ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\frac{a}{2R+r-r_a}=\prod\frac{a}{2R+r-r_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\sum ar_a=F\sum\frac{a}{s-a}=\frac{F}{\prod(s-a)}\left(\sum\left(a(s-b)(s-c)\right)\right)=$$
$$=\frac{F}{sr^2}\left(s^2(a+b+c)-2s(ab+bc+ca)+3abc\right)$$
$$=\frac{r.s}{sr^2}\left(2s^3-2s(s^2+r^2+4Rr)+12Rrs\right)=\frac{2s(2Rr-r^2)}{r}=2s(2R-r)\ (1)$$

$$\sum a(r_b+r_c)=\left(\sum a\right)\left(\sum r_a\right)-\sum ar_a=$$
$$=2s(4R+r)-2s(2R-r)=2s(2R+2r)\ (2)$$

$$\sum ar_br_c=F^2\sum\frac{a}{(s-b)(s-c)}=\frac{F^2}{\prod(s-a)}\sum a(s-a)=$$
$$=\frac{r^2s^2}{sr^2}\left(s(a+b+c)-\sum a^2\right)=s(2s^3-2(s^2-r^2-4Rr)=2s(4Rr+r^2)\ (3)$$

$$\sum a(2R+r-r_b)(2R+r-r_c)=$$
$$=\sum a(4R^2+4Rr-2R(r_b+r_c)+r^2-r(r_b+r_c)+r_br_c)=$$
$$=4R^2\sum a+4Rr\sum a-2R\sum a(r_b+r_c)+r^2\sum a-r\sum a(r_b+r_c)+\sum ar_br_c$$

$$\overset{using\ (2)\&(3)}{=}2s(2Rr)=4Rrs=abc$$

$$\sum\frac{a}{2R+r-r_a}=\frac{\sum a(2R+r-r_b)(2R+r-r_c)}{\prod(2R+r-r_a)}=\frac{abc}{\prod(2R+r-r_a)}=$$

$$=\frac{a}{2R+r-r_a}\cdot\frac{b}{2R+r-r_b}\frac{c}{2R+r-r_c}=\prod\frac{a}{2R+r-r_a}$$