ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{R-r-\sqrt{R(R-2r)}}{r}\le\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{r_a}{h_a}+1=\frac{rpa}{2rp(p-a)}+1\ (p\to\text{semi}-\text{perimeter})=\frac{a}{b+c-a}+1$$

$$=\frac{b+c}{2(p-a)}=\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\Rightarrow\frac{r_a}{h_a}+1\overset{①}{=}\frac{c}{c-s}$$

$$\left(\text{where } c=\cos\frac{B-C}{2},s=\sin\frac{A}{2}\right)$$

$$\text{Again},\frac{R\pm\sqrt{R(R-2r)}}{r}=\frac{1}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{R.\sqrt{1-4sc+4s^2}}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore\frac{R\pm\sqrt{R(R-2r)}}{r}\overset{②}{=}\frac{1}{2s(c-s)}\pm\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}$$

$$\text{Now, ① and ②}\Rightarrow\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}\Leftrightarrow$$

$$\frac{c}{c-s}\le\frac{1}{2s(c-s)}+\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow\frac{2sc-1}{2s(c-s)}\le\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow$$

$$2sc-1\overset{(*)}{\le}\sqrt{1-4sc+4s^2}\left(\because c-s=\frac{b+c}{a}.\sin\frac{A}{2}-\sin\frac{A}{2}=\frac{b+c-a}{a}.\sin\frac{A}{2}>0\right)$$

and if $2sc-1<0$, then : $(*)$ is trivially true and so, we now focus on :

$2sc-1\ge0$ and then : $(*)\Leftrightarrow4s^2c^2-4sc+1\le1-4sc+4s^2$

$$\Leftrightarrow4s^2(c^2-1)\le0\to\text{true}\because c^2=\cos^2\frac{B-C}{2}\le1\Rightarrow(*)\text{ is true }\forall\,\Delta\,ABC$$

$$\therefore\frac{r_a}{h_a}\le\frac{R-r+\sqrt{R(R-2r)}}{r}$$

$$\text{Also, ① and ②}\Rightarrow\frac{r_a}{h_a}\ge\frac{R-r-\sqrt{R(R-2r)}}{r}\Leftrightarrow$$

$$\frac{c}{c-s}\ge\frac{1}{2s(c-s)}-\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\Leftrightarrow\frac{\sqrt{1-4sc+4s^2}}{2s(c-s)}\ge\frac{1-2sc}{2s(c-s)}$$

$\Leftrightarrow\sqrt{1-4sc+4s^2}\overset{(**)}{\ge}1-2sc$ and if $1-2sc<0$, then : $(**)$ is trivially true and so, we now focus on : $1-2sc\ge0$ and then : $(**)\Leftrightarrow$

$$1-4sc+4s^2\ge4s^2c^2-4sc+1\Leftrightarrow4s^2(1-c^2)\ge0$$

$$\rightarrow \text{true} \because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (**) \text{ is true } \forall \Delta ABC$$

$$\therefore \frac{r_a}{h_a} \ge \frac{R - r - \sqrt{R(R-2r)}}{r} \text{ and hence,}$$

$$\frac{R - r - \sqrt{R(R-2r)}}{r} \le \frac{r_a}{h_a} \le \frac{R - r + \sqrt{R(R-2r)}}{r} \ \forall \Delta ABC \ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We have } \frac{r_a}{h_a} = \frac{a}{2(s-a)} = \frac{1}{2}\left(\frac{s}{s-a} - 1\right)$$

$$= \frac{1}{2}\left(\frac{r_a}{r} - 1\right), \text{ so the desired inequality is equivalent to}$$

$$2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)}.$$

We have

$$4 \le (r_b + r_c)\left(\frac{1}{r_b} + \frac{1}{r_c}\right) = (4R + r - r_a)\left(\frac{1}{r} - \frac{1}{r_a}\right) \Leftrightarrow 4rr_a \le (4R + r - r_a)(r_a - r)$$

$$\Leftrightarrow {r_a}^2 - 2(2R - r)r_a + r(4R + r) \le 0$$

$$\Leftrightarrow \left(r_a - 2R + r + 2\sqrt{R(R-2r)}\right)\left(r_a - 2R + r - 2\sqrt{R(R-2r)}\right) \le 0$$

$$\Leftrightarrow 2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)}.$$

which completes the proof. Equality holds iff $b = c$.