ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $r_a = 3r$, the following relationship holds :

$$w_a^2 + m_a^2 + p_a^2 = h_a^2 + g_a^2 + n_a^2 + \frac{21r(R-2r)}{8}$$

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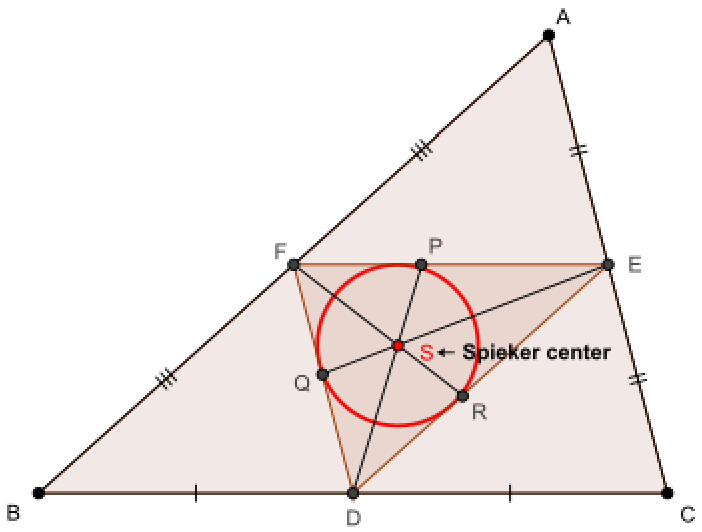

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum_{cyc}\left(\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)\right) - \sum_{cyc}\frac{a^4}{16} = \frac{1}{16}\left(2\sum_{cyc}a^2b^2 - \sum_{cyc}a^4\right)$$

$$= \frac{16r^2s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \boxed{\overset{(i)}{=}} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\boxed{\overset{(*)}{=}}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\ \boxed{\overset{(**)}{=}}\ \frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\ \boxed{\overset{(ii)}{=}}\ \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)$$

Now, $b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$

$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$

$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$

$$= (2s+a)\left(b^2-bc+c^2\right)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4} =$$

$$(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}-\frac{a(b-c)^2}{4}$$

$$(a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore b^3+c^3-abc+a\left(4m_a^2\right) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right) \Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Also, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$ *and* $b^2(s-b)+c^2(s-c) = ag_a^2+a(s-b)(s-c)$ *and via summation, we get* :

$$\left(b^2+c^2\right)(2s-b-c) = an_a^2+ag_a^2+2a(s-b)(s-c) \Rightarrow 2a\left(b^2+c^2\right) = 2a\left(n_a^2+g_a^2\right)+a(a+b-c)(c+a-b) \Rightarrow 2\left(b^2+c^2\right) = 2\left(n_a^2+g_a^2\right)+a^2-(b-c)^2 \Rightarrow 2\left(b^2+c^2\right)-a^2+(b-c)^2 = 2\left(n_a^2+g_a^2\right) \Rightarrow 4m_a^2+(b-c)^2$$

$$= 2\left(n_a^2+g_a^2\right) \Rightarrow 2(b-c)^2+4s(s-a) = 2\left(n_a^2+g_a^2\right)$$

$$\Rightarrow n_a^2+g_a^2 \overset{(*)}{=} (b-c)^2+2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2+a(s-b)(s-c)$

$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a) = an_a^2+a\left(s^2-s(2s-a)+bc\right) \Rightarrow s\left(b^2+c^2\right)-2sbc$$

$$= an_a^2+a\left(as-s^2\right) \Rightarrow s\left(b^2+c^2-a^2-2bc\right) = an_a^2-as^2 \Rightarrow an_a^2 = as^2+$$

$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)}{bc}$$
$$=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2\overset{\circledast\circledast}{=}s(s-a)+\frac{s}{a}(b-c)^2$$
Via $\circledast$ and $\circledast\circledast$, we get: $g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)=(s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\Rightarrow g_a^2\overset{\circledast\circledast\circledast}{=}s(s-a)-\frac{s-a}{a}(b-c)^2$$
$\therefore$ via $(\bullet\bullet\bullet)$, $\circledast\circledast$ and $\circledast\circledast\circledast$, $w_a^2+m_a^2+p_a^2-h_a^2-g_a^2-n_a^2$
$$=s(s-a)-\frac{s(s-a)}{(2s-a)^2}.(b-c)^2+s(s-a)+\frac{(b-c)^2}{4}+s(s-a)$$
$$+\frac{s(3s+a)}{(2s+a)^2}.(b-c)^2-s(s-a)+\frac{s(s-a)}{a^2}.(b-c)^2+\frac{s-a}{a}.(b-c)^2-s(s-a)$$
$$-\frac{s}{a}.(b-c)^2\Rightarrow w_a^2+m_a^2+p_a^2-h_a^2-g_a^2-n_a^2$$
$$\overset{(\blacksquare)}{=}\frac{64s^6-64s^5a-48s^4a^2+36s^2a^4+4sa^5-3a^6}{4a^2(4s^2-a^2)^2}.(b-c)^2$$
Now, $r_a=3r\Rightarrow\frac{rs}{s-a}=3r\Rightarrow 2s\overset{(\blacksquare\blacksquare)}{=}3a\Rightarrow 2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$
$$=3.4R\sin\frac{A}{2}\cos\frac{A}{2}\Rightarrow\sin\frac{A}{2}+\cos\frac{B-C}{2}=3\sin\frac{A}{2}$$
$$\Rightarrow 2S\overset{(\blacksquare\blacksquare\blacksquare)}{=}C\ \left(S=\sin\frac{A}{2},C=\cos\frac{B-C}{2}\right)$$
Via $(\blacksquare)$ and $(\blacksquare\blacksquare)$, we have : $w_a^2+m_a^2+p_a^2-h_a^2-g_a^2-n_a^2=$
$$\frac{729a^6-486a^6-243a^6+81a^6+6a^6-3a^6}{4a^2(9a^2-a^2)^2}.16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$
$$=\frac{21}{4}.R^2S^2(1-C^2)$$ $\therefore$ via $(\blacksquare\blacksquare\blacksquare)$, we have :
$$w_a^2+m_a^2+p_a^2-h_a^2-g_a^2-n_a^2\overset{\circledast}{=}\frac{21}{16}.R^2C^2(1-C^2)$$
Finally, $\frac{21r(R-2r)}{8}=\frac{21.4R^2.\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.(1-4SC+4S^2)}{8}$
$$=\frac{21.2R^2.S(C-S).(1-4SC+4S^2)}{8}\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}\frac{21.2R^2.\frac{C}{2}\left(C-\frac{C}{2}\right).(1-2C^2+C^2)}{8}$$
$$\Rightarrow\frac{21r(R-2r)}{8}\overset{\circledast\circledast}{=}\frac{21}{16}.R^2C^2(1-C^2)$$ $\therefore$ $\circledast$ and $\circledast\circledast$
$$\Rightarrow w_a^2+m_a^2+p_a^2=h_a^2+g_a^2+n_a^2+\frac{21r(R-2r)}{8}\ \forall\,\Delta ABC\text{ with }r_a=3r\ (QED)$$