

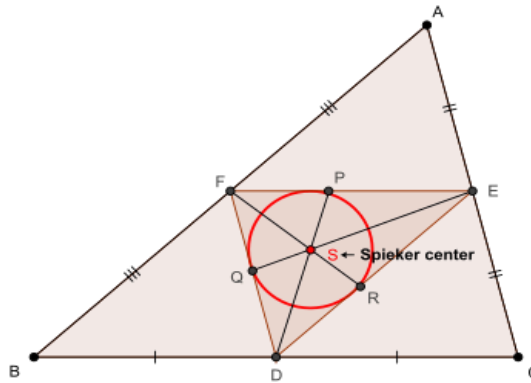
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In any ΔABC , the following relationship holds :

$$s \sum_{\text{cyc}} \frac{\sqrt{4r^2 + (b-c)^2}}{p_a} \geq \sum_{\text{cyc}} \left(a \sqrt{\frac{w_a}{g_a}} \right)$$

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Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
and inradius of $\Delta DEF = r'$ (say)

$$\begin{aligned} \text{Now, } 16[DEF]^2 &= 2 \sum_{\text{cyc}} \left(\left(\frac{a^2}{4} \right) \left(\frac{b^2}{4} \right) \right) - \sum_{\text{cyc}} \frac{a^4}{16} = \frac{1}{16} \left(2 \sum_{\text{cyc}} a^2 b^2 - \sum_{\text{cyc}} a^4 \right) \\ &= \frac{16r^2 s^2}{16} \Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2} \right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1) \end{aligned}$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\angle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A - C}{2} \\ \Rightarrow 2AS^2 &\stackrel{(i)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} \\ &\quad - \left(\frac{2r}{2 \sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A - C}{2} \end{aligned}$$

$$\begin{aligned}
 & \text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 &= \frac{r}{2} \left(4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 &= Rr \left(2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 &= Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right) \\
 &= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 &= \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 &= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 &= \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 &\Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 &\quad \boxed{(*)} = \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 & \text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 &= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \boxed{(**)} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 & \text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 &= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s} \\
 &= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \boxed{(ii)} = \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 & \text{Via sine law on } \triangle AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}} \\
 &\Rightarrow c\sin\alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{****}{=} \frac{r(a+c)}{2AS} \\
 & \text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 & \quad \text{via (***) and ****)} \Rightarrow \frac{p_a(a+b+a+c)}{2} = s \Rightarrow p_a = \frac{4s}{2s+a} AS \\
 & \Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s} \\
 & \quad \therefore p_a^2 \boxed{(\odot)} = \frac{2s}{(2s+a)^2} (b^3+c^3-abc+a(4m_a^2)) \\
 & \text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)
 \end{aligned}$$

$$\begin{aligned}
 &= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2) \\
 &= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2) \\
 &= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right) \\
 &= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\
 &= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\
 &= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4} = \\
 &\quad 4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + \\
 (2s+a) \cdot \frac{(y+z)((z+x) + (x+y) - 2(y+z))}{4} - \frac{a(b-c)^2}{4} \\
 &\quad (a = y+z, b = z+x, c = x+y) \\
 &= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4} \\
 &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4} \\
 &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4} \\
 &= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2} \\
 \therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\bullet\bullet)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2} \\
 \therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right) \\
 &= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right) \\
 &= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2 \\
 &= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right) \Rightarrow p_a^2 \stackrel{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \\
 \text{Now, } m_a n_a \stackrel{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \stackrel{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \\
 \left(s(s-a) + \frac{(b-c)^2}{4} \right) \left(s(s-a) + \frac{s(b-c)^2}{a} \right) \stackrel{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right)^2 \\
 + \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} \cdot \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right) \\
 \Leftrightarrow s(s-a)(b-c)^2 \left(\frac{s}{a} + \frac{1}{4} \right) + \frac{s(b-c)^4}{4a} \stackrel{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} + \\
 2s(s-a) \cdot \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a) \cdot \frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2} \\
 \Leftrightarrow s(s-a) \left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9} \right) +
 \end{aligned}$$

$$\begin{aligned} & \left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2} \right) (b-c)^2 \stackrel{?}{\geq} 0 \quad (\because (b-c)^2 \geq 0) \\ & \Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} + \\ & \frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4} \cdot (b-c)^2 \stackrel{?}{\geq} 0 \\ & \Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} + \\ & \frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4} \cdot (b-c)^2 \end{aligned}$$

$$\stackrel{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow \textcircled{*}$$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$
and $b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$ and via summation, we get :

$$\begin{aligned} (b^2 + c^2)(2s - b - c) &= an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2 + c^2) = \\ & 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2) = 2(n_a^2 + g_a^2) + \\ a^2 - (b-c)^2 &\Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 \\ &= 2(n_a^2 + g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2 + g_a^2) \end{aligned}$$

$$\Rightarrow n_a^2 + g_a^2 \stackrel{\textcircled{(*)}}{=} (b-c)^2 + 2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$
 $\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc$
 $= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$

$$s(2bc \cos A - 2bc) = as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)}{bc}$$

$$= as^2 - as \left(\frac{a^2 - (b-c)^2}{a} \right) \Rightarrow n_a^2 = s \left(s - \frac{a^2 - (b-c)^2}{a} \right)$$

$$\Rightarrow n_a^2 \stackrel{\textcircled{(**)}}{=} s(s-a) + \frac{s}{a} (b-c)^2$$

Via $(*)$ and $(**)$, we get: $g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$

$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$

$$= (s-a)^2 + 4(s-b)(s-c) \left(\frac{s}{a} - 1 \right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$

$$= (s-a) \left(s-a + \frac{a^2 - (b-c)^2}{a} \right) \Rightarrow g_a^2 \stackrel{\textcircled{(***)}}{=} (s-a) \left(s - \frac{(b-c)^2}{a} \right)$$

$$\therefore (**), (***) \Rightarrow n_a^2 g_a^2 = s(s-a) \left(s-a + \frac{(b-c)^2}{a} \right) \left(s - \frac{(b-c)^2}{a} \right)$$

$$= s(s-a) \left(s(s-a) + s \frac{(b-c)^2}{a} - \frac{(b-c)^2}{a} (s-a) - \frac{(b-c)^4}{a^2} \right)$$

$$\Rightarrow n_a^2 g_a^2 \stackrel{\textcircled{(***)}}{=} s(s-a) \left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2} \right)$$

$$\begin{aligned}
 & \text{Again, } m_a^2 w_a^2 = \frac{(b-c)^2 + 4s(s-a)}{4} \cdot \frac{4bcs(s-a)}{(b+c)^2} \\
 & \Rightarrow m_a^2 w_a^2 \stackrel{(\bullet\bullet)}{=} s(s-a) \frac{bc}{(b+c)^2} \left((b-c)^2 + 4s(s-a) \right) \\
 & \quad \therefore (\bullet), (\bullet\bullet) \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2 \\
 & = s(s-a) \left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2} - \frac{bc}{(b+c)^2} \left((b-c)^2 + 4s(s-a) \right) \right) \\
 & = s(s-a) \left(\frac{s(s-a) + (b-c)^2 \left(\frac{a^2 - (b-c)^2}{a^2} \right)}{-\frac{bc}{(b+c)^2} \left((b-c)^2 + (b+c)^2 - a^2 \right)} \right) \\
 & = s(s-a) \left(s(s-a) - bc + (a^2 - (b-c)^2) \left(\frac{(b-c)^2}{a^2} + \frac{bc}{(b+c)^2} \right) \right) \\
 & = \frac{s(s-a)}{4} \left(((b+c)^2 - a^2 - 4bc) + (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} \right) \right) \\
 & = \frac{s(s-a)}{4} \left((b-c)^2 - a^2 + (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} \right) \right) \\
 & = \frac{s(s-a)}{4} (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} - 1 \right) \\
 & = \frac{s(s-a)}{4} \cdot 4(s-b)(s-c) \left(\frac{4(b-c)^2}{a^2} - \frac{(b-c)^2}{(b+c)^2} \right) = r^2 s^2 (b-c)^2 \left(\frac{4}{a^2} - \frac{1}{(b+c)^2} \right) \\
 & = r^2 s^2 (b-c)^2 \left(\frac{2}{a} + \frac{1}{b+c} \right) \left(\frac{2b+2c-a}{a(b+c)} \right) \geq 0 \\
 & \Rightarrow n_a^2 g_a^2 \geq m_a^2 w_a^2 \Rightarrow n_a g_a \geq m_a w_a \rightarrow (\ast)(\ast)
 \end{aligned}$$

Now, via (\ast) and $(\ast)(\ast)$, $m_a n_a \cdot n_a g_a \geq p_a^2 \cdot m_a w_a \Rightarrow \frac{n_a^2}{p_a^2} \geq \frac{w_a}{g_a} \Rightarrow \frac{n_a}{p_a} \geq \sqrt{\frac{w_a}{g_a}} \rightarrow (\ast)(\ast)(\ast)$

We have proved : $an_a^2 = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$$\Rightarrow a^2 n_a^2 = a^2 s^2 - sa(a^2 - (b-c)^2)$$

$$\Rightarrow a^2 n_a^2 = a^2 s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} 4r^2 s^2 + s^2 (b-c)^2$$

$$\Leftrightarrow a^2 s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} s(s-a)(a^2 - (b-c)^2) + s^2 (b-c)^2$$

$$= a^2 s^2 - sa^3 - s(s-a)(b-c)^2 + s^2 (b-c)^2$$

$$= a^2 s^2 - sa^3 - s^2 (b-c)^2 + sa(b-c)^2 + s^2 (b-c)^2$$

$$\Leftrightarrow a^2 s^2 - sa^3 + sa(b-c)^2 \stackrel{?}{=} a^2 s^2 - sa^3 + sa(b-c)^2 \rightarrow \text{true}$$

$$\therefore \frac{n_a^2}{(b-c)^2 + 4r^2} = \frac{s^2}{a^2} \Rightarrow \frac{s}{a} \cdot \frac{\sqrt{4r^2 + (b-c)^2}}{p_a} = \frac{n_a}{p_a} \stackrel{\text{via } (\ast)(\ast)(\ast)}{\geq} \sqrt{\frac{w_a}{g_a}}$$

$$\therefore s \cdot \frac{\sqrt{4r^2 + (b-c)^2}}{p_a} \geq a \cdot \sqrt{\frac{w_a}{g_a}} \text{ and analogs}$$

$$\Rightarrow s \sum_{\text{cyc}} \frac{\sqrt{4r^2 + (b-c)^2}}{p_a} \geq \sum_{\text{cyc}} \left(a \cdot \sqrt{\frac{w_a}{g_a}} \right) \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$