ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$r_a + r = \frac{rp}{p-a} + \frac{rp}{p} = \frac{r(b+c)}{p-a} \ (p \to \text{semi} - \text{perimeter})$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\therefore r_a + r \overset{①}{=} 4Rsc \left(\text{where } c = \cos\frac{B-C}{2}, s = \sin\frac{A}{2}\right)$$

$$\text{Again, } 2R \pm 2\sqrt{R(R-2r)} \overset{②}{=} 2R \pm 2R.\sqrt{1-4sc+4s^2}$$

Now, ① and ② $\Rightarrow r_a \le 2R - r + 2\sqrt{R(R-2r)}$

$\Leftrightarrow 4Rsc \le 2R + 2R.\sqrt{1-4sc+4s^2} \Leftrightarrow 2sc - 1 \overset{(*)}{\le} \sqrt{1-4sc+4s^2}$ and if $2sc - 1 < 0$, then : $(*)$ is trivially true and so, we now focus on :

$$2sc - 1 \ge 0$$

and then : $(*) \Leftrightarrow 4s^2c^2 - 4sc + 1 \le 1 - 4sc + 4s^2 \Leftrightarrow 4s^2(c^2-1) \le 0 \to$ true

$$\because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*) \text{ is true } \forall \Delta ABC \therefore r_a \le 2R - r + 2\sqrt{R(R-2r)}$$

Also, ① and ② $\Rightarrow r_a \ge 2R - r - 2\sqrt{R(R-2r)}$

$\Leftrightarrow 4Rsc \ge 2R - 2R.\sqrt{1-4sc+4s^2} \Leftrightarrow \sqrt{1-4sc+4s^2} \overset{(**)}{\ge} 1 - 2sc$ and if $1 - 2sc < 0$, then : $(**)$ is trivially true and so, we now focus on : $1 - 2sc \ge 0$

and then : $(**) \Leftrightarrow 1 - 4sc + 4s^2 \ge 4s^2c^2 - 4sc + 1 \Leftrightarrow 4s^2(1-c^2) \ge 0 \to$ true

$$\because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (**) \text{ is true } \forall \Delta ABC \therefore r_a \ge 2R - r - 2\sqrt{R(R-2r)}$$

and hence

$$2R - r - 2\sqrt{R(R-2r)} \le r_a \le 2R - r + 2\sqrt{R(R-2r)} \ \forall \Delta ABC \ (QED)$$

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Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$4 \leq (r_b + r_c)\left(\frac{1}{r_b} + \frac{1}{r_c}\right) = (4R + r - r_a)\left(\frac{1}{r} - \frac{1}{r_a}\right) \Leftrightarrow 4rr_a \leq (4R + r - r_a)(r_a - r)$$

$$\Leftrightarrow {r_a}^2 - 2(2R - r)r_a + r(4R + r) \leq 0$$

$$\Leftrightarrow \left(r_a - 2R + r + 2\sqrt{R(R - 2r)}\right)\left(r_a - 2R + r - 2\sqrt{R(R - 2r)}\right) \leq 0$$

$$\Leftrightarrow 2R - r - 2\sqrt{R(R - 2r)} \leq r_a \leq 2R - r + 2\sqrt{R(R - 2r)}.$$

Equality holds iff $b = c$.