ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $\widehat{A} \geq 90^\circ$, the following relationship holds :

$$m_a h_a \geq w_a^2$$

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$$m_a h_a \geq w_a^2 \Leftrightarrow \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\geq \left(s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}\right)^2$$

$$\Leftrightarrow -\frac{s^2(s-a)^2(b-c)^2}{a^2}+s(s-a).\frac{(b-c)^2}{4}-\frac{s(s-a)(b-c)^4}{4a^2}$$

$$\geq -\frac{2s^2(s-a)^2(b-c)^2}{(b+c)^2}+\frac{s^2(s-a)^2(b-c)^4}{(b+c)^4} \text{ and } \because (b-c)^2 \geq 0 \therefore \text{ it suffices}$$

$$\text{to prove : } \frac{1}{4}-\frac{s(s-a)}{a^2}-\frac{(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^2}\left(2-\frac{(b-c)^2}{(b+c)^2}\right)>0$$

$$\Leftrightarrow \frac{a^2-(b+c)^2+a^2-(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc)>0$$

$$\text{and } \because a^2 \geq b^2+c^2 \therefore \text{ it suffices to prove :}$$

$$\frac{2(b^2+c^2)-(b+c)^2-(b-c)^2}{4a^2}+\frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc)>0$$

$$\Leftrightarrow \frac{s(s-a)}{(b+c)^4}.(b^2+c^2+6bc)>0 \to \text{true} \therefore m_a h_a \geq w_a^2 \ \forall \ \Delta \text{ ABC with } \widehat{A} \geq 90^\circ,$$

$$'' = '' \text{ iff } \widehat{B} = \widehat{C} < 45^\circ \text{ (QED)}$$