ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$2m_a \ge n_a + \frac{r_b r_c}{n_a} \ge 2\sqrt{r_b r_c}$$

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$$2m_a \overset{?}{\ge} n_a + \frac{r_b r_c}{n_a} \Leftrightarrow (2m_a n_a)^2 \overset{?}{\ge} \left(n_a^2 + s(s-a)\right)^2$$

$$\Leftrightarrow (4s(s-a) + (b-c)^2)\left(s(s-a) + \frac{s}{a}(b-c)^2\right) \overset{?}{\ge} \left(2s(s-a) + \frac{s}{a}(b-c)^2\right)^2$$

$$\left(\begin{array}{c} \because n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2; \\ \text{Reference : ''About Nagel and Gergonne cevians } - \text{Bogdan Fustei''} \end{array}\right)$$

$$\Leftrightarrow 4s^2(s-a)^2 + \frac{4s^2(s-a)}{a}(b-c)^2 + s(s-a)(b-c)^2 + \frac{s}{a}(b-c)^4 \overset{?}{\ge}$$

$$4s^2(s-a)^2 + \frac{4s^2(s-a)}{a}(b-c)^2 + \frac{s^2}{a^2}(b-c)^4$$

$$\Leftrightarrow (b-c)^2\left(s(s-a) - \frac{s(s-a)}{a^2}(b-c)^2\right) \overset{?}{\ge} 0$$

$$\Leftrightarrow (b-c)^2\frac{s(s-a)}{a^2}.4(s-b)(s-c) \overset{?}{\ge} 0 \to \text{true} \therefore 2m_a \ge n_a + \frac{r_b r_c}{n_a} \text{ and}$$

$$n_a + \frac{r_b r_c}{n_a} \overset{?}{\ge} 2\sqrt{r_b r_c} \Leftrightarrow n_a^2 + s(s-a) \overset{?}{\ge} 2\sqrt{r_b r_c}.n_a \to \text{true via AM} - \text{GM}$$

$$\therefore n_a + \frac{r_b r_c}{n_a} \ge 2\sqrt{r_b r_c} \text{ and so, } 2m_a \ge n_a + \frac{r_b r_c}{n_a} \ge 2\sqrt{r_b r_c}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$