times L \times (-L)^2 = eL^3$$

$$L = \lim_{x\to 0^+} \frac{x - \ln(1+x)}{x^2}$$

$$\textit{We have: } 1 - y^2 < 1 < 1 + y^3 \quad \forall y > 0 \Rightarrow$$

$$1 - y < \frac{1}{1+y} < 1 - y + y^2 \Rightarrow$$

$$\int_0^x (1-y)\,dy < \int_0^x \frac{1}{1+y}\,dy < \int_0^x (1 - y + y^2)\,dy \Rightarrow$$

$$x - \frac{x^2}{2} < \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \quad \forall x > 0 \Rightarrow$$

$$\frac{1}{2}-\frac{x}{3}<\frac{x-\ln(1+x)}{x^2}<\frac{1}{2}\Rightarrow\frac{1}{2}<\lim_{x\to0^+}\frac{x-\ln(1+x)}{x^2}<\frac{1}{2}\Rightarrow$$

By squeeze theorem: $L=\frac{1}{2}\Rightarrow\Omega=eL^3=\frac{e}{8}$

SOLUTION 5.12

Solution by Yubian Andres Bedoya Henao-Medellin-Colombia

Let $f(n)=\sum_{k=1}^{n}\frac{1}{k^2}$ *and* $L=\frac{\pi^2}{6}\Rightarrow\lim_{n\to\infty}f(n)=L$

$$\Rightarrow\Omega=\lim_{n\to\infty}n\left[f(n)^L-L^{f(n)}\right]=?\Rightarrow\Omega=\lim_{n\to\infty}\frac{f(n)^L-L^{f(n)}}{\frac{1}{n}}\to\frac{0}{0}\quad\textit{(LH)}$$

$$=\lim_{n\to\infty}\frac{Lf(n)^{L-1}f'(n)-L^{f(n)}\ln(L)f'(n)}{-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}n^2f'(n)\left[L^{f(n)}\ln(L)-Lf(n)^{L-1}\right]=L^L(\ln(L)-1)\lim_{n\to\infty}n^2f'(n)$$

$$\therefore f'(n)=\frac{d}{dn}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{d}{dn}\sum_{k=1}^{n}\int_0^1\int_0^1(xy)^{k-1}\,dx\,dy=\frac{d}{dn}\int_0^1\int_0^1\frac{1-(xy)^n}{1-xy}\,dx\,dy$$

$$=-\int_0^1\int_0^1\frac{(xy)^n\ln(xy)}{1-xy}\,dx\,dy\quad(z=x^n\ \ w=y^n)$$

$$=-\int_0^1\int_0^1\frac{zw\ln(zw)^{\frac{1}{n}}}{1-(zw)^{\frac{1}{n}}}\,\frac{(zw)^{\frac{1}{n}}dz\,dw}{n^2zw}=\frac{1}{n^3}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw$$

$$\therefore\lim_{n\to\infty}n^2f'(n)=\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw$$

$$=\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\ln(zw)-\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw=-\lim_{n\to\infty}\frac{1}{n}\int_0^1\int_0^1\left(\frac{\ln(zw)}{1-(zw)^{\frac{1}{n}}}\right)dz\,dw\quad\textit{(L'H)}$$

$$=\lim_{n\to\infty}\int_0^1\int_0^1\left(\frac{\left(\frac{1}{n^2}\right)\ln(zw)}{\left(\frac{1}{n^2}\right)(zw)^{\frac{1}{n}}\ln(zw)}\right)dz\,dw=\lim_{n\to\infty}\int_0^1\int_0^1\left(\frac{1}{(zw)^{\frac{1}{n}}}\right)dz\,dw=\int_0^1\int_0^1dz\,dw=1$$

$$\Rightarrow\Omega=L^L(\ln(L)-1)=\left(\frac{\pi^2}{6}\right)^{\frac{\pi^2}{6}}\left[\ln\left(\frac{\pi^2}{6}\right)-1\right]$$

SOLUTION 5.13

Solution by Zaharia Burghelea-Romania

$$\Phi = \sum_{n=0}^{\infty} \frac{(2n)!!}{(2n)!}\left[\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1} dx\right]$$

Since $\sum_{k=0}^{\infty}\binom{n+1}{k}x^{-2k} = (1+x^{-2})^{n+1}$

We have that:

$$\left(\sum_{k=1}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1} = \left(\frac{x^2}{x^2+1}\right)^{n+1}\frac{1}{x^{2(n+1)}} = \frac{1}{(x^2+1)^{n+1}}$$

$$I_n = \int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\binom{n+1}{k}\frac{x^{2(n+1)}}{x^{2k}}\right)^{-1} dx = \int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{n+1}}$$

Substituting $x^2 = t$ ***we get:*** $I_n = \int_{-\infty}^{\infty}\frac{t^{-\frac{1}{2}}}{(t+1)^{n+1}}dt =$

$$= B\left(\frac{1}{2}, n+\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{n!}\cdot\frac{(2n)!\sqrt{\pi}}{4^n n!} = \pi\frac{(2n)!}{4^n(n!)^2}$$

Where $B(x,y)$ ***and*** $\Gamma(x)$ ***are Euler's beta and gamma function.***

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

Also, since $(2n)!! = [2n][2(n-1)][2(n-2)]\cdots 2 = 2^n n!$.

$$\Phi = \pi\sum_{n=0}^{\infty}\frac{2^n n!}{(2n)!}\cdot\frac{(2n)!}{4^n(n!)^2} = \pi\sum_{n=0}^{\infty}\frac{1}{2^n n!} = \pi\sqrt{e}$$

SOLUTION 5.14

Solution by Pierre Mounir-Cairo-Egypt

Given: $\omega(n) = \sum_{i=1}^{n}\left[\frac{i^2+i+1}{i^2-i+1}\right]$

Find: $\Omega = \lim_{n\to\infty}\left\{\ln(3n+1) - \sum_{k=1}^{n}\frac{1}{\omega(k)}\right\}$

$$\omega(n) = \sum_{i=1}^{n}\left[1+\frac{2i}{i^2-i+1}\right] = n + \sum_{i=1}^{n}\left[\frac{2i}{i^2-i+1}\right]$$

We have: $0 < \frac{2i}{i^2-i+1} < 1\ \forall i \geq 3$

$$\therefore \left[\frac{2i}{i^2-i+1}\right] = 0\ \forall i \geq 3$$

$$\omega(n) = n + [2] + \left[\frac{4}{3}\right] + \sum_{i=3}^{n}\left[\frac{2i}{i^2 - i + 1}\right] = n + 3$$

$$\Omega = \lim_{n\to\infty}\left\{ln(3n+1) - \sum_{k=1}^{n}\frac{1}{k+3}\right\} = \lim_{n\to\infty}\left\{ln(3n+1) - ln\, n + \left(ln\, n - \sum_{k=1}^{n}\frac{1}{k}\right) + 1 + \frac{1}{2} + \frac{1}{3}\right\}$$

$$= \lim_{n\to\infty} ln\left(3 + \frac{1}{n}\right) + \lim_{n\to\infty}\left(ln\, n - \sum_{k=1}^{n}\frac{1}{k}\right) + \frac{11}{6} = \ln 3 - \gamma + \frac{11}{6}$$

SOLUTION 5.15

Solution by Pierre Mounir-Cairo-Egypt

By Jensen's inequality, we have:

$$\frac{\sqrt{k} + \sqrt{k+2}}{2} < \sqrt{\frac{k + (k+2)}{2}} \Rightarrow \sqrt{k} + \sqrt{k+2} < 2\sqrt{k+1} \Rightarrow$$

(We could've proved it using $\left(\sqrt{k+2} - \sqrt{k}\right)^2 > 0$***)***

$$\sqrt{k} + \sqrt{k+1} + \sqrt{k+2} < 3\sqrt{k+1} \Rightarrow$$

$$\sqrt{k} = \frac{\sqrt{k} + \sqrt{k} + \sqrt{k}}{3} < \frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3} < \sqrt{k+1} < \sqrt{k} + 1$$

$$\because x < y \Rightarrow [x] < [y]$$

$$\therefore \left[\sqrt{k}\right] < \left[\frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3}\right] < \left[\sqrt{k} + 1\right] = \left[\sqrt{k}\right] + 1$$

$$\because \left[\frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3}\right] \in \mathbb{Z} \Rightarrow \left[\frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3}\right] = \left[\sqrt{k}\right]$$

$$\therefore \Omega = \lim_{n\to\infty}\left\{ln(2n+1) - \sum_{k=1}^{n}\frac{1}{k\left[\sqrt{k}\right]}\left[\frac{\sqrt{k} + \sqrt{k+1} + \sqrt{k+2}}{3}\right]\right\}$$

$$= \lim_{n\to\infty}\left\{ln(2n+1) - ln\, n + ln\, n - \sum_{k=1}^{n}\frac{1}{k}\right\} = \lim_{n\to\infty} ln\left(2 + \frac{1}{n}\right) + \lim_{n\to\infty}\left\{ln\, n - \sum_{k=1}^{n}\frac{1}{k}\right\}$$

$$= ln\, 2 - \gamma$$

SOLUTION 5.16

Solution by Remus Florin Stanca-Romania

$$x_{n+1} = \frac{x_n^2 + \frac{1}{3}x_n}{\left(\sqrt[3]{x_n^2 + \frac{1}{3}x_n + \frac{1}{27}}\right)^2 + \sqrt[3]{x_n^2 + \frac{1}{3}x_n + \frac{1}{27}}\cdot\frac{1}{3} + \frac{1}{9}} \quad \textit{(1)}$$

We prove by using the Mathematical induction that $x_n > 0$*:*

1) we prove that $P(0)$*: "*$x_0 > 0$*" is true (true)*

2) we suppose that $P(n)$*: "*$x_n > 0$*" is true*

3) we prove that $P(n+1)$*: "*$x_{n+1} > 0$*" is true by using* $P(n)$*:*

From the fact that $P(n)$ *is true and from (1) we obtain that* $x_{n+1} > 0 \succ$

$\succ P(n+1)$ *is true (proved)* $\succ x_n > 0; \forall n \in \mathbb{N}$

$$x_{n+1} - x_n = \sqrt[3]{x_n^2 + \frac{1}{3}x_n + \frac{1}{27}} - \frac{1}{3} - x_n = \frac{1}{3} \cdot \frac{\sqrt[3]{27x_n^2 + 9x_n + 1} - 1 - 3x_n}{1}$$

Let $x > 0 \succ 27x^3 > 0 \succ 27x^3 + 1 + 27x^2 + 9x > 27x^2 + 9x + 1 \succ$

$\Rightarrow 3x + 1 > \sqrt[3]{27x^2 + 9x + 1} \Rightarrow x_{n+1} - x_n < 0 \Rightarrow (x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence*

$x_n > 0$ *and* x_n *decreasing* $\Rightarrow \exists l \in \mathbb{R}$ *such that* $\lim_{n\to\infty}(x_n) = l$

$$\succ l = \sqrt[3]{l^2 + \frac{1}{3}l + \frac{1}{27}} - \frac{1}{3} \succ 3l = \sqrt[3]{27l^3 + 9l + 1} - 1 \succ 27l^3 + 1 + 27l^2 + 9l =$$

$$= 27l^3 + 9l + 1 \succ l = 0 \succ \lim_{n\to\infty}(x_n) = 0$$

$$\lim_{n\to\infty}(nx_n) = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} \overset{Stolz\ Cesaro}{=} \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt[3]{x_n^2 + \frac{x_n}{3} + \frac{1}{27}} - \frac{1}{3}} - \frac{1}{x_n}}$$

$$\lim_{n\to\infty} \frac{3}{\sqrt[3]{27x_n^2 + 9x_n + 1} - 1} - \frac{1}{x_n} =$$

$$= \lim_{n\to\infty} 3 \cdot \frac{\left(\sqrt[3]{27x_n^2 + 9x_n + 1}\right)^2 + \sqrt[3]{27x_n^2 + 9x_n + 1} + 1}{27x_n^2 + 9x_n} - \frac{27x_n + 9}{27x_n^2 + 9x_n} = l_1$$

$$= -3 + \lim_{n\to\infty} \frac{\left(\sqrt[3]{27x_n^2 + 9x_n + 1}\right)^2 + \sqrt[3]{27x_n^2 + 9x_n + 1} + 1}{9x_n^2 + 3x_n} - \frac{3}{9x_n^2 + 3x_n}$$

$$y_n = 27x_n^2 + 9x_n$$

$$l_1 = -3 + \lim_{n\to\infty} \frac{\left(\sqrt[3]{y_n + 1}\right)^2 - 1 + \sqrt[3]{y_n + 1} - 1}{y_n} =$$

$$= -3 + \lim_{n\to\infty} 3 \frac{\left(\sqrt[3]{y_n + 1} + 1\right) \cdot \frac{y_n}{\left(\sqrt[3]{y_n + 1}\right)^2 + \sqrt[3]{y_n + 1} + 1} + \frac{y_n}{\left(\sqrt[3]{y_n + 1}\right)^2 + \sqrt[3]{y_n + 1} + 1}}{y_n}$$

$$= -3+3=0 \Rightarrow \Omega = \frac{1}{0_+} = \infty \Rightarrow \Omega = \infty$$

is 0_+ ***because*** $\frac{1}{x_{n+1}} - \frac{1}{x_n} > 0$ ***(x_n decreasing) and*** $n+1 > n$

SOLUTION 5.17

Solution by Togrul Ehmedov-Baku-Azerbaijan

$$\Omega(n) = \int_0^{2\pi} \ln(n^2 - 2n\cos x + 1)\,dx = 2\int_0^{\pi} \ln(n^2 - 2n\cos x + 1)dx$$

$$= 2\int_0^{\pi} \ln(n^2 + 2n\cos x + 1)dx$$

$$\Omega(n) = \int_0^{\pi} \ln(n^2 - 2n\cos x + 1)\,dx + \int_0^{\pi} \ln(n^2 + 2n\cos x + 1)\,dx$$

$$= \int_0^{\pi} \ln((n^2+1)^2 - 4n^2\cos^2 x)\,dx = 2\int_0^{\frac{\pi}{2}} \ln((n^2+1)^2 - 4n^2\cos^2 x)\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \ln((1+n^2)^2\sin^2 x + (1-n^2)^2\cos^2 x)\,dx$$

$$= 2\left[\frac{\pi}{2}\ln\frac{|(1+n^2)| + |1-n^2|}{2}\right] = 2 * 2\pi\ln(n) = 4\pi\ln(n), n \geq 1$$

$$\Omega = \lim_{n\to\infty}\left(1 + \frac{\Omega(n)}{4\pi}\right)^{\ln(n+1)} = \lim_{n\to\infty}\left(1 + \frac{4\pi\ln(n)}{4\pi}\right)^{\ln(n+1)}$$

$$= \lim_{n\to\infty}(1+\ln(n))^{\ln(n+1)} = \textit{infinity}$$

SOLUTION 5.18

Solution by Remus Florin Stanca-Romania

$$\Omega = \lim_{n\to\infty}\frac{x_n^b - b^{x_n}}{\frac{1}{n}} \overset{Stolz\ Cesaro}{\underset{\frac{0}{0}}{=}} \lim_{n\to\infty}\frac{x_{n+1}^b - b^{x_{n+1}} - x_n^b + b^{x_n}}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty}\frac{b^{x_{n+1}} + x_n^b - b^{x_n} - x_{n+1}^b}{x_{n+1} - x_n} n(n+1)(x_{n+1} - x_n) =$$

$$= a \cdot \lim_{n\to\infty}\frac{b^{x_{n+1}} + x_n^b - b^{x_n} - x_{n+1}^b}{x_{n+1} - x_n} = a \cdot \left(\lim_{n\to\infty}\frac{b^{x_{n+1}} - b^{x_n}}{x_{n+1} - x_n} + \lim_{n\to\infty}\frac{x_n^b - x_{n+1}^b}{x_{n+1} - x_n}\right) \quad \textit{(1)}$$

$$Let\ l_1=\lim_{n\to\infty}\frac{b^{x_{n+1}}-b^{x_n}}{x_{n+1}-x_n}=\lim_{n\to\infty}\frac{b^{x_n}(b^{x_{n+1}-x_n}-1)}{x_{n+1}-x_n}=b^b\ln(b)\quad (2)$$

$$Let\ l_2=\lim_{n\to\infty}\frac{x_n^b-x_{n+1}^b}{x_{n+1}-x_n}=\lim_{n\to\infty}\frac{x_{n+1}^b\left(\left(\frac{x_n}{x_{n+1}}\right)^b-1\right)}{x_{n+1}-x_n}=b^b\cdot\lim_{n\to\infty}\frac{\left(\left(\frac{x_n}{x_{n+1}}\right)^b-1\right)}{\frac{x_n}{x_{n+1}}-1}\cdot\left(\frac{x_n}{x_{n+1}}-1\right)\cdot\frac{1}{x_{n+1}-x_n}$$

$$It's\ known\ that\ \lim_{x\to1}\frac{x^a-1}{x-1}=a\Rightarrow l_2=-b^b\lim_{n\to\infty}b\cdot\frac{1}{b}=b^b\quad (3)$$

$$\overset{(1);(2);(3)}{\Rightarrow}\ \Omega=a\left(b^b\ln(b)-b^b\right)=ab^b\ln\left(\frac{b}{e}\right)\Rightarrow\Omega=ab^b\ln\left(\frac{b}{e}\right)$$

SOLUTION 5.19

Solution by Shafiqur Rahman-Bangladesh

$$x_n=\sum_{i=1}^{n}\left[\frac{\sqrt{i}-i}{\sqrt{i+\sqrt{i}}-i}\right]=0+3+\sum_{i=3}^{n}\left[\frac{\sqrt{i}-i}{\sqrt{i+\sqrt{i}}-i}\right]=3+n-2=n+1$$

$$\lim_{n\to\infty}\left(\frac{1+x_n^2\ln\left(\frac{1+x_n}{x_n}\right)}{x_n}\right)^{x_n}=\lim_{n\to\infty}\left(\frac{1+(n+1)^2\ln\left(1+\frac{1}{n+1}\right)}{n+1}\right)^{n+1}=$$

$$=\lim_{n\to\infty}\left(\frac{1+(n+1)^2\left(\frac{1}{n+1}-\frac{1}{2(n+1)^2}+0\left(\frac{1}{(n+1)^3}\right)\right)}{n+1}\right)^{n+1}$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{2(n+1)}+0\left(\frac{1}{(n+1)^2}\right)\right)^{n+1}=\lim_{n\to\infty}e^{(n+1)\ln\left(1+\frac{1}{2(n+1)}+0\left(\frac{1}{(n+1)^2}\right)\right)}=$$

$$=\lim_{n\to\infty}e^{(n+1)\left(\frac{1}{2(n+1)}+0\left(\frac{1}{(n+1)^2}\right)\right)}=\lim_{n\to\infty}e^{\left(\frac{1}{2}+0\left(\frac{1}{n+1}\right)\right)}$$

$$\therefore\lim_{n\to\infty}\left(\frac{1+x_n^2\ln\left(\frac{1+x_n}{x_n}\right)}{x_n}\right)^{x_n}=\sqrt{e}$$

SOLUTION 5.20

Solution by Kelvin Hong-Rawang-Malaysia

$$\Omega=\int_0^{\frac{\pi^2}{4}}\frac{1}{1+\tan\sqrt{x}+\cot\sqrt{x}}dx\overset{u^2=x}{=}\int_0^{\frac{\pi}{2}}\frac{2u\,du}{1+\tan u+\cot u}$$

$$\overset{u=\frac{\pi}{2}-t}{=}\int_0^{\frac{\pi}{2}}\frac{\pi-2t}{1+\tan t+\cot t}dt=\int_0^{\frac{\pi}{2}}\frac{\pi}{1+\tan t+\cot t}-\frac{2t}{1+\tan t+\cot t}dt$$

$$=\int_0^{\frac{\pi}{2}}\frac{\pi}{1+\tan t+\cot t}dt-I=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan t+\cot t}dt$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\tan t}{\tan^2 t+\tan t+1}dt\overset{k=\tan t}{=}\frac{\pi}{2}\int_0^{\infty}\frac{k}{(k^2+k+1)(k^2+1)}dk$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{1}{k^2+1}-\frac{1}{k^2+k+1}dk=\frac{\pi}{2}\int_0^{\infty}\frac{1}{k^2+1}-\frac{4}{3}\cdot\frac{1}{\left(\frac{2k+1}{\sqrt{3}}\right)^2+1}dk$$

$$=\frac{\pi}{2}\left[\tan^{-1}k-\frac{4}{3}\cdot\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2k+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$=\frac{\pi}{2}\left[\tan^{-1}k-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2k+1}{\sqrt{3}}\right)\right]_0^{\infty}$$

$$=\frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right)-\frac{2}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right]=\frac{\pi}{2}\left[\frac{\pi}{2}-\frac{2\pi}{3\sqrt{3}}\right]=\left(\frac{9-4\sqrt{3}}{36}\right)\pi^2$$

SOLUTION 5.21

Solution by Tran Hong-Vietnam

$$\textit{Let } u=\sqrt{ab}, v=\frac{a+b}{2}\Rightarrow 0<u\le v<\frac{\pi}{2}$$

$$f(v)=\left(\int_0^u\sqrt[3]{x}\cos x\,dx\right)\left(\int_0^v\sqrt[3]{x}\sin x\,dx\right)-\left(\int_0^u\sqrt[3]{x}\sin x\,dx\right)\left(\int_0^v\sqrt[3]{x}\cos x\,dx\right)$$

$$\Rightarrow f'(v)=\sqrt[3]{v}\sin v\cdot\int_0^u\sqrt[3]{x}\cos x\,dx-\sqrt[3]{v}\cos v\int_0^u\sqrt[3]{x}\sin x\,dx$$

$$=\sqrt[3]{v}\left(\sin v\int_0^u\sqrt[3]{x}\cos x\,dx-\cos v\int_0^u\sqrt[3]{x}\sin x\,dx\right)$$

$$g(v)=\sin v\int_0^u\sqrt[3]{x}\cos x\,dx-\cos v\int_0^u\sqrt[3]{x}\sin x\,dx$$

$$\Rightarrow g'(v) = \cos v \int_0^u \sqrt[3]{x} \cos x \, dx + \sin x \int_0^u \sqrt[3]{x} \sin x \, dx > 0$$

$$g(0) \cdot g\left(\frac{\pi}{2}\right) < 0 \Rightarrow \exists! \, v_0 \in \left(0, \frac{\pi}{2}\right) : g(v_0) = 0 \Rightarrow f(v) \geq f(v_0) \geq f(u) = 0$$

SOLUTION 5.22

Solution by Avishek Mitra-India

$$\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(c)}{\Gamma(c)} = \psi(a) + \psi(b) + \psi(c)$$

$$= \psi(a+1) + \psi(b+1) + \psi(c+1) - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$$

$$= \log a + \log b + \log c - \frac{1}{2a} - \frac{1}{2b} - \frac{1}{2c} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \cdot a^{2n}} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \cdot b^{2n}} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \cdot c^{2n}}$$

Now, given $\Omega = \frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(c)}{\Gamma(c)} + \frac{ab+bc+ca}{2abc}$

$$= \log(abc) - \frac{1}{2a} - \frac{1}{2b} - \frac{1}{2c} + \frac{ab+bc+ca}{2abc} - S_1 - S_2 - S_3$$

$$= \log(abc) - S_1 - S_2 - S_3$$

as $a, b, c > 1$ ***and*** $a + b + c = 6$***, for equality we may take*** $a = b = c = 2$***, by putting values***

$$\Leftrightarrow \Omega = \log(2 \cdot 2 \cdot 2) - S_1 - S_2 - S_3 = 3\log 2 - (S_1 + S_2 + S_3)$$

Clearly $\Omega < 3\log 2$ ***(proved)***

SOLUTION 5.23

Solution by Soumitra Mandal-Chandar Nagore-India

$$\ln\left(\int_0^{\frac{\pi}{2}} \left(\frac{8^{\sin x}}{3^{\sin x} + 4^{\sin x}} + \frac{27^{\sin x}}{2^{\sin x} + 4^{\sin x}} + \frac{64^{\sin x}}{2^{\sin x} + 3^{\sin x}}\right) dx\right)$$

$$\overset{\substack{HOLER'S \\ INEQUALITY}}{\geq} \ln\left(\int_0^{\frac{\pi}{2}} \frac{(2^{\sin x} + 3^{\sin x} + 4^{\sin x})^3}{6(2^{\sin x} + 3^{\sin x} + 4^{\sin x})} dx\right) = \ln\left(\frac{1}{6}\int_0^{\frac{\pi}{2}} \left(\sum_{cyc} 2^{\sin x}\right)^2 dx\right)$$

$$\overset{AM \geq GM}{\geq} \ln\left(\frac{9}{6}\int_0^{\frac{\pi}{2}} \left(\sqrt[3]{2^{\sin x} \cdot 3^{\sin x} \cdot 4^{\sin x}}\right)^2 dx\right) = \ln\left(\frac{3}{2}\int_0^{\frac{\pi}{2}} (4!)^{\frac{2}{3}\sin x} dx\right)$$

$$\geq \ln\left(\frac{3}{2}\int_0^{\frac{\pi}{2}}(4!)^{\frac{2}{3}\sin x}\cdot \cos x\,dx\right) = \ln\left(\frac{3}{2}\left[\frac{(4!)^{\frac{2}{3}\sin x}}{\frac{2}{3}\ln(4!)}\right]_{x=0}^{x=\frac{\pi}{2}}\right) = \ln\left(\frac{9\left((4!)^{\frac{2}{3}}-1\right)}{4\ln(4!)}\right)$$

SOLUTION 5.24

Solution by proposer

First, we will prove three claims as a start.

Claim 0.1 For any $a_i > 0$ $(i = 1, \dots, n)$ and $a_1 a_2 \dots a_n = 1$, we have

$$\sum_{i=1}^{n}\left(a_i - \sqrt{a_i}\right) > 0$$

Proof. we have by AM-GM inequality:

$$\sum_{i=1}^{n} a_i \geq n \qquad (1)$$

hence, by Cauchy Schwartz's inequality and (1), we then have

$$\left(\sum_{i=1}^{n} a_i\right)^2 \geq n\cdot\sum_{i=1}^{n} a_i = (1+1+\dots+1+1)\cdot\sum_{i=1}^{n} a_i \geq \left(\sum_{i=1}^{n} a_i\right)^2$$

and this ends the proof.

Claim 0.2. f has a fixed point $(\neq 0)$.

Proof.

Let $\phi(x) = f(x) - x$ and define the fixed points of g and h as follows $h(a) = a$ and $g(b) = b$. Thus, since ϕ is continuous on $[0, 1]$ and

$$\phi(a)\cdot\phi(b) = (f(a)-a)(f(b)-b) = \big(f(a)-h(a)\big)\big(f(b)-g(b)\big) < 0$$

hence f also has a fixed point $(\neq 0)$.

Claim 0.3. There are n distinct numbers $a_i \in (0, 1)$ with $i = 1, 2, \dots, n$ such that

$$\prod_{i=1}^{n} f'(\alpha_i) = 1\ \left(\forall i \neq j;\ \alpha_i \neq \alpha_j\right).$$

Proof. Let $\psi(x) = f_n(x) - x$ with $f_n(x) = f\left(f\left(f(\dots f(x)..)\right)\right)$ [n - times]

Now since ψ is continuously defferentiable function and $\psi(0) = \psi(\gamma)$ (γ is the fixed point of f). Therefore, there is $\beta \in (0, 1)$ such that

$$\psi'(\beta) = 0 \Leftrightarrow \prod_{i=1}^{n} f'(\alpha_i) = 1$$

with $a_i = f_i(\beta)$ and because of iv) the $a_i{}'s$ must be distincts.

Now, let's go back to our problem and put $a_i = f'(\alpha_i)$ in Claim 0.1 we get the desired result immediately, and we are done.

SOLUTION 5.25

Solution by Said Ibnja – Marrakesh – Morocco

i. Define f on $(0,\infty)$ so that:

$$(\forall n \in N^*); f(x) = \frac{e^x}{\sqrt[n]{2}} - e^{x+1} + 1$$

Now, since f is derivable on $(0,\infty)$ we then find that

$$f'(x) = e^x\left(\frac{1}{\sqrt[n]{2}} - e\right) < 0$$

that is

$$\forall x > 0, f(x) < f(0) < 0 \Rightarrow \forall x > 0, \frac{e^x}{\sqrt[n]{2}} < e^{x+1} - 1.$$

Now, let's go back to our problem and set $x = k - 1$ for any $k = 2, 3, \dots, n+1$ and then we get after multiplying these inequalities the desired result, and we are done.

Solution 2 by proposer

we will prove ii) and then we conclude the second.

We begin by recalling some well-known results.

1. $\forall x \geq 0, e^x \geq x + 1.$

2. $(\forall\, k \in N^*), a^k + k - 1 \geq ka^{k-1}$

3. $(\forall n \in N), 2(1 + 2 + 3 + \dots + n) = n(n+1)$

4. $\forall x > 0, \sin(x) < x$

5. $\sum_{k=1}^{n} \sin\left(\frac{k}{n}\right) > \frac{(11n-1)(n+1)}{24n}$ ***. Now, using 1. and 2. to get that***

$$e^k + a^k - 2 \geq a^k + k - 1 \geq ka^{k-1}$$

thus,

$$\prod_{k=1}^{n}\left(e^k + a^k - 2\right) \geq n!\, a^{\sum_{k=1}^{n} k - n} \geq 2^{n-1} \cdot a^{\frac{11n^2 - 14n - 1}{24}}$$

where the last step follows from 3., 4., 5. and the fact that $n! \geq 2^{n-1}$, hence proved

ii). It suffices to take $a = e$ in the previous inequality and the desired result follows immediately .

SOLUTION 5.26

Solution by Omran Kouba-Damascus-Syria

For $\alpha \geq 2$ prove that $\sum_{k\geq 1}(\zeta(\alpha k) - 1) \leq \frac{3}{4}$ where ζ is the Riemann zeta function.

Clearly the function $\alpha \to \sum_{k\geq 1}(\zeta(\alpha k) - 1)$ is decreasing on $[2, \infty)$ so

$$\sum_{k\geq 1}(\zeta(\alpha k) - 1) \leq \sum_{k\geq 1}(\zeta(2k) - 1) = \sum_{k\geq 1}\left(\sum_{j\geq 2}\frac{1}{j^{2k}}\right) = \sum_{j\geq 2}\left(\sum_{k\geq 1}\frac{1}{j^{2k}}\right) = \sum_{j\geq 2}\frac{1}{j^2 - 1}$$

$$= \frac{1}{2}\sum_{j\geq 2}\left(\frac{2j-1}{j(j-1)} - \frac{2j+1}{(j+1)j}\right) = \frac{3}{4}$$

SOLUTION 5.27

Solution by Marian Ursărescu-Romania

Let $f(a) = \frac{\ln(1 + a\cos x)}{\cos x}$ is a continuous function in $a \Rightarrow \Omega'(a) = \int_0^\pi \frac{1}{1 + a\cos x}dx$

$$\left.\begin{array}{r} Let\ \tan\dfrac{x}{2} = t \Rightarrow x = 2\arctan t \Rightarrow dx = \dfrac{2}{1+t^2}dt \\ x = 0 \Rightarrow t = 0; x = \pi \Rightarrow t = \infty \end{array}\right\} \Rightarrow \Omega'(a) = \int_0^\infty \frac{1}{1 + a\dfrac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}dt$$

$$= 2\int_0^\infty \frac{1}{1 + t^2 + a - at^2}dt = 2\int_0^\infty \frac{1}{(1-a)t^2 + 1 + a}dt = \frac{2}{1-a}\int_0^\infty \frac{1}{t^2 + \left(\sqrt{\dfrac{1+a}{1-a}}\right)^2}dt =$$

$$= \frac{2}{1-a} \cdot \frac{1}{\sqrt{\dfrac{1+a}{1-a}}}\arctan\left.\frac{t}{\sqrt{\dfrac{1+a}{1-a}}}\right|_0^\infty = \frac{\pi}{\sqrt{1-a^2}} \Rightarrow$$

$$\left.\begin{array}{r} \Omega(a) = \pi\int\frac{1}{\sqrt{1-a^2}}da = \pi\arcsin a + c \\ But\ \Omega(a) = 0 \Rightarrow c = 0 \end{array}\right\} \Rightarrow \Omega(a) = \pi\arcsin a \Rightarrow \textit{we must show:}$$

$\sum(\arcsin a)^2 \geq \sum \arcsin a \cdot \arcsin b$*, which its true because* $\sum x^2 \geq \sum xy$

SOLUTION 5.28

Solution by Kelvin Hong-Rawang-Malaysia

Let $f(z)=\frac{1}{[(\pi^2+2)z^2+2\pi z+2\pi^2]}$*, the only poles of* f *are simple, which are*

$$z_{1,2}=-\frac{\pi}{\pi^2+2}\pm\frac{\pi}{\pi^2+2}\sqrt{2\pi^2+3i}\{Im(z_1)>Im(z_2)\}$$

Note that $z_1+z_2=\frac{2\pi}{(\pi^2+2)}, z_1-z_2=\frac{i2\pi\sqrt{2\pi^2+3}}{\pi^2+2}$

$$I=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}=-\sum Res_{z=z_1,z_2}\frac{\pi\cot(\pi z)}{(\pi^2+2)z^2+2\pi z+2\pi^2}$$

$$=-\sum\left[\lim_{z\to z_1,z_2}\frac{\pi\cot(\pi z)}{2(\pi+2)z+2\pi}\right]=-\left[\frac{\pi\cot(\pi z_1)}{2\pi\sqrt{2\pi^2+3i}}-\frac{\pi\cot(\pi z_2)}{2\pi\sqrt{2\pi^2+3i}}\right]$$

$$=\frac{i}{2\sqrt{2\pi^2+3}}(\cot(\pi z_1)-\cot(\pi z_2))$$

Similarly,

$$J=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{n^2+(\pi n)^2+(n+\pi)^2+\pi^2}=-\sum Res_{z=z_1,z_2}\frac{\pi\csc(\pi z)}{(\pi^2+2)z^2+2\pi z+2\pi^2}$$

$$=\frac{i}{2\sqrt{2\pi^2+3}}(\csc(\pi z_1)-\csc(\pi z_2))$$

After some cancellation, we have:

$$\frac{I}{J}=\frac{\cot(\pi z_1)-\cot(\pi z_2)}{\csc(\pi z_1)-\csc(\pi z_2)}=\frac{\sin[\pi(z_2-z_1)]}{\sin(\pi z_2)-\sin(\pi z_1)}$$

$$=\frac{2\sin\left[\frac{\pi(z_2-z_1)}{2}\right]\cos\left[\frac{\pi(z_2-z_1)}{2}\right]}{2\cos\left[\frac{\pi(z_1+z_2)}{2}\right]\sin\left[\frac{\pi(z_2-z_1)}{2}\right]}=\frac{\cos\left(\frac{\pi^2}{\pi^2+2}\sqrt{2\pi^2+3i}\right)}{\cos\left(\frac{\pi^2}{\pi^2+2}\right)}$$

$$=\frac{\cosh\left(\frac{\pi^2\sqrt{2\pi+3}}{\pi^2+2}\right)}{\cos\left(\frac{\pi^2}{\pi^2+2}\right)},\therefore p=\frac{\pi^2}{\pi^2+2}\sqrt{2\pi^2+3}, q=\frac{\pi^2}{\pi^2+2}$$

SOLUTION 5.29

Solution by Shafiqur Rahman-Bangladesh

$$\tan^2\left(\frac{\pi}{8}\right)\tan^2\left(\frac{\pi}{16}\right)\tan^2\left(\frac{\pi}{32}\right)\tan^2\left(\frac{\pi}{64}\right)=$$

$$=\left(\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}\right)\left(\frac{2-2\cos\frac{\pi}{8}}{2+2\cos\frac{\pi}{8}}\right)\left(\frac{2-2\cos\frac{\pi}{16}}{2+2\cos\frac{\pi}{16}}\right)\left(\frac{2-2\cos\frac{\pi}{32}}{2+2\cos\frac{\pi}{32}}\right)$$

$$=\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\left(\frac{2-\sqrt{2+2\cos\frac{\pi}{4}}}{2+\sqrt{2+2\cos\frac{\pi}{4}}}\right)\left(\frac{2-\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}\right)\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos\frac{\pi}{4}}}}}\right)$$

$$\therefore \tan^2\left(\frac{\pi}{8}\right)\tan^2\left(\frac{\pi}{16}\right)\tan^2\left(\frac{\pi}{32}\right)\tan^2\left(\frac{\pi}{64}\right)=$$

$$=\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\cdot\left(\frac{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)$$

SOLUTION 5.30

Solution by Feti Sinani-Kosovo

We have to prove that

$$\int_0^1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}dx=-\frac{4}{\pi}+\frac{\pi}{64}(63-7\pi^4)$$

Since that series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}dx$ ***uniformly converges from Weistras criterion***

$$\int_0^1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}x^{n-1}dx=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{(2n-1)!}\int_0^1 x^{n-1}\,dx=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}(n-1)^7}{n(2n-1)!}$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}}{(2n-1)!}\left(n^6-7n^5+21n^4-35n^3+35n^2-21n+7-\frac{1}{n}\right)$$

Using the Taylor series of $\sin x$***, we get:***

$$S(x)=\sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!},|x|<+\infty\Rightarrow x\sin x=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{(2n-1)!}$$

$$\Rightarrow S_2(x)=\sin x+x\cos x=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nx^{2n-1}}{(2n-1)!}$$

$$S_3(x)=\sin x+3x\cos x-x^2\cos x=2^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2x^{2n-1}}{(2n-1)!}$$

$$S_4(x)=\sin x+7x\cos x-6x^2\sin x-x^3\cos x=2^3\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^3x^{2n-1}}{(2n-1)!}$$

$$S_5(x)=\sin x+15x\cos x-25x^2\sin x-10x^3\cos x+x^4\sin x=2^4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^4x^{2n-1}}{(2n-1)!}$$

$$S_6(x)=\sin x+31x\cos x-90x^2\sin x-65x^3\cos x+15x^4\sin x+x^4\cos x=$$

$$=2^5\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^5x^{2n-1}}{(2n-1)!}$$

$$S_7(x)=\sin x+63x\cos x-301x^2\sin x-350x^3\cos x+140x^4\sin x+21^5\cos x$$

$$-x^6\sin x=$$

$$=2^5\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^5x^{2n-1}}{(2n-1)!},\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n-1}}{n(2n-1)!}=\frac{2}{x}(1-\cos x)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\pi^{2n-1}}{(2n-1)!}\left(n^6-7n^5+21n^4-35n^3+35n^2-21n+7-\frac{1}{n}\right)=$$

$$=\frac{1}{2^6}S_7(\pi)-\frac{7}{2^5}S_6(\pi)+\frac{21}{2^4}S_5(\pi)-\frac{35}{2^3}S_4(\pi)+\frac{35}{2^2}S_3(\pi)-\frac{21}{2}S_2(\pi)+7S(\pi)-\frac{4}{\pi}=$$

$$=\frac{1}{64}\left(-63\pi+350\pi^3-21\pi^5\right)-\frac{7}{32}\left(-31\pi+65\pi^3-\pi^5\right)+\frac{21}{16}(-15\pi+10\pi^3)-$$

$$-\frac{35}{8}(-7\pi+\pi^3)+\frac{35}{4}(-3\pi)+\frac{21}{2}\pi-\frac{4}{\pi}=\frac{63\pi}{64}-\frac{7\pi^5}{64}-\frac{1}{\pi}=\frac{\pi}{64}(64-7\pi^2)-\frac{4}{\pi}$$

SOLUTION 5.31

Solution by Kamel Benaicha-Algeirs-Algerie

$$S=\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}$$

On a: $S=\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{2^{2n}}\left\{\frac{1}{n}-\frac{2}{2n+1}\right\}=\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{n2^{2n}}-2\sum_{n=1}^{+\infty}\frac{\zeta(2n)}{(2n+1)2^{2n}}$ *(A)*

Nous savons que: $\sum_{k=0}^{+\infty}\zeta(2k)x^{2k}=-\frac{\pi x}{2}\cot(\pi x)\,,|x|<1$ *(B)*

Integrant cette relation de 0 *a* $\frac{1}{2}$*, on trouve que:*

$$\sum_{k=0}^{+\infty}\zeta(2k)\frac{1}{(2k+1)2^{2k+1}}=-\int_0^{\frac{1}{2}}\frac{\pi x}{2}\cot(\pi x)\,dx$$

$$=-\frac{1}{2}x\ln(\sin(x))\Big|_0^{\frac{1}{2}}+\frac{1}{2}\int_0^{\frac{1}{2}}\ln(\sin(\pi x))\,dx=\frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\ln(\sin(x))\,dx=-\frac{1}{4}\ln(2)$$

$\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{(2k+1)2^{2k}}=-\frac{1}{2}\ln(2)+\frac{1}{2}=-\frac{1}{2}\ln\left(\frac{2}{e}\right)$ *(I)* $\left(\zeta(0)=-\frac{1}{2}\right)$

Posons dans (B): $x=e^{-t},\sum_{k=1}^{+\infty}\zeta(2k)e^{-2kt}=-\frac{\pi}{2}e^{-t}\cot(\pi e^{-t})+\frac{1}{2}$ *(2)*

Par integration de (2) (en t de $\ln(2)$ *a* $(+\infty)$*), on obtient:*

$-\sum_{k=1}^{+\infty}\zeta(2k)\frac{x^{2k}}{2k}\Big|_0^{\frac{1}{2}}=\left\{\frac{\pi}{2}\int\cot(\pi x)\,dx+\frac{1}{2}t\right\}\Big|_{\ln(2)}^{+\infty}$ *(avec* $t=-\ln(x)$*)*

$$=\left\{\frac{1}{2}\ln(\sin(\pi x))-\frac{1}{2}\ln(x)\right\}\Big|_0^{\frac{1}{2}}=\frac{1}{2}\ln\left(\frac{\sin(\pi x)}{x}\right)\Big|_0^{\frac{1}{2}}-\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{2k2^{2k}}$$

$$=\frac{1}{2}\{\ln(2)-\ln(\pi)\}$$

$\sum_{k=1}^{+\infty}\zeta(2k)\frac{1}{k2^{2k}}=\ln\left(\frac{\pi}{2}\right)$ *(3)*

D'apres (1) et (3) et la reformulation (A) de (S), on trouve que:

$$S=\ln\left(\frac{\pi}{2}\right)+\ln\left(\frac{2}{e}\right)=\ln\left(\frac{\pi}{e}\right)$$

$\therefore\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}=\ln\left(\frac{\pi}{e}\right)$ *(II)*

$$s=\sum_{n=1}^{+\infty}\left\{\frac{\zeta(2n)}{n}-\frac{\zeta(2n+1)}{4^n}\right\}\frac{1}{2n+1}$$

On a trouve que:

$\sum_{n=1}^{+\infty}\frac{\zeta(2n)4^{-n}}{n(2n+1)}=\ln\left(\frac{\pi}{e}\right)$ *(II)*

On a $\Psi(x+1)=-\gamma-\sum_{n=1}^{+\infty}\zeta(n+1)\,(-x)^n$

$$= -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)\, x^{2n} + \sum_{n=0}^{+\infty} \zeta(2n+2)\, x^{2n+1} \quad (C)$$

Integrant la relation (C), on trouve:

$$\ln\left(\Gamma\left(\frac{1}{2}+1\right)\right) = -\frac{\gamma}{2} - \sum_{n=1}^{+\infty} \zeta(2n+1) \frac{1}{(2n+1)2^{2n+1}} +$$

$$+\sum_{n=1}^{+\infty} \zeta(2n) \frac{1}{(2n)2^{2n}} \left(entre\ \left(0\ et \frac{1}{2}\right)\right)$$

$$\ln\left(\frac{1}{2}\sqrt{\pi}\right) = -\frac{\pi}{2} + \frac{1}{2}\sum_{n=1}^{+\infty} \left\{\frac{\zeta(2n)}{n2^{2n}} - \frac{\zeta(2n+1)}{(2n+1)2^{2n}}\right\} \quad (4)$$

D'autre part:

$$\int_0^1 \int_0^t \sum_{n=1}^{+\infty} \zeta(2n) x^{2n-1}\, dx\, dt = \sum_{n=1}^{+\infty} \frac{\zeta(2n)}{n(2n+1)} = \frac{1}{2}\ln(2) + \ln\left(\frac{\pi}{e}\right)$$

$$\sum_{n=1}^{+\infty} \frac{\zeta(2n)}{n(2n+1)} = \ln(2) + \ln\left(\frac{\pi}{e}\right) \quad (5)$$

$$\sum_{n=1}^{+\infty} \frac{\zeta(2n)}{n2^{2n}} = \ln\left(\frac{\pi}{2}\right) \quad (3)$$

D'apre les relations (3), (4), on trouve

$$\ln\left(\frac{1}{2}\right) + \frac{1}{2}\ln(\pi) = -\frac{\gamma}{2} + \frac{1}{2}\ln\left(\frac{\pi}{2}\right) - \sum_{n=1}^{+\infty} \frac{\zeta(2n+1)}{2(2n+1)2^{n}}$$

$$-\sum_{n=1}^{+\infty} \frac{\zeta(2n+1)}{(2n+1)4^{n}} = \gamma - \ln(2) \quad (6)$$

La summation de (5) et (6) donne:

$$\sum_{n=1}^{+\infty} \left\{\frac{\zeta(2n)}{n} - \frac{\zeta(2n+1)}{4^{n}}\right\} \frac{1}{2n+1} = \gamma + \ln\left(\frac{\pi}{e}\right)$$

SOLUTION 5.32

Solution by Shafiqur Rahman-Bangladesh

$$\Omega(x) = 2\sum_{k=0}^{\infty} \frac{1}{2k+1} \tanh^{2k+1}\left(\frac{1}{2\Gamma(x)}\right) = 2\tanh^{-1}\left(\tanh\left(\frac{1}{2\Gamma(x)}\right)\right) = \frac{1}{\Gamma(x)}$$

Now,

$$\int_{\frac{1}{2}}^{\frac{3}{2}} e^{-\Gamma(x)}\, \Psi_0(x)\big(1+\Omega(x)\big)dx = \int_{\frac{1}{2}}^{\frac{3}{2}} e^{-\Gamma(x)} \left(1+\frac{1}{\Gamma(x)}\right)\frac{\Gamma'(x)}{\Gamma(x)}\, dx =$$

$$= -\int_{\frac{1}{2}}^{\frac{3}{2}} \left(\frac{d}{dx}\left(e^{-\Gamma(x)}\frac{1}{\Gamma(x)}\right) + e^{-\Gamma(x)}\frac{d}{dx}\left(\frac{1}{\Gamma(x)}\right)\right)dx = -\left[\frac{e^{-\Gamma(x)}}{\Gamma(x)}\right]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{e^{-\sqrt{\pi}}}{\sqrt{\pi}} - \frac{e^{-\frac{\sqrt{\pi}}{2}}}{\frac{\sqrt{\pi}}{2}}$$

$$\therefore \int_{\frac{1}{2}}^{\frac{3}{2}} e^{-\Gamma(x)}\Psi_0(x)\left(1+\Omega(x)\right)dx = \frac{e^{-\sqrt{\pi}} - 2e^{-\frac{\sqrt{\pi}}{2}}}{\sqrt{\pi}}$$

SOLUTION 5.33

Solution by Avishek Mitra-India

$$\int (4\cot^3 x + \cot^2 x + \cot x - 2)\, e^x dx$$

$$= \int [4\cot x\,(\csc^2 x - 1) + (\csc^2 x - 1) + \cot x - 2] e^x\, dx$$

$$= \int [4\cot x \cdot \csc^2 x - 3\cot x - 2\csc^2 x + 3\csc^2 x - 3)]\, e^x dx$$

$$= \int e^x[-3(1+\cot x) + 3\csc^2 x]\, dx - \int e^x(2\csc^2 x - 4\cot x \cdot \csc^2 x)\, dx$$

$$= -3e^x(\cot x + 1) - 2e^x\csc^2 x + c$$

By applying: $\int e^x[f(x) + f'(x)] = e^x f(x) + c$

$$= -3e^x(\cot x + 1) - 2e^x(\cot^2 x + 1) + c = -e^x(5 + 3\cot x + 2\cot^2 x) + c \quad \textit{(Answer)}$$

SOLUTION 5.34

Solution by Marian Ursărescu-Romania

Let $f: [0, x] \to \mathbb{R}, a < x < 1, f(x) = \sqrt[m]{1+x}$***. From Lagrange theorem*** $\Rightarrow$

$\Rightarrow \exists c \in (0, x)$ ***such that*** $\frac{f(x)-f(0)}{x} = f'(c)$

$$\Rightarrow \frac{\sqrt[m]{1+x}-1}{x} = \frac{1}{m\sqrt[m]{(1+c)^{m-1}}} \Rightarrow \sqrt[m]{1+x} - 1 = \frac{x}{m\sqrt[m]{(1+c)^{m-1}}} \quad \textit{(1)}$$

$$c \in (0, x) \Rightarrow 0 < c < x \Rightarrow 1 < 1 + c < 1 + x \Rightarrow$$

$$1 < (1+c)^{m-1} < (1+x)^{m-1} \Rightarrow \frac{1}{(1+x)^{m-1}} < \frac{1}{(1+c)^{m-1}} < 1 \Rightarrow$$

$$\frac{1}{\sqrt[m]{(1+x)^{m-1}}} < \frac{1}{\sqrt[m]{(1+c)^{m-1}}} < 1 \Rightarrow \frac{x}{m\sqrt[m]{(1+x)^{m-1}}} < \frac{x}{m\sqrt[m]{(1+c)^{m-1}}} < \frac{x}{m} \quad \textit{(2)}$$

From (1)+(2) $\Rightarrow \frac{x}{m\sqrt[m]{(1+x)^{m-1}}} < \sqrt[m]{1+x} - 1 < \frac{x}{m}$

$$x=\frac{1^{m-1}}{n^m},\frac{2^{m-1}}{n^m},\dots,\frac{n^{m-1}}{n^m}\Rightarrow$$

$$\sum_{k=1}^{n}\frac{\frac{k^{m-1}}{n^m}}{m\sqrt[m]{\left(1+\frac{1}{n}\right)^{m-1}}}<\sum_{k=1}^{n}\frac{\frac{k^{m-1}}{n^m}}{m\sqrt[m]{\left(1+\frac{k^{m-1}}{n^m}\right)^{m-1}}}<\sum_{k=1}^{n}\left(\sqrt[m]{1+\frac{k^{m-1}}{n^m}}-1\right)<\frac{1}{m}\sum_{k=1}^{n}\frac{k^{m-1}}{n^m}$$

Because $\frac{k^{m+1}}{n^m}<\frac{n^{m-1}}{n^m}$ *(3)*

But $\lim_{n\to\infty}\frac{\sum_{k=1}^{n}k^{m-1}}{n^m}\overset{C.S.}{=}\lim_{n\to\infty}\frac{(n+1)^{m-1}}{(n+1)^m-n^m}==\lim_{n\to\infty}\frac{C_{m-1}^{0}n^{m-1}+\cdots}{C_m^1n^{m-1}+\cdots}=\frac{1}{m}\Rightarrow$

$$\lim_{n\to\infty}\frac{1}{m}\sum_{k=1}^{n}\frac{k^{m-1}}{n^m}=\frac{1}{m^2}\quad (4)$$

Similarly, for the left hand, and $\sqrt[m]{\left(1+\frac{1}{n}\right)^m}\to 1$ *(5)*

For (3)+(4)+(5) $\Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt[m]{1+\frac{k^{m-1}}{n^m}}-1\right)=\frac{1}{m^2}$

SOLUTION 5.35

Solution by Tran Hong-Vietnam

We have: $\log t\le t-1,\forall t\ge 1$ *(*)*

In fact:

Let $\varphi(t)=t-1-\log t\quad (t\ge 1)$

$$\varphi'(t)=1-\frac{1}{t}=\frac{t-1}{t}\ge 0\Rightarrow \varphi(t)\nearrow[1;+\infty)$$

$\Rightarrow \varphi(t)\ge\varphi(1)=0\Rightarrow$ *(*) true.*

Now, let $m=\int_a^b f(x)\,dx\;\;(f(x)\ge 1\Rightarrow m\ge b-a)$

$$RHS=3(b-a)^2m+m^3$$

$$LHS\overset{(*)}{\le}4(b-a)^3+6(b-a)^2\left\{\int_a^b[f(x)-1]\,dx\right\}$$

$$=4(b-a)^3+6(b-a)^2[m-(b-a)]=6(b-a)^2m-2(b-a)^3$$

Must show that:

$$m^3+3(b-a)^2m\ge 6(b-a)^2m-2(b-a)^3$$

$$\Leftrightarrow m^3-3(b-a)^2m+2(b-a)^3\ge 0\quad (**)$$

Let $f(m)=m^3-3(b-a)^2m+2(b-a)^3\quad (m\ge b-a)$

$$\because f'(m) = 3m^2 - 3(b-a)^2 \geq 3(b-a)^2 - 3(b-a)^2 = 0$$

$$\Rightarrow f(m) \nearrow [b-a; +\infty]$$

$$\Rightarrow f(m) \geq f(b-a) = (b-a)^3 - 3(b-a)^3 + 2(b-a)^3 = 0$$

$\Rightarrow$ ***(**) true*** $\Rightarrow LHS \leq RHS.$

SOLUTION 5.36

Solution by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = x^2 + 2(\tan^{-1} x) - \frac{2x}{1+x^2}$ ***for all*** $x \in \mathbb{R}$

$$f'(x) = 2x + \frac{4x^2}{(1+x^2)^2}, f''(x) = 2 + \frac{8x}{(1+x^2)^2} - \frac{32x^2}{(1+x^2)^3}$$

For $f'(\alpha) = 0 \Rightarrow \alpha = 0$ ***then*** $f''(\alpha) = 2 > 0$***. Hence*** f ***attains minimum at*** $x = 0$***.***

$$\therefore f(x) \geq f(0) = 0 \Rightarrow x^2 + 2(\tan^{-1} x) \geq \frac{2x}{1+x^2}$$

$$\Rightarrow \int_a^b x^2 dx + 2\int_a^b \tan^{-1} x\, dx \geq \int_a^b \frac{2x}{1+x^2} dx$$

$$\Rightarrow \frac{b^3 - a^3}{3} + 2\int_a^b \tan^{-1} x\, dx \geq \ln\left(\frac{1+b^2}{1+a^2}\right)$$

$$\therefore b^3 + 6\int_a^b \tan^{-1} x\, dx \geq 3\ln\left(\frac{1+b^2}{1+a^2}\right) + a^3$$

SOLUTION 5.37

Solution by Ravi Prakash-New Delhi-India

For $c_k = (1)(2!)^{\frac{1}{2}}(3!)^{\frac{1}{3}} \dots (k!)^{\frac{1}{k}}\ \forall k \geq 1$

$$c_1 = 1 \leq \frac{2}{2}$$

For $k \geq 2, (k!)^{\frac{1}{k}} \leq \frac{1+2+\cdots+k}{k} = \frac{k+1}{2}$

$$\therefore c_k < (1)\left(\frac{3}{2}\right)\left(\frac{4}{2}\right)\dots\left(\frac{k+1}{2}\right)\ \forall k \geq 2$$

$$\Rightarrow \frac{c_k}{(k+1)!} < \frac{1}{2^k}\ \forall k \geq 2 \Rightarrow \sum_{k=1}^{n} \frac{c_k}{(k+1)!} < \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} < 1$$

That is, $\sum_{k=1}^{n} \frac{c_k}{(k+1)!}$ ***is bounded by 1.***

Also, $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \to \infty$ ***as*** $n \to \infty$***.***

$As\ 0 < \frac{1}{H_n}\sum_{k=1}^{n}\frac{c_k}{(k+1)!} < \frac{1}{H_n}\ and\ \frac{1}{H_n} \to 0\ as\ n \to \infty,\ we\ get\ \lim_{n\to\infty}\frac{1}{H_n}\sum_{k=1}^{n}\frac{c_k}{(k+1)!} = 0$

SOLUTION 5.38

Solution by Avishek Mitra -West Bengal-India

$$(1-x^2)^n = 1 - \sum_{k=1}^{n}(-1)^{k-1}\cdot C_k \cdot x^{2k}$$

$$\Rightarrow \sum_{k=1}^{n}(-1)^{k-1}\cdot {}^nC_k \cdot \int_0^1 x^{2k}\,dx = 1 - \int_0^1 (1-x^2)^n\,dx$$

$$Let\ x^2 = z \Rightarrow 2x\,dx = dz$$

$$\Rightarrow \sum_{k=1}^{n}\frac{(-1)^{k-1}\cdot {}^nC_k}{(2k+1)} = 1 - \frac{1}{2}\int_0^1 z^{\left(\frac{1}{2}-1\right)}\cdot(1-z)^{(n+1)-1}dz = 1 - \frac{1}{2}\beta\left(\frac{1}{2}, n+1\right)$$

$$= 1 - \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}+1\right)} = 1 - \frac{1}{2}\cdot\frac{\sqrt{\pi}\cdot\Gamma(n+1)}{\left(n+\frac{1}{2}\right)\cdot\frac{\Gamma(2n+1)\cdot\sqrt{\pi}}{2^{2n}\Gamma(n+1)}}$$

$$= 1 - \frac{(n!)^2\cdot 2^{2n}}{(2n+1)(2n)!} = L$$

$$\Rightarrow Now,\ \lim_{n\to\infty} L = 1 - \lim_{n\to\infty}\frac{\left[\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n\right]^2\cdot 4^n}{(2n+1)\sqrt{4\pi n}\cdot\left(\frac{2n}{e}\right)^{2n}} = 1 - \lim_{n\to\infty}\frac{2\pi n}{2(2n+1)\cdot\sqrt{\pi n}} = 1 - \lim_{n\to\infty}\frac{\sqrt{\pi n}}{(2n+1)}$$

$$= 1 - \lim_{n\to\infty}\frac{\sqrt{\pi}}{2}\cdot\frac{1}{2\sqrt{n}} = 1 - 0 = 1$$

$$Hence\ P = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{(-1)^{k}.{}^nC_k}{(2k+1)}\right]^{\sqrt{n}} \Leftrightarrow 1^\infty\ form$$

$$\therefore P = e^{\lim_{n\to\infty}\left[1-\frac{1}{2}\beta\left(\frac{1}{2},n+1\right)-1\right]\sqrt{n}}$$

$$= e^{\left[-\lim_{n\to\infty}\frac{\sqrt{\pi n}\cdot\sqrt{n}}{(2n+1)}\right]} = e^{\left[-\lim_{n\to\infty}\frac{\sqrt{\pi}}{\left(2+\frac{1}{n}\right)}\right]} = e^{-\frac{\sqrt{\pi}}{2}}\ \textit{(Answer)}$$

SOLUTION 5.39

Solution by Shafiqur Rahman-Bangladesh

Method 1

$$\frac{a+1}{(b+1)(b+2)}+\frac{(a+1)(a+2)}{(b+1)(b+2)(b+3)}+\cdots=\sum_{n=1}^{\infty}\frac{(a+1)(a+2)\dots(a+n)}{(b+1)(b+2)(b+3)\dots(b+n+1)}$$

$$=$$

$$=\frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a+1)}\sum_{n=1}^{\infty}\beta(a+n+1,b-a+1)$$

$$=\frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a+1)}\int_0^1 x^{a+1}(1-x)^{b-a}\left(\sum_{n=1}^{\infty}x^{n-1}\right)dx=$$

$$=\frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a+1)}\int_0^1 x^{a+1}(1-x)^{b-a-1}dx=$$

$$=\frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(b-a+1)}\cdot\frac{\Gamma(a+2)\Gamma(b-a)}{\Gamma(b+2)}$$

$$\therefore\frac{a+1}{(b+1)(b+2)}+\frac{(a+1)(a+2)}{(b+1)(b+2)(b+3)}+\cdots=\frac{a+1}{(b-a)(b+1)}$$

Method 2

$$\frac{a+1}{(b+1)(b+2)}+\frac{(a+1)(a+2)}{(b+1)(b+2)(b+3)}+\cdots=$$

$$=\frac{1}{b-a}\sum_{n=1}^{\infty}\frac{(a+1)(a+2)\dots(a+n)(b+n+1)-(a+n+1)}{(b+1)(b+2)(b+3)\dots(b+n+1)}$$

$$=\frac{1}{b-a}\sum_{n=1}^{\infty}\left(\frac{(a+1)(a+2)\dots(a+n)}{(b+1)(b+2)\dots(b+n)}-\frac{(a+1)(a+2)\dots(a+n)(a+n+1)}{(b+1)(b+2)\dots(b+n)(b+n+1)}\right)=$$

$$=\frac{1}{b-a}\left(\frac{a+1}{b+1}-\lim_{n\to\infty}\frac{(a+1)(a+2)\dots(a+n)(a+n+1)}{(b+1)(b+2)\dots(b+n)(b+n+1)}\right)$$

$$\therefore\frac{a+1}{(b+1)(b+2)}+\frac{(a+1)(a+2)}{(b+1)(b+2)(b+3)}+\cdots=\frac{a+1}{(b-a)(b+1)}$$

SOLUTION 5.40

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } I_k=\int_k^{k+1}\sqrt{(k+1-x)(x-k)}\,dx$$

$$\textit{Put } x-k=t$$

$$I_k=\int_0^1\sqrt{t(1-t)}\,dt=\int_0^1\sqrt{\frac{1}{4}-\left(\frac{1}{2}-t\right)^2}\,dt$$

$$Put\ \frac{1}{2}-t=\frac{1}{2}\sin\theta$$

$$I_k=\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}\sqrt{\frac{1}{4}-\frac{1}{4}\sin^2\theta}\left(-\frac{1}{2}\right)\cos\theta\,d\theta=\frac{7}{4}\int_0^{\frac{\pi}{2}}\cos\theta\cos\theta\,d\theta=\frac{2}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\pi$$

Next,

$$\sin^2\left(\frac{\pi k}{2n}\right)+\cos\left(\frac{\pi k}{2n}\right)\geq\sin^2\left(\frac{\pi}{2n}\right)+\cos^2\left(\frac{\pi k}{2n}\right)=1$$

$$and\ \sin\left(\frac{\pi k}{2n}\right)+\cos\left(\frac{\pi k}{2n}\right)+1<3$$

Therefore,

$$a_k=\frac{\sin^2\left(\frac{\pi k}{2n}\right)+\cos\left(\frac{\pi k}{2n}\right)}{\sin\left(\frac{\pi k}{2n}\right)+\cos\left(\frac{\pi k}{2n}\right)+1}\left(\frac{k}{n}I_n\right)\geq\frac{1}{3}\frac{k\pi}{n}$$

Now,

$$\sum_{k=1}^{n}a_k\geq\frac{\pi}{3n}\sum_{k=1}^{n}k=\frac{\pi(n+1)}{6}\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n}a_k=\infty$$

SOLUTION 5.41

Solution by Nassim Nicholas Taleb-USA

$\int_i^{i+1}\sqrt{(i+1-x)(x-i)}\,dx$, with $u=x-i$ becomes $\int_0^1\sqrt{(1-u)u}\,du$ which is a semicircle with radius $\frac{1}{2}$. So, using the standard formula,

$$\int_0^1\sqrt{(1-u)u}\,du=\frac{1}{2}\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{8}$$

Hence

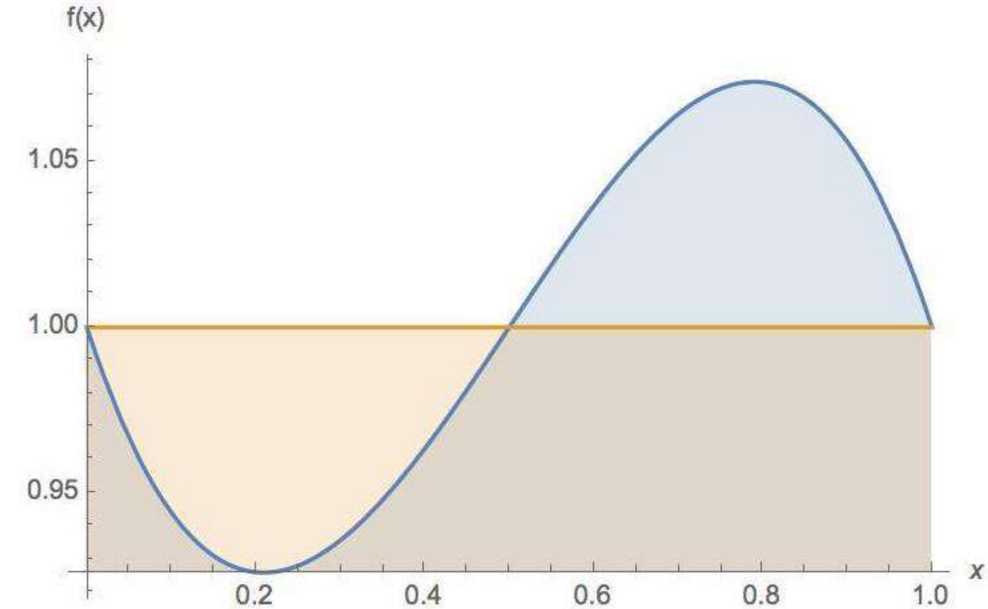
$\Omega=\frac{\pi}{8}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)$. Rewriting, $f\left(\frac{i}{n}\right)=\frac{\left(1+2\cos\left[\frac{i\pi}{2n}\right]-\cos\left[\frac{i\pi}{n}\right]\right)}{\left(1+\cos\left[\frac{i\pi}{2n}\right]+\sin\left[\frac{i\pi}{2n}\right]\right)}$

Let $x=\frac{i}{n}$, in $(0,1]$. $f(x)=\frac{\left(1+2\cos\left[\frac{x\pi}{2}\right]-\cos[x\pi]\right)}{\left(1+\cos\left[\frac{x\pi}{2}\right]+\sin\left[\frac{x\pi}{2}\right]\right)}$. Since $f(0)=f\left(\frac{1}{2}\right)=f(1)=1$

We can see symmetry around $\frac{1}{2}$, with $f(x)\geq 1$ for $x>\frac{1}{2}$ and lower otherwise, so

$$\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\to\int_0^1 f(x)dx=1$$

$$\boldsymbol{\Omega = \frac{\pi}{16}}$$



FAMOUS INEQUALITIES AND IDENTITIES

SOLUTIONS

SOLUTION 6.01

Proof by Rozeta Atanasova - Skopje – Macedonia

$$\textit{Let } f(x) = \sqrt{sinx}$$

$$\Rightarrow \forall\, x \in (0,\pi)\; f'(x) = -\frac{(1+sin^2 x)}{4(sin\,x)^{\frac{3}{4}}} < 0 \Rightarrow \textit{according to Jensen's inequality}$$

$$LHS \le 3\cdot\sqrt{sin\left(\frac{A+B+C}{3}\right)} = 3\cdot\left(\frac{3}{4}\right)^{\frac{1}{4}} = RHS$$

SOLUTION 6.02

Proof by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c} \le \frac{abc}{r} \quad (1) \quad \textit{(Anderson's Inequality)}$$

$$r_a = \frac{\Delta}{s-a}, r_b = \frac{\Delta}{s-b}, r_c = \frac{\Delta}{s-c}$$

$$\textit{(1)} \Leftrightarrow \frac{a^3(s-a)+b^3(s-b)+c^3(s-c)}{\Delta} \le \frac{abc}{r}$$

$$\Leftrightarrow s(a^3+b^3+c^3)-(a^4+b^4+c^4) \le \frac{abc}{r}(rs)$$

$$\Leftrightarrow \left(\frac{a+b+c}{2}\right)(a^3+b^3+c^3)-(a^4+b^4+c^4) \le \frac{abc(a+b+c)}{2}$$

$$\Leftrightarrow (a+b+c)(a^3+b^3+c^3)-2(a^4+b^4+c^4) \le abc(a+b+c)$$

$$\Leftrightarrow a^4+b^4+c^4+a^3(b+c)+b^3(c+a)+c^3(a+b)-2(a^4+b^4+c^4) \le$$

$$\le abc(a+b+c)$$

$$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c) \ge a^3(b+c)+b^3(c+a)+c^3(a+b)$$

The above is Schur inequality for $t=2$

SOLUTION 6.03

Solution by Soumava Chakraborty – Kolkata – India

Given inequality $\Leftrightarrow$

$$\left(2\sum ab-\sum a^2\right)^{\frac{3}{2}}\le 4(4Rrs)+8\left(3\sqrt{3}-4\right)\frac{s\prod(s-a)}{s}$$

$$\Leftrightarrow (16Rr+4r^2)^{\frac{3}{2}}\le 16Rrs+8\left(3\sqrt{3}-4\right)r^2s$$

$$\Leftrightarrow (4Rr+r^2)^{\frac{3}{2}}\le 2Rrs+\left(3\sqrt{3}-4\right)r^2s=sr\left\{2R+\left(3\sqrt{3}-4\right)+r\right\}$$

$$\Leftrightarrow s^2r^2\left\{2R+\left(3\sqrt{3}-4\right)r\right\}^2\ge (4Rr+r^2)^3=r^3(4R+r)^3$$

$$\Leftrightarrow s^2\left\{2R+\left(3\sqrt{3}-4\right)r\right\}^2\ge r(4R+r)^3 \quad \textit{(1)}$$

$$\textit{LHS of (1)} \overset{Gerretsen}{\ge} (16Rr-5r^2)\left\{2R+\left(3\sqrt{3}-4\right)r\right\}^2$$

it is sufficient to prove:

$$(16R-5r)\left\{2R+\left(3\sqrt{3}-4\right)r\right\}^2\ge (4R+r)^3$$

$$\Leftrightarrow \left\{4R^2+\left(43-24\sqrt{3}\right)r^2+4\left(3\sqrt{3}-4\right)Rr\right\}(16R-5r)\ge 64R^3+48R^2r+12Rr^2+r^3$$

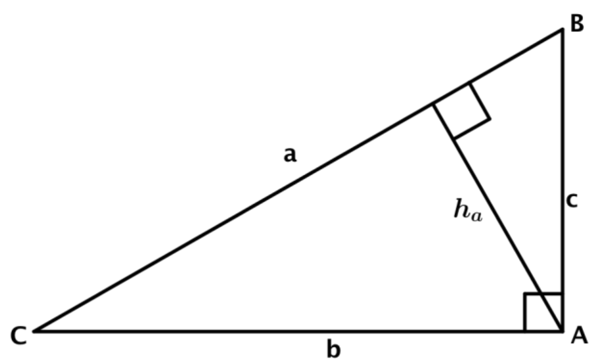
$$\Leftrightarrow \left(16\sqrt{3}-27\right)R^2-\left(37\sqrt{3}-63\right)Rr-\left(18-10\sqrt{3}\right)r^2\ge 0$$

$$\Leftrightarrow (R-2r)\left\{\underbrace{\left(16\sqrt{3}-27\right)}_{>0}R+\underbrace{\left(9-5\sqrt{3}\right)}_{>0}r\right\}\ge 0 \textit{which is true} \because R\ge 2r \textit{ (Euler) (Proved)}$$

SOLUTION 6.04

Proof by Kunihiko Chikaya-Tokyo-Japan

$$h_a\le b+c-\left(\sqrt{2}-\tfrac{1}{2}\right)a \quad (*)$$



$$2[ABC]=ah_a=bc$$

$$(*)\Leftrightarrow \frac{bc}{a}\le b+c-\left(\sqrt{2}-\tfrac{1}{2}\right)a \Leftrightarrow \left(\frac{b+c}{a}-1\right)^2\le 3-2\sqrt{2}$$

$$(a^2=b^2+c^2)$$

$$0<\frac{b+c}{a}-1\le\sqrt{2}-1\Leftrightarrow 0<\frac{b+c}{a}\le\sqrt{2}$$

$$\frac{b+c}{a}=\frac{b+c}{\sqrt{b^2+c^2}}\le\frac{\sqrt{2}\sqrt{b^2+c^2}}{\sqrt{b^2+c^2}}=\sqrt{2}$$

SOLUTION 6.05

Proof by Soumava Chakraborty-Kolkata-India

$$\frac{2b^2+2c^2-a^2}{bc}+\frac{2c^2+2a^2-b^2}{ca}+\frac{2a^2+2b^2-c^2}{ab}\ge 9$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)-\sum a^3\ge 9abc$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)\ge (a+b+c)^3-3(a+b)(b+c)(c+a)+9abc$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)\ge 8s^3-3\left(2abc+\sum a^2b+\sum ab^2\right)+9abc$$

$$\Leftrightarrow 5\left(\sum a^2b+\sum ab^2\right)\ge 8s^3+3abc$$

$$\Leftrightarrow 5\{ab(a+b)+bc(b+c)+ca(c+a)\}\ge 8s^3+3abc$$

$$\Leftrightarrow 5\{ab(2s-c)+bc(2s-a)+ca(2s-b)\}\ge 8s^3+3abc$$

$$\Leftrightarrow 10s(ab+bc+ca)\ge 8s^3+18abc\Leftrightarrow 5s\{s^2+r(4R+r)\}\ge 4s^3+9\cdot 4R(rs)$$

$$(abc=4Rrs)$$

$$\Leftrightarrow s^3+5rs(4R+r)\ge 36Rrs\Leftrightarrow s^2+5r(4R+r)\ge 36Rr$$

$$\Leftrightarrow s^2\ge 16Rr-5r^2$$ ***- true (Gerretsen)***

SOLUTION 6.06

Proof by Soumava Chakraborty-Kolkata-India

In $\Delta ABC,\frac{ma^2}{bc}+\frac{mb^2}{ca}+\frac{mc^2}{ab}\ge\frac{9}{4}$ ***(Bager's Inequality - 1)***

$$\Leftrightarrow\frac{2b^2+2c^2-a^2}{bc}+\frac{2c^2+2a^2-b^2}{ca}+\frac{2a^2+2b^2-c^2}{ab}\ge 9$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)-\sum a^3\ge 9abc$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)\ge (a+b+c)^3-3(a+b)(b+c)(c+a)+9abc$$

$$\Leftrightarrow 2\left(\sum a^2b+\sum ab^2\right)\ge 8S^3-3\left(2abc+\sum a^2b+\sum ab^2\right)+9abc$$

$$\Leftrightarrow 5\left(\sum a^2b+\sum ab^2\right)\ge 8S^3+3abc$$

$$\Leftrightarrow 5\{ab(a+b)+bc(b+c)+ca(c+a)\}\ge 8S^3+3abc$$

$$\Leftrightarrow 5\{ab(2S-c)+bc(2S-a)+ca(2S-b)\} \geq 9S^3+3abc$$

$$\Leftrightarrow 10S(ab+bc+ca) \geq 8S^3+18abc \Leftrightarrow 5s\{s^2+r(4R+r)\} \geq 4S^3+9\cdot 4R(rs)$$

$$(abc=4RS)$$

$$\Leftrightarrow S^3+5rs(4R+r) \geq 36Rrs \Leftrightarrow S^2+5r(4R+r) \geq 36Rr$$

$$\Leftrightarrow S^2 \geq 16Rr-5r^2 \text{ – true (Gerretsen)}$$

SOLUTION 6.07

Proof by Adil Abdullayev – Baku – Azerbaidian

$$\sum_{cyc} h_a^2 \leq \sum_{cyc} w_a^2 \leq s^2 \leq \sum_{cyc} m_a^2 \leq \sum_{cyc} r_a^2.$$

Lemma 1.

$$w_a^2 \leq r_b r_c \leq m_a^2$$

Lemma 2.

$$\sum_{cyc} m_a^2 = \frac{3}{4}\cdot\sum_{cyc} a^2 = \frac{3}{2}(s^2-r^2-4Rr)$$

$$\sum_{cyc} r_a^2 = (4R+r)^2-2s^2.$$

Lemma 3.

$$\sum_{cyc} r_b r_c = s^2.$$

$$h_a \leq w_a \Rightarrow \sum_{cyc} h_a^2 \leq \sum_{cyc} w_a^2$$

$$w_a^2 \leq r_b r_c \leq m_a^2 \Rightarrow \sum_{cyc} w_a^2 \leq \sum_{cyc} r_b r_c \leq \sum_{cyc} m_a^2 \Leftrightarrow\Leftrightarrow \sum_{cyc} w_a^2 \leq s^2 \leq \sum_{cyc} m_a^2.$$

$$\sum_{cyc} m_a^2 \leq \sum_{cyc} r_a^2 \Leftrightarrow 7s^2 \leq 32R^2+28Rr+5r^2.$$

$$7s^2 \overset{GERRETSEN}{\leq} 28R^2+28Rr+21r^2 \overset{?}{\leq} 32R^2+28Rr+5r^2 \Leftrightarrow R \geq 2r.$$

SOLUTION 6.08

Proof by Soumitra Mandal-Chandar Nagore-India

We know, $\cos A+\cos B+\cos C \leq \frac{3}{2}$

We need to prove,

$$\prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right)\cdot 27\prod_{cyc}(\tan A+\tan B)\geq 8\left(\sum_{cyc}\tan A\right)^3$$

$$\therefore \prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right)\cdot 27\prod_{cyc}(\tan A+\tan B)$$

$$=\prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right)\cdot 27\prod_{cyc}\frac{(\sin A\cos B+\cos A\sin B)}{\cos A\cos B}$$

$$=\prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right)\cdot 27\prod_{cyc}\frac{\sin(A+B)}{\cos A\cos B}$$

$$=27\prod_{cyc}(1-\cos A)\cdot\frac{1}{(\cos A\cos B\cos C)^2}\cdot\prod_{cyc}\tan A$$

Now, we will prove,

$$27\prod_{cyc}(1-\cos A)\cdot\frac{1}{(\cos A\cos B\cos C)^2}\geq 8\prod_{cyc}\tan^2 A$$

$$\Leftrightarrow 27\prod_{cyc}(1-\cos A)\geq 8\prod_{cyc}\sin^2 A=8\prod_{cyc}(1-\cos^2 A)$$

$$\Leftrightarrow \frac{27}{8}\geq\prod_{cyc}(1+\cos A)\ \ldots.(1)$$

Now, applying A.M $\geq$ G.M

$$\left(\frac{3+\cos A+\cos B+\cos C}{3}\right)^3\geq\prod_{cyc}(1+\cos A)\Rightarrow\left(\frac{3+\frac{3}{2}}{3}\right)^3\geq\prod_{cyc}(1+\cos A)$$

$\therefore \frac{27}{8}\geq\prod_{cyc}(1+\cos A)$ ***(established statement (1))***

$\therefore 27\prod_{cyc}(1-\cos A)\cdot\frac{1}{(\cos A\cos B\cos C)^2}\geq 8\prod_{cyc}\tan^2 A$ ***(established)***

$$\therefore 27\prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right)\cdot\prod_{cyc}(\tan A+\tan B)\geq 8\prod_{cyc}\tan^3 A=8\left(\sum_{cyc}\tan A\right)^3$$

Again, applying A.M $\geq$ G.M,

$$\left(\frac{1}{3}\sum_{cyc}(\tan A+\tan B)\right)^3\geq\prod_{cyc}(\tan A+\tan B)$$

$$\therefore \frac{8\left(\sum_{cyc}\tan A\right)^3}{27\prod_{cyc}(\tan A+\tan B)} \geq 1 \ \ \textit{(established)}$$

$$\therefore \prod_{cyc}\left(\frac{1-\cos A}{\cos A}\right) \geq \frac{8\left(\sum_{cyc}\tan A\right)^3}{27\prod_{cyc}(\tan A+\tan B)} \geq 1$$

SOLUTION 6.09

Proof by Soumava Chakraborty-Kolkata-India

$$\cos A\cos B\cos C = \frac{b^2+c^2-a^2}{2bc}\cdot\frac{c^2+a^2-b^2}{2ca}\cdot\frac{a^2+b^2-c^2}{2ab}$$

$$\textit{Numerator} = \left(\sum a^2-2a^2\right)\left(\sum a^2-2b^2\right)\left(\sum a^2-2c^2\right)$$

$$= \left(\sum a^2\right)^3 - 2\left(\sum a^2\right)^2\left(\sum a^2\right) + 4\left(\sum a^2\right)\left(\sum a^2b^2\right) - 8a^2b^2c^2$$

$$= -\left(\sum a^2\right)^3 + 4\left(\sum a^2\right)\left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} - 128R^2r^2S^2$$

$$= \left(\sum a^2\right)\left\{4\left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 16s\,abc\right\} - 128R^2r^2S^2$$

$$= 4\left(\sum a^2\right)\{(s^2+4Rr+r^2)^2-(s^2-4Rr-r^2)^2-16Rrs^2\} - 128R^2r^2S^2$$

$$= 4\left(\sum a^2\right)\{2S^2(8Rr+2r^2)-16Rrs^2\} - 128R^2r^2S^2$$

$$= 32r^2S^2(s^2-4Rr-r^2) - 128R^2r^2S^2 \overset{(2)}{=} 32r^2S^2(s^2-4Rr-r^2-4R^2)$$

$$\textit{(1), (2)} \Rightarrow \cos A\cos B\cos C = \frac{32r^2S^2(S^2-4R^2-4Rr-r^2)}{128R^2r^2S^2}$$

$$\overset{(3)}{=} \frac{s^2-4R^2-4Rr-r^2}{4R^2} \ \therefore k = \frac{2R^2}{s^2-4R^2-4Rr-r^2}$$

$$\therefore \frac{11-2k}{2(k-1)} = \frac{11-\frac{4R^2}{S^2-4R^2-4Rr-r^2}}{\frac{4R^2}{S^2-4R^2-4Rr-r^2}-2} \overset{(4)}{=} \frac{11S^2-48R^2-44Rr-11r^2}{12R^2+8Rr+2r^2-2s^2}$$

$$\textit{Now, } 2s^2 \overset{Gerretsen}{\leq} 8R^2+8Rr+6r^2 \overset{?}{<} 12R^2+8Rr+2r^2$$

$$\Leftrightarrow 4R^2 \overset{?}{>} 4r^2 \Leftrightarrow R \overset{?}{>} r \rightarrow \textit{true}$$

$$\therefore 2S^2 < 12R^2+8Rr+2r^2 \Rightarrow 12R^2+8Rr+2r^2-2s^2 > 0 \quad \textit{(5)}$$

Now, if $11s^2-48R^2-44Rr-11r^2 \leq 0$***, then***

(4), (5) $\Rightarrow \frac{11-2k}{2(k-1)} \leq 0 \Rightarrow \frac{r}{R} > 0 \geq \frac{11-2k}{2(k-1)} \Rightarrow$ ***Banica's inequality holds true***

Let us now consider $11S^2 - 48R^2 - 44Rr - 11r^2 > 0$

i.e. $S^2 > \frac{4R^2}{11} + (2R+r)^2$*. Then, (5)* $\Rightarrow \frac{11-2k}{2(k-1)} > 0$

$$\therefore \frac{r}{R} \geq \frac{11-2k}{2(k-1)} = \frac{11S^2 - 48R^2 - 44Rr - 11r^2}{12R^2 + 8Rr + 2r^2 - 2S^2}$$

$$\Leftrightarrow 12R^2r + 8Rr^2 + 2r^3 - 2s^2r \geq 11s^2R - 48R^3 - 44R^2r - 11Rr^2$$

$$\Leftrightarrow (11R+2r)S^2 \leq 48R^3 + 56R^2r + 19Rr^2 + 2r^3 \quad (6)$$

Now, LHS of (6) $\overset{Gerretsen}{\leq} (11R+2r)(4R^2+4Rr+3r^2)$

$$\overset{?}{\leq} 48R^3 + 56R^2r + 19Rr^2 + 2r^3 \Leftrightarrow 2R^3 + 2R^2r - 11Rr^2 - 2r^3 \overset{?}{\geq} 0$$

$\Leftrightarrow (R-2r)(2R^2+6Rr+r^2) \overset{?}{\geq} 0 \rightarrow$ *true,*

$\because R \geq 2R$ *(Euler)*$\Rightarrow$ *(6) is true* $\Rightarrow$ *Banica's inequality is true*

SOLUTION 6.10

Proof by Adil Abdullayev-Baku-Azerbaidian

$$\sum_{cyc} h_a = \frac{p^2 + r^2 + 4Rr}{2R} \leq 2R + 5r \Leftrightarrow p^2 \leq 4R^2 + 6Rr - r^2.$$

$$p^2 \overset{GERRETSEN}{\leq} 4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2 \Leftrightarrow R \geq 2r$$

SOLUTION 6.11

Proof by Kevin Soto Palacios – Huarmey – Peru

Por la desigualdad de Cauchy:

$$\left(\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C}\right)\left(\sqrt{\cos^2 A} + \sqrt{\cos^2 B} + \sqrt{\cos^2 C}\right) \geq$$

$$\geq \left(\sqrt[4]{\sin A \cos^2 A} + \sqrt[4]{\sin B \cos^2 B} + \sqrt[4]{\sin C \cos^2 C}\right)^2$$

AHORA BIEN:

$$\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \leq \sqrt{3(\sin A + \sin B + \sin C)} \leq$$

$$\leq \sqrt{3 \sin\left(\frac{A+B+C}{3}\right)} \leq \sqrt{3 \times \frac{3\sqrt{3}}{2}} = \frac{3\sqrt[4]{3}}{2}$$

$$\sqrt{\cos^2 A} + \sqrt{\cos^2 B} + \sqrt{\cos^2 C} = \cos A + \cos B + \cos C \leq 3\cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2}$$

Por transitividad se obtiene:

$$\left(\frac{3\sqrt[4]{3}}{2}\right)\left(\frac{3}{2}\right) \geq \left(\sqrt[4]{\sin A\cos^2 A} + \sqrt[4]{\sin B\cos^2 B} + \sqrt[4]{\sin C\cos^2 C}\right)^2$$

$$\Rightarrow \sqrt{\left(\frac{3\sqrt[4]{3}}{2}\right)\left(\frac{3}{2}\right)} = 3\sqrt[8]{\frac{3}{64}} \geq \sqrt[4]{\sin A\cos^2 A} + \sqrt[4]{\sin B\cos^2 B} + \sqrt[4]{\sin C\cos^2 C}$$

SOLUTION 6.12

Proof by Ravi Prakash-New Delhi-India

$$x\cos A + y\cos B + z\cos C \leq \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)$$

Divide by xyz and write it as

$$x^2 + y^2 + z^2 \cdot 2(yz\cos A + zx\cos B + xy\cos C) \geq 0$$

where $\frac{1}{x} = x, \frac{1}{y} = y, \frac{1}{z} = z$

$$LHS = [x - (z\cos B + y\cos C)]^2 + t$$

where

$$t = y^2 + z^2 - 2yz\cos A - (z\cos B + y\cos C)^2$$

It is sufficient to show $t \geq 0$

But

$$t = y^2\sin^2 C + z^2\sin^2 B - 2yz\{-\cos(B + C) + \cos B\cos C\}$$

$$= (y\sin C - z\sin B)^2 \geq 0$$

SOLUTION 6.13

Proof by Soumava Chakraborty – Kolkata – India

In $\Delta\ ABC, \sum a^2 = 2H, \sum ab = K,$

Beatty's Inequality $\Rightarrow \frac{(K-H)(3K-5H)}{12} \leq S^2 \leq \frac{(K-H)^2}{12}$

$$H = S^2 - 4Rr - r^2, K = S^2 + 4Rr + r^2$$

$$K - H = 8Rr + 2r^2$$

$$3K - 5H = -2S^2 + 32Rr + 8r^2$$

$$\frac{(K-H)(3K-5H)}{12} \leq S^2 \Leftrightarrow \frac{4(4Rr + r^2)(-S^2 + 16Rr + 4r^2)}{12} \leq r^2S^2$$

$$\Leftrightarrow -S^2(4Rr + r^2) + (16Rr + 4r^2)(4Rr + r^2) - 3r^2S^2 \leq 0$$

$$\Leftrightarrow s^2(4Rr+4r^2) \ge 4r^2(4R+r)^2 \quad (1)$$

Gerretsen $\Rightarrow S^2 \ge 16Rr - 5r^2$. *LHS of (1)* $\ge (16Rr-5r^2)(4Rr+4r^2)$

it suffices to show: $(16R-5r)(R+r) \ge (4R+r)^2$

$$\Leftrightarrow 16R^2 + 11Rr - 5r^2 \ge 16R^2 + 8Rr + r^2$$

$$\Leftrightarrow 3Rr \ge 6r^2 \Leftrightarrow R \ge 2r \to \text{true (Euler)} \ \frac{(K-H)(3K-5H)}{12} \le S^2$$

$$\text{Again, } S^2 \le \frac{(K-H)^2}{12} \Leftrightarrow \frac{(4R+r)^2}{3} \ge S^2$$

$$\Leftrightarrow S \le \frac{4R+r}{\sqrt{3}} \Leftrightarrow \sqrt{3}S \le 4R+r \to \text{true (Trucht's Inequality)} \ S^2 \le \frac{(K-H)^2}{12}$$

SOLUTION 6.14

Proof by Pirkuliyev Rovsen-Sumgait-Azerbaijan

It is known that

$$m_a^2 = \frac{2(b^2+c^2)-a^2}{4} \text{ and } \frac{a^2}{4S^2} = \frac{1}{h_a^2} \Rightarrow$$

$$\Rightarrow \frac{16S^2m_a^2}{h_a^2} = a^2(2b^2+2c^2-a^2) \quad (*)$$

$$\text{we have } (2a^2-b^2-c^2)^2 \ge 0 \Rightarrow$$

$$\Rightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge 6a^2b^2+6a^2c^2-3a^4 \Rightarrow$$

$$\Rightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge$$

$$\ge 3a^2(2b^2+2c^2-a^2) \overset{(*)}{=} \frac{48S^2m_a^2}{h_a^2}$$

$$\text{and } (a^2+b^2+c^2)^2 \ge \frac{48S^2m_a^2}{h_a^2} \Rightarrow a^2+b^2+c^2 \ge 4\sqrt{3}S \cdot \frac{m_a}{h_a} \Rightarrow$$

$$\Rightarrow a^2+b^2+c^2 \ge 4\sqrt{3}S \cdot max\left(\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right)$$

SOLUTION 6.15

Proof by Kevin Soto Palacios – Huarmey – Peru

$$2\sum ab - \sum a^2 \ge 4\left(2\sum \tan\left(\frac{A+B}{4}\right) - \sqrt{3}\right)S \ge 4\sqrt{3}S$$

Desde que:

$$ab = 2S\csc C,\ bc = 2S\csc A,\ ac = 2S\csc B$$

$$a^2+b^2+c^2 = 4S(\cot A + \cot B + \cot C)$$

$$4S(\csc A+\csc B+\csc C)-4S(\cot A+\cot B+\cot C)\geq 4\left(2\sum\tan\left(\frac{A+B}{4}\right)-\sqrt{3}\right)S$$

$$\geq 4\sqrt{3}S$$

$$\Rightarrow(\csc A-\cot A)+(\csc B-\cot B)+(\csc C-\cot C)+\sqrt{3}\geq 2\sum\tan\left(\frac{A+B}{4}\right)$$

$$\Rightarrow\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\sqrt{3}\geq 2\tan\left(\frac{A+B}{4}\right)+2\tan\left(\frac{B+C}{4}\right)+2\tan\left(\frac{A+C}{4}\right)$$

Desigualdad Popoviciu:

Sea f una function a partir de un intervalo $I\subseteq\mathbb{R}\times\mathbb{R}$.

Si f es covexa, entonces para cualesquiera tres puntos: x_1,x_2,x_3 de I, se cumple lo siguiente:

$$f(x_1)+f(x_2)+f(x_3)+3f\left(\frac{x_1+x_2+x_3}{3}\right)$$

$$\geq 2f\left(\frac{x_2+x_3}{2}\right)+2f\left(\frac{x_1+x_3}{2}\right)+2f\left(\frac{x_1+x_2}{2}\right)$$

Desde que:

$$f(A)=\tan\frac{A}{2},f(B)=\tan\frac{B}{2},f(C)=\tan\frac{C}{2},\qquad \frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$$

$$\Rightarrow\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\sqrt{3}\geq 2\tan\left(\frac{A+B}{4}\right)+2\tan\left(\frac{B+C}{4}\right)+2\tan\left(\frac{A+C}{4}\right)\text{...(LQQD)}$$

$$4\left(2\sum\tan\left(\frac{A+B}{4}\right)-\sqrt{3}\right)S\geq 4\sqrt{3}S$$

$$\sum tan\left(\frac{A+B}{4}\right)\geq\sqrt{3}$$

$$\sum tan\left(\frac{A+B}{4}\right)\underbrace{\geq}_{Jensen}3tan\left(\frac{2\left(\frac{A+B+C}{4}\right)}{3}\right)=3tan\frac{\pi}{6}=\sqrt{3}$$

SOLUTION 6.16

Proof by Myagmarsuren Yadamsuren-Ulanbataar-Mongolia

$$a^2+b^2+c^2\underbrace{\geq}_{LHS}\sqrt{48S^2+8r(4R+r)\sum(a-b)^2+\left(\sum(a-b)^2\right)^2}\underbrace{\geq}_{RHS}4\sqrt{3}S$$

1) Lemma: $\sqrt{3}p\underset{proof\ GERRETSEN}{\overset{\leq}{\Downarrow}}4R+r:\Rightarrow 8\sqrt{3}\cdot r\cdot p\leq 8r\cdot(4R+r)$

$$8\sqrt{3}\cdot S\leq 8r\cdot(4r+r)$$

$$\sqrt{48S^2+8r\cdot(4R+r)\cdot\sum(a-b)^2+\left(\sum(a-b)^2\right)^2}\geq$$

$$\geq\sqrt{\left(4\sqrt{3}S\right)^2+8\sqrt{3}\cdot S\cdot\left(\sum(a-b)^2\right)+\left(\sum(a-b)^2\right)^2}=$$

$$=\sqrt{\left(4\sqrt{3}\cdot S+\left(\sum(a-b)^2\right)\right)^2}=4\sqrt{3}S+\sum(a-b)^2$$

$$RHS: 4\sqrt{3}S+\sum\underbrace{(a-b)^2}_{\geq 0}\geq 4\sqrt{3}S$$

SOLUTION 6.17

Proof by Soumava Chakraborty-Kolkata-India

$$s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$$

(Blundon – Gerretsen's Inequality)

Let's first prove $S^2\leq\frac{R(4R+r)^2}{2(2R-r)}$

Now, $S^2\leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}$

(Baric Triangle Inequality → Rouche's Inequality)

it suffices to prove:

$$2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\leq\frac{R(4R+r)^2}{2(2R-r)}$$

$$\Leftrightarrow (t-2)(8t^2-12t+1)\geq 4(t-2)(2t-1)\sqrt{t^2-2t}$$

where $t=\frac{R}{r}$

$$\Leftrightarrow (t-2)^2\{(8t^2-12t+1)^2-16(2t-1)^2(t^2-2t)\}\geq 0$$

$\Leftrightarrow (t-2)^2(16t^2+8t+1)\geq 0$ → ***true;*** $t\geq 2$

$$S^2\leq\frac{R(4R+r)^2}{2(2R-r)}$$

Now, let's prove: $\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2$

$$\Leftrightarrow 16R^3+Rr^2+8R^2r\leq 16R^3+8R^2r+4Rr^2-6r^3$$

$\Leftrightarrow 3Rr^2-6r^3\geq 0\Rightarrow R\geq 2r$ → ***true***

$$\frac{R(4R+r)^2}{2(2R-r)} \le 4R^2+4Rr+3r^2$$

SOLUTION 6.18

Proof by Kevin Soto Palacios-Huarmey-Peru

Tener presente lo siguiente en un triángulo:

$$abc = 4RS$$

$$S = pr \rightarrow abc = 4Rpr$$

La desigualdad es equivalente:

$$4Rpr \le 8R^2r + \left(12\sqrt{3}-16\right)Rr^2$$

$$\Rightarrow p \le 2R + \left(3\sqrt{3}-4\right)r \quad \textit{(Blundon's Inequality)}$$

De la desigualdad de Gerretsen:

$$p^2 \le 4R^2+4Rr+3r^2 \rightarrow 4R^2+4Rr+3r^2 \le \left(2R+\left(3\sqrt{3}-4\right)r\right)^2$$

$$4R^2+4Rr+3r^2 \le 4R^2+4Rr\left(3\sqrt{3}-4\right)+\left(43-24\sqrt{3}\right)r^2$$

$$\Rightarrow 4Rr\left(3\sqrt{3}-5\right)+\left(40-24\sqrt{3}\right)r^2 \ge 0$$

$$\Rightarrow R \ge 2r \rightarrow 4Rr\left(3\sqrt{3}-5\right)+\left(40-24\sqrt{3}\right)r^2 \ge 8r^2\left(3\sqrt{3}-5\right)+\left(40-24\sqrt{3}\right)r^2 \ge 0$$

SOLUTION 6.19

Proof by Soumava Chakraborty-Kolkata-India

$$\textit{Given inequality} \Leftrightarrow 2\sum \tan A \ge \frac{18R}{\sqrt[3]{8R^3 \sin A \sin B \sin C}}$$

$$\Leftrightarrow \sum(\tan A+\tan B) \ge \frac{9}{\sqrt[3]{\sin A \sin B \sin C}} \quad \textit{(1)}$$

$$\textit{Now, } \tan A+\tan B = \frac{\sin A}{\cos A}+\frac{\sin B}{\cos B} = \frac{\sin(A+B)}{\cos A\cos B} = \frac{\sin C}{\cos A\cos B}$$

$$\textit{Similarly, } \tan B+\tan C = \frac{\sin A}{\cos B\cos C}, \tan C+\tan A = \frac{\sin B}{\cos C\cos A}$$

$$\therefore \sum(\tan A+\tan B) = \frac{\sin C}{\cos A\cos B}+\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos C\cos A}$$

$$\underset{(2)}{\overset{A-G}{\ge}} 3\sqrt[3]{\frac{\sin A\sin B\sin C}{\cos^2 A\cos^2 B\cos^2 C}}$$

(1), (2) ⇒ it suffices to prove:

$$\sqrt[3]{\frac{\prod \sin A}{\prod \cos^2 A}} \geq \frac{3}{\sqrt[3]{\prod \sin A}} \Leftrightarrow \sqrt[3]{\frac{\prod \sin^2 A}{\prod \cos^2 A}} \geq 3 \Leftrightarrow \prod \tan^2 A \geq 27$$

$$\Leftrightarrow \prod \tan A \geq 3\sqrt{3} \Leftrightarrow \sum \tan A \geq 3\sqrt{3} \text{ (3)}$$

$$\because f(x) = \tan x \ \forall x \in \left(0, \frac{\pi}{2}\right) \text{ is convex,}$$

$$\therefore \text{Jensen} \Rightarrow \sum \tan A \geq 3 \tan\left(\frac{A+B+C}{3}\right) = 3\sqrt{3} \Rightarrow \text{(3) is true (Proved)}$$

SOLUTION 6.20

Proof by Adil Abdullayev-Baku-Azerbaidjian

$$64s^2S^2 \leq 27 \cdot 16R^2S^2 \Leftrightarrow 4s^2 \leq 27R^2.$$

SOLUTION 6.21

Proof by Adil Abdullayev – Baku – Azerbaidian

BRETSCHNEIDER THEOREM $\Rightarrow$

$$\Rightarrow AC^2 \cdot BD^2 = AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2 \cdot AB \cdot CD \cdot AD \cdot BC \cdot \cos(A + C) =$$

$$= (AB \cdot CD - AD \cdot BC)^2 + 2 \cdot AB \cdot CD \cdot AD \cdot BC \cdot (1 - \cos(A + C)) \geq$$

$$\geq (AB \cdot CD - AD \cdot BC)^2 \Rightarrow AC^2 \cdot BD^2 \geq (AB \cdot CD - AD \cdot BC)^2 \Rightarrow$$

$$LHS \geq RHS.$$

SOLUTION 6.22

Solution by Hamza Mahmood-Lahore-Pakistan

$$\text{Let } f(x) = \ln \sin x, x \in (0, \pi)$$

$$f'(x) \text{ is concave on } (0, \pi)\text{. By Jensen's inequality,}$$

$$f\left(\frac{\alpha+\beta+\gamma}{3}\right) \geq \frac{1}{3}\big(f(\alpha) + f(\beta) + f(\gamma)\big) \text{ where } \alpha, \beta, \gamma \text{ are the angles of } \Delta\ ABC$$

$$\Rightarrow \ln \sin\left(\frac{\pi}{3}\right) \geq \frac{1}{3} \ln(\sin\alpha \sin\beta \sin\gamma) \Rightarrow \sin^3\left(\frac{\pi}{3}\right) \geq \sin\alpha \sin\beta \sin\gamma$$

$$\Rightarrow 3\frac{\sqrt{3}}{8} \geq \sin\alpha \sin\beta \sin\gamma \Rightarrow 3\frac{\sqrt{3}}{8} \geq \frac{2S}{bc} \cdot \frac{2S}{ac} \cdot \frac{2S}{ab} \Rightarrow 3\frac{\sqrt{3}}{8} \geq \frac{8S^3}{(abc)^2}$$

$$\Rightarrow (abc)^2 \geq \frac{64S^3}{3\sqrt{3}} \Rightarrow (abc)^2 \geq \left(\frac{4S}{\sqrt{3}}\right)^3$$

SOLUTION 6.23

Proof by Mehmet Sahin-Ankara-Turkey

$$b+c-a>\frac{a(b+c)-bc}{4R}\Leftrightarrow(b+c-a)4R>a(b+c)-bc$$

$$(a+b+c=2s)$$

$$\Leftrightarrow(b+c-a)4R>a(b+c)-\left(\frac{b+c}{2}\right)^2\Leftrightarrow2(s-a)4R>a(2s-a)-\left(\frac{2s-a}{2}\right)^2$$

$$\Leftrightarrow 8Rs-8Ra>2sa-a^2-s^2+a\cdot s-\frac{a^4}{4}$$

$$\Leftrightarrow sa^2-4as+4s^2+40Rs-40Ra>0?$$

$$\Leftrightarrow 4\cdot s^2-s(4a-40R)+(sa^2-40Ra)>0?$$

$$\Leftrightarrow \Delta_s<0\Leftrightarrow\Delta_s=(4a-40R)^2-4\cdot4\cdot(sa^2-4aRa)<0\ (?)$$

$$=-4a^2+20Ra+100R^2<0\quad(?)\Leftrightarrow a^2-Ra+5R^2>0\quad(?)$$

$$\Delta_a=R^2-20R^2=-19R^2<0$$

SOLUTION 6.24

Proof by Soumitra Moukherjee - Chandar Nagore – India

Case 1: let the triangle be acute angled triangle.

$$\textit{Let } f(x)=csc^n\frac{x}{2}\textit{ for all } x\in\left(0,\frac{\pi}{2}\right),f'(x)=-\frac{n}{2}csc^n\frac{x}{2}cot\frac{x}{2}$$

$$f''(x)=\frac{n^2}{4}csc^n\frac{x}{2}cot^2\frac{x}{2}+\frac{n}{4}csc^{n+2}\frac{x}{2}>0$$

$$\textit{so applying Jensen's Inequality, }\frac{\sum_{cyc}csc^n\frac{A}{2}}{3}\ge csc^n\left(\frac{\pi}{6}\right)=2^n$$

$$\sum_{cyc}csc^n\frac{A}{2}\ge3\cdot2^n$$

Case 2: let the triangle be obtused angle triangle

$$\frac{a}{b+c}=\frac{sin\,A}{sin\,B+sin\,C}=\frac{sin\frac{A}{2}}{cos\left(\frac{B-C}{2}\right)}\ge sin\frac{A}{2}\Rightarrow csc\frac{A}{2}\ge\frac{b+c}{a}$$

$$\sum_{cyc}csc^n\frac{A}{2}\ge\sum_{cyc}\left(\frac{b+c}{a}\right)^n\ge3\sqrt[3]{\prod_{cyc}\left(\frac{b+c}{a}\right)^n}\ge3\cdot2^n$$

SOLUTION 6.25

Proof by Soumava Chakraborty-Kolkata-India

$$\sqrt{sin\frac{A}{2}sin\frac{B}{2}}+\sqrt{sin\frac{B}{2}sin\frac{C}{2}}+\sqrt{sin\frac{C}{2}sin\frac{A}{2}}\le\frac{3}{2}\quad\textit{(Child)}$$

$$LHS \overset{C-B-S}{\leq} \sqrt{\sum \sin\frac{A}{2}}\sqrt{\sum \sin\frac{A}{2}} = \sum \sin\frac{A}{2} \overset{Jensen}{\leq} 3\sin\frac{A}{6} = \frac{3}{2}$$

SOLUTION 6.26

Proof by Adil Abdullayev – Baku – Azerbaidian

$$\frac{1}{\sin\frac{\alpha}{2}} + \frac{1}{\sin\frac{\beta}{2}} + \frac{1}{\sin\frac{\gamma}{2}} \geq 6$$

$$LHS \overset{AM-GM}{\geq} 3 \cdot \sqrt[3]{\frac{1}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}} = 3 \cdot \sqrt[3]{\frac{4R}{r}} \overset{Euler}{\geq} 3 \cdot \sqrt[3]{8} = 6$$

SOLUTION 6.27

Proof by Soumava Pal-Kolkata-India

$$\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}} \geq 3\sqrt[3]{\frac{1}{\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2}}$$

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{1}{\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)^2} = \frac{16 \cdot R^2}{r^2} \geq 16 \cdot 2^2 = 64$$

$$\left(\frac{R}{r} \geq 2\right)$$

$$\Rightarrow \frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}} + \frac{1}{\sin\frac{B}{2}\sin\frac{C}{2}} + \frac{1}{\sin\frac{C}{2}\sin\frac{A}{2}} \geq 3\sqrt[3]{64} = 12$$

SOLUTION 6.28

Proof 1 by Ravi Prakash - New Delhi – India

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1, b_2, b_3 \in \mathbb{R}.$$

$$a_1 \leq b_1, a_1 + a_2 \leq b_1 + b_2, a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$$

$$a_1^2 + a_2^2 + a_3^2 =$$

$$= a_1^2 + a_2(a_1 + a_2 - a_1) + a_3(a_1 + a_2 + a_3 - a_1 - a_2) =$$

$$= a_1(a_1 - a_2) + (a_1 + a_2)(a_2 - a_3) + a_3(a_1 + a_2 + a_3) \leq$$

$$\leq b_1(a_1 - a_2) + (b_1 + b_2)(a_2 - a_3) + (b_1 + b_2 + b_3)a_3 =$$

$$= \mathbf{a_1b_1 + a_2(b_1 + b_2 - b_1) + a_3(b_1 + b_2 + b_3 - b_1 - b_2) =}$$

$$= \mathbf{a_1b_1 + a_2b_2 + a_3b_3 \le \sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

[CS inequality]

$$\Rightarrow \sqrt{\mathbf{a_1^2 + a_2^2 + a_3^2}} \le \sqrt{\mathbf{b_1^2 + b_2^2 + b_3^2}}$$

$$\mathbf{a_1^2 + a_2^2 + a_3^2 \le b_1^2 + b_2^2 + b_3^2}$$

SOLUTION 6.29

Proof by Adil Abdullayev – Baku – Azerbaidian

$$LHS \le RHS \Leftrightarrow \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right) \le \frac{9R}{2r} \Leftrightarrow$$

$$\Leftrightarrow A := \left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)^2\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}\right)^2 \le \frac{81R^2}{4r^2}$$

$$A \overset{C-B-S}{\le} 3(a + b + c)3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \le \frac{81R^2}{4r^2} \Leftrightarrow$$

$$2p \cdot \frac{p^2 + r^2 + 4Rr}{4prR} \le \frac{9r^2}{4r^2} \Leftrightarrow p^2 \le \frac{9R^3}{2r} - r^2 - 4Rr$$

$$p^2 \overset{GERRETSEN}{\le} 4R^2 + 4Rr + 3r^2 \le \frac{9R^3}{2r} - r^2 - 4Rr \Leftrightarrow 9R^3 - 8r(R + r)^2 \ge 0$$

$$t := \frac{R}{r} \Rightarrow t \ge 2.\ \ 9t^3 - 8(t + 1)^2 \ge 0 \Leftrightarrow (t - 2)(9t^2 + 10t + 4) \ge 0 \Leftrightarrow EULER$$

SOLUTION 6.30

Proof by Seyran Ibrahimov-Maasilli-Azerbaidjian

$$a \ne b \ne c$$

$$\left|\frac{a}{a - b} + \frac{b}{b - c} + \frac{c}{c - a}\right| > \sqrt{6} - 1$$

$$|a - b| < c$$

$$|b - c| < a$$

$$|c - a| < b$$

$$\left|\frac{a}{a - b} + \frac{b}{b - c} + \frac{c}{c - a}\right| > \left|\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right| = \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \ge 3 > \sqrt{6} - 1$$

SOLUTION 6.31

Proof by Ngo Dinh Tuan-Quang Nam-Da Nang-VietNam

$$\textit{According to Heron Formula: } S = \sqrt{p(p-a)(p-b)(p-c)}$$

$$\Rightarrow 4S = \sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}$$

$$\textit{We have inequality } \sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \le 3\sqrt{3}abc$$

$$\Leftrightarrow a^2b^2c^2 \ge \frac{1}{27}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$

$$\textit{Let} \begin{cases} a+b-c=x \\ b+c-a=y \\ c+a-b=z \end{cases} \Rightarrow \begin{cases} a+b+c=x+y+z \\ a=\frac{x+z}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2} \end{cases}$$

$$\textit{Need to prove that: } (x+z)^2(x+y)^2(y+z)^2 \ge \frac{64}{27}xyz(x+y+z)$$

$$\textit{Because: } (x+y+z)(xy+yz+zx) \le \frac{9}{8}(x+z)(x+y)(y+z)$$

$$\Rightarrow LHS \ge \frac{64}{81}(x+y+z)(x+y+z)(xy+yz+zx)^2 \ge \frac{64}{81}(x+y+z)3\sqrt[3]{xyz}3\sqrt[3]{(xyz)^2}$$

$$= RHS$$

SOLUTION 6.32

Proof by Soumitra Moukherjee - Chandar Nagore – India

$$\sum_{cyc}\frac{a}{b+c} = \sum_{cyc}\frac{a^2}{ab+ac} \ge \frac{(a+b+c)^2}{2(ab+bc+ca)} \textit{ [Applying Bergstrom's Ineq]}$$

$$\textit{we will prove, } \frac{(a+b+c)^2}{2(ab+bc+ca)} \ge \frac{3\sqrt{3\left(\sum_{cyc}a^2\right)}}{2(a+b+c)}$$

$$\Rightarrow \frac{p^2}{q} \ge \frac{3\sqrt{3(p^2-2q)}}{p} \textit{ where } p=a+b+c \textit{ and } q=ab+bc+ca$$

$$\Rightarrow p^6 \ge 27q^2(p^2-2q) \Rightarrow p^6 - 27q^3 - 27q^2(p^2-3q) \ge 0$$

$$\Rightarrow (p^2-3q)(p^4+3p^2q-18q^2) \ge 0$$

$$\Rightarrow (p^2-3q)^2(p^2+6q) \ge 0, \textit{ which is true again,}$$

$$3\left(\sum_{cyc}a^2\right) \ge \left(\sum_{cyc}a\right)^2 \Rightarrow \frac{3\sqrt{3\left(\sum_{cyc}a^2\right)}}{2\left(\sum_{cyc}a\right)} \ge \frac{3}{2}$$

$$\sum_{cyc}\frac{a}{b+c} \ge \frac{\left(\sum_{cyc}a\right)^2}{2\left(\sum_{cyc}ab\right)} \ge \frac{3\sqrt{3\left(\sum_{cyc}a^2\right)}}{2\left(\sum_{cyc}a\right)} \ge \frac{3}{2}$$

SOLUTION 6.33

Solution by Adil Abdullayev-Baku-Azerbaidian

$$a\cos\alpha + b\cos\beta + c\ cos\gamma \le s \Leftrightarrow R(\sin 2\alpha + \sin 2\beta + 2\sin\gamma) \le s \Leftrightarrow$$

$$\Leftrightarrow R\cdot\frac{2sr}{R^2} \le s \Leftrightarrow 2r \le R$$

SOLUTION 6.34

Proof by Adil Abdullayev – Baku – Azerbaijan

$$9r(4R+r) \le 3s^2 \le (4R+r)^2$$

$$9r(4R+r) \le 3s^2 \Leftrightarrow 9\cdot\frac{r_a r_b r_c}{s^2}\cdot(r_a+r_b+r_c) \le 3s^2 \Leftrightarrow$$

$$\Leftrightarrow 3r_a r_b r_c(r_a+r_b+r_c) \le (r_a r_b + r_a r_c + r_b r_c)^2 \Leftrightarrow$$

$$\Leftrightarrow 3(xy+xz+yz)^2 \le (x+y+z)^2.$$

$$3s^2 \le (4R+r)^2 \Leftrightarrow 3(r_a r_b + r_a r_c + r_b r_c) \le (r_a+r_b+r_c)^2.$$

$$x^3 - (4R+r)x^2 + s^2 x - rs^2 = 0 \Leftrightarrow$$

$$x_1 = r_a, x_2 = r_b, x_3 = r_c.$$

$$\textit{ROLLE THEOREM} \Rightarrow 3x^2 - 2(4R+r)x + s^2 = 0$$

$$D \ge 0 \Leftrightarrow 4(4R+r)^2 - 12s^2 \ge 0 \Leftrightarrow 3s^2 \le (4R+r)^2.$$

$$t^3 - 2st^2 + \left(s^2 + r(4R+r)\right)t - 4srR = 0 \Leftrightarrow$$

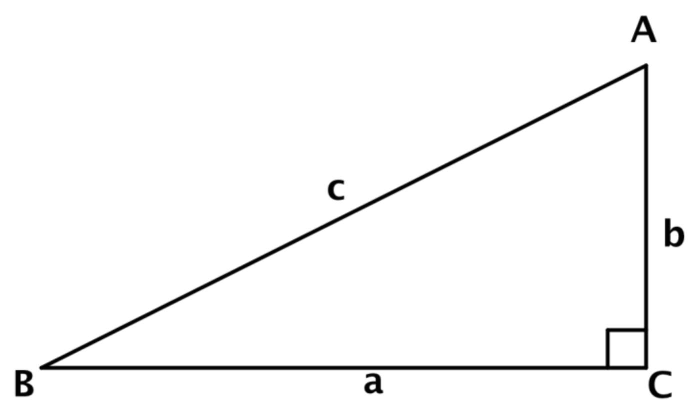
$$t_1 = a, t_2 = b, t_3 = c.$$

$$\textit{ROLLE THEOREM} \Rightarrow 3t^2 - 4st + s^2 + r(4R+r) = 0$$

$$D \ge 0 \Leftrightarrow 16s^2 - 12\left(s^2 + r(4R+r)\right) \ge 0 \Leftrightarrow s^2 \ge 3r(4R+r)$$

SOLUTION 6.35

Proof by Kunihiko Chikaya-Tokyo-Japan



r: ***radius of incircle,*** R: ***radius of circumcirle***

$$r=\frac{a+b-c}{2}\cdot R=\frac{c}{2}$$

$$\frac{R}{r}=\frac{c}{a+b-c}\geq\frac{c}{\sqrt{2}\sqrt{a^2+b^2}-c}\Leftarrow a^2+b^2=c^2$$

$$=\frac{c}{(\sqrt{2}-1)c}=1+\sqrt{2}$$

SOLUTION 6.36

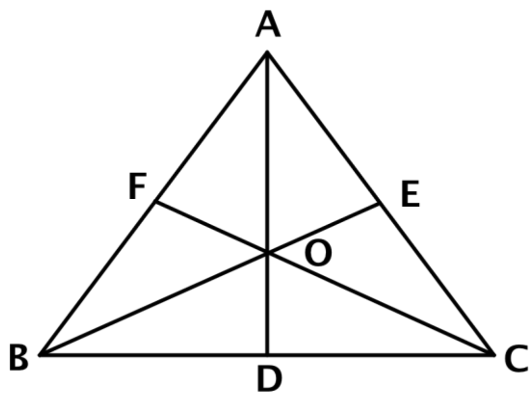
Proof by Soumitra Moukherjee-Chandar Nagore-India

$$R+r\leq max\{h_a,h_b,h_c\}=max\left\{\frac{2\Delta}{a},\frac{2\Delta}{b},\frac{2\Delta}{c}\right\}=\frac{2\Delta}{min\{a,b,c\}}$$

$$\Leftrightarrow min\{a,b,c\}\,(R+r)\leq 2\Delta \qquad (1)$$

Now, $R+r=R\left(\sum_{cyc}cos\,A\right)=OD+OE+OF$, ***where***

D,E ***and*** F ***are the mid – points of*** BC,CA ***and*** AB ***respectively and*** O ***is the circumcentre.***



Now, we have,

$$min\{a,b,c\}\,(R+r)\leq a\cdot OD+b\cdot OE+c\cdot OF$$

$= 2$ ***(area of*** $\Delta\ BOC$***)*** $+2$***(area of*** ΔCOA***)+(area of*** ΔAOB***)*** $=2\Delta$.

$$R+r\leq max(h_a,h_b,h_c)$$

SOLUTION 6.37

Proof by Soumava Pal – Kolkata – India

Without loss of generality:

$a\geq b\geq c$ ***(1)***

$$\Rightarrow A\geq B\geq C$$

$\Rightarrow\frac{A}{2}\geq\frac{B}{2}\geq\frac{C}{2}\Rightarrow sin\frac{A}{2}\geq sin\frac{B}{2}\geq sin\frac{C}{2}$ ***(2)***

$(\frac{A}{2},\frac{B}{2},\frac{C}{2}\in\left(0,\frac{\pi}{2}\right)$ *and* $\sin x$ *is increasing in* $\left(0,\frac{\pi}{2}\right))$

Applying Chebyshev's Inequality in (1) and (2)

$$3\sum_{cycl}a\sin\frac{A}{2}\geq\left(\sum_{cycl}a\right)\left(\sum_{cycl}\sin\frac{A}{2}\right)\Rightarrow\frac{\left(\sum_{cycl}a\sin\frac{A}{2}\right)}{\left(\sum_{cycl}\sin\frac{A}{2}\right)}\geq\left(\frac{\sum_{cycl}a}{3}\right)=\frac{2s}{3}$$

$$c+a>b,\qquad a+b>c$$

$$\left.\begin{array}{l}b+c>a\\ \sin\frac{A}{2}>0\end{array}\right\}\Rightarrow(b+c)\sin\frac{A}{2}>a\sin\frac{A}{2}$$

$$\Rightarrow(a+b+c)\sin\frac{A}{2}>2a\sin\frac{A}{2}\Rightarrow2s\sin\frac{A}{2}>2a\sin\frac{A}{2}$$

$$\Rightarrow2s\sin\frac{A}{2}>2a\sin\frac{A}{2}\Rightarrow s\sin\frac{A}{2}>a\sin\frac{A}{2}$$

Similarly: $s\sin\frac{B}{2}>b\sin\frac{B}{2},s\sin\frac{C}{2}>c\sin\frac{C}{2}\Rightarrow$

$$\Rightarrow s\sum_{cycl}\sin\frac{A}{2}>\sum_{cycl}a\sin\frac{A}{2}\Rightarrow s>\frac{\sum_{cycl}a\sin\frac{A}{2}}{\sum_{cycl}\sin\frac{A}{2}}$$

SOLUTION 6.38

Proof by Soumava Chakraborty-Kolkata-India

$$r^2+\sum r_a^2\geq7R^2\text{ (Gerasimov's Inequality)}$$

$$\Leftrightarrow r^2+(4R+r)^2-2s^2\geq7R^2\Leftrightarrow2s^2\leq9R^2+8Rr+2r^2$$

Now, $2s^2\leq8R^2+8Rr+6r^2$ *(Gerretsen)*

It suffices to show $8R^2+8Rr+6r^2\leq9R^2+8Rr+2r^2$

$$\Leftrightarrow4r^2\leq R^2\Leftrightarrow2r\leq R\rightarrow\text{true (Euler)}$$

SOLUTION 6.39

Proof by Adil Abdullayev – Baku – Azerbaidjian

$$LHS\geq(ab)(bc)+(bc)(ca)+(ca)(ab)=8p^2Rr\overset{Euler}{\gtrsim}16p^2r^2\Leftrightarrow R\geq2r$$

SOLUTION 6.40

Proof by Soumava Chakraborty – Kolkata – India

In any $\Delta\,ABC,16r^2s^2\overset{(b)}{\leq}\sum a^2b^2\overset{(a)}{\leq}4R^2s^2$

(Goldstone's inequality)

Let's first prove that: $\sin\omega \geq \frac{r}{R}$,

$$\sin\omega \overset{(1)}{\geq} \frac{r}{R} \Leftrightarrow \frac{R^2}{r^2} \geq \csc^2\omega = \sum \csc^2 A$$

$$\Leftrightarrow \frac{R^2}{r^2} \geq \sum(1+\cot^2 A) = 3 + \sum\cot^2 A = 1 + \left(\sum\cot^2 A + 2\sum\cot A\cot B\right)$$

$$= 1 + \left(\sum\cot A\right)^2 = 1 + \left(\frac{\sum a^2}{4\Delta}\right)^2 = 1 + \frac{\left(\sum a^2\right)^2}{16r^2s^2} \Leftrightarrow \frac{R^2-r^2}{r^2} \geq \frac{\left(\sum a^2\right)^2}{16r^2s^2} \Leftrightarrow R^2 - r^2 \geq \frac{\left(\sum a^2\right)^2}{16s^2} \quad \textit{(2)}$$

We shall now prove that: $R^2 - r^2 \geq \frac{\sum a^3}{8s}$ *(3)*

$$\Leftrightarrow R^2 - r^2 \geq \frac{\sum a^3 - 3abc + 3abc}{8s}$$

$$= \frac{(2s)(\sum a^2 - \sum ab) + 12Rrs}{8s} = \frac{s^2 - 6Rr - 3r^2}{4}$$

$$\Leftrightarrow s^2 \leq 4R^2 + 6Rr - r^2$$

Gerretsen $\Rightarrow s^2 \leq 4R^2 + 4Rr + 3r^2$

$\therefore$ *in order to prove (3), it suffices to prove that:*

$4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2 \Leftrightarrow R \geq 2r \rightarrow$ *true* $\Rightarrow$ *(3) is true.*

$\therefore$ *to prove (2) and hence (1), it suffices to prove*

$$\frac{\sum a^3}{8s} \geq \frac{(\sum a^2)^2}{16s^2} \Leftrightarrow \left(\sum a\right)\left(\sum a^3\right) \geq \left(\sum a^2\right)^2$$

$$\Leftrightarrow ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \geq 0,$$

which is true $\Rightarrow$ *(2) and* $\therefore$ *(1) is true*

$\therefore \sin\omega \geq \frac{r}{R} \Rightarrow \frac{2\Delta}{\sqrt{\sum a^2b^2}} \geq \frac{r}{R} \Rightarrow 4R^2s^2 \geq \sum a^2b^2 \Rightarrow$ *(a) is true*

Again, $\sum a^2b^2 \geq abc(a+b+c) = 4Rrs(2s) = 8s^2Rr \geq 16s^2r^2 \ (\because R \geq 2r)$

$\Rightarrow$ *(b) is true (Proved)*

SOLUTION 6.41

Proof 1 by Adil Abdullayev-Baku-Azerbaidian

$$\sum_{cyc} m_a \leq 4R + r$$

Lemma

$$\sum_{cyc} m_a \leq 2p - \left(6\sqrt{3} - 9\right)r$$

$$\sum_{cyc} m_a \le 2p - (6\sqrt{3}-9)r \overset{BLUNDON}{\le} 2(2R + (3\sqrt{3}-4)r) - (6\sqrt{3}-9)r =$$

$$= 4R + r.$$

$$(h_a + h_b + h_c)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) \ge 9 \Leftrightarrow (h_a + h_b + h_c) \cdot \frac{1}{r} \ge 9 \Leftrightarrow$$

$$\Leftrightarrow h_a + h_b + h_c \ge 9r.$$

$$(m_a + m_b + m_c)^2 = \sum_{cyc} m_a^2 + \sum_{cyc} 2m_a m_b \le \frac{3}{4}\sum_{cyc} a^2 + \sum_{cyc}\left(\frac{ab}{2} + c^2\right) =$$

$$= 4p^2 - 12Rr - 3r^2 \overset{GERRETSEN}{\le} 16R^2 + 16Rr + 12r^2 - 12Rr - 3r^2 =$$

$$= 16R^2 + 4Rr + 9r^2 = (4R + r)^2 - 4r(R - 2r) \overset{EYLER}{\le} (4R + r)^2.$$

$$4R + r \overset{EYLER}{\le} \frac{9R}{2}$$

SOLUTION 6.42

Solution by Kevin Soto Palacios –Huarmey- Peru

→ Tener presentre que la dseigualdad de Gerretsen: $p^2 \ge 16Rr - 5r^2$

De la desigualdad, lo reempazamos:

$p^2 \ge \frac{r(4R+r)^2}{R+r}$*, por trantividad:* $(16R - 5r)(R + r) \ge 16R^2 + 8Rr + r^2$

$16R^2 + 11Rr - 5r^2 \ge 16R^2 + 8Rr + r^2 \to 3Rr \ge 6r^2 \to R \ge 2r$ ***(Des. Euler)***

SOLUTION 6.43

Solution by Marian Ursărescu-Romania

We can choose $\Delta\, ABC$ ***with circumcenter O in origin of axis.***

Let $t_A, t_B, t_C \in \mathbb{C}$ ***so that*** $A(t_A), B(t_B), C(t_C)$

$$S_{ABC} = \left|\frac{i}{4}\begin{vmatrix} t_A & \overline{t_A} & 1 \\ t_B & \overline{t_B} & 1 \\ t_C & \overline{t_C} & 1 \end{vmatrix}\right| \quad (1)$$

But $t_{H_1} = t_P + t_A + t_B, z_{H_2} = z_P + t_B + t_C, z_{H_3} = z_P + t_A + t_C \Rightarrow$

$$S_{H_1H_2H_3} = \left|\frac{i}{4}\begin{vmatrix} t_p + t_A + t_B & \overline{t_P} + \overline{t_A} + \overline{t_B} & 1 \\ t_P + t_B + t_C & \overline{t_P} + \overline{t_B} + \overline{t_C} & 1 \\ t_P + t_A + t_C & \overline{t_P} + \overline{t_A} + \overline{t_C} & 1 \end{vmatrix}\right| \quad (2)$$

Now, we use these properties:

$$\begin{vmatrix} a'_{11}+a''_{11} & a_{12} & a_{13} \\ a'_{21}+a''_{21} & a_{22} & a_{23} \\ a'_{31}+a''_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a''_{11} & a_{12} & a_{13} \\ a''_{21} & a_{22} & a_{23} \\ a''_{31} & a_{32} & a_{33} \end{vmatrix} \quad (3)$$

$$\text{and} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{vmatrix} \quad (4)$$

From (1)+(2)+(4) $\Rightarrow S_{ABC} = S_{H_1H_2H_3}$

SOLUTION 6.44

Proof by Nirapada Pal-Jhargram-India

$$\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m} \overset{AM-GM}{\geq} 3\frac{1}{[abc]^{\frac{m}{3}}} \overset{AM-GM}{\geq} \frac{3}{\left[\frac{(a+b+c)}{3}\right]^m}$$

$$\geq \frac{3^{1+m}}{[3\sqrt{3}R]^m} \quad \textit{[since } a+b+c \leq 3\sqrt{3}R\textit{]} = \frac{3}{R^m(\sqrt{3})^m} = \frac{(\sqrt{3})^{2-m}}{R^m}$$

SOLUTION 6.45

Proof by Soumava Chakraborty-Kolkata-India

$$LHS = \frac{1^{m+1}}{(ax+by)^m}+\frac{1^{m+1}}{(bx+cy)^m}+\frac{1^{m+1}}{(cx+ay)^m} \overset{Radon}{\geq} \frac{3^{m+1}}{(a+b+c)^m(x+y)^m}$$

$$\begin{pmatrix} \because ax+by, bx+cy, cx+ay > 0, \\ as\ x,y > 0\ and\ a,b,c, > 0 \end{pmatrix}$$

$$\overset{Mitrinovic}{\geq} \frac{3^{m+1}}{(3\sqrt{3}R)^m(x+y)^m} = \frac{(\sqrt{3})^{2-m}}{(x+y)^mR^m} = RHS$$

SOLUTION 6.46

Proof by Adil Abdullayev – Baku – Azerbaidian

$$m_am_bm_c(h_a+h_b+h_c) \geq h_ah_bh_c(m_a+m_b+m_c)$$

$$LHS \geq RHS \Leftrightarrow \frac{h_a+h_b+h_c}{h_ah_bh_c} \geq \frac{m_a+m_b+m_c}{m_am_bm_c} \Leftrightarrow \sum_{cyc}\frac{1}{h_ah_b} \geq \sum_{cyc}\frac{1}{m_am_b} \dots (A)$$

$$h_ah_b \leq m_am_b \Leftrightarrow \frac{1}{h_ah_b} \geq \frac{1}{m_am_b} \Rightarrow (A) \Leftrightarrow LHS \geq RHS$$

SOLUTION 6.47

Proof by Soumitra Mandal-Chandar Nagore-India

We will prove, $\frac{\log x}{x^3-1} < \frac{x+1}{3(x^3+x)}$ *for all* $x \in (1,\infty)$

Let $f(x) = \frac{1}{3}\left(x-\frac{1}{x}\right)\left(1+\frac{x}{x^2+1}\right) - \log x$ *for all* $x \in (1,\infty)$

$$\therefore f'(x) = \frac{1}{3}\left(1+\frac{1}{x^2}\right)\left(1+\frac{x}{x^2+1}\right) - \frac{(x^2-1)^2}{3x(x^2+1)^2} - \frac{1}{x}$$

$$= \frac{(x^2+1)^2(x^2+x+1) - x(x^2-1)^2 - 3x(x^2+1)^2}{3(x^3+x)^2} =$$

$$= \frac{(x^2+1)^3 - 3x(x^2+1)^2 + 4x^3}{3(x^3+x)^2}$$

$$= \frac{(x-1)(x^2-2x^4+x^3-x^2+2x-1)}{3(x^3+x)^2} = \frac{(x-1)^2(x^4-x^3-x+1)}{3(x^3+x)^2}$$

$= \frac{(x-1)^4(x^2+x+1)}{3(x^3+x)^2} \geq 0$. *So,* $f'(x) \geq 0$ *for all* $x \in (1,\infty)$. *Hence,* f *is increasing on* $(1,\infty)$. $\therefore f(x) > f(1) = 0 \Rightarrow \frac{1}{3}\left(x-\frac{1}{x}\right)\left(1+\frac{x}{x^2+1}\right) \geq \log x$

$\therefore \frac{\log x}{x^3-1} < \frac{x+1}{3(x^3+x)}$. *Now putting* $= \sqrt[3]{\frac{a}{b}}$. *So,*

$$\frac{a-b}{\log a - \log b} > \frac{a\sqrt[3]{b}+b\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}} \quad \textit{(Proved)}$$

SOLUTION 6.48

Proof by Ravi Prakash - New Delhi – India

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) < 2\sin\left(\frac{\theta}{2}\right) \quad \left[\because 0 < \cos\left(\frac{\theta}{2}\right) < 1\right] < 2\left(\frac{\theta}{2}\right) = \theta$$

Also, $\sin\theta + \tan\theta > 2\theta$ *for* $0 < \theta < \frac{\pi}{2}$

$$\therefore \theta < \sin\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right) < \frac{\theta}{2} + \tan\left(\frac{\theta}{2}\right) < \tan\left(\frac{\theta}{2}\right) + \tan\left(\frac{\theta}{2}\right) = 2\tan\left(\frac{\theta}{2}\right)$$

Also

$$2\tan\frac{\theta}{2} < \frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^4\left(\frac{\theta}{2}\right)} = \sqrt{\frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)} \cdot \frac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)}}$$

$$\Rightarrow 2\tan\left(\frac{\theta}{2}\right) < \sqrt{\sin\theta\tan\theta} < \frac{1}{2}(\sin\theta + \tan\theta) < \frac{1}{2}(\tan\theta + \tan\theta) = \tan\theta$$

SOLUTION 6.49

Solution by Omran Kouba-Damascus-Syria

First, let us define $a_n = \ln\left(n+\frac{1}{2}\right) - \ln\left(n-\frac{1}{2}\right) - \frac{1}{n}$*. Note that*

$$a_n = \int_{-\frac{1}{2}}^{-\frac{1}{2}} \left(\frac{1}{n+t} - \frac{1}{n}\right) dt = -\frac{1}{n}\int_{-\frac{1}{2}}^{-\frac{1}{2}} \frac{t}{t+n} dt = -\frac{1}{n}\left(\int_0^{\frac{1}{2}} \frac{-t}{-t+n} dt + \int_0^{\frac{1}{2}} \frac{t}{t+n} dt\right) =$$

$$= \frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-t^2} dt. \text{ So,}$$

$$\frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-0} dt < a_n < \frac{1}{n}\int_0^{\frac{1}{2}} \frac{2t^2}{n^2-\frac{1}{4}} dt$$

Equivalently $\frac{1}{12n^3} < a_n < \frac{1}{12n\left(n^2-\frac{1}{4}\right)}$*. Using the trivial inequalities:*

$$\frac{1}{n^2} - \frac{1}{(n+1)^2} < \frac{2}{n^3}, \frac{2}{n\left(n^2-\frac{1}{4}\right)} < \frac{1}{\left(n-\frac{1}{2}\right)^2} - \frac{1}{\left(n+\frac{1}{2}\right)^2}$$

We conclude that $\frac{1}{24}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) < a_n < \frac{1}{24}\left(\frac{1}{\left(n-\frac{1}{2}\right)^2} - \frac{1}{\left(n+\frac{1}{2}\right)^2}\right)$*. Consequently*

$$\frac{1}{24(n+1)^2} < \sum_{k=n+1}^{\infty} a_k < \frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1)$$

Now,

$$\sum_{k=1}^{n} a_k = \ln\left(n+\frac{1}{2}\right) + \ln 2 - \sum_{k=1}^{n} \frac{1}{k}$$

So, $\sum_{k=1}^{n} a_k = \ln 2 - \gamma$*. Thus,* $\sum_{k=n+1}^{n} a_k \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \gamma$*. Combining this with (1) we get:*

$$\frac{1}{24(n+1)^2} < \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n+\frac{1}{2}\right) - \gamma < \frac{1}{24\left(n+\frac{1}{2}\right)^2} \quad (1)$$

Which is stronger than the proposed inequality.

Observation by authors : The inequality (1) was also discovered by the romanian mathematician TANASE NEGOI few years ago and it was published as a Math Note in GMA.

SOLUTION 6.50

Proof by Adil Abdullayev-Baku-Azerbadjan

Gerretsen :

$$p^2 \geq 16Rr - 5r^2$$

$$\left.\begin{array}{l} x = 2p - 3a \\ y = 2p - 3b \\ z = 2p - 3c \end{array}\right\} \Rightarrow x + y + z = 0 \Rightarrow$$

$$yz \cdot a^2 + xz \cdot b^2 + xy \cdot c^2 \leq 0 \Leftrightarrow p^2 \geq 16Rr - 5r^2.$$

SOLUTION 6.51

Solution by Soumitra Moukherjee - Chandar Nagore – India

Case I: When n is an even number i.e. $n = 2k$

$$x^2 + y^2 + z^2 \geq 2(-1)^{2k+1}(yz\cos nA + zx\cos nB + xy\cos nC)$$

$$\Leftrightarrow x^2 + y^2 + z^2 \geq -2yx\cos nA - 2zx\cos nB - 2xy\cos nC$$

$$\Leftrightarrow x^2 + 2x(z\cos nB + y\cos nC) + (z\cos nB + y\cos nC)^2 +$$

$$+y^2 + x^2 + 2yz\cos nA \geq (z\cos nB + y\cos nC)^2$$

$$\Leftrightarrow (x + z\cos nB + y\cos nC)^2 + y^2 + z^2 + 2yz\cos nA \geq$$

$$\geq z^2(1 - \sin^2 nB) + y^2(1 - \sin^2 nC) + 2yz\cos nB\cos nC$$

$$\Leftrightarrow (x + z\cos nB + y\cos nC)^2 + y^2\sin^2 nC + z^2\sin^2 nB +$$

$$+2yz\{\cos(x - nB - nC) - \cos nB\cos nC\} \geq 0$$

$$\Leftrightarrow (x + z\cos nB + y\cos nC)^2 + (y\sin nC - z\sin nB)^2 \geq 0$$

which is true

Case II: Let n be an odd integer i.e. $n = 2k + 1$

$$x^2 + y^2 + z^2 \geq 2(-1)^{2k+1}(yz\cos nA + zx\cos nB + xy\cos nC)$$

$$\Leftrightarrow x^2 + y^2 + z^2 \geq 2\sum_{cyc} yz\cos nA$$

$$\Leftrightarrow x^2 - 2x(z\cos nB + y\cos nC) + (z\cos nB + y\cos nC)^2$$

$$+y^2 + z^2 + 2yz\cos nA \geq (z\cos nB + y\cos nC)^2$$

$$\Leftrightarrow (x - z\cos nB - y\cos nC)^2 + y^2 + z^2 + 2yz\cos nA$$

$$\geq z^2(1 - \sin^2 nB) + y^2(1 - \sin^2 nC) + 2zy\cos nB\cos nC$$

$$\Leftrightarrow (x - z\cos nB - y\cos nC)^2 + (z\sin nB - y\sin nC)^2 \geq 0$$

which is true

Considering Case I and Case II,

$$x^2 + y^2 + z^2 \geq 2(-1)^{n+1}(yz\cos nA + zx\cos nB + xy\cos nC)$$

proved

SOLUTION 6.52

Proof by Soumava Pal – Kolkata – India

$$12\sqrt{3} > 20\left\{\left(12\sqrt{3}\right)^2 = 144 \times 3 = 432 > 400 = 20^2\right\}$$

$$\left(12\sqrt{3}r - 20r\right) > 0$$

$$\Rightarrow (R - 2r)\left(12\sqrt{3} - 20r\right) \geq 0 \quad \textit{(}R \geq 2r \textit{ by Euler)}$$

$$\Rightarrow 12\sqrt{3}Rr - 24\sqrt{3}r^2 + 40r^2 - 20Rr \geq 0$$

$$\Rightarrow \left(43 - 24\sqrt{3}\right)r^2 - 3r^2 + \left(12\sqrt{3} - 16\right)Rr - 4Rr \geq 0$$

$$\Rightarrow \left(43 - 24\sqrt{3}\right)r^2 + 4\left(3\sqrt{3} - 4\right)Rr \geq 4Rr + 3r^2$$

$$\Rightarrow 4R^2 + \left(43 - 24\sqrt{3}\right)r^2 + 4\left(3\sqrt{3} - 4\right)Rr \geq 4R^2 + 4Rr + 3r^2 \geq s^2 \textit{ (Gerretsen)}$$

$$\Rightarrow \left(2R + \left(3\sqrt{3} - 4\right)r\right)^2 \geq s^2 \Rightarrow 2R + \left(3\sqrt{3} - 4\right)r \geq s$$

SOLUTION 6.53

Proof by Soumava Chakraborty-Kolkata-India

$$\forall m, n > 0, \textit{ let } A(m,n) = \frac{m+n}{2}, G(m,n) = \sqrt{mn},$$

$$L(m,n) = \frac{m-n}{\ln m - \ln n}. \textit{ We have, } \sqrt[3]{G^2A} < L \textit{ (EB Leach, MC Scholander)}$$

$$\textit{Now, } A(e^x, e^{-x}) = \cosh x, G(e^x, e^{-x}) = 1, L(e^x, e^{-x}) = \sinh x$$

$$\textit{Applying } \sqrt[3]{G^2A} < L, \textit{ we get } \sqrt[3]{\cosh x} < \frac{\sinh x}{x} \textit{ (proved)}$$

SOLUTION 6.54

Proof by Kevin Soto Palacios-Huarmey-Peru

Tener presente lo siguiente:

$$w_a \leq \sqrt{p(p-a)}, w_b \leq \sqrt{p(p-b)}$$

La desigualdad es equivalente:

$$\sqrt{p(p-a)} + \sqrt{p(p-b)} + m_c \leq p\sqrt{3}$$

$$\left(\sqrt{p(p-a)} + \sqrt{p(p-b)}\right)^2 \leq \left(p(p-a) + p(p-b)\right)(1+1)$$

$$\left(\sqrt{p(p-a)} + \sqrt{p(p-b)}\right)^2 \leq \left(p^2 + p(p-a-b)\right)(2)$$

$$\left(\sqrt{p(p-a)} + \sqrt{p(p-c)}\right)^2 \leq \left(2p^2 - 2\left(\frac{a+b+c}{2}\right)\left(\frac{c-a-b}{2}\right)\right)$$

$$\textit{Pero: } m_c^2 \geq \frac{(b+c+a)(a+b-c)}{4}, -m_c^2 \leq \frac{(a+b+c)(c-a-b)}{4}$$

$$\left(\sqrt{p(p-a)}+\sqrt{p(p-b)}\right)^2 \leq (2p^2-2m_c^2) \rightarrow$$

$$\rightarrow \sqrt{p(p-a)}+\sqrt{p(p-b)} \leq \sqrt{2p^2-2m_c^2}$$

$$\sqrt{p(p-a)}+\sqrt{p(p-b)}+m_c \leq \sqrt{2p^2-2m_c^2}+m_c$$

Demostraremos que:

$$\sqrt{2p^2-2m_c^2}+m_c \leq p\sqrt{3} \rightarrow \sqrt{2p^2-2m_c^2} \leq \sqrt{3}p-m_c \Leftrightarrow \sqrt{3}p > m_c$$

Elevando al cuadrado la expresión:

$$\left(\sqrt{2p^2-2m_c^2}\right)^2 \leq \left(\sqrt{3}p-m_c\right)^2 \rightarrow 2p^2-2m_c^2 \leq 3p^2+m_c^2-2m_cp\sqrt{3} \rightarrow$$

$$\rightarrow p^2-2m_cp\sqrt{3}+3m_c^2 \geq 0 \rightarrow \left(p-\sqrt{3}m_c\right)^2 \geq 0$$

SOLUTION 6.55

Proof by Kevin Soto Palacios – Huarmey-Peru

$$\textit{1. } 2p^2 \leq 2(2R+r)^2+R^2$$

De la desigualdad Gerretsen:

$$p^2 \leq 4R^2+4Rr+3r^2$$

$$2p^2 \leq 8R^2+8Rr+6r^2 \leq 2(4R^2+4Rr+r^2)+R^2$$

$$\rightarrow 8R^2+8Rr+6r^2 \leq 9R^2+8Rr+2r^2 \Leftrightarrow R^2 \geq 4r^2 \rightarrow R \geq 2r \;\; \textit{(Desigualdad de Euler)}$$

$$\textit{2. } 6r(4R+r) \leq 2p^2$$

Por último, desde que:

$$p^2 \geq 16Rr-5r^2 \rightarrow 2p^2 \geq 32Rr-10r^2 \geq 24Rr+6r^2 \Leftrightarrow 8Rr \geq 16R^2 \rightarrow R \geq 2r$$

(Desigualdad de Euler)

SOLUTION 6.56

Proof by Soumava Chakraborty-Kolkata-India

$$\frac{9r}{2\Delta} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{9R}{4\Delta} \;\; \textit{(Leuenberger's Inequality)}$$

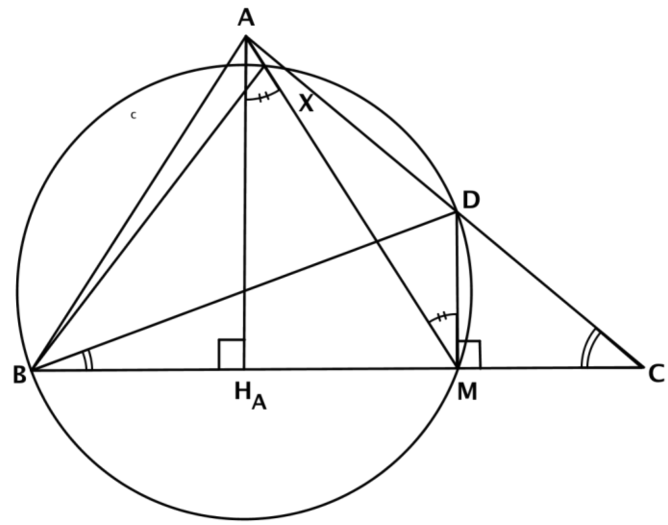
$$\textit{AM-HM} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c} = \frac{9r}{2Sr} = \frac{9r}{2\Delta}$$

$$9R^2 \geq \sum a^2 \geq \sum ab \Rightarrow \frac{9R^2}{abc} \geq \frac{\sum ab}{abc} \Rightarrow \sum \frac{1}{a} \leq \frac{9R^2}{4R\Delta} = \frac{9R}{4\Delta}$$

$$\frac{9r}{2\Delta} \leq \sum \frac{1}{a} \leq \frac{9R}{4\Delta}$$

SOLUTION 6.57

Proof by Soumava Pal – Kolkata – India



Without loss of generality

$$\angle B > \angle C$$

$$AM \rightarrow \textit{median}$$

$$AH_A \rightarrow \textit{altitude}$$

D is a point on AC, such that $\angle ABD = \angle B - \angle C$

$$\Rightarrow \angle DBC = \angle C \Rightarrow BD = DC$$

Also M midpoint of BC in isosceles $\Delta\ DBC$.

$$DM \perp BC (AH_A \perp BC) \Rightarrow DM \parallel AH_A \quad (1)$$

$$\frac{h_a}{m_a} = \cos\angle H_aAM = \cos\angle AMD \quad (\angle H_aAM = \angle AMD \text{ From (1)})$$

Drow circumcircle of $\Delta\ BMD$. Now $\angle BAC$ is acute, so A with lie outside the circle, since BD is diameter of the circle ($\angle BMD = 90°$).

Let the circle intersect AM at X. Since $\angle BAD$ is acute, X lies between A and M.

$BXDM$ is cyclic $\Rightarrow \angle XBD = \angle XMD = \angle H_AAM$

$$\Rightarrow \cos\angle XBD = \cos\angle H_aAM = \frac{h_a}{m_a}$$

$$\text{Now } \angle XBD < \angle ABD = B - C$$

$$\Rightarrow \cos\angle XBD > \cos(B - C) \Rightarrow \frac{h_a}{m_a} > \cos(B - C)$$

Equality holds if $\angle B = \angle C$

SOLUTION 6.58

Proof by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{a}{m_a} \overset{m_a \geq h_a}{\leq} \sum \frac{a}{h_a} = \frac{1}{2S} \cdot (a^2 + b^2 + c^2) \leq \frac{9R^2}{2S}$$

$$\sum \frac{a}{m_a} \overset{Chebyshev}{\geq} \frac{1}{3} \cdot (a + b + c) \cdot \left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right) \overset{RHS}{=}$$

$$= \frac{2}{3} \cdot p \cdot \left(\frac{1^2}{m_a} + \frac{1^2}{m_b} + \frac{1^2}{m_c}\right) \geq \frac{2}{3} \cdot p \cdot \frac{9}{m_a + m_b + m_c} =$$

$$= \frac{6p}{m_a + m_b + m_c} \geq \frac{6p}{\sum \sqrt{p(p-a)}} = \frac{6p}{\sqrt{p} \cdot \sum \sqrt{p-a}} \geq$$

$$\overset{CBS}{\geq} \frac{6p}{\sqrt{p} \cdot \sqrt{3 \cdot p}} \geq \frac{6 \cdot 3\sqrt{3} \cdot r}{\sqrt{3} \cdot p} = \frac{18r}{p}$$

SOLUTION 6.59

Proof by Soumava Chakraborty-Kolkata-India

In $\Delta\ ABC, \frac{2}{R} \leq \sqrt[4]{\frac{27}{S^2}} \leq \frac{1}{r}$ ***(Makowski's inequality)***

$$\sqrt[4]{\frac{27}{S^2}} \geq \frac{2}{R} \Leftrightarrow \frac{27}{S^2} \geq \frac{16}{R^4} \Leftrightarrow S^2 \leq \frac{27}{16} R^4$$

Isoperimetric inequality $\Rightarrow S \leq \frac{\sqrt{3}}{36}(a + b + c)^2$

$$\Rightarrow S \leq \frac{\sqrt{3}}{36}(4s^2) = \frac{\sqrt{3}S^2}{9} \Rightarrow S^2 \leq \frac{3s^4}{81} \Rightarrow S^2 \leq \frac{3}{81}\left(\frac{3\sqrt{3}R}{2}\right)^4 \left(s \leq \frac{3\sqrt{3}}{2} R \rightarrow Mitrinovic\right)$$

$= \frac{3}{81} \cdot \frac{81 \cdot 9}{16} R^4 = \frac{27}{16} R^4$ ***(Proof)***

$$\sqrt[4]{\frac{27}{S^2}} \leq \frac{1}{r} \Leftrightarrow \frac{27}{S^2} \leq \frac{1}{r^4} \Leftrightarrow \frac{27}{r^2 S^2} \leq \frac{1}{r^4} \Leftrightarrow S^2 \geq 27r^2$$

$S^2 \geq 16Rr - 5r^2$ ***(Gerretsen)***

it suffices to prove $16Rr - 5r^2 \geq 27r^2 \Leftrightarrow 16Rr \geq 32r^2 \Leftrightarrow R \geq 2r$***, which is true***

(Proof of 2)

SOLUTION 6.60

Proof by Marian Dincă-Romania

Use reverse Bernoulli inequality:

$$(1 - x)^y \leq 1 - xy, x \in (0, 1), y \in (0, 1)$$

The reverse Bernoulli inequality is equivalent Bernoulli inequality:

$$(1-a)^b \geq 1-ab, \text{ for } b \geq 1 \text{ and } a \in (0,1)$$

$$\text{Let } b = \frac{1}{y} \geq 1 \text{ and } a = xy \in (0,1)$$

$$\text{We obtain: } (1-xy)^{\frac{1}{y}} \geq 1-(xy)\cdot\frac{1}{y} = 1-x$$

$$(1-xy)^{\frac{1}{y}} \geq 1-x \Leftrightarrow 1-xy \geq (1-x)^y$$

Proof Lemma:

$$\text{Let } a = 1-x, b = 1-y, x, y \in (0,1)$$

$$a^b = (1-x)^{1-y} = \frac{1-x}{(1-x)^y} \geq \frac{1-x}{1-xy} = \frac{a}{1-(1-a)(1-b)} =$$

$$= \frac{a}{1-1+a+b-ab} = \frac{a}{a+b-ab}$$

SOLUTION 6.61

Proof by Hung Nguyen Viet-HaNoi City-VietNam

Without loss of generality we can assume that $a \geq b \geq c$. This implies

$$\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$$

Hence according to Chebyshev's inequality and using some known familiar inequalities, we have

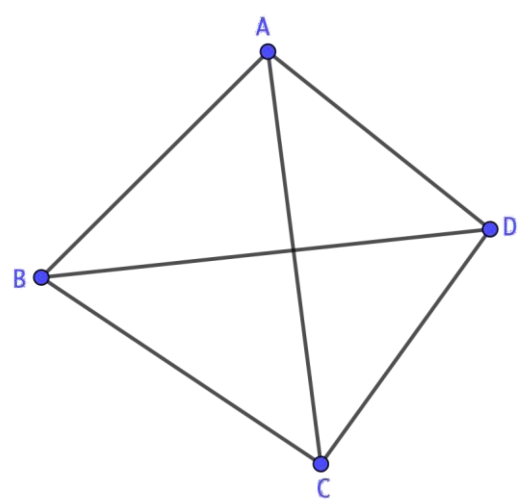
$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b} \geq \frac{1}{3}(a^n+b^n+c^n)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$$

$$\geq \frac{1}{3}(a^n+b^n+c^n)\cdot\frac{9}{2(a+b+c)} \geq \left(\frac{a+b+c}{3}\right)^n\cdot\frac{9}{2(a+b+c)}$$

$$= \left(\frac{2s}{3}\right)^n\cdot\frac{9}{4s} = \left(\frac{2}{3}\right)^{n-2}\cdot s^{n-1}$$

SOLUTION 6.62

Proof by Marian Ursărescu – Romania



We use Crelle – Van – Staudt identity: $6RV = T$, where

$$16T^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b) \Rightarrow$$

$$36R^2V^2 = T^2 \Rightarrow 36R^2V^2 = \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16}$$

$$\Rightarrow (a+b-c)(b+c-a)(c+a-b) = \frac{36\cdot 16\cdot R^2V^2}{a+b+c} \quad (1)$$

From (1) we must show:

$$\frac{36\cdot 16\cdot R^2V^2}{a+b+c} \geq 72V^2 \Leftrightarrow \frac{8R^2}{a+b+c} \geq 1 \Leftrightarrow$$

$$8R^2 \geq AB\cdot CD + AC\cdot BD + AD\cdot BC \Leftrightarrow$$

$$16R^2 \geq 2AB\cdot CD + AC\cdot BD + AB\cdot BC \quad (2)$$

But in any tetrahedron we have:

$$16R^2 \geq AB^2 + AC^2 + AD^2 + BC^2 + CD^2 + BD^2 \quad (3)$$

From (2)+(3) we must show:

$$AB^2 + AC^2 + AD^2 + BC^2 + CD^2 + BD^2 \geq 2AB\cdot CD + AC\cdot BD + AD\cdot BC$$

$$\Leftrightarrow (AB-CD)^2 + (AC-BD)^2 + (AB-BC)^2 \geq 0$$

(true, with equality for echifacial tetrahedron)

SOLUTION 6.63

Proof by Nguyen Hung Viet – Hanoi – Vietnam

By Cauchy – Schwarz inequality we have

$$\sum_{i=1}^{n} \frac{a_ib_i}{a_i+b_i} = \sum_{i=1}^{n}\left(\frac{a_ib_i}{a_i+b_i} - a_i\right) + \sum_{i=1}^{n} a_i$$

$$= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} \frac{a_i^2}{a_i+b_i}$$

$$\leq \sum_{i=1}^{n} a_i - \frac{(a_1+\cdots+a_n)^2}{(a_1+\cdots+a_n)+(b_1+\cdots+b_n)}$$

$$= \frac{(a_1+\cdots+a_n)(b_1+\cdots+b_n)}{(a_1+\cdots+a_n)+(b_n+\cdots+b_n)}.$$

This completes the proof.

SOLUTION 6.64

Proof by Soumava Chakraborty-Kolkata-India

$$a\,cos\frac{A}{2}+b\,cos\frac{B}{2}+c\,cos\frac{C}{2}\geq 3\sqrt[3]{abc}\sqrt[3]{cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}}\quad (AM\geq GM)$$

$$=3\sqrt[3]{abc}\sqrt[3]{\frac{s(s-a)s(s-b)s(s-c)}{a^2b^2c^2}}=3\sqrt[3]{abc}\sqrt[3]{\frac{s}{abc}\sqrt{s(s-a)(s-b)(s-c)}}=3\sqrt[3]{s\Delta}$$

$$=3\sqrt[3]{s^2r}\text{ it suffices to show }3\sqrt[3]{s^2r}\geq 9r$$

$$\Leftrightarrow s^2r\geq 27r^3\Leftrightarrow s^2\geq 27r^2.\text{ Now, }s^2\geq 16Rr-5r^2\text{ (Gerretsen)}$$

It suffices to prove $16Rr-5r^2\geq 27r^2\Leftrightarrow R\geq 2r$ *(true)*

$$a\,cos\frac{A}{2}+b\,cos\frac{B}{2}+c\,cos\frac{C}{2}\geq 9r$$

Also, $a\,cos\frac{A}{2}+b\,cos\frac{B}{2}+c\,cos\frac{C}{2}\overset{C-B-S}{\leq}\sqrt{a^2+b^2+c^2}\sqrt{\sum cos^2\frac{A}{2}}\leq\sqrt{9R^2}\sqrt{\frac{9}{4}}=\frac{9R}{2}$

SOLUTION 6.65

Proof by Soumitra Mandal-Chandar Nagore-India

Let $f(x)=x-\sin x\,(\cos x)^{-\frac{1}{3}}$ *for all* $0\leq x\leq\frac{\pi}{2}$

$$f'(x)=1-(\cos x)^{\frac{2}{3}}-\frac{\sin^2 x\,(\cos x)^{-\frac{4}{3}}}{3}$$

$$f''(x)=\frac{4}{9}\sin x\,(\cos x)^{-\frac{7}{3}}(\cos^2 x-1),\text{ hence }f''(x)\leq 0, f'(x)\leq f'(0)$$

$$\Rightarrow f(x)\leq f(0)=0\Rightarrow\cos x\leq\left(\frac{\sin x}{x}\right)^3\text{ (proved)}$$

SOLUTION 6.66

Proof by Soumitra Moukherjee - Chandar Nagore – India

LEMMA: If $A_1A_2\ldots A_n(n\geq 3)$ ***is a convex polygon,*** M ***is point inside the polygon then:***

$$\sum_{k=1}^{n}\frac{a_k}{d_k^2}\geq\frac{2n^2}{s}tan^2\frac{\pi}{n}$$

Proof:

$$\sum_{k=1}^{n}\frac{a_k}{d_k^2}=\sum_{k=1}^{n}\frac{1}{a_k}\left(\frac{a_k}{d_k}\right)^2.$$

Using Radon's Inequality

$$\sum_{k=1}^{n}\frac{a_k}{d_k^2}\geq\frac{\sum\left(\frac{a_k}{d_k}\right)^2}{\sum_{k=1}^{n}a_k}\geq\frac{\left(2n\tan\frac{\pi}{n}\right)^2}{2s}=\frac{2n^2\tan^2\frac{\pi}{n}}{s}$$

hence,

$$s\left(\sum_{k=1}^{n}\frac{a_k}{d_k^2}\right)\geq 2n^2\tan^2\frac{\pi}{n}\Rightarrow\frac{2s^2}{r^2}\geq 2n^2\tan^2\frac{\pi}{n}\Rightarrow s\geq nr\tan\frac{\pi}{n}$$

SOLUTION 6.67

Proof by Soumava Pal – Kolkata – India

$$a+b+c\geq 3\sqrt[3]{abc} \text{ (by AM-GM inequality)}$$

$$\Rightarrow\left(\frac{a+b+c}{3}\right)^3\geq abc$$

Putting $a=x+y, b=y+z, c=z+x$, we get,

$$8\left(\frac{x+y+z}{3}\right)^3\geq(x+y)(y+z)(z+x)$$

$$\left(\frac{x+y+z}{3}\right)^3\geq\frac{(x+y)(y+z)(z+x)}{8}$$

SOLUTION 6.68

Proof by Adil Abdullayev-Baku-Azerbadjan

$$a^2+b^2+c^2\geq\frac{36}{35}\left(\frac{abc}{s}+s^2\right)\Leftrightarrow 2(s^2-r^2-4Rr)\geq\frac{36}{35}\left(\frac{4Rrs}{s}+s^2\right)\Leftrightarrow$$

$$\Leftrightarrow 34s^2\geq 424Rr+70r^2.$$

Gerretsen $\Rightarrow s^2\geq 16Rr-5r^2\Leftrightarrow 34s^2\geq 544Rr-170r^2.$

$$544Rr-170r^2\geq 424Rr+70r^2\Leftrightarrow R\geq 2r.$$

SOLUTION 6.69

Proof by Adil Abdullayev – Baku – Azerbadjan

$$\sqrt[3]{\frac{R}{2S^2}}\leq\frac{1}{3}\cdot\frac{1}{r}\Leftrightarrow 2p^2r^2\geq 27r^3R\Leftrightarrow 2p^2\geq 27Rr.$$

Gerretsen $\Rightarrow p^2\geq 16Rr-5r^2\Leftrightarrow 2p^2\Leftrightarrow 32Rr-5r\cdot 2r\geq 32Rr-5Rr=27Rr.$

SOLUTION 6.70

Proof by Soumitra Moukherjee-Chandar Nagore-India

$$(a+b+c)^2-2(p^2+r^2+4Rr)\le 8R^2+\frac{4\Delta}{3\sqrt{3}}\Leftrightarrow 4p^2-2p^2-2r^2-8Rr\le 8R^2+\frac{4\Delta}{3\sqrt{3}}$$

$$\Leftrightarrow 2p^2\le 2r^2+8Rr+8R^2+\frac{4\Delta}{3\sqrt{3}}\Leftrightarrow p^2\le r^2+4Rr+4R^2+\frac{2\Delta}{3\sqrt{3}}$$

We need to prove, $p^2\le 4R^2+4Rr+3r^2\le r^2+4Rr+4R^2+\frac{2\Delta}{3\sqrt{3}}$

$\Leftrightarrow 2r^2\le \frac{2\Delta}{3\sqrt{3}}\Leftrightarrow 3\sqrt{3}r^2\le \Delta\Leftrightarrow 3\sqrt{3}r^2\le rp$***, where*** $\Delta=rp$.

$\Leftrightarrow 3\sqrt{3}r\le p\Leftrightarrow 6r\sqrt{3}\le a+b+c$***, which is true.*** $a^2+b^2+c^2\le 8R^2+\frac{4\Delta}{3\sqrt{3}}$ ***(proved)***

SOLUTION 6.71

Proof by Adil Adullayev-Baku-Azerbaidian

Lemma. $a^2+b^2+c^2\ge 4S\sqrt{3}$.

$$3\cdot\sum_{cyc}(a^2)^2\ge\left(\sum_{cyc}a^2\right)^2\ge 16S^2\cdot 3\Leftrightarrow LHS\ge RHS.$$

SOLUTION 6.72

Por Herón:

$$4F=\sqrt{(a+b+c)(b+c-a)(a+c-b)(b+a-c)}$$

$$16f^2=((b+a)^2-c^2)(c^2-(a-b)^2)$$

$$16f^2=(c^2((a+b)^2+(a-b^2)-(a^2-b^2)^2-c^4$$

$$16f^2=(c^2(2a^2+2b^2)+2a^2b^2-a^4-b^4-c^4)\to 16f^2$$

$$=-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2a^2c^2$$

$$16F^2=-A^4-B^4-C^4+2A^2B^2+2B^2C^2+2A^2C^2$$

Por CauChy:

$$16Ff+2a^2A^2+2b^2B^2+2c^2C^2\le$$

$$\le\sqrt{16f^2+2a^4+2b^3+c^4}\sqrt{16F^2+2A^4+2B^4+2C^4}$$

$$\sqrt{16f^2+2a^4+2b^3+c^4}\sqrt{16F^2+2A^4+2B^4+2C^4}\le(a^2+b^2+c^2)(A^2+B^2+C^2)$$

$$16Ff+2a^2A^2+2b^2B^2+2c^2C^2\le(a^2+b^2+c^2)(A^2+B^2+C^2)$$

SOLUTION 6.73

Proof by Kevin Soto Palacios – Peru

$$\prod\left(\frac{1-\cos A}{\cos A}\right) \geq \frac{8(\tan A+\tan B+\tan C)^3}{27(\tan A+\tan B)(\tan B+\tan C)(\tan A+\tan C)}$$

Recordar lo siguiente:

$$\tan x+\tan y=\frac{\sin(x+y)}{\cos x\cos y}, 1-\cos 2x=2\sin^2 x, \tan A+\tan B+\tan C$$
$$=\tan A\tan B\tan C$$

$$27\left(\frac{\sin C}{\cos A\cos B}\right)\left(\frac{\sin A}{\cos B\cos C}\right)\left(\frac{\sin B}{\cos A\cos C}\right)\prod\left(\frac{1-\cos A}{\cos A}\right) \geq 8\frac{\sin^3 A\sin^3 B\sin^3 C}{\cos^3 A\cos^3 B\cos^3 C}$$

$$27\frac{\sin A\sin B\sin C}{\cos^3 A\cos^3 B\cos^3 C}8\prod\sin^2\frac{A}{2} \geq 8\frac{\sin^3 A\sin^3 B\sin^3 C}{\cos^3 A\cos^3 B\cos^3 C} \to 27\prod\sin^2\frac{A}{2}$$
$$\geq \prod\sin^2 A$$

$$\Rightarrow 3\sqrt{3}\prod\sin\frac{A}{2} \geq \prod\sin A \to 3\sqrt{3} \geq 8\prod\cos\frac{A}{2} \to \frac{3\sqrt{3}}{8} \geq \frac{p}{4R} \to \frac{3\sqrt{3}}{2}R \geq p$$

2. $\frac{8(\tan A+\tan B+\tan C)^3}{27(\tan A+\tan B)(\tan B+\tan C)(\tan A+\tan C)} \geq 1$

Sea: $x=\tan A, y=\tan B, z=\tan c$

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(x+z)$$

$$(x+y)(y+z)(z+x)=xy(x+y)+yz(y+z)+zx(z+x)+2xyz$$

$$\Rightarrow 8(x+y+z)^3 \geq 27(x+y)(y+z)(x+z)$$

$$\Rightarrow 8(x^3+y^3+z^3)+24(x+y)(y+z)(x+z) \geq 27(x+y)(y+z)(z+x)$$

$$\Rightarrow 8(x^3+y^3+z^3) \geq 3(x+y)(y+z)(z+x)$$

$\Rightarrow$ **Utilizaremos:** $x^3+y^3 \geq xy(x+y), y^3+z^3 \geq yz(y+z), z^3+x^3 \geq xz(x+z)$

$$\Rightarrow 8(x^3+y^3+z^3) \geq 4xy(x+y)+4yz(y+z)+4zx(z+x) \geq 3(x+y)(y+z)(z+x)$$

$$\Rightarrow 4xy(x+y)+4yz(y+z)+4zx(z+x)$$
$$\geq 3xy(x+y)+3yz(y+z)+3zx(z+x)+6xyz$$

$\Rightarrow xy(x+y)+yz(y+z)+zx(z+x) \geq 6xyz$, **Dividiendo** $\div (xyz)$

$\Rightarrow \frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y} \geq 6$ **(Válido par:** $MA \geq MG$**)**

SOLUTION 6.74

Proof by Kevin Soto Palacios – Huarmey – Peru

Tener presente lo siguiente:

$$HA+HB+HC=2R+2r$$

$$IA+IB+IC\le 2R+2r\le 3r$$ **(INEQUALITY IN TRIANGLE 34-www.ssmrmh.ro)**

$$IA+IB+IC=r\left(\csc\frac{A}{2}+\csc\frac{B}{2}+\csc\frac{C}{2}\right)\ge 6r$$

1. $2r\le\frac{1}{3}(2R+r)\Leftrightarrow R\ge 2r\to\frac{1}{3}(2R+r)\le R\Leftrightarrow 2r\le R$

2. $2r\le\frac{r}{2}+\frac{1}{4}(IA+IB+IC)\Leftrightarrow IA+IB+IC\ge 6r\to$

$$\to\frac{r}{2}+\frac{1}{4}(IA+IB+IC)\le R\Leftrightarrow IA+IB+IC\le 3R\ \wedge r\le\frac{R}{2}$$

SOLUTION 6.75

Proof by Adil Abdullayev – Baku – Azerbaidian

$$\sum_{cyc}\frac{2\sqrt{s(s-a)}}{a}\ge 3\sqrt{3}$$

$$\left.\begin{matrix}a=x+y\\ b=y+z\\ c=z+x\end{matrix}\right\}\Rightarrow\sum_{cyc}\frac{\sqrt{(x+y+z)x}}{y+z}\ge\frac{3\sqrt{3}}{2}\quad\ldots(A)$$

Homogen $\Rightarrow x+y+z=3$. **(A)** $\Leftrightarrow\sum_{cyc}\frac{\sqrt{x}}{3-x}\ge\frac{3}{2}$ **... (B)**

$$x^2+\sqrt{x}+\sqrt{x}\overset{AM-GM}{\ge}3\cdot\sqrt[3]{x^2\cdot\sqrt{x}\cdot\sqrt{x}}=3x\Leftrightarrow\frac{\sqrt{x}}{3-x}\ge\frac{x}{2}\ldots\text{(C)}$$

$$\sum_{cyc}\frac{\sqrt{x}}{3-x}\overset{(C)}{\ge}\sum_{cyc}\frac{x}{2}=\frac{3}{2}\Leftrightarrow\text{(B)}$$

SOLUTION 6.76

Proof by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC:

$$a^4+b^4+c^4+abc(a+b+c)\ge a^3(b+c)+b^3(a+c)+c^3(a+b)$$

$$\Rightarrow(a^4-a^3b-a^3c+a^2bc)+(b^4-b^3a-b^3c+b^2ac)+$$

$$+(c^4-c^3a-c^3b+c^2ab)\ge 0$$

$$\Rightarrow a^2(a^2-ab-ac+bc)+b^2(b^2-ba-bc+ac)+$$

$$+c^2(c^2-ca-cb+ab)\ge 0$$

$$\Rightarrow a^2\big(a(a-b)-c(a-b)\big)+b^2\big(b(b-a)-c(b-a)\big)+$$

$$+c^2\big(c(c-a)-b(c-a)\big)\ge 0$$

$$\Rightarrow a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b)\ge 0$$ ***(Desigualdad Schur)***

SOLUTION 6.77

Proof by Hamza Mahmood-Lahore-Pakistan

Since, $x, y, z > 0 \Rightarrow$ **there exists** $a, b, c > 0$ **such that** $x = a^6, y = b^6, z = c^6$

$$\frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} = \frac{a^6}{b^6} + \frac{b^3}{c^3} + \frac{c^2}{a^2}$$

$$= \frac{a^6}{b^6} + \frac{b^3}{2c^3} + \frac{b^3}{2c^3} + \frac{c^2}{3a^2} + \frac{c^2}{3a^2} + \frac{c^2}{3a^2} \geq 6\left(\frac{a^6}{b^6} \cdot \frac{b^3}{2c^3} \cdot \frac{b^3}{2c^3} \cdot \frac{c^2}{3a^2} \cdot \frac{c^2}{3a^2} \cdot \frac{c^2}{3a^2}\right)^{\frac{1}{6}} =$$

$$= 6\left(\frac{a^6 b^6 c^6}{2^2 \cdot 3^3 \cdot b^6 c^6 a^6}\right)^{\frac{1}{6}}$$

$$\Rightarrow \frac{a^6}{b^6} + \frac{b^3}{c^3} + \frac{c^2}{a^2} \geq 6\left(\frac{1}{2^2 \cdot 3^3}\right)^{\frac{1}{6}} \qquad (A)$$

Now Since $2^{10} > 2^5 = 32 > 27 \Rightarrow 3^3 \Rightarrow \frac{2^6}{2^2} \cdot 2^6 > 3^3 \Rightarrow \frac{2^6}{2^2} \cdot 2^6 \cdot \left(\frac{3^6}{3^3} \cdot \frac{1}{2^6}\right) >$

$$> 3^3\left(\frac{3^6}{3^3} \cdot \frac{1}{2^6}\right) \Rightarrow \frac{2^6}{2^2} \cdot \frac{3^6}{3^3} > \frac{3^6}{2^6} \Rightarrow 6^6\left(\frac{1}{2^2 \cdot 3^3}\right) > \left(\frac{3}{2}\right)^6 \Rightarrow 6\left(\frac{1}{2^2 \cdot 3^3}\right)^{\frac{1}{6}} > \frac{3}{2} \qquad (B)$$

From (A) and (B),

$$\frac{a^6}{b^6} + \frac{b^3}{c^3} + \frac{c^2}{a^2} > \frac{3}{2}$$

Therefore, $\frac{x}{y} + \sqrt{\frac{y}{z}} + \sqrt[3]{\frac{z}{x}} > \frac{3}{2}, x, y, z > 0$

SOLUTION 6.78

Proof by Rovsen Pirkuliyev – Sumgait – Azerbaidjian

Denote $a + b + c = x$

Using AM-GM $\Rightarrow$

$$\sqrt{\frac{b+c}{a}} \leq \frac{\frac{b+c}{a} + 1}{2} = \frac{x}{2a} \Rightarrow \sqrt{\frac{a}{b+c}} = \frac{2a}{x}$$

$$\sqrt{\frac{c+a}{b}} \leq \frac{\frac{c+a}{b} + 1}{2} = \frac{x}{2b} \Rightarrow \sqrt{\frac{b}{c+a}} \geq \frac{2b}{x}$$

and $\sqrt{\frac{c}{a+b}} \geq \frac{2c}{x}$

$$Hence\ \sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+c}}+\sqrt{\frac{c}{a+b}}\geq\frac{2a}{x}+\frac{2b}{x}+\frac{2c}{x}=2$$

Equality is possible

$$\frac{b+c}{a}=\frac{a+c}{b}=\frac{a+b}{c}=1\Rightarrow impossible$$

$$\sum\sqrt{\frac{a}{b+c}}>2$$

SOLUTION 6.79

Proof by Soumitra Mandal-Chandar Nagore-India

We know,

$$\tan A=\frac{abc}{R}\cdot\frac{1}{b^2+c^2-a^2},\tan B=\frac{abc}{R}\cdot\frac{1}{a^2+c^2-b^2}\ and$$

$$\tan C=\frac{abc}{R}\cdot\frac{1}{a^2+b^2-c^2}.\ So,$$

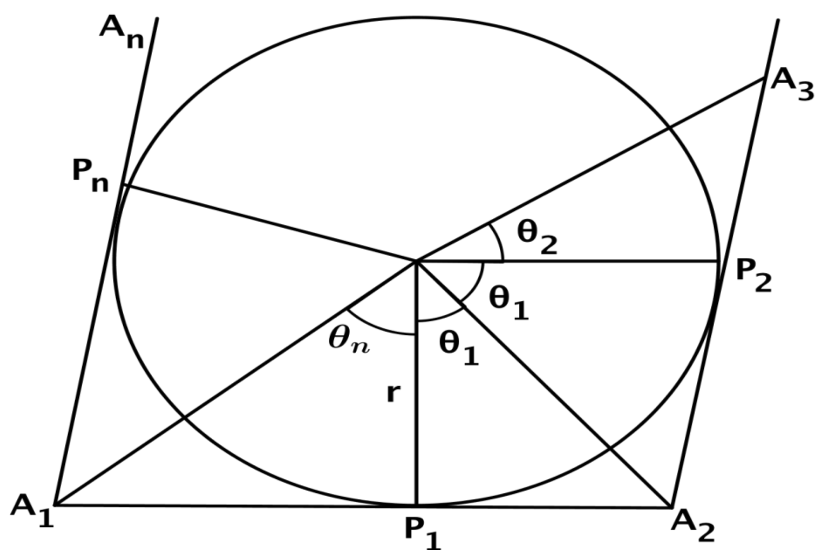
$$\therefore\sum_{cyc}\tan A=\frac{abc}{R}\sum_{cyc}\frac{1}{a^2+b^2-c^2}$$

$$\geq\frac{3abc}{R}\sqrt[3]{\prod_{cyc}\frac{1}{(a^2+b^2-c^2)}}\geq\frac{3abc}{R}\sqrt[3]{\frac{1}{a^2b^2c^2}}\left[\because a^2b^2c^2\geq\prod_{cyc}(a^2+b^2-c^2)\right]$$

$$=\frac{3}{R}\sqrt[3]{abc}=3\sqrt[3]{abc/R^3}=3\sqrt[3]{\frac{4\Delta}{R^2}}\ \ [\because abc=4R\Delta]\ (proved)$$

SOLUTION 6.80

Proof by Ravi Prakash - New Delhi – India



Suppose A_1A_2 touch the circle at P_1, A_2A_3 at P_2 etc.

Note $A_2P_1 = A_2P_2 = r\tan\theta_1$, $A_1P_n = P_1A_1 = r\tan\theta_n$

Now,

$$2S = A_1A_2 + A_2A_3 + \cdots + A_nA_1 =$$

$$= (r\tan\theta_1 + r\tan\theta_n) + (r\tan\theta_1 + r\tan\theta_2) + \cdots + (r\tan\theta_{n-1} + r\tan\theta_n) =$$

$$= 2r[\tan\theta_1 + \tan\theta_2 + \cdots + \tan\theta_n]$$

where $\theta_1 + \cdots + \theta_n = \pi$, and $\theta_i > 0$

$$\Rightarrow \frac{2S}{n} \geq 2r\tan\left(\frac{\theta_1 + \theta_2 + \cdots + \theta_n}{n}\right) \Rightarrow S \geq rn\tan\left(\frac{\pi}{n}\right)$$

[$\tan\theta$ is a convex function on $\left(0, \frac{\pi}{2}\right)$]

Also, $\Delta_1 = arc\,(OA_1A_2) = \frac{1}{2}(A_1A_2)r = \frac{1}{2}(P_1A_1 + P_1A_2)r$

Similarly for other triangles.

$$F = \sum_{k=1}^{n} \Delta_k = \frac{1}{2}r\sum_{k=1}^{n}(P_kA_k + P_kA_{k+1}) = \frac{1}{2}r(2s) = rs$$

where s = semiperimeter of polygon.

We have:

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{n} = \frac{4s^2}{n} = \frac{4}{nr}(F)(s)$$

But $s \geq nr\tan\left(\frac{\pi}{n}\right)$ [see (1)]

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq 4F\tan\left(\frac{\pi}{n}\right)$$

SOLUTION 6.81

Proof by Adil Abdullayev-Baku-Azerbaidian

$$a \leq b \leq c \Rightarrow \min(a,b,c) = a \quad \max(a,b,c) = c.$$

$$R = \frac{abc}{4 \cdot \frac{1}{2}} = \frac{abc}{2} \Rightarrow abc = 2R$$

$$\sum_{cyc} \sin A = \frac{a+b+c}{2R} = \frac{a+b+c}{abc}$$

$$a \leq \frac{a^2 + b^2 + c^2}{abc \cdot \frac{a+b+c}{abc}} \leq c \Leftrightarrow a^2 + ab + ac \leq a^2 + b^2 + c^2 \leq ac + bc + c^2$$

SOLUTION 6.82

Solution by Soumitra Moukherjee-Chandar Nagore-India

Let $f: [a, a+1] \to \mathbb{R}, g: [a, a+1] \to \mathbb{R}$ ***be two functions defined as***

$$f(x) = x^{\sqrt{x}} \text{ for all } x \in [a, a+1] \text{ and } g(x) = \sqrt{x} \text{ for all } x \in [a, a+1].$$

Now, f ***and*** g ***are both continuous on*** $[a, a+1]$,

f ***and*** g ***are both differentiable on*** $[a, a+1]$,

then by Cauchy Mean Value Theorem

$$\frac{f(a+1)-f(a)}{g(a+1)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \text{ where } \xi \in (a, a+1).$$

$$\frac{(a+1)^{\sqrt{a+1}} - a^{\sqrt{a}}}{\sqrt{a+1} - \sqrt{a}} = \xi^{\sqrt{\xi}}(2 + \ln \xi)$$

$$\Rightarrow \frac{\sqrt{a+1} - \sqrt{a}}{(a+1)^{\sqrt{a+1}} - a^{\sqrt{a}}} = \frac{1}{\xi^{\sqrt{\xi}}(2 + \ln \xi)} < \frac{1}{2\xi^{\sqrt{\xi}}} < \frac{1}{2a^{\sqrt{a}}}$$

$$\frac{\sqrt{a+1} - \sqrt{a}}{(a+1)^{\sqrt{a+1}} - a^{\sqrt{a}}} < \frac{1}{2a^{\sqrt{a}}}$$

SOLUTION 6.83

Proof by Dan Radu Seclaman-Romania

Let $a, b, c \in [0, \infty)$, ***such that*** $a + b + c = 3$.

a) Prove that: $(1-a)(1-b)(1-c) + 2 \geq 2abc$.

b) Find the maximum value of the following expression:

$$E(a, b, c) = 2(a^3 + b^3 + c^3) + 15(ab + bc + ca) + 6abc.$$

a) We have $(1-a)(1-b)(1-c) = -2 + ab + bc + ca - abc$.

But $(a+b+c)(ab+bc+ca) \geq 9abc$, ***wherefrom it follows***

$ab + bc + ca \geq 3abc$ ***and so*** $(1-a)(1-b)(1-c) \geq 2abc - 2$

(we have equality if $a = b = c = 1$***).***

b) As $x^3 + y^3 + z^3 = 3xyz + (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, ***for any*** $x, y, z \in \mathbb{R}$,

and $a + b + c = 3$, ***we obtain (taking into account point a)):***

$$(a-1)^3 + (b-1)^3 + (c-1)^3 = 3(a-1)(b-1)(c-1) \leq 6 - 6abc.$$

So: $\sum a^3 - 3\sum a^2 + 6 \leq 6 - 6abc$. ***Because*** $\sum a^2 = 9 - 2\sum ab$, ***we obtain that:***

$$a^3 + b^3 + c^3 + 6(ab + bc + ca + abc) \leq 27. \textit{ (1)}$$

As $\sum a^3 = 3abc + 3(9 - 3\sum ab) \leq 30 - 9\sum ab$*, we deduce that*

$a^3 + b^3 + c^3 + 9(ab + bc + ca) \leq 30$. *(2) (we took into account that*

$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1$*)*

Adding the relationships (1) and (2) we obtain that $E(a, b, c) \leq 57$ *with equality if and only if* $a = b = c = 1$*. So* $\max_{a,b,c\geq 0} E(a, b, c) = 57$ *and its realized for* $a = b = c = 1$*.*

SOLUTION 6.84

Proof by Soumitra Mandal-Chandar Nagore-India

Elementary results, if $X \in \mathcal{M}_n(\mathbb{R})$ *then* $I_n \cdot X = X$

and $I_n^2 = I_n$*. Now,* $(A - B)^2 = O_n \Rightarrow A^2 + B^2 = AB + BA = 2AB [\because AB = BA]$

$$\therefore I_n - a(A + B) + a^2AB = I_n^2 - a(I_nA + I_nB) + \frac{a^2}{2}(A^2 + B^2)$$

$$= \frac{1}{2}(I_n - aA)^2 + \frac{1}{2}(I_n - aB)^2, \therefore \det(I_n - a(A + B) + a^2AB)$$

$$= \det\left(\frac{1}{2}(I_n - aA)^2 + \frac{1}{2}(I_n - aB)^2\right)$$

$$= \frac{1}{2^n}\det((I_n - aA)^2 + (I_n - aB)^2) [\therefore \det(aX) = a^n \det(X)]$$

$$= \frac{1}{2^n}\det\{(I_n - aA + i(I_n - aB))(I_n - aA - i(I_n - aB))\} \textit{ [where } i = \sqrt{-1}]$$

$$= \frac{1}{2^n}\det\left\{(I_n - aA + i(I_n - aB))\overline{(I_n - aA + i(I_n - aB))}\right\}$$

$$= \frac{1}{2^n}\det\{I_n - aA + i(I_n - aA)\} \cdot \det\left\{\overline{I_n - aA + i(I_n - aB)}\right\}$$

$$= \frac{1}{2^n}[\det\{I_n - aA + i(I_n - aB)\}]^2 \geq 0 \left[\because \det(X) = \det(\overline{X})\right]$$

$$\therefore \det(I_n - a(A + B) + a^2AB) \geq 0 \textit{ (proved)}$$

SOLUTION 6.85

Proof by Mehmet Sahin-Ankara-Turkey

$$\sum m_a^2 = \frac{3}{4}(a^2 + b^2 + c^2), \sum r_a = r + 4R, \sum a^2 \leq 9R^2$$

$$\frac{m_a^2 + m_b^2 + m_c^2}{r_a + r_b + r_c} = \frac{\frac{3}{4}(a^2 + b^2 + c^2)}{r + 4R} \leq \frac{\frac{3}{4} \cdot 9R^2}{1 + 4R} \leq 2R - r$$

$$\Leftrightarrow 27R^2 \leq 4(r + 4R)(2R - 1) \Leftrightarrow 5R^2 - 8Rr - 4r^2 \geq 0$$

$$\Leftrightarrow (R-2r)(5R+2r) \geq 0 \Leftrightarrow R \geq 2r \quad \textit{(Euler)}$$

SOLUTION 6.86

Proof by Soumava Chakraborty-Kolkata-India

$$\sqrt{2} \overset{(1)}{<} \frac{\sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab}}{\frac{a+b}{2} - \sqrt{ab}} \overset{(2)}{<} 2$$

$$\text{Let } \sqrt{\frac{a^2+b^2}{2}} = Q, \frac{a+b}{2} = A, \sqrt{ab} = G$$

$$(2) \Leftrightarrow \frac{Q-G}{A-G} < 2 \text{ (of course, } Q > A > G) \Leftrightarrow Q - G < 2A - 2G \Leftrightarrow Q + G < 2A \Leftrightarrow$$

$$\Leftrightarrow \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} < a + b \Leftrightarrow \frac{a^2+b^2}{2} + ab + \sqrt{2ab(a^2+b^2)} < a^2 + b^2 + 2ab \Leftrightarrow$$

$$\Leftrightarrow (a^2+b^2) + 2ab - 2\sqrt{2ab(a^2+b^2)} > 0 \Leftrightarrow \left(\sqrt{a^2+b^2} - \sqrt{2ab}\right)^2 > 0 \to \textit{true} \Rightarrow$$

$$\Rightarrow \text{(2) is true} \therefore Q + G < 2A \to \text{(2a)}$$

$$(1) \Leftrightarrow \frac{(Q-G)^2}{(A-G)^2} > 2 \Leftrightarrow Q^2 + G^2 - 2QG > 2(A^2 + G^2 - 2AG) \Leftrightarrow \frac{a^2+b^2}{2} + ab - 2\sqrt{\frac{ab(a^2+b^2)}{2}} >$$

$$> 2 \cdot \frac{(a+b)^2}{4} + 2ab - 2(a+b)\sqrt{ab} \Leftrightarrow (a+b)\sqrt{ab} - ab > \sqrt{\frac{ab(a^2+b^2)}{2}} \Leftrightarrow$$

$$\Leftrightarrow (a+b) - \sqrt{ab} > \sqrt{\frac{a^2+b^2}{2}} \Leftrightarrow 2A - G > Q \to \text{true by (2a)} \Rightarrow \text{(1) is true (Done)}$$

SOLUTION 6.87

Proof by Ravi Prakash-New Delhi-India

Let's take O as the origin,

$$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c} \text{ then } |\vec{a}| = |\vec{b}| = |\vec{c}| = R,$$

where R is circumcentre of triangle.

$$\text{Centroid of triangle is } G\left(\frac{1}{3}\left(\vec{a} + \vec{b} + \vec{c}\right)\right)$$

But G divides OH in the ratio $1:2$. Thus,

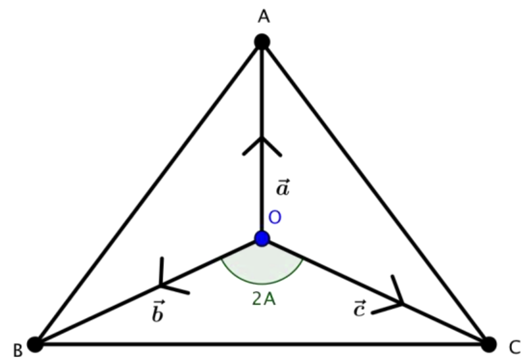
$$\overrightarrow{OH} = \vec{a} + \vec{b} + \vec{c}$$

$$\text{Also, } \overrightarrow{OI} = \frac{a\vec{a} + b\vec{b} + c\vec{c}}{a+b+c}$$

$$\textit{Area of } \Delta\ OIH = \frac{1}{2}\left|\overrightarrow{OI} \times \overrightarrow{OH}\right|. \textit{ Now, } \overrightarrow{OI} \times \overrightarrow{OH} =$$

$$= \frac{1}{2s}\left(a\vec{a} + b\vec{b} + c\vec{c}\right) \times \left(\vec{a} + \vec{b} + \vec{c}\right)$$

$$= \frac{1}{2s}\left[(a-b)\left(\vec{a} \times \vec{b}\right) + (b-c)\left(\vec{b} \times \vec{c}\right) + (c-a)\left(\vec{c} \times \vec{a}\right)\right]$$



$$a = BC, b = CA, c = AB$$

But $\vec{b} \times \vec{c} = R^2 \sin 2A\, \hat{n}$ *where* $\hat{n}$ *is unit normal to plane containing* A, B, C

Thus, $\overrightarrow{OI} \times \overrightarrow{OH} = \frac{R^2\Delta}{2s}\hat{n}$

where

$$\Delta = (a-b)\sin 2C + (b-c)\sin 2A + (c-a)\sin 2B$$

$$= \frac{1}{R}\left[(b-c)a\cos A + (c-a)b\cos B + (a-b)c\cos C\right]$$

$$= \frac{1}{2Rabc}\left[\begin{matrix} a^2(b-c)(b^2+c^2-a^2) + b^2(c-a)(c^2+a^2-b^2) \\ +c^2(a-b)(a^2+b^2-c^2) \end{matrix}\right]$$

$$= \frac{1}{2Rabc}\left[\begin{matrix} a^2b^2(b-c+c-a) + a^2c^2(b-c+a-b) + \\ +b^2c^2(c-a+a-b) - a^4(b-c) - b^4(c-a) \\ -c^4(a-b) \end{matrix}\right]$$

$$= \frac{1}{2Rabc}\left[a^2b^2(b-a) + c^2a^2(a-c) + b^2c^2(c-b) - a^4(b-c) - b^4(c-a) - c^4(a-b)\right]$$

$$= \frac{1}{2Rabc}\left[\Delta_1 - \Delta_2\right]$$

where

$$\Delta_1 = \begin{vmatrix} a^2b^2 & a^2c^2 & b^2c^2 \\ c & b & a \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{a^2b^2c^2}\begin{vmatrix} a^2b^2c^2 & a^2b^2c^2 & a^2b^2c^2 \\ c^3 & b^3 & a^3 \\ c^2 & b^2 & a^2 \end{vmatrix}$$

$$=\begin{vmatrix}1 & 0 & 0\\ c^3 & b^3-c^3 & a^3-b^3\\ c^2 & b^2-c^2 & a^2-b^2\end{vmatrix}=\begin{vmatrix}b^3-c^3 & a^3-b^3\\ b^2-c^2 & a^2-b^2\end{vmatrix}$$

$$=(b-c)(a-b)\begin{vmatrix}b^2+c^2+bc & a^2+b^2+ab\\ b+c & a+b\end{vmatrix}$$

$$R_1\to R_1-bR_2$$

$$\Delta_1=(b-c)(a-b)\begin{vmatrix}c^2 & a^2\\ b+c & a+b\end{vmatrix}=(b-c)(a-b)\begin{vmatrix}c^2-a^2 & a^2\\ c-a & a+b\end{vmatrix}$$

$$=(b-c)(a-b)(c-a)\begin{vmatrix}c+a & a^2\\ 1 & a+b\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(ab+bc+ca)$$

$$\text{and } \Delta_2=\begin{vmatrix}a^4 & b^4 & c^4\\ a & b & c\\ 1 & 1 & 1\end{vmatrix}=\begin{vmatrix}a^4-b^4 & b^4-c^4 & c^4\\ a-b & b-c & c\\ 0 & 0 & 1\end{vmatrix}$$

$$=(a-b)(b-c)\begin{vmatrix}a^3+a^2b+ab^2+b^3 & b^3+b^2c+bc^2+c^3\\ 1 & 1\end{vmatrix}$$

$$=(a-b)(b-c)[a^3-c^3+b(a^2-c^2)+b^2(a-c)]$$

$$=(a-b)(b-c)(a-c)[a^2+b^2+c^2+ab+bc+ca]$$

$$\text{Thus, } \Delta_1-\Delta_2$$

$$=(a-b)(b-c)(c-a)[ab+bc+ca+a^2+b^2+c^2+ab+bc+ca]$$

$$=(a-b)(b-c)(c-a)(2s)^2=4(a-b)(b-c)(c-a)s^2$$

$$\therefore \Delta=\frac{4(a-b)(b-c)(c-a)s^2}{2Rabc}$$

$$\Rightarrow\left|\overrightarrow{OI}\times\overrightarrow{OH}\right|=\frac{4(a-b)^2(b-c)^2(c-a)^2s^4}{R^2(abc)^2}\cdot\frac{R^4}{4s^2}=\frac{(a-b)^2(b-c)^2(c-a)^2s^2R^2}{(abc)^2}$$

$$\text{But } R=\frac{abc}{4S}$$

$$\therefore\left|\overrightarrow{OI}\times\overrightarrow{OH}\right|^2=\frac{(a-b)^2(b-c)^2(c-a)^2s^2}{16S^2}=\frac{(a-b)^2(b-c)^2(c-a)^2}{16r^2}$$

$$\text{Hence } (area\ of\ \Delta\ OIH)^2=\frac{1}{64r^2}(a-b)^2(b-c)^2(c-a)^2$$

SOLUTION 6.88

Proof by Soumitra Moukherjee - Chandar Nagore – India

$$\sum_{cyc}\frac{a}{xb+yc}=\sum_{cyc}\frac{a^2}{xab+yca}\geq$$

$$\geq \frac{(a+b+c)^2}{(x+y)(ab+bc+ca)} \text{ [Applying Bergstrom's Inequality]}$$

$$\frac{(a+b+c)^2}{(x+y)(ab+bc+ca)} \geq \frac{3\sqrt{3\left(\sum_{cyc} a^2\right)}}{(x+y)(a+b+c)}$$

$$\frac{p^2}{q} \geq \frac{3\sqrt{3(p^2-2q)}}{p} \text{ where } p = a+b+c \text{ and } q = ab+bc+ca$$

$$p^3 \geq 3q\sqrt{3(p^2-2q)} \Leftrightarrow p^6 \geq 27q^2(p^2-2q) \Leftrightarrow p^6 - 27q^3 - 27q^2(p^2-3q) \geq 0$$

$$\Leftrightarrow (p^2-3q)(p^4+3p^2q-18q^2) \geq 0 \Leftrightarrow (p^2-3q)^2(p^2+6q) \geq 0, \text{ which is true}$$

$$\text{again, } 3\left(\sum_{cyc} a^2\right) \geq \left(\sum_{cyc} a\right)^2 \Rightarrow \frac{3\sqrt{3\left(\sum_{cyc} a^2\right)}}{(x+y)\left(\sum_{cyc} a\right)} \geq \frac{3}{x+y}$$

$$\sum_{cyc} \frac{a}{xb+yc} \geq \frac{(a+b+c)^2}{(x+y)(ab+bc+ca)} \geq \frac{3\sqrt{3\left(\sum_{cyc} a^2\right)}}{(x+y)(a+b+c)} \geq \frac{3}{x+y}$$

SOLUTION 6.89

Proof by Nirapada Pal-Jhargram-India

$$\frac{1}{\sin B} + \frac{1}{\sin C} \overset{AHM}{\geq} \frac{4}{\sin B + \sin C} = \frac{2}{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \frac{2}{\sin\left[\frac{\pi}{2}-\frac{A}{2}\right]\cos\frac{B-C}{2}} \geq \frac{2}{\cos\frac{A}{2}} \text{ since } \cos\frac{B-C}{2} \leq 1$$

SOLUTION 6.90

Proof by George Apostolopoulos-Messolonghi-Greece

From Cauchy – Schwarz Inequality, we have

$$\left(\sqrt{bc(s-a)} + \sqrt{ca(s-b)} + \sqrt{ab(s-c)}\right)^2 \leq$$

$$\leq (bc+ca+ab)(s-a+s-b+s-c) \leq$$

$$(a^2+b^2+c^2)\cdot s \leq 9R^2 \cdot s$$

Namely:

$$\sqrt{bc(s-a)} + \sqrt{ca(s-b)} + \sqrt{ab(s-c)} \leq 3R\cdot\sqrt{s}$$

Equality holds when the triangle is equilateral.

SOLUTION 6.91

Proof by Ravi Prakash-New Delhi-India

We have

$$(a-k)^n=\sum_{r=0}^{n}(-1)^r\binom{n}{r}a^{n-r}k^r \Rightarrow \sum_{k=0}^{n}(-1)^k\binom{n}{k}(a-k)^n=$$

$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\sum_{r=0}^{n}(-1)^r\binom{n}{r}a^{n-r}k^r\right]=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}(-1)^r\left\{\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^r\right\}$$

We have

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k \Rightarrow n(1+x)^{n-1}=\sum_{k=1}^{n}\binom{n}{k}x^{k-1}k \Rightarrow nx(1+x)^{n-1}=\sum_{k=1}^{n}\binom{n}{k}x^k k$$

$$n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2}=\sum_{k=0}^{n}k^2\binom{n}{k}x^{k-1}$$

$$\Rightarrow nx(1+x)^{n-1}+n(n-1)x^2(1+x)^{n-2}=\sum_{k=1}^{n}k^2\binom{n}{k}x^k$$

$$\Rightarrow n(1+x)^{n-1}+3n(n-1)x(1+x)^{n-2}+n(n-1)(n-2)x^2(1+x)^{n-3}=$$

$$=\sum_{k=1}^{n}k^3\binom{n}{k}x^{k-1}$$

Repeating above procedure r times, we get

$$\sum_{k=1}^{n}(-1)^k(k^r)\binom{n}{k}=0; 1\le r\le n-1 \Rightarrow \sum_{k=0}^{n}(-1)^k k^r\binom{n}{k}=0; 1\le r\le n-1$$

Also, $\sum_{k=1}^{n}(-1)^k k^n\binom{n}{k}=(-1)^n(n!)$

And $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$

Thus,

$$\sum_{k=0}^{n}(-1)^k(1-k)^n\binom{n}{k}=\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\left[\sum_{k=0}^{n}(-1)^k k^r\binom{n}{k}\right]+(-1)^n\binom{n}{n}\sum_{k=0}^{n}(-1)^k k^n\binom{n}{k}$$

$$=\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(0)+(-1)^n(1)(-1)^n n!=n!$$

SOLUTION 6.92

Solution by Adil Abdulayev-Baku-Azerdbajan

$$r_a^2+r_b^2+r_c^2+r^2=16R^2-(a^2+b^2+c^2)\ge 7R^2 \Leftrightarrow$$

$$\Leftrightarrow LHS \geq 7R^2 - r^2 \geq 7R^2 - \frac{R^2}{4} = \frac{27R^2}{4}.$$

SOLUTION 6.93

Proof by Soumava Chakraborty – Kolkata – India

$$4R + r \geq s\sqrt{3} \Rightarrow 3s^2 \leq 16R^2 + 8Rr + r^2$$

$$\textit{Gerretsen} \Rightarrow 3s^2 \leq 12R^2 + 12Rr + 9r^2$$

$$\textit{It suffices to prove } 12R^2 + 12Rr + 9r^2 \leq 16R^2 + 8Rr + r^2$$

$$\Leftrightarrow 4R^2 - 4Rr - 8r^2 \geq 0 \Leftrightarrow R^2 - Rr - 2r^2 \geq 0$$

$$\Leftrightarrow (R + r)(R - 2r) \geq 0 \textit{ which is true}$$

$$R \geq 2r$$

SOLUTION 6.94

Proof by Kevin Soto Palacios –Huarmey- Peru

$$14Rr \leq p^2 + r^2 \Leftrightarrow \text{Gerretsen: } p^2 \geq 16Rr - 5r^2 \rightarrow$$

$$\rightarrow p^2 + r^2 \geq 16Rr - 4r^2 \geq 14Rr \rightarrow R \geq 2r$$

SOLUTION 6.95

Proof by Kevin Soto Palacios-Huarmey-Peru

De la desigualdad Weizenbock (Refinamiento de Pohoata)

$$a^2x + b^2y + c^2z \geq 4\sqrt{xy + yz + xz}S \rightarrow x, y, z \geq 0$$

$$\textit{Sea: } x = \frac{m}{n+p}, y = \frac{n}{m+p}, z = \frac{p}{m+n}$$

La desigualdad es equivalente:

$$\frac{m}{n+p}a^2 + \frac{n}{m+p}b^2 + \frac{p}{m+n}c^2 \geq 4\sqrt{\frac{m}{n+p} \cdot \frac{n}{m+p} + \frac{n}{m+p} \cdot \frac{p}{m+n} + \frac{m}{m+p} \cdot \frac{p}{m+n}}S$$

Por desigualdad de Cauchy:

$$\frac{m^2}{m(n+p)} \cdot \frac{n^2}{n(m+p)} + \frac{n^2}{n(m+p)} \cdot \frac{p^2}{p(m+n)} + \frac{m^2}{m(m+p)} \cdot \frac{p^2}{p(m+n)} \geq$$

$$\geq \frac{(mn + np + mp)^2}{\sum mn(p+n)(p+m)}$$

$$\frac{(mn + np + mp)^2}{\sum mn(p+n)(p+m)} = \frac{\sum (mn)^2 + 2mnp(m+n+p)}{\sum p^2mn + \sum (mn)^2 + \sum n^2pm + \sum m^2pn} =$$

$$= \frac{\sum (mn)^2 + 2mnp(m+n+p)}{3mnp(m+n+p) + \sum (mn)^2} \geq \frac{3}{4}$$

Por la tanto:

$$\frac{m}{n+p}a^2 + \frac{n}{m+n}b^2 + \frac{p}{m+n}c^2 \geq 4\sqrt{\frac{m}{n+p}\cdot\frac{n}{m+p} + \frac{n}{m+p}\cdot\frac{p}{m+n} + \frac{m}{m+p}\cdot\frac{p}{m+n}}$$

$$\geq 4\sqrt{\frac{3}{4}}S = 2\sqrt{3}S$$

SOLUTION 6.96

Proof by Soumitra Mandal – Kolkata – India

$$\textit{Let } f(x) = \sin x \textit{ for all } x \in \left(0, \frac{\pi}{2}\right)$$

$$f''(x) = -\sin x \leq 0 \textit{ for all } x \in \left(0, \frac{\pi}{2}\right)$$

$$\sum_{A,B,C} \sin A \leq 3\sin\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{2}$$

$$0 < A, B, C < \frac{\pi}{2} \Rightarrow \frac{2}{\pi} > \frac{1}{\pi - A}, \frac{1}{\pi - B}, \frac{1}{\pi - C} > \frac{1}{\pi}$$

$$\sum_{A,B,C} \frac{\sin A}{\pi - A} < \frac{2}{\pi}\left(\sum_{cyc} \sin A\right) = \frac{3\sqrt{3}}{\pi}$$

$$\sum_{A,B,C} \frac{\sin A}{\pi - A} \geq \sum_{cyc} \frac{A - \frac{A^2}{3}}{\pi - A}$$

$$\textit{[since, } \sin x \geq x - \frac{x^2}{3}\textit{]}$$

$$\textit{Let } \varphi(x) = \frac{x - \frac{x^2}{3}}{\pi - x} \textit{ for all } x \in \left(0, \frac{\pi}{2}\right)$$

$$\varphi''(x) = \frac{1 - \frac{2\pi}{3}}{(\pi - x)^2} + \frac{\left(1 - \frac{2x}{3}\right)(\pi - x) + 2\left(x - \frac{x^2}{3}\right)}{(\pi - x)^3} > 0 \textit{ fro all } x \in \left(0, \frac{\pi}{2}\right)$$

$$\frac{1}{3}\left(\sum_{cyc} \varphi(A)\right) \geq \varphi\left(\frac{A+B+C}{3}\right) = \frac{\frac{\pi}{3} - \frac{\pi^2}{27}}{\pi - \frac{\pi}{3}}$$

$$\sum_{cyc} \varphi(A) > \frac{3}{\pi}$$

$$\frac{3}{\pi} < \sum_{A,B,C} \frac{\sin A}{\pi - A} < \frac{3\sqrt{3}}{\pi}$$

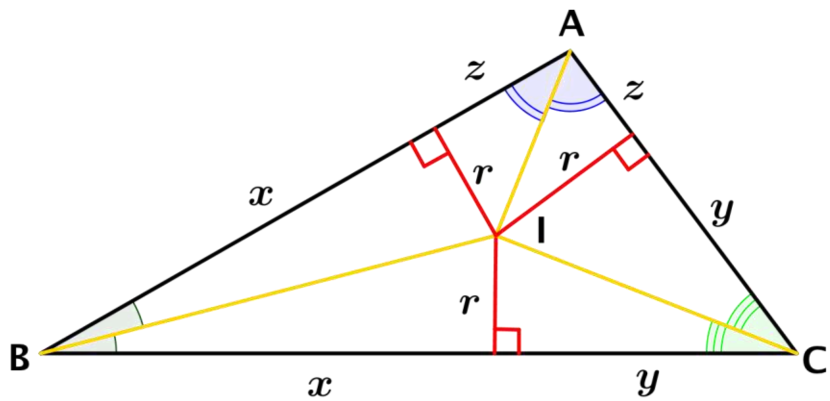
SOLUTION 6.97

Proof by Saptak Bhattacharya-Kolkata-India

To show

$$\sum \sqrt{\sin A} \geq \sum \sqrt{\frac{r}{R}\cot\frac{A}{2}} \Leftrightarrow \sum \sqrt{\frac{a}{2R}} \geq \sum \sqrt{\frac{r\cot\frac{A}{2}}{R}}$$

$$\Leftrightarrow \sum \sqrt{\frac{a}{2}} \geq \sum \sqrt{r\cot\frac{A}{2}}$$



By Ravi transformation;

$I =$ ***Incentre;*** $a = x + y;\ b = y + z;\ c = z + x;\ r\cot\frac{A}{2} = z;$

To show, $\sum \sqrt{\frac{x+y}{2}} \geq \sum \sqrt{x}$ ***. Put*** $x = l^2;\ y = m^2;\ z = n^2;$

Now,

$$\sum \sqrt{\frac{l^2 + m^2}{2}} \geq \sum \frac{l + m}{2} = \sum l$$

Thus,

$$\sum \sqrt{\frac{x + y}{2}} \geq \sum \sqrt{x}$$

SOLUTION 6.98

Proof by Soumava Chakraborty-Kolkata-India

$$x \sin A + y \sin B + z \sin C \leq \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)$$

$$\forall\, x, y, z \in \mathbb{R}, xyz > 0 \rightarrow \textit{Vasic's Inequality}$$

$$x \sin A + y \sin B + z \sin C$$

$$\overset{C-B-S}{\leq} \sqrt{x^2 + y^2 + z^2}\sqrt{\sin^2 A + \sin^2 B + \sin^2 C} \leq \frac{3}{2}\sqrt{x^2 + y^2 + z^2},$$

$$\left(\sum \sin^2 A \leq \frac{9}{4}\right) \quad (*)$$

$$(*) \sum \sin^2 A = \frac{(a^2 + b^2 + c^2)}{4R^2} \leq \frac{(9R^2)}{4R^2} = \frac{9}{4}$$

So, if we can show:

$$\frac{3}{2}\sqrt{x^2 + y^2 + z^2} \leq \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right) \quad (1)$$

we are done.

$$(1) \Leftrightarrow \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \geq \sqrt{3(x^2 + y^2 + z^2)} \Leftrightarrow \frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz} \geq \sqrt{3(x^2 + y^2 + z^2)}$$

$$\Leftrightarrow x^2y^2 + y^2z^2 + z^2x^2 \geq xyz\sqrt{3(x^2 + y^2 + z^2)}$$

$$(xyz > 0)$$

$$\Leftrightarrow (x^2y^2 + y^2z^2 + z^2x^2)^2 \geq 3x^2y^2z^2(x^2 + y^2 + z^2)$$

$$\Leftrightarrow x^4y^4 + y^4z^4 + z^4x^4 \geq x^2y^2z^2(x^2 + y^2 + z^2) \quad (2)$$

$$\textit{Let } x^2y^2 = u, y^2z^2 = v, z^2x^2 = w$$

$$u^2 + v^2 + w^2 \geq w + vw + wu$$

$$x^4 - y^4 + y^4z^4 + z^4x^4 \geq x^2y^2z^2(x^2 + y^2 + z^2) \Rightarrow (2) \textit{ is true}$$

SOLUTION 6.99

Proof by Kevin Soto Palacios-Huarmey-Peru

$$\Rightarrow 3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq \left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) + \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right) + 3$$

$$\Rightarrow 2\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right) + 3$$

Realizamos lo siguientes cambios de variables:

$$a^2 = x + z \geq 0,\ b^2 = y + z \geq 0, c^2 = x + y \geq 0 \ \Leftrightarrow (x, y, z) \geq 0$$

La desigualdad es equivalente:

$$\Rightarrow 2\left(\frac{x+z}{y+z} + \frac{y+z}{x+y} + \frac{x+y}{x+z}\right) \geq \left(\frac{y+z}{x+z} + \frac{x+y}{y+z} + \frac{x+z}{x+y}\right) + 3$$

$$\Rightarrow 2(x+z)^2(x+y) + 2(y+z)^2(x+z) + 2(x+y)^2(y+z) \geq$$
$$\geq (y+z)^2(x+y) + (x+y)^2(x+z) + (x+z)^2(y+z) + 3\prod(x+y)$$

$$\Rightarrow 2\sum(x^2 + z^2 + 2xz)(x+y) \geq$$

$$\geq \sum(y^2 + z^2 + 2yz)(x+y) + 3\sum xy(x+y) + 6xyz$$

$$\Rightarrow 2\sum(x^2 + z^2)\,(x+y) + 4\sum xz(x+y) \geq$$
$$\geq \sum(y^2 + z^2)(x+y) + 2\sum yz(x+y) + 3\sum xy(x+y) + 6xyz$$

$$\Rightarrow 2\sum(x^2 + z^2)\,(x+y) + 4\sum x^2z + 12xyz \geq$$

$$\geq \sum(y^2 + z^2)\,(x+y) + 2\sum y^2z + 3\sum xy(x+y) + 12xyz$$

$$2\sum(x^2 + z^2)(x+y) = 2\sum x^3 + 4xz^2 + 4zy^2 + 4yx^2 + 2yz^2 + 2y^2x + 2x^2z$$

$$\sum(y^2 + z^2)(x+y) = \sum x^3 + 2xy^2 + 2yz^2 + 2zx^2 + xz^2 + yx^2 + zy^2$$

$$3\sum xy(x+y) = 3x^2y + 3y^2x + 3y^2z + 3z^2y + 3z^2x + 3xz^2$$

$$4\sum x^2z = 4x^2z + 4z^2y + 4y^2x$$

$$2\sum y^2z = 2y^2z + 2x^2y + 2z^2x$$

$$\Rightarrow 2\ \sum(x^2 + z^2)\,(x+y) - \sum(y^2 + z^2)\,(x+y) =$$
$$= \sum x^3 + 3xz^2 + 3zy^2 + 3yx^2$$

$$\Rightarrow \sum x^3 + 3xz^2 + 3zy^2 + 3yx^2 - 2\sum y^2z = \sum x^3 + xz^2 + zy^2 + yx^2$$

La desigualdad es equivalente:

$$\Rightarrow \sum x^3 + xz^2 + zy^2 + yx^2 + 4x^2z + 4z^2y + 4y^2x \geq 3\sum xy(x+y)$$

$$\Rightarrow \sum x^3 - 2z^2x - 2y^2z - 2x^2y + x^2z + z^2y + y^2x \geq 0$$

$$\Rightarrow (x^3 - 2x^2y + y^2x) + (y^3 - 2y^2z + z^2y) + (z^3 - 2z^2x + x^2z) \geq 0$$

$$\Rightarrow x(x-y)^2 + y(y-z)^2 + z(z-x)^2 \geq 0 \quad \textit{(LQQD)}$$

SOLUTION 6.100

Proof by Marian Dinca-Romania

$$R + r = R(\cos A + \cos B + \cos C) \text{ – Carnot identity}$$

$$a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$$

$$\sin^2 A + \sin^2 B + \sin^2 C \geq (\cos A + \cos B + \cos C)^2$$

$$3 - \cos^2 A - \cos^2 A - \cos^2 C \geq$$

$$\geq \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A$$

$$3 \geq 2(\cos^2 A + \cos^2 B + \cos^2 C) + 2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A$$

$$3 \geq 2(1 - 2\cos A\cos B\cos C) + 2\cos A\cos B + 2\cos B\cos C + 2\cos C\cos A$$

$$\frac{1}{2} \geq \cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C$$

$$\text{Let } \frac{\pi}{2} \geq A \geq B \geq C \Rightarrow A \geq \frac{\pi}{3}, 0 \leq \cos A \leq \frac{1}{2}$$

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C =$$

$$= \cos B\cos C\,(1 - 2\cos A) + \cos A\,(\cos B + \cos C)$$

$$\cos B\cos C = \frac{\cos(B+C) + \cos(B-C)}{2} \leq \frac{\cos(B+C) + 1}{2} = \cos^2\left(\frac{B+C}{2}\right) =$$

$$= \sin^2\left(\frac{A}{2}\right) = \frac{1 - \cos A}{2}$$

$$\cos B + \cos C + \cos A \leq \frac{3}{2} \quad \text{(well – known)}$$

$$\cos B + \cos C \leq \frac{3}{2} - \cos A$$

$$\cos B\cos C\,(1 - 2\cos A) + \cos A\,(\cos A + \cos C) \leq$$

$$\leq \left(\frac{1-\cos A}{2}\right)\cdot(1-2\cos A)+\cos A\left(\frac{3}{2}-\cos A\right)$$

$$=\frac{1-3\cos A+2\cos^2 A}{2}+\frac{3\cos A-2\cos^2 A}{2}=\frac{1}{2}$$

SOLUTION 6.101

Solution by Kevin Soto Palacios – Huarmey-Peru

Recordar las siguientes fórmulas:

$$\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4S}$$

$$sen\frac{A}{2}sen\frac{B}{2}sen\frac{C}{2}=\frac{4(p-a)(p-b)(p-c)}{abc}=\frac{r}{R},S=pr$$

La desigualdad es equivalente:

$$\frac{\frac{a+b+c}{2R}}{\frac{a^2+b^2+c^2}{4S}}\leq\frac{3}{2}\rightarrow\frac{(a+b+c)4S}{2R(a^2+b^2+c^2)}\leq\frac{3}{2}\rightarrow\frac{(a+b+c)4pr}{(a^2+b^2+c^2)R}\leq 3$$

$$\rightarrow(a+b+c)(a+b+c)8(p-a)(p-b)(p-c)\leq 3abc(a^2+b^2+c^2)$$

$$\rightarrow(a+b+c)^2(b+c-a)(a+c-b)(a+b-c)\leq 3abc(a^2+b^2+c^2)$$

$$\rightarrow \textit{Sea: } a=x+z,b=x+y,c=y+z$$

$$\rightarrow 4(x+y+z)^2 8xyz\leq 3(x+y)(x+z)(y+z)2(x^2+y^2+z^2+xy+xz+yz)$$

$$\rightarrow \textit{Se puede observar claramente que:}$$

$$(x+y)(y+z)(z+x)\geq 8xyz\rightarrow \textit{Válido: } MA\geq MG$$

Por la que quelda demonstrar:

$$6(x^2+y^2+z^2+xy+xz+yz)\geq 4(x+y+z)^2\Leftrightarrow$$

$$\Leftrightarrow(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

SOLUTION 6.102

Proof by Soumitra Mandal-Chandar Nagore-India

$$\textit{Let } A(x)=\frac{\sin 2x}{2x}+1-2x\cot x \textit{ and } B(x)=x^4 \textit{ for all } x\in\left(0,\frac{\pi}{2}\right)$$

$$\textit{Let } \alpha,\beta\in\left(0,\frac{\pi}{2}\right) \textit{ such that } \alpha\leq\beta \textit{ then } \frac{A(\beta)}{B(\beta)}-\frac{A(\alpha)}{B(\alpha)}$$

$$=\frac{\frac{\sin 2\beta}{2\beta}+1-2\beta\cot\beta}{\beta^4}-\frac{\frac{\sin 2\alpha}{2\alpha}+1-2\alpha\cot\alpha}{\alpha^4}$$

$$= \frac{1}{(\alpha\beta)^4}\left[\alpha^4 \frac{\sin 2\beta}{2\beta} - \beta^4 \frac{\sin 2\alpha}{2\alpha} - (\beta^4 - \alpha^4) - 2\alpha^4\beta\cot\beta + 2\beta^4\alpha\cot\alpha\right]$$

$$= \frac{1}{(\alpha\beta)^4}\left[\alpha^4\left(1 - \frac{2}{3}\beta^2 + \cdots\right) - \beta^2\left(1 - \frac{2}{3}\alpha^4 + \cdots\right) + (\alpha^4 - \beta^4) + 2\beta^4\left(1 - \frac{\alpha^2}{3} + \cdots\right)\right.$$

$$\left. - 2\alpha^4\left(1 - \frac{\beta^2}{3} + \cdots\right)\right] \le 0$$

Hence for $\alpha \le \beta, \frac{A(\alpha)}{B(\alpha)} \ge \frac{A(\beta)}{B(\beta)}$ hence $\frac{A(x)}{B(x)}$ is a decreasing function,

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{A(x)}{B(x)} = \frac{16}{\pi^4}$$

$$\therefore \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \frac{16}{\pi^4}x^3 \tan x$$

SOLUTION 6.103

Proof by Thanasis Xenos-Greece

$$\frac{\ln x}{x^3 - 1} < \frac{1}{3} \cdot \frac{x+1}{x^3 + x} \overset{x>1}{\Leftrightarrow} 3(x^3 + x)\ln x - (x+1)(x^3 - 1) < 0$$

$$f(x) = 3(x^3 + x)\ln x - (x+1)(x^3 - 1), x \ge 1$$

$$f'(x) = 3(3x^2 + 1)\ln x - 4(x^3 - 1)$$

$$f''(x) = \frac{1}{x} \cdot (18\,x^2 \ln x + 2x^2 + 3 - 18x^3)$$

$$g(x) = 18x^2 \ln x + 9x^2 + 3 - 12x^3, x \ge 1$$

$$g'(x) = 36x(\ln x - x + 1) \le 0, \ln x \le x - 1$$

$$g \downarrow [1, +\infty)$$

$$x > 1 \Rightarrow g(x) < g(1) = 0 \Rightarrow f''(x) < 0 \Rightarrow f' \downarrow [1, +\infty)$$

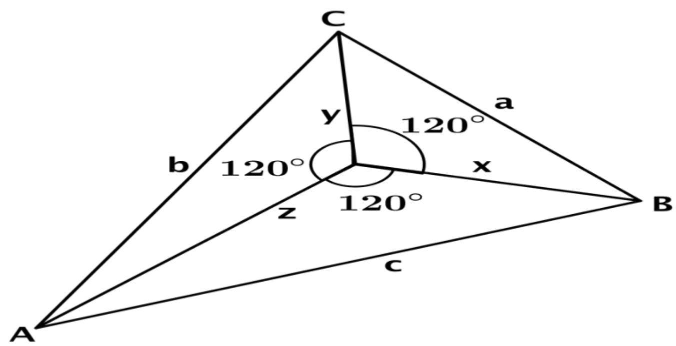
$$x > 1 \Rightarrow f'(x) < f'(1) = 0 \Rightarrow f \downarrow [1, \infty)$$

$$x > 1 \Rightarrow f(x) < f(1) = 0$$

SOLUTION 6.104

Proof by Soumava Pal – Kolkata – India

Considering a triangle with sides x, y, included angle $120°$, a triangle with sides y, z included angle $120°$ and another with sides z, x, included angle $120°$, we get.



$$\Delta\, ABC=\frac{1}{2}(xy+yz+zx)\sin 120^\circ=\frac{\sqrt{3}}{4}\left(\sum xy\right)\Rightarrow \sum xy=\frac{4\Delta}{\sqrt{3}}\quad (1)$$

$$\text{by cosine rule}\begin{cases} x^2+xy+y^2=a^2 & (2)\\ y^2+yz+z^2=b^2 & (3)\\ z^2+zx+x^2=c^2 & (4)\end{cases}$$

From isoperimetric inequality for triangles

$$\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}}\ge \Delta\Rightarrow a^2b^2c^2\ge\left(\frac{4\Delta}{\sqrt{3}}\right)^3$$

(substituting values from (1), (2), (3), (4) gives required inequality)

SOLUTION 6.105

Proof by Adil Abdullayev-Baku-Azerbaidian

$$\textit{Lemma. } 16Rr-5r^2+2(R-2r)\left(R-r-\sqrt{R^2-2Rr}\right)\le p^2\le$$

$$\le 4R^2+4Rr+3r^2-2(R-2r)\left(R-r-\sqrt{R^2-2Rr}\right).$$

$$\textit{EULER}\Rightarrow 0<\frac{r}{R-r}\le 1$$

$$R-r-\sqrt{R^2-2Rr}=(R-r)\left(1-\sqrt{1-\frac{r^2}{(R-r)^2}}\right)\overset{Bernoulli}{\ge}$$

$$\ge\frac{1}{2}(R-r)\left(\frac{r}{R-r}\right)^2=\frac{r^2}{2(R-r)}$$

SOLUTION 6.106

Solution by Soumava Chakraborty-Kolkata-India

$$a,b>0\Rightarrow\sqrt{\frac{a^2+b^2}{2}}+\frac{2}{\frac{1}{a}+\frac{1}{b}}\overset{(1)}{\ge}\frac{a+b}{2}+\sqrt{ab}$$

$$\textit{(1)} \Leftrightarrow \sqrt{\frac{a^2+b^2}{2}}-\sqrt{ab} \geq \frac{a+b}{2}-\frac{2ab}{a+b} \Leftrightarrow \frac{\frac{a^2+b^2}{2}-ab}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}} \geq \frac{(a+b)^2-4ab}{2(a+b)}$$

$$\Leftrightarrow \frac{(a-b)^2}{2\left(\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}\right)}-\frac{(a-b)^2}{2(a+b)} \geq 0 \Leftrightarrow (a-b)^2\left(\frac{1}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}}-\frac{1}{a+b}\right) \geq 0$$

$$\Leftrightarrow \frac{1}{\sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}}-\frac{1}{a+b} \geq 0 (\because (a-b)^2 \geq 0) \Leftrightarrow a+b \geq \sqrt{\frac{a^2+b^2}{2}}+\sqrt{ab}$$

$$\Leftrightarrow a^2+b^2+2ab \geq \frac{a^2+b^2}{2}+ab+2\sqrt{\frac{ab(a^2+b^2)}{2}}$$

$$\Leftrightarrow \frac{a^2+b^2}{2}+ab \geq 2\sqrt{\frac{ab(a^2+b^2)}{2}} \Leftrightarrow (a+b)^2 \geq 4\sqrt{\frac{ab(a^2+b^2)}{2}}$$

$$\Leftrightarrow (a+b)^4 \geq 8ab(a^2+b^2) \Leftrightarrow a^4+b^4+6a^2b^2 \geq 4a^3b+4ab^3$$

$$\Leftrightarrow (a^2+b^2)^2+(2ab)^2-2(a^2+b^2)(2ab) \geq 0$$

$$\Leftrightarrow (a^2+b^2-2ab)^2 \geq 0 \Leftrightarrow (a-b)^4 \geq 0 \rightarrow \textit{true (Proved)}$$

SOLUTION 6.107

Proof by Soumava Chakraborty-Kolkata-India

Using Goldstone's inequality,

$$4R^2s^2 \geq \sum a^2b^2 \Rightarrow \frac{1}{2Rs} \leq \frac{1}{\sqrt{\sum a^2b^2}} \Rightarrow \frac{rs}{2Rs} \leq \frac{\Delta}{\sqrt{\sum a^2b^2}}$$

$$\Rightarrow \frac{2\Delta}{\sqrt{\sum a^2b^2}} \geq \frac{r}{R} \Rightarrow \sin\omega \geq \frac{r}{R} \Rightarrow \frac{R}{r} \geq \frac{1}{\sin\omega} \quad \textit{(1)}$$

$$\textit{Now, } \frac{R}{r} \geq \frac{(a+b)(b+c)(c+a)}{16RS} \quad \textit{(2)} \Leftrightarrow \frac{R}{r} \geq \frac{2abc+\sum ab(2s-c)}{16Rrs}$$

$$\Leftrightarrow 16R^2s \geq 8Rrs+2s\left(\sum ab\right)-12Rrs$$

$$\Leftrightarrow 8R^2 \geq s^2+4Rr+r^2-2Rr = s^2+2Rr+r^2 \Leftrightarrow s^2 \leq 8R^2-2Rr-r^2$$

$$\textit{Gerretsen} \Rightarrow s^2 \leq 4R^2+4Rr+3r^2$$

∴ to prove (2), it suffices to prove that:

$$4R^2+4Rr+3r^2 \leq 8R^2-2Rr-r^2$$

$$\Leftrightarrow 4R^2-6Rr-4r^2 \geq 0 \Leftrightarrow 2R^2-3Rr-2r^2 \geq 0$$

$$\Leftrightarrow (R-2r)(2R+r)\geq 0 \rightarrow \textit{true}, \because R\geq 2r \textit{ (Euler)} \Rightarrow \textit{(2) is true.}$$

$$\textit{Now, } \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{2(4ab^2+4bc^2+4ca^2)}{3\cdot 4abc}$$

$$\leq \frac{b(a+b)^2+c(b+c)^2+a(c+a)^2}{3\cdot 2abc}\ \left(\because 4ab \underset{etc}{\leq} (a+b)^2\right)$$

$$=\frac{(\sum a^2b+\sum ab^2)+\sum ab^2+\sum a^3}{3\cdot 2abc}\leq \frac{(\sum a^3+3abc)+\sum ab^2+\sum a^3}{3\cdot 2abc}\ \textit{(Schur)}$$

$$\leq \frac{(\sum a^3+3abc)+\sum a^3+\sum a^3}{3\cdot 2abc}$$

$$\left(\begin{array}{c} a^3+b^3+b^3\geq 3ab^2, b^3+c^3+c^3 \overset{A-G}{\geq} 3bc^2, c^3+a^3+a^3 \overset{A-G}{\geq} 3ca^2 \\ \Rightarrow 3\sum a^3 \geq 3\sum ab^2 \Rightarrow \sum ab^2 \leq \sum a^3 \end{array}\right)$$

$$=\frac{3\sum a^3+3abc}{3\cdot 2ab}=\frac{\sum a^3+abc}{2abc}$$

$$\therefore \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\leq \frac{\sum a^3+abc}{2abc}\quad \textit{(3)}$$

$$\textit{Now } \frac{R}{r}\geq \frac{\sum a^3+abc}{2abc}\quad \textit{(4)}$$

$$\Leftrightarrow \frac{R}{r}\geq \frac{\sum a^3-3abc+4abc}{2abc}\Leftrightarrow \frac{R}{r}\geq \frac{2s(\sum a^2-\sum ab)+16Rrs}{8Rrs}$$

$$\Leftrightarrow 4R^2\geq \sum a^2-\sum ab+8Rr \Leftrightarrow 4R^2\geq s^2-12Rr-3r^2+8Rr$$

$$\Leftrightarrow s^2\leq 4R^2+4Rr+3r^2 \rightarrow \textit{true by Gerretsen} \Rightarrow \textit{(4) is true}$$

$$\textit{(3) and (4)} \Rightarrow \frac{R}{r}\geq \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ \textit{(5)}$$

(1), (2) and (5) ⇒

$$\frac{R}{r}\geq \max\left\{\frac{1}{\sin\omega}, \frac{(a+b)(b+c)(c+a)}{16RS}, \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\right\}$$

SOLUTION 6.108

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

Put $A=\frac{\frac{a-b}{\ln a-\ln b}-\sqrt{ab}}{\frac{a+b}{2}-\sqrt{ab}}$. ***We need to prove that*** $0<A<\frac{1}{3}$

1) LEMMA: $\frac{a-b}{\ln a-\ln b}>\sqrt{ab}$ ***when*** $a,b>0$ ***and*** $a\neq b$

We have $\frac{a-b}{\ln a-\ln b}>\sqrt{ab}\Rightarrow \frac{\ln a-\ln b}{a-b}<\frac{1}{\sqrt{ab}}$

$$\Rightarrow \frac{\ln\left(\frac{a}{b}\right)}{\frac{a}{b}-1} < \sqrt{\frac{b}{a}} \quad (1)$$

Put $\frac{a}{b} = t \ (t > 0, t \neq 1)$ *, we have (1)* $\Rightarrow \frac{\ln t}{t-1} < \frac{1}{\sqrt{t}}$ *(2)*

Put $f(t) = \ln t - \frac{t-1}{\sqrt{t}}$

$f'(t) = \frac{-(\sqrt{t}-1)^2}{2\sqrt{t^3}} < 0 \Rightarrow f(t)$ *is decreasing function* $\Rightarrow f(t) < f(1)$ *when* $t > 1$ *and* $f(t) > f(1)$ *when* $t < 1 \Rightarrow f(t) < 0$ *when* $t > 1$ *and* $f(t) > 0$ *when* $t < 1$.

1.1.) If $t > 1$. *We have (2)* $\Rightarrow \ln t < \frac{t-1}{\sqrt{t}}$ *(True)*

1.2) If $t < 1$. *We have (2)* $\Rightarrow \ln t > \frac{t-1}{\sqrt{t}}$ *(True)*

$\Rightarrow$ *(1) true* $\Rightarrow \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$

Applying the lemma $\Rightarrow \frac{a-b}{\ln a - \ln b} > \sqrt{ab}$ *(since* $0 < \frac{\ln a - \ln b}{a-b} < \frac{1}{\sqrt{ab}}$*)*

On the other hand, by AM-GM inequality, we have $\frac{a+b}{2} - \sqrt{ab} > 0$ *(since* $a \neq b$*)*

2) We need to prove that $A < \frac{1}{3} \Rightarrow$

$$\Rightarrow \frac{3(a-b)}{\ln a - \ln b} - 3\sqrt{ab} < \frac{a+b}{2} - \sqrt{ab} \Rightarrow \frac{3(a-b)}{\ln a - \ln b} < \frac{a+b}{2} + 2\sqrt{ab}$$

$$\Rightarrow \frac{3\left(\frac{a}{b}-1\right)}{\ln\left(\frac{a}{b}\right)} < \frac{\frac{a}{b}+1}{2} + 2\sqrt{\frac{a}{b}} \quad (3)$$

Put $\frac{a}{b} = t \ (t > 0, t \neq 1)$*, we have (3)* $\Rightarrow \frac{3(t-1)}{\ln t} < \frac{t+1}{2} + 2\sqrt{t}$ *(4)*

Put $g(t) = \frac{t+1}{2} + 2\sqrt{t} - \frac{3(t-1)}{\ln t}$

$$g'(t) = \frac{1}{\sqrt{t}} + \frac{1}{2} + \frac{3(t-1) - 3t \cdot \ln t}{t \cdot \ln^2 t} = \frac{2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t}{2t \cdot \ln^2 t}$$

Put $h(t) = 2\sqrt{t} \cdot \ln^2 t + t \cdot \ln^2 t + 6(t-1) - 6t \cdot \ln t$

$$h'(t) = \frac{\ln t \cdot \left(-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t\right)}{\sqrt{t}}$$

$h'(t) = 0 \Rightarrow \ln t = 0$ *(5) or* $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t = 0$ *(6)*

(5): $\ln t = 0 \Rightarrow t = 1$

(6): $-4\sqrt{t} + 4 + \left(\sqrt{t}+1\right) \cdot \ln t = 0 \Rightarrow \ln t = \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

Put $y(t) = \ln t - \frac{4(\sqrt{t}-1)}{\sqrt{t}+1}$

$y'(t) = \frac{(\sqrt{t}-1)^2}{t(\sqrt{t}+1)^2} > 0 \Rightarrow y(x)$ ***is increasing function*** $\Rightarrow y(x) = 0$ ***has at most 1 root***

On the other hand, we have $y(1) = 0 \Rightarrow t = 1$ ***is the root of (6)***

So $h'(t) = 0 \Rightarrow t = 1$***. So we have***

2.1) $g'(t) < 0$ ***when*** $t < 1$

So when $t < 1 \Rightarrow g(t)$ ***is decreasing function*** $\Rightarrow g(t) > \lim_{t\to 1^+} g(t) \Rightarrow g(t) > 0$

2.2) $g'(t) > 0$ ***when*** $t > 1$

So when $t < 1 \Rightarrow g(t)$ ***is an increasing function*** $\Rightarrow g(t) > \lim_{t\to 1^+} g(t)$

So, $g(t) > 0\ \forall t > 0$

$\Rightarrow$ ***(4) true*** $\Rightarrow$ ***(3)*** $\Rightarrow A < \frac{1}{3} \Rightarrow$ ***Q.E.D***

MISCELANEOUS PROBLEMS

SOLUTIONS

SOLUTION 7.01

Solution by proposer:

Denoting:

$$y_1 = \cos\frac{2\pi}{13}\cos\frac{3\pi}{13}$$

$$y_2 = -\cos\frac{4\pi}{13}\cos\frac{6\pi}{13}$$

$$y_3 = -\cos\frac{\pi}{13}\cos\frac{5\pi}{13}$$

We can express y_1, y_2, y_3 under the form of sums in the following way:

$$y_1 = \frac{1}{2}\left(\cos\frac{\pi}{13} + \cos\frac{5\pi}{13}\right)$$

$$y_2 = \frac{1}{2}\left(-\cos\frac{2\pi}{13} + \cos\frac{3\pi}{13}\right)$$

$$y_3 = \frac{1}{2}\left(-\cos\frac{4\pi}{13} - \cos\frac{6\pi}{13}\right)$$

The equation that have the roots y_1, y_2, y_3 is:

$$y^3 - S_1'y^2 + S_2'y - S_3' = 0, \text{ where:}$$

$$S_1' = y_1 + y_2 + y_3$$

$$S_2' = y_1y_2 + y_2y_3 + y_3y_1$$

$$S_3' = y_1y_2y_3.$$

We calculate S_1':

$$S_1' = y_1 + y_2 + y_3 = \frac{1}{2}\left(\cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} + \cos\frac{5\pi}{13} - \cos\frac{6\pi}{13}\right)$$

We calculate the sum:

$$u = \cos\frac{\pi}{13} - \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} + \cos\frac{5\pi}{13} - \cos\frac{6\pi}{13}$$

By multiplyling both memebers with $sin\frac{\pi}{13}$ and transforming the products into sums, we obtain $u = \frac{1}{2}$. It follows immediateley that $S_1' = \frac{1}{4}$.

For the calculus of S_2', we will first calculate the product y_1y_2. Using the expressions of y_1 and y_2 as sums, transforming the products that appear in sums and reducing to the first cadran, we obtain: $y_1y_2 = \frac{1}{4}(-y_1 - 2y_2 - y_3)$. Analogous, we have:

$$y_2y_3 = \frac{1}{4}(-y_2 - 2y_3 - y_1)$$

$$y_3y_1 = \frac{1}{4}(-y_3 - 2y_1 - y_2).$$

It follows $S_2' = y_1y_2 + y_2y_3 + y_3y_1 = -S_1' = -\frac{1}{4}$.

We calculate S_3':

$$S_3' = y_1y_2y_3 = cos\frac{\pi}{13}cos\frac{2\pi}{13}cos\frac{3\pi}{13}cos\frac{4\pi}{13}cos\frac{5\pi}{13}cos\frac{6\pi}{13}.$$

We calculate the product:

$$v = cos\frac{\pi}{13}cos\frac{2\pi}{13}cos\frac{3\pi}{13}cos\frac{4\pi}{13}cos\frac{5\pi}{13}cos\frac{6\pi}{13}.$$

By multiplying both members with $sin\frac{\pi}{13}$ and by repeatedly using the double angle sinus formula, as well as reducing to the first cadran, we easily obtain: $v = \frac{1}{64}$.

So, $S_3' = \frac{1}{64}$.

So, we can write the equation that has the roots y_1, y_2, y_3, which is:

$$64y^3 - 16y^2 - 16y - 1 = 0 \qquad (1)$$

We consider the third degree equation:

$$x^3 - S_1x^2 + S_2x - S_3 = 0 \qquad (2)$$

with the roots x_1, x_2, x_3 in which:

$$S_1 = x_1 + x_2 + x_3$$

$$S_2 = x_1x_2 + x_2x_3 + x_3x_1$$

$$S_3 = x_1x_2x_3.$$

We have the relationships, which can be easily verified:

$$x_1^3 + x_2^3 + x_3^3 = S_1^3 - 3S_1S_2 + 3S_3$$

$$x_1^3x_2^3 + x_2^3x_3^3 + x_3^3x_1^3 = S_2^3 - 3S_1S_2S_3 + 3S_3^2$$

$$x_1^3x_2^3x_3^3 = S_3^3.$$

The equation that admits as roots the cubs of the roots of previous equation (2) is:

$$y^3 - S_1'y^2 + S_2'y - S_3' = 0 \qquad (3)$$

where:

$$y_1 = x_1^3, y_2 = x_2^3, y_3 = x_3^3$$

$$S_1' = y_1 + y_2 + y_3 = S_1^3 - 3S_1S_2 + 3S_3$$

$$S_2' = y_1y_2 + y_2y_3 + y_3y_1 = S_2^3 - 3S_1S_2S_3 + 3S_3^2$$

$$S_3' = y_1y_2y_3 = S_3^3.$$

Returning to equation (1), we can write:

$$S_1' = \frac{1}{4}, S_2' = -\frac{1}{4}, S_3' = \frac{1}{64}.$$

We put the conditions:

$$\begin{cases} S_1^3 - 3S_1S_2 + 3S_3 = \dfrac{1}{4} \\ S_2^3 - 3S_1S_2S_3 + 3S_3^2 = -\dfrac{1}{4} \\ S_3^3 = \dfrac{1}{64} \end{cases}$$

We will solve this equation system in real numbers sets. We are specially interested in the value of S_1.We immediately obtain: $S_3 = \frac{1}{4}$.Replacing in the other equations, it follows:$\begin{cases} S_1^3 - 3S_1S_2 = -\frac{1}{2} \\ 16S_2^3 - 12S_1S_2 = -7 \end{cases}$

From the first equation of the previous system, we take out S_2.We have: $S_2 = \frac{2S_1^3+1}{6S_1}$.

Replacing in the other equation, we obtain:

$$2\left(2S_1^3 + 1\right)^3 - 54S_1^3\left(2S_1^3 + 1\right) + 189S_1^3 = 0.$$

We put $S_1^3 = t$ and it follows the equation: $16t^3 - 84t^2 + 147t + 2 = 0.$

A typical analysis of this equation (using for example Rolle's sequence) take us to the conclusion that the equation admits a real root t_1 and additionaly, $t_1 \in (-1, 0)$.

This real root has the value: $t_1 = \frac{7-3\sqrt[3]{13}}{4}$

which can be easily verified, by writing: $3\sqrt[3]{13} = 7 - 4t_1$

and by rising to the third power.

It follows: $S_1 = \sqrt[3]{\frac{7-3\sqrt[3]{13}}{4}}$*.So, we obtain:*

$$S_1 = x_1 + x_2 + x_3 = \sqrt[3]{y_1} + \sqrt[3]{y_2} + \sqrt[3]{y_2} =$$

$$= \sqrt[3]{\frac{1}{2}\left(\cos\frac{\pi}{13} + \cos\frac{5\pi}{13}\right)} + \sqrt[3]{\frac{1}{2}\left(-\cos\frac{2\pi}{13} + \cos\frac{2\pi}{13}\right)} + \sqrt[3]{\frac{1}{2}\left(-\cos\frac{4\pi}{13} - \cos\frac{6\pi}{13}\right)} =$$

$$= \sqrt[3]{\frac{7-3\sqrt[3]{13}}{4}}$$

and the proposed equality in the enunciation of the problem is proved.

SOLUTION 7.02

Solution by Michael Sterghiou-Greece

$$\Omega(n) = \sqrt[n]{\left(\log_n\left(\frac{n!}{(n-2)!}\right)^2\right)^2} + \log_n\left(\sqrt{\binom{2n}{3}}\right) \quad (1)$$

$n \geq 2$ *else (1) is not defined.*

(1) is written as: $(2\log_n[n(n-1)])^{\frac{2}{n}} + \frac{1}{2}\log_n\left[\frac{2}{3}n(n-1)(2n-1)\right]$

or $\underbrace{[2(1+\log_n(n-1))]^{\frac{2}{n}}}_{\Omega_1(n)} + \underbrace{\frac{1}{2}\left(\log_n\frac{2}{3} + 1 + \log_n(n-1) + \log_n(2n-1)\right)}_{\Omega_2(n)}$ *(2)*

$\Omega_1(n) < [2\cdot(1+1)]^{\frac{2}{n}} = 16^{\frac{1}{n}}$ *(as* $\log_n(n-1) < 1$*). Also,*

$\Omega_1(n) > 1$ $\left(2^{\frac{2}{n}} > 1 \text{ and } (1+\log_n(n-1)) > 1\right)$ *so,* $1 < \Omega_1(n) < 16^{\frac{1}{n}}$

For $n > 9 \rightarrow 1 < \Omega_1(n) < \frac{4}{3}$

$$\Omega_2(n) > 1 \text{ and } \left.\begin{matrix} \log_n\frac{2}{3} < 0 \quad n \geq 2 \\ \log_n(n-1) < 1 \quad n \geq 2 \\ \log_n(2n-1) < \frac{4}{3} \quad n \geq 6 \\ 1 \leq 1 \quad n \geq 2 \end{matrix}\right\} \Omega_2(n) < \frac{1}{2}\left(0 + 2 + \frac{4}{3}\right) = \frac{5}{3}$$

Therefore for $n > 9: 2 < \Omega_1(n) + \Omega_2(n) = \Omega(n) < \frac{4}{3} + \frac{5}{3} = 3$

and $\Omega(n)$ *cannot be natural. By trial and error for all* $n: 2 \leq n \leq 9$ *we conclude that only* $\Omega(2) = 3 \in \mathbb{N}$*. [Answer:* $n = 2$*]*

SOLUTION 7.03

Solution by Pierre Mounir-Cairo-Egypt

$$f(x) = -\ln\left(x + \sqrt{x^2+m^2}\right) \Rightarrow f^{(1)}(x) = -\frac{1}{\sqrt{x^2+m^2}} \Rightarrow$$

$$\sqrt{x^2+m^2}f^{(1)} = -1 \Rightarrow \sqrt{x^2+m^2}f^{(2)} + \frac{xf^{(1)}}{\sqrt{x^2+m^2}} = 0 \Rightarrow$$

$$(x^2+m^2)f^{(2)} + xf^{(1)} = 0 \quad \textit{(differentiating n times)}$$

$$(x^2+m^2)f^{(n+2)} + n(2x)f^{(n+1)} + \frac{n(n-1)}{2}(2)f^{(n)} + xf^{(n+1)} + n(1)f^{(n)} = 0 \Rightarrow$$

$$\frac{f^{(n+2)}(x)}{f^{(n)}(x)} = -\frac{(2n+1)x}{x^2+m^2} \times \frac{f^{(n+1)}(x)}{f^{(n)}(x)} - \frac{n^2}{x^2+m^2} \Rightarrow$$

$$\lambda = \lim_{x\to 0}\frac{f^{(n+2)}(x)}{f^{(n)}(x)} = -\frac{n^2}{m^2} \quad (f^{(n)}(0) \text{ is defined } \forall n \in \mathbb{N})$$

Note: $f(x)$ ***has infinite continuous derivatives*** $\in C^{\infty}$

$$\therefore \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\frac{n^2}{m^2} \times \frac{(n+1)}{n^5} + \sum_{k=1}^{\infty}\frac{(k+m+1)}{(k+m)^3 m^2}\right]$$

$$= \sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{m}\frac{(n+1)}{n^3} + \sum_{n=m+1}^{\infty}\frac{(n+1)}{n^3}\right] \quad (n = k+m)$$

$$= \sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{\infty}\frac{(n+1)}{n^3}\right] = \sum_{m=1}^{\infty}\frac{1}{m^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{n^3}\right] = \frac{\pi^2}{6}\left[\frac{\pi^2}{6} + \zeta(3)\right]$$

SOLUTION 7.04

Solution by Shafiqur Rahman-Bangladesh

$$\sum_{j=0}^{n}\frac{(-1)^j}{1+j}\binom{k}{j}\binom{k-1-j}{n-j} = \frac{k!}{n!\,(k-1-n)!}\sum_{j=0}^{n}\frac{(-1)^j}{(1+j)(k-j)}\binom{n}{j} =$$

$$= \frac{k!}{(k+1)n!\,(k-n-1)!}\sum_{j=0}^{n}\left[\frac{(-1)^j\binom{n}{j}}{1+j} + \frac{(-1)^j\binom{n}{j}}{k-j}\right]$$

$$= \frac{k!}{(k+1)n!\,(k-n-1)!}\int_0^1\left[(1-x)^n + (-1)^n x^{k-n-1}(1-x)^n\right]dx =$$

$$= \frac{k!}{(k+1)n!\,(k-n-1)!}\left[\frac{1}{n+1} + (-1)^n\frac{n!\,(k-n-1)!}{k!}\right]$$

$$\therefore \sum_{j=0}^{n}\frac{(-1)^j}{1+j}\binom{k}{j}\binom{k-1-j}{n-j} = \frac{1}{k+1}\left[\frac{k!}{(n+1)!\,(k-n-1)!} + (-1)^n\right] =$$

$$= \frac{1}{k+1}\left[(-1)^n + \binom{k}{n+1}\right]$$

SOLUTION 7.05

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } z = x + iy$$

$$z^2 = x^2 - y^2 + 2ixy$$

$$\textit{Now, } |z^2 - 2| = |4z + i| \Rightarrow \left|(x^2 - y^2 - 2) + 2ixy\right|^2 = |4x + (4y + 1)i|^2$$

$$\Rightarrow (x^2 - y^2 - 2)^2 + 4x^2y^2 = 16x^2 + (4y + 1)^2$$

$$\Rightarrow (x^2 - y^2)^2 + 4 - 4(x^2 - y^2) + 4x^2y = 16(x^2 + y^2) + 8y + 1$$

$$\Rightarrow (x^2 + y^2)^2 - 20(x^2 + y^2) + 3 = -8y^2 + 8y$$

$$\Rightarrow (x^2 + y^2 - 10)^2 = 97 - 8y^2 + 8y = 97 - 8\left(\left(y - \frac{1}{2}\right)^2 - \frac{1}{4}\right) = 99 - 8\left(y - \frac{1}{2}\right)^2$$

$$< 100$$

$$\Rightarrow |x^2 + y^2 - 10| < 10 \Rightarrow \left||z|^2 - 10\right| < 10 \Rightarrow |z|^2 - 10 \le \left||z|^2 - 10\right| < 10$$

$$\Rightarrow |z|^2 < 20 \Rightarrow |z| < 2\sqrt{5}$$

SOLUTION 7.06

Solution 2 by Soumava Chakraborty-Kolkata-India

$$(x + y)^{x^n + y^n} \overset{(1)}{=} (x + 1)^{x^n}(y + 1)^{y^n}$$

$$\textit{(1)} \Leftrightarrow (x^n + y^n)\ln(x + y) = x^n \ln(x + 1) + y^n \ln(y + 1)$$

$$\Leftrightarrow x^n \ln\left(\frac{x + y}{x + 1}\right) + y^n \ln\left(\frac{x + y}{y + 1}\right) \overset{(1)}{=} 0$$

$$\because x \ge 1 \therefore x + y \ge y + 1 \Rightarrow \frac{x + y}{y + 1} \ge 1$$

$$\Rightarrow \ln\left(\frac{x + y}{y + 1}\right) \ge 0 \Rightarrow y^n \ln\left(\frac{x + y}{y + 1}\right) \overset{(i)}{\ge} 0 \ (\because y^n \ge 1)$$

$$\textit{Also, } \because y \ge 1 \therefore x + y \ge x + 1 \Rightarrow \frac{x+y}{x+1} \ge 1$$

$$\Rightarrow \ln\left(\frac{x + y}{x + 1}\right) \ge 0 \Rightarrow x^n \ln\left(\frac{x + y}{x + 1}\right) \overset{(ii)}{\ge} 0 \ \ (\because x^n \ge 1)$$

(i)+(ii)$\Rightarrow$ *LHS of (1)*≥ 0*, equality if* $x = y = 1$

and $\because$ *LHS* $= 0 \therefore x = y = 1$ *(Answer)*

SOLUTION 7.07

Solution by Lazaros Zachariadis-Thessaloniki-Greece

$$\underbrace{1+\sin x+\cos x}_{\substack{\downarrow \\ LHS}} = \underbrace{(1+\sin x)(\sin x)^{\cos x}+(1+\cos x)(\cos x)^{\sin x}}_{\substack{\downarrow \\ RHS}}$$

$$RHS = (1+\sin x)\big(1+(\sin x-1)\big)^{\cos x}+(1+\cos x)\big(1+(\cos x-1)\big)^{\sin x}$$

$$\overset{Bernoulli}{\leq} (1+\sin x)(1+\cos x\cdot\sin x-\cos x)+(1+\cos x)(1+\cos x\sin x-\sin x)$$

$$= 1+\sin x-\cos^3 x+1+\cos x-\sin^3 x$$

$$= (1+\sin x+\cos x)-(\cos^3 x+\sin^3 x)+1$$

$$= LHS-(\cos^3 x+\sin^3 x)+1$$

So, $RHS = LHS$ *if* $\cos^3 x+\sin^3 x = 1$

$$x = 2k\pi, k\in\mathbb{Z} \vee x = 2k\pi+\frac{\pi}{2}, k\in\mathbb{Z}$$

SOLUTION 7.08

Solution by Tran Hong-Vietnam

Set $x := x-n, y = 1 \Rightarrow f(x)\geq 0, \forall x\in\mathbb{R}$ *(*)*

Let $y = \frac{1}{n} \Rightarrow f\left(x+\frac{1}{n}\right)\geq\left(1+\frac{1}{n}\right)^n f(x), \forall x\in\mathbb{R}$ *(1)*

Set: $x := x+\frac{1}{n} \Rightarrow f\left(x+\frac{2}{n}\right)\geq\left(1+\frac{1}{n}\right)^n f\left(x+\frac{1}{n}\right), \forall x\in\mathbb{R}$ *(2)*

$$\overset{(1),(2)}{\Rightarrow} f\left(x+\frac{2}{n}\right)\geq\left(1+\frac{1}{n}\right)^{2n} f(x), \forall x\in\mathbb{R}$$

By induction we have: $f\left(x+\frac{k}{n}\right)\geq\left(1+\frac{1}{n}\right)^{kn} f(x), \forall x\in\mathbb{R}, k\in\mathbb{N}$

Let $k = n \Rightarrow f(x+1)\geq\left(1+\frac{1}{n}\right)^{n^2} f(x), \forall x\in\mathbb{R}$ *(3)*

Suppose exists $x_0\in\mathbb{R}$ *such that* $f(x_0)\neq 0 \Rightarrow f(x_0)>0$ *(because (*)).*

From (3) we let n *from to* ∞

$$f(x_0+1)\geq\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n^2} f(x_0) = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^n f(x_0) = \lim_{n\to\infty} e^n = +\infty$$

But $f(t_0+1)$ *is real number* $\Rightarrow$ *contradiction* $\Rightarrow f(x) = 0, \forall x\in\mathbb{R}$*.*

SOLUTION 7.09

Solution 2 by Marian Ursărescu-Romania

More general: $1<a<b \Rightarrow f(ax) = f(bx)+x^2$*, let* $bx = t \Rightarrow x = \frac{t}{b} \Rightarrow$

$$f\left(\frac{a}{b}t\right) = f(t)+\frac{1}{b^2}t^2, \text{ now } \frac{a}{b} = \alpha_1, \alpha\in(0,1) \Rightarrow$$

$$\left.\begin{aligned} f(\alpha t)-f(t)&=\frac{1}{b^2}t^2\\ f(\alpha^2 t)-f(\alpha t)&=\frac{1}{b^2}\alpha^2t^2\\ &\vdots\\ f(\alpha^n t)-f(\alpha^{n-1}t)&=\frac{1}{b^2}\alpha^{2(n-1)}t^2\end{aligned}\right\}\Rightarrow$$

$$f(\alpha^4 t)-f(t)=\frac{1}{b^2}t^2\left(1+\alpha^2+\cdots+\alpha^{2(n-1)}\right)\Rightarrow\lim_{n\to\infty}f(\alpha^n t)-f(t)=\lim_{n\to\infty}\frac{1}{b^2}t^2\frac{\alpha^{2n}-1}{\alpha^2-1}\Rightarrow$$

$$f\left(\lim_{n\to\infty}\alpha^n t\right)-f(t)=\frac{1}{b^2}t^2\frac{1}{1-\alpha^2}\Rightarrow$$

$$f(0)-f(t)=\frac{1}{b^2}\frac{t^2}{1-\frac{a^2}{b^2}}\Rightarrow f(0)-f(t)=\frac{t^2}{b^2-a^2}$$

Let $f(0)=c\Rightarrow f(t)=c-\frac{t^2}{(b-a)(b+a)}$. *In our case* $a=2018, b=2019$

$$f(x)=c-\frac{x^2}{4037}$$

SOLUTION 7.10

Solution by Ravi Prakash-New Delhi-India

$$2017f'(x)+2018f(x)\le 2019\Rightarrow f'(x)+\frac{2018}{2017}f(x)\le\frac{2019}{2017}$$

Multiplying both sides by $e^{\frac{2018x}{2017}}$ ***to obtain:***

$$\frac{d}{dx}\left[e^{\frac{2018x}{2017}}f(x)\right]\le\frac{2019}{2017}e^{\frac{2018x}{2017}}\Rightarrow\frac{d}{dx}\left[e^{\frac{2018x}{2017}}\left(f(x)-\frac{2019}{2018}\right)\right]\le 0$$

$\Rightarrow F(x)=e^{\frac{2018x}{2017}}\left(f(x)-\frac{2019}{2018}\right)$ ***decreases on*** $[0,1]$

But $F(0)=F(1)=0$

$\therefore F(x)$ ***must be constant on*** $[0,1]\Rightarrow F(x)=F(0)=0\Rightarrow f(x)=\frac{2019}{2018}\forall x\in[0,1]$

SOLUTION 7.11

Solution by Kevin Soto Palacios – Peru

$$\frac{2\sqrt{ab}}{a+b}\le 1\quad\textit{(A)}$$

$$\frac{4ab}{(a+b)^2}\le 1\quad\textit{(B)}$$

$$x=\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\ \ \textit{(C)}\to MA\ge MG: x\ge 2,\ x^2-2=\frac{a}{b}+\frac{b}{a}\quad\textit{(D)}$$

La desigualdad es equivalente:

$$\mathbf{1+1+\frac{a}{4b}+\frac{b}{4a}+\frac{1}{2}+\frac{1}{2}\sqrt{\frac{a}{b}}+\frac{1}{2}\sqrt{\frac{b}{a}}\leq 2\left(\frac{a}{b}+\frac{b}{a}\right)}$$

$$\mathbf{2+\frac{1}{4}(x^2-2)+\frac{1}{2}(x+1)\leq 2(x^2-2)}$$

$$\mathbf{\Rightarrow 2\cdot 4+(x^2-2)+2(x+1)\leq 8(x^2-2)\Rightarrow 7x^2-2x-24\geq 0\to}$$

$$\mathbf{\to (x-2)(7x+12)\geq 0} \text{ (La desigualdad se mantiene)}$$

SOLUTION 7.12

Solution by Myagmarsuren Yadamsuren-Mongolia

$$\mathbf{|(a_1+\cdots+a_n)+(b_1+\cdots+b_n)| \overset{Cauchy-Schwarz}{\leq}}$$

$$\mathbf{\leq \left|\sqrt{(1^2+1^2)((a_1+\cdots+a_n)^2+(b_1+\cdots+b_n)^2)}\right| =}$$

$$\mathbf{= \left|\sqrt{2}\cdot\sqrt{(a_1+\cdots+a_n)^2+(b_1+\cdots+b_n)^2}\right| = \left|\sqrt{2}\big((a_1+\cdots+a_n)+(b_1+\cdots b_n)i\big)\right| =}$$

$$\mathbf{=\sqrt{2}\cdot|(a_1+b_1i)+\cdots+(a_n+b_ni)|\leq}$$

$$\mathbf{\leq\sqrt{2}\cdot(|a_1+ib_1|+\cdots+|a_n+ib_n|)=\sqrt{2}\cdot\left(\sum_{i=1}^{n}|z_i|\right)}$$

SOLUTION 7.13

Solution by Amit Dutta-Jamshedpur-India

$$\mathbf{\frac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}} \quad \text{(1)}$$

$$\mathbf{\frac{\tan^{-1}y}{\cot^{-1}y}=e^{\frac{4}{\pi}(\tan^{-1}x-\cot^{-1}x)}} \quad \text{(2)}$$

Equality (1)$\Rightarrow \mathbf{\frac{\tan^{-1}x}{\cot^{-1}x}=e^{\frac{4}{\pi}(\tan^{-1}y-\cot^{-1}y)}}$

RHS$\mathbf{>0\Rightarrow LHS=\frac{\tan^{-1}x}{\cot^{-1}x}>0\Rightarrow \tan^{-1}x>0\Rightarrow x>0}$

Similarly, from equality (2) $\mathbf{\Rightarrow y>0}$***. So,*** $\mathbf{x,y>0}$

Taking logarithm on both sides of equation (1)

$$\log(\tan^{-1}x) - \log(\cot^{-1}x) = \frac{4}{\pi}(\tan^{-1}y - \cot^{-1}y) \quad \textit{(3)}$$

Taking log on both sides of equation (2)

$$\log(\tan^{-1}y) - \log(\cot^{-1}y) = \frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x)$$

$$or\ \frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x) = \log(\tan^{-1}y) - \log(\cot^{-1}y) \quad \textit{(4)}$$

Equality (3)+(4)

$$\log(\tan^{-1}x) - \log(\cot^{-1}x) + \frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x)$$

$$= \log(\tan^{-1}y) - \log(\cot^{-1}y) + \frac{4}{\pi}(\tan^{-1}y - \cot^{-1}y)$$

Now, let $F(t) = \log(\tan^{-1}t) - \log(\cot^{-1}t) + \frac{4}{\pi}(\tan^{-1}t - \cot^{-1}t)$

$\Rightarrow$ *Equation (3)+(4)* $\Rightarrow F(x) = F(y)$

Now, differentiation $F(t)$ w.r.t t

$$F'(t) = \frac{1}{\tan^{-1}t\,(1+t^2)} + \frac{1}{\cot^{-1}t\,(1+t^2)} + \frac{4}{\pi}\left(\frac{1}{1+t^2} + \frac{1}{1+t^2}\right)$$

$$F'(t) = \frac{1}{(1+t^2)}\left\{\frac{1}{\tan^{-1}t} + \frac{1}{\cot^{-1}t}\right\} + \frac{8}{\pi(1+t^2)}$$

Using power mean inequality

$$\frac{(\tan^{-1}t)^{-1} + (\cot^{-1}t)^{-1}}{2} \geq \left(\frac{\tan^{-1}t + \cot^{-1}t}{2}\right)^{-1} \geq \left(\frac{\pi}{4}\right)^{-1} \quad \left\{\tan^{-1}t + \cot^{-1}t = \frac{\pi}{2}\right\}$$

$$\Rightarrow \frac{1}{\tan^{-1}t} + \frac{1}{\cot^{-1}t} \geq \frac{8}{\pi} \quad \textit{(5)}$$

$\Rightarrow F'(t) \geq \frac{8}{\pi(1+t^2)} + \frac{8}{\pi(1+t^2)} \geq \frac{16}{\pi(1+t^2)} > 0 \Rightarrow F(t)$ *is strictly increasing function*

But $F(x) = F(y) \Rightarrow x = y$

Equation (1) $\Rightarrow \frac{\tan^{-1}x}{\cot^{-1}x} = e^{\frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x)}\ \{\because x = y\}$

Taking log on both sides

$$\ln(\tan^{-1}x) - \ln(\cot^{-1}x) = \frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x)$$

Let $G(x) = \ln(\tan^{-1}x) - \ln(\cot^{-1}x) - \frac{4}{\pi}(\tan^{-1}x - \cot^{-1}x)$

$$G'(x) = \frac{1}{\tan^{-1}x\,(1+x^2)} + \frac{1}{\cot^{-1}x\,(1+x^2)} - \frac{4}{\pi}\left\{\frac{1}{1+x^2} + \frac{1}{1+x^2}\right\}$$

$$G'(x) = \frac{1}{(1+x^2)}\left\{\frac{1}{\tan^{-1}x} + \frac{1}{\cot^{-1}x} - \frac{8}{\pi}\right\}$$

From (V) $\Rightarrow \frac{1}{\tan^{-1}x} + \frac{1}{\cot^{-1}x} \geq \frac{8}{\pi} \Rightarrow G'(x) \geq 0 \Rightarrow G(x)$ *is an increasing function*

So, $G(x) = 0$ *can have only one real root. Also, we can see that real root exists only when*

$\tan^{-1}x = \cot^{-1}x \Rightarrow x = 1 \Rightarrow x = 1$ *is the only possible real root.*

SOLUTION 7.14

Solution by Ravi Prakash –New Delhi-India:

For $a, b, c, d > 0$*:*

$$\frac{a}{b}\cdot\frac{d}{c} + \frac{bc}{ad} - \frac{a^2+b^2}{ab}\cdot\frac{cd}{c^2+d^2} - \frac{c^2+d^2}{cd}\cdot\frac{ab}{a^2+b^2} \geq 0$$

$$\Leftrightarrow \frac{a^2d^2+b^2c^2}{abcd} - \frac{(a^2+b^2)^2c^2d^2 + (c^2+d^2)^2a^2b^2}{abcd(a^2+b^2)(c^2+d^2)} \geq 0$$

$$\Leftrightarrow (a^2d^2+b^2c^2)(a^2+b^2)(c^2+d^2) - [(a^2+b^2)^2c^2d^2 + a^2b^2(c^2+d^2)^2] \geq 0$$

$$\Leftrightarrow (a^2d^2+b^2c^2)[a^2c^2+b^2c^2+a^2d^2+b^2d^2]$$

$$-(a^4+b^4+2a^2b^2)c^2d^2 - (c^4+d^4+2c^2d^2)a^2b^2 \geq 0$$

$$\Leftrightarrow a^4c^2d^2 + a^2b^2c^4 + a^2b^2c^2d^2 + b^4c^4 +$$

$$+a^4d^4 + a^2b^2c^2d^2 + a^2b^2d^4 + b^4c^2d^2 -$$

$$-[a^4c^2d^2 + b^4c^2d^2 + 2a^2b^2c^2d^2 + a^2b^2c^4 + a^2b^2d^4 + 2a^2b^2c^2d^2] \geq 0$$

$$\Leftrightarrow (b^2c^2 - a^2d^2)^2 \geq 0$$

which is true. Consider

$$\left(\sum_{i=1}^{n}\frac{x_i}{y_i}\right)\left(\sum_{i=1}^{n}\frac{y_i}{x_i}\right)-\left(\sum_{i=1}^{n}\frac{x_i^2+y_i^2}{x_iy_i}\right)\left(\sum_{i=1}^{n}\frac{x_iy_i}{x_i^2+y_i^2}\right)=$$

$$=\sum_{i=1}^{n}\frac{x_i}{y_i}\cdot\frac{y_i}{x_i}+\sum_{i<j}\frac{x_i}{y_i}\cdot\frac{y_j}{x_j}+\sum_{i>j}\frac{x_i}{y_i}\cdot\frac{y_j}{x_j}$$

$$-\sum_{i=1}^{n}\frac{x_i^2+y_i^2}{x_iy_i}\cdot\frac{x_iy_i}{x_i^2+y_i^2}-\sum_{i<j}\frac{x_i^2+y_i^2}{x_iy_i}\cdot\frac{x_jy_j}{x_j^2+y_j^2}-\sum_{i>j}\frac{x_i^2+y_i^2}{x_iy_i}\cdot\frac{x_jy_j}{x_j^2+y_j^2}=$$

$$=n-n+\sum_{i<j}\left(\frac{x_iy_j}{y_ix_j}+\frac{x_j}{y_j}\cdot\frac{y_i}{x_i}-\frac{x_1^2+y_i^2}{x_iy_i}\cdot\frac{x_jy_j}{x_j^2+y_j^2}-\frac{x_j^2+y_j^2}{x_jy_j}\cdot\frac{x_iy_i}{x_i^2+y_i^2}\right)$$

$$=\sum_{i<j}\left(x_j^2y_i^2-x_i^2y_j^2\right)^2\frac{1}{x_ix_jy_iy_j\left(x_i^2+y_i^2\right)\left(x_j^2+y_j^2\right)}\geq 0$$

SOLUTION 7.15

Solution by Daniel Sitaru-Romania:

$$\frac{\sum_{i=1}^{n}a_i+a_k}{n+1}\overset{AM-HM}{\geq}\frac{n+1}{\sum_{i=1}^{n}\frac{1}{a_i}+\frac{1}{a_k}},k\in\overline{1,n}$$

$$\frac{\sum_{i=1}^{n}\frac{1}{a_i}+\frac{1}{a_k}}{n+1}\geq\frac{n+1}{1+a_k}$$

$$\frac{1}{n+1}\left(\sum_{k=1}^{n}\sum_{i=1}^{n}\frac{1}{a_i}+\sum_{k=1}^{n}\frac{1}{a_k}\right)\geq(n+1)\sum_{k=1}^{n}\frac{1}{1+a_k}$$

$$\frac{(n+1)\sum_{k=1}^{n}\frac{1}{a_k}}{n+1}\geq(n+1)\sum_{k=1}^{n}\frac{1}{1+a_k}$$

$$\sum_{k=1}^{n}\frac{1}{a_k}\geq(n+1)\sum_{k=1}^{n}\frac{1}{1+a_k}$$

$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}}\geq n+1$$

SOLUTION 7.16

Solution by Soumava Pal – Kolkata – India

$$P_k = \frac{k}{\sum_{i=1}^{k}\left(\frac{1}{a+(i-1)r}\right)} < \frac{\sum_{i=1}^{k}(a+(i-1)r)}{k} =$$

(by AM – HM)

$$= \frac{\frac{k}{2}(2a+(k-1)r)}{k} = a + \frac{(k-1)r}{2} = S_k$$

$$\sum_{k=1}^{n} P_k < \sum_{k=1}^{n} S_k =$$

$$= \sum_{k=1}^{n}\left(a + \frac{(k-1)r}{2}\right) = na + \frac{n(n-1)}{4}r < 2na + n(n-1)r =$$

$$= n(2a+(n-1)r)$$

$$\Rightarrow \sum_{k=1}^{n}\left(\frac{k}{\sum_{r=1}^{k}\left(\frac{1}{a+(i-1)r}\right)}\right) < (2a+(n-1)r)n$$

SOLUTION 7.17

Solution by Saptak Bhattacharya-Kolkata-India

We have

$$\frac{\left(\sum \frac{1}{a_i}\right)^n + \left(\sum \frac{1}{b_i}\right)^n}{2} \geq \frac{\left\{\sum\left(\frac{1}{a_i}+\frac{1}{b_i}\right)\right\}^n}{2^n} \geq 4^n\left(\sum \frac{1}{(a_i+b_i)}\right)^n$$

Thus, enough to show that

$$\left(\frac{\sum \frac{1}{(a_i+b_i)}}{n}\right)^n \geq \frac{1}{\prod(a_i+b_i)}$$

which clearly holds by AM ≥ GM

SOLUTION 7.18

Solution by Redwane El Mellass-Casablanca-Morroco

$$\textit{Let } f\left(0 < x < \frac{\pi}{2}\right) = \frac{\sin(x)}{x} \textit{ and } g(t \geq 0) = \sin(t) - t + \frac{t^3}{6}$$

$$\therefore f'(x) = \frac{\cos(x)\,(x - \tan(x))}{x^2} < 0(\tan(x) > x)$$

$$\therefore g'(t) = \cos(t) - 1 + \frac{t^2}{2} \textit{ and } g''(t) = t - \sin(t) \geq 0(\sin(t) \leq t)$$

$$\textit{So } g'(t) \geq g'(0) = 0 \Rightarrow g(t) \geq g(0) = 0 \ \textit{ we get } f(0 < x \leq 1) \geq \sin(1)$$

$$\textit{and } g(1) > 0 \Rightarrow \sin(1) > \frac{5}{6}. \textit{ So } f(0 < x \leq 1) > \frac{5}{6}.$$

$$\therefore \sum_{k=1}^{n} \frac{1}{a_k} \geq \frac{n^2}{\sum_{k=1}^{n} a_k} \geq n \Rightarrow 0 < \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}} \leq 1$$

$$\Rightarrow \left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \sin\left(\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}\right) + \frac{n}{\pi} = n \left(\frac{\sin\left(\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}\right)}{\frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}}} + \frac{1}{\pi} \right) >$$

$$> n\left(\frac{5}{6} + \frac{1}{\pi}\right) > n\left(\frac{5}{6} + \frac{1}{6}\right) = n.$$

SOLUTION 7.19

Solution by Soumitra Mandal - Chandar Nagore – India

$$(n+1)\left(\frac{1}{2a_1 + a_2 + \cdots + a_n} + \frac{1}{a_1 + 2a_2 + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + 2a_n}\right)$$

$$\leq \frac{1}{n+1}\left(\frac{2}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) + \frac{1}{n+1}\left(\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{1}{a_n}\right) + \cdots$$

$$+ \frac{1}{n+1}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{2}{a_n}\right) = \frac{1}{n+1}\left(\sum_{k=1}^{n} \frac{n+1}{a_k}\right) = \sum_{k=1}^{n} \frac{1}{a_k}$$

SOLUTION 7.20

Solution by Rozeta Atanasova – Skopje – Macedonia

$$\textit{WLOG let } a \geq b \geq c \Rightarrow$$

$$(a^4 + b^4 + c^4)\left(\frac{1}{a^4} + \frac{1}{a^4} + \frac{1}{c^4}\right) \geq \textit{(Rearrangement inequality)}$$

$$\geq (a^2b^2 + a^2c^2 + b^2c^2)\left(\frac{1}{a^2b^2} + \frac{1}{a^2c^2} + \frac{1}{b^2c^2}\right) \geq \quad \textit{(CSB inequality)}$$

$$\geq \left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)^2 = \left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right) \geq \textit{(Rearrangement inequality)}$$

$$\geq \left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)\left(\frac{a}{a} + \frac{c}{c} + \frac{b}{b}\right) = 3\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$$

SOLUTION 7.21

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

We have:

$$2a^6 - a^5 - 3a^3 + a^2 + 1 = 2a^5(a-1) + a^4(a-1) + a^3(a-1) - 2a^2\,(a-1) -$$
$$a(a-1) - (a-1)$$
$$= (a-1)\left(2a^5 + a^4 + a^3 - 2a^2 - a - 1\right)$$
$$= (a-1)^2(2a^4 + 3a^3 + 4a^2 + 2a + 1) \geq 0 \ \ (a > 0 \text{ and } (a-1)^2 \geq 0)$$
$$\Rightarrow 2a^6 - a^5 - 3a^3 + a^2 + 1 \geq 0 \Leftrightarrow 2a^6 - a^5 + a^2 + 1 \geq 3a^3 \Leftrightarrow$$
$$\Leftrightarrow 2a^6 - a^5 + b^4 + a^2 + 1 \geq 3a^3 + b^4$$
$$\Leftrightarrow \frac{1}{2a^6 - a^5 + b^4 + a^2 + 1} \leq \frac{1}{3a^3 + b^4} \Leftrightarrow \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} \leq \frac{ab}{3a^3 + b^4} \quad (1)$$

By inequality AM – GM for 4 positive real numbers:

$$3a^3 + b^4 = a^3 + a^3 + a^3 + b^4 \geq 4\sqrt[4]{a^3 \cdot a^3 \cdot a^3 \cdot b^4} = 4\sqrt[4]{a^9 \cdot b^4} =$$
$$= 4a^2b\sqrt[4]{a} \Leftrightarrow \frac{ab}{3a^3 + b^4} \leq \frac{ab}{4a^2b\sqrt[4]{a}} = \frac{1}{4a\sqrt[4]{a}}$$

Therefore (1) and by AM-GM:

$$\Rightarrow \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} \leq \frac{1}{4a\sqrt[4]{a}} \leq \frac{1}{4a} \cdot \frac{1}{4}\left(\frac{1}{a} + 1 + 1 + 1\right) = \frac{1}{16a}\left(\frac{1}{a} + 3\right)$$

Similar: $\frac{bc}{2b^6 - b^5 + c^4 + b^2 + 1} \leq \frac{1}{16b}\left(\frac{1}{b} + 3\right); \frac{ca}{2c^6 - c^5 + a^4 + c^2 + 1} \leq \frac{1}{16c}\left(\frac{1}{c} + 3\right)$

Therefore: $\Rightarrow P = \frac{ab}{2a^6 - a^5 + b^4 + a^2 + 1} + \frac{bc}{2b^6 - b^5 + c^4 + b^2 + 1} + \frac{ca}{2c^6 - c^5 + a^4 + c^2 + 1} \leq$

$$\leq \frac{1}{16a}\left(\frac{1}{a} + 3\right) + \frac{1}{16b}\left(\frac{1}{b} + 3\right) + \frac{1}{16c}\left(\frac{1}{c} + 3\right)$$
$$\Leftrightarrow P \leq \frac{1}{16}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{3}{16}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad (2)$$

I have $a^2 + b^2 + c^2 = 3abc$ ***and inequality:*** $(x + y + z) \geq \sqrt{3(xy + yz + zx)}$ ***with:***

$$x = \frac{a}{bc}, y = \frac{b}{ca}, z = \frac{c}{ab}:$$

$$3 = \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq \sqrt{3\left(\frac{a}{bc} \cdot \frac{b}{ca} + \frac{b}{ca} \cdot \frac{c}{ab} + \frac{c}{ab} \cdot \frac{a}{bc}\right)} = \sqrt{3\left(\frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2}\right)} \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \quad (3)$$

Other let (3) and inequality AM-GM. I have:

$$3 \geq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \left(\frac{1}{a^2} + 1\right) + \left(\frac{1}{b^2} + 1\right) + \left(\frac{1}{c^2} + 1\right) - 3 \geq \frac{2}{a} + \frac{2}{b} + \frac{2}{c} - 3 \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 3 \quad (4)$$

$$\textit{Let (2), (3), (4):} \Rightarrow P \le \frac{1}{16}\cdot 3 + \frac{3}{16}\cdot 3 = \frac{12}{16} = \frac{3}{4} \Rightarrow P \le \frac{3}{4} \Rightarrow P_{Max} = \frac{3}{4}$$

$$\textit{Equality occurs if:} \begin{cases} a, b, c > 0; a^2 + b^2 + c^2 = 3abc \\ a - 1 = b - 1 = c - 1 = 0 \\ a^3 = b^4; b^3 = c^4; c^3 = a^4 \\ \frac{1}{a} = \frac{1}{b} = \frac{1}{c} = 1; \frac{a}{bc} = \frac{b}{ca} = \frac{c}{ab} \end{cases} \Leftrightarrow a = b = c = 1.$$

Maximum of P be: $\frac{3}{4}$ *then* $a = b = c = 1.$

SOLUTION 7.22

Solution by Ravi Prakash-New Delhi-India

$$\textit{Let } S = \prod_{k=1}^{n} (x_k)^k \Rightarrow \ln s = \sum_{k=1}^{n} k \ln(x_k)$$

$$\textit{Let } F = \sum_{k=1}^{n} k \ln(x_k) + \lambda\left(\sum_{j=1}^{n} x_i - n^2\right)$$

$$\Rightarrow \frac{\partial F}{\partial x_i} = \frac{i}{x_i} + \lambda \quad (i = 1, 2, \dots, n)$$

$$\textit{Set } \frac{\partial F}{\partial x_i} = 0 \Rightarrow -\lambda = \frac{x_i}{i} \quad (i = 1, 2, \dots, n)$$

Thus,

$$\frac{x_1}{1} = \frac{x_2}{2} = \cdots = \frac{x_n}{n} = -\lambda \Rightarrow x_1 = -\lambda, x_2 = -2\lambda, \dots, x_n = -n\lambda$$

Now,

$$n^2 = \sum_{j=1}^{n} x_j = (-\lambda)\sum_{j=1}^{n} (j) \Rightarrow -\lambda = \frac{2n^2}{n(n+1)} = \frac{2n}{n+1}$$

Thus,

$$x_k = -k\lambda = \frac{2nk}{n+1}$$

$$\max S = \left(\frac{2n}{n+1}\right)^{1+2+\cdots+n} \prod_{k=1}^{n} (k^k) = \left(\frac{2n}{n+1}\right)^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} (k^k)$$

SOLUTION 7.23

Solution by Khalef Ruhemi-Iordania

$$\textit{Find } \sum_{k=1}^{k=n} \psi(k)\textit{, use } \int_0^1 \frac{1 - x^{k-1}}{1 - x}\cdot dx = \psi(k) + \gamma; \gamma\textit{: Euler's constant}$$

$$\therefore I_n := \sum_{k=1}^{k=n} \psi(k) = \sum_{k=1}^{k=n} \left(-\gamma + \int_0^1 \left(\frac{1 - x^{k-1}}{1 - x}\right) dx\right)$$

$$= \sum_{k=1}^{k=n} -\gamma + \sum_{k=1}^{k=n} \int_0^1 \frac{1 - x^{k-1}}{1 - x} \cdot dx = -\gamma \sum_{k=1}^{k=n} 1 + \int_0^1 \left(\frac{1}{1 - x} \left(\sum_{k=1}^{k=n} 1 - \sum_{k=1}^{k=n} x^{k-1} \right) \right) dx$$

$$\textit{use } \sum_{k=1}^{k=n} 1 = n, \sum_{k=1}^{k=n} x^{k-1} = \frac{x^n - 1}{x - 1}$$

$$\textit{Then } I_n = -n\gamma + \int_0^1 \left(\frac{1}{1-x} \right) \left(n - \frac{(x^n - 1)}{(x-1)} \right) dx$$

$$\therefore I_n = -n\gamma + \int_0^1 \left(\frac{1}{1 - x} \right) \left(n + \frac{x^n - 1}{1 - x} \right) dx = -n\gamma + \int_0^1 \underbrace{\frac{(x^n - 1 + n - nx)}{(1 - x)^2}}_{(Integrating\ by\ parts)} dx$$

$$\therefore I_n = -n\gamma + \frac{(x^n - 1 + n - nx)}{(1 - x)} \Bigg|_0^{x \to 1} - \int_0^1 \left(\frac{1}{1 - x} \right) (nx^{n-1} - n) dx$$

$$\textit{Since } \lim_{x \to 1} \left(\frac{x^n - 1 + n - nx}{1 - x} \right) = \lim_{x \to 1} \left(\frac{nx^{n-1} - n}{-1} \right) = \frac{n - n}{-1} = 0$$

$$\textit{Then, } I_n = -n\gamma + 1 - n + n \cdot \int_0^1 \left(\frac{1 - x^{n-1}}{1 - x} \right) dx$$

$$= -n\gamma + 1 - n + n \left(\frac{\Gamma'(n)}{\Gamma(n)} + \gamma \right) = -n\gamma + 1 - n + n\gamma + n\psi(n)$$

$$\therefore I_n = 1 - n + n\psi(n), \textit{ But } \psi(n) = -\gamma + H_{n-1}$$

$$I_n = 1 - n + n(-\gamma + H_{n-1}) = 1 - n - n\gamma + nH_{n-1}$$

$$\therefore \sum_{k=1}^{k=n} \psi(k) = nH_{n-1} - n\gamma - (n - 1). \textit{ But } H_{n-1} = H_n - \frac{1}{n}$$

$$\therefore \sum_{k=1}^{k=n} \psi(k) = n \left(H_n - \frac{1}{n} \right) - n\gamma - n + 1 = nH_n - 1 - n\gamma - n + 1$$

$$= nH_n - n\gamma - n$$

$$\therefore \sum_{k=1}^{k=n} \psi(k) = nH_n - n\gamma - n. \textit{ Take } n = 2017$$

$$\Rightarrow \sum_{k=1}^{u=2017} \psi(k) = (2017)H_{2017} - (2017)\gamma - (2017) = aH_b - c\gamma - d$$

$$\therefore a = 2017 = b = c = d$$

$$\therefore a + b + c + d = (4)(2017) = 8068$$

SOLUTION 7.24

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x}{y+z}+\frac{y}{z+x}=\frac{x(z+x)+y(y+z)}{(y+z)(z+x)}=\frac{z(x+y)+x^2+y^2}{(y+z)(z+x)}$$

$$\overset{Chebyshev}{\geq}\frac{z(x+y)+\frac{1}{2}(x+y)^2}{(y+z)(z+x)}=\frac{(x+y)(z+x+y+z)}{2(y+z)(z+x)}$$

$$\overset{A-G}{\geq}\frac{2(x+y)\sqrt{(y+z)(z+x)}}{2(y+z)(z+x)}=\frac{x+y}{\sqrt{(y+z)(z+x)}}\underset{(1)}{\overset{G\leq A}{\geq}}\frac{2(x+y)}{x+y+2z}$$

$$\therefore LHS\geq\frac{2(x+y)}{x+y+2z}+2\sqrt{\frac{x+y+2z}{2(x+y)}}\ \textit{(using (1))}$$

$$=\frac{2(x+y)}{x+y+2z}+\sqrt{\frac{x+y+2z}{2(x+y)}}+\sqrt{\frac{x+y+2z}{2(x+y)}}\overset{A-G}{\geq}3\sqrt[3]{\frac{2(x+y)(x+y+2z)}{2(x+y)(x+y+2z)}}=3$$

$$\therefore \textit{required min value}=3$$

SOLUTION 7.25

Solution by Geanina Tudose – Romania

We can denote

$$\log_a b=x$$

$$\log_b c=y$$

$$\log_c a=z$$

$$\Omega(a,b,c)=\Omega(x,y,z)=\sum_{cyc}\frac{x^2+xy+y^2}{x+y}$$

Where $x,y,z>0$ ***subject to*** $xyz=1$

We have $\Omega(x,y,z)=\sum\frac{(x+y)^2-xy}{x+y}=\sum_{cyc}\left[(x+y)-\frac{xy}{x+y}\right]$

From $HM\leq AM$ ***we have*** $-\frac{xy}{x+y}\geq-\frac{x+y}{4}$

$$\Rightarrow\Omega(x,y,z)\geq\sum_{cyc}(x+y)-\frac{x+y}{4}=\sum_{cyc}\frac{3(x+y)}{4}=$$

$$=\frac{3}{2}(x+y+z)\overset{AM\geq GM}{\geq}\frac{3}{2}\cdot 3\sqrt[3]{xyz}=\frac{9}{2}$$

$\Omega(a,b,c)=\Omega(x,y,z)\geq\frac{9}{2}$ ***(min. value attained for*** $x=y=z=1$ ***i.e.*** $a=b=c$***)***

SOLUTION 7.26

Solution by Hoang Le Nhat Tung – Hanoi – Vietnam

By inequality AM-GM. We have:

$$\frac{x^3}{y\sqrt{x^3+8}}=\frac{x^3}{y\sqrt{(x+2)(x^2-2x+4)}}\geq\frac{x^3}{y\cdot\left(\frac{x+2+x^2-2x+4}{2}\right)}=\frac{2x^3}{y(x^2-x+6)}$$

Similar: $\frac{y^3}{z\sqrt{y^3+8}}\geq\frac{2y^3}{z(y^2-y+6)}$; $\frac{z^3}{x\sqrt{z^3+8}}\geq\frac{2z^3}{x(z^2-z+6)}$

Therefore:

$$\Rightarrow P=\frac{x^3}{y\sqrt{x^3+8}}+\frac{y^3}{z\sqrt{y^3+8}}+\frac{z^3}{x\sqrt{z^3+8}}\geq 2\left(\frac{x^3}{y(x^2-x+6)}+\frac{y^3}{z(y^2-y+6)}+\frac{z^3}{x(z^2-z+6)}\right)\quad (1)$$

Other, by inequality CBS:

$$\frac{x^3}{y(x^2-x+6)}+\frac{y^3}{z(y^2-y+6)}+\frac{z^3}{x(z^2-z+6)}=\frac{x^4}{xy(x^2-x+6)}+\frac{y^4}{yz(y^2-y+6)}+\frac{z^4}{zx(z^2-z+6)}\geq$$

$$\geq\frac{(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)}\quad (2)$$

Then (1), (2): $\Rightarrow P\geq\frac{2(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)}$ ***(3)***

We will prove that: $\frac{2(x^2+y^2+z^2)^2}{xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)}\geq 1$ ***(4)***

$$(4)\Leftrightarrow 2(x^2+y^2+z^2)^2\geq xy(x^2-x+6)+yz(y^2-y+6)+zx(z^2-z+6)$$

$$\Leftrightarrow 2(x^4+y^4+z^4)+4(x^2y^2+y^2z^2+z^2x^2)\geq(x^3y+y^3z+z^3x)-$$

$$-(x^2y+y^2z+z^2x)+6(xy+yz+zx)$$

$$\Leftrightarrow 6(x^4+y^4+z^4)+12(x^2y^2+y^2z^2+z^2x^2)\geq 3(x^3y+y^3z+z^3x)-$$

$$-3(x^2y+y^2z+z^2x)+18(xy+yz+zx)$$

$$\Leftrightarrow 6(x^4+y^4+z^4)+12(x^2y^2+y^2z^2+z^2x^2)\geq$$

$$\geq 3(x^3y+y^3z+z^3x)-(x+y+z)(x^2y+y^2z+z^2x)+$$

$$+2(x+y+z)^2(xy+yz+zx)$$

(Because $3=x+y+z$ ***and*** $18=2(x+y+z)^2$***)***

$$\Leftrightarrow 6(x^4+y^4+z^4)+12(x^2y^2+y^2z^2+z^2x^2)+$$

$$+\left(x^3y+y^3z+z^3x+x^2y^2+y^2z^2+z^2x^2+xyz(x+y+z)\right)\geq$$

$$\geq 3(x^3y+y^3z+z^3x)+2(x^2+y^2+z^2+2xy+2yz+2zx)(xy+yz+zx)$$

$$\Leftrightarrow 6(x^4+y^4+z^4)+12(x^2y^2+y^2z^2+z^2x^2)+$$

$$+\left(x^3y+y^3z+z^3x+x^2y^2+y^2z^2+z^2x^2+xyz(x+y+z)\right)\geq$$

$$\geq 5(x^3y+y^3z+z^3x)+2(xy^3+yz^3+zx^3)+4(x^2y^2+y^2z^2+z^2x^2)+$$
$$+10xyz(x+y+z)$$
$$\Leftrightarrow 6(x^4+y^4+z^4)+9(x^2y^2+y^2z^2+z^2x^2)\geq 4(x^3y+y^3z+z^3x)+$$
$$+2(xy^3+yz^3+zx^3)+9xyz(x+y+z) \quad (5)$$

By AM-GM I have:

$$x^4+y^4+z^4=\frac{x^4+x^4+x^4+y^4}{4}+\frac{y^4+y^4+y^4+z^4}{4}+\frac{z^4+z^4+z^4+x^4}{4}\geq$$
$$\geq \frac{4x^3y}{4}+\frac{4y^3z}{4}+\frac{4z^3x}{4}$$
$$\Rightarrow x^4+y^4+z^4\geq x^3y+y^3z+z^3x \Leftrightarrow 4(x^4+y^4+z^4)\geq 4(x^3y+y^3z+z^3x) \quad (6)$$
$$x^4+y^4+z^4=\frac{x^4+y^4+y^4+y^4}{4}+\frac{y^4+z^4+z^4+z^4}{4}+\frac{z^4+x^4+x^4+x^4}{4}\geq$$
$$\geq \frac{4xy^3}{4}+\frac{4yz^3}{4}+\frac{4zx^3}{4}$$
$$\Rightarrow x^4+y^4+z^4\geq xy^3+yz^3+zx^3 \Leftrightarrow 2(x^4+y^4+z^4)\geq 2(xy^3+yz^3+zx^3) \quad (7)$$
$$x^2y^2+y^2z^2+z^2x^2=\frac{x^2(y^2+z^2)}{2}+\frac{y^2(z^2+x^2)}{2}+\frac{z^2(x^2+y^2)}{2}\geq$$
$$\geq \frac{x^2\cdot 2yz}{2}+\frac{y^2\cdot 2zx}{2}+\frac{z^2\cdot 2xy}{2}$$
$$\Rightarrow x^2y^2+y^2z^2+z^2x^2\geq xyz(x+y+z) \Leftrightarrow 9(x^2y^2+y^2z^2+z^2x^2)\geq 9xyz(x+y+z)$$

(8)

Then (6), (7), (8):

$$\Rightarrow 6(x^4+y^4+z^4)+9(x^2y^2+y^2z^2+z^2x^2)\geq 4(x^3y+y^3z+z^3x)+$$
$$+2(xy^3+yz^3+zx^3)+9xyz(x+y+z)$$

$\Rightarrow$ ***Inequality (5) True*** $\Rightarrow$ ***(4) True***

Then (3), (4): $\Rightarrow P\geq 1\Rightarrow P_{Min}=1$***. Equality occurs if:***

$$\Leftrightarrow \begin{cases} x,y,z>0; x+y+z=3 \\ x+2=x^2-2x+4; y+2=y^2-2y+4; z+2=z^2-2z+4 \\ \dfrac{x^2}{xy(x^2-x+6)}=\dfrac{y^2}{yz(y^2-y+6)}=\dfrac{z^2}{zx(z^2-z+6)} \\ x=y=z>0 \end{cases}$$

Therefore Minimum of P ***is:*** 1 ***then*** $x=y=z=1$***.***

SOLUTION 7.27

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo x, y, z números reales no negativos de tal manera que

$$x + y + z = 1$$

Hallar el máximo y mínimo valor de

$$E = (y+z)\sqrt{(1+x)} + (z+x)\sqrt{(1+y)} + (x+y)\sqrt{1+z}$$

Para hallar el máximo valor

Aplicamos la desigualdad de Cauchy

$$E = (y+z)\sqrt{(1+x)} + (z+x)\sqrt{(1+y)} + (x+y)\sqrt{1+z} \le$$

$$\le \sqrt{(y+z+z+x+y)\big((y+z)(1+x)+(z+x)(1+y)+(x+y)(1+z)\big)}$$

$$= 2\sqrt{2(x+y+z)\big(2(x+y+z)+2(xy+yz+zx)\big)} =$$

$$= \sqrt{2\big(2+2(xy+yz+zx)\big)} \le \sqrt{2\left(2+\frac{2(x+y+z)^2}{3}\right)} = \sqrt{2\left(2+\frac{2}{3}\right)} = \sqrt{\frac{16}{3}} = \frac{4\sqrt{3}}{3}$$

La igualdad se alcanza cuando $x = y = z = \frac{1}{3}$

Para hallar el mínimo valor

Como $x, y, z \ge 0 \Leftrightarrow \sqrt{1+x} \ge 1, \sqrt{1+y} \ge 1, \sqrt{1+z} \ge 1$

$$\Rightarrow E = (y+z)\sqrt{(1+x)} + (z+x)\sqrt{(1+y)} + (x+y)\sqrt{1+z} \ge$$

$$\ge (y+z) + (z+x) + (x+y) = 2(x+y+z) = 2$$

La igualdad se alcanza cuando $x = 1, y = z = 0$ y sus permutaciones.

SOLUTION 7.28

Solution by Nguyen Van Nho-Nghe An-Vietnam

***We have:** $\sum xy \sum \frac{x}{y} \ge \left(\sum x^2\right)^2 = 9 \to \sum \frac{x}{y} \ge \frac{9}{\sum xy}$ **and** $\sum xy \le \frac{(\sum x)^2}{3} = 3.$*

$$Q = \sum\left(\frac{1}{x(2y^2-yz+2z^2)} + \frac{x}{3y}\right) + \frac{5}{3}\sum\frac{x}{y} \overset{AM-GM}{\ge} \sum\frac{2}{3y(2y^2-yz+2z^2)} + \frac{15}{\sum xy}$$

$$\overset{AM-GM}{\ge} \sum\frac{4}{3y+2y^2-yz+2z^2} + \frac{15}{\sum xy} \ge \frac{36}{3\sum y + \sum(2y^2-yz+2z^2)} + \frac{15}{\sum xy}$$

$$= \sum\frac{36}{9+4(\sum x)^2 - 9\sum xy} + \frac{15}{\sum xy} = \sum\frac{4}{5-\sum xy} + \frac{9}{\sum xy} + \frac{6}{\sum xy}$$

$$\geq \frac{(2+3)^2}{5-\sum xy+\sum xy}+\frac{6}{\sum xy} \geq 5+\frac{6}{3}=7$$

$$Q=7 \leftrightarrow x=y=z=1. \text{ So: } \min Q=7.$$

SOLUTION 7.29

Solution by Daniel Sitaru-Romania

$$\frac{\sum_{cyc} \sin\frac{A}{2}\sin\frac{B}{2}}{\sum_{cyc}\frac{w_a+w_b}{h_c}}=\frac{\sum_{cyc}\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)}{bc\cdot ac}}}{\frac{1}{2S}\sum_{cyc} c(w_a+w_b)}=$$

$$=\frac{2S}{\sqrt{abcs}}\cdot\frac{S\cdot\sum_{cyc}\sqrt{\frac{s-c}{c}}}{\sum_{cyc}(b+c)w_a}=\frac{2S^2}{\sqrt{sabc}}\cdot\frac{\sum_{cyc}\sqrt{\frac{s-a}{a}}}{2\sum_{cyc}\sqrt{bcs(s-a)}}=$$

$$=\frac{2S^2}{2\sqrt{sabc}}\cdot\frac{1}{\sqrt{sabc}}\cdot\frac{\sum_{cyc}\sqrt{\frac{s-a}{a}}}{\sum_{cyc}\sqrt{\frac{s-a}{a}}}=\frac{S^2}{sabc}=\frac{S^2}{4RSs}=\frac{S}{4Rs}=\frac{rs}{4Rs}=\frac{r}{4R}$$

SOLUTION 7.30

Solution by Tran Hong-Vietnam

We have: $xyz>0$. ***Must show that:***

$$\left(\frac{xy}{z}\right)^4+\left(\frac{yz}{x}\right)^4+\left(\frac{zx}{y}\right)^4 \geq xyz\sqrt[4]{27(x^4+y^4+z^4)}$$

$$a=\frac{xy}{z};b=\frac{yz}{x};c=\frac{zx}{y} \Rightarrow abc=xyz>0; x^2=ac>0; y^2=ab>0; z^2=bc>0;$$

$$\Rightarrow a,b,c>0$$

$$a^4+b^4+c^4 \geq abc\sqrt[4]{27(a^2c^2+a^2b^2+b^2c^2)}$$

$$\Leftrightarrow (a^4+b^4+c^4)^4 \geq 27(abc)^4(a^2b^2+b^2c^2+c^2a^2) \quad (*)$$

(*) true because:

$$\therefore (a^4+b^4+c^4)^3 \overset{(Cauchy)}{\geq} \left\{3\sqrt[3]{(abc)^4}\right\}^3=27(abc)^4$$

$$\therefore a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2$$

$$\Rightarrow \textit{Equality} \Leftrightarrow a=b=c \Rightarrow (x,y,z)\in\{(a,a,a);(-a,-a,a);(a,-a,-a);(-a,a,-a)\}$$

$$\therefore a^4-4a^3+6a^2-4a+1=0 \quad (a>0)$$

$$\Leftrightarrow (a-1)^4 = 0 \Leftrightarrow a = 1$$

$$\Rightarrow (x,y,z) \in \{(1,1,1);(-1,-1,1);(1,-1,-1);(-1,1,-1)\}$$

SOLUTION 7.31

Solution by Djeeraj Badera-India

$A \in M_3(\mathbb{R})$ ***then characteristics polynomial has highest degree*** 3

$\therefore$ ***We have to find a polynomial their eigen values***

$\therefore \det(A^2+2A+2I_3)=0$ $\therefore$ ***then polynomial is*** $x^2+2x+2=0$

It has two different eigen values $(-1+i)$ ***and*** $(-1-i)$

[by solving quadratic equation] . Here $|A+I|=0$

$\therefore$ ***one eigen value of*** A ***is*** -1 $\therefore$ ***characteristic polynomial is***

$$=(x+1)(x^2+2x+2)=x^3+2x^2+2x+x^2+2x+2=x^3+3x^2+4x+2$$

$\therefore$ ***then*** $\det(A)=$ ***product of eigen value*** $=-2$

SOLUTION 7.32

Solution by Ravi Prakash-New Delhi-India

$$\alpha = \begin{vmatrix} \frac{1}{a+x} & \frac{1}{b+x} & \frac{1}{c+x} \\ \frac{1}{a+y} & \frac{1}{b+y} & \frac{1}{c+y} \\ \frac{1}{a+z} & \frac{1}{b+z} & \frac{1}{c+z} \end{vmatrix}$$

$$C_3 \to C_3 - C_2, C_2 \to C_2 - C_1,$$

$$\alpha = \begin{vmatrix} \frac{1}{a+x} & \frac{a-b}{(a+x)(b+x)} & \frac{b-c}{(b+x)(c+x)} \\ \frac{1}{a+y} & \frac{a-b}{(b+y)(a+y)} & \frac{b-c}{(b+y)(c+y)} \\ \frac{1}{a+z} & \frac{a-b}{(a+z)(b+z)} & \frac{b-c}{(b+z)(c+z)} \end{vmatrix}$$

$$= \frac{(a-b)(b-c)\alpha_1}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)}$$

where

$$\alpha_1 = \begin{vmatrix} (b+x)(c+x) & c+x & a+x \\ (b+y)(c+y) & c+y & a+y \\ (b+z)(c+z) & c+z & a+z \end{vmatrix}$$

$$C_3 \to C_3 - C_2, C_1 \to C_1 - bC_2$$

$$\alpha_1 = \begin{vmatrix} x(c+x) & c+x & a-c \\ y(c+y) & c+y & a-c \\ z(c+z) & c+z & a-c \end{vmatrix}$$

Therefore $(a-c)$ common from C_3 and use $C_2 \to C_2 - cC_3$

$$\alpha_1 = (a-c)\begin{vmatrix} x(c+x) & x & 1 \\ y(c+y) & y & 1 \\ z(c+z) & z & 1 \end{vmatrix}$$

$$C_1 \to C_1 - cC_2$$

$$\alpha_1 = (a-c)\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = -(a-c)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\therefore |Num\ of\ \alpha| = |\beta\gamma|$$

Denominator of $\alpha = (a+x)(b+x)(c+x)(a+y)(b+y)(c+y)(a+z)(b+z)(c+z)$

$$\le \left(\frac{a+x+b+x+c+x+a+y+b+y+c+y+a+z+b+z+c+z}{9}\right)^9$$

$$= \left(\frac{a+b+c+x+y+z}{3}\right)^9$$

$$\Rightarrow 3^9 \text{ (Denominator of } \alpha) \le (a+b+c+x+y+z)^9$$

Thus, $3^9|\alpha| = \frac{3^9|Num\ of\ \alpha|}{Den\ of\ \alpha} \ge \frac{|\beta\gamma|}{(a+b+c+x+y+z)^9}$

SOLUTION 7.33

Solution by Marian Ursărescu-Romania

Equation $\Leftrightarrow 2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} = 4\sqrt{3} + 9\sqrt{2}$

If $x < 0 \Rightarrow 2^x \cdot 3^{\frac{1}{x}} < 1$ *and* $3^x \cdot 2^{\frac{1}{x}} < 1 \Rightarrow$

$2^x \cdot 3^{\frac{1}{x}} + 3^x \cdot 2^{\frac{1}{x}} < 2 \Rightarrow$ *equation can't have negative solutions*

Let $x > 0$; $x = \frac{1}{2}$ *and* $x = 2$ *are solutions for this equation. We've proved that this are its only solutions.*

Let $p: (0, +\infty) \to \mathbb{R}; p(x) = a^x b^{\frac{1}{x}}, a, b > 1$

We show that p *is strictly increasing for* $\left(\sqrt{\log_a b}, +\infty\right)$

and strictly decreasing for $\left(0, \sqrt{\log_a b}\right)$ *(1)*

p strictly increasing for $\left(\sqrt{\log_a b}, +\infty\right) \Leftrightarrow \forall x_1, x_2 > \sqrt{\log_a b}$

Such that $x_1 < x_2 \Rightarrow p(x_1) < p(x_2) \Leftrightarrow$

$$a^{x_1} b^{\frac{1}{x_1}} < a^{x_2} b^{\frac{1}{x_2}} \Leftrightarrow b^{\frac{x_2 - x_1}{x_1 x_2}} < a^{x_2 - x_1} \Leftrightarrow$$

$b < a^{x_1 x_2}$ (because $a, b > 1$ and $x_1 < x_2$) $\Leftrightarrow$

$\log_a b < x_1 x_2$, relation which is true because $x_1, x_2 > \sqrt{\log_a b}$

Similarly, for $\left(0, \sqrt{\log_a b}\right)$

Let $p_1(x) = 2^x \cdot 3^{\frac{1}{x}}$ and $p_2(x) = 3^x \cdot 2^{\frac{1}{x}}$

For (1)$\Rightarrow$ p_1 it is increasing for $\left(\sqrt{\log_2 3}, +\infty\right)$ and strictly decreasing for $\left(0, \sqrt{\log_2 3}\right)$ (2)

For (2)$\Rightarrow$ p_2 it is strictly increasing for $\left(\sqrt{\log_3 2}, +\infty\right)$ and strictly decreasing for $\left(0, \sqrt{\log_3 2}\right)$. Because $\log_3 2 < \log_2 3 \Rightarrow p_1(x) + p_2(x)$ it is strictly decreasing for $\left(0, \sqrt{\log_3 2}\right) \Rightarrow$ for this interval the equation

$p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ has a unique solution $x = \frac{1}{2}$.

$p_1(x) + p_2(x)$ it is strictly increasing for $\left(\sqrt{\log_2 3}, +\infty\right) \Rightarrow$ for this interval the equation

$p_1(x) + p_2(x) = 4\sqrt{3} + 9\sqrt{2}$ has a unique solution $x = 2$.

For internal $\left(\sqrt{\log_3 2}, \sqrt{\log_2 3}\right)$, $p_1(x) + p_2(x) < 4\sqrt{3} + 9\sqrt{2} \Rightarrow$ the only solutions are

$$x = \frac{1}{2}, x = 2$$

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