ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$2\sum\frac{r_a+r_b}{h_c}=\sum\frac{h_a+h_b+r_a+r_b}{r_c}$$

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$$\textit{Known results}:$$

$$\sum a^2=2(s^2-r^2-4Rr)\ (1),\sum a^3=2s(s^2-3r^2-6Rr)\ (2),\sum a=2s\ (3)$$

$$2\sum\frac{r_a+r_b}{h_c}=2\sum\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{c}}=\sum\frac{c(2s-a-b)}{(s-a)(s-b)}=$$

$$=\sum\frac{c^2}{(s-a)(s-b)}=\frac{1}{\prod(s-a)}\sum c^2(s-c)=$$

$$=\frac{1}{sr^2}\left(s\sum c^2-\sum c^3\right)\overset{(1),(2),(3)}{=}\frac{2s}{sr^2}2r(R+r)=\frac{4(R+r)}{r}\ (A)$$

$$\sum\frac{h_a+h_b+r_a+r_b}{r_c}=\sum\frac{\frac{2F}{a}+\frac{2F}{b}+\frac{F}{s-a}+\frac{F}{s-b}}{\frac{F}{s-c}}=$$

$$=\sum\left(\frac{2}{a}+\frac{2}{b}+\frac{c}{(s-a)(s-b)}\right)(s-c)=2\sum\frac{(a+b)c(s-c)}{abc}+\sum\frac{c(s-c)^2}{\prod(s-a)}=$$

$$=2\sum\frac{(2s-c)c(s-c)}{4Rrs}+\sum\frac{c(s^2-2sc+c^2)}{sr^2}=$$

$$=2\sum\frac{(2s^2c-3sc^2+c^3)}{4Rrs}+\sum\frac{(s^2c-2sc^2+c^3)}{sr^2}=$$

$$=\frac{2}{4Rrs}\left(2s^2\sum c-3s\sum c^2+\sum c^3\right)+\frac{1}{sr^2}\left(s^2\sum c-2s\sum c^2+\sum c^3\right)=$$

$$\overset{(1),(2),(3)}{=}\frac{2}{4Rrs}(12Rrs)+\frac{1}{sr^2}\left(s(4Rr-2r^2)\right)=6+\frac{4R-2r}{r}=\frac{4(R+r)}{r}\ \ (B)$$

$$\textit{From }(A)\&(B)\textit{we get }2\sum\frac{r_a+r_b}{h_c}=\sum\frac{h_a+h_b+r_a+r_b}{r_c}$$